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On the infinity flavor of Heegaard Floer homology and the integral cohomology ring

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Abstract. For a three-manifold Y and torsion Spin^c structure $\mathfrak{s}$, Ozsváth and Szabó construct a spectral sequence with E^2 term an exterior algebra over $H^1(Y;\mathbb{Z})$ converging to $HF^\infty(Y,\mathfrak{s})$. They conjecture that the differentials are completely determined by the integral triple cup product form. In this paper, we prove that $HF^\infty(Y,\mathfrak{s})$ is in fact determined by the cohomology ring when $\mathfrak{s}$ is torsion. Furthermore, we give a complete calculation of such $HF^\infty(Y,\mathfrak{s})$, with mod 2 coefficients, in the case where $b_1(Y)$ is 3 or 4.

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1. Introduction

Throughout the previous decade, Heegaard Floer theory has been a very useful and calculable machine in low-dimensional topology. It includes invariants for closed three- and four-manifolds, as well as for knots and links. Similarly, manifolds with boundary, singular knots, and contact structures can be studied as well. One of the most effective computational tools in Heegaard Floer theory is the integral surgery formula (Theorem 1.1 of [18]), which converts the Heegaard Floer complex for a nullhomologous knot K in a closed, oriented 3-manifold Y into the Heegaard Floer homology of Dehn surgeries on K.

Given a Heegaard splitting of Y along a surface Σ, Heegaard Floer homology is defined to be the Lagrangian Floer homology of certain tori in the symmetric product of Σ. The Heegaard Floer homology of Y splits as a direct sum over the set of Spin^c structures on Y. Different flavors of Heegaard Floer homology twist the differential by a count of the intersection number of a holomorphic disk with a codimension-two submanifold of the symmetric product determined by some choice of basepoint(s) on the surface.

While many new results in low-dimensional topology have come from calculations of these groups, one flavor, HF^∞, has the simplest structure. Still, it has many useful applications. For example, studying the absolute grading on HF^∞ allows one to define a powerful invariant, the correction term d (Definition 4.1 in [11]). This has had numerous applications, including a lower bound for the four-ball genus of a knot (Theorem 1.5 in [19]). Furthermore, in [11], Ozsváth and Szabó use properties of HF^∞ to find new restrictions on intersection forms for four-manifolds.

In fact, HF^∞ has been calculated for three-manifolds with b_1 at most 2 in Theorem 10.1 of [14]. In this case, it is completely determined by the integral cohomology ring. Also, Mark [7] has obtained results in this direction, gaining information about HF^∞ from a complex $C^\infty_*(Y)$ with differential given completely by the cup product structure. If one calculates HF^∞ with the U variable formally completed (or in other words, coefficients in $\mathbb{Z}[[U, U^{-1}])$, it is shown in Section 2 of [6] that these groups vanish for any non-torsion Spin^c structure $\mathfrak{s}$. Therefore, we are only concerned with torsion Spin^c structures in this paper.

In Theorem 10.12 of [14], it is shown that for each torsion Spin^c structure $\mathfrak{s}$ there exists a coefficient system such that the Heegaard Floer homology with twisted coefficients, $\underline{HF^\infty}(Y, \mathfrak{s})$, is isomorphic to $\mathbb{Z}[U, U^{-1}]$ as $\mathbb{Z}[U, U^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[H^1(Y;\mathbb{Z})]$-modules, where $H^1(Y;\mathbb{Z})$ acts trivially on $\mathbb{Z}[U, U^{-1}]$. There is therefore a universal coefficients spectral sequence with E^2 term $\Lambda^*(H^1(Y;\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$ converging to $HF^\infty(Y, \mathfrak{s})$. We refer the reader to Proposition 16 of [7] for more details on the construction of this spectral sequence.

We do need to recall how the gradings work in this spectral sequence. More specifically, the universal coefficients spectral sequence identifies $E^2_{i,*}$, for i even, with $\Lambda^*(H^1(Y;\mathbb{Z}))$. Since multiplication by U induces an isomorphism between $E^2_{i,*}$ and $E^2_{i-2,*}$, we see that $E^2_{i,*}$ vanishes for odd i. This implies that $d_k\colon E^k_{i,j} \to E^k_{i+k-1,j-k}$ automatically vanishes if k is even. Therefore, the E^2 and E^3 pages are isomorphic. Furthermore, in Conjecture 4.10 of [12], Ozsváth and Szabó propose that the rest of the behavior of the spectral sequence is easily computed from the integral cohomology ring on Y. In order to state their conjecture more precisely, we first need a definition.

Definition 1.1. For a closed, oriented three-manifold, the *integral triple cup product form*, μ_Y, is the three-form on $H^1(Y;\mathbb{Z})$ given by

$$\mu_Y(a \wedge b \wedge c) = \langle a \cup b \cup c, [Y] \rangle.$$

Conjecture 1.2 (Ozsváth–Szabó). *The differential* $d_3\colon \Lambda^j(H^1(Y;\mathbb{Z})) \otimes U^i \to \Lambda^{j-3}(H^1(Y;\mathbb{Z})) \otimes U^{i-1}$ *is given by*

$$d_3(\alpha \otimes U^i) = \iota_{\mu_Y}(\alpha) \otimes U^{i-1}. \tag{1}$$

In other words, d_3 *is essentially contraction by the integral triple cup product form. Furthermore, all higher differentials vanish.* (*For notational purposes, we will omit the* U*'s in the domain and range*).

Note that if this conjecture is true, knowing the integral triple cup product form on Y allows a complete calculation of $HF^\infty(Y, \mathfrak{s})$. The goal of this paper is to present a few partial results in this direction. This work will be used in [5] to completely calculate $HF^\infty(Y, \mathfrak{s}; \mathbb{Z}/2\mathbb{Z})$. From now on all of our coefficients for Heegaard Floer homology will be $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. For compatibly with $\mathbb{F}$-coefficients, we will take integral triple cup products and then reduce mod 2, as opposed to taking the triple cup products in $H^*(Y; \mathbb{F})$. Therefore, when referring to Conjecture 1.2, we will mean with mod 2 coefficients in this sense.

Theorem 1.3. *$HF^\infty(Y, \mathfrak{s})$ is completely determined by the integral cohomology ring. In other words, if $H^*(Y_1; \mathbb{Z}) \cong H^*(Y_2; \mathbb{Z})$ as graded rings and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are torsion* Spin^c *structures on Y_1 and Y_2 respectively, then $HF^\infty(Y_1, \mathfrak{s}_1)$ and $HF^\infty(Y_2, \mathfrak{s}_2)$ are isomorphic as relatively-graded $\mathbb{F}[U, U^{-1}]$-modules.*

Theorem 1.4. *If $b_1(Y) = 3$, then Conjecture 1.2 holds.*

Theorem 1.5. *For $b_1(Y) = 4$, $HF^\infty(Y, \mathfrak{s})$ agrees with the prediction for the homology given by Conjecture 1.2.*

Remark 1.6. The analogues of the above theorems were previously known in monopole Floer homology (see Chapter IX in [4]).

We now outline the arguments given for the proofs in this paper. In order to calculate $HF^\infty(Y, \mathfrak{s})$ in general, we prove that it suffices to consider any manifold which can be obtained from Y by a sequence of nonzero surgeries on nullhomologous knots. This is done by showing that such a sequence of surgeries does not affect the integral triple cup product form or HF^∞. Furthermore, we show that we only need to calculate HF^∞ in the case of $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}^n$, by showing that in each torsion Spin^c structure, $HF^\infty(Y, \mathfrak{s})$ behaves as HF^∞ of a manifold which is some "version" of Y with H_1 torsion-free. Since these torsionless versions will have a different number of torsion Spin^c structures, as an abuse of notation, we will say that two three-manifolds Y and Y' have the same HF^∞ if for all torsion $\mathfrak{s}_Y$ and $\mathfrak{s}_{Y'}$, $HF^\infty(Y, \mathfrak{s}_Y)$ is isomorphic to $HF^\infty(Y', \mathfrak{s}_{Y'})$.

