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Entropy on Riemann surfaces and the Jacobians of finite covers

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Abstract. This paper characterizes those pseudo-Anosov mappings whose entropy can be detected homologically by taking a limit over finite covers. The proof is via complex-analytic methods. The same methods show the natural map $\mathcal{M}_g \to \prod \mathcal{A}_h$, which sends a Riemann surface to the Jacobians of all of its finite covers, is a contraction in most directions.

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1. Introduction

Let $f : S \to S$ be a pseudo-Anosov mapping on a surface of genus g with n punctures. It is well-known that the topological entropy $h(f)$ is bounded below in terms of the spectral radius of $f^* : H^1(S, \mathbb{C}) \to H^1(S, \mathbb{C})$; we have

$$\log \rho(f^*) \le h(f).$$

If we lift f to a map $\tilde{f} : \widetilde{S} \to \widetilde{S}$ on a finite cover of S, then its entropy stays the same but the spectral radius of the action on homology can increase. We say the entropy of f can be *detected homologically* if

$$h(f) = \sup \log \rho(\tilde{f}^* : H^1(\widetilde{S}) \to H^1(\widetilde{S})),$$

where the supremum is taken over all finite covers to which f lifts.

In this paper we will show:

Theorem 1.1. *The entropy of a pseudo-Anosov mapping f can be detected homologically if and only if the invariant foliations of f have no odd-order singularities in the interior of S.*

The proof is via complex analysis. Hodge theory provides a natural embedding $\mathcal{M}_g \to \mathcal{A}_g$ from the moduli space of Riemann surfaces into the moduli space of

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Abelian varieties, sending X to its Jacobian. Any characteristic covering map from a surface of genus h to a surface of genus g, branched over n points, provides a similar map

$$\mathcal{M}_{g,n} \to \mathcal{M}_h \to \mathcal{A}_h. \tag{1.1}$$

It is known that the hyperbolic metric on a Riemann surface X can be reconstructed using the metrics induced from the Jacobians of its finite covers ([Kaz]; see the Appendix). Similarly, it is natural to ask if the Teichmüller metric on $\mathcal{M}_{g,n}$ can be recovered from the Kobayashi metric on $\mathcal{A}_h$, by taking the limit over all characteristic covers $\mathcal{C}_{g,n}$. We will show such a construction is impossible.

Theorem 1.2. *The natural map* $\mathcal{M}_{g,n} \to \prod_{\mathcal{C}_{g,n}} \mathcal{A}_h$ *is not an isometry for the Kobayashi metric, unless* $\dim \mathcal{M}_{g,n} = 1$.

It is an open problem to determine if the Kobayashi and Carathéodory metrics on moduli space coincide when $\dim \mathcal{M}_{g,n} > 1$ (see e.g. [FM], Problem 5.1). An equivalent problem is to determine if Teichmüller space embeds holomorphically and isometrically into a (possibly infinite) product of bounded symmetric domains. Theorem 1.2 provides some support for a negative answer to this question.

Here is a more precise version of Theorem 1.2, stated in terms of the lifted map

$$\mathcal{T}_{g,n} \to \mathcal{T}_h \xrightarrow{J} \mathfrak{H}_h$$

from Teichmüller space to Siegel space determined by a finite cover.

Theorem 1.3. *Suppose the Teichmüller mapping between a pair of distinct points* $X, Y \in \mathcal{T}_{g,n}$ *comes from a quadratic differential with an odd order zero. Then*

$$\sup d(J(\widetilde{X}), J(\widetilde{Y})) < d(X, Y),$$

where the supremum is taken over all compatible finite covers of X *and* Y.

Conversely, if the Teichmüller map from X to Y has only even order singularities, then there is a double cover such that $d(J(\widetilde{X}), J(\widetilde{Y})) = d(X, Y)$ (cf. [Kra]). In particular, the complex geodesics generated by squares of holomorphic 1-forms map isometrically into $\mathcal{A}_g$. The only directions contracted by the map $\mathcal{M}_g \to \prod \mathcal{A}_h$ are those identified by Theorem 1.3.

