Orbifold quantum D-modules associated to weighted projective spaces

Martin A. Guest and Hironori Sakai

Abstract. We construct in an abstract fashion (without using Gromov–Witten invariants) the orbifold quantum cohomology of weighted projective space, starting from a certain differential operator. We obtain the product, grading, and intersection form by making use of the associated self-adjoint D-module and the Birkhoff factorization procedure. The method extends in principle to the more difficult case of Fano hypersurfaces in weighted projective space, where Gromov–Witten invariants have not yet been computed, and we illustrate this by means of an example originally studied by A. Corti. In contrast to the case of weighted projective space itself or the case of a Fano hypersurface in projective space, a "small cell" of the Birkhoff decomposition plays a role in the calculation.

1. Introduction

The weighted projective space

$$\mathbb{P}(w_0, \ldots, w_n) = \mathbb{C}^{n+1} - \{0\} \,/\, \mathbb{C}^*, \ \ z \cdot (z_0, \ldots, z_n) = (z^{-w_0} z_0, \ldots, z^{-w_n} z_n)$$

provides a simple test case (see [3], [2], [13]) for the recently developed theories of orbifold cohomology and orbifold quantum cohomology. Direct geometrical calculations are difficult, but mirror symmetry suggests an alternative and very effective approach: Corti and Golyshev conjectured (see [6], [5]) that the structure constants can be read off from

$$T_w - q = \prod_{i=0}^{n} (w_i \hbar \partial)(w_i \hbar \partial - \hbar) \ldots (w_i \hbar \partial - (w_i - 1)\hbar) \, - \, q,$$

where $\partial = q \frac{d}{dq}$; this is an ordinary differential operator of order $s = \sum_{i=0}^{n} w_i$.

This generalizes the well-known quantum differential equation of projective space $\mathbb{C}P^n = \mathbb{P}(1, \ldots, 1)$. Namely, the equation $((\hbar\partial)^{n+1} - q)y = 0$ is a scalar form of the system

$$
\hbar\partial \begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_n \end{pmatrix},
$$

where the matrix is interpreted as that of quantum multiplication by the generator $p \in H^2\mathbb{C}P^n$ with respect to the standard cohomology basis $1, p, \ldots, p^n$. Thus,

$$
p \circ p^i = \begin{cases} p^{i+1} & \text{if } 0 \le i < n \\ q & \text{if } i = n \end{cases}
$$

from which all quantum products $p^i \circ p^j$ can be computed.

The conjecture of Corti and Golyshev was proved in [3], by extending to orbifold quantum cohomology a method of Givental for quantum cohomology. The method has three steps. First, a basis of solutions of the quantum differential equation is written down – the I-function. Then, the orbifold version of Givental's Mirror Theorem shows that the I-function is equal to the J-function, a certain generating function for Gromov–Witten invariants. This is the most substantial ingredient, but specific properties of weighted projective spaces are not required. Finally, the structure constants for the orbifold quantum product are extracted from this J-function by a method which involves repeated differentiation.

The first goal of this paper is to give a straightforward version (alluded to in the introduction to [3]) of the last step, using the Birkhoff factorization method of [8]. The simplifying feature is that we use the differential equation (D-module) directly, rather than its solution (I-function).

The second goal is to study in its own right the differential operator $T_w - q$, or rather, the D-module $D^\hbar/(T_w - q)$ (where $D^\hbar$ is a certain ring of differential operators). We show how to extract from this D-module an "abstract quantum cohomology ring" with a product operation, grading, and nondegenerate pairing. Then we observe that this coincides with the usual orbifold quantum cohomology. It is remarkable that such a simple differential operator contains all relevant geometrical information, which is complicated and non-intuitive even in the case of $\mathbb{P}(w_0, \ldots, w_n)$.

The third and main goal (Section 5) is to indicate how our method extends to hypersurfaces in weighted projective spaces. This generalizes the method of [15] for hypersurfaces in projective spaces. It presents a new feature: instead of the "big cell" of the Birkhoff decomposition, in general a "small cell" is needed. Alternatively, this method can be interpreted as the Gram–Schmidt orthogonalization procedure together with a "big cell factorization". As a nontrivial example, we apply the method to a hypersurface of degree 3 in $\mathbb{P}(1, 1, 1, 2)$, where the orbifold quantum cohomology

has been computed geometrically by Corti. However, we are not able to give general conditions which ensure that our method works, and we must leave this as a problem for the interested reader.

The first author is very grateful to Alessio Corti for explaining the conjecture and the basic ideas of orbifold quantum cohomology; the idea for extracting the structure constants of $\mathbb{P}(w_0, \ldots, w_n)$ from the differential operator was originally worked out with him in 2006, and Alessio also explained the geometry behind the hypersurface example in Section 5. He also thanks Hiroshi Iritani for many essential explanations and comments on an earlier version, and Josef Dorfmeister for discussions on the Birkhoff decomposition.

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2. Notation for orbifold cohomology

We write $\mathbb{P}(w_0, \ldots, w_n) = \mathbb{P}(w)$ from now on. As far as possible we shall follow the notation of [3] for orbifold cohomology. That paper and its references contain more detailed information.

First, let

$$F = \{\tfrac{i}{w_j} \mid 0 \le i \le w_j - 1,\ 0 \le j \le n\}$$

$$= \{f_1, \ldots, f_k\} \quad \text{where } 0 = f_1 < f_2 < \cdots < f_k < f_{k+1} \stackrel{\text{def}}{=} 1 .$$

Let $u_1, \ldots, u_k$ be the "multiplicities" of the fractions $f_1, \ldots, f_k$ as elements of F. We write

$$s = u_1 + \cdots + u_k = w_0 + \cdots + w_n.$$

The positive integer u_i can also be described as the cardinality of the set

$$S_{f_i} = \{j \mid w_j f_i \in \mathbb{Z}\} \subseteq \{0, \ldots, n\}.$$

The orbifold cohomology of $\mathbb{P}(w)$ may be defined as a vector space by

$$H^*_{\text{orbi}} \mathbb{P}(w) = \bigoplus_{i=1}^{k} H^* \mathbb{P}(V_{f_i}),$$

where

$$V_{f_i} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_j = 0 \text{ if } j \notin S_{f_i}\} \cong \mathbb{C}^{u_i}.$$

This can be equipped with a commutative associative multiplicative operation called the orbifold cup product. Using this product, we obtain a $\mathbb{C}$-basis

$$\mathbf{1}_{f_i}, \mathbf{1}_{f_i}\, p, \ldots, \mathbf{1}_{f_i}\, p^{u_i - 1}$$

of the subspace $H^*\mathbb{P}(V_{f_i})$, where $p \in H^2\mathbb{P}(V_0)$ and $\mathbf{1}_{f_i}$ denotes the canonical generator of $H^0\mathbb{P}(V_{f_i})$. When $i = 1$ we have $f_1 = 0$, $u_1 = n + 1$, and generators $\mathbf{1}_0, \mathbf{1}_0\,p, \ldots, \mathbf{1}_0\,p^n$; we shall just write $1, p, \ldots, p^n$ in this case. The element 1 is the identity element of the orbifold cohomology ring.

There is also a natural grading, in which

$$|\mathbf{1}_{f_i}\,p^j| = |\mathbf{1}_{f_i}| + |p^j| = 2\,\mathrm{age}\,\mathbf{1}_{f_i} + 2j.$$

Here, $\mathrm{age}\,\mathbf{1}_{f_i} = (u_1 + \cdots + u_{i-1}) - f_i s = \langle -w_0\,f_i \rangle + \cdots + \langle -w_n\,f_i \rangle$ where $\langle r \rangle = r - \max\{i \in \mathbb{Z} \mid i \le r\}$. The orbifold cohomology has a nondegenerate symmetric "intersection pairing" $(\,,\,)$, which generalizes the Poincaré pairing for ordinary cohomology.

We record the following properties for later use.

Lemma 2.1. (1) $f_i + f_j = 1$ *if* $i + j = k + 2$.

(2) $u_i = u_j$ *if* $i + j = k + 2$.

(3) $u_2 + \cdots + u_i = u_{k+2-i} + \cdots + u_k$ *for* $2 \le i \le k$.

Proof. The involution $f \mapsto 1 - f$ preserves $F \cup \{1\}$. It maps $f_1 < \cdots < f_{k+1}$ to $1 - f_{k+1} < \cdots < 1 - f_1$, so these sequences must coincide. This proves (1), then (2) and (3) follow immediately. $\square$

3. The structure constants: statement of results

As mentioned in the introduction, a key role is played by the s-th order differential operator

$$T_w - q = \prod_{i=0}^{n}(w_i\hbar\partial)(w_i\hbar\partial - \hbar)\ldots(w_i\hbar\partial - (w_i - 1)\hbar) - q$$

$$= w^w \hbar^s \prod_{i=0}^{n} \partial(\partial - \tfrac{1}{w_i})\ldots(\partial - \tfrac{w_i-1}{w_i}) - q,$$

where $s = \sum_{i=0}^{n} w_i$, $w^w = \prod_{i=0}^{n} w_i^{w_i}$, and $\partial = q\frac{d}{dq}$.

