Outer automorphisms of free Burnside groups

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Abstract. In this paper, we study some properties of the outer automorphism group of free Burnside groups of large odd exponent. In particular, we prove that it contains free and free abelian subgroups.

Introduction

The free Burnside group of rank r and exponent n, denoted by $\mathbf{B}_r(n)$, is the quotient of the free group $\mathbf{F}_r$ by the subgroup $\mathbf{F}_r^n$ generated by the n-th powers of all its elements. In 1902, W. Burnside asked whether $\mathbf{B}_r(n)$ had to be finite of not (see [9]). For a long time, one only knew that for some small exponents $\mathbf{B}_r(n)$ was finite (for $n = 2$ see [9], $n = 3$ [9] and [22], $n = 4$ [31] and $n = 6$ [21]). In 1968, P. S. Novikov and S. I. Adian achieved a breakthrough (see [25], [26] and [27]). Using the small cancellation theory developed by V. A. Tartakovskiĭ [33] and M. Greendlinger [17], [18], [19], they proved that for large odd exponents, $\mathbf{B}_r(n)$ is infinite. Thanks to a diagrammatic formulation of small cancellation, A. Y. Ol'shanskiĭ simplified the proof of P. S. Novikov and S. I. Adian [28]. Recently, T. Delzant and M. Gromov gave a more geometrical proof of the same theorem [15]. These results not only provide examples of infinite Burnside groups, they also help to study many of their properties (solution for the word-problem, description of finite subgroups,...). Other information about the history of the Burnside problems can be found in [20].

The next step to understand Burnside groups is to study their automorphisms. In this paper, we are interested in the following questions. What kind of outer automorphisms of $\mathbf{B}_r(n)$ have infinite order? Does $\mathrm{Out}(\mathbf{B}_r(n))$ contain relevant subgroups like free groups or free abelian groups? To that end, we focus on the canonical map $\mathrm{Out}(\mathbf{F}_r) \to \mathrm{Out}(\mathbf{B}_r(n))$.

Using the work of P. S. Novikov and S. I. Adian, E. A. Cherepanov proved that the automorphism φ of $\mathbf{F}_2 = \mathbf{F}(a, b)$, defined by $\varphi(a) = ab$ and $\varphi(b) = a$, induces

an outer automorphism of infinite order of $\mathbf{B}_r(n)$ (see [10] and Proposition 1.1). Our first theorem provides a large class of automorphisms of the free group having the same property.

Theorem 1 (see Theorem 1.3). *Let φ be an automorphism of $\mathbf{F}_r$. Assume that φ is hyperbolic, i.e. the semi-direct product $\mathbf{F}_r \rtimes_\varphi \mathbb{Z}$ defined by φ is a hyperbolic group. There exists an integer n_0 such that for all odd integers n larger than n_0, φ induces an outer automorphism of infinite order of $\mathbf{B}_r(n)$.*

All proofs dealing with free Burnside groups have the same weakness: they involve a presentation of $\mathbf{B}_r(n)$ which is not stable under automorphisms. Our work tries to regain a little symmetry: we build a sequence of groups (H_k) such that φ induces an automorphism of H_k for all k and $\varinjlim H_k = \mathbf{B}_r(n)$. To that end, we start with $H_0 = \mathbf{F}_r$ and, at each step, we construct H_{k+1} as a small cancellation quotient of H_k. Some difficulties appear during this process. Assume that ρ is one of the relations defining the first quotient $\mathbf{F}_r = H_0 \twoheadrightarrow H_1$. Since we want φ to induce an automorphism of H_1, the elements $\varphi^m(\rho)$ for all $m \in \mathbb{N}$ have to belong to the set of relations. However the small cancellation theory only deals with relations having more or less the same length. In our case, the relations $\varphi^m(\rho)$ may have very different lengths. To avoid this problem, we encode the information concerning the automorphism in a larger group: $\mathbf{F}_r \rtimes_\varphi \mathbb{Z}$. Thus the elements $\varphi^m(\rho)$ become conjugates of ρ and do not need to be added to the set of relations. We shall now use the fact that the group $\mathbf{F}_r \rtimes \mathbb{Z}$ is hyperbolic. In 1991, A.Y. Ol'shanskiĭ provided a generalisation of the Novikov–Adian theorem (see [29]). Given a torsion-free, hyperbolic group G, he proved that for large odd exponent n the quotient G/G^n is infinite. This result was recovered by T. Delzant and M. Gromov in [15]. We would like to apply the same techniques to $G = \mathbf{F}_r \rtimes \mathbb{Z}$. However we must take care not to kill all n-th powers of G. Indeed, if we did so, the automorphism obtained at the end of the construction would have finite order dividing n. That is why we propose an extension of the Delzant–Gromov construction where the relations are chosen in a normal subgroup of $\mathbf{F}_r \rtimes \mathbb{Z}$. This construction works in a more general situation. It leads to our main theorem:

Main theorem. *Let $1 \to H \to G \to F \to 1$ be a short exact sequence of groups. Assume that H is non-trivial, finitely generated, G is hyperbolic, non-elementary, torsion-free and F is torsion-free. There exists an integer n_0 such that for all odd integers n larger than n_0, the canonical map $F \to \mathrm{Out}(H)$ induces an injective homomorphism $F \hookrightarrow \mathrm{Out}(H/H^n)$.*

Theorem 1 is obtained by applying the main theorem to the short exact sequence $1 \to \mathbf{F}_r \to \mathbf{F}_r \rtimes \mathbb{Z} \to \mathbb{Z} \to 1$. The work of M. Bestvina, M. Feighn and M. Handel (see [5]) provides examples of hyperbolic extensions of free groups by free groups. Using this result, we obtain our second theorem.

Theorem 2 (see Theorem 1.8). *Let $r \geqslant 3$. There exists an integer n_0 such that for all odd integers n larger than n_0, the group $\mathrm{Out}\,(\mathbf{B}_r(n))$ contains a subgroup which is isomorphic to $\mathbf{F}_2$.*

The strategy to embed abelian subgroups in $\mathrm{Out}\,(\mathbf{B}_r(n))$ is a little different. We do not apply the main theorem to an appropriate hyperbolic extension of the free group. We construct a family of automorphisms of $\mathbf{F}_r$ which already commute in $\mathrm{Aut}\,(\mathbf{F}_r)$ and check "by hand" that they do not satisfy any other relation in $\mathrm{Out}\,(\mathbf{B}_r(n))$. This yields the following result.

Theorem 3 (see Theorem 1.10). *Let $r \geqslant 1$. There exists an integer n_0 such that for all odd integers n larger than n_0, the groups $\mathrm{Out}\,(\mathbf{B}_{2r}(n))$ and $\mathrm{Out}\,(\mathbf{B}_{2r+1}(n))$ contain a subgroup which is isomorphic to $\mathbb{Z}^r$.*

Hyperbolic automorphisms induce automorphisms of infinite order of free Burnside groups of large exponent. But they are not the only ones. For instance, the automorphism φ, studied by E. A. Cherepanov and characterized by $\varphi(a) = ab$ and $\varphi(b) = a$ is not hyperbolic. Indeed, φ^2 fixes the commutator $[a^{-1}, b^{-1}]$. The semi-direct product $\mathbf{F}_r \rtimes_\varphi \mathbb{Z}$ contains therefore a subgroup which is isomorphic to $\mathbb{Z}^2$. We wonder if there exists a criterion to decide whether an automorphism of $\mathbf{F}_r$ induces an outer automorphism of infinite order of $\mathbf{B}_r(n)$ for some large exponent or not. In particular, is there a link between this property and the growth of the automorphism? Section 1.2 gives a partial answer. We prove that a polynomially growing automorphism always induces an automorphism of finite order of $\mathbf{B}_r(n)$.

Outline of the paper.   In Section 1 we explain the consequences of the main theorem. In particular, we provide examples of automorphisms of infinite order of $\mathbf{B}_r(n)$. We also construct free and free abelian subgroups of $\mathrm{Out}\,(\mathbf{B}_r(n))$. Section 2 deals with the proof of the main theorem. To begin with, we recall the geometrical point of view on the small cancellation theory developed by T. Delzant and M. Gromov. We also improve some results of [15] which are necessary to control the small cancellation parameters in our situation. Then, we prove an induction lemma (Lemma 2.16) which is the fundamental step of the induction process used in Section 3 to prove the main theorem.

