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Injective modules and amenable groups

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Abstract. We show that a locally compact group is amenable if and only if it admits a (non-zero) injective Banach module that is reflexive as a Banach space.

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1. Introduction

Let A be a Banach algebra. By a left A-module we shall always mean a Banach left A-module satisfying $\|ax\| \leq \|a\|\,\|x\|$ whenever $a \in A$ and $x \in X$, and a morphism of left A-modules will be a bounded linear map commuting with the respective actions. X is called injective, cf. [H], III.1.14, p. 136, if for any morphism ι of left A-modules admitting a bounded linear left inverse, ℓ, and any morphism λ_0 from Y_0 into X, there is a morphism λ from Y into X satisfying $\lambda_0 = \lambda \circ \iota$,

$$\begin{array}{ccccc} Y_0 & \xrightarrow{\ \iota\ } & Y & \xrightarrow{\ \ell\ } & Y_0, \qquad \ell \circ \iota = \mathrm{id}_{Y_0}. \\ {\scriptstyle\lambda_0}\downarrow & \swarrow{\scriptstyle\lambda} & & & \\ X & & & & \end{array}$$

Let the essential part, X_e, of a left A-module X be defined as the closed linear hull of the set of products ax, $a \in A$, $x \in X$. X is called non-zero if $X_e \neq 0$, essential if $X_e = X$, and reflexive if X is reflexive as a Banach space. In case X is reflexive and A has a bounded two-sided approximate unit (of norm $\leq c$), there is an A-module morphism (of norm $\leq c$) projecting X onto X_e. The Banach space dual, X^*, of X becomes a right A-module under the action defined by $\langle x, x^*a\rangle = \langle ax, x^*\rangle$, for $x^* \in X^*$, $a \in A$, and $x \in X$.

Choosing a left invariant Haar measure on the locally compact group G we obtain the Banach algebra $L^1(G)$. It is well known that every essential left $L^1(G)$-module is a left G-module such that, for any $x \in X$, the mapping $s \mapsto sx$ is continuous from G into X and $\|sx\| = \|x\|$, $s \in G$, the respective actions being related by the formula $ax = \int a(s)sx\,ds$, for $a \in L^1(G)$ and $x \in X$. This same formula defines on any such left G-module an essential left $L^1(G)$-action.

Letting G act by left translation on $L^p(G)$, $1 < p < \infty$, $L^p(G)$ becomes an essential reflexive left $L^1(G)$-module. H. G. Dales, M. Daws, H. L. Pham and P. Ramsden recently showed the following theorem, [DDPR], Theorem 9.6.

Theorem ([DDPR]). *Let G be a locally compact group, and $1 < p < \infty$. If the left $L^1(G)$-module $L^p(G)$ is injective, then G is amenable.* □

Employing F. J. Yeadon's method, [Y], for establishing the existence of a trace in a finite von Neumann algebra, we show

Proposition. *Let G be a locally compact group. If G admits a non-zero injective Banach left $L^1(G)$-module that is reflexive as a Banach space, then G is amenable.*

Combining this with known results we obtain the following characterization of compact and amenable groups, in good correspondence with Helemskii's philosophy, cf. e.g. [H], p. 262.

Corollary. *Let G be a locally compact group.*

a) *If G admits a non-zero projective left $L^1(G)$-module that is reflexive as a Banach space, then G is compact; if, conversely, G is compact then every essential left $L^1(G)$-module is projective.*
b) *If G admits a non-zero flat left $L^1(G)$-module that is reflexive as a Banach space, then G is amenable; if, conversely, G is amenable then every left $L^1(G)$-module is flat.*

These results are equally valid for uniformly bounded, left or right Banach $L^1(G)$-modules. For the notion of the injective tensor product, $\check{\otimes}$, of Banach spaces we refer to the monograph of J. Cigler, V. Losert and P. Michor, [CLM]. The proof of the Proposition starts immediately after this introduction.

2. The auxiliary module $C^{bu}(G) \check{\otimes} X$

The G-action on $C^{bu}(G) \check{\otimes} X$ and the morphism ι below were already considered by P. Ramsden, [Ra], Chapter 5, p. 21; cf. also Chapter 9 of [DDPR].