In this section we state without explanation how the structure constants of orbifold quantum cohomology – in particular, of orbifold cohomology itself – may be extracted from the differential operator $T_w - q$. A systematic explanation will be given in the next section.

Using the formula $\partial q^{-1} = q^{-1}(\partial - 1)$, we may factorize the differential operator $q^{-1}T_w$ in the following way:

$$q^{-1}T_w = \underbrace{m_k q^{-\Delta_k}(\hbar\partial)^{u_k}}_{k\text{th factor}} \underbrace{m_{k-1}q^{-\Delta_{k-1}}(\hbar\partial)^{u_{k-1}}}_{k-1\text{th factor}} \cdots \underbrace{m_1 q^{-\Delta_1}(\hbar\partial)^{u_1}}_{1\text{st factor}}$$

where

$$\Delta_i = f_{i+1} - f_i, \quad m_i = \prod_{j \in S_{f_i}} w_j,$$

for $1 \le i \le k$. Thus we have $\prod_{i=1}^{k} m_i = w^w$ and $\sum_{i=1}^{k} \Delta_i = 1$. We shall need the following symmetry properties later on, which follow directly from Lemma 2.1:

Lemma 3.1. (1) $\Delta_i = \Delta_j$ if $i + j = k + 1$.

  (2) $m_i = m_j$ if $i + j = k + 2$.

Let us rewrite the factorization above as

$$q^{-1} T_w = \tfrac{1}{r_s}\, \hbar \partial \ \tfrac{1}{r_{s-1}}\, \hbar \partial \ \ \ldots \ \ \tfrac{1}{r_1} \hbar \partial$$

where we use the following notation:

Definition 3.2. For $1 \le \alpha \le s$,

$$r_\alpha = \begin{cases} \frac{1}{m_i} q^{\Delta_i} & \text{if } \alpha = u_1 + \cdots + u_i \\ 1 & \text{otherwise.} \end{cases}$$

The result of [3] may be stated as follows:

Theorem 3.3. *Denote by $c_0, \ldots, c_{s-1}$ the additive basis*

$$1, p, \ldots, p^{u_1-1};\ \mathbf{1}_{f_2}, \mathbf{1}_{f_2}\, p, \ldots, \mathbf{1}_{f_2}\, p^{u_2-1};\ \ldots;\ \mathbf{1}_{f_k}, \mathbf{1}_{f_k}\, p, \ldots, \mathbf{1}_{f_k}\, p^{u_k-1}$$

of $H^*_{\mathrm{orbi}} \mathbb{P}(w)$. Then the matrix of orbifold quantum multiplication by p with respect to this basis is given by

$$\begin{pmatrix} 0 & & & & r_s \\ r_1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & r_{s-1} & 0 \end{pmatrix}.$$

That is, we have $p \circ c_i = r_{i+1} c_{i+1}$ for $0 \le i < s - 1$ and $p \circ c_{s-1} = r_s c_0$. In particular, p is a cyclic element of this ring.

The orbifold structure constants (giving the product structure of $H^*_{\mathrm{orbi}} \mathbb{P}(w)$) are obtained by setting $q = 0$ in the above matrix. Although the matrix itself gives only the products involving p, all other products can be deduced.

4. Direct approach from the D-module

The structure constants in Theorem 3.3 were computed in [3] from the I-function (i.e. solution of the differential equation $(T_w - q)y = 0$) and by establishing a mirror theorem in the style of Givental. In this section we discuss a somewhat different procedure: we construct "abstract orbifold quantum cohomology" from $T_w - q$ itself. To prove that our abstract orbifold quantum cohomology agrees with the usual orbifold quantum cohomology, it is still necessary to appeal to the mirror theorem, so in this sense our procedure relates only to the extraction of information from the differential equation. However, our procedure gives a very direct way of obtaining the orbifold degrees and orbifold Poincaré pairing as well as the structure constants.

We follow [8] and chapter 6 of [9], although the orbifold case presents some new features. Let us consider the $D^\hbar$-module

$$\mathcal{M} = D^\hbar/(T_w - q)$$

where $D^\hbar$ denotes the ring of (ordinary) differential operators generated by $\hbar\partial$, and $(T_w - q)$ denotes the left ideal generated by $T_w - q$. As coefficient ring we can take the ring of functions which are polynomial in $q^{\pm 1/l}$, where

$$l = \text{lowest common multiple of } w_0, \dots, w_n,$$

and which are holomorphic in $\hbar$ in a neighbourhood of $\hbar = 0$.

The $D^\hbar$-module $\mathcal{M}$ is free of rank s over the coefficient ring. With respect to the natural basis $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$, the matrix of the action of ∂ is of the form

$$\Omega = \tfrac{1}{\hbar}\omega + \theta^{(0)} + \hbar\theta^{(1)} + \hbar^2\theta^{(2)} + \cdots.$$

More precisely, if we identify $\mathcal{M}$ with a space of meromorphic sections of the trivial bundle, we may regard $1, \hbar\partial, \dots, (\hbar\partial)^{s-1}$ as a local basis of sections, and the action of ∂ on $\mathcal{M}$ defines a connection on the bundle, with local connection matrix Ω.

If we replace $\hbar\partial$ by an abstract (commutative) variable p, then set $\hbar = 0$, we may construct from $\mathcal{M}$ a commutative ring generated by p which is subject to the relation $w^w p^s - q$, and which has $1, p, \dots, p^{s-1}$ as an additive $\mathbb{C}[q^{\pm 1/l}]$-basis. That is, we have the "abstract orbifold quantum cohomology ring"

$$QA = \mathbb{C}[p, q^{\pm 1/l}]/(w^w p^s - q).$$

In order to define "abstract orbifold Gromov–Witten invariants" (structure constants) we shall introduce a ring A, the "abstract orbifold cohomology ring", such that QA and $A \otimes \mathbb{C}[q^{\pm 1/l}]$ are isomorphic as $\mathbb{C}[q^{\pm 1/l}]$-modules. A choice of basis will give a specific isomorphism $\delta\colon QA \cong A \otimes \mathbb{C}[q^{\pm 1/l}]$, hence a new $A \otimes \mathbb{C}[q^{\pm 1/l}]$-valued product operation

$$a \circ b = \delta\left(\delta^{-1}(a)\delta^{-1}(b)\right)$$

on A. Our main task will be the construction of a basis such that the product has the expected properties of the orbifold quantum product.

For this, the fundamental step is to transform $1, \hbar\partial, \ldots, (\hbar\partial)^{s-1}$ to a new basis, with respect to which the connection matrix has the form

$$\widehat{\Omega} = \tfrac{1}{\hbar}\hat{\omega}$$

where $\hat{\omega}$ is independent of $\hbar$. In the case of a Fano manifold, the transformation procedure is explained in detail in chapter 6 of [9]. It involves a Birkhoff factorization $L = L_-L_+$ of a matrix-valued function L such that $\Omega = L^{-1}dL$, after which one defines $\widehat{\Omega} = (L_-)^{-1}dL_-$. The basis $1, \hbar\partial, \ldots, (\hbar\partial)^{s-1}$ is transformed to the new basis $L_+^{-1} \cdot 1, L_+^{-1} \cdot \hbar\partial, \ldots, L_+^{-1} \cdot (\hbar\partial)^{s-1}$, where $L_+^{-1} \cdot (\hbar\partial)^i$ means $\sum_{j=0}^{s-1}(L_+)_{ji}^{-1}(\hbar\partial)^j$.

In general it is difficult to carry out such Birkhoff factorizations explicitly, or even to know whether they exist. Here, however, L_+ can be found by the method of [1], [9]. The effectiveness of our approach comes from the fact that only the factor L_+ is needed (not the more complicated factor L_-, which is equivalent to the I-function).

In the case of weighted projective spaces themselves (though not for hypersurfaces), the differential operator factorization given in Section 3 provides a short cut for the computation of L_+. Namely, we introduce directly a new basis $P_0, \ldots, P_{s-1}$ by defining

$$P_0 = 1 \quad \text{and} \quad P_i = \frac{1}{r_i}\hbar\partial P_{i-1}$$

for $1 \le i \le s-1$. Fortuitously, with respect to this basis, the matrix of ∂ already has the form $\frac{1}{\hbar}\hat{\omega}$, so L_+ may be read off by regarding the above basis as $L_+^{-1} \cdot 1, L_+^{-1} \cdot \hbar\partial, \ldots, L_+^{-1} \cdot (\hbar\partial)^{s-1}$. We have $L_+ = Q_0(I + \hbar Q_1 + \cdots + \hbar^{k-2}Q_{k-2})$ where

$$Q_0 = \begin{pmatrix} \frac{1}{m_0}q^{f_1}I & & & \\ & \frac{1}{m_0m_1}q^{f_2}I & & \\ & & \ddots & \\ & & & \frac{1}{m_0\ldots m_{k-1}}q^{f_k}I \end{pmatrix},$$

$m_0 = 1$, and where $Q_1, \ldots, Q_{k-2}$ are (easily computed) constant matrices.