We then use a theorem of Cochran, Gerges, and Orr [1] which constructs an explicit class of "model manifolds". Their results show that any Y with torsion-free first homology can be related to a model manifold by a sequence of ±1-surgeries on nullhomologous knots. Therefore, we will have that Y and the model manifold have the same HF^∞. For $b_1 = 3$ and 4, we will explicitly write down these models and calculate HF^∞ simply based on knowledge of $HF^\infty(\mathbb{T}^3, \mathfrak{s}_0)$ (calculated in [11]) and the integer surgery formula for knots of [18].

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2. Eliminating torsion

The goal of this section is to reduce the general calculation of $HF^\infty(Y,\mathfrak{s})$ to the case where $H_1(Y;\mathbb{Z})$ is torsion-free. The idea is to construct a sufficiently nice surgery presentation for Y and then argue that we can remove each knot surgery that is not contributing to $b_1(Y)$ without changing either the integral triple cup product form or HF^∞. The notion of isomorphism for the integral triple cup product form ($\mu_Y \cong \mu_{Y'}$) is an isomorphism $\phi\colon H^1(Y;\mathbb{Z}) \to H^1(Y';\mathbb{Z})$ such that

$$\phi(\mu_Y(a \wedge b \wedge c)) = \mu_{Y'}(\phi(a) \wedge \phi(b) \wedge \phi(c)).$$

Let's do an example to see the idea of removing torsion from H_1.

Example 2.1. Fix a closed, oriented three-manifold Y and consider $Y\#S^3_n(K)$ for some K and $n \neq 0$. Notice that the integral triple cup product form of $Y\#S^3_n(K)$ is isomorphic to that of Y. Similarly, the connect-sum formula for HF^∞ (Theorem 6.2 in [14]) and the calculation of HF^∞ for rational homology spheres (Theorem 10.1 in [14]) give $HF^\infty(Y\#S^3_n(K),\mathfrak{s}_Y\#\mathfrak{s}_K) \cong HF^\infty(Y,\mathfrak{s}_Y)$ for any choice of Spin^c structures on Y and $S^3_n(K)$. Thus, to calculate HF^∞ for $Y\#S^3_n(K)$ it suffices to study Y instead. We have now, for our calculations, removed $S^3_n(K)$ from $Y\#S^3_n(K)$, and thus removed a factor of $\mathbb{Z}/n\mathbb{Z}$ from H_1.

We want to generalize this procedure in order to remove all of the torsion in H_1.

Proposition 2.2. *Perform n-surgery on a nullhomologous knot K in Y for some nonzero integer n. The resulting manifold, $Y_n(K)$, and Y have isomorphic integral triple cup product forms.*

Proof. We simply use the result of Cochran, Gerges, and Orr on rational surgery equivalence (Theorem 5.1 of [1]), which states that two three-manifolds will have isomorphic integral triple cup product forms if and only if there is a sequence of non-longitudinal surgeries on rationally nullhomologous knots relating the two. □

The following proposition is made as an observation in Section 4.1 of [12].

Proposition 2.3 (Ozsváth–Szabó). *Fix a torsion* Spin^c *structure $\mathfrak{s}$ on Y and a nonzero integer n. Let $\mathfrak{s}_K$ be a torsion* Spin^c *structure on $Y_n(K)$ which agrees with $\mathfrak{s}$ on $Y - K$. Then we have that $HF^\infty(Y,\mathfrak{s})$ and $HF^\infty(Y_n(K),\mathfrak{s}_K)$ are isomorphic.*

We will give a proof of this in Section 4.

To remove the torsion from H_1, we need a sufficiently nice surgery presentation to try to generalize the argument from Example 2.1. However, since a surgery presentation might not consist of all pairwise-split components, we have to find the next best thing. The idea is to represent Y by surgery on a link in S^3 where the components have pairwise linking number 0. Such a link is called *homologically split*. The following lemma tells us that we can do this if we are willing to slightly change the manifold. The proof can be found at the end of this paper.

Lemma 2.4 (Manolescu). *Let Y be a closed, oriented* 3*-manifold. There exist finitely many nonzero integers, $m_1, \dots, m_k$, such that there exists a homologically split surgery presentation for $Y \# L(m_1, 1) \# \dots \# L(m_k, 1)$.*

Proposition 2.5. *For all Y, there exists a three-manifold M given by* 0*-surgery on a homologically split link such that Y and M have the same HF^∞ and isomorphic triple cup product forms. In particular, $H_1(M; \mathbb{Z})$ is torsion-free.*

Proof. By applying Lemma 2.4, we may connect-sum Y with the necessary lens spaces such that the resulting manifold is presented by $S^3_\Lambda(L)$, where L is a homologically split link. We know that connect sums with lens spaces do not change HF^∞ or the integral triple cup product form. Since each nonzero surgery in the presentation will now be performed on a nullhomologous knot, Proposition 2.2 (respectively Proposition 2.3) shows that the triple cup product form (respectively HF^∞) of the 3-manifold obtained by surgery on the sublink of L consisting of components that are 0-framed will be isomorphic to the triple cup product form (respectively HF^∞) for $S^3_\Lambda(L)$. If we take M to be surgery on the 0-framed components of L, then $H_1(M)$ will clearly be torsion-free since the linking matrix for this presentation will be the 0 matrix. Therefore, this is the desired manifold. □

This is the method of removing torsion from $H_1(Y; \mathbb{Z})$. Observe that a manifold with torsion-free H_1 has a unique torsion Spin^c structure.

3. Model manifolds

Following [1], we will call two 3-manifolds, Y_1 and Y_2, *surgery equivalent* if there is a finite sequence of ± 1-surgeries on nullhomologous knots, beginning in Y_1 and terminating at Y_2.

We can rephrase the work of the previous section by saying that if Y_1 and Y_2 are surgery equivalent, then they have isomorphic triple cup product forms and the same HF^∞.

Theorem 3.1 (Cochran–Gerges–Orr (Corollary 3.5 of [1])). *Let $H_1(Y_1;\mathbb{Z}) \cong \mathbb{Z}^n$. Suppose that Y_1 and Y_2 have isomorphic integral triple cup product forms. Then Y_1 and Y_2 are surgery equivalent.*

It is important to note that this is not necessarily true if H_1 has torsion. A counterexample can be exhibited by taking Y_1 as $\#_{i=1}^3 L(5,1)$ and Y_2 as 5-surgery on each component of the Borromean rings (Example 3.15 of [1]).

Since both Y_1 and Y_2 have $b_1 = 0$, we know they must have the same HF^∞. Therefore, HF^∞ cannot quite detect the subtlety seen by singular cohomology with certain coefficient rings, as Y_1 and Y_2 can be distinguished by their triple cup product forms over $\mathbb{Z}/5\mathbb{Z}$. However, for the rest of the paper, we will always assume our triple cup product forms are integral.

Proof of Theorem 1.3. Theorem 3.1 and Proposition 2.5 prove that the integral triple cup product form determines HF^∞. A little more work allows the statement for the integral cohomology ring. If the integral cohomology rings of Y_1 and Y_2 are isomorphic (grading preserving), then the integral triple cup product form of Y_1 is isomorphic to either that of Y_2 or $-Y_2$. Note that if we apply Proposition 2.5 to both Y_1 and Y_2, then the resulting manifolds, M_1 and M_2, will also have isomorphic cohomology rings. Furthermore, we have not affected the integral triple cup product forms or HF^∞. Thus, we may assume Y_1 and Y_2 do not have torsion in H_1. If Y_1 and Y_2 have isomorphic triple cup product forms, then we are clearly done by the theorem. On the other hand, if Y_1 and $-Y_2$ have isomorphic triple cup product forms, then we apply Corollary 3.8 of [1] to see that Y_2 is surgery equivalent to $-Y_2$. This completes the proof. □

In the case of $b_1 = 3$ or $b_1 = 4$, we can explicitly see what the set of surgery equivalence classes is that we are dealing with. The following is calculated in Example 3.3 in [1].

Theorem 3.2 (Cochran–Gerges–Orr). *For each Y with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}^3$, there exists a unique $n \geq 0$ such that Y is surgery equivalent to the manifold M_n with Kirby diagram shown in Figure 1.*

We will call the component that spirals n times Z_n. It is useful to note that $M_0 = \#_{i=1}^3 S^2 \times S^1$ and $M_1 = \mathbb{T}^3$. Calculating HF^∞ for each M_n is what suffices to prove Theorem 1.4. Furthermore, it turns out that calculating $b_1 = 3$ combined with the following proposition is sufficient to understand $b_1 = 4$ as well.