2.1. Let G be a locally compact group, and X be an essential Banach left $L^1(G)$-module, with $sx, s \in G, x \in X$, denoting the corresponding G-action. We let G act on the Banach space, $C^{bu}(G)$, of uniformly continuous bounded functions on G by left translation $(L_s\varphi)(t) = \varphi(s^{-1}t)$, $s \in G, \varphi \in C^{bu}(G)$, so that the injective tensor product $C^{bu}(G) \check{\otimes} X$ becomes a continuous isometric Banach left G-module under the action $s(\varphi \otimes x) = L_s\varphi \otimes sx$, $s \in G$, $\varphi \otimes x \in C^{bu}(G) \check{\otimes} X$.

The morphism $\iota\colon X \to C^{bu}(G) \check{\otimes} X$ is defined by $\iota x = 1_G \otimes x$, $x \in X$, 1_G the function constant one on G, and for any $s \in G$ the bounded linear map $\ell\colon C^{bu}(G) \check{\otimes} X \to X$, $\ell(\varphi \otimes x) = \varphi(s)x$, $\varphi \in C^{bu}(G)$, $x \in X$, is left inverse to ι.

In case the essential left $L^1(G)$-module X is injective, setting $Y_0 = X$, $Y = C^{bu}(G) \check{\otimes} X$, and $\lambda_0 = \mathrm{id}_X$ in the diagram on p. 1023 yields a morphism λ of $L^1(G)$-modules left inverse to ι,

$$X \xrightarrow{\iota} C^{bu}(G) \check{\otimes} X \xrightarrow{\lambda} X.$$

Since λ commutes also with the G-actions, the map λ enjoys the following properties:

(i) λ is linear and bounded;

(ii) $\lambda(L_s\varphi \otimes sx) = s\lambda(\varphi \otimes x)$;

(iii) $\lambda(1_G \otimes x) = 1$,

whenever $s \in G$, $\varphi \in C^{bu}(G)$, and $x \in X$.

2.2. Remark Instead of $C^{bu}(G)$ we could also take $L^\infty(G)$, Corollary 3.7 below equally applying to it. By using the module $C^{bu}(G) \check{\otimes} X$, suggested by the referee, however, we shall obtain: *If an arbitrary topological group G admits a non-zero relatively injective Banach left G-module X that is reflexive as a Banach space, then G is amenable.* For the relevant notions we refer to N. Monod's Lecture Notes, [M], Definition 4.1.2, p. 32, and the definition preceding 5.1.4, p. 46.

3. Weakly compact operators on $C(K) \check{\otimes} X$

The formulation of the main lemma, (3.5) below, is due to the referee.

3.1. Let K be a compact Hausdorff space, and X be a Banach space. It is well known that the dual space of the injective tensor product $C(K) \check{\otimes} X = C(K, X)$ is isometrically isomorphic to the Banach space, $I(C(K), X^*)$, of integral operators v from $C(K)$ into X^*, and that this again is isometrically isomorphic to the Banach space, $bvrca(B(K), X^*)$, of regular countably additive vector measures m of bounded variation on the Borel σ-algebra, $B(K)$, of K with values in X^*,

$$(C(K) \check{\otimes} X)^* = I(C(K), X^*) = bvrca(B(K), X^*),$$

the correspondence between v and m being given by $m(A) = \tilde{v}(c_A)$, $A \in B(K)$, where $\tilde{v}\colon C(K)^{**} \to X^*$ denotes the unique weak*-weak* continuous extension of v

and c_A the characteristic function of A. The variation, $|m|$, of $m \in bvrca(B(K), X^*)$, defined as

$$|m|(A) = \sup \sum \|m(A_i)\| \quad (A \in B(K)),$$

the supremum being taken over all finite Borel partitions (A_i) of A, is a regular finite positive Borel measure on K. Defining the norm of $m \in bvrca(B(K), X^*)$ by $\|m\| = |m|(K)$, we have $\|m\| = I(v)$, the integral norm of $v \in I(C(K), X^*)$ corresponding to m. – The theorems involved in this discussion are due to I. Singer, [S]; cf. also VI.3.Theorem 3, p. 162, and VI.3.Theorem 12, p. 169, in [DU], and, in particular, Satz 1 in Losert's Thesis, [L], p. 7.