1. Automorphisms of Burnside groups

Remark. In this paper, we are interested in outer automorphisms of free Burnside groups. One question still unanswered is the following: given an automorphism of the free group $\mathbf{F}_r$, does it induce an automorphism of infinite order of $\mathbf{B}_r(n)$? Note that every element of $\mathbf{B}_r(n)$ has finite order. In particular, every inner automorphism of $\mathbf{B}_r(n)$ has finite order. It follows that an element of Aut $(\mathbf{B}_r(n))$ has finite order if and only if so has its image in Out $(\mathbf{B}_r(n))$.

1.1. Examples of automorphisms of infinite order. Using the work of P. S. Novikov and S. I. Adian (see [25], [26] and [27]), we exhibit a first example of outer automorphism of infinite order of $\mathbf{B}_r(n)$. This example was already studied by E. A. Cherepanov in [10].

Proposition 1.1 (see [10], Theorem 1). *Let $\{a, b\}$ be a generating set of the free group $\mathbf{F}_2$. Let φ be the automorphism of $\mathbf{F}_2$ defined by $\varphi(a) = ab$ and $\varphi(b) = a$. There exists an integer n_0 such that for all odd integers n larger than n_0, φ induces an automorphism of infinite order of $\mathbf{B}_2(n)$.*

Proof. We consider the sequence of iterated images of a by φ.

$$\begin{aligned}
\varphi^0(a) &= a, & \varphi^4(a) &= abaababa, \\
\varphi^1(a) &= ab, & \varphi^5(a) &= abaababaabaab, \\
\varphi^2(a) &= aba, & \varphi^6(a) &= abaababaabaababaababa, \\
\varphi^3(a) &= abaab, & & \ldots.
\end{aligned}$$

This sequence converges to a right infinite positive word

$$\varphi^\infty(a) = abaababaabaababaababa \ldots$$

which has the following property. For every word u in $\{a, b\}$, u^4 is not a subword of $\varphi^\infty(a)$ (see [24]). Let n be an odd integer larger than 10 000. In order to prove that the free Burnside group of large exponent is infinite, P. S. Novikov and S .I. Adian use the following fact: if m is a non-trivial reduced word in $\{a, b\}$ which does not contain a subword that equals a fourth power, then m defines a non-trivial element of $\mathbf{B}_2(n)$ (see [1], IV. 2.16, or Statement 1 in [2]). In particular, $(\varphi^p(a))$ induces a sequence of pairwise distinct elements of $\mathbf{B}_2(n)$. It follows that φ induces an automorphism of infinite order of $\mathbf{B}_2(n)$. $\qquad\square$

We now wish to investigate a large class of automorphisms of free groups: the hyperbolic ones. We prove that they induce automorphisms of infinite order of free Burnside groups.

Definition 1.2. Let G be a hyperbolic group. An automorphism φ of G is hyperbolic if the semi-direct product $G \rtimes_\varphi \mathbb{Z}$ defined by φ is hyperbolic.

Example. Let Σ be the fundamental group of a compact surface S of genus larger than 2. Thanks to Thurston's hyperbolisation Theorem, every pseudo-Anosov homeomorphism of S induces a hyperbolic automorphism of Σ (see [30]).

There exist many characterizations of hyperbolic automorphisms. Let us endow G with the word metric $|\,.\,|$ relative to a generating set. M. Bestvina and M. Feighn proved in [3] that an automorphism φ of G is hyperbolic if and only if there exist $\lambda > 1$ and $m \in \mathbb{N}$ such that for all $g \in G$

$$\lambda \, |g| \leqslant \max \left\{ |\varphi^m(g)| \,, |\varphi^{-m}(g)| \right\}.$$

On the other hand, an automorphism of a free group is hyperbolic, if and only if it has no non-trivial periodic conjugacy classes (see [5] and [8]). Note that the automorphism φ studied in Proposition 1.1 is not hyperbolic: φ^2 fixes the commutator $[a^{-1}; b^{-1}]$. More generally, Aut $(\mathbf{F}_2)$ does not contain hyperbolic elements. Every automorphism φ of $\mathbf{F}_2$ is indeed induced by a homeomorphism of the punctured torus. Therefore φ^2 has to fix the conjugacy class of $\mathbf{F}_2$ corresponding to the boundary of the torus.

Theorem 1.3. *Let $r \geqslant 3$. Let φ be a hyperbolic automorphism of $\mathbf{F}_r$. There exists an integer n_0 such that for all odd integers n larger than n_0, φ induces an outer automorphism of infinite order of $\mathbf{B}_r(n)$.*

Proof. By definition, the group $\mathbf{F}_r \rtimes_\varphi \mathbb{Z}$ is hyperbolic. It follows that the short exact sequence $1 \to \mathbf{F}_r \to \mathbf{F}_r \rtimes_\varphi \mathbb{Z} \to \mathbb{Z} \to 1$ satisfies the assumptions of the main theorem (see page 790). Thus there exists an integer n_0 such that for all odd integers n larger than n_0, the map $\mathbb{Z} \to \text{Out} \, (\mathbf{F}_r)$ induces an injective homomorphism $\mathbb{Z} \hookrightarrow \text{Out} \, (\mathbf{B}_r(n))$. However, the morphism $\mathbb{Z} \to \text{Out} \, (\mathbf{F}_r)$ is by construction the one that maps m to the outer automorphism induced by φ^m. Consequently, φ induces an outer automorphism of infinite order of $\mathbf{B}_r(n)$. $\qquad\square$

1.2. Polynomially growing automorphisms of free groups. We provide now examples of automorphisms of infinite order of $\mathbf{F}_r$ which induce automorphisms of finite order of $\mathbf{B}_r(n)$. Given a conjugacy class x of $\mathbf{F}_r$, we denote by $[x]$ the length of any cyclically reduced word representing x. Let Φ be an outer automorphism of $\mathbf{F}_r$. We look at the action of Φ on the conjugacy classes of $\mathbf{F}_r$.

Definition 1.4. The automorphism Φ grows polynomially if for every conjugacy class x of $\mathbf{F}_r$, the sequence $([\Phi^p(x)])$ grows polynomially.

Proposition 1.5 (see [23]). *Let Φ be a polynomially growing outer automorphism of $\mathbf{F}_r$. Up to replacement of Φ with a power of Φ, one of the following assertion holds.*

(i) *There exist $\varphi \in \mathrm{Aut}\,(\mathbf{F}_r)$ representing Φ and a non-trivial free decomposition $F_1 * F_2$ of $\mathbf{F}_r$ which is invariant under φ.*

(ii) *There exist $\varphi \in \mathrm{Aut}\,(\mathbf{F}_r)$ representing Φ, a non-trivial free decomposition $F_1 * \langle t \rangle$ of $\mathbf{F}_r$ and an element f of F_1 such that F_1 is invariant under φ and $\varphi(t) = tf$.*

Theorem 1.6. *Let $r \geqslant 1$. Let Φ be a polynomially growing outer automorphism of $\mathbf{F}_r$. For all positive integers n, Φ induces an outer automorphism of finite order of $\mathbf{B}_r(n)$.*

Proof. The proof is by induction on the rank r of the free group. The outer automorphism group of $\mathbb{Z}$ is reduced to $\{\mathrm{id}, -\mathrm{id}\}$. Hence the theorem holds for rank one. Let $r \geqslant 1$. Assume that the theorem holds for any rank smaller or equal to r. Let Φ be a polynomially growing outer automorphism of $\mathbf{F}_{r+1}$ and n a positive integer. Following Proposition 1.5, we distinguish two cases.

First case. There exist an automorphism $\varphi \in \mathrm{Aut}\,(\mathbf{F}_{r+1})$ representing a power of Φ and a non-trivial free decomposition $F_1 * F_2$ of $\mathbf{F}_{r+1}$ invariant under φ. We denote by φ_i the restriction of φ to F_i. By induction, there exists an integer p_i such that $\varphi_i^{p_i}$ induces the identity of F_i / F_i^n. It follows that $\varphi^{p_1 p_2}$ is trivial in $\mathrm{Aut}\,(\mathbf{B}_{r+1}(n))$. Therefore Φ induces an outer automorphism of finite order of $\mathbf{B}_{r+1}(n)$.