Proposition 3.3 (Cochran–Gerges–Orr (Corollary 3.7 of [1])). *If $H_1(Y) \cong \mathbb{Z}^4$, then Y is surgery equivalent to $M_n \# S^2 \times S^1$ for some $n \geq 0$.*

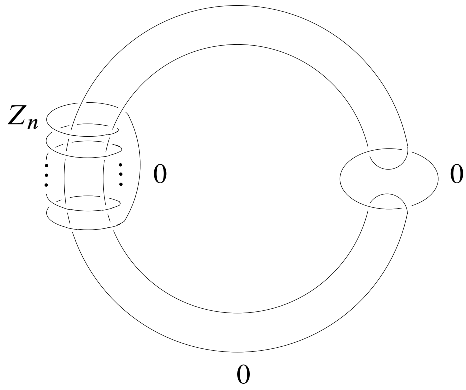

Figure 1. Surgery presentation of M_n.

In fact, there is an explicit way to produce a 3-manifold with $H_1(Y) \cong \mathbb{Z}^n$ in each surgery equivalence class by a construction similar to the M_n (see Corollary 3.5 in [1]).

4. Review of the surgery formula

In this section we review the integer surgery formula for knots, Theorem 1.1 of [18], with the perspective and notation of [6]. We will assume the reader has some familiarity with the constructions of Heegaard Floer homology for three-manifolds and knots ([15] and [14] respectively). For convenience, we will assume that Y is an integer homology sphere; this is solely for the purpose of having one Spin^c structure to keep track of. This construction will apply for any torsion Spin^c structure on any three-manifold with the appropriate bookkeeping. Finally, we will assume all diagrams are admissible and stabilized as needed.

Let K be a nullhomologous knot in Y and fix $\mathfrak{s}_0$ to be the torsion Spin^c structure on Y. Knowledge of the knot Floer complex will be used to calculate the Heegaard Floer homology of surgeries on K.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ be a doubly-pointed Heegaard diagram for K in Y. Note that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ are each singly-pointed diagrams for Y, and thus no longer contain any information about the knot. Recall that K determines a $\mathbb{Z}$-valued Alexander grading A on the elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ satisfying

$$A(x) - A(y) = n_z(\phi) - n_w(\phi) \tag{2}$$

for $\phi \in \pi_2(x, y)$, which can canonically be made absolute (see Section 3.3 of [13]). Similarly, since $\mathfrak{s}_0$ is torsion, for any pointed Heegaard diagram for Y, there is an absolute $\mathbb{Q}$-valued Maslov grading M satisfying

$$M(x) - M(y) = \mu(\phi) - 2n_p(y), \tag{3}$$

where p is the chosen basepoint and again $\phi \in \pi_2(x, y)$ (this is due to Theorem 7.1 in [16]). Recall that multiplication by U lowers A by 1 and M by 2.

We can now define a CFK-like complex with differential twisted by the Alexander grading. For notation, let $x \vee y = \max\{x, y\}$.

Definition 4.1. Fix $s \in \mathbb{Z}$. $\mathfrak{A}_s$ is the chain complex over $\mathbb{F}[U, U^{-1}]$ freely-generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. This is equipped with the differential

$$\partial_s(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{(A(x)-s)\vee 0 - (A(y)-s)\vee 0 + n_w(\phi)} y \tag{4}$$

for $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

While this complex first arises in Theorem 1.1 of [18], the explicit formulation for ∂_s can be found in Section 4.2 of [6]. This is also where the reader can find an explicit description of the relative $\mathbb{Z}$-grading and a proof that $(\partial_s)^2 = 0$.

We will use CF_p to denote the chain complex (or sometimes just the chain group) $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, p)$ for some basepoint p on Σ and ∂ for its differential. Note that CF_w and CF_z correspond to $\mathfrak{A}_{+\infty}$ and $\mathfrak{A}_{-\infty}$ respectively. We now describe chain maps relating $\mathfrak{A}_s$, CF_z, and CF_w given by

$$\begin{array}{ccc} & \mathfrak{A}_s & \\ {\scriptstyle \mathcal{I}_s^{-K}} \swarrow & & \searrow {\scriptstyle \mathcal{I}_s^{K}} \\ CF_z & \xrightarrow{\ \mathcal{D}^{-K}\ } & CF_w. \end{array}$$

First, the diagonal maps are the *inclusions*

$$\mathcal{I}_s^K(x) = U^{(A(x)-s)\vee 0} x, \quad \mathcal{I}_s^{-K}(x) = U^{(s-A(x))\vee 0} x.$$

After stabilizing the diagram if necessary, the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ can be transformed into $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ by a sequence of basepoint-avoiding isotopies and handleslides, since they both represent Y. Choose such a sequence of moves and let the *destabilization*, $\mathcal{D}^{-K}$, denote the induced chain homotopy equivalence of CF^∞ as described by the proof of Theorem 1.1 in [15].

It turns out that the choice of Heegaard moves does not affect $\mathcal{D}^{-K}$ on the level of homology as long as the path that z follows to w is fixed (see Remark 4.15 in [6]). We will ignore the concern of paths as one can insist on using a good set of trajectories (Definition 6.27 in [6]) to eliminate this concern. We define the destabilization $\mathcal{D}^K$ to be the identity map. The reason for this is that we can relate the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ to itself by performing no isotopies or handleslides.

Proposition 4.2. *The inclusion maps, $\mathcal{I}_s^{\pm K}$, are quasi-isomorphisms which preserve the relative Maslov grading. Furthermore, $\mathfrak{D}^{-K}$ is a quasi-isomorphism and its induced map on homology preserves absolute gradings.*

Proof. The inclusions are quasi-isomorphisms because they are bijective chain maps. That they preserve the relative grading is shown in Section 7.1 of [6].

The map $\mathfrak{D}^{-K}$ is known to be a quasi-isomorphism which preserves relative gradings (Theorem 1.1 of [15]). Theorem 7.1 of [16] proves that the absolute grading on Heegaard Floer homology is solely an invariant of a given three-manifold and torsion Spin^c structure. Furthermore, $\mathfrak{D}^{-K}$ induces a relatively-graded quasi-isomorphism between the subcomplexes, CF^-, which are generated by elements with only non-negative powers of U (Theorem 11.1 of [15]). Similarly, HF^- inherits this absolute grading. The key observation is that HF^- always has an element of maximal grading, because multiplication by U lowers the grading in $\mathbb{F}[U]$ by 2; however, we know the value of this maximal grading is independent of Heegaard diagram. If $\mathfrak{D}^{-K}$ did not preserve the absolute grading, then the induced map on HF^- could not be a relatively-graded isomorphism. □

Remark 4.3. This is the key point where we are making use of the infinity flavor. In general, the $\mathcal{I}$ maps will not be quasi-isomorphisms for other flavors of Heegaard Floer homology. On the other hand, $\mathfrak{D}^{-K}$ is always a quasi-isomorphism, regardless of flavor.

Following [6], let $\Phi_s^{-K} = \mathfrak{D}^{-K} \circ \mathcal{I}_s^{-K}$ and $\Phi_s^K = \mathfrak{D}^K \circ \mathcal{I}_s^K = \mathcal{I}_s^K$.

Remark 4.4. Lemma 7.12 in [6] shows that Φ_0^{-K} and Φ_0^K shift the gradings by the same amount. Therefore, $\Phi_0^{-K} + \Phi_0^K$ is a homogeneous map.

We are ready to define the integer surgery formula for knots.

Definition 4.5. For each $s \in \mathbb{Z}$, let $\mathfrak{B}_s = CF_w$. Consider the chain map

$$\Psi_n^K \colon \prod_{s\in\mathbb{Z}} \mathfrak{A}_s \longrightarrow \prod_{s\in\mathbb{Z}} \mathfrak{B}_s, \quad (s,x) \longmapsto (s, \Phi_s^K(x)) + (s+n, \Phi_s^{-K}(x)).$$

The mapping cone of Ψ_n^K, $C(\Psi_n^K)$, is called the *surgery formula.*

Remark 4.6. There exists a correspondence between the mod n equivalence classes of $\mathbb{Z}$ and the Spin^c structures on $Y_n(K)$ (see Section 2 of [18] for more details). When $n = 0$, the unique torsion Spin^c structure on $Y_0(K)$ corresponds to $s = 0$.