We shall need the following two lemmas.

3.2 Lemma ([Gro], Théorème 2). *Let K be a compact Hausdorff space. A bounded subset C of $C(K)^*$ is relatively weakly compact if and only if for every sequence (A_n) of pairwise disjoint open subsets of K we have*

$$\lim_n \mu(A_n) = 0$$

uniformly for μ in C. □

3.3 Lemma. *Let K be a compact Hausdorff space, and X be a Banach space. If D is a relatively weakly compact subset of $(C(K) \check{\otimes} X)^*$, then the set, $|D|$, of variations of its corresponding vector measures is relatively weakly compact in $C(K)^*$.*

Proof. Let D be a relatively weakly compact subset of $(C(K) \check{\otimes} X)^*$. Using the identification in (3.1), we may assume D to be relatively weakly compact in $bvrca(B(K), X^*)$; being a closed subspace of the Banach space $bvca(B(K), X^*)$ of all countably additive measures of bounded variation, it is relatively weakly compact also there. Theorem 1.ii) in [B], p. 288, yields a finite positive measure ν on $B(K)$ such that the set $|D| = \{|m| : m \in D\}$ is ν-equicontinuous. For any sequence (A_n) of disjoint Borel subsets of K, $\lim \nu(A_n) = 0$ therefore implies $\lim |m|(A_n) = 0$ uniformly for m in D. The elements of the set $|D|$ being all regular, its relative weak compactness in $C(K)^*$ results now, for instance, from (3.2). □

3.4. Let X and Y be Banach spaces, and $u\colon C(K) \check{\otimes} X \to Y$ a bounded linear map with adjoint $u^*\colon Y^* \to (C(K) \check{\otimes} X)^* = I(C(K), X^*)$. Any pair of elements (x, y^*) in $X \times Y^*$ defines an element u_{x,y^*} of $C(K)^*$ by

$$u_{x,y^*}(\varphi) = \langle u(\varphi \otimes x), y^* \rangle, \quad \varphi \in C(K),\ x \in X,\ y^* \in Y^*.$$

Denoting by $(u^* y^*)^\sim\colon B(K) \to X^*$ the vector measure corresponding to $u^* y^*\colon C(K) \to X^*$, we deduce from

$$u_{x,y^*}(\varphi) = \langle \varphi \otimes x, u^* y^* \rangle = \langle x, u^* y^*(\varphi) \rangle, \quad \varphi \in C(K),$$

that

$$u_{x,y^*}(A) = \langle x, (u^* y^*)^{\sim}(A)\rangle, \quad A \in B(K),$$

for all $x \in X$, $y^* \in Y^*$.

3.5 Lemma. *Let K be a compact Hausdorff space, X and Y be Banach spaces, and u be a weakly compact linear map from $C(K) \check{\otimes} X$ into Y. Then the set*

$$\{u_{x,y^*} : \|x\| \leq 1, \|y^*\| \leq 1\}$$

is relatively weakly compact in $C(K)^*$.

Proof. Let (A_n) be a sequence of pairwise disjoint open subsets of K, and $\varepsilon > 0$. As $u^* : Y^* \to (C(K) \check{\otimes} X)^*$ is equally weakly compact, the image, $u^*(OY^*)$, of the unit ball of Y^* is relatively weakly compact in $(C(K) \check{\otimes} X)^*$, and so is the set, $|u^*(OY^*)|$, of variations of its corresponding vector measures in $C(K)^*$, by (3.3). Lemma (3.2) furnishes an index n_0 such that

$$|(u^* y^*)^{\sim}|(A_n) \leq \varepsilon \quad (\|y^*\| \leq 1, \ n \geq n_0),$$

implying, for all $x \in X$ and $y^* \in Y^*$ of norm ≤ 1,

$$\begin{aligned} |u_{x,y^*}(A_n)| &= |\langle x, (u^* y^*)^{\sim}(A_n)\rangle| \\ &\leq \|x\| \|(u^* y^*)^{\sim}(A_n)\| \\ &\leq |(u^* y^*)^{\sim}|(A_n) \\ &\leq \varepsilon \quad (n \geq n_0), \end{aligned}$$

thus proving the assertion, again by (3.2). □

3.6. Each of the following conditions on X and Y assures the weak compactness of any bounded linear map from $C(K) \check{\otimes} X$ into Y:

(a) X is arbitrary and Y reflexive;

(b) X^* has the Radon–Nikodym property and Y is weakly sequentially complete, cf. [G];

(c) X is a C^*-algebra and Y is weakly sequentially complete, cf. [ADG], Theorem 4.2, p. 449.

3.7 Corollary. *Let G be a locally compact group, X a reflexive Banach space, and u a bounded linear map from $C^{bu}(G) \check{\otimes} X$ into X. Then the set*

$$\{u_{x,x^*} : \|x\| \leq 1, \|x^*\| \leq 1\}$$

is relatively weakly compact in $C^{bu}(G)^*$.

Proof. $C^{bu}(G)$ being a commutative C^*-algebra with unit, there exist a compact Hausdorff space K and an isomorphism from $C^{bu}(G)$ onto $C(K)$ so that (3.5) applies. □

3.8 Remark (by the referee). In case X is reflexive (and therefore X and X^* enjoy the Radon–Nikodym property), one can deduce (3.5) directly from the vector-valued version of Grothendieck's criterion (3.2), as stated in the middle of p. 117 in [DU].

4. Proof of the Proposition

Let G be a locally compact group and X a non-zero injective left $L^1(G)$-module, reflexive as a Banach space. Since $L^1(G)$ possesses bounded approximate units, the essential part of X – being $L^1(G)$-module complemented in X – is equally injective, and reflexive, so that we may assume X from the outset to be essential itself. Let then $\lambda\colon C^{bu}(G)\check{\otimes} X \to X$ be a map satisfying (2.1) (i), (ii), (iii). For any fixed pair $(x,x^*)\in X\times X^*$, $\langle x,x^*\rangle = 1$, the element λ_{x,x^*} in $C^{bu}(G)^*$, $\lambda_{x,x^*}(\varphi)=\langle\lambda(\varphi\otimes x),x^*\rangle$, $\varphi\in C^{bu}(G)$, enjoys the following two properties:

(iv) $\lambda_{x,x^*}(1_G)=1$;

(v) $\{L_s^*\lambda_{x,x^*} : s\in G\}$ is relatively weakly compact in $C^{bu}(G)^*$.

(iv) follows immediately from (2.1.iii); to see (v), we use (2.1.ii) to compute, with $\varphi\in C^{bu}(G)$ and $s\in G$,

$$\begin{aligned} L_s^*\lambda_{x,x^*}(\varphi) &= \lambda_{x,x^*}(L_s\varphi)\\ &= \langle\lambda(L_s\varphi\otimes x),x^*\rangle\\ &= \langle\lambda(L_s\varphi\otimes ss^{-1}x),x^*\rangle\\ &= \langle s\lambda(\varphi\otimes s^{-1}x),x^*\rangle\\ &= \langle\lambda(\varphi\otimes s^{-1}x),\ x^*s\rangle\\ &= \lambda_{s^{-1}x,x^*s}(\varphi)\quad (s\in G,\ \varphi\in C^{bu}(G)). \end{aligned}$$

Since $\|s^{-1}x\|=\|x\|$ and $\|x^*s\|=\|x^*\|$, $s\in G$, the assertion now follows from (3.7).

It ensues that the closed convex hull, C, of $\{L_s^*\lambda_{x,x^*} : s\in G\}$ is a weakly compact convex subset of $C^{bu}(G)^*$. Being invariant under the group of linear isometries L_s^*, $s\in G$, Ryll-Nardzewski's fixed point theorem yields an element M of C satisfying $L_s^*M=M$, $s\in G$, and, in virtue of (iv), $M(1_G)=1$. Decomposing M into its selfadjoint parts and these into their positive ones, we obtain, possibly after rescaling, a positive linear functional on $C^{bu}(G)$, left invariant and taking the value one at the constant function 1_G, thus establishing the amenability of G; cf. [Gr], Theorem 2.2.1, p. 26. □

5. Proof of the Corollary

For the definition of projective and flat Banach modules over a Banach algebra we refer to [H], III.1.14, p. 136, and [H], VII.1.2, p. 239, respectively. Rather than reproducing them here, we note only that every projective module is flat, and that a module X is flat if and only if its dual module, X^*, is injective, cf. [H], VII.1.14, p. 243.