For future reference, we explain how (a modification of) the algorithm of [1] and Section 6.6 of [9] produces this answer. First, by definition, the factor $L_+(q, \hbar) = Q_0(q)(I + \hbar Q_1(q) + \hbar^2 Q_2(q) + \cdots)$ satisfies the ordinary differential equation

$$\tfrac{1}{\hbar}\hat{\omega} \; (= \tfrac{1}{\hbar}Q_0\omega Q_0^{-1}) = L_+\Omega L_+^{-1} + L_+ dL_+^{-1}.$$

In the situation of [1] and [9], L_+ is determined uniquely by the initial condition $L_+|_{q=0} = I$, and there is a natural homogeneity condition on L_+ which reduces the computation of L_+ to a finite algebraic algorithm. The present situation is similar, but L_+ must be normalized in a different way.

Let us make the Ansatz that Q_0 is of the above diagonal form. This is natural as $\mathrm{diag}(q^{f_1}I, \ldots, q^{f_k}I)$ arises from the Frobenius method for solving the original o.d.e., and, as we shall see in Corollary 4.3 below, the coefficients $\frac{1}{m_0}, \ldots, \frac{1}{m_0 \ldots m_{k-1}}$ have the effect of producing the "expected" pairing matrix

$$\begin{pmatrix} M_1^{-1} & & & \\ & M_2^{-1} & & \\ & & \ddots & \\ & & & M_k^{-1} \end{pmatrix}, \quad M_i = \begin{pmatrix} & & m_i \\ & \reflectbox{$\ddots$} & \\ m_i & & \end{pmatrix}.$$

Furthermore, let us assume that each Q_i is homogeneous and polynomial in $q^{1/l}$. Then the differential equation again reduces to a system of algebraic equations for $Q_1, Q_2, \ldots$ and it is easy to show that there is a unique solution.

We shall use the above basis $P_0, \ldots, P_{s-1}$ to construct in turn a product operation, a grading, and a pairing.

1. *The product.* Let us group the basis elements of $\mathcal{M}$ as follows:

$$(\hbar\partial)^i \quad \text{for } 0 \le i \le u_1 - 1,$$
$$(\hbar\partial)^i m_1 q^{-\Delta_1} (\hbar\partial)^{u_1} \quad \text{for } 0 \le i \le u_2 - 1,$$

$$\ldots$$

$$(\hbar\partial)^i m_1 q^{-\Delta_1} (\hbar\partial)^{u_1} \ldots m_{k-1} q^{-\Delta_{k-1}} (\hbar\partial)^{u_{k-1}-1} \quad \text{for } 0 \le i \le u_k - 1.$$

Replacing $\hbar\partial$ by p here, and introducing the notation

$$\mathbf{1}_{f_{i+1}} = m_1 \ldots m_i q^{-\Delta_1 - \cdots - \Delta_i} p^{u_1 + \cdots + u_i},$$

we obtain a corresponding basis

$$1, \ p, \ \ldots, \ p^{u_1-1};$$
$$\mathbf{1}_{f_2}, \ \mathbf{1}_{f_2} p, \ \ldots, \ \mathbf{1}_{f_2} p^{u_2-1};$$
$$\vdots$$
$$\mathbf{1}_{f_k}, \ \mathbf{1}_{f_k} p, \ \ldots, \ \mathbf{1}_{f_k} p^{u_k-1}$$

of QA. The vector space spanned (over $\mathbb{C}$) by these basis elements will be denoted A. By definition, the action of p on $A \otimes \mathbb{C}[q^{\pm 1/l}]$ is given (with respect to this basis) by the matrix of Theorem 3.3. As 1 is a cyclic element, this action extends to a product operation on $A \otimes \mathbb{C}[q^{\pm 1/l}]$, that is, it allows us to define the product of any two elements $\mathbf{1}_{f_i} p^j, \mathbf{1}_{f_k} p^l$. We denote this product by $\mathbf{1}_{f_i} p^j \circ \mathbf{1}_{f_k} p^l$, and regard $A \otimes \mathbb{C}[q^{\pm 1/l}]$ as the abstract orbifold quantum cohomology ring of $\mathbb{P}(w)$. We obtain

a subring $A \otimes \mathbb{C}[q^{1/l}]$, and by putting $q = 0$ we obtain a product operation on A, which we regard as the abstract orbifold cohomology.

2. *The grading.* The differential operator $T_w - q$ is homogeneous of weight $2s$, if we assign weights as follows: $|\hbar| = 2$, $|\partial| = 0$, $|q| = 2s$. The differential operators $P_0, \ldots, P_{s-1}$ are also homogeneous. Indeed, from the formula for $P_{u_1 + \cdots + u_i}$, its weight is

$$
\begin{aligned}
|P_{u_1 + \cdots + u_i}| &= 2(u_1 + \cdots + u_i) - 2s(\Delta_1 + \cdots + \Delta_i) \\
&= 2(u_1 + \cdots + u_i) - 2s f_{i+1} \\
&= 2 \operatorname{age} \mathbf{1}_{f_{i+1}}.
\end{aligned}
$$

It follows that our product operation satisfies

$$
|\mathbf{1}_{f_i} p^j \circ \mathbf{1}_{f_k} p^l| = |\mathbf{1}_{f_i} p^j| + |\mathbf{1}_{f_k} p^l|
$$

and $| \ |$ coincides with the usual orbifold quantum cohomology grading.

3. *Self-adjointness and the pairing.* We shall obtain a natural identification of the $D^\hbar$-module $\mathcal{M} = D^\hbar/(T_w - q)$ with a "dual" $D^\hbar$-module; this will give us a pairing on $\mathcal{M}$, and a nondegenerate symmetric $\mathbb{C}[q^{\pm 1/l}]$-linear pairing on $A \otimes \mathbb{C}[q^{\pm 1/l}]$. This pairing will turn out to be a $\mathbb{C}[q^{\pm 1/l}]$-linear extension of a $\mathbb{C}$-linear pairing on A. We shall use the notation of Section 6.3 of [9].

First, the $D^\hbar$-module $\mathcal{M}^*$ is defined to be the space of $\mathcal{F}$-module homomorphisms $\mathcal{M} \to \mathcal{F}$, where $\mathcal{F}$ is the coefficient ring. The $D^\hbar$-module structure is given by

$$
(\hbar \cdot \pi)(P) = \hbar \pi(P), \quad (\partial \cdot \pi)(P) = -\pi(\partial \cdot P) + q \frac{\partial}{\partial q} \pi(P)
$$

for $\pi \in \mathcal{M}^*$.

Next, we denote by $\bar{\mathcal{M}}^*$ the $D^\hbar$-module obtained from $\mathcal{M}^*$ by reversing the sign in the action of $\hbar$. That is, $\bar{\mathcal{M}}^* = \mathcal{M}^*$ (as $\mathcal{F}$-modules), but with action of $D^\hbar$ derived in the obvious way from $\hbar \odot \pi = -\hbar \pi$, $\partial \odot \pi = \partial \cdot \pi$.

Let $\delta_0, \ldots, \delta_{s-1}$ be the basis of $\bar{\mathcal{M}}^* = \mathcal{M}^*$ (over $\mathcal{F}$) which is dual to the basis $1, \hbar \partial, \ldots, (\hbar \partial)^{s-1}$ of $\mathcal{M}$. The key technical result we need is:

Proposition 4.1. (1) δ_n *is a cyclic element of* $\bar{\mathcal{M}}^*$ *(that is,* $D^\hbar \odot \delta_n = \bar{\mathcal{M}}^*$*)*.

(2) $(T_w - q) \odot \delta_n = 0$.

(3) *The map* $\mathcal{M} \to \bar{\mathcal{M}}^*$, $[P] \mapsto [P \odot \delta_n]$ *is an isomorphism of* $D^\hbar$-*modules.*

It should be noted that the operator $T_w - q$ is self-adjoint only in the special case $\mathbb{P}(w) = \mathbb{C}P^n$, even though $\mathcal{M} = D^\hbar/(T_w - q)$ is always a self-adjoint $D^\hbar$-module.

Proof. Let $P_0^*, \ldots, P_{s-1}^*$ be the basis of $\bar{\mathcal{M}}^*$ which is dual to $P_0, \ldots, P_{s-1}$. For readability we shall omit square brackets throughout this proof. Note that $P_i^* = \delta_i$ for $i = 0, \ldots, n$.

We claim that

$$
P_\alpha \odot \delta_n = \begin{cases} P^*_{n-\alpha} = \delta_{n-\alpha} & \text{when } 0 \le \alpha < u_1 = n+1, \\ \frac{m_1}{m_{i+1}} P^*_{s+n-\alpha} & \text{when } u_1 + \cdots + u_i \le \alpha < u_1 + \cdots + u_{i+1}, \\ P^*_n = \delta_n & \text{when } \alpha = s \text{ (we define } P_s \text{ below).} \end{cases}
$$

Assuming this, the first two formulae (for $\alpha = 0, \ldots, s-1$) prove (1). In the third formula $P^*_n = \delta_n$, P_s means $\frac{1}{r_s}\hbar\partial \frac{1}{r_{s-1}}\hbar\partial \ldots \frac{1}{r_1}\hbar\partial$, which is $q^{-1}T_w$, so this gives (2). The third statement is an immediate consequence of (1) and (2) (cf. Section 6.3 of [9]).