Theorem 1.1 follows from Theorem 1.3 by taking X and Y to be points on the Teichmüller geodesic stabilized by the mapping-class f. It would be interesting to find a direct topological proof of Theorem 1.1.

As a sample application, let $\beta \in B_n$ be a pseudo-Anosov braid whose monodromy map $f : S \to S$ (on the n-times punctured plane) has an odd order singularity. Then Theorem 1.1 implies the image of β under the Burau representation satisfies

$$\log \sup_{|q|=1} \rho(B(q)) < h(f).$$

Indeed, $\rho(B(q))$ at any d-th root of unit is bounded by $\rho(\tilde{f}^*)$ on a $\mathbb{Z}/d$ cover S [Mc2]. This improves a result in [BB]. Similar statements hold for other homological representations of the mapping–class group.

Notes and references. For C^∞ diffeomorphisms of a compact smooth manifold, one has $h(f) \geq \log \sup_i \rho(f^*|H^i(X))$ [Ym], and equality holds for holomorphic maps on Kähler manifolds [Gr]. The lower bound $h(f) \geq \log \rho(f^*|H^1(X))$ also holds for homeomorphisms [Mn]. For more on pseudo-Anosov mappings, see e.g. [FLP], [Bers] and [Th].

A proof that the inclusion of $\mathcal{T}_{g,n}$ into universal Teichmüller space is a contraction, based on related ideas, appears in [Mc1].

2. Odd order zeros

We begin with an analytic result, which describes how well a monomial z^k of odd order can be approximated by the square of an analytic function.

Theorem 2.1. *Let $k \geq 1$ be odd, and let $f(z)$ be a holomorphic function on the unit disk Δ such that $\int |f(z)|^2 = 1$. Then*

$$\left| \int_\Delta f(z)^2 \left(\frac{\bar{z}}{|z|} \right)^k \right| \leq C_k = \frac{\sqrt{k+1}\sqrt{k+3}}{k+2} < 1.$$

Here the integral is taken with respect to Lebesgue measure on the unit disk.

Proof. Consider the orthonormal basis $e_n(z) = a_n z^n, n \geq 0, a_n = \sqrt{n+1}/\sqrt{\pi}$, for the Bergman space $L^2_\alpha(\Delta)$ of analytic functions on the disk with $\|f\|_2^2 = \int |f(z)|^2 < \infty$. With respect to this basis, the nonzero entries in the matrix of the symmetric bilinear form $Z(f, g) = \int f(z)g(z)\bar{z}^k/|z|^k$ are given by

$$Z(e_n, e_{k-n}) = a_n a_{k-n} \int_\Delta |z|^k = \frac{2\sqrt{n+1}\sqrt{k-n+1}}{k+2}.$$

In particular, $Z(e_i, e_i) = 0$ for all i (since k is odd), and $Z(e_i, e_j) = 0$ for all $i, j > k$.

Note that the ratio above is less than one, by the inequality between the arithmetic and geometric means, and it is maximized when $n < k/2 < n+1$. Thus the maximum of $|Z(f, f)|/\|f\|^2$ over $L^2_\alpha(\Delta)$ is achieved when $f = e_n + e_{n+1}, n = (k-1)/2$, at which point it is given by C_k. □

3. Siegel space

In this section we describe the Siegel space of Hodge structures on a surface S, and its Kobayashi metric.

Hodge structures. Let S be a closed, smooth, oriented surface of genus g. Then $H^1(S) = H^1(S, \mathbb{C})$ carries a natural involution $C(\alpha) = \bar{\alpha}$ fixing $H^1(S, \mathbb{R})$, and a natural Hermitian form

$$\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_S \alpha \wedge \bar{\beta}$$

of signature (g, g). A *Hodge structure* on $H^1(S)$ is given by an orthogonal splitting

$$H^1(S) = V^{1,0} \oplus V^{0,1}$$

such that $V^{1,0}$ is positive-definite and $V^{0,1} = C(V^{1,0})$. We have a natural norm on $V^{1,0}$ given by $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$.