5.1. Proof of Corollary a. Let X be a non-zero projective left $L^1(G)$-module that is reflexive as a Banach space. Since X_e is module-complemented in X, X_e is also projective, and reflexive, so that G is compact, by [R1], 1.4, p. 316. (It is shown there that a locally compact group is already compact, if it admits a non-zero essential projective left $L^1(G)$-module X whose dual Banach space, X^*, is weakly sequentially complete or norm separable.) The second statement is also proved there, [R1], 1.2, p. 316. □

The second part of Corollary b is equally well known. In [H], VII.2.29, p. 257, it is deduced from the vanishing of the Tor functor over an amenable algebra, or can be seen, more directly, from B. E. Johnson's original definition, [J], p. 60, as follows.

5.2 Lemma ([H]). *Let A be an amenable Banach algebra. Then all Banach* (*left, right, or bi-*) *modules over A are flat.*

Proof. We shall show only that the dual right module, X^*, of a left A-module X is injective. Replacing X with X^* in the diagram defining injectivity on p. 1023, and taking ι and λ_0 as morphisms of right A-modules, we consider $\lambda_0 \circ \ell$ as element of the Banach space, $L(Y, X^*)$, of bounded linear maps from Y into X^*. Turning it into an A-bimodule by $(aT)(y) = T(ya)$ and $(Ta)(y) = (Ty)a$, for $a \in A$, $T \in L(Y, X^*)$, $y \in Y$, we obtain a bounded linear map $D\colon A \to L(Y, X^*)$, $Da = a(\lambda_0 \circ \ell) - (\lambda_0 \circ \ell)a$, $a \in A$, whose values vanish on the closed submodule ιY_0 of Y, thus defining a new map, $D_0\colon A \to L(Y/\iota Y_0, X^*)$, by the formula $(D_0 a)(\pi y) = (Da)(y)$, $a \in A$, $y \in Y$, π denoting the canonical morphism from Y onto $Y/\iota Y_0$. Endowing the projective tensor product $Y/\iota Y_0 \mathbin{\hat{\otimes}} X$ with A-actions $a(\pi y \otimes x) = \pi y \otimes ax$ and $(\pi y \otimes x)a = \pi ya \otimes x$, the Banach space $L(Y/\iota Y_0, X^*) = (Y/\iota Y_0 \mathbin{\hat{\otimes}} X)^*$, cf. [CLM], II.1.7, p. 54, becomes a dual A-bimodule and D_0 a derivation so that, by the amenability of A, $D_0 a = aS - Sa$, $a \in A$, for some $S \in L(Y/\iota Y_0, X^*)$. Comparing with the definition of D_0 yields

$$a(\lambda_0 \circ \ell - S \circ \pi) = (\lambda_0 \circ \ell - S \circ \pi)a \quad (a \in A),$$

such that $\lambda = \lambda_0 \circ \ell - S \circ \pi$ is a morphism extending λ_0 along ι. Hence X^* is injective and X flat. □

5.3. Proof of Corollary b. Let X be a non-zero flat left $L^1(G)$-module, reflexive as a Banach space. Then X^* is a non-zero injective right $L^1(G)$-module and equally reflexive, implying the amenability of G by the Proposition. If, conversely, the group G is amenable, then the Banach algebra $L^1(G)$ is amenable, [J], Theorem 2.5, p. 32, so that every left $L^1(G)$-module is flat by the lemma above. □

6. An open problem

Let $\mathcal{M}$ be a von Neumann algebra admitting a non-zero injective normal Banach left module, reflexive as a Banach space. Does this entail the injectivity of $\mathcal{M}$? Cf. [R2], in particular Corollary 2.6, p. 2533.

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