We therefore use $C(\Psi_n^K, [s])$ to represent the subcomplex generated by the $\mathfrak{A}_{s'}$ and $\mathfrak{B}_{s'}$ with $s' \equiv s \pmod n$.

Remark 4.7. If $n \neq 0$, then each $C(\Psi_n^K, [s])$ admits a relative $\mathbb{Z}$-grading. If $n = 0$, then $C(\Psi_0^K, [0])$ also admits a relative $\mathbb{Z}$-grading. This is explicitly described in Section 7 of [6].

Theorem 4.8 (Ozsváth–Szabó (Theorem 1.1 of [18])). *Fix an integer n. If $n = 0$, then we assume that $s = 0$ as well. The mapping cone $C(\Psi_n^K, [s])$ is quasi-isomorphic to $CF^\infty(Y_n(K), \mathfrak{s})$, where $\mathfrak{s}$ corresponds to $[s]$ as described in Remark 4.6.*

The wary reader will note that Theorem 4.8 is not proved for the infinity flavor in [18]; furthermore, the argument there does not quite work for HF^∞. In order to actually prove Theorem 4.8 for HF^∞, one must complete with respect to the variable U; in other words, the proof requires working with $\mathbb{F}[[U, U^{-1}]$-coefficients instead. This is in fact what is done in Theorem 1.1 of [6] to prove a more general version of this theorem for links. The reason is that in order to prove that the integer surgery formula calculates the Heegaard Floer homology of surgery on a knot, one must sum over infinitely many cobordism maps with increasing powers of U, which may be all nonzero in CF^∞; therefore, one must work over $\mathbb{F}[[U, U^{-1}]$ to make sense of these sums.

However, we would like to show that for torsion Spin^c structures on the surgered manifold, it suffices to use the surgery formula with $\mathbb{F}[U, U^{-1}]$-coefficients. We define $\mathbf{CF}^\infty$, $\mathbf{HF}^\infty$, and $C(\underline{\Psi}_n^K, [s])$ to be the analogous constructions with $\mathbb{F}[[U, U^{-1}]$-coefficients instead.

Lemma 4.9. *As an $\mathbb{F}[U, U^{-1}]$-module, $\mathbb{F}[[U, U^{-1}]$ is flat.*

Proof. All of the following steps can be found in a standard commutative algebra text (see, for example, [8]). The field of fractions of $\mathbb{F}[U, U^{-1}]$ is $\mathbb{F}(U)$ (the rational functions in one variable over $\mathbb{F}$). Since localization is exact, $\mathbb{F}(U)$ is flat over $\mathbb{F}[U, U^{-1}]$. Furthermore, $\mathbb{F}(U)$ is a subfield of $\mathbb{F}[[U, U^{-1}]$. Note that every field is flat over a subfield since it is a vector space over the subfield. Therefore, $\mathbb{F}[[U, U^{-1}]$ is flat over $\mathbb{F}[U, U^{-1}]$ by transitivity of flatness. □

Lemma 4.10. *We have that $H_*(C(\Psi_n^K, [s]))$ is isomorphic to $HF^\infty(Y_n(K), \mathfrak{s})$ as long as s is 0 when $n = 0$. In particular, Theorem 4.8 is true as stated.*

Proof. The first thing we point out is that for any Y and torsion $\mathfrak{s}_0$, $HF^\infty(Y, \mathfrak{s}_0)$ is always a finitely generated, free $\mathbb{F}[U, U^{-1}]$-module. This is because for torsion Spin^c structures, U lowers the relative $\mathbb{Z}$-grading on $CF^\infty(Y, \mathfrak{s}_0)$ by 2.

Since $\mathbb{F}[[U, U^{-1}]$ is flat over $\mathbb{F}[U, U^{-1}]$, we have that

$$HF^\infty(Y_n(K), \mathfrak{s}) \otimes_{\mathbb{F}[U,U^{-1}]} \mathbb{F}[[U, U^{-1}] \cong \mathbf{HF}^\infty(Y_n(K), \mathfrak{s}).$$

Because both HF^∞ and $\mathbf{HF}^\infty$ are free and finitely generated over their respective base rings, it is now clear how to recover HF^∞ from $\mathbf{HF}^\infty$.

Let us consider the case $n \neq 0$. In order to do this, we look ahead in this section at the specifics of the proof of Proposition 2.3. This in fact gives a direct proof that $H_*(C(\Psi_n^K, [s]))$ and $HF^\infty(Y, \mathfrak{s}_0)$ are isomorphic; note that this isomorphism goes to Y, the non-surgered manifold. Repeating the proof with $\mathbb{F}[[U, U^{-1}]$-coefficients shows that $H_*(C(\underline{\Psi}_n^K, [s])) \cong \mathbf{HF}^\infty(Y, \mathfrak{s}_0)$. By applying Theorem 4.8 for $\mathbb{F}[[U, U^{-1}]$-coefficients (the case which is proved in [6]), we see that $\mathbf{HF}^\infty(Y, \mathfrak{s}_0) \cong \mathbf{HF}^\infty(Y_n(K), \mathfrak{s})$. Since we can recover HF^∞ from $\mathbf{HF}^\infty$, we have $HF^\infty(Y, \mathfrak{s}_0) \cong HF^\infty(Y_n(K), \mathfrak{s})$. We can therefore pass through these various isomorphisms to obtain that

$$H_*(C(\Psi_n^K, [s])) \cong HF^\infty(Y, \mathfrak{s}_0) \cong HF^\infty(Y_n(K), \mathfrak{s}),$$

which is what we needed to show.

The case when $n = 0$ is easier. Since $C(\Psi_n^K, [0])$ is finitely generated (its chain group is $\mathfrak{A}_0 \oplus CF_w$), we have that

$$C(\Psi_n^K, [0]) \otimes_{\mathbb{F}[U,U^{-1}]} \mathbb{F}[[U, U^{-1}] \cong C(\underline{\Psi}_n^K, [0]).$$

By Theorem 4.8 for $\mathbb{F}[[U, U^{-1}]$-coefficients and Lemma 4.9,

$$\begin{aligned} H_*(C(\Psi_n^K, [0])) \otimes_{\mathbb{F}[U,U^{-1}]} \mathbb{F}[[U, U^{-1}] &\cong H_*(C(\underline{\Psi}_n^K, [0])) \\ &\cong \mathbf{HF}^\infty(Y_0(K), \mathfrak{s}) \\ &\cong HF^\infty(Y_0(K), \mathfrak{s}) \otimes_{\mathbb{F}[U,U^{-1}]} \mathbb{F}[[U, U^{-1}]. \end{aligned}$$

By the same grading arguments used previously, now applied to Remark 4.4, we have that $H_*(C(\Psi_0^K, [0]))$ is free and finitely generated. This allows us to recover the desired isomorphism. □

In light of this technical interlude, we are content to work with $\mathbb{F}[U, U^{-1}]$-coefficients for the rest of the paper.

To give some practice with the integer surgeries formula, we will use it to prove Proposition 2.3. We remark that the technique here will be useful in the sequel [5].

Proof of Proposition 2.3. Again, for notational convenience, we assume that Y is an integer homology sphere. Furthermore, we work with $n > 0$; the proof for $n < 0$ is essentially the same. Fix a Spin^c structure, $\mathfrak{s}_K$, that agrees with $\mathfrak{s}_0$ on $Y - K$. The idea is to show that for some s, $H_*(\mathfrak{A}_s) \cong HF^\infty(Y_n(K), \mathfrak{s}_K)$. Since Proposition 4.2 implies that $H_*(\mathfrak{A}_s)$ is isomorphic to $HF^\infty(Y, \mathfrak{s}_0)$, this will complete the proof.