The set of all possible Hodge structures forms the *Siegel space* $\mathfrak{H}(S)$. To describe this complex symmetric space in more detail, fix a splitting $H^1(S) = W^{1,0} \oplus W^{0,1}$. Then for any other Hodge structure $V^{1,0} \oplus V^{0,1}$, there is a unique operator

$$Z \colon W^{1,0} \to W^{0,1}$$

such that $V^{1,0} = (I + Z)(W^{1,0})$. This means $V^{1,0}$ coincides with the graph of Z in $W^{1,0} \oplus W^{0,1}$.

The operator Z is determined uniquely by the associated bilinear form

$$Z(\alpha, \beta) = \langle \alpha, CZ(\beta) \rangle$$

on $W^{1,0}$, and the condition that $V^{1,0} \oplus V^{0,1}$ is a Hodge structure translates into the conditions

$$Z(\alpha, \beta) = Z(\beta, \alpha) \quad \text{and} \quad |Z(\alpha, \alpha)| < 1 \text{ if } \|\alpha\| = 1. \tag{3.1}$$

Since the second inequality above is an open condition, the tangent space at the base point $p \sim W^{1,0} \oplus W^{0,1}$ is given by

$$\mathrm{T}_p \mathfrak{H}(S) = \{\text{symmetric bilinear maps } Z \colon W^{1,0} \times W^{1,0} \to \mathbb{C}\}.$$

Comparison maps. Any Hodge structure on $H^1(S)$ determines an isomorphism

$$V^{1,0} \cong H^1(S, \mathbb{R}) \tag{3.2}$$

sending α to $\Re(\alpha) = (\alpha + C(\alpha))/2$. Thus $H^1(S, \mathbb{R})$ inherits a norm and a complex structure from $V^{1,0}$.

Put differently, (3.2) gives a *marking* of $V^{1,0}$ by $H^1(S,\mathbb{R})$. By composing one marking with the inverse of another, we obtain the real-linear *comparison map*

$$T = (I + Z)(I + CZ)^{-1} : W^{1,0} \to V^{1,0} \tag{3.3}$$

between any pair of Hodge structures. It is characterized by $\Re(\alpha) = \Re(T(\alpha))$.

Symmetric matrices. The classical Siegel domain is given by

$$\mathfrak{H}_g = \{Z \in \mathrm{M}_g(\mathbb{C}) \ : \ Z_{ij} = Z_{ji} \text{ and } I - Z\overline{Z} \gg 0\}.$$

(cf. [Sat], Chapter II.7). It is a convex, bounded symmetric domain in $\mathbb{C}^N$, $N = g(g+1)/2$. The choice of an orthonormal basis for $W^{1,0}$ gives an isomorphism $Z \mapsto Z(\omega_i, \omega_j)$ between $\mathfrak{H}(S)$ and $\mathfrak{H}_g$, sending the basepoint p to zero.

The Kobayashi metric. Let $\Delta \subset \mathbb{C}$ denote the unit disk, equipped with the metric $|dz|/(1-|z|^2)$ of constant curvature -4. The *Kobayashi metric* on $\mathfrak{H}(S)$ is the largest metric such that every holomorphic map $f : \Delta \to \mathfrak{H}(S)$ satisfies $\|Df(0)\| \le 1$. It determines both a norm on the tangent bundle and a distance function on pairs of points [Ko].

Proposition 3.1. *The Kobayashi norm on* $\mathrm{T}_p\mathfrak{H}(S)$ *is given by*

$$\|Z\|_K = \sup\{Z(\alpha,\alpha)| \ : \ \|\alpha\| = 1\},$$

and the Kobayashi distance is given in terms of the comparison map (3.3) *by*

$$d(V^{1,0}, W^{1,0}) = \log \|T\|.$$

Proof. Choosing a suitable orthonormal basis for $W^{1,0}$, we can assume that

$$Z(\omega_i, \omega_j) = \lambda_i \delta_{ij}$$

with $\lambda_1 \ge \lambda_2 \ge \cdots \lambda_g \ge 0$. Since $\mathfrak{H}_g$ is a convex symmetric domain, the Kobayashi norm at the origin and the Kobayashi distance satisfy