To prove the claim, we shall make use of

$$(*) \qquad\qquad\qquad \hbar\partial P_\alpha = r_{\alpha+1} P_{\alpha+1},$$

$$(**) \qquad\qquad\qquad \hbar\partial \odot P^*_\alpha = r_\alpha P^*_{\alpha-1}$$

and the value of r_α given in Definition 3.2.

The case $0 \le \alpha < u_1 = n+1$. Since $r_0 = \cdots = r_n = 1$, from $(**)$ we have $P_\alpha \odot \delta_n = (\hbar\partial)^\alpha \odot P^*_n = P^*_{n-\alpha}$.

The case $u_1 + \cdots + u_i \le \alpha < u_1 + \cdots + u_{i+1}$. We shall prove this by induction on $i = 0, 1, \ldots, k-1$ (regarding the previous case as $i = 0$).

(i) If $\alpha = u_1 + \cdots + u_i$ for some $i \ge 1$, we have

$$
\begin{aligned}
P_\alpha \odot \delta_n &= m_i q^{-\Delta_i} \hbar\partial P_{\alpha-1} \odot \delta_n && \text{by } (*), \text{ as } r_\alpha = m_i^{-1} q^{\Delta_i} \\
&= m_i q^{-\Delta_i} \hbar\partial \odot \frac{m_1}{m_i} P^*_{s+n-(\alpha-1)} && \text{(inductive hypothesis)} \\
&= m_1 q^{-\Delta_i} r_{s+n-\alpha+1} P^*_{s+n-\alpha} && \text{by } (**).
\end{aligned}
$$

Now, $s + n - \alpha + 1 = s + u_1 - (u_1 + \cdots + u_i) = s - (u_{k+2-i} + \cdots + u_k)$ (by Lemma 2.1) $= u_1 + \cdots + u_{k+1-i}$. (This argument applies only if $i \ge 2$, but the case $i = 1$ is obvious.) Hence

$$
r_{s+n-\alpha+1} = r_{u_1+\cdots+u_{k+1-i}} = \frac{1}{m_{k+1-i}} q^{\Delta_{k+1-i}} = \frac{1}{m_{i+1}} q^{\Delta_i}
$$

by Lemma 3.1. We obtain $P_\alpha \odot \delta_n = \frac{m_1}{m_{i+1}} P^*_{s+n-\alpha}$.

(ii) If $u_1 + \cdots + u_i \le \alpha < u_1 + \cdots + u_{i+1}$ for some i, then

$$
\begin{aligned}
P_\alpha \odot \delta_n &= \hbar\partial P_{\alpha-1} \odot \delta_n && \text{by } (*), \text{ as } r_\alpha = 1 \\
&= \hbar\partial \odot \frac{m_1}{m_{i+1}} P^*_{s+n-(\alpha-1)} && \text{(inductive hypothesis)} \\
&= \frac{m_1}{m_{i+1}} r_{s+n-\alpha+1} P^*_{s+n-\alpha} && \text{by } (**).
\end{aligned}
$$

Here we have $s + n - \alpha + 1 = u_1 + \cdots + u_{k+1-i} - l$ with $0 < l < u_{i+1} = u_{k+1-i}$ (from Lemma 2.1), so $r_{s+n-\alpha+1} = 1$. We obtain $P_\alpha \odot \delta_n = \frac{m_1}{m_{i+1}} P^*_{s+n-\alpha}$ again.

The case $\alpha = s$. We have

$$
\begin{aligned}
P_s \odot \delta_n &= m_k q^{-\Delta_k} \hbar \partial P_{s-1} \odot \delta_n && \text{by } (*) \\
&= m_k q^{-\Delta_k} \hbar \partial \odot \frac{m_1}{m_k} P^*_{n+1} && \text{(inductive hypothesis)} \\
&= m_1 q^{-\Delta_k} r_{n+1} P^*_n && \text{by } (**).
\end{aligned}
$$

Here we have $r_{n+1} = r_{u_1} = \frac{1}{m_1} q^{\Delta_1}$, and $\Delta_1 = \Delta_k$ by Lemma 3.1, so we conclude that $P_s \odot \delta_n = \delta_n$. $\qquad\square$

The natural composition $\mathcal{M} \times \mathcal{M} \to \bar{\mathcal{M}}^* \times \mathcal{M} \to \mathcal{F}$, making use of the above isomorphism $\mathcal{M} \to \bar{\mathcal{M}}^*$, defines a pairing. We normalize it as follows:

Definition 4.2. $\langle\!\langle P, Q \rangle\!\rangle = \frac{1}{w_0 \ldots w_n} (P \odot \delta_n)(Q) \ (= \frac{1}{m_1} (P \odot \delta_n)(Q))$.

Corollary 4.3. *We have (from the formula for $P_\alpha \odot \delta_n$ in the proof of Proposition 4.1)*

$$
\langle\!\langle P_\alpha, P_\beta \rangle\!\rangle = \begin{cases} \frac{1}{m_1} \delta_{n-\alpha, \beta} & \text{if } 0 \le \alpha < u_1, \\ \frac{1}{m_{i+1}} \delta_{s+n-\alpha, \beta} & \text{if } u_1 + \cdots + u_i \le \alpha < u_1 + \cdots + u_{i+1}, i \ge 1. \end{cases}
$$

With this normalization, the induced pairing on A agrees with the usual Poincaré intersection pairing on the cohomology of $\mathbb{P}(w)$; it is known from [12] that $(1, p^n) = 1/(w_0 \ldots w_n)$. The induced pairing on $A \otimes \mathbb{C}[q^{\pm 1/l}]$ satisfies the Frobenius property (see Section 6.5 of [9]). Hence, by the cyclic property, it agrees with the orbifold quantum Poincaré intersection pairing.

This concludes our construction of an abstract orbifold quantum product, grading, and pairing directly from $T_w - q$, and our verification that they agree with the usual ones.

Example 4.4. $\mathbb{P}(1, 2, 3)$. We have $w_0 = 1, w_1 = 2, w_2 = 3$ and $s = 1+2+3 = 6$. The differential operator is

$$
\begin{aligned}
T_w - q &= \hbar \partial \ 2\hbar\partial(2\hbar\partial - \hbar) \ 3\hbar\partial(3\hbar\partial - \hbar)(3\hbar\partial - 2\hbar) - q \\
&= 2^2 3^3 \hbar^6 \partial^3 (\partial - \tfrac{1}{3})(\partial - \tfrac{1}{2})(\partial - \tfrac{2}{3}) - q.
\end{aligned}
$$

This has order 6, and it is homogeneous of weight 12, where $|\hbar| = 2, |q| = 12$.

We have
$$F = \{\tfrac{0}{1}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}\} = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}\},$$

so $u_1 = 3$, $u_2 = 1$, $u_3 = 1$, $u_4 = 1$. It is convenient to display all relevant data in the following diagram:

	$w_0 = 1$	$w_1 = 2$	$w_2 = 3$	
$S_{f_1} = \{0, 1, 2\},\ f_1 = 0$	$\tfrac{0}{1}$	$\tfrac{0}{2}$	$\tfrac{0}{3}$	$\Delta_1 = \tfrac{1}{3}, m_1 = 6$
$S_{f_2} = \{2\},\ f_2 = \tfrac{1}{3}$			$\tfrac{1}{3}$	$\Delta_2 = \tfrac{1}{6}, m_2 = 3$
$S_{f_3} = \{1\},\ f_3 = \tfrac{1}{2}$		$\tfrac{1}{2}$		$\Delta_3 = \tfrac{1}{6}, m_3 = 2$
$S_{f_4} = \{2\},\ f_4 = \tfrac{2}{3}$			$\tfrac{2}{3}$	$\Delta_4 = \tfrac{1}{3}, m_4 = 3$

In the central 4×3 block, the number of entries in the ith row is u_i, and the number of entries in the $j+1$th column is w_j.

The factorization is
$$q^{-1} T_w = 3q^{-\frac{1}{3}}(\hbar\partial)^1 2q^{-\frac{1}{6}}(\hbar\partial)^1 3q^{-\frac{1}{6}}(\hbar\partial)^1 6q^{-\frac{1}{3}}(\hbar\partial)^3.$$

The bases of $\mathcal{M}$ and A constructed above are:

$$1,\ \hbar\partial,\ (\hbar\partial)^2, \qquad\qquad\qquad 1,\ p,\ p^2,$$
$$6q^{-\frac{1}{3}}(\hbar\partial)^3, \qquad\qquad\qquad\qquad \mathbf{1}_{\frac{1}{3}},$$
$$3q^{-\frac{1}{6}}(\hbar\partial)\, 6q^{-\frac{1}{3}}(\hbar\partial)^3, \qquad\qquad \mathbf{1}_{\frac{1}{2}},$$
$$2q^{-\frac{1}{6}}\hbar\partial\, 3q^{-\frac{1}{6}}\hbar\partial\, 6q^{-\frac{1}{3}}(\hbar\partial)^3, \qquad \mathbf{1}_{\frac{2}{3}}.$$

The matrix of structure constants (quantum multiplication by p) with respect to this basis is
$$\begin{pmatrix} 0 & & & & & \tfrac{1}{3}q^{\frac{1}{3}} \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \tfrac{1}{6}q^{\frac{1}{3}} & 0 & & \\ & & & \tfrac{1}{3}q^{\frac{1}{6}} & 0 & \\ & & & & \tfrac{1}{2}q^{\frac{1}{6}} & 0 \end{pmatrix}.$$