Fix an s whose mod n equivalence class corresponds to $\mathfrak{s}_K$. Recall that Theorem 4.8 tells us $H_*(C(\Psi_n^K, [s])) \cong HF^\infty(Y_n(K), \mathfrak{s}_K)$. Consider the subcomplex of $C(\Psi_n^K, [s])$ given by

$$C_{>s} = \prod_{\substack{s' > s \\ s' \equiv s \ (\mathrm{mod}\ n)}} \mathfrak{A}_{s'} \oplus \prod_{\substack{s' > s \\ s' \equiv s \ (\mathrm{mod}\ n)}} \mathfrak{B}_{s'}.$$

We claim that this complex is acyclic. Equip $C_{>s}$ with the filtration $\mathcal{F}_>(x) = -s'$ for $x \in \mathfrak{A}_{s'}$ or $x \in \mathfrak{B}_{s'}$. The only components of the differential that do not lower the filtration level are $\partial_{s'}$, ∂, and Φ^K. Therefore, the associated graded splits as a product of complexes of the form

$$(\mathfrak{A}_{s'}, \partial_{s'}) \xrightarrow{\Phi^K} (\mathfrak{B}_{s'}, \partial).$$

By Proposition 4.2, these are all acyclic. Therefore, $C_{>s}$ is acyclic as well.

Construct the subcomplex

$$C_{<s} = \prod_{\substack{s' \leq s-n \\ s' \equiv s \ (\mathrm{mod}\ n)}} \mathfrak{A}_{s'} \oplus \prod_{\substack{s' \leq s \\ s' \equiv s \ (\mathrm{mod}\ n)}} \mathfrak{B}_{s'}.$$

Note that if we take $C(\Psi^K_n, [s])$ and remove $C_{<s}$ and $C_{>s}$, we are left solely with $\mathfrak{A}_s$, since there can be only one integer in the interval $(s-n, s]$ that corresponds to $\mathfrak{s}_K$. Thus, the proof will be complete if we can show that $C_{<s}$ is also acyclic. This follows by the same argument as before, except now we use the filtration

$$\mathcal{F}_<(x) = \begin{cases} s' & \text{if } x \in \mathfrak{A}_{s'}, \\ s' - n & \text{if } x \in \mathfrak{B}_{s'}. \end{cases}$$

This time the associated graded splits into the complexes

$$(\mathfrak{A}_{s'}, \partial_{s'}) \xrightarrow{\Phi^{-K}} (\mathfrak{B}_{s'+n}, \partial).$$

Again, by Proposition 4.2, these are acyclic. Thus, $C_{<s}$ is acyclic. □

For the remainder of the paper we will only be working with 0-surgery on K in Y with H_1 torsion-free; more specifically we will restrict to the unique torsion Spin^c structure on $Y_0(K)$, $\mathfrak{s}$, which agrees with the unique torsion Spin^c structure on Y, $\mathfrak{s}_0$, on $Y - K$. Most importantly, we will restrict the surgery formula to ignore all nontorsion Spin^c structures. In other words, we will study the mapping cone of Ψ^K, where

$$\Psi^K = \Phi^K_0 + \Phi^{-K}_0 : \mathfrak{A}_0 \longrightarrow CF_w.$$

Note that our constructions for the surgery formula must be restricted to be compatible with $\mathfrak{s}_0$; in other words we are restricting $\mathfrak{A}_0$ and CF_w to be generated only by the elements of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ that correspond to $\mathfrak{s}_0$. Note that we are now doing away with the $\mathfrak{B}_s$ notation, since there is only one copy of CF_w to keep track of. Furthermore, we will eliminate the s index from the Φ and $\mathcal{I}$ maps. We let $H_*(CF_p) = \mathcal{K}_p$ and $H_*(\mathfrak{A}_0) = \mathcal{K}_{z,w}$. It is important to note that from the surgery formula, $HF^\infty(Y_0(K), \mathfrak{s})$ is a free $\mathbb{F}[U, U^{-1}]$-module with

$$\operatorname{rk} HF^\infty(Y_0(K), \mathfrak{s}) = \operatorname{rk} \mathcal{K}_w + \operatorname{rk} \mathcal{K}_{z,w} - 2 \operatorname{rk}(\Psi^K_*). \tag{5}$$

We will further abuse notation; from now on, the symbol for any of the chain maps defined previously will refer to the induced map on homology, unless otherwise specified.

5. Example: $\mathbb{T}^3$

Recall that we are interested in calculating the Heegaard Floer homology of the manifolds M_n given in Figure 1. The main goal of this section is to understand the simplest nontrivial example, $M_1 = \mathbb{T}^3$. From Figure 1, we can represent M_n by 0-surgery on the knot Z_n in $S^2 \times S^1 \# S^2 \times S^1$ and can therefore apply the surgery formula. For M_1, this in fact gives 0-surgery on the Borromean rings, which is $\mathbb{T}^3$. The Heegaard Floer homology of $\mathbb{T}^3$ has already been calculated to have rank 6 in Proposition 1.9 of [11]. Analyzing this result via the surgery formula will allow us to deduce valuable information for the remaining M_n. But first, let us specialize to the case of $b_1 = 3$ for the universal coefficients spectral sequence with E^3 page $\Lambda^*(H^1(Y;\mathbb{Z})) \otimes \mathbb{Z}[U, U^{-1}]$ converging to $HF^\infty(Y, \mathfrak{s})$ mentioned in the introduction.

Let's study the differentials $d_k \colon E^k_{i,j} \to E^k_{i+k-1,j-k}$. Since each $E^2_{i,j}$ is a copy of $\Lambda^j(H^1(Y;\mathbb{Z}))$, the E_2 page is supported entirely in the region $0 \leq j \leq b_1(Y)$. Therefore, for $b_1 = 3$ the spectral sequence must collapse after d_3. In fact, the only possibly nontrivial component of d_3 maps from $\Lambda^3(H^1)$ to $\Lambda^0(H^1)$, each of which has rank 1. Therefore, to calculate d_3 for $b_1 = 3$, it suffices to find HF^∞. If HF^∞ has rank 8, then $d_3 \equiv 0$, and if HF^∞ has rank 6, then $d_3(\phi^1 \wedge \phi^2 \wedge \phi^3) = 1$.

Before dealing with M_1, we note that $M_0 = \#_{i=1}^3 S^2 \times S^1$ has $\operatorname{rk} HF^\infty(M_0) = 8$ by the connect-sum formula. Thus, this corresponds to d_3 being identically 0 in Equation (1). For $\mathbb{T}^3$, Conjecture 1.2 predicts that the map $d_3 \colon \Lambda^3(H^1) \to \Lambda^0(H^1)$ should be nonzero, which agrees with $\operatorname{rk} HF^\infty(\mathbb{T}^3, \mathfrak{s}_0) = 6$. We now want to use this fact to understand the map $\mathcal{D}^{-Z_1}$ in detail. We will ignore the underlying choice of Heegaard diagram for Z_1, since this will not show up in our calculations.

The best way to understand the calculation is via matrix representations, so we must pick out the right bases for $\mathcal{K}_{z,w}$, $\mathcal{K}_z$, and $\mathcal{K}_w$.

Let's fix our knot K. Define the map $\Theta^K : CF_z \to CF_w$ by $\Theta^K(x) = U^{A_K(x)}x$.

Proposition 5.1. $\Theta^K \circ \mathcal{I}^{-K} = \mathcal{I}^K$.

Proof. Add the powers of U together. □

This proposition shows that Θ^K must be a chain map and, like the inclusion maps, this is a quasi-isomorphism.