$$\|Z\|_K = r \quad \text{and} \quad d(0, Z) = \frac{1}{2}\log\frac{1+r}{1-r},$$

where $r = \inf\{s > 0 : Z \in s\mathfrak{H}_g\}$ (see [Ku]). Clearly $r = \lambda_1 = \sup |Z(\alpha,\alpha)|/\|\alpha\|^2$, and by (3.3), we have

$$\|T\|^2 = \|T(\sqrt{-1}\,\omega_1)\|^2 = \left\| \frac{\omega_1}{1-\lambda_1} + \frac{\lambda_1\bar{\omega}_1}{1-\lambda_1} \right\|^2 = \frac{1+\lambda_1}{1-\lambda_1},$$

which gives the expressions above. □

4. Teichmüller space

This section gives a functorial description of the derivative of the map from Teichmüller space to Siegel space.

Markings. Let $\overline{S}$ be a compact oriented surface of genus g, and let $S \subset \overline{S}$ be a subsurface obtained by removing n points.

Let $\mathrm{Teich}(S) \cong \mathcal{T}_{g,n}$ denote the Teichmüller space of Riemann surfaces marked by S. A point in $\mathrm{Teich}(S)$ is specified by a homeomorphism $f\colon S \to X$ to a Riemann surface of finite type. This means there is a compact Riemann surface $\overline{X} \supset X$ and an extension of f to a homeomorphism $\bar{f}\colon \overline{S} \to \overline{X}$.

Metrics. Let $Q(X)$ denote the space of holomorphic quadratic differentials on X such that

$$\|q\|_X = \int_X |q| < \infty.$$

There is a natural pairing $(q, \mu) \mapsto \int_X q\mu$ between the space $Q(X)$ and the space $M(X)$ of L^∞-measurable Beltrami differentials μ. The tangent and cotangent spaces to Teichmüller space at X are isomorphic to $M(X)/Q(X)^\perp$ and $Q(X)$ respectively.

The Teichmüller and Kobayashi metrics on $\mathrm{Teich}(S)$ coincide [Roy1], [Hub], Chapter 6. They are given by the norm

$$\|\mu\|_T = \sup\left\{\left|\int q\mu\right| \;:\; \|q\|_X = 1\right\}$$

on the tangent space at X; the corresponding distance function

$$d(X, Y) = \inf \frac{1}{2} \log K(\phi)$$

measures the minimal dilatation $K(\phi)$ of a quasiconformal map $\phi\colon X \to Y$ respecting their markings.

Hodge structure. The periods of holomorphic 1-forms on X serve as classical moduli for X. From a modern perspective, these periods give a map

$$J\colon \mathrm{Teich}(S) \to \mathfrak{H}(\overline{S}) \cong \mathfrak{H}_g,$$

sending X to the Hodge structure

$$H^1(\overline{S}) \cong H^1(\overline{X}) \cong H^{1,0}(\overline{X}) \oplus H^{0,1}(\overline{X}).$$

Here the first isomorphism is provided by the marking $\bar{f}\colon \overline{S} \to \overline{X}$. We also have a natural isomorphism between $H^{1,0}(\overline{X})$ and the space of holomorphic 1-forms $\Omega(\overline{X})$. The image $J(X)$ encodes the complex analytic structure of the Jacobian variety $\mathrm{Jac}(\overline{X}) = \Omega(\overline{X})^*/H_1(\overline{X}, \mathbb{Z})$. (It is does not depend on the location of the punctures of X.)

Proposition 4.1. *The derivative of the period map sends $\mu \in M(X)$ to the quadratic form $Z = DJ(\mu)$ on $\Omega(\overline{X})$ given by*

$$Z(\alpha, \beta) = \int_{\overline{X}} \alpha\beta\mu.$$

This is a basis-free reformulation of Ahlfors' variational formula [Ah], §5; see also [Ra], [Roy2] and Proposition 1 of [Kra]. Note that $\alpha\beta \in Q(X)$.

5. Contraction

This section brings finite covers into play, and establishes a uniform estimate for contraction of the mapping $\mathcal{T}_{g,n} \to \mathcal{T}_h \to \mathfrak{H}_h$.