Second case. There exist an automorphism $\varphi \in \mathrm{Aut}\,(\mathbf{F}_{r+1})$ representing a power of Φ, a free decomposition $F_1 * \langle t \rangle$ of $\mathbf{F}_{r+1}$ and an element f of F_1 such that F_1 is invariant under φ and $\varphi(t) = tf$. We denote by φ_1 the restriction of φ to F_1. By induction, there exists an integer p_1 such that $\varphi_1^{p_1}$ induces the identity of F_1 / F_1^n. On the other hand, for all integers q, $\varphi^q(t)$ equals $tf\varphi_1(f)\varphi_1^2(f)\ldots\varphi_1^{q-1}(f)$. It follows that the equality below holds in $\mathbf{B}_{r+1}(n)$:

$$\varphi^{np_1}(t) = t\big[f\varphi_1(f)\varphi_1^2(f)\ldots\varphi_1^{p_1-1}(f)\big]^n = t.$$

Hence φ^{np_1} is trivial in $\mathrm{Aut}\,(\mathbf{B}_{r+1}(n))$. Therefore Φ induces an outer automorphism of finite order of $\mathbf{B}_{r+1}(n)$. $\qquad\square$

1.3. Subgroups of $\mathrm{Out}\,(\mathbf{B}_r(n))$. We are now interested in relevant subgroups that can be embedded in $\mathrm{Out}\,(\mathbf{B}_r(n))$. We start with free subgroups. The following result is due to M. Bestvina, M. Feighn and M. Handel

Theorem 1.7 (see [5], Theorem 5.2). *Let $r \geqslant 3$. Let φ_1 and φ_2 two automorphisms of $\mathbf{F}_r$. We assume that the outer automorphisms induced by φ_1 and φ_2 are irreducible, do not have common powers and neither have a nontrivial periodic conjugacy class.*

There exists an integer m such that φ_1^m and φ_2^m generate a free group. Moreover, the semi-direct product $\mathbf{F}_r \rtimes \mathbf{F}_2$ defined by φ_1^m and φ_2^m is hyperbolic.

Theorem 1.8. *Let $r \geqslant 3$. There exists an integer n_0 such that for all odd integers n larger than n_0, Out $(\mathbf{B}_r(n))$ contains a subgroup which is isomorphic to $\mathbf{F}_2$.*

Proof. Theorem 1.7 provides a hyperbolic extension of $\mathbf{F}_r$ by $\mathbf{F}_2$. In other words, $1 \to \mathbf{F}_r \to \mathbf{F}_r \rtimes \mathbf{F}_2 \to \mathbf{F}_2 \to 1$ is a short exact sequence such that $\mathbf{F}_r \rtimes \mathbf{F}_2$ is hyperbolic. The result follows from the main theorem. $\square$

We are now looking for free abelian subgroups of Out $(\mathbf{B}_r(n))$. Let G_1 and G_2 be two torsion-free groups. We denote by G the free product $G_1 * G_2$. Since G_1 and G_2 are torsion-free, so is G (see [32]). Moreover, for all integers n, $G^n \cap G_1 = G_1^n$.

Lemma 1.9. *Let n be an integer. Let φ be an automorphism of G which stabilizes the factor G_1. We assume that φ induces an automorphism of finite order of G/G^n. Then, the restriction of φ to G_1 induces also an automorphism of finite order of G_1/G_1^n.*

Proof. Since φ induces an automorphism of finite order of G/G^n, there exists $k \in \mathbb{N}^*$ such that for all $g \in G$, $\varphi^k(g)g^{-1} \in G^n$. However G_1 is invariant under φ. Thus for all $g \in G_1$, $\varphi^k(g)g^{-1} \in G^n \cap G_1 = G_1^n$. It follows that the restriction of φ to G_1 induces an automorphism of finite order of G_1/G_1^n. $\square$

Theorem 1.10. *Let $r \geqslant 2$. There exists an integer n_0 such that for all odd integers n larger than n_0, Out $(\mathbf{B}_{2r}(n))$ and Out $(\mathbf{B}_{2r+1}(n))$ contain a subgroup which is isomorphic to $\mathbb{Z}^r$.*

Proof. We denote by φ the automorphism of $\mathbf{F}_2$ studied in Proposition 1.1. There exists an integer n_0 such that for all odd integers n larger than n_0, φ induces an automorphism of infinite order of $\mathbf{B}_2(n)$. Let n be an odd integer larger than n_0. We consider $\mathbf{F}_{2r}$ as a free product $F_1 * \cdots * F_r$ of r copies of $\mathbf{F}_2$. For all $i \in \{1, \ldots, r\}$, we define an automorphism φ_i of $\mathbf{F}_{2r}$ as follows.

  (i) The restriction of φ_i to F_i is φ.

 (ii) The restriction of φ_i to any other factor is the identity.

By construction, the φ_i's generate an abelian subgroup of Aut $(\mathbf{F}_{2r})$ and all the more of Out $(\mathbf{B}_{2r}(n))$. We now study the relations between the φ_i's in Out $(\mathbf{B}_{2r}(n))$. Consider r integers $k_1, \ldots, k_r$ such that $\psi = \varphi_1^{k_1} \ldots \varphi_r^{k_r}$ induces an inner automorphism of $\mathbf{B}_{2r}(n)$. By Lemma 1.9, φ^{k_i}, which is the restriction of ψ to F_i, induces an automorphism of finite order of $F_i/F_i^n = \mathbf{B}_2(n)$. This forces k_i to equal zero. Hence the φ_i's generate a subgroup of Out $(\mathbf{B}_{2r}(n))$ which is isomorphic to $\mathbb{Z}^r$. For Out $(\mathbf{B}_{2r+1}(n))$ we apply the same argument with the following free factorization: $\mathbf{F}_{2r+1} = F_1 * \cdots * F_r * \mathbb{Z}$. $\square$

2. Small cancellation theory

In this section we will be concerned with the small cancellation theory. We expose the geometrical point of view developed by T. Delzant and M. Gromov in [15] and used in Section 3 to prove the main theorem.

2.1. Hyperbolic spaces. Let X be a proper and geodesic space. The distance between two points x and x' of X is denoted by $|x - x'|_X$ (or simply $|x - x'|$). Although it may not be unique, we denote by $[x, x']$ a geodesic joining x and x'. We assume that X is δ-hyperbolic (in the sense of Gromov) i.e. for all $x, y, z, t \in X$,

$$|x - y| + |z - t| \leqslant \max \{|x - z| + |y - t|, |x - t| + |y - z|\} + 2\delta.$$

The boundary at infinity of X will be denoted by ∂X (see Chapter 2 of [11]). A subset Y of X is α-quasi-convex if every geodesic of X joining two points of Y lies in the α-neighbourhood of Y, denoted by $Y^{+\alpha}$.

Lemma 2.1 (see [15], Lemma 2.1.5, or [12], Corollary 1.2.2). *Let x, x', y and y' be four points of X. Let u be a point of $[x, x']$ such that $|u - x| > |x - y| + 8\delta$ and $|u - x'| > |x' - y'| + 8\delta$. Then u belongs to the 8δ-neighbourhood of $[y, y']$.*

Proposition 2.2 (see [15], Lemma 2.2.2, or [12], Proposition 1.2.4). *Let Y and Z be two α-quasi-convex subsets of X. For all $A \geqslant 0$*

$$\operatorname{diam} \left(Y^{+A} \cap Z^{+A}\right) \leqslant \operatorname{diam} \left(Y^{+\alpha+10\delta} \cap Z^{+\alpha+10\delta}\right) + 2A + 20\delta.$$

Let g be an isometry of X. In order to measure its action on X, we define two translation lengths. By the *translation length* $[g]_X$ (or simply $[g]$) we mean

$$[g] = \inf_{x \in X} |gx - x|.$$

The *asymptotic translation length* $[g]_X^\infty$ (or simply $[g]^\infty$) is

$$[g]^\infty = \lim_{n \to +\infty} \frac{1}{n} |g^n x - x|.$$

These two lengths satisfy the following inequality (see [11], Chapter 10, Proposition 6.4):

$$[g]^\infty \leqslant [g] \leqslant [g]^\infty + 32\delta \quad \text{for all } g \in G.$$

The *axis* A_g of g, defined as follows, is a 40δ-quasi-convex subset of X (see [15], Proposition 2.3.3):

$$A_g = \{x \in X / |gx - x| \leqslant \max \{[g], 40\delta\}\}.$$

The isometry g is *hyperbolic* if its asymptotic translation length is positive. In this case, g fixes exactly two points of ∂X denoted by g^- and g^+. The cylinder of $\{g^-, g^+\}$, denoted by Y_g, is defined to be the set of points of X which are 10δ-close to some geodesic joining g^- and g^+. It is a g-invariant, 8δ-quasi-convex subset of X.