These products determine all others, and we obtain the following orbifold quantum

multiplication table:

	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{2}{3}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{1}{2}}$	$\mathbf{1}_{\frac{2}{3}}$
p		p^2	$\frac{1}{6}q^{\frac{1}{3}}\mathbf{1}_{\frac{1}{3}}$	$\frac{1}{3}q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{2}}$	$\frac{1}{2}q^{\frac{1}{6}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}$
p^2			$\frac{1}{18}q^{\frac{1}{2}}\mathbf{1}_{\frac{1}{2}}$	$\frac{1}{6}q^{\frac{1}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{6}q^{\frac{1}{2}}$	$\frac{1}{3}q^{\frac{1}{3}}p$
$\mathbf{1}_{\frac{1}{3}}$				$\frac{1}{3}q^{\frac{1}{3}}$	$q^{\frac{1}{6}}p$	$2p^2$
$\mathbf{1}_{\frac{1}{2}}$					$3p^2$	$q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{3}}$
$\mathbf{1}_{\frac{2}{3}}$						$\frac{2}{3}q^{\frac{1}{6}}\mathbf{1}_{\frac{1}{2}}$

Orbifold cohomology products are obtained by setting $q = 0$ in this table. Note that p generates the orbifold quantum cohomology, but not the orbifold cohomology. Ages and degrees are as shown below:

age $\mathbf{1}_0 = 0$	$\|1\| = 0$	$\|p\| = 2$	$\|p^2\| = 4$
age $\mathbf{1}_{\frac{1}{3}} = 1$			$\|\mathbf{1}_{\frac{1}{3}}\| = 2$
age $\mathbf{1}_{\frac{1}{2}} = 1$		$\|\mathbf{1}_{\frac{1}{2}}\| = 2$	
age $\mathbf{1}_{\frac{2}{3}} = 1$			$\|\mathbf{1}_{\frac{2}{3}}\| = 2$

Finally, the pairing on $\mathcal{M}$ is given by $\langle\!\langle P_i, P_j \rangle\!\rangle = \frac{1}{6}$ if $i + j = 2$, $\langle\!\langle P_3, P_5 \rangle\!\rangle = \langle\!\langle P_5, P_3 \rangle\!\rangle = \frac{1}{3}$, and $\langle\!\langle P_4, P_4 \rangle\!\rangle = \frac{1}{2}$ (with all other products zero). □

Example 4.5. $\mathbb{P}(1, 1, 3)$. In this case we have orbifold classes with fractional degrees. We just state the results, as the calculations are very similar to those in the previous example. First, the data is

	$w_0 = 1$	$w_1 = 1$	$w_2 = 3$	
$S_{f_1} = \{0, 1, 2\},\ f_1 = 0$	$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{3}$	$\Delta_1 = \frac{1}{3}, m_1 = 3$
$S_{f_2} = \{2\},\ f_2 = \frac{1}{3}$			$\frac{1}{3}$	$\Delta_2 = \frac{1}{3}, m_2 = 3$
$S_{f_3} = \{2\},\ f_3 = \frac{2}{3}$			$\frac{2}{3}$	$\Delta_3 = \frac{1}{3}, m_3 = 3$

and we have

$$q^{-1}T_w = q^{-1}3^3\hbar^5\partial^3(\partial - \tfrac{1}{3})(\partial - \tfrac{2}{3}) = 3q^{-\frac{1}{3}}(\hbar\partial)^1 3q^{-\frac{1}{3}}(\hbar\partial)^1 3q^{-\frac{1}{3}}(\hbar\partial)^3.$$

The orbifold quantum multiplication table is

	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{2}{3}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{\frac{2}{3}}$
p		p^2	$\frac{1}{3}q^{\frac{1}{3}}\mathbf{1}_{\frac{1}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}$
p^2			$\frac{1}{9}q^{\frac{2}{3}}\mathbf{1}_{\frac{2}{3}}$	$\frac{1}{9}q^{\frac{2}{3}}$	$\frac{1}{3}q^{\frac{1}{3}}p$
$\mathbf{1}_{\frac{1}{3}}$				$\frac{1}{3}q^{\frac{1}{3}}p$	p^2
$\mathbf{1}_{\frac{2}{3}}$					$\mathbf{1}_{\frac{1}{3}}$

where $1, p, p^2, \mathbf{1}_{\frac{1}{3}}, \mathbf{1}_{\frac{2}{3}}$ correspond to $1, \hbar\partial, (\hbar\partial)^2, 3q^{-\frac{1}{3}}(\hbar\partial)^3, 3q^{-\frac{1}{3}}(\hbar\partial)3q^{-\frac{1}{3}}(\hbar\partial)^3$. We have

age $\mathbf{1}_0 = 0$	$|1| = 0$	$|p| = 2$	$|p^2| = 4$
age $\mathbf{1}_{\frac{1}{3}} = \frac{4}{3}$			$\|\mathbf{1}_{\frac{1}{3}}\| = \frac{8}{3}$
age $\mathbf{1}_{\frac{2}{3}} = \frac{2}{3}$			$\|\mathbf{1}_{\frac{2}{3}}\| = \frac{4}{3}$

and the pairing is given by $\langle\!\langle P_i, P_j \rangle\!\rangle = \frac{1}{3}$ if $i + j = 2$, $\langle\!\langle P_3, P_4 \rangle\!\rangle = \langle\!\langle P_4, P_3 \rangle\!\rangle = \frac{1}{3}$ (with all other products zero).   □

5. Hypersurfaces in weighted projective space

Based on the toric approach to mirror symmetry, Corti and Golyshev conjectured that the orbifold quantum cohomology of a (quasismooth) hypersurface

$$X^d \subseteq \mathbb{P}(w)$$

of degree d is governed by the differential operator

$$\hbar^s \prod_{i=0}^{n}(w_i\partial)(w_i\partial - 1)\dots(w_i\partial - (w_i-1)) - q\hbar^d(d\partial + 1)\dots(d\partial + (d-1))(d\partial + d).$$

(this operator appears in Section 7.3 of [6] without the $\hbar$ factors; also in [5] for the Calabi–Yau case $s = d$, where the $\hbar$ factors cancel out). The method of [3] gives evidence for this conjecture in the Fano case, i.e. when $s > d$. We shall always assume that $s > d$, although our approach applies also when $s = d$ (cf. Section 6.7 of [9]).

Since $\partial q = q(\partial + 1)$, we have

$$q \hbar^d d^d (\partial + \tfrac{1}{d}) \ldots (\partial + \tfrac{d-1}{d})(\partial + \tfrac{d}{d}) = \hbar^d d^d (\partial - \tfrac{d-1}{d}) \ldots (\partial - \tfrac{1}{d})(\partial - \tfrac{0}{d})q,$$

which shows that both summands of

$$w^w \hbar^s \prod_{i=0}^{n} \partial(\partial - \tfrac{1}{w_i}) \ldots (\partial - \tfrac{w_i-1}{w_i}) \; - \; q\hbar^d d^d (\partial + \tfrac{1}{d}) \ldots (\partial + \tfrac{d-1}{d})(\partial + \tfrac{d}{d})$$

can be written with a factor of $\hbar\partial$ on the left. Cancelling this factor, we obtain an operator of order $s - 1$ (in terms of $D^\hbar$-modules, we quotient out by the trivial $D^\hbar$-module $D^\hbar/(\hbar\partial)$). We call[1] this operator $T_{w_1,\ldots,w_n} - qS_{d-1}$:

$$\underbrace{w^w \hbar^{s-1} \prod_{i=1}^{n} \partial(\partial - \tfrac{1}{w_i}) \ldots (\partial - \tfrac{w_i-1}{w_i})}_{T_{w_1,\ldots,w_n}} \; - \; q \underbrace{\hbar^{d-1} d^d (\partial + \tfrac{1}{d}) \ldots (\partial + \tfrac{d-1}{d})}_{S_{d-1}}$$

Here we have assumed that $w_0 = 1$. To simplify notation, we shall also assume that $w_1, \ldots, w_n$ *are such that no further left-cancellations of the above type are possible.* It follows that the $D^\hbar$-module

$$\mathcal{M} = D^\hbar/(T_{w_1,\ldots,w_n} - qS_{d-1})$$

is irreducible. In the general case, an irreducible $D^\hbar$-module is obtained by left-cancelling all common factors (see [6]), and our method can be applied to that.

Observe that the case $d = 1$ gives $T_{w_1,\ldots,w_n} - q$, which is the operator associated with $\mathbb{P}(w_1, \ldots, w_n)$, as expected. The case $w_1 = \cdots = w_n = 1$ (hence $s = n + 1$) gives $(\hbar\partial)^n - qS_{d-1}$, which is the operator associated with a degree d hypersurface in $\mathbb{C}P^n$, denoted by M^d_{n+1} in [15].