Jacobians of finite covers. A finite connected covering space $S_1 \to S_0$ determines a natural map

$$P\colon \operatorname{Teich}(S_0) \to \operatorname{Teich}(S_1)$$

sending each Riemann surface to the corresponding covering space $X_1 \to X_0$. By taking the Jacobian of X_1, we obtain a map $J \circ P\colon \operatorname{Teich}(S_0) \to \mathfrak{H}(\overline{S}_1)$.

Let $q_0 \in Q(X_0)$ be a holomorphic quadratic differential with a zero of odd order k, say at $p \in X_0$. Let $\mu = \bar{q}_0/|q_0| \in M(X_0)$; then $\|\mu\|_T = 1$. Let $\pi\colon X_1 \to X_0$ denote the natural covering map, and let $q_1 = \pi^*(q_0)$.

We will show that $J(X_1)$ cannot change too rapidly under the unit deformation μ of X_0. Indeed, if $J(X_1)$ were to move at nearly unit speed, then $\pi^*(\mu) = \bar{q}_1/|q_1|$ would pair efficiently with α^2 for some unit-norm $\alpha \in \Omega(\overline{X}_1)$, which is impossible because of the many odd-order zeros of q_1.

To make a quantitative estimate, choose a holomorphic chart $\phi\colon (\Delta, 0) \to (X_0, p)$ such that $\phi^*(\mu) = z^k/|z|^k \, d\bar{z}/dz$. Let $U = \phi(\Delta)$, and let

$$m(U) = \inf\{\|q\|_U \;:\; q \in Q(X_0),\; \|q\|_X = 1\}.$$

(Here $\|q\|_U = \int_U |q|$.) Since $Q(X_0)$ is finite-dimensional, we have $m(U) > 0$.

Theorem 5.1. *The image Z of the vector $[\mu]$ under the derivative of $J \circ P$ satisfies*

$$\|Z\|_K \le \delta < 1 = \|\mu\|_T,$$

where $\delta = \max(1/2, 1 - (1 - C_k)m(U)/2)$ does not depend on the finite cover $S_1 \to S_0$.

Proof. The derivative of P sends μ to $\pi^*(\mu)$. By Proposition 3.1, to show $\|Z\|_K \le \delta$ it suffices to show that

$$|Z(\alpha, \alpha)| = \left| \int_{X_1} \alpha^2 \pi^* \mu \right| \le \delta$$

for all $\alpha \in \Omega(\overline{X}_1)$ with $\|\alpha^2\|_{X_1} = 1$. Setting $q = \pi_*(\alpha^2)$, we also have

$$|Z(\alpha,\alpha)| = \left|\int_{X_0} q\mu\right| \leq \|q\|_{X_0},$$

so the proof is complete if $\|q\|_{X_0} \leq 1/2$. Thus we may assume that

$$\|\alpha^2\|_V \geq \|q\|_U \geq m(U)\|q\|_{X_0} \geq m(U)/2,$$

where $V = \pi^{-1}(U) = \bigcup_1^d V_i$ is a finite union of disjoint disks. Using the coordinate charts $V_i \cong U \cong \Delta$ and Theorem 2.1, we find that on each of these disks we have

$$\left|\int_{V_i} \alpha^2 \pi^*(\mu)\right| = \left|\int_\Delta \alpha(z)^2 \left(\frac{z}{|z|}\right)^k\right| \leq C_k \|\alpha^2\|_{V_i}.$$

Summing these bounds and using the fact that $\|\alpha^2\|_{(X_1 - V)} + \|\alpha^2\|_V = 1$, we obtain

$$\left|\int_{X_1} \alpha^2 \pi^*(\mu)\right| \leq \|\alpha^2\|_{(X_1-V)} + C_k\|\alpha^2\|_V \leq 1 - \frac{(1-C_k)m(U)}{2} \leq \delta. \qquad \square$$

6. Conclusion

It is now straightforward to establish the results stated in the Introduction.

