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On the values of G-functions

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À la mémoire de Philippe Flajolet

Abstract. In this paper we study the set $\mathbf{G}$ of values at algebraic points of analytic continuations of G-functions (in the sense of Siegel). This subring of $\mathbb{C}$ contains values of elliptic integrals, multiple zeta values, and values at algebraic points of generalized hypergeometric functions ${}_{p+1}F_p$ with rational coefficients. Its group of units contains non-zero algebraic numbers, π, $\Gamma(a/b)^b$ and $B(x,y)$ (with $a, b \in \mathbb{Z}$ such that $a/b \notin \mathbb{Z}$, and $x, y \in \mathbb{Q}$ such that $B(x,y)$ exists and is non-zero). We prove that for any $\xi \in \mathbf{G}$, both $\operatorname{Re} \xi$ and $\operatorname{Im} \xi$ can be written as $f(1)$, where f is a G-function with rational coefficients of which the radius of convergence can be made arbitrarily large. As an application, we prove that quotients of elements of $\mathbf{G} \cap \mathbb{R}$ are exactly the numbers which can be written as limits of sequences a_n/b_n, where $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are G-functions with rational coefficients. This result provides a general setting for irrationality proofs in the style of Apéry for $\zeta(3)$, and gives answers to questions asked by T. Rivoal in "Approximations rationnelles des valeurs de la fonction Gamma aux rationnels : le cas des puissances", Acta Arith. 142 (2010), no. 4, 347–365.

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1. Introduction

The purpose of this text is to study the set of values of G-functions at algebraic numbers. Let us recall the following definition, which essentially goes back to Siegel [30].

Definition 1. A G-function f is a formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the coefficients a_n are algebraic numbers and there exists $C > 0$ such that:

(i) the maximum of the moduli of the conjugates of a_n is $\leq C^{n+1}$.

(ii) there exists a sequence of integers d_n, with $|d_n| \leq C^{n+1}$, such that $d_n a_m$ is an algebraic integer for all $m \leq n$.

(iii) $f(z)$ satisfies a homogeneous linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$. [1]

Throughout this paper we fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$; all algebraic numbers and all convergent series are considered in $\mathbb{C}$.

G-functions occur frequently in analysis, number theory, geometry and physics: for example, algebraic functions over $\overline{\mathbb{Q}}(z)$ which are holomorphic at 0, polylogarithms, Gauss' hypergeometric function with rational parameters, are G-functions. The exponential function is not a G-function but an E-function (that is, it satisfies the requirements of Definition 1 if a_n is replaced with $a_n/n!$ in the expansion of $f(z)$).

In Definition 1, condition (i) ensures that any non-polynomial G-function has finite non-zero radius of convergence at $z = 0$. Condition (iii) implies that in fact the coefficients a_n, $n \geq 0$, all belong to a same number field. Classical references on G-functions are the books [1] and [17].

Siegel's goal was to find conditions ensuring that E and G-functions take irrational or transcendental values at algebraic points: the picture is very well understood for E-functions but largely unknown for G-functions. The main tool to study the nature of values of G-functions is inexplicit Padé-type approximation (see [3], [12], [14], [22]). In an explicit form, Padé approximation is also behind Apéry's celebrated proof [7] of the irrationality of $\zeta(3)$, and similar results in specific cases (see for instance [9], [19]).

In this paper, we study the following set.

Definition 2. Let $\mathbf{G}$ denote the set of all values $f(\alpha)$, where f is a G-function and $\alpha \in \overline{\mathbb{Q}}$. More precisely, all values at α of analytic continuations of f are considered, as soon as they are finite.

This subset of $\mathbb{C}$ is a subring (this can be seen as a consequence of Theorem 1 below). It contains $\overline{\mathbb{Q}}$, and also (see §2.2 for proofs) multiple zeta values, elliptic integrals, and values at algebraic points of generalized hypergeometric functions ${}_{p+1}F_p$ with rational coefficients. André proved in [1], p. 123, that the units of the ring of G-functions are exactly the algebraic functions which are holomorphic and don't vanish at the origin. The description of the units of $\mathbf{G}$ is an interesting open problem whose solution is not as simple as for functions, for we show in §2.2 that the group of units of $\mathbf{G}$ contains not only the non-zero algebraic numbers but also π, the values of the Gamma function $\Gamma(a/b)^b$ and that of Euler's Beta function $B(x, y)$ (with $a, b \in \mathbb{Z}$ such that $a/b \notin \mathbb{Z}$, and $x, y \in \mathbb{Q}$ such that $B(x, y)$ exists and is non-zero). On the other hand, there is no explicit interesting number for which we

[1] All differential equations considered in this text are homogeneous and consequently we will no longer mention the term "homogeneous".

are able to prove that it is not in $\mathbf{G}$;[2] it is likely that e, Euler's constant γ, $\Gamma(a/b)$ (with a, b integers such that $a/b \notin \mathbb{Z}$) or Liouville numbers do not belong to $\mathbf{G}$.

A conjecture of Bombieri and Dwork predicts a strong relationship between differential equations satisfied by G-functions and Picard–Fuchs equations satisfied by periods of families of algebraic varieties defined over $\overline{\mathbb{Q}}$. See the precise formulation given by André in [1], p. 7, who proved half of the conjecture in [1], pp. 110–111. Christol [13] also conjectured that globally bounded G-functions are diagonals of rational functions, which are known to satisfy Picard–Fuchs equations. This raises the question of a connection between the set $\mathbf{G}$ and the set $\mathcal{P}$ of periods considered by Kontsevich and Zagier [26]; all elements of $\mathcal{P}$ we have thought of belong also to $\mathbf{G}$. However $1/\pi$ is conjectured not to belong to $\mathcal{P}$, so that $\mathbf{G}$ is presumably distinct from $\mathcal{P}$. However, a natural problem is the determination of the link between $\mathbf{G}$ and $\mathcal{P}[1/\pi]$ (see the discussion at the end of § 2.2).

Our main result is the following.

Theorem 1. *A complex number ξ belongs to* $\mathbf{G}$ *if, and only if, its real and imaginary parts can be written as $f(1)$, where f is a G-function with rational coefficients of which the radius of convergence can be made arbitrarily large.*

One of the consequences of this theorem is that the set of values of G-functions $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{Q}$ at points $z \in \mathbb{Q}$ inside the disk of convergence (respectively at points where this series is absolutely convergent, respectively convergent) is equal to $\mathbf{G} \cap \mathbb{R}$.

The main tool in the proof of Theorem 1 is André–Chudnovski–Katz's theorem (stated as Theorem 6 in §4.1 below), which provides for any G-function f and any $\zeta \in \overline{\mathbb{Q}}$ a local basis $(g_1, \ldots, g_\mu)$ of solutions around ζ of a minimal differential equation satisfied by f. Expanding an analytic continuation of f in this basis yields *connection constants* $\varpi_1, \ldots, \varpi_\mu \in \mathbb{C}$ such that $f(z) = \sum_{j=1}^{\mu} \varpi_j g_j(z)$. As a step towards Theorem 1, we prove the following result which is of independent interest:

Theorem 2. *The connection constants $\varpi_1, \ldots, \varpi_\mu$ belong to* $\mathbf{G}$.