Proposition 2.3. *Let g be a hyperbolic isometry of X. We denote by σ a geodesic joining g^- and g^+, the points of ∂X fixed by g. Let Y be a α-quasi-convex subset of X. If Y is g-invariant, then σ is contained in the $(\alpha + 8\delta)$-neighbourhood of Y. In particular, σ is contained in the 48δ-neighbourhood of A_g.*

Proof. Let x be a point of σ. We write d for the distance between x and Y. Let y be a point of Y such that $|x - y| \leq d + \delta$. Since g is hyperbolic, there is an integer m such that $|g^m x - g^{-m} x| > 2d + 48\delta$ (see [11], Chapter 10, Lemme 6.5). We denote by p_+ (respectively p_-) a projection of $g^m x$ (respectively $g^{-m} x$) on σ, i.e. a point of σ such that $|g^m x - p_+| = d(g^m x, \sigma)$ (respectively $|g^{-m} x - p_-| = d(g^{-m} x, \sigma)$). The geodesics σ and $g^m \sigma$ have the same extremities. It follows that they are 8δ-close (see [11], Chapter 2, Proposition 2.2). In particular, $|g^m x - p_+| \leq 8\delta$. In the same way, we have $|g^{-m} x - p_-| \leq 8\delta$. Note that x lies on the subgeodesic of σ delimited by p_- and p_+. Indeed, if it was not the case we should have

$$
\begin{aligned}
|g^{-m} x - g^m x| &\leq |p_- - p_+| + 16\delta \leq ||x - p_-| - |x - p_+|| + 16\delta \\
&\leq ||x - g^{-m} x| - |x - g^m x|| + 32\delta \\
&\leq 32\delta.
\end{aligned}
$$

Contradiction. On the other hand, we have

$$
|x - p_+| \geq |x - g^m x| - 8\delta \geq \frac{1}{2}|g^{-m} x - g^m x| - 8\delta > d + 16\delta.
$$

Hence

$$
|x - p_+| > |g^m y - g^m x| + |g^m x - p_+| + 8\delta \geq |g^m y - p_+| + 8\delta.
$$

In the same way, we have $|x - p_-| > |g^{-m} y - p_-| + 8\delta$. By Lemma 2.1, the point x is 8δ-close to $[g^{-m} y, g^m y]$. However $g^{-m} y$ and $g^m y$ belong to Y which is α-quasi-convex. Therefore the distance between x and Y is smaller than $\alpha + 8\delta$. $\square$

Definition 2.4. Let P be a set of isometries of X. The *injectivity radius* denoted by $r_{\mathrm{inj}}(P, X)$ and the *maximal overlap* denoted by $\Delta(P, X)$ are the following quantities

$$
r_{\mathrm{inj}}(P, X) = \inf\{[g]^\infty / g \in P^*\},
$$

$$
\Delta(P, X) = \sup\{\operatorname{diam}\left(Y_g^{+20\delta} \cap Y_{g'}^{+20\delta}\right) / g, g' \in P^*,\ g \neq g'\},
$$

where P^* denotes the set of hyperbolic elements of P.

Let G be a group acting properly, co-compactly, by isometries on X. An element g of G is either *elliptic* (in particular, it has finite order) or *hyperbolic* (see [11], Chapter 9). A subgroup of G is called *elementary* if it is virtually cyclic. Since G is a hyperbolic group, every non-elementary subgroup of G contains a copy of $\mathbf{F}_2$, the free group of rank 2 (see [16], Chapter 8, Theorem 37). Given a hyperbolic element g of G, the normalizer of $\langle g \rangle$ is elementary (see [11], Chapter 10, Corollary 7.2). The group G satisfies the *small centralizers hypothesis* if G is non-elementary and every elementary subgroup of G is cyclic. The next lemma will be needed in Section 2.5.

Lemma 2.5. *We assume that G satisfies the small centralizers hypothesis. Let R be a subset of G stable by conjugation and inverse, whose elements are hyperbolic and not proper powers. There exists a subset R_0 of R, stable by conjugation satisfying the followings:*

(i) *for all $h \in R$ one and only one element of $\{h, h^{-1}\}$ belongs to R_0,*

(ii) *any two distinct elements of R_0 generate a non-elementary subgroup.*

Proof. To prove (i) it is sufficient to show that an element h of R cannot be conjugate to its inverse. Suppose, contrary to our claim, that there exist $h \in R$ and $g \in G$ such that $ghg^{-1} = h^{-1}$. Thus, g belongs to the normalizer of h, which is elementary (see [11], Chapter 10, Proposition 1). In particular, g and h generate an elementary subgroup of G. Since G satisfies the small centralizers hypothesis, g and h commute. It follows that $h = h^{-1}$. Hence h is not hyperbolic, a contradiction.

Let us now prove (ii). Let $h_1, h_2 \in R_0$. Assume that they generate an elementary subgroup. G satisfying the small centralizers hypothesis, h_1 and h_2 commute. Since h_1 and h_2 are not proper powers, they are either equal or inverse. However R_0 does not contain an element and its inverse. Hence $h_1 = h_2$. $\qquad\square$

Definition 2.6. Let G be a group acting properly co-compactly by isometries on a proper, geodesic, δ-hyperbolic space X. The invariant $A(G, X)$ is the upper bound of $\operatorname{diam}(A_g^{+50\delta} \cap A_{g'}^{+50\delta})$, where g and g' are two elements of G which generate a non-elementary subgroup and whose translation lengths are smaller than 100δ.

Remark. If there is no such g and g' we adopt the convention that $A(G, X) = +\infty$. However in our study, G will be non-elementary and the translation lengths of its generators small in comparison to δ. Therefore $A(G, X)$ will always be finite.

This invariant $A(G, X)$ was used by T. Delzant and M. Gromov to study hyperbolic groups satisfying the small centralizers hypothesis (see §2.4 in [15]).

Remark. The objects defined previously depend implicitly on the hyperbolicity constant δ (e.g. Y_g, A_g, and $A(G, X)$). Although the notation does not make this dependency explicit, we should keep in mind that it plays an important role. For instance, we have the following lemma:

Lemma 2.7. *Let X be proper, geodesic, δ-hyperbolic space and G a group acting properly, co-compactly by isometries on X. Let λ be a positive number. Then $A(G, \lambda X) = \lambda A(G, X)$, where λX is the $\lambda \delta$-hyperbolic space X endowed with the rescaled metric $\lambda \mid . \mid_X$.*

Proof. Let g be an element of G. Its translation lengths satisfy $[g]_{\lambda X} = \lambda [g]_X$ and $[g]_{\lambda X}^\infty = \lambda [g]_X^\infty$. Since λX is a $\lambda \delta$-hyperbolic space, the axis of g in λX is exactly the image in λX of the axis A_g of g in X. We will denote it by λA_g. Let g and g' be two elements of G that do not generate an elementary group and whose asymptotic translation lengths in λX are smaller than $100 \lambda \delta$. In particular, we have $[g]_X^\infty, [g']_X^\infty \leqslant 100 \delta$. By definition of $A(G, X)$, we have

$$\text{diam} \left(\lambda A_g^{+50\lambda\delta} \cap \lambda A_{g'}^{+50\lambda\delta} \right) = \lambda \, \text{diam} \left(A_g^{+50\delta} \cap A_{g'}^{+50\delta} \right)$$
$$\leqslant \lambda A(G, X).$$

After taking the upper bound for all g and g', we obtain $A(G, \lambda X) \leqslant \lambda A(G, X)$. In the same way, $A(G, \lambda X) \geqslant \lambda A(G, X)$. This establishes the desired equality. $\quad\square$

2.2. Small cancellation theorem. In the remainder of Section 2 we require X to be a proper, geodesic, simply-connected, δ-hyperbolic space and G a group acting properly, co-compactly, by isometries on X.