Proof of Theorem 1.3. Assume the Beltrami coefficient of the Teichmüller mapping between $X, Y \in \mathcal{T}_{g,n}$ has the form $\mu = k\bar{q}/q$, where $q \in Q(X)$ has an odd order zero. Then the same is true for the tangent vectors to the Teichmüller geodesic γ joining X to Y. Theorem 5.1 then implies that $D(J \circ P)|_\gamma$ is contracting by a factor $\delta < 1$ independent of P, and therefore

$$d(J \circ P(X), J \circ P(Y)) = d(J(\widetilde{X}), J(\widetilde{Y})) < \delta \cdot d(X,Y). \qquad \square$$

Proof of Theorem 1.2. The contraction of $\mathcal{M}_{g,n} \to \prod_{\mathcal{C}_{g,n}} \mathcal{A}_h$ in some directions is immediate from the uniformity of the bound in Theorem 1.3, using the fact that the Kobayashi metric on a product is the sup of the Kobayashi metrics on each term, and that there exist $q \in Q(X)$ with simple zeros whenever $X \in \mathcal{M}_{g,n}$ and $\dim \mathcal{M}_{g,n} > 1$. $\square$

Proof of Theorem 1.1. Let $f : S_0 \to S_0$ be a pseudo-Anosov mapping. If f has only even order singularities, then its expanding foliation is locally orientable, and hence there is a double cover $\widetilde{S} \to \widetilde{S}$ such that $\log \rho(\tilde{f}^*) = h(f)$.

Now suppose f has an odd-order singularity. Let $X_0 \in \mathrm{Teich}(S_0)$ be a point on the Teichmüller geodesic stabilized by the action of f on $\mathrm{Teich}(S_0)$. Then $h(f) = d(f \cdot X_0, X_0) > 0$ (see e.g. [FLP] and [Bers]).

Let $\tilde{f}: S_1 \to S_1$ be a lift of f to a finite covering of S_0, and let $X_1 = P(X_0) \in \mathrm{Teich}(S_1)$. Using the marking of X_1 and the isomorphism $H^1(X_1, \mathbb{R}) \cong H^{1,0}(X_1)$, we obtain a commutative diagram

$$\begin{array}{ccc} H^1(S_1,\mathbb{R}) & \xrightarrow{\tilde{f}^*} & H^1(S_1,\mathbb{R}) \\ \downarrow & & \downarrow \\ H^{1,0}(\overline{X}_1) & \xrightarrow{T} & H^{1,0}(\overline{X}_1) \end{array}$$

where T is the comparison map between $J(X_1)$ and $J(\tilde{f} \cdot X_1)$ (see equation (3.3)). Then Theorem 1.3 and Proposition 3.1 yield the bound

$$\log \rho(\tilde{f}^*) \le \log \|T\| = d(J(X_1), \tilde{f} \cdot J(X_1)) \le \delta d(X_0, f \cdot X_0) = \delta h(f),$$

where $\delta < 1$ does not dependent on the finite covering $S_1 \to S_0$. Consequently, $\sup \log \rho(\tilde{f}^*) < h(f)$. □

Appendix. The hyperbolic metric via Jacobians of finite covers

Let $X = \Delta/\Gamma$ be a compact Riemann surface, presented as a quotient of the unit disk by a Fuchsian group Γ. Let $Y_n \to X$ be an ascending sequence of finite Galois covers which converge to the universal cover, in the sense that

$$Y_n = \Delta/\Gamma_n, \quad \Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots, \quad \text{and} \quad \bigcap \Gamma_i = \{e\}. \tag{A.1}$$

The Bergman metric on Y_n (defined below) is invariant under automorphisms, so it descends to a metric β_n on X. This appendix gives a short proof of:

Theorem A.1 (Kazhdan). *The Bergman metrics inherited from the finite Galois covers $Y_n \to X$ converge to a multiple of the hyperbolic metric; more precisely, we have*

$$\beta_n \to \frac{\lambda_X}{2\sqrt{\pi}}$$

uniformly on X.

The argument below is based on [Kaz], §3; for another, somewhat more technical approach, see [Rh].