Lemma 5.2. Θ^K *preserves absolute Maslov gradings.*

Proof. We study $\Phi^K + \Phi^{-K} = (\Theta^K + \mathcal{D}^{-K}) \circ \mathcal{I}^{-K}$ on the chain level. We know that the invertible map $\mathcal{I}^{-K}$ preserves relative gradings by Proposition 4.2. Applying Remark 4.4 and factoring out $\mathcal{I}^{-K}$ shows $\Theta^K + \mathcal{D}^{-K}$ must be a homogeneous map that preserves gradings. However, $\mathcal{D}^{-K}$ preserves the absolute grading by Proposition 4.2. Therefore, Θ^K must preserve absolute gradings as well. □

As before, we will now use Θ^K to denote the induced map on homology. Observe that $\mathcal{K}_{z,w} \cong \mathcal{K}_z \cong \mathcal{K}_w \cong \mathbb{F}[U, U^{-1}] \otimes H^*(\mathbb{T}^2)$, by applying the connect-sum formula to $S^2 \times S^1 \# S^2 \times S^1$. We can choose ordered $\mathbb{F}$-bases (x_1, x_2) for $(\mathcal{K}_z)_0$ and (y_1, y_2) for $(\mathcal{K}_z)_1$. The key point about this choice is that the pairs live in adjacent Maslov gradings. This clearly gives an ordered $\mathbb{F}[U, U^{-1}]$-basis for the entire module. Furthermore, we use Θ^K to push this basis over to $\mathcal{K}_w$ to obtain a basis with the same properties. By Proposition 4.2, $\mathcal{D}^{-K}$ is represented by a matrix (we keep the same ordering between the bases) of the form

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}, \quad a, b, c, d, e, f, g, h \in \mathbb{F}.$$

Choose a basis for $\mathcal{K}_{z,w}$ such that $\mathcal{I}^{-K}$ can be represented by the identity. The next thing that we would like to understand is the matrix representation of $\mathcal{I}^K$.

Lemma 5.3. *With respect to these bases, $\mathcal{I}^K$ is represented by the identity.*

Proof. Because the representation for $\mathcal{I}^{-K}$ is the identity, Proposition 5.1 guarantees $\mathcal{I}^K$ and Θ^K will be represented by the same matrix. However, we know that Θ^K is represented by the identity by construction. □

We now specialize to the case of $K = Z_1$. Consider the collection of matrices

$$X = \left\{ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

Proposition 5.4. *The map $\mathcal{D}^{-Z_1}$ is represented by a matrix in X.*

Proof. Note that the rank of $\Phi^{Z_1} + \Phi^{-Z_1}$ must be precisely 1. This follows from Equation (5) as $HF^\infty(\mathbb{T}^3, \mathfrak{s}_0)$ has rank 6 and both $\mathcal{K}_w$ and $\mathcal{K}_{z,w}$ have rank 4. Since $\Phi^{Z_1} + \Phi^{-Z_1}$ is represented by

$$\begin{pmatrix} a+1 & b & 0 & 0 \\ c & d+1 & 0 & 0 \\ 0 & 0 & e+1 & f \\ 0 & 0 & g & h+1 \end{pmatrix},$$

exactly three of the two-by-two blocks must be identically 0 and the other must have rank 1. It is easy to check that each of the matrices in X have this property. Either $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right)$ must be the identity. Without loss of generality, we assume $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now, the possible blocks $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F})$ that don't appear in matrices in X are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Direct calculation shows that $\Phi^{Z_1} + \Phi^{-Z_1}$ would have either rank 0 or rank 2 in any of these cases, which would be a contradiction. Repeating the arguments with the top-left and bottom-right blocks switched discounts all of the other matrices not in X. ☐

Remark 5.5. We note that Proposition 5.4 does not apply to every knot in $S^2 \times S^1 \# S^2 \times S^1$. Doing 0-surgery on the split unknot, Z_0, to get $\#_{i=1}^3 S^2 \times S^1$, which has rank 8, shows that $\Phi^{Z_0} = \Phi^{-Z_0}$. This in fact means that after this choice of bases, $\mathcal{D}^{-Z_0}$ must be the identity.

After choosing bases analogously, it remains to analyze $\mathcal{D}^{-Z_n}$ to yield the calculation for M_n ($n \geq 2$). To do this, we rephrase the computation as an iteration of what we've done for $\mathbb{T}^3$ using a technique we call composing knots.

6. Composing knots and the calculation for M_n

Recall that given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, any two points on $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ determine a knot, K, in Y. Now, suppose there are instead 3 distinct points, z, u, and w. Then the pairs of basepoints, (z, u), (u, w), (z, w), determine three knots. We want to consider Heegaard diagrams containing this information. We will ignore concerns with orientations, since these will not arise in our setting. Finally, knots will always be nullhomologous.

Definition 6.1. A *Heegaard diagram* for (K, K_1, K_2) in Y is a Heegaard diagram for Y, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, equipped with 3 distinct basepoints z, u, and w, in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$, such that (z, u), (u, w), and (z, w) determine K_1, K_2, and K respectively.

Proposition 6.2. *Consider a Heegaard diagram for* (K, K_1, K_2). *We have that* $\mathcal{D}^{-K} = \mathcal{D}^{-K_2} \circ \mathcal{D}^{-K_1}$.

Proof. The map $\mathcal{D}^{-K_1}$ is induced by a sequence of Heegaard moves taking $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u)$ and $\mathcal{D}^{-K_2}$ comes from a sequence of moves from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, u)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$. Therefore, the composition of isotopies and handleslides goes from $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ and gives us a choice of $\mathcal{D}^{-K}$. □

Remark 6.3. In this setup, the concatenation of a good set of trajectories from z to u and from u to w gives a good set of trajectories from z to w, so there are still no concerns with our choice of paths.

Thus, since most of the complexity in the knot surgery formula for HF^∞ comes from the map $\mathcal{D}^{-K}$, having a Heegaard diagram for (K, K_1, K_2) and an understanding of each $\mathcal{D}^{-K_i}$ should make the computation more manageable. This is the approach we will use for the rest of the M_n. However, we must first establish that such things exist and more importantly, derive a way of relating this information to the M_n.

Lemma 6.4. *Suppose K_1 and K_2 are knots in Y where $K_1 \cap K_2$ is an embedded connected interval. Then if K is the knot obtained from $(K_1 \cup K_2) - K_1 \cap K_2$ (see Figure 2), there exists a Heegaard diagram for (K, K_1, K_2).*

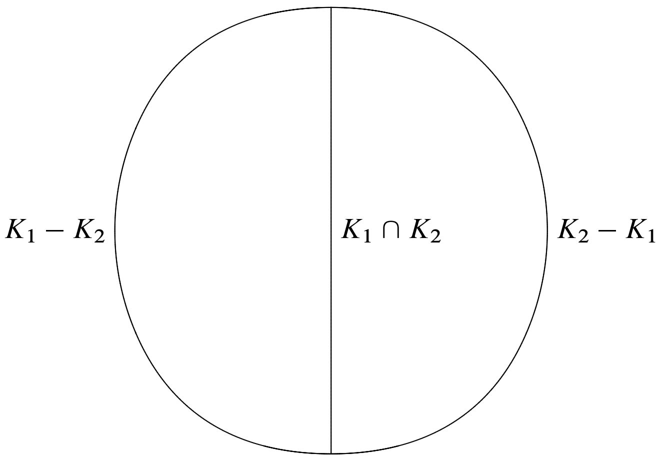

Figure 2. Each simple cycle corresponds to a knot.

Proof. The idea follows the construction of Heegaard diagrams for knots in [17]. Begin with a self-indexing Morse function, $h\colon S^3 \to [0, 3]$, with exactly two critical points. Note that traversing a flow from index 0 to index 3 and then another flow in "reverse" gives a knot. Thus, three flow lines give three knots in a natural way as before (see Figure 3).

Choose a small neighborhood, U, of three flow lines between the two points. Identify a neighborhood of $K_1 \cup K_2$ in Y, N, with U such that each knot gets

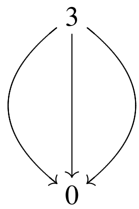

Figure 3. Three flow lines forming three knots.

mapped to the union of two of the three flows. We will now use h to refer to the induced Morse function on N, with index 0 and index 3 critical points, p and q. Extend h to a Morse function f on all of Y such that it is still self-indexing. If there were no other index 0 or index 3 critical points, then we could construct the desired Heegaard diagram simply by choosing the three basepoints to be where the three flow lines pass through the Heegaard surface, $f^{-1}(3/2)$. The idea is to cancel any critical points of index 0 or 3 outside of N, without affecting $f|_N$.