In this section, by extending the method of Section 4, we shall give a method to extract an "abstract orbifold quantum product". We emphasize that this is a method whose success is not guaranteed. Moreover, the (genuine) quantum product is not yet known for hypersurfaces, in general. Nevertheless, we can give a nontrivial example (Example 5.2) where the Gromov–Witten invariants have been proposed by Corti ([4]), and our method is consistent with his results.

---

[1] We do not write T_w here; the abbreviation T_w always means $T_{w_0,\ldots,w_n}$.

As in Section 2, we define

$$F = \{\tfrac{i}{w_j} \mid 0 \le i \le w_j - 1, \ \ 1 \le j \le n\}$$
$$= \{f_1, \dots, f_k\}$$

and denote by $u_1, \dots, u_k$ the multiplicities of $f_1, \dots, f_k$. However, $u_1 = n$ here. We use the notation Δ_i, m_i as in Section 3. Thus, we have a factorization

$$q^{-1} T_{w_1, \dots, w_n} = \tfrac{1}{r_{s-1}} \hbar\partial \ \tfrac{1}{r_{s-2}} \hbar\partial \ \ \dots \ \ \tfrac{1}{r_1} \hbar\partial$$

and we can introduce $P_0 = 1$ and $P_i = \tfrac{1}{r_i}\hbar\partial P_{i-1}$ for $1 \le i \le s-2$. The equivalence classes of the operators $P_0, \dots, P_{s-2}$ form a basis of the $D^\hbar$-module $D^\hbar/(T_{w_1, \dots, w_n} - q S_{d-1})$.

As in Section 4, the action of ∂ defines a connection on the bundle whose space of sections is $\mathcal{M}$. However, when $d \ge 2$, the connection matrix Ω with respect to the basis $P_0, \dots, P_{s-2}$ is not of the form $\tfrac{1}{\hbar}\omega$. To achieve this form (which is the starting point for the construction of a product operation) we must construct a new basis.

It will be convenient to construct such a basis in two steps.

Step 1. The method of [15] produces a basis $\widehat{P}_0, \dots, \widehat{P}_{s-2}$ with respect to which the connection matrix has the form $\tfrac{1}{\hbar}\hat{\omega}$. Let us review that method here. As in our discussion of the Birkhoff factorization method in Section 4, the new basis is given by $L_+^{-1} \cdot P_0, L_+^{-1} \cdot P_1, \dots, L_+^{-1} \cdot P_{s-2}$, for a certain "gauge transformation" $L_+ = Q_0(I + \hbar Q_1 + \cdots)$. In contrast to the situation of Section 4, there is no short cut to finding L_+ here. However, L_+ can be found as the unique solution of the ordinary differential equation

$$\tfrac{1}{\hbar}\hat{\omega} = L_+ \Omega L_+^{-1} + L_+ d L_+^{-1}$$

which is homogeneous and polynomial in $q^{1/l}$, and which satisfies the initial condition $L_+|_{q^{1/l}=0} = I$. As in [15] it can be proved that this reduces to a system of algebraic equations for $Q_0, Q_1, \dots$ which can be solved by an explicit algorithm. (In the situation of Section 4 we would have $L_+ = I$, as we are now starting from the basis given by $P_i = \tfrac{1}{r_i}\hbar\partial \cdots \tfrac{1}{r_1}\hbar\partial$. The L_+ of Section 4 converts $1, \hbar\partial, \dots, (\hbar\partial)^{s-2}$ to $P_0, \dots, P_{s-2}$, which could have been described as Step 0.)

Example 5.1. $X^3 \subseteq \mathbb{P}(1,1,1,1,1) = \mathbb{C}P^4$. In the notation of [15] this is M_5^3. As this example is worked out in detail in Examples 3.6, 5.4, 6.24, 6.36 of [9] we shall just summarize the results of the calculations.

First, we have the differential operator

$$q^{-1} T_{1,1,1,1} - S_2 = q^{-1}(\hbar\partial)^4 - 3^3 \hbar^2 (\partial + \tfrac{1}{3})(\partial + \tfrac{2}{3}).$$

With respect to the basis $P_0 = 1$, $P_1 = \hbar\partial$, $P_2 = (\hbar\partial)^2$, $P_3 = (\hbar\partial)^3$ the connection matrix is

$$\Omega = \tfrac{1}{\hbar} \begin{pmatrix} & & & 6q\hbar^2 \\ 1 & & & 27q\hbar \\ & 1 & & 27q \\ & & 1 & \end{pmatrix}.$$

The gauge transformation $L_+ = Q_0(I + \hbar Q_1)$ can be found by solving the o.d.e. $\frac{1}{\hbar}\hat{\omega} = L_+\Omega L_+^{-1} + L_+ dL_+^{-1}$ subject to $L_+|_{q=0} = I$. This gives

$$Q_0 = \begin{pmatrix} 1 & & 6q & \\ & 1 & & 21q \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} & & & 6q \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

The new basis is $\hat{P}_0 = 1$, $\hat{P}_1 = \hbar\partial$, $\hat{P}_2 = (\hbar\partial)^2 - 6q$, $\hat{P}_3 = (\hbar\partial)^3 - 21q\hbar\partial - 6\hbar q$, and the matrix of the connection form is

$$\hat{\Omega} = \tfrac{1}{\hbar} \begin{pmatrix} & 6q & & 36q^2 \\ 1 & & 15q & \\ & 1 & & 6q \\ & & 1 & \end{pmatrix}$$

with respect to this basis. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

The basis $\hat{P}_0, \dots, \hat{P}_{s-2}$ allows us to construct a product operation as in Section 4. Unfortunately, this product does not necessarily satisfy the Frobenius property. In general, therefore, it is necessary to modify the basis further, and this will be Step 2. As preparation for this, we begin with a brief review of the Birkhoff decomposition.

Motivation for Step 2 The Birkhoff decomposition (Theorem 8.1.2 of [14]) can be written

$$\Lambda\mathrm{GL}_{s-1}\mathbb{C} = \bigcup_{\gamma \in \check{T}} \Lambda_-\mathrm{GL}_{s-1}\mathbb{C}\, \gamma\, \Lambda_+\mathrm{GL}_{s-1}\mathbb{C},$$

where $\check{T}$ denotes the set of homomorphisms from S^1 to the diagonal matrices in $\mathrm{GL}_{s-1}\mathbb{C}$. If $\gamma(\hbar) = \mathrm{diag}(\hbar^{a_0}, \dots, \hbar^{a_{s-2}})$ is restricted to the set of homomorphisms satisfying $a_0 \leq \dots \leq a_{s-2}$, then the decomposition is a disjoint union. The "big cell" is the piece given by $\gamma = I$; it is a dense open subset of the identity component of $\Lambda\mathrm{GL}_{s-1}\mathbb{C}$. The "small cells" (where $\gamma \neq I$) have finite codimension in $\Lambda\mathrm{GL}_{s-1}\mathbb{C}$.

The term "cell" is used here because the decomposition is equivalent to the $\Lambda_-\mathrm{GL}_{s-1}\mathbb{C}$-orbit decomposition

$$\Lambda\mathrm{GL}_{s-1}\mathbb{C}/\Lambda_+\mathrm{GL}_{s-1}\mathbb{C} = \bigcup_{\gamma \in \check{T}} \Lambda_-\mathrm{GL}_{s-1}\mathbb{C}\, [\gamma]$$

of the Grassmannian $Gr^{(s-1)} \cong \Lambda \mathrm{GL}_{s-1}\mathbb{C}/\Lambda_+\mathrm{GL}_{s-1}\mathbb{C}$ (see Section 8.3 of [14]). It is analogous to the cell decomposition, or cell-bundle[2] decomposition, of a finite-dimensional generalized flag manifold given by the orbits of a parabolic subgroup. The main point is that a "small cell" $\Lambda_-\mathrm{GL}_{s-1}\mathbb{C}\,[\gamma]$ is diffeomorphic to a proper unipotent subgroup Λ_-^γ of $\Lambda_-\mathrm{GL}_{s-1}\mathbb{C}$ (Theorem 8.6.3 of [14]). This shows that any map L which takes values in $\Lambda_-\mathrm{GL}_{s-1}\mathbb{C}\,\gamma\,\Lambda_+\mathrm{GL}_{s-1}\mathbb{C}$ (and therefore admits at least one factorization $L = L_-\gamma L_+$) has a *most economical* factorization

$$L = L_-^c\,\gamma\,L_+^c.$$

The same phenomenon occurs for finite-dimensional generalized flag manifolds. The simplest example is $\mathbb{C}P^n$: the i-dimensional cell $\mathbb{C}^i$ can be described as an orbit of the $\frac{1}{2}(n+1)(n+2)$-dimensional group of upper triangular matrices in $\mathrm{GL}_{n+1}\mathbb{C}$, but most economically as an orbit of a certain i-dimensional unipotent subgroup (see Chapter 14, part III, of [7]).

Step 2 will amount to extracting the economical factor L_-^c from $L_-\gamma$. More precisely, by Theorem 8.6.3 of [14], we can write

$$L = L_-\gamma L_+ = L_-^c L_-^f\gamma L_+ = L_-^c\gamma L_+^f L_+$$

where L_-^f denotes the "superfluous factor"; this is a polynomial in $\hbar^{-1}$ and satisfies $L_-^f\gamma = \gamma L_+^f$ where L_+^f is polynomial in $\hbar$. Thus, Step 1 uses the gauge transformation L_+^{-1} to convert $P_0, \ldots, P_{s-2}$ to a provisional basis $\widehat{P}_0, \ldots, \widehat{P}_{s-2}$, then Step 2 uses a further gauge transformation $(\gamma L_+^f)^{-1}$ to convert $\widehat{P}_0, \ldots, \widehat{P}_{s-2}$ to the desired basis $\widetilde{P}_0, \ldots, \widetilde{P}_{s-2}$.

Step 2. As in Proposition 4.1, it can be proved that

$$(T_{w_1,\ldots,w_n} - qS_{d-1}) \odot \delta_{n-1} = 0.$$

However the map

$$\mathcal{M} \to \bar{\mathcal{M}}^*, \quad [P] \mapsto [P \odot \delta_{n-1}]$$

is not in general an isomorphism of $D^\hbar$-modules, and it is at this point that we need the homomorphism γ. Let us assume that

(H1) there exist integers $a_0 \leq \cdots \leq a_{s-2}$ with the property that

$$\hbar^{-a_0}\widehat{P}_0 \odot \delta_{n-1}, \ldots, \hbar^{-a_{s-2}}\widehat{P}_{s-2} \odot \delta_{n-1}$$

have *minus* the weighted degrees of the elements $\hbar^{-a_0}\widehat{P}_0, \ldots, \hbar^{-a_{s-2}}\widehat{P}_{s-2}$ (not necessarily in the same order).

---

[2]The cell decompositions here arise from Morse functions; the cell-bundle decompositions arise from Morse–Bott functions.

As in Definition 4.2, we can define a pairing by

$$\langle\!\langle P, Q \rangle\!\rangle = \frac{d}{w_1 \dots w_n} (P \odot \delta_{n-1})(Q).$$

The normalization of Section 4 is modified by the factor d here, to take account of the degree of the hypersurface. We shall assume further that

(H2) there exists a basis with respect to which the matrix of $\langle\!\langle\ ,\ \rangle\!\rangle$ is a nondegenerate symmetric matrix independent of $q, \hbar$.

That is, the pairing $\langle\!\langle\ ,\ \rangle\!\rangle$ extends a nondegenerate symmetric $\mathbb{C}$-linear pairing on a complex vector space of dimension $s - 1$. Finally (from the motivation above) we seek a map $G = L_-^f \gamma = \gamma L_+^f$ such that the gauge transformation G^{-1} converts $\hat{\Omega}$ to a connection form $\tilde{\Omega} = \frac{1}{\hbar} \tilde{\omega}$ where $\tilde{\omega}$ is independent of $\hbar$. This $\tilde{\omega}$ is the connection matrix with respect to the basis $\widetilde{P}_0 = G^{-1} \cdot \hat{P}_0, \dots, \widetilde{P}_{s-2} = G^{-1} \cdot \hat{P}_{s-2}$. Such a G necessarily satisfies

(∗) $$\frac{1}{\hbar}\tilde{\omega} = G\frac{1}{\hbar}\hat{\omega}G^{-1} + GdG^{-1}$$

and we shall assume that

(H3) there exists in some neighbourhood of $q^{1/l} = 0$ a solution $G = L_-^f \gamma$ of the differential equation (∗), where $\gamma(\hbar) = \mathrm{diag}(\hbar^{a_0}, \dots, \hbar^{a_{s-2}})$.

We discuss suitable normalizations of such solutions (initial conditions) later.

Assuming (H1)–(H3) (which we shall verify in our main example), we can attempt to define a product operation as in Section 4. Let A be the vector space with basis denoted by the symbols

$$1,\ p,\ \dots,\ p^{u_1-1};$$
$$\mathbf{1}_{f_2},\ \mathbf{1}_{f_2}p,\ \dots,\ \mathbf{1}_{f_2}p^{u_2-1};$$
$$\vdots$$
$$\mathbf{1}_{f_k},\ \mathbf{1}_{f_k}p,\ \dots,\ \mathbf{1}_{f_k}p^{u_k-1}$$

We define QA to be $A \otimes \mathbb{C}[q^{\pm 1/l}]$, and we define a $\mathbb{C}[p, q^{\pm 1/l}]$-module action on QA by specifying that the matrix of multiplication by p is $\tilde{\omega}$. As we shall see, in contrast to the situation of the previous section, this $\mathbb{C}[p, q^{\pm 1/l}]$-module action does not in general allow us to obtain a product structure on QA, because the action of p is not necessarily cyclic.

Example 5.2. $X^3 \subseteq \mathbb{P}(1, 1, 1, 2)$. We have $w_0 = w_1 = w_2 = 1$, $w_3 = 2$ and $s = 5$, $d = 3$. The differential operator is

$$q^{-1}T_{1,1,2} - S_2 = q^{-1}2^2\hbar^4\partial^3(\partial - \tfrac{1}{2}) - 3^3\hbar^2(\partial + \tfrac{1}{3})(\partial + \tfrac{2}{3}).$$

We have
$$F = \{\tfrac{0}{1}, \tfrac{0}{1}, \tfrac{0}{2}, \tfrac{1}{2}\} = \{0, \tfrac{1}{2}\},$$

so $u_1 = 3, u_2 = 1$. As in Section 4 we can display the data as follows:

	$w_1 = 1$	$w_2 = 1$	$w_3 = 2$	
$S_{f_1} = \{1, 2, 3\},\ f_1 = 0$	$\tfrac{0}{1}$	$\tfrac{0}{1}$	$\tfrac{0}{2}$	$\Delta_1 = \tfrac{1}{2},\ m_1 = 2$
$S_{f_2} = \{3\},\ f_2 = \tfrac{1}{2}$			$\tfrac{1}{2}$	$\Delta_2 = \tfrac{1}{2},\ m_2 = 2$

The factorization of $q^{-1} T_{1,1,2}$ is

$$q^{-1} T_{1,1,2} = 2q^{-\frac{1}{2}} (\hbar\partial) 2q^{-\frac{1}{2}} (\hbar\partial)^3 = \tfrac{1}{r}(\hbar\partial)\tfrac{1}{r}(\hbar\partial)^3,$$

where $r = \tfrac{1}{2}q^{\frac{1}{2}}$. Thus, our starting point is the basis

$$P_0 = 1,\ \ P_1 = \hbar\partial,\ \ P_2 = (\hbar\partial)^2,\ \ P_3 = \frac{1}{r}(\hbar\partial)^3.$$

We have $|r| = 2$, so the degrees of these basis elements are $0, 2, 4, 4$. With respect to this basis we have

$$\Omega = \tfrac{1}{\hbar}\begin{pmatrix} 1 & & & 6\hbar^2 r \\ & 1 & & 27\hbar r \\ & & 1 & 27r \\ & & & r \end{pmatrix}.$$

Step 1. The gauge transformation L_+^{-1} is given by

$$L_+ = \begin{pmatrix} 1 & & 12r^2 & \\ & 1 & & 30r \\ & & 1 & \\ & & & 1 \end{pmatrix}\left(I + \hbar\begin{pmatrix} & & & 12r \\ & & & \\ & & & \\ & & & \end{pmatrix}\right).$$

Application of L_+^{-1} produces the new basis

$$\widehat{P}_0 = P_0,\ \ \ \widehat{P}_1 = P_1,\ \ \ \widehat{P}_2 = P_2 - 12r^2 P_0,\ \ \ \widehat{P}_3 = P_3 - 30rP_1 - 12\hbar rP_0.$$

With respect to this basis, we have

$$\widehat{\Omega} = \tfrac{1}{\hbar}\hat{\omega} = \tfrac{1}{\hbar}\begin{pmatrix} 1 & 12r^2 & & -36r^3 \\ & 1 & 18r^2 & \\ & & 1 & -3r \\ & & & r \end{pmatrix}.$$

We omit the details of this calculation, which is similar to those in [15].

Step 2. We have to verify (H1)–(H3). For γ we take $\gamma(\hbar) = (1,1,1,\hbar)$. The degrees of $\gamma^{-1}\widehat{P}_0, \gamma^{-1}\widehat{P}_1, \gamma^{-1}\widehat{P}_2, \gamma^{-1}\widehat{P}_3$ are $0, 2, 4, 2$, and the degrees of $(\gamma^{-1}\widehat{P}_0)\odot\delta_2, (\gamma^{-1}\widehat{P}_1)\odot\delta_2, (\gamma^{-1}\widehat{P}_2)\odot\delta_2, (\gamma^{-1}\widehat{P}_3)\odot\delta_2$ are $-4, -2, 0, -2$, so (H1) is satisfied.

To verify (H3), we note that G^{-1} must be of the form

$$G^{-1} = \gamma^{-1}Z = \gamma^{-1}(Z_0 + \tfrac{1}{\hbar}Z_1 + \tfrac{1}{\hbar^2}Z_2),$$

where Z is homogeneous with respect to a basis with degrees $0, 2, 4, 2$, i.e. the entries of the matrix function Z have the degrees shown below:

0	2	4	2
−2	0	2	0
−4	−2	0	−2
−2	0	2	0

Equating the coefficients of each power of $\hbar$ in the above differential equation ($*$) gives a collection of equations for the coefficients of Z_0, Z_1, Z_2 and $\tilde{\omega}$. With the initial condition $Z|_{r=0} = I$, the unique solution is

$$Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -2r & 1 \end{pmatrix} + \tfrac{1}{\hbar}\begin{pmatrix} & & -6r^2 & 3r \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

More generally, the initial condition $Z|_{q=0} = \mathrm{diag}(1,1,1,y)$ leads to the solution

$$Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -2r & y \end{pmatrix} + \tfrac{1}{\hbar}\begin{pmatrix} & & -6r^2 & 3ry \\ & & & \\ & & & \\ & & & \end{pmatrix}.$$

The new basis $(\widetilde{P}_i = G^{-1}\cdot\widehat{P}_i)$ produced by Step 2 is, therefore,

$$\widetilde{P}_0 = 1, \quad \widetilde{P}_1 = \hbar\partial, \quad \widetilde{P}_2 = \widehat{P}_2 - \tfrac{2r}{\hbar}\widehat{P}_3 - \tfrac{6r^2}{\hbar}\widehat{P}_1, \quad \widetilde{P}_3 = \tfrac{y}{\hbar}\widehat{P}_3 + \tfrac{3ry}{\hbar}\widehat{P}_1.$$

The connection matrix with respect to this basis is

$$\widetilde{\Omega} = \tfrac{1}{\hbar}\tilde{\omega} = \tfrac{1}{\hbar}\begin{pmatrix} & 12r^2 & & \\ 1 & & 12r^2 & \tfrac{3yr}{2} \\ & 1 & & \\ & \tfrac{2r}{y} & & \end{pmatrix}.$$

Finally we verify condition (H2) by explicit calculation of $\langle\!\langle\ ,\ \rangle\!\rangle$ with respect to the new basis:

$$\left(\langle\!\langle \widetilde{P}_i, \widetilde{P}_j \rangle\!\rangle\right)_{0\le\alpha,\beta\le 3} = \begin{pmatrix} & & & \frac{3}{2} \\ & & \frac{3}{2} & \\ & \frac{3}{2} & & \\ & & & \frac{9}{8}y^2 \end{pmatrix} \quad (= S, \text{ say}).$$

Regarding the normalization of the solution, we achieve the analogue

$$\begin{pmatrix} & & & \frac{d}{m_1} \\ & & \frac{d}{m_1} & \\ & \frac{d}{m_1} & & \\ & & & \frac{1}{m_2} \end{pmatrix} = \begin{pmatrix} & & & \frac{3}{2} \\ & & \frac{3}{2} & \\ & \frac{3}{2} & & \\ & & & \frac{1}{2} \end{pmatrix}$$

of the Ansatz for $Q_0(I + \hbar Q_1 + \cdots)$ in Section 4 if we take $y = \frac{2}{3}$.

This allows us to define an action of p (abstract orbifold quantum multiplication by p) on $A \otimes \mathbb{C}[r^{\pm 1}]$, where A is the vector space whose $\mathbb{C}$-basis vectors are denoted by $1, p, p^2, \mathbf{1}_{\frac{1}{2}}$. The matrix of the action with respect to this basis is, by definition, the matrix $\tilde{\omega}$. As in Section 4, we may also introduce a grading by defining age $\mathbf{1}_{f_i} = \frac{1}{2}|\widetilde{P}_{u_1+\cdots+u_{i-1}}|$, and this gives:

age $\mathbf{1}_0 = 0$	$\|1\| = 0$	$\|p\| = 2$	$\|p^2\| = 4$
age $\mathbf{1}_{\frac{1}{2}} = 1$			$\|\mathbf{1}_{\frac{1}{2}}\| = 2$

The action is compatible with this grading (i.e. the action of p increases degree by 2).

We also have the pairing $(\ ,\ )$ whose matrix is S. The Frobenius condition $(p \circ a, b) = (a, p \circ b)$ (for any $a, b \in A$) is satisfied by construction (see the discussion following Definition 6.14 of [9]); in matrix terms this is $S^{-1}\tilde{\omega}^t S = \tilde{\omega}$. We remark that this holds for any value of y, not just $y = \frac{2}{3}$.

The module action reproduces the first two rows of the following table of orbifold quantum products obtained by Corti ([4]):

	1	p	p^2	$\mathbf{1}_{\frac{1}{2}}$
1	1	p	p^2	$\mathbf{1}_{\frac{1}{2}}$
p		$p^2 + 12r^2 + 3r\mathbf{1}_{\frac{1}{2}}$	$12r^2 p$	rp
p^2			$108r^4 + 36r^3\mathbf{1}_{\frac{1}{2}}$	$12r^3$
$\mathbf{1}_{\frac{1}{2}}$				$\frac{1}{3}p^2 - 3r\mathbf{1}_{\frac{1}{2}}$

Furthermore, S agrees with the matrix of the orbifold Poincaré pairing from [4].

If it is assumed that the module action extends to a commutative associative abstract quantum product operation which satisfies the Frobenius condition $(c \circ a, b) = (a, c \circ b)$ (for any $a, b, c \in A$), then it follows from the first two rows of the table that

$$p^2 \circ p^2 = 108r^4 + 36r^3 \mathbf{1}_{\frac{1}{2}} + xr^3(r - \mathbf{1}_{\frac{1}{2}}),$$

$$p^2 \circ \mathbf{1}_{\frac{1}{2}} = 12r^3 - \tfrac{x}{3}r^2(r - \mathbf{1}_{\frac{1}{2}}),$$

$$\mathbf{1}_{\frac{1}{2}} \circ \mathbf{1}_{\frac{1}{2}} = \tfrac{1}{3}p^2 + \tfrac{x}{9}r(r - \mathbf{1}_{\frac{1}{2}})$$

for some real scalar x. It follows that $(\mathbf{1}_{\frac{1}{2}} \circ \mathbf{1}_{\frac{1}{2}}, \mathbf{1}_{\frac{1}{2}}) = -\tfrac{3}{2}r + \tfrac{x}{9}r(r - \mathbf{1}_{\frac{1}{2}})$. The condition $x = 0$ is equivalent to

$$(\mathbf{1}_{\frac{1}{2}} \circ \mathbf{1}_{\frac{1}{2}}, \mathbf{1}_{\frac{1}{2}}) = -\tfrac{3}{2}r,$$

and Corti computed this as a Gromov–Witten invariant. The ambiguity involving $r - \mathbf{1}_{\frac{1}{2}}$ is unavoidable in our construction as the second row of the table already tells us that $p \circ (r - \mathbf{1}_{\frac{1}{2}}) = 0$. However, for any x we do obtain an abstract quantum product operation which satisfies the Frobenius condition. $\qquad \square$

Returning to the general theory, let us mention an alternative interpretation of our method, which explains informally our assumptions (H1)–(H3). The significance of (H1) is that it is a necessary condition for the natural pairing $\langle\!\langle \ , \ \rangle\!\rangle$ to be "flat", i.e. for condition (H2). Having such a flat pairing is, in turn, a necessary condition for being able to carry out the Gram–Schmidt orthonormalization procedure, which is what (H3) accomplishes. From the Birkhoff factorization point of view, our method utilizes L_+ rather than L_-, as we have already mentioned; more accurately, it utilizes the transformation "$P_i \mapsto L_+^{-1} \cdot P_i$" in the D-module, which is essentially the Gram–Schmidt process.

It may appear at first sight that this could be done in many inequivalent ways. However (with suitable initial conditions, as in Example 5.2), the final basis $\widetilde{P}_0, \ldots, \widetilde{P}_{s-2}$ is unique, and this may be explained as follows. Step 2 involves a Birkhoff factorization of the form "$L = L_- \gamma L_+$". The Frobenius property is satisfied if and only if L_- is a twisted loop with respect to the involution defined by S, i.e. $S^{-1}(L_-^t)^{-1}S = L_-(-\hbar)$ (Section 6.5 of [9]). Now, if there exists *some* twisted L, for example, from any Gram–Schmidt orthonormalization, and γ is twisted, then L_- must also be twisted, as the Birkhoff decomposition is valid also for the twisted loop group. By the uniqueness of the (normalized) Birkhoff decomposition, we always obtain the same L_-. Thus, any Gram–Schmidt orthonormalization followed by a Birkhoff factorization produces the same $\widetilde{P}_0, \ldots, \widetilde{P}_{s-2}$.

Thus, the role of the loop γ is to compensate for the non-flatness of the pairing $\langle\!\langle \ , \ \rangle\!\rangle$. It does this by modifying the original $D^\hbar$-module $\mathcal{M}$ (with basis $P_0, \ldots, P_{s-2}$)

to a new $D^\hbar$-module with basis $\widetilde{P}_0, \ldots, \widetilde{P}_{s-2}$, which is a submodule of $\mathcal{M} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar, \hbar^{-1}]$. This phenomenon is related to the failure of the action of p to be cyclic, in the hypersurface case. We thank Hiroshi Iritani for emphasizing to us the significance of this, cf. [11].

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Martin A. Guest, Department of Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan

E-mail: martin@waseda.jp

Hironori Sakai, Mathematisches Institut, WWU Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail: sakai@blueskyproject.net