Let P be a set of hyperbolic elements of G. We assume that P is stable by conjugation and contains only a finite number of conjugacy classes. We denote by N the (normal) subgroup of G generated by P. Our goal is to study the quotient $\bar{G} = G/N$. The small cancellation parameters $\Delta(P, X)$ and $r_{\text{inj}}(P, X)$, defined in the previous section, respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory. We are interested in situations where the ratios $\frac{\delta}{r_{\text{inj}}(P,X)}$ and $\frac{\Delta(P,X)}{r_{\text{inj}}(P,X)}$ are very small (see Theorem 2.8 below). To that end, we build a space $\bar{X}$ with an action of $\bar{G}$. We only recall the main steps of this construction. For more details we refer the reader to [12] and [13].

Fix $r_0 > 0$. Its value will be made precise in the small cancellation theorem (see Theorem 2.8). Let $\rho \in P$. We endow Y_ρ with the length metric $\mid . \mid_\rho$ induced by the restriction of $\mid . \mid_X$ to Y_ρ. The *cone over* Y_ρ denoted by $C_\rho(r_0)$ (or simply C_ρ) is the topological quotient of $Y_\rho \times [0, r_0]$ by the equivalence relation which identifies the points $(y, 0)$ for all $y \in Y_\rho$. Given two points $x = (y, r)$ and $x' = (y', r')$ of C_ρ the following formula defines a distance on C_ρ (see [7], Chapter I.5, Proposition 5.9 (1)):

$$\cosh(|x, x'|) = \cosh r \cosh r' - \sinh r \sinh r' \cos \left(\min \left\{ \pi, \frac{|y - y'|_\rho}{\sinh r_0} \right\} \right).$$

The *cone-off over X relatively to P*, denoted by $\dot{X}_P(r_0)$ (or simply $\dot{X}$), is obtained by attaching for all $\rho \in P$ the cone C_ρ to X along Y_ρ. The distances $\mid . \mid_X$ and $\mid . \mid_{C_\rho}$

induce a metric on $\dot{X}$ (see [12], Proposition 3.1.7). We extend by homogeneity the action of G on X in an action of G on $\dot{X}$. Let $x = (y, r) \in C_\rho$ and $g \in G$. We define gx to be the point of $C_{g\rho g^{-1}} = gC_\rho$ given by $gx = (gy, r)$. The group G acts by isometries on $\dot{X}$ (see [12], Lemma 4.3.1). The space $\bar{X}_P(r_0)$ (or simply $\bar{X}$) is the quotient of $\dot{X}$ by N.

Theorem 2.8 (Small cancellation theorem, see [15], Theorem 5.5.2, or [12], Theorem 4.2.2). *There exist positive numbers δ_0, δ_1, Δ_0 and $r_0 \geqslant 10^5 \delta_1$ satisfying the following properties.*

Let X be a proper, geodesic, simply-connected, δ-hyperbolic space. Let G be a group acting properly, co-compactly, by isometries on X. Let P be a set of hyperbolic elements of G. We assume that P is stable by conjugation and contains only a finite number of conjugacy classes.

If $\delta \leqslant \delta_0$, $\Delta(P, X) \leqslant \Delta_0$ and $r_{\mathrm{inj}}(P, X) \geqslant 3\pi \sinh r_0$, then $\bar{X}_P(r_0)$ is proper, geodesic, simply-connected and δ_1-hyperbolic. Moreover $\bar{G}$ acts properly, co-compactly, by isometries on it .

Remark. The fact that the constants r_0, δ_0, δ_1 and Δ_0 do not depend on X, P or G is very important in order to iterate the small cancellation construction.

2.3. Estimation of an injectivity radius. From now on we suppose that X, G and P satisfy the assumptions of the small cancellation theorem. In particular, $\delta \leqslant \delta_0$, $\Delta(P, X) \leqslant \Delta_0$ and $r_{\mathrm{inj}}(P, X) \geqslant 3\pi \sinh r_0$, where δ_0, Δ_0 and r_0 are the constants given by Theorem 2.8. We also assume now that G satisfies the small centralizers hypothesis, i.e. G is non-elementary and all elementary subgroups of G are cyclic (see Section 2.1). We are now interested in the properties of the action of $\bar{G}$ on $\bar{X}$. This will allow us to iterate the small cancellation process. We need in particular an estimation of $A(\bar{G}, \bar{X})$ and some injectivity radius. We already know that $\bar{X}$ is δ_1-hyperbolic (see Theorem 2.8).

For all $\rho \in P$, we write E_ρ for the subgroup of G which stabilizes Y_ρ. It is an elementary subgroup of G (see [11], Chapter 10, Proposition 7.1). Let ν and π be the respective maps $\nu \colon \dot{X} \to \bar{X}$ and $\pi \colon G \twoheadrightarrow \bar{G}$. The space $\bar{X}$ is obtained by attaching cones of large radius to $\nu(X)$. This provides a kind of Margulis' decomposition. The cones play the role of the thick part: the translation length of a hyperbolic element of $\bar{G}$ in a cone is very large. The following lemma illustrates this fact.

Lemma 2.9 (see [15], Lemme 5.9.3). *Let $\bar{g}$ be an element of $\bar{G}$ such that $[\bar{g}] \leqslant 200\delta_1$. Assume that $\bar{g}$ does not belong to $\bar{E}_\rho = \pi(E_\rho)$ for all $\rho \in P$. Then $A_{\bar{g}}$ is contained in $\nu(X)^{+100\delta_1}$ and $A_{\bar{g}} \cap \nu(X)$ is non-empty.*

To study $\nu(X)^{+100\delta_1}$, which is an analogue of the thin part of the Margulis decomposition, we use the fact that the map $\nu(X) \to \bar{X}$ is a local quasi-isometry:

Lemma 2.10 (see [12], Proposition 3.1.8). *Let x and x' be two points of X. First $|x-x'|_{\dot{X}} \leqslant |x-x'|_X$. Moreover, if $|x-x'|_{\dot{X}} \leqslant \frac{r_0}{2}$ then $|x-x'|_X \leqslant \frac{2\pi \sinh r_0}{r_0} |x-x'|_{\dot{X}}$.*

Using this point of view, T. Delzant and M. Gromov proved the following result.

Proposition 2.11 (see [15], Lemme. 5.10.1). *Let $\bar{C}$ be a $50\delta_1$-quasi-convex subset of $\nu(X)^{+100\delta_1}$. There exists a subset C of $\dot{X}$ such that*

  (i) *the map $\nu \colon \dot{X} \to \bar{X}$ induces an isometry from C onto $\bar{C}$,*

  (ii) *the projection $\pi \colon G \to \bar{G}$ induces an isomorphism between $\mathrm{Stab}(C)$ and $\mathrm{Stab}(\bar{C})$ which are respectively the stabilizers of C and $\bar{C}$.*

Proposition 2.12. *Let H be a normal subgroup of G containing P and $\bar{H}$ its image by the projection $\pi \colon G \to \bar{G}$. Then $r_{\mathrm{inj}}(\bar{H}, \bar{X}) \geqslant \min\{\kappa l, \delta_1\}$, where κ stands for $\frac{3r_0}{40\pi \sinh r_0}$ and l is the smallest asymptotic translation length of a hyperbolic element of H that does not belong to any E_ρ.*

Remark. This lemma is a refinement of Lemma 5.11.1 proved by T. Delzant and M. Gromov in [15]. They gave indeed a lower bound for $r_{\mathrm{inj}}(\bar{G}, \bar{X})$. Four our purpose, we need a more accurate result. We provide here an estimation of the injectivity radius of a normal subgroup of $\bar{G}$.