We would like to emphasize that analytic continuation (and its properties encompassed in André–Chudnovski–Katz's theorem) is the main tool in our approach. As the referee pointed out to us, it would be interesting to find a connection with other methods used in similar contexts, including Dèbes–Zannier's [15] or Euler's for accelerating convergent series; however we did not find any. For instance, Euler's binomial transform $\sum_{n\geq 0}(-1)^n a_n = \sum_{n\geq 0}\big(\sum_{k=0}^{n}(-1)^k \binom{k}{n} a_k\big)2^{-n-1}$ is involutive and therefore it cannot be used to obtain series with arbitrarily large radius of convergence.

[2] Since the set $\mathbf{G}$ is countable, there are complex numbers outside $\mathbf{G}$ but the real difficulty is to exhibit such a number by an effective process leading to an analytic expression like a series or an integral for example.

As an application of Theorem 1, we answer questions asked in [28], p. 351, where the second author introduced the notion of rational G-approximations to a real number. This corresponds to assertion (ii) in the next result, which provides a characterization of numbers admitting rational G-approximations.

Given a subring $\mathbb{A}$ of $\mathbb{C}$, we denote by $\mathrm{Frac}(\mathbb{A})$ the field of fractions of $\mathbb{A}$, namely the subfield of $\mathbb{C}$ consisting in all elements ξ/ξ' with $\xi, \xi' \in \mathbb{A}$, $\xi' \neq 0$.

Theorem 3. *Let $\xi \in \mathbb{R}^\star$. The following statements are equivalent:*

(i) *We have $\xi \in \mathrm{Frac}(\mathbf{G}) \cap \mathbb{R} = \mathrm{Frac}(\mathbf{G} \cap \mathbb{R})$.*

(ii) *There exist two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of rational numbers such that the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are G-functions, $b_n \neq 0$ for any n large enough and $\lim_{n\to+\infty} a_n/b_n = \xi$.*

(iii) *For any $R \geq 1$ there exist two G-functions $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$, with rational coefficients and radius of convergence $= 1$, such that $A(z) - \xi B(z)$ has radius of convergence $> R$.*

Remark. When $\xi \in \mathbf{G}$, we can take $b_n = 1$ in (ii). However, it is not clear to us if this is also the case for other elements $\xi \in \mathrm{Frac}(\mathbf{G})$, in particular because it is doubtful that $\mathbf{G}$ itself is a field.

Apéry has proved [7] that $\zeta(3) \notin \mathbb{Q}$ by constructing sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ essentially as in (iii), such that $b_n \in \mathbb{Z}$ and $\mathrm{lcm}(1, 2, \dots, n)^3 a_n \in \mathbb{Z}$. Since $\zeta(3) = \mathrm{Li}_3(1)$ (where the polylogarithms defined by $\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{1}{n^s} z^n$, $s \geq 1$, are G-functions), we have $\zeta(3) \in \mathbf{G}$. Theorem 3 provides a general setting for such irrationality proofs and one may wonder if, given a real irrational number $\xi \in \mathrm{Frac}(\mathbf{G})$, there exists a proof *à la Apéry* that ξ is irrational. In particular, this would be a strategy to prove the following conjecture (see §7.2 below):

Conjecture 1. *No $\xi \in \mathrm{Frac}(\mathbf{G})$ can be a Liouville number.*

Our approach does not yield (at least for now) any actual result towards this conjecture, because the denominators of the coefficients of the G-functions we construct grow too fast. It would be interesting to control them in some way.

The paper is organized as follows. We introduce some notation in §2.1, and state slight generalizations of Theorems 1 and 3, namely Theorems 4 and 5. We prove in §2.2 that the numbers mentioned above actually belong to $\mathbf{G}$. Then we start proving Theorems 4 and 5 by gathering some lemmas in §§2.3 and 2.4. In §3, we prove that the conclusion of Theorem 1 holds for algebraic numbers and their logarithms. In §4, we review some classical results concerning the properties of differential equations satisfied by G-functions (namely Theorem 6, due to André, Chudnovski and Katz). We also prove in this section that connection constants belong to $\mathbf{G}$, and the conclusion of Theorem 1 holds for them (see Theorem 7). This result, along with the analytic continuation properties of G-functions deduced from Theorem 6, is used to prove Theorem 4 in §5. In §6, we present the proof of Theorem 5: the main tool is the results of *Singularity Analysis* due to Flajolet and Odlyzko [21], described in details in the book [20]. Finally, we mention in §7 a few problems suggested by our results: what can be said about the case of E-functions and about Diophantine perspectives.

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2. Background of the proofs

2.1. Notation and results. In this section we introduce some notation that will be used throughout this text. We also state Theorems 4 and 5, which are slight generalizations of Theorems 1 and 3 respectively.

The letter $\mathbb{K}$ will always stand for a (finite or infinite) algebraic extension of $\mathbb{Q}$, embedded into $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Definition 3. Given an algebraic extension $\mathbb{K}$ of $\mathbb{Q}$, we denote by $\mathbf{G}_{\mathbb{K}}^{\text{a.c.}}$ the set of all values, at points in $\mathbb{K}$, of multivalued analytic continuations of G-functions with Taylor coefficients at 0 in $\mathbb{K}$.

For any G-function f with coefficients in $\mathbb{K}$ and any $\alpha \in \mathbb{K}$, we consider all values of $f(\alpha)$ obtained by analytic continuation, as in the definition of $\mathbf{G}$ in the introduction; obviously $\mathbf{G} = \mathbf{G}_{\overline{\mathbb{Q}}}^{\text{a.c.}}$. If α is a singularity of f, then we consider also these values if they are finite. Of course $f(\alpha z)$ is also a G-function with coefficients in $\mathbb{K}$ so that we may restrict ourselves to values at the point 1. By Abel's theorem, $\mathbf{G}_{\mathbb{K}}^{\text{a.c.}}$ contains all convergent series $\sum_{n=0}^{\infty} a_n \alpha^n$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a G-function with coefficients in $\mathbb{K}$ and $\alpha \in \mathbb{K}$.

Definition 4. Given an algebraic extension $\mathbb{K}$ of $\mathbb{Q}$, we denote by $\mathbf{G}_{\mathbb{K}}^{\text{cv}}$ the set of all $\xi \in \mathbb{C}$ such that, for any $R \geq 1$, there exists a G-function f with Taylor coefficients at 0 in $\mathbb{K}$ and radius of convergence $> R$ such that $\xi = f(1)$.