If such critical points exist, we rescale the Morse function in a neighborhood of p and q so as to not affect the critical points, but make $h(p) = -\epsilon$ and $h(q) = 3+\epsilon$ (and thus the same for f). Now, remove the balls $\{f > 3+\epsilon/2\}$ and $\{f < -\epsilon/2\}$ around the index 0 and index 3 critical points from N, to obtain a cobordism $W\colon S^2 \to S^2$. In the terminology of [9], this is a self-indexing Morse function on the triad (W, S^2, S^2). Since each manifold in the triad is connected, we know that for each index 0 critical point, there is a corresponding index 1 with a single flow line traveling to the index 0. This pair can be canceled such that the Morse function will not be changed outside of a neighborhood of the flow line between them. We want to see that by perhaps choosing a smaller neighborhood, N', of the knots inside of N, this flow line does not hit N'. This must be the case because if no such neighborhood existed, by compactness, this flow line would have to intersect K_1 or K_2. But these are flows of f themselves, so the two lines cannot intersect.

Hence, we can alter f to remove the index 0/1 pair without affecting $f|_{N'}$. By repeating this argument and an analogous one for index 2/3 pairs, we can remove all of the critical points of index 0 and 3 in W in this fashion. This says, after rescaling the function on the neighborhoods of p and q back to their original values, the new Morse function is self-indexing on Y with exactly one index 0 and one index 3 critical point, and furthermore, still agrees with h when restricted to a small enough neighborhood of the knots. This is exactly what we want to give the desired Heegaard diagram. □

Consider the link in the Kirby diagram for M_n, Figure 1. Since Z_n is the knot which we will apply the surgery formula to, we would like a way to decompose Z_n and apply Lemma 6.4.

Proposition 6.5. *For each n, there exists a Heegaard diagram for* (Z_n, Z_1, Z_{n-1}) *in* $S^2 \times S^1 \# S^2 \times S^1$.

Proof. Let us first study Figure 4. Here we have attached an arc to Z_n at two points (the large black dots). This creates two additional knots as follows. Note that one can travel two different paths from the bottom attachment point to the top attachment point; we may either wind in an upward spiral once around the two vertical strands or follow the path that begins by winding downward $n-1$ times. Beginning at the top attachment point, following the attaching arc to the bottom point, and finally traversing one of the two winding paths back to the top point gives either Z_1 or Z_{n-1}. We are now in the position to apply Lemma 6.4 to Z_n, Z_1, and Z_{n-1}. □

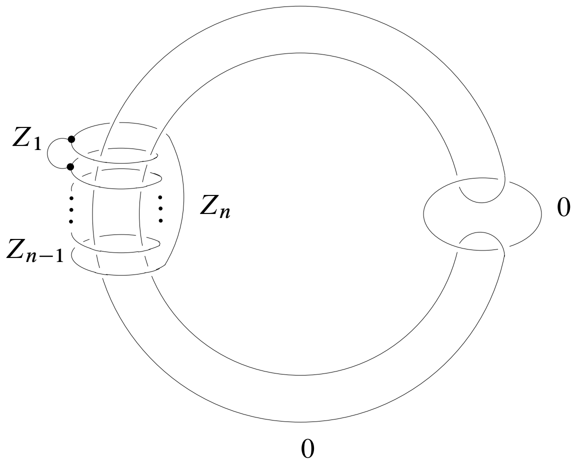

Figure 4. Splitting of Z_n into Z_{n-1} and Z_1.

When applying the surgery formula for $\mathbb{T}^3$, it was critical to use the map Θ^K to make all of the inclusions consistently identity matrices. The following lemma will allow us to do this in general.

Lemma 6.6. *Consider a Heegaard diagram for* (K, K_1, K_2). *Then* $\Theta^K = \Theta^{K_2} \circ \Theta^{K_1}$.

Proof. Consider the Alexander gradings for the three knots in the diagram.

$$\begin{aligned} A_K(x) - A_K(y) &= n_z(\phi) - n_w(\phi) \\ &= n_z(\phi) - n_u(\phi) + n_u(\phi) - n_w(\phi) \\ &= A_{K_1}(x) - A_{K_1}(y) + A_{K_2}(x) - A_{K_2}(y) \end{aligned}$$

for each $\phi \in \pi_2(x, y)$. Therefore, the relative Alexander grading for K is the sum of the relative Alexander gradings for K_1 and K_2. Thus, the absolute Alexander grading for K is the sum of the absolute Alexander gradings for K_1 and K_2 plus an additional constant. Therefore, $\Theta^K = U^\ell \cdot \Theta^{K_2} \circ \Theta^{K_1}$, for some $\ell \in \mathbb{Z}$. By Proposition 5.2, the Θ maps preserve absolute Maslov gradings, so we know that $\ell = 0$. □

Fix a Heegaard diagram as given by Proposition 6.5. We now will choose the proper bases as in the $\mathbb{T}^3$ example. Figure 5 will provide a useful visual reference for the upcoming proposition.

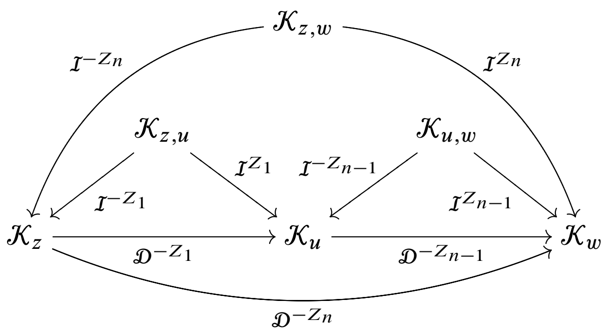

Figure 5. The setup that appears in Proposition 6.7.

Proposition 6.7. *Following Section 5, choose bases for $\mathcal{K}_{z,u}$, $\mathcal{K}_z$, and $\mathcal{K}_u$, such that the inclusions and Θ^{Z_1} are given by the identity and the map $\mathcal{D}^{-Z_1}$ is a matrix in X. Now, choose bases for $\mathcal{K}_w$ and $\mathcal{K}_{u,w}$ such that the inclusions and $\Theta^{Z_{n-1}}$ are the identity. There exists a basis for $\mathcal{K}_{z,w}$ such that $\mathcal{I}^{-Z_n}$, $\mathcal{I}^{Z_n}$, and Θ^{Z_n} are given by the identity.*

Proof. Clearly we can fix a basis for $\mathcal{K}_{z,w}$ such that $\mathcal{I}^{-Z_n}$ is the identity. Now, we combine the fact that $\mathcal{I}^{Z_n} = \Theta^{Z_n} \circ \mathcal{I}^{-Z_n}$ with $\Theta^{Z_n} = \Theta^{Z_{n-1}} \circ \Theta^{Z_1} = I$, to get the required result. □

Remark 6.8. These constructions could be generalized to any number of basepoints (and the corresponding larger number of induced knots), but we only need three basepoints for our purposes.

Although $\mathcal{D}^{-Z_{n-1}}$ is not necessarily represented by an element of X in this diagram, we do know that it comes in the form of $A \oplus B$ for $A, B \in \mathrm{GL}_2(\mathbb{F})$, since $\mathcal{D}^{-Z_{n-1}}$ preserves absolute gradings.

Remark 6.9. While the individual matrix representations may seem to depend on the choice of Heegaard diagram, if $\mathcal{D}^{-K} = I$, this is independent of the diagram as long as the bases are chosen such that $\mathcal{I}^K = \Theta^K = I$. A similar statement based on the work of Section 5 can be made about $\mathfrak{D}^{-K}$ being in X regardless of diagram.

We are now ready for the calculation of the maps $\mathfrak{D}^{-Z_n}$ for all n.

Theorem 6.10. *Begin with a diagram for (Z_{2n+1}, Z_1, Z_{2n}). After a choice of bases given by Proposition 6.7 we have that $\mathfrak{D}^{-Z_{2n}}$ is the identity and $\mathfrak{D}^{-Z_{2n+1}}$ is a matrix in X for all $n \geq 0$.*

Proof. For $n = 0$, we know that the map $\mathfrak{D}^{-Z_0}$ must be the identity in order to have rk $HF^\infty(\#_{i=1}^3 S^2 \times S^1) = 8$. Similarly, from our computation for $\mathbb{T}^3$, we have seen that $\mathfrak{D}^{-Z_1}$ is in X. Thus, the base case is established. For the induction step, note that as soon as $\mathfrak{D}^{-Z_{2n}}$ is the identity, we can compose with $\mathfrak{D}^{-Z_1}$ to get that $\mathfrak{D}^{-Z_{2n+1}}$ is of type X. Thus, we only need to find $\mathfrak{D}^{-Z_{2n}}$.