Proof. Since H is a normal subgroup containing the relations P, $\pi^{-1}(\bar{H}) = H$. Suppose the assertion of the lemma is false. There exists a hyperbolic element $\bar{h}$ of $\bar{H}$ such that $[\bar{h}]^\infty < \min\{\kappa l, \delta_1\}$. Fix an integer m such that $7\delta_1 \leqslant m \min\{\kappa l, \delta_1\} \leqslant 8\delta_1$. By [11], Chapter 10, Proposition 6.4,

$$[\bar{h}^m] \leqslant m[\bar{h}]^\infty + 32\delta_1 < m \min\{\kappa l, \delta_1\} + 32\delta_1 \leqslant 40\delta_1.$$

Since $[h^m] \leqslant 200\delta_1$, the axis $A_{\bar{h}^m}$ of $\bar{h}^m$ in $\bar{X}$, is a $40\delta_1$-quasi-convex (see [15], Proposition 2.3.3), which is contained in the $100\delta_1$-neighbourhood of $\nu(X)$ (see Lemma 2.9). By Proposition 2.11, there exists a subset C of X such that

  (i) *the map $\nu \colon \dot{X} \to \bar{X}$ induces an isometry from C onto $A_{\bar{h}^m}$,*

  (ii) *the map $\pi \colon G \to \bar{G}$ induces an isomorphism from $\mathrm{Stab}(C)$ onto $\mathrm{Stab}(A_{\bar{h}^m})$.*

However $\bar{h}$ belongs to $\mathrm{Stab}(A_{\bar{h}^m})$. We denote by h its preimage in $\mathrm{Stab}(C)$. Since $\bar{h}$ is hyperbolic, h is necessarily a hyperbolic element of H that does not belong to E_ρ for all $\rho \in P$. By assumption, $[h]_X^\infty \geqslant l$.

On the other hand by Lemma 2.9, $A_{\bar{h}^m} \cap \nu(X) \neq \emptyset$. Fix $\bar{x} \in A_{\bar{h}^m} \cap \nu(X)$ and denote by x its preimage in $C \cap X$. As the map $\nu \colon C \to A_{\bar{h}^m}$ is an equivariant isometry we have $|h^m x - x|_{\dot{X}} = |\bar{h}^m \bar{x} - \bar{x}|_{\bar{X}}$. Recall that $\bar{x}$ belongs to $A_{\bar{h}^m}$, the axis of $\bar{h}^m$. It follows that

$$|h^m x - x|_{\dot{X}} = |\bar{h}^m \bar{x} - \bar{x}|_{\bar{X}} \leqslant \max\{[\bar{h}^m], 40\delta_1\} \leqslant 40\delta_1.$$

By Lemma 2.10, $|h^m x - x|_X$ is smaller than $\frac{2\pi \sinh r_0}{r_0} |h^m x - x|_{\dot{X}}$. Consequently,

$$ml \leqslant m[h]_X^\infty \leqslant [h^m]_X \leqslant |h^m x - x|_X \leqslant \frac{80\pi \delta_1 \sinh r_0}{r_0} = \frac{6\delta_1}{\kappa}.$$

In particular, $7\delta_1 \leqslant m \min\{\kappa l, \delta_1\} \leqslant m\kappa l \leqslant 6\delta_1$, a contradiction. $\qquad\square$

2.4. Other properties of $\overline{G}$ and $\overline{X}$. In this section, we review some results obtained by T. Delzant and M. Gromov in [15].

Proposition 2.13 (see [15], Lemme 5.9.5). *The constant $A(\overline{G}, \overline{X})$ satisfies the following inequality.*
$$A(\overline{G}, \overline{X}) \leqslant A(G, X) + 1000\delta_1 e^{350\delta_1}.$$

Proposition 2.14 (see [15], Lemme 5.10.2 and Lemme 5.10.3). *Assume that every element of P is an odd power of an element of G which is not a proper power. Then $\overline{G}$ satisfies the following properties.*

(i) *Every elementary subgroup of $\overline{G}$ is cyclic.*

(ii) *Let $\overline{F}$ be a finite subgroup of $\overline{G}$. Either $\overline{F}$ is the image of a finite subgroup of G, or there exists $\rho \in P$ such that $\overline{F}$ is a subgroup of $\overline{E}_\rho = \pi(E_\rho)$.*

Proposition 2.15 (see [15], Theorem 5.7.1). *The Euler characteristic of $\overline{G}$ satisfies $\chi(\overline{G}, \mathbb{Q}) = \chi(G, \mathbb{Q}) + |P/G|$, where $|P/G|$ denotes the number of conjugacy classes of P.*

2.5. An induction lemma. One should think of the next lemma as a step of the iterative procedure involved in the proof of the main theorem. We recall that the invariant $A(G, X)$ stands for the maximal overlap between the axes of two small hyperbolic elements of G (see Definition 2.6). The injectivity radius $r_{\mathrm{inj}}(H, X)$ denotes the smallest asymptotic translation length of a hyperbolic element of H (see Definition 2.4).

Lemma 2.16 (Induction lemma). *There exist positive numbers δ_1, Δ_1, l_1, l_2, l_3 and an integer n_0 satisfying the following properties. Let n be an odd integer larger than n_0. Let X be a proper, geodesic, simply-connected, δ_1-hyperbolic space. Let G be a group acting properly, co-compactly, by isometries on X and H a normal subgroup of G. Let R be the set of hyperbolic elements of H, which are not proper powers in G and whose asymptotic translation lengths are smaller than l_1. Let N be the normal subgroup of G generated by $\{h^n \,/\, h \in R\}$, $\overline{G}$ the quotient G/N and $\overline{H}$ the image of H by the canonical map $\pi \colon G \to \overline{G}$. We assume that*

(i) *G satisfies the small centralizers hypothesis; moreover the order of every finite subgroup of G divides n,*

(ii) $A(G, X) \leqslant \Delta_1$ and $r_{\mathrm{inj}}(H, X) \geqslant \frac{l_2}{\sqrt{n}}$,

(iii) $\chi(G, \mathbb{Q}) + \frac{1}{2} |R/G| > 0$, where $\chi(G, \mathbb{Q})$ is the Euler characteristic of G and $|R/G|$ the number of conjugacy classes in R.

   Then, there exists a proper, geodesic, simply-connected, δ_1-hyperbolic space $\overline{X}$ on which $\overline{G}$ acts properly, co-compactly, by isometries. Moreover, $\overline{G}$, $\overline{H}$ and $\overline{X}$ satisfy the points (i) and (ii), $\chi(\overline{G}, \mathbb{Q}) > 0$ and

$$[\pi(g)]_{\overline{X}}^{\infty} \pi(g) \leqslant \frac{l_3}{\sqrt{n}} [g]_X^{\infty} \quad \text{for all } g \in G.$$

Remark. If G, H, X and n satisfy the hypothesis of the previous lemma – including hypotheses (i), (ii) and (iii) – we will say that (G, H, X) *satisfies the induction assumptions for exponent n*. The induction lemma says in particular that if (G, H, X) satisfies the induction assumptions for exponent n, so does $(\overline{G}, \overline{H}, \overline{X})$.

   The remainder of this section will be devoted to the proof of the induction lemma.

Proof. The positive numbers r_0, δ_0, δ_1, and Δ_0 are given by the small cancellation theorem (see Theorem 2.8). The constant $\kappa = \frac{3r_0}{40\pi \sinh r_0}$ is the one that appears in Proposition 2.12. We define a rescaling parameter $L_n = \sqrt{\frac{n\kappa\delta_1}{\pi \sinh r_0}}$. The sequence (L_n) is increasing and tends to infinity. Up to chose n_0 large enough, we may assume that for all $n \geqslant n_0$

$$\frac{\delta_1}{L_n} \leqslant \delta_0, \tag{1}$$

$$\frac{2000\delta_1 e^{350\delta_1} + 176\delta_1}{L_n} \leqslant \min\left\{\Delta_0, 1000\delta_1 e^{350\delta_1}\right\}, \tag{2}$$

$$\frac{3\kappa\delta_1}{L_n} \leqslant \delta_1. \tag{3}$$

Note that n_0 only depends on δ_0, δ_1, Δ_0 and r_0. We now define the following constants:

$$\Delta_1 = 2000\delta_1 e^{350\delta_1}, \quad l_1 = 3\delta_1, \quad l_2 = 3\sqrt{\kappa\delta_1\pi \sinh r_0} \quad \text{and} \quad l_3 = \sqrt{\frac{\pi \sinh r_0}{\kappa\delta_1}}.$$

   Let n be an odd integer larger than n_0. We assume that (G, H, X) satisfies the induction assumptions for exponent n. In particular, R is the set of hyperbolic elements of H, which are not a proper powers in G and whose asymptotic translation lengths are smaller than l_1. For the remainder of the proof of Lemma 2.16 we consider the action of G on the rescaled space $\frac{1}{L_n} X$. By (1), this space is δ-hyperbolic with

$\delta = \frac{\delta_1}{L_n} \leqslant \delta_0$. As explained in the remark on page 798, we should keep in mind that the hyperbolicity constant we are now always working with is δ.