Metrics. We begin with some definitions. Let $\Omega(X)$ denote the Hilbert space of holomorphic 1-forms on a Riemann surface X such that

$$\|\omega\|_X^2 = \int_X |\omega|^2 < \infty.$$

The area form of the *Bergman metric* on X is given by

$$\beta_X^2 = \sum |\omega_i|^2, \tag{A.2}$$

where (ω_i) is any orthonormal basis of $\Omega(X)$. Equivalently, the Bergman length of a tangent vector $v \in \mathrm{T}X$ is given by

$$\langle \beta_X, v\rangle = \sup_{\omega \neq 0} \frac{|\omega(v)|}{\|\omega\|_X}. \tag{A.3}$$

This formula shows that inclusions are contracting: if Y is a subdomain of X, then $\beta_Y \geq \beta_X$.

Now suppose X is a compact surface of genus $g > 0$. Then (A.2) shows its Bergman area is given by

$$\int_X \beta_X^2 = \dim \Omega(X) = g. \tag{A.4}$$

In this case β_X is also the pullback, via the Abel–Jacobi map, of the natural Kähler metric on the Jacobian of X.

Finally suppose $X = \Delta/\Gamma$. Then the hyperbolic metric of constant curvature -1,

$$\lambda_\Delta = \frac{2|dz|}{1-|z|^2},$$

descends to give the *hyperbolic metric* λ_X on X. Using the fact that $\|dz\|_\Delta = \pi$, it is easy to check that $4\pi\beta_\Delta^2 = \lambda_\Delta^2$.

Proof of Theorem A.1. We will regard the Bergman metric β_n on Y_n as a Γ_n-invariant metric on Δ. It suffices to show that $\beta_n/\beta_\Delta \to 1$ uniformly on Δ.

Let g and g_n denote the genus of X and Y_n respectively, and let d_n denote the degree of Y_n/X; then $g_n - 1 = d_n(g-1)$. By (A.1), the injectivity radius of Y_n tends to infinity. In particular, there is a sequence $r_n \to 1$ such that $\gamma(r_n\Delta)$ injects into Y_n for any $\gamma \in \Gamma$. Since inclusions are contracting, this shows

$$\beta_n \leq (1+\epsilon_n)\beta_\Delta \tag{A.5}$$

where $\epsilon_n \to 0$.

Next, note that both β_n and β_Δ are Γ-invariant, so they determine metrics on X. By (A.4), we have

$$\int_X \beta_n^2 = \frac{1}{d_n}\int_{Y_n} \beta_n^2 = \frac{g_n}{d_n} \to (g-1) = \int_X \beta_\Delta^2$$

(since $\int_X \lambda_X^2 = 2\pi(2g-2)$ by Gauss–Bonnet). Together with (A.5), this implies

$$\int_X |\beta_n - \beta_\Delta|^2 \to 0. \tag{A.6}$$

To show $\beta_n \to \beta_\Delta$ uniformly, consider any sequence $p_n \in \Delta$ and let $x \in [0,1]$ be a limit point of $(\beta_n/\beta_\Delta)(p_n)$. It suffices to show $x = 1$.

Passing to a subsequence and using compactness of X, we can assume that $p_n \to p \in \Delta$ and that $\beta_n(p_n) \to x\beta_\Delta(p)$. By changing coordinates on Δ, we can also assume $p = 0$. By (A.6) we can find $q_n \to 0$ such that $\beta_n(q_n) \to \beta_\Delta(0)$. Then by (A.3), there exist Γ_n-invariant holomorphic 1-forms $\omega_n(z)\,dz$ on Δ such that $\int_{Y_n} |\omega_n|^2 = 1$ and

$$|\omega_n(q_n)| = \beta_n(q_n) \to \beta_\Delta(0) = \frac{|dz|}{\pi}.$$

Since ω_n is holomorphic and $\int_{r_n\Delta} |\omega_n|^2 < 1$, the equation above easily implies that $|\omega_n| \to |dz|/\pi$ uniformly on compact subsets of Δ. But we also have

$$\beta_n(p_n) \geq |\omega_n(p_n)| \to \beta_\Delta(0),$$

and thus $\beta_n(p_n) \to \beta_\Delta(0)$ and hence $x = 1$. □

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