By hypothesis, $\mathfrak{D}^{-Z_{2n-1}} \in X$. The first case we consider is if $\mathfrak{D}^{-Z_1}$ and $\mathfrak{D}^{-Z_{2n-1}}$ were to be represented by two different elements of X when considering bases chosen for (Z_{2n}, Z_1, Z_{2n-1}). If this were to happen, then the product of the matrices, which gives a representative for $\Phi^{-Z_{2n}}$, has the property that its sum with the identity, $\Phi^{Z_{2n}}$, has rank at least 2. However, this is impossible by the rank bounds coming from the spectral sequence. Therefore, both $\mathfrak{D}^{-Z_{2n-1}}$ and $\mathfrak{D}^{-Z_1}$ are represented by the same matrix. But, every element of X squares to the identity. $\mathfrak{D}^{-Z_{2n}}$ must then be the identity. □

Proof of Theorem 1.4. We apply Theorem 6.10 to see that the rank of $\Phi^{-Z_{2n}} + \Phi^{Z_{2n}}$ is equal to that of $\Phi^{-Z_0} + \Phi^{Z_0}$. Therefore, $HF^\infty(M_{2n}, \mathfrak{s}_0)$ and $HF^\infty(M_0, \mathfrak{s}_0)$ are isomorphic by Equation (5). Similarly, we see that $\Phi^{-Z_{2n+1}} + \Phi^{Z_{2n+1}}$ and $\Phi^{-Z_1} + \Phi^{Z_1}$ have the same rank. Thus, $HF^\infty(M_{2n+1}, \mathfrak{s}_0) \cong HF^\infty(M_1, \mathfrak{s}_0)$. But, this shows exactly that d_3 must satisfy $x_1 \wedge x_2 \wedge x_3 \mapsto \langle x_1 \cup x_2 \cup x_3, [Y] \rangle \pmod 2$ by the discussion at the beginning of Section 5. □

7. Calculations for $b_1 = 4$

Recall from Proposition 3.3 that if $b_1(Y) = 4$, then Y has integral triple cup product form isomorphic to that of $M_n \# S^2 \times S^1$ for some n. We then choose a basis for $H^1(Y; \mathbb{Z})$, $\{x_1, x_2, x_3, x_4\}$, with the property that $\iota_{\mu_Y}(x_1 \wedge x_2 \wedge x_3) = n$ and $\iota_{\mu_Y}(x_i \wedge x_j \wedge x_4) = 0$ for all i and j.

Theorem 7.1. *Let $\mathfrak{s}$ be torsion. If n is even, $HF^\infty(Y, \mathfrak{s})$ has rank 16. For n odd, $HF^\infty(Y, \mathfrak{s})$ has rank 12.*

Proof. As before, we simply need to calculate HF^∞ for $M_n \# S^2 \times S^1$. By the connect sum formula, $HF^\infty(Y, \mathfrak{s}) \cong HF^\infty(M_n, \mathfrak{s}_0) \otimes (\mathbb{F}[U, U^{-1}])^2$. Therefore, applying the work of the previous section gives the result. □

Proof of Theorem 1.5. To see that the homology agrees with the differential coming from the conjecture, we just need to study the predicted differential d_3. If n is even, then we have the result, since both homologies are rank 16, as $d_3 \equiv 0$.

Now consider the case where n is odd. Note that d_3 is identically 0 on each $\Lambda^i(H^1(Y;\mathbb{Z}))$ except for $x_1 \wedge x_2 \wedge x_3$ and $x_1 \wedge x_2 \wedge x_3 \wedge x_4$. Therefore, the kernel of d_3 has rank 14 and the rank of d_3 is 2. This gives the desired rank of 12. □

8. Proof of the existence of homologically split surgery presentations

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We start with a discussion of some results from algebra. A *lattice* is a free $\mathbb{Z}$-module of finite rank, together with a nondegenerate symmetric bilinear form taking values in $\mathbb{Z}$. A lattice S is called *odd* if there exists $t \in S$ with $t \cdot t \in \mathbb{Z}$ being odd. By $S_1 \oplus S_2$ we denote the orthogonal direct sum of two lattices.

The bilinear form of a lattice S determines an embedding of S into $S^* = \mathrm{Hom}(S,\mathbb{Z})$. The factor group $A_S = S^*/S$ is a finite Abelian group. It comes naturally equipped with a bilinear form

$$b_S\colon A_S \times A_S \to \mathbb{Q}/\mathbb{Z}, \quad b_S(t_1 + S, t_2 + S) = t_1 \cdot t_2 + \mathbb{Z},$$

called the *discriminant-bilinear form* of S.

The following results are taken from the literature; see [3], [2], [20], [10]:

Theorem 8.1 (Kneser–Puppe, Durfee). *Two lattices S_1 and S_2 have isomorphic discriminant-bilinear forms if and only if there exist unimodular lattices L_1, L_2 such that $S_1 \oplus L_1 \cong S_2 \oplus L_2$.*

Theorem 8.2 (Milnor). *Let S be an indefinite, unimodular, odd lattice. Then $S \cong m\langle 1\rangle \oplus n\langle -1\rangle$ for some $m, n \geq 1$.*

We say that two lattices S_1, S_2 are *stably equivalent* if there exist nonnegative integers m_1, n_1, m_2, n_2 such that the stabilized lattices

$$S_1' = S_1 \oplus m_1\langle 1\rangle \oplus n_1\langle -1\rangle,$$

$$S_2' = S_2 \oplus m_2\langle 1\rangle \oplus n_2\langle -1\rangle$$

are isomorphic.

Note that for any lattice S, the direct sum $S \oplus \langle 1\rangle \oplus \langle -1\rangle$ is indefinite and odd. Therefore, an immediate consequence of Theorems 8.1 and 8.2 is:

Corollary 8.3. *Two lattices are stably equivalent if and only if they have isomorphic discriminant-bilinear forms.*

Observe that we can restate Theorem 8.2 by saying that all unimodular lattices are stably diagonalizable. This is not the case for general lattices. Indeed, Corollary 8.3 shows that a lattice is stably diagonalizable if and only if its discriminant-bilinear form comes from a diagonal lattice. Wall [21] classified nonsingular bilinear forms on finite Abelian groups and showed that any such form can appear as a discriminant-bilinear form of a lattice; see also Proposition 1.8.1 of [10]. The classification contains non-diagonal forms. As a consequence, for example, the lattice of rank two given by the matrix

$$H_2 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

is not stably diagonalizable.

Following the classification scheme for discriminant-bilinear forms (see Proposition 1.8.2 of [10]), we see that given any discriminant-bilinear form A_S, there exists A_L coming from a (not necessarily unimodular) diagonal lattice, L, such that $A_S \oplus A_L$ is isomorphic to $A_{L'}$, where L' is also a diagonal lattice. Applying Corollary 8.3 we obtain the following result:

Proposition 8.4. *For any lattice S, there exists a diagonal lattice L (not necessarily unimodular), such that $S \oplus L$ is diagonalizable.*

For example, $H_2 \oplus \langle 2 \rangle$ is isomorphic to $\langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle$.

Remark 8.5. Any degenerate symmetric bilinear form over $\mathbb{Z}$ can be expressed as a direct sum of a non-degenerate form and some zeros. Hence, the result of Proposition 8.4 applies to all symmetric bilinear forms (not necessarily non-degenerate).

Proof of Lemma 2.4. Let Y be a 3-manifold. We represent it by surgery on S^3 along a framed link, with linking matrix S. Handleslides and stabilizations correspond to elementary operations (integral changes of basis and direct sums with $\langle \pm 1 \rangle$) on the bilinear form of S. Since a connect sum with $L(m,1)$ can be presented by m-surgery on a split unknot, this corresponds to a direct sum of the linking matrix of Y with the diagonal lattice $\langle m \rangle$. Proposition 8.4 and Remark 8.5 complete the proof. $\square$

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