By Lemma 2.5, there exists a subset R_0 of R, stable by conjugation satisfying the followings:

(i) for all $h \in R$ one and only one element of $\{h, h^{-1}\}$ belongs to R_0,

(ii) any two distinct elements of R_0 generate a non-elementary subgroup.

Let us denote by P the set of relations $P = \{h^n, h \in R_0\}$.

Lemma 2.17. *The set P satisfies the following assumptions:*

$$\Delta\left(P, \frac{1}{L_n}X\right) \leqslant \Delta_0 \quad and \quad r_{\text{inj}}\left(P, \frac{1}{L_n}X\right) \geqslant 3\pi \sinh r_0.$$

Proof. Let h_1 and h_2 be two elements of R_0 such that $h_1^n \neq h_2^n$. By Proposition 2.3, $Y_{h_j^n}$ is contained in $A_{h_j}^{+58\delta}$. Thus Proposition 2.2 gives

$$\text{diam}\left(Y_{h_1^n}^{+20\delta} \cap Y_{h_2^n}^{+20\delta}\right) \leqslant \text{diam}\left(A_{h_1}^{+78\delta} \cap A_{h_2}^{+78\delta}\right)$$
$$\leqslant \text{diam}\left(A_{h_1}^{+50\delta} \cap A_{h_2}^{+50\delta}\right) + 176\delta.$$

If h_1 and h_2 generated an elementary subgroup, by Lemma 2.5 we should have $h_1^n = h_2^n$. Hence h_1 and h_2 generate a non-elementary subgroup. On the other hand, $[h_1]^\infty$ and $[h_2]^\infty$ are smaller than $\frac{l_1}{L_n} = \frac{3\delta_1}{L_n} = 3\delta$. By definition of $A(G, X)$,

$$\text{diam}\left(Y_{h_1^n}^{+20\delta} \cap Y_{h_2^n}^{+20\delta}\right) \leqslant A\left(G, \frac{1}{L_n}X\right) + 176\,\delta$$
$$\leqslant \frac{1}{L_n}A(G, X) + 176\,\delta$$
$$\leqslant \frac{\Delta_1 + 176\delta_1}{L_n} = \frac{2000\delta_1 e^{350\delta_1} + 176\delta_1}{L_n}.$$

By (2), $\Delta\left(P, \frac{1}{L_n}X\right)$ is smaller than Δ_0.

The injectivity radius of H on $\frac{1}{L_n}X$ is larger than

$$\frac{1}{L_n}r_{\text{inj}}(H, X) \geqslant \frac{1}{L_n}\frac{l_2}{\sqrt{n}} = \sqrt{\frac{\pi \sinh r_0}{n\kappa\delta_1}}\frac{3\sqrt{\kappa\delta_1\pi \sinh r_0}}{\sqrt{n}} = \frac{3\pi \sinh r_0}{n}.$$

In particular, $[h^n]^\infty \geqslant 3\pi \sinh r_0$ for all $h \in R_0$. Therefore, $r_{\text{inj}}\left(P, \frac{1}{L_n}X\right) \geqslant 3\pi \sinh r_0$. $\qquad \square$

On account of the previous lemma, we can now apply the small cancellation theorem (Theorem 2.8) to the rescaled space $\frac{1}{L_n}X$, G and P. We obtain a proper, geodesic, simply-connected, δ_1-hyperbolic space $\bar{X}$ on which $\bar{G} = G/ \ll P \gg$ acts properly, co-compactly, by isometries.

Lemma 2.18. *Every elementary subgroup of $\bar{G}$ is cyclic, either infinite or finite with order dividing n.*

Proof. All elements of P are odd powers of elements of G which are not proper powers. By Proposition 2.14, all elementary subgroups of $\bar{G}$ are cyclic. Assume now that $\bar{F}$ is a finite subgroup of $\bar{G}$. According to the same proposition, we distinguish two cases.

  (i) $\bar{F}$ is the image of a finite subgroup of G. However, the order of every finite subgroup of G divides n. Thus the order of $\bar{F}$ divides n.

 (ii) There exists $h \in R_0$ such that $\bar{F}$ is a subgroup of $\bar{E}_{h^n} = \pi(E_{h^n}) = \langle \pi(h) \rangle$, whose order divides n. $\qquad\square$

Lemma 2.19. *The constant $A(\bar{G}, \bar{X})$ is bounded above by Δ_1. The injectivity radius $r_{\mathrm{inj}}(\bar{H}, \bar{X})$ is bounded below by $\frac{l_2}{\sqrt{n}}$.*

Proof. By Proposition 2.13, $A(\bar{G}, \bar{X}) \leqslant A\big(G, \frac{1}{L_n}X\big) + 1000\delta_1 e^{350\delta_1}$. However, using inequality (2), we obtain

$$A\left(G, \frac{1}{L_n}X\right) \leqslant \frac{1}{L_n} A(G, X) \leqslant \frac{\Delta_1}{L_n} = \frac{2000\delta_1 e^{350\delta_1}}{L_n} \leqslant 1000\delta_1 e^{350\delta_1}.$$

Hence $A(\bar{G}, \bar{X}) \leqslant 2000\delta_1 e^{350\delta_1} = \Delta_1$.

Let g be a hyperbolic element of H, which does not belong to $E_{h^n} = \langle h \rangle$ for all $h \in R_0$. Its asymptotic translation length in $\frac{1}{L_n}X$ is larger than $\frac{l_1}{L_n} = \frac{3\delta_1}{L_n}$. By Proposition 2.12 and inequality (3),

$$r_{\mathrm{inj}}\left(\bar{H}, \bar{X}\right) \geqslant \min\left\{\frac{3\kappa\delta_1}{L_n}, \delta_1\right\} = \frac{3\kappa\delta_1}{L_n} = \frac{3\sqrt{\kappa\delta_1\pi \sinh r_0}}{\sqrt{n}} = \frac{l_2}{\sqrt{n}}. \qquad\square$$

Lemma 2.20. *The Euler characteristic of $\bar{G}$ satisfies*

$$\chi(\bar{G}, \mathbb{Q}) = \chi(G, \mathbb{Q}) + \frac{1}{2}|R/G| > 0,$$

where $|R/G|$ is the number of conjugacy classes of R. In particular, $\bar{G}$ is non-elementary.

Proof. Consider now $h_1, h_2 \in R_0$ and $u \in G$ such that $h_1^n = u h_2^n u^{-1}$. Since R_0 is stable by conjugation, h_1 and $u h_2 u^{-1}$ are two elements of R_0 which generate an elementary subgroup. By Lemma 2.5, $h_1 = u h_2 u^{-1}$. In other words, there are as many conjugacy classes in P as in R_0. By construction, there are twice as many conjugacy classes in R as in R_0. Thus $|P/G| = |R_0/G| = \frac{1}{2} |R/G|$. The result follows from Proposition 2.15. $\square$

Lemma 2.21. *For all $g \in G$, we have $[\pi(g)]_{\bar{X}}^{\infty} \leqslant \frac{l_3}{\sqrt{n}} [g]_X^{\infty}$.*

Proof. By Lemma 2.10, the map $\frac{1}{L_n} X \to \bar{X}$ contracts the distances. Thus for all $g \in G$,

$$[\pi(g)]_{\bar{X}}^{\infty} \leqslant \frac{1}{L_n} [g]_X^{\infty} = \sqrt{\frac{\pi \sinh r_0}{n \kappa \delta_1}} [g]_X^{\infty} = \frac{l_3}{\sqrt{n}} [g]_X^{\infty}. \qquad \square$$

The previous lemma completes the proof of the induction lemma. $\square$

3. Proof of the main theorem

Recall the statement of the main theorem.