For any $R \geq 1$, we denote by $\mathbf{G}_{R,\mathbb{K}}^{\text{cv}}$ the set of all $\xi = f(1)$ where f is a G-function with Taylor coefficients at 0 in $\mathbb{K}$ and radius of convergence $> R$. In this way we have $\mathbf{G}_{\mathbb{K}}^{\text{cv}} = \bigcap_{R\geq 1} \mathbf{G}_{R,\mathbb{K}}^{\text{cv}}$, and also $\mathbf{G}_{R,\mathbb{K}}^{\text{cv}} \subset \mathbf{G}_{\mathbb{K}}^{\text{a.c.}}$ for any $R \geq 1$.

With this notation, Theorem 1 reads $\mathbf{G}^{\text{a.c.}}_{\overline{\mathbb{Q}}} = \mathbf{G}^{\text{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\text{cv}}_{\mathbb{Q}} = \mathbf{G}^{\text{cv}}_{\mathbb{Q}(i)}$. Actually we prove that $\mathbf{G}^{\text{a.c.}}_{\mathbb{K}}$ is independent from $\mathbb{K}$, so that it is always equal to $\mathbf{G} = \mathbf{G}^{\text{a.c.}}_{\overline{\mathbb{Q}}}$. Concerning $\mathbf{G}^{\text{cv}}_{\mathbb{K}}$, there is an obvious remark: if $\mathbb{K} \subset \mathbb{R}$ then $\mathbf{G}^{\text{cv}}_{\mathbb{K}} \subset \mathbb{R}$. Apart from this, $\mathbf{G}^{\text{cv}}_{\mathbb{K}}$ is independent from $\mathbb{K}$, and equal (up to taking real parts) to $\mathbf{G}$. Our result reads as follows.

Theorem 4. *Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$. Then:*

- *We have* $\mathbf{G}^{\text{a.c.}}_{\mathbb{K}} = \mathbf{G} = \mathbf{G}^{\text{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\text{cv}}_{\mathbb{Q}}$.
- *If* $\mathbb{K} \not\subset \mathbb{R}$ *then* $\mathbf{G}^{\text{cv}}_{\mathbb{K}} = \mathbf{G} = \mathbf{G}^{\text{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\text{cv}}_{\mathbb{Q}}$; *if* $\mathbb{K} \subset \mathbb{R}$ *then* $\mathbf{G}^{\text{cv}}_{\mathbb{K}} = \mathbf{G} \cap \mathbb{R} = \mathbf{G}^{\text{cv}}_{\mathbb{Q}}$.

In particular this result contains the fact that $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbf{G}^{\text{cv}}_{\mathbb{Q}}$ and $\overline{\mathbb{Q}} \subset \mathbf{G}^{\text{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\text{cv}}_{\mathbb{Q}}$; this will be proved in §3.1. Another consequence of this theorem is that the set of values of G-functions $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \in \mathbb{K}$ at points $z \in \mathbb{K}$ inside the disk of convergence (respectively at points where this series is absolutely convergent, respectively convergent) is equal to $\mathbf{G}^{\text{cv}}_{\mathbb{K}}$ (so that it is equal to either $\mathbf{G}$ or $\mathbf{G} \cap \mathbb{R}$).

We also generalize Theorem 3 as follows.

Theorem 5. *Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$, and $\xi \in \mathbb{C}^\star$. Then the following statements are equivalent:*

(i) *We have* $\xi \in \mathrm{Frac}(\mathbf{G}^{\text{cv}}_{\mathbb{K}})$.

(ii) *There exist two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of elements of $\mathbb{K}$ such that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are G-functions, $b_n \neq 0$ for infinitely many n and $a_n - \xi b_n = o(b_n)$.*

(iii) *For any $R \geq 1$ there exist two G-functions $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$, with coefficients $a_n, b_n \in \mathbb{K}$ and radius of convergence $= 1$, such that $A(z) - \xi B(z)$ has radius of convergence $> R$ and $a_n, b_n \neq 0$ for any n sufficiently large.*

When $\mathbb{K} = \mathbb{Q}$, this is a refinement of Theorem 3 because assumption (ii) of Theorem 3 implies assumption (ii) of Theorem 5, and (iii) of Theorem 5 implies (iii) of Theorem 3 (see also Lemma 2 below). The point in assertion (ii) of Theorem 5 is that b_n may vanish for infinitely many n; by asking $a_n - \xi b_n = o(b_n)$ we require that $a_n = 0$ as soon as $b_n = 0$ and n is sufficiently large.

2.2. Examples and connection to periods. In this section, we prove that the numbers mentioned in the introduction belong to $\mathbf{G}$, and give some hints on the connection with periods. This section is independent from the rest of the paper, except that we assume here that $\mathbf{G}$ is a ring.

Many examples of G-functions are provided by the generalized hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \dots (\alpha_k)_n}{(1)_n(\beta_1)_n \dots (\beta_{k-1})_n} z^n$$

with rational coefficients α's and β's, and $(x)_n = x(x+1)\dots(x+n-1)$. Special cases are the polylogarithmic functions $\mathrm{Li}_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$ $(k \geq 1)$ and $\arctan(z) = \sum_{n\geq 0}(-1)^n \frac{z^n}{2n+1}$. We deduce in particular that $\pi = 4\arctan(1)$ and the values of the Riemann zeta function $\zeta(k) = \mathrm{Li}_k(1)$ are in $\mathbf{G}$ for any integer $k \geq 2$. Catalan's constant $\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2}$ is also in $\mathbf{G}$.

Other examples of G-functions are the multiple polylogarithms

$$\sum_{n_1 > \dots > n_s \geq 1} \frac{z^{n_1}}{n_1^{k_1} \dots n_s^{k_s}}$$

where the k's are positive integers. This is a consequence of the fact that for $s = 1$, we have a polylogarithm from which we obtain the multiple series by a succession of integrations and multiplications by $1/z$ or $1/(1-z)$; this process does not leave the set of G-functions. As a consequence, multiple zeta values $\zeta(k_1, \dots, k_s) = \sum_{n_1 > \dots > n_s \geq 1} \frac{1}{n_1^{k_1} \dots n_s^{k_s}}$ (with $k_1 \geq 2$) are in $\mathbf{G}$.

It could seem more surprising that $1/\pi$ is also in $\mathbf{G}$, a fact proved by each one of the following identities:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{(1-2n)2^{4n+1}}, \quad \frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3 (42n+5)}{2^{12n+4}}.$$

The first identity is a direct translation of the identity $E(1) = 1$ where $E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\mathrm{d}t$ is Legendre's complete elliptic function of the second kind. The second identity is due to Ramanujan and it also has an elliptic interpretation. Both series are in fact values of generalized hypergeometric series, hence $1/\pi \in \mathbf{G}$.

In particular, π and the non-zero algebraic numbers are units of $\mathbf{G}$. These numbers do not span the whole group of units, as we now proceed to prove. Euler's Beta function is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. It is well-known that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, which provides the meromorphic continuation of B to $\mathbb{C}^2$; we recall that $\pi = B(\frac{1}{2}, \frac{1}{2})$.

Proposition 1. (i) *For all rational numbers x, y such that $B(x, y)$ is defined and non-zero, the number $B(x, y)$ is a unit of* $\mathbf{G}$.

(ii) *For any integers $a, b \geq 1$, we have*

$$\Gamma\Big(\frac{a}{b}\Big)^b = (a-1)! \prod_{j=1}^{b-1} B\Big(\frac{a}{b}, \frac{ja}{b}\Big)$$

and $\Gamma\big(\frac{a}{b}\big)^b$ is a unit of $\mathbf{G}$.

Remark. a) To sum up, the group of units of $\mathbf{G}$ contains the algebraic numbers and the numbers $B(x, y)$ where $x, y \in \mathbb{Q}$ (as soon as they are defined and non-zero). We don't know if this provides a complete list of generators of this group.

b) Chudnovski proved in 1974 that $\Gamma(1/3)$, respectively $\Gamma(1/4)$, and π are algebraically independent over $\overline{\mathbb{Q}}$. Hence one needs other transcendental generators than π in the group of units of $\mathbf{G}$.

c) This proposition is a transposition in our context of a discussion in André's book [6], pp. 211–212, where he shows that the numbers $\Gamma(a/b)^b$ are periods (in the geometric sense).

Proof. (i) We first show that $B(x, y) \in \mathbf{G}$ for all rational numbers $0 < x, y \leq 1$. Clearly, $B(x, y)$ is well defined in this case and

$$\begin{aligned} B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t &= \int_0^1 \sum_{n=0}^{\infty} (-1)^n \binom{y-1}{n} t^{n+x-1}\mathrm{d}t \\ &= \sum_{n=0}^{\infty} (-1)^n \binom{y-1}{n} \int_0^1 t^{n+x-1}\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \frac{\binom{y-1}{n}}{n+x}. \end{aligned}$$

Since $(-1)^n\binom{y-1}{n}$ is positive, permuting the series and integral is licit. Moreover, $\frac{\binom{y-1}{n}}{n+x} = \mathcal{O}(1/n^{y+1})$ so that the final series converges absolutely and is the value at $z = 1$ of a G-function. This proves that $B(x, y) \in \mathbf{G}$ in this case.

From now on, we let $x, y \in \mathbb{Q}$ and we assume that $x, y, x + y \notin \mathbb{Z}$ (otherwise the conclusion is easier to prove). Then $B(x, y)$ is defined and non-zero. There exist two integers M, N such that $0 < x + M, y + N \leq 1$, and the functional equations

$$B(x,y) = \frac{x+y}{x} B(x+1,y), \quad B(x,y) = \frac{x+y}{y} B(x,y+1)$$

yield $B(x, y) = R_{M,N}(x, y)B(x + M, y + N)$ with $R_{M,N}(x, y) \in \mathbb{Q}(x, y)$. Since $B(x + M, y + N)$ is in $\mathbf{G}$ by the previous case, it follows that $B(x, y) \in \mathbf{G}$.

To prove that $1/B(x, y)$ is also in $\mathbf{G}$, we use the reflection formula $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ to get

$$\frac{1}{B(x,y)} = \frac{\sin(\pi x)\sin(\pi y)}{\sin \pi(x+y)} \cdot \frac{1-x-y}{\pi} \cdot B(1-x, 1-y).$$

Now $B(1-x, 1-y) \in \mathbf{G}$ by the case above, $\frac{(1-x-y)\sin(\pi x)\sin(\pi y)}{\sin \pi(x+y)}$ is an algebraic number (hence in $\mathbf{G}$) and $1/\pi \in \mathbf{G}$, so that $\frac{1}{B(x,y)} \in \mathbf{G}$.

(ii) We have

$$\prod_{j=1}^{b-1} B\Big(\frac{a}{b}, \frac{ja}{b}\Big) = \prod_{j=1}^{b-1} \frac{\Gamma\big(\frac{a}{b}\big)\Gamma\big(\frac{aj}{b}\big)}{\Gamma\big(\frac{a(j+1)}{b}\big)} = \Gamma\Big(\frac{a}{b}\Big)^{b-1}\frac{\Gamma\big(\frac{a}{b}\big)}{\Gamma(a)},$$

from which we obtain the claimed identity. Moreover, for any integer $j \geq 1$, $B\big(\frac{a}{b}, \frac{ja}{b}\big)$ is obviously defined and non-zero, hence is a unit of $\mathbf{G}$ by (i). Thus, this is also the case of $\Gamma\big(\frac{a}{b}\big)^b$. $\square$

To conclude this section, we mention some remarks (due to the referee) towards the determination of the link between $\mathbf{G}$ and $\mathcal{P}[1/\pi]$, where $\mathcal{P}$ is the ring of periods (in Kontsevich and Zagier's sense [26]); in particular a natural question is whether $\mathbf{G} = \mathcal{P}[1/\pi]$ or not.

Bombieri–Dwork's conjecture suggests that $\mathbf{G}$ might be contained in $\mathcal{P}[1/\pi]$. Indeed, this conjecture predicts that any G-function is solution of an extension of sub-quotients of Picard–Fuchs equations. It is not clear that such an extension is motivic, but for a Picard–Fuchs equation the G-matrix solution $Y(z)$ is the quotient $P(z)P(0)^{-1}$ of two period matrices. Since the determinant of $P(0)$ is an algebraic number times a power of π (see [2]), the inclusion $\mathbf{G} \subset \mathcal{P}[1/\pi]$ would follow.

Towards the converse inclusion, it is possible to prove that if a one-parameter Picard–Fuchs equation doesn't have 0 as a singularity then the special values of its solutions can be expressed in terms of G-functions which are solutions of the same equation.

In view of this discussion, it would be very interesting to refine Theorem 1 by ensuring that 0 isn't a singularity of the minimal differential equation of the G-function f we construct (such that $f(1)$ is a given $\xi \in \mathbf{G}$). However our proof does not provide this refinement directly and new ideas are necessary to do that.

2.3. General properties of the ring $\mathbf{G}_{\mathbb{K}}^{\mathbf{cv}}$. The set of G-functions satisfies a number of structural properties. It is a ring and even a $\overline{\mathbb{Q}}[z]$-algebra; it is stable by differentiation and the Hadamard product of two G-functions (obtained by pointwise multiplication of the coefficients) is again a G-function. These properties will be used throughout the text, as well as the fact that algebraic functions over $\overline{\mathbb{Q}}(z)$ which are holomorphic at $z = 0$ are G-functions: this is a consequence of Eisenstein's theorem [3] and the fact that an algebraic function over $\overline{\mathbb{Q}}(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}[z]$.

The following property is useful too:

[3] which states that for any power series $\sum_{n=0}^{\infty} a_n z^n$ algebraic over $\overline{\mathbb{Q}}(z)$, there exists a positive integer D such that $D^n a_n$ is an algebraic integer for any n.

Lemma 1. *Consider a G-function $\sum_{n=0}^{\infty} a_n z^n$. Then the series*

$$\sum_{n=0}^{\infty} \overline{a_n} z^n, \quad \sum_{n=0}^{\infty} \mathrm{Re}(a_n) z^n \quad \textit{and} \quad \sum_{n=0}^{\infty} \mathrm{Im}(a_n) z^n$$

are also G-functions.

Proof. The series $\sum_{n=0}^{\infty} a_n z^n$ satisfies a linear differential equation $Ly = 0$ with coefficients in $\overline{\mathbb{Q}}[z]$, hence $\sum_{n=0}^{\infty} \overline{a_n} z^n$ satisfies the linear differential equation $\bar{L}y = 0$ where $\bar{L}$ is obtained from L by replacing each coefficient $\sum_{k=0}^{d} p_k z^k$ with $\sum_{k=0}^{d} \overline{p_k} z^k$. Furthermore, the moduli of the conjugates of $\overline{a_n}$ and their common denominators obviously grow at most geometrically. Hence, $\sum_{n=0}^{\infty} \overline{a_n} z^n$ is a G-function.

For $\sum_{n=0}^{\infty} \mathrm{Re}(a_n) z^n$ and $\sum_{n=0}^{\infty} \mathrm{Im}(a_n) z^n$, we write $2\mathrm{Re}(a_n) = a_n + \overline{a_n}$, $2i\,\mathrm{Im}(a_n) = a_n - \overline{a_n}$ and use the fact that the sum of two G-functions is also a G-function. □

The following lemma includes the easiest properties of $\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$; especially (i) will be used very often without explicit reference.

Lemma 2. *Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$.*

(i) $\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$ *is a ring and it contains* $\mathbb{K}$.

(ii) *If* $\mathbb{K}$ *is invariant under complex conjugation then:*

- $\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$ *is invariant under complex conjugation.*
- $\mathbf{G}_{\mathbb{K}\cap\mathbb{R}}^{\mathrm{cv}} = \mathbf{G}_{\mathbb{K}}^{\mathrm{cv}} \cap \mathbb{R}$.
- $\mathbb{R} \cap \mathrm{Frac}(\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}) = \mathrm{Frac}(\mathbf{G}_{\mathbb{K}\cap\mathbb{R}}^{\mathrm{cv}}) = \mathrm{Frac}(\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}} \cap \mathbb{R})$.

(iii) $\mathbf{G}_{\mathbb{Q}(i)}^{\mathrm{cv}} = \mathbf{G}_{\mathbb{Q}}^{\mathrm{cv}}[i] = \mathbf{G}_{\mathbb{Q}}^{\mathrm{cv}} + i\mathbf{G}_{\mathbb{Q}}^{\mathrm{cv}}$, *and more generally if* $\mathbb{K} \subset \mathbb{R}$ *then* $\mathbf{G}_{\mathbb{K}(i)}^{\mathrm{cv}} = \mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}[i] = \mathbf{G}_{\mathbb{K}}^{\mathrm{cv}} + i\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$.

Proof. (i) The properties of G-functions ensure that the sum and product of two G-functions with coefficients in $\mathbb{K}$ and radii of convergence $> R \geq 1$ are G-functions with coefficients in $\mathbb{K}$ and radii of convergence $> R$. Moreover algebraic constants are G-functions with infinite radius of convergence.

(ii) Using Lemma 1 and the fact that $\mathbb{K}$ is invariant under complex conjugation, if $\sum_{n=0}^{\infty} a_n z^n$ is a G-function with coefficients in $\mathbb{K}$ and radii of convergence $> R \geq 1$ then so is $\sum_{n=0}^{\infty} \overline{a_n} z^n$: this proves that $\mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$ is invariant under complex conjugation.

The inclusion $\mathbf{G}_{\mathbb{K}\cap\mathbb{R}}^{\mathrm{cv}} \subset \mathbf{G}_{\mathbb{K}}^{\mathrm{cv}} \cap \mathbb{R}$ is obvious. Conversely, if $\xi \in \mathbb{R} \cap \mathbf{G}_{\mathbb{K}}^{\mathrm{cv}}$ then for any $R \geq 1$ we have $\xi = \sum_{n=0}^{\infty} a_n$ where $\sum_{n=0}^{\infty} a_n z^n$ is a G-function with coefficients in $\mathbb{K}$ and radius of convergence $> R$. Then $\sum_{n=0}^{\infty} \mathrm{Re}(a_n) z^n$ is also a G-function (by

Lemma 1); it has coefficients in $\mathbb{K} \cap \mathbb{R}$ (because $\mathrm{Re}(a_n) = \frac{1}{2}(a_n + \overline{a_n})$) and radius of convergence $> R$. Therefore $\xi = \sum_{n=0}^{\infty} \mathrm{Re}(a_n) \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}\cap\mathbb{R}}$.

Finally, the inclusion $\mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} \cap \mathbb{R}) \subset \mathbb{R} \cap \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}})$ is trivial. The converse is trivial too if $\mathbb{K} \subset \mathbb{R}$; otherwise let $\xi, \xi' \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ be such that $\xi' \neq 0$ and $\xi/\xi' \in \mathbb{R}$. Multiplying if necessary by a non-real element of $\mathbb{K}$, we may assume $\xi, \xi' \notin i\mathbb{R}$. Then we have $\xi/\xi' = (\xi + \bar{\xi})/(\xi' + \bar{\xi}') \in \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} \cap \mathbb{R})$.

(iii) Assume $\mathbb{K} \subset \mathbb{R}$. Since $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ is a ring and $i^2 = -1 \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$, we have $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}[i] = \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$. This is obviously a subset of $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$. Conversely, $\mathbb{K}(i)$ is invariant under complex conjugation (because $\mathbb{K} \subset \mathbb{R}$) so that for any $\xi \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$ we have $\mathrm{Re}(\xi) = \frac{1}{2}(\xi + \bar{\xi}) \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)} \cap \mathbb{R} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ by (ii). Since $i \in \mathbb{K}(i) \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$ we have $\mathrm{Im}(\xi) = -i(\xi - \mathrm{Re}(\xi)) \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)} \cap \mathbb{R} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$, using (ii) again. Finally $\xi = \mathrm{Re}(\xi) + i\,\mathrm{Im}(\xi) \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$. $\square$

The following lemma is a consequence of Lemma 7 proved in §3 below; of course the proof of Lemma 7 does not use Lemma 3, hence there is no circularity.

Lemma 3. *Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$.*

(i) *We have $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$, and $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ is a $(\overline{\mathbb{Q}} \cap \mathbb{R})$-algebra.*

(ii) *If $\mathbb{K} \not\subset \mathbb{R}$ then $\overline{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$, and $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ is a $\overline{\mathbb{Q}}$-algebra.*

Proof. (i) By Lemma 7, we have $\overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} \cap \mathbb{R}$; this is equal to $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$ by Lemma 2. The inclusion $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ is trivial since $\mathbb{Q} \subset \mathbb{K}$.

(ii) Since $\mathbb{K} \not\subset \mathbb{R}$, there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha + i\beta \in \mathbb{K}$ and $\beta \neq 0$; since $\alpha - i\beta$ is also algebraic, we have $\alpha, \beta \in \overline{\mathbb{Q}}$. Therefore we can write $i = \frac{1}{\beta}((\alpha + i\beta) - \alpha)$ with $\frac{1}{\beta}, \alpha \in \overline{\mathbb{Q}} \cap \mathbb{R} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ (by (i)). Since $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$ is a ring which contains $\alpha + i\beta$, this yields $i \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$, so that (using Lemma 2 and the trivial inclusion $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$) $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$. Using the inclusion $\overline{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$ proved in Lemma 7, this concludes the proof of (ii). $\square$

To conclude this section, we state and prove the following lemma, which is very useful for constructing elements of $\mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}}$. Recall that $\mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}}$ is the set of all $\xi = f(1)$ where f is a G-function with coefficients in $\mathbb{K}$ and radius of convergence $> R$.

Lemma 4. *Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$. Let $\zeta \in \mathbb{K}$, and $g(z)$ be a G-function in the variable $\zeta - z$, with coefficients in $\mathbb{K}$ and radius of convergence $\geq r > 0$. Then $g(z_0) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}}$ for any $R \geq 1$ and any $z_0 \in \mathbb{K}$ such that $|z_0 - \zeta| < r/R$.*

Proof. Letting $f(z) = g\big(\zeta + z(z_0 - \zeta)\big)$, we have $f(1) = g(z_0)$ and f is a G-function with coefficients in $\mathbb{K}$ and radius of convergence $> R$. $\square$

2.4. Miscellaneous lemmas. We gather in this section two lemmas which are neither difficult nor specific to G-functions, but very useful.

Lemma 5. *Let $\mathbb{A}$ be a subring of $\mathbb{C}$. Let $S \subset \mathbb{N}$ and $T \subset \mathbb{Q}$ be finite subsets. For any $(s,t) \in S \times T$, let $f_{s,t}(z) = \sum_{n=0}^{\infty} a_{s,t,n} z^n \in \mathbb{A}[[z]]$ be a function holomorphic at 0, with Taylor coefficients in $\mathbb{A}$. Let Ω denote an open subset of $\mathbb{C}$, with 0 in its boundary, on which a continuous determination of the logarithm is chosen. Then there exist $c \in \mathbb{A}$, $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Q}$ such that, as $z \to 0$ with $z \in \Omega$,*

$$\sum_{s\in S}\sum_{t\in T} (\log z)^s z^t f_{s,t}(z) = c\,(\log z)^{\sigma} z^{\tau}(1+o(1)). \tag{2.1}$$

Proof. Let $T+\mathbb{N} = \{t+n, t \in T, n \in \mathbb{N}\}$. For any $s \in S$ and any $\theta \in T+\mathbb{N}$, let $c_{s,\theta} = \sum_{t\in T} a_{s,t,\theta-t}$ where we let $a_{s,t,\theta-t} = 0$ if $\theta - t \notin \mathbb{N}$. Then the left-hand side of (2.1) can be written, for $z \in \Omega$ sufficiently close to 0, as an absolutely converging series $\sum_{\theta\in T+\mathbb{N}}\sum_{s\in S} c_{s,\theta}(\log z)^s z^{\theta}$. If $c_{s,\theta} = 0$ for any (s,θ) then (2.1) holds with $c = 0$. Otherwise we denote by τ the minimal value of θ for which there exists $s \in S$ with $c_{s,\theta} \neq 0$, and by σ the largest $s \in S$ such that $c_{s,\tau} \neq 0$. Then (2.1) holds with $c = c_{\sigma,\tau} \in \mathbb{A}$. □

The following result will be used in the proof of Theorem 5.

Lemma 6. *Let $\omega_1, \dots, \omega_t$ be pairwise distinct complex numbers, with $|\omega_1| = \cdots = |\omega_t| = 1$. Let $\kappa_1, \dots, \kappa_t \in \mathbb{C}$ be such that $\lim_{n\to+\infty} \kappa_1\omega_1^n + \cdots + \kappa_t\omega_t^n = 0$. Then $\kappa_1 = \cdots = \kappa_t = 0$.*

Proof. For any $n \geq 0$, let $\delta_n = \det M_n$ where

$$M_n = \begin{pmatrix} \omega_1^n & \omega_2^n & \dots & \omega_t^n \\ \omega_1^{n+1} & \omega_2^{n+1} & \dots & \omega_t^{n+1} \\ \vdots & \vdots & & \vdots \\ \omega_1^{n+t-1} & \omega_2^{n+t-1} & \dots & \omega_t^{n+t-1} \end{pmatrix}.$$

Let $C_{i,n}$ denote the i-th column of M_n. Since $C_{i,n} = \omega_i^n C_{i,0}$ we have $|\delta_n| = |\omega_1^n \dots \omega_t^n \delta_0| = |\delta_0| \neq 0$ because δ_0 is the Vandermonde determinant built on the pairwise distinct numbers $\omega_1, \dots, \omega_t$. Now assume that $\kappa_j \neq 0$ for some j. Then for computing δ_n we can replace $C_{j,n}$ with $\frac{1}{\kappa_j}\sum_{i=1}^{t} \kappa_i C_{i,n}$; this implies $\lim_{n\to+\infty} \delta_n = 0$, in contradiction with the fact that $|\delta_n| = |\delta_0| \neq 0$. □

3. Algebraic numbers and logarithms as values of G-functions

An important step for us is to show that algebraic numbers are values of G-functions with coefficients in $\mathbb{Q}(i)$ (and, more precisely, that they satisfy the conclusion of

Theorem 1). Despite quite general results in related directions, this fact does not seem to have been proved in the literature in the full form we need. Eisenstein [31] showed that the G-function (of hypergeometric type)

$$\sum_{n=0}^{\infty}(-1)^n \frac{\binom{5n}{n}}{4n+1} a^{4n+1}$$

is a solution of the quintic equation $x^5 + x = a$, provided that $|a| \leq 5^{-5/4}$ (to ensure the convergence of the series). Eisenstein's formula can be proved using Lagrange's inversion formula. More generally, given a polynomial $P(x) \in \mathbb{C}[x]$, it is known that multivariate series can be used to find expressions of the roots of P in terms of its coefficients p_j. For example in [32], it is shown that these roots can be formally expressed as A-hypergeometric series evaluated at rational powers of the p_j's. (A-hypergeometric series are an example of multivariate G-functions.) It is not clear how such a representation could be used to prove Lemma 7 below: beside the multivariate aspect, the convergence of the series imposes some conditions on the p_j's and their exponents are not integers in general. Our proof is more in Eisenstein's spirit.

Lemma 7. *Let $\alpha \in \overline{\mathbb{Q}}$, and $Q(X) \in \mathbb{Q}[X]$ be a non-zero polynomial of which α is a simple root. For any $u \in \mathbb{Q}(i)$ such that $Q'(u) \neq 0$, the series*

$$\Phi_u(z) = u + \sum_{n=1}^{\infty}(-1)^n \frac{Q(u)^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\left(\frac{x-u}{Q(x)-Q(u)} \right)^n \right)_{|x=u} z^n$$

is a G-function with coefficients in $\mathbb{Q}(i)$; it satisfies the equation

$$Q(\Phi_u(z)) = (1-z)Q(u).$$

For any $R \geq 1$, if u is close enough to α then the radius of convergence of Φ_u is $> R$ and $\alpha = \Phi_u(1) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{Q}(i)}$.

Accordingly we have $\overline{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$.

Remarks. a) The proof can be made effective, i.e., given α, Q and R, we can compute $\varepsilon(\alpha, Q, R)$ such that for any $u \in \mathbb{Q}(i)$ with $|\alpha - u| < \varepsilon(\alpha, Q, R)$, we have $\Phi_u(1) = \alpha$ and the radius of convergence of Φ_u is $> R$.

b) Using Lemma 2 (ii), we deduce that any real algebraic number is in $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$.

We also need a similar property for values of the logarithm.

Lemma 8. *Let $\alpha \in \overline{\mathbb{Q}}^{\star}$. For any determination of the logarithm, the number* $\log(\alpha)$ *belongs to $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$.*

3.1. Algebraic numbers

Proof of Lemma 7. If $\deg Q = 1$ then $\Phi_u(z) = u + (\alpha - u)z$ so that Lemma 7 holds trivially. From now on we assume $\deg Q \geq 2$. Then $\frac{Q(X)-Q(u)}{X-u}$ is a non-constant polynomial with coefficients in $\mathbb{Q}(i)$; its value at $X = u$ is $Q'(u) \neq 0$ so that the coefficients of $\Phi_u(z)$ are well-defined and belong to $\mathbb{Q}(i)$. If $Q(u) = 0$ then $\Phi_u(z) = u$ and the result is trivial, so that we may assume $Q(u) \neq 0$ and define the polynomial function

$$z_u(t) = 1 - \frac{Q(t+u)}{Q(u)} \in \mathbb{Q}(i)[t]$$

so that $z_u(0) = 0$ and $z_u'(0) = -\frac{Q'(u)}{Q(u)} \neq 0$. Hence $z_u(t)$ can be locally inverted around $t = 0$ and its inverse $t_u(z) = \sum_{n\geq 1} \phi_n(u) z^n$ is holomorphic at $z = 0$.

The Taylor coefficients of t_u can be computed by means of the Lagrange inversion formula [20], p. 732, which in this case gives $\Phi_u(z) = u + t_u(z)$. By definition of $t_u(z)$, this implies $Q(\Phi_u(z)) = (1-z)Q(u)$. Therefore Φ_u is an algebraic function hence it is a G-function.

Now let

$$\phi_n(u) = \frac{(-Q(u))^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left(\left(\frac{x-u}{Q(x)-Q(u)} \right)^n \right)_{|x=u}$$

denote, for $n \geq 1$, the coefficient of z^n in $\Phi_u(z)$. Then for any $n \geq 1$ we have

$$\phi_n(u) = \frac{Q(u)^n}{2i\pi} \int_{\mathscr{C}} \frac{\mathrm{d}z}{(Q(u)-Q(z))^n} \tag{3.1}$$

where $\mathscr{C}$ is a closed path surrounding u but no other roots of the polynomial $Q(X) - Q(u)$. This enables us to get an upper bound on the growth of the coefficients $\phi_n(u)$. Let us denote by $\beta_1(u) = u, \beta_2(u), \dots, \beta_d(u)$ the roots (repeated according to their multiplicities) of the polynomial $Q(X) - Q(u)$, with $d = \deg Q \geq 2$. We take u close enough to α so that $\beta_2(u), \dots, \beta_d(u)$ are also close to the other roots $\alpha_2, \dots, \alpha_d$ of the polynomial $Q(X)$. Since α is a simple root of $Q(X)$, we have $\alpha \notin \{\alpha_2, \dots, \alpha_d\}$. We can then choose the smooth curve $\mathscr{C}$ in (3.1) independent from u such that the distance from $\mathscr{C}$ to any one of $u, \beta_2(u), \dots, \beta_d(u)$ is $\geq \varepsilon > 0$ with ε also independent from u, in such a way that u lies inside $\mathscr{C}$ and $\beta_2(u), \dots, \beta_d(u)$ outside $\mathscr{C}$.[4] It follows in particular that, for any $z \in \mathscr{C}$, $|Q(u) - Q(z)| \geq \rho$ for some $\rho > 0$ independent from u. Hence $\max_{z\in\mathscr{C}} \left| \frac{1}{Q(u)-Q(z)} \right| \leq \frac{1}{\rho}$. From the Cauchy integral in (3.1), we deduce that

$$|\phi_n(u)| \leq \frac{|\mathscr{C}|}{2\pi} \cdot \frac{|Q(u)|^n}{\rho^n}, \tag{3.2}$$

[4] We do so because we want to use a curve $\mathscr{C}$ that does not depend of u, whereas the poles of the integrand move with u.

where $|\mathscr{C}|$ is the length of $\mathscr{C}$. Let $R \geq 1$. Since $Q(u) \to Q(\alpha) = 0$ as $u \to \alpha$, we deduce that the radius of convergence of $\Phi_u(z)$ is $> R$ provided that u is sufficiently close to α (namely as soon as $R|Q(u)| < \rho$). Then the series $\Phi_u(1)$ is absolutely convergent and we have

$$|\Phi_u(1) - u| = \left| \sum_{n=1}^{\infty} \phi_n(u) \right| \leq \frac{|\mathscr{C}|}{2\pi} \sum_{n=1}^{\infty} \frac{|Q(u)|^n}{\rho^n} = \mathcal{O}\big(|Q(u)|\big). \tag{3.3}$$

Therefore $\Phi_u(1)$ can be made arbitrarily close to u, and accordingly arbitrarily close to α. Now for any z inside the disk of convergence of Φ_u we have $Q(\Phi_u(z)) = (1-z)Q(u)$, so that $\Phi_u(1)$ is a root of $Q(X)$. If it is sufficiently close to α, it has to be α. This completes the proof of Lemma 7. □

3.2. Logarithms of algebraic numbers

Proof of Lemma 8. Throughout this proof, we will always consider the determination of $\log z$ of which the imaginary part belongs to $(-\pi, \pi]$ (but the result holds for any determination because $i\pi = \log(-1) \in \mathbf{G}^{\text{cv}}_{\mathbb{Q}(i)}$).

Using the formula $\log(\alpha) = n \log(\alpha^{1/n})$ with n sufficiently large, we may assume that α is arbitrarily close to 1; in particular the imaginary part of $\log \alpha$ gets arbitrarily close to 0.

Letting $Q(X)$ denote the minimal polynomial of α, we keep the notation in the proof of Lemma 7, and write $\alpha = \Phi_u(1) = u + u\Psi_u(1)$ where $u \in \mathbb{Q}(i)$ is close enough to α, $\Psi_u(1)$ is in $\mathbf{G}^{\text{cv}}_{\mathbb{Q}(i)}$ and $\Psi_u(0) = 0$. By Equation (3.2), the radius of convergence at $z = 0$ of the G-function $\Psi_u(z)$ can be taken arbitrarily large provided that $u \in \mathbb{Q}(i)$ is close enough to α. We have

$$\log(\alpha) = \log(\alpha/u) + \log(u) = \log\big(1 + \Psi_u(1)\big) + \log(u),$$

because all logarithms in this equality have imaginary parts arbitrarily close to 0. Let $R \geq 1$; we shall prove, if u is close enough to 1, that both $\log(1 + \Psi_u(1))$ and $\log(u)$ belong to $\mathbf{G}^{\text{cv}}_{R,\mathbb{Q}(i)}$.

a) Provided that u is close enough to α, reasoning as in Equation (3.3) we get $|\Psi_u(z)| < 1$ for all z in a disk of center 0 and radius $> R$. Hence for such a u, the radius of convergence of the Taylor series of $\log(1 + \Psi_u(z))$ at $z = 0$ is $> R \geq 1$. To see that it is a G-function with coefficients in $\mathbb{Q}(i)$, we observe that $\frac{d}{dz} \log\big(1 + \Psi_u(z)\big) = \frac{\Psi_u'(z)}{1+\Psi_u(z)}$ is an algebraic function holomorphic at the origin: its Taylor series is a G-function $\sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}(i)[[z]]$. Therefore $\log(1 + \Psi_u(z)) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \in \mathbb{Q}(i)[[z]]$; this is a G-function because the set of G-functions is stable under Hadamard product and both $\sum_{n=0}^{\infty} a_n z^{n+1}$ and $\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1}$ are G-functions. Whence, $\log(1 + \Psi_u(1)) \in \mathbf{G}^{\text{cv}}_{R,\mathbb{Q}(i)}$.

b) It remains to prove that $\log(u) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{Q}(i)}$ for any $u \in \mathbb{Q}(i)$ sufficiently close to 1. Let $a, b \in \mathbb{Q}$ be such that $u = a + ib$. Then we have

$$\log(u) = \frac{1}{2}\log(a^2+b^2) + i \arctan\left(\frac{b}{a}\right).$$

Now $\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ and $\arctan(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$ are G-functions with rational coefficients and radius of convergence $= 1$, and we may assume that $|a^2+b^2-1| < 1/R$ and $|b/a| < 1/R$. Then $\log(u) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{Q}(i)}$ (see Lemma 4). □

4. Analytic continuation and connection constants

4.1. Properties of differential equations of G-functions. Let $\mathbb{K}$ be an algebraic extension of $\mathbb{Q}$, and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{K}[[z]]$ be a G-function with coefficients $a_n \in \mathbb{K}$. Let L be a minimal differential equation with coefficients in $\mathbb{K}[z]$ of which $f(z)$ is a solution. We denote by $\xi_1, \dots, \xi_p \in \mathbb{C}$ the singularities of L (throughout this paper, we will consider only points at finite distance). For any $i \in \{1, \dots, p\}$, let Δ_i be a closed broken line from ξ_i to the point at infinity; we assume $\Delta_i \cap \Delta_j = \emptyset$ for any $i \neq j$, and let $\mathscr{D} = \mathbb{C} \setminus (\Delta_1 \cup \dots \cup \Delta_p)$: this is a simply connected open subset of $\mathbb{C}$. In most cases we shall take for Δ_i a closed half-line starting at ξ_i.

The differential equation $Ly = 0$ has holomorphic solutions on $\mathscr{D}$, and these solutions make up a $\mathbb{C}$-vector space of dimension equal to the order of L; a basis of this vector space will be referred to as a *basis of solutions of* L.

Let ζ be a singularity of L. Then for any sufficiently small open disk D centered at ζ, the intersection $D \cap \mathscr{D}$ is equal to D with a ray removed; let us choose a determination of the logarithm of $\zeta - z$, denoted by $\log(\zeta - z)$, for $z \in D \cap \mathscr{D}$ (in such a way that it is holomorphic in z). If $\zeta \in \mathscr{D}$ is not a singularity of L, the function $\log(\zeta - z)$ will cancel out in what follows.

We shall use the following theorem (see [4], p. 719, for a discussion).

Theorem 6 (André, Chudnovski, Katz). *Let $\mathbb{K}$ denote an algebraic extension of $\mathbb{Q}$. Consider a minimal differential equation L of order μ, with coefficients in $\mathbb{K}[z]$ and admitting a solution at $z = 0$ which is a G-function in $\mathbb{K}[[z]]$. Let $\mathscr{D}$, ξ_1,..., ξ_p be as above. Then L is fuchsian with rational exponents at each of its singularities, and for each point $\zeta \in \mathscr{D} \cup \{\xi_1, \dots, \xi_p\}$ there is a basis of solutions $(g_1(z), \dots, g_\mu(z))$ of L, holomorphic on $\mathscr{D}$, with the following properties:*

- *There exists an open disk D centered at ζ and functions $F_{s,t,j}(z)$, holomorphic at 0, such that for any $j \in \{1, \dots, \mu\}$ and any $z \in D \cap \mathscr{D}$:*

$$g_j(z) = \sum_{s \in S_j} \sum_{t \in T_j} \big(\log(\zeta - z)\big)^s (\zeta - z)^t F_{s,t,j}(\zeta - z)$$

where $S_j \subset \mathbb{N}$ and $T_j \subset \mathbb{Q}$ are finite subsets.

- *If $\zeta \in \mathbb{K}$ then the functions $F_{s,t,j}(z)$ are G-functions with coefficients in $\mathbb{K}$.*
- *If ζ is not a singularity of L then $S_j = T_j = \{0\}$ for any j, so that $g_1(z), \dots, g_\mu(z)$ are holomorphic at $z = \zeta$.*

This theorem is usually stated in a more precise form, namely

$$(g_1(z), \dots, g_\mu(z)) = \big(f_1(\zeta - z), f_2(\zeta - z), \dots, f_\mu(\zeta - z)\big) \cdot (\zeta - z)^{C_\zeta}$$

where the functions $f_j(z)$ are holomorphic at 0 and C_ζ is an upper triangular matrix, and a similar formulation holds for the singularity at infinity, where one replaces $\zeta - z$ by $1/z$. However this precise version won't be used in this paper.

4.2. Statement of the theorem on connection constants. Let $\mathbb{K}$, f, L and $\mathscr{D}$ be as in §4.1. Let $(g_1, \dots, g_\mu)$ denote a basis of the $\mathbb{C}$-vector space of holomorphic solutions on $\mathscr{D}$ of the differential equation $Ly = 0$; here μ is the order of L. Since $f \in \mathbb{K}[[z]]$ satisfies $Lf = 0$ and is holomorphic on a small open disk centered at 0, it can be analytically continued to $\mathscr{D}$ and expanded in the basis $(g_1, \dots, g_\mu)$:

$$f(z) = \sum_{j=1}^{\mu} \varpi_j g