Main theorem. *Let $1 \to H \to G \to F \to 1$ be a short exact sequence of groups. Assume that H is non-trivial, finitely generated, G is hyperbolic, non-elementary, torsion-free and F is torsion-free. There exists an integer n_0 such that for all odd integers n larger than n_0, the canonical map $F \to \mathrm{Out}\,(H)$ induces an injective homomorphism $F \hookrightarrow \mathrm{Out}\,(H/H^n)$.*

Proof. The constants δ_1, Δ_1, l_1, l_2, l_3 and n_0 are given by the induction lemma (see Lemma 2.16). Up to increase n_0, we may also assume that $\frac{l_3}{\sqrt{n_0}} < 1$. Let $1 \to H \to G \to F \to 1$ be a short exact sequence of groups, which satisfies the hypotheses of the theorem. The strategy is to build by induction a family of short exact sequences $1 \to H_k \to G_k \to F \to 1$ with an action of G_k on a hyperbolic space X_k, such that the direct limit $\varinjlim H_k$ is the Burnside group H/H^n.

Initialization. Put $H_0 = H$ and $G_0 = G$. Let X_0 be a proper, geodesic, simply-connected, hyperbolic space on which G acts properly, co-compactly, by isometries. Take for instance the Rips polyhedron of G (see Chapter 5 in [11]). We can assume, by rescaling X_0 if necessary, that

- X_0 is δ_1-hyperbolic,
- $A\,(G_0, X_0) \leqslant \Delta_1$ (see Lemma 2.7),

- $\{h \in H / [h]^\infty \leqslant l_1, h$ is not a proper power in $G\}$ contains a number of conjugacy classes in G bounded below by $-2\chi(G, \mathbb{Q})$.
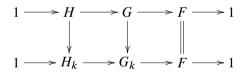
The group G is hyperbolic and torsion free. Since H is a non-trivial subgroup of G, it contains a hyperbolic element. In particular, the injectivity radius of H is positive (see [14]). Thus, up to increase one more time n_0, we may assume that $r_{\mathrm{inj}}(H_0, X_0) \geqslant \frac{l_2}{\sqrt{n_0}}$. Hence (G_0, H_0, X_0) satisfies the induction assumptions for exponent n_0.

Let n be an odd integer larger than n_0. (G_0, H_0, X_0) satisfies *a fortiori* the induction assumptions for exponent n.

Induction. Let (G_k, H_k, X_k) satisfying the induction assumptions for exponent n. We denote by R_k the set of hyperbolic elements of H_k which are not proper powers in G_k and whose asymptotic translation lengths are smaller than l_1. Let N_k be the normal subgroup of G_k generated by $\{h^n / h \in R_k\}$, G_{k+1} the quotient G_k / N_k and H_{k+1} the image of H_k by the canonical map $\pi_k \colon G_k \to G_{k+1}$. By the induction lemma, there exists a metric space X_{k+1} such that $(G_{k+1}, H_{k+1}, X_{k+1})$ satisfies the induction assumptions for the exponent n. In this way, we obtain two sequences of groups (H_k) and (G_k) whose properties we want to study now.

Properties of H_k and G_k

Lemma 3.1. *For every integer k, there exists a map $G_k \to F$ such that the following diagram is commutative. Moreover its rows are short exact sequences.*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\
  &   & \downarrow &   & \downarrow &   & \| &   &   \\
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1
\end{array}
$$

Proof. Following the construction of H_k and G_k, the proof is by induction on k. The result is obvious for $k = 0$. Assume the lemma holds for k. The group N_k being generated by elements of H_k, $N_k \subset H_k$. Consequently, the short exact sequence $1 \to H_k \to G_k \to F \to 1$ induces maps such that the following diagram commutes and its rows are short exact sequences.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1 \\
  &   & \downarrow &   & \pi_k \downarrow &   & \| &   &   \\
1 & \longrightarrow & H_{k+1} & \longrightarrow & G_{k+1} & \longrightarrow & F & \longrightarrow & 1
\end{array}
$$

Thus the lemma holds for $k + 1$. $\qquad\square$

Our next goal is to compare the groups H/H^n and $\varinjlim H_k$. To shorten notations, we let h stand for an element of H as well as its images in H_k, $\varinjlim H_k$ or H/H^n.

Lemma 3.2. *The kernel of the canonical map $H \to \varinjlim H_k$ is exactly H^n, the subgroup of H generated by all n-th powers.*

Proof. Let h be an element of $H \setminus \{1\}$. By the induction lemma,

$$[h]^\infty_{X_k} \leqslant \left(\frac{l_3}{\sqrt{n}} \right)^k [h]^\infty_{X_0} \leqslant \left(\frac{l_3}{\sqrt{n_0}} \right)^k [h]^\infty_{X_0}$$

for all integers k. However, we chose n_0 in such a way that $\frac{l_3}{\sqrt{n_0}} < 1$. It follows that there is an integer k such that $[h]^\infty_{X_k} < \frac{l_2}{\sqrt{n}}$. By construction, the injectivity radius of H_k on X_k is larger than $\frac{l_2}{\sqrt{n}}$ (point (ii) of Lemma 2.16). Therefore h is an elliptic isometry. In particular, h has finite order dividing n in H_k (point (i) of Lemma 2.16). Hence H^n is contained in the kernel of $H \to \varinjlim H_k$.

On the other hand, at each step of the construction, the kernel of the map $H_k \to H_{k+1}$ is generated by n-th powers of elements of H_k. It follows that the kernel of the morphism $H \to \varinjlim H_k$ is contained in H^n. $\qquad\square$

Lemma 3.3. *The groups H/H^n and $\varinjlim H_k$ are isomorphic.*

Proof. This lemma follows from the previous one and from the fact that the map $H \to \varinjlim H_k$ is onto. $\qquad\square$

Lemma 3.4. *Let f be a non-trivial element of F. Let g be a preimage of f by the map $G \to F$. The conjugation by g defines an automorphism of H which induces a non-trivial outer automorphism of H/H^n.*

Proof. Let S be a finite generating set of H. We denote by φ the automorphism of H defined by $\varphi(h) = ghg^{-1}$ for all $h \in H$. Assume, contrary to our claim, that φ induces an inner automorphism of H/H^n. There exists $l \in H$ such that for all $h \in H$ $\varphi(h)$ and lhl^{-1} have the same image in H/H^n. According to Lemma 3.3, H/H^n and $\varinjlim H_k$ are isomorphic. Since S is finite, there is an integer k such that for all $s \in S$ $\varphi(s)$ equals lsl^{-1} in H_k. However S is a generating set of H. Thus $\varphi(h) = ghg^{-1}$ is equal to lhl^{-1} in H_k for all $h \in H$. Lemma 3.1 yields the following commutative diagram.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1
\end{array}
$$

Hence, the image of $l^{-1}g$ in G_k commutes with every element of H_k. By construction, the injectivity radius of H_k on X_k is positive. Therefore, it contains a hyperbolic element h. Thus h and $l^{-1}g$ generate an abelian subgroup of G_k, which has to be cyclic. There exists $(p, q) \in \mathbb{Z}^* \times \mathbb{Z}$ such that $\left(l^{-1}g\right)^p = h^q$ in G_k. Using the commutative diagram, we push this identity in F and obtain $f^p = 1$. Since F is torsion-free, f is trivial. This contradicts our assumption. $\qquad\square$

End of the proof of the main theorem. The map $F \to \mathrm{Out}\,(H)$ can be constructed as follows. Let f be an element of F and g a preimage of f by $G \to F$. The image of f by the map $F \to \mathrm{Out}\,(H)$ is exactly the outer automorphism of H induced by the conjugation by g in G. The previous lemma is hence an equivalent formulation of the following fact: the map $F \to \mathrm{Out}\,(H)$ induces an injective homomorphism $F \hookrightarrow \mathrm{Out}\,(H/H^n)$. This remark completes the proof of the main theorem. $\qquad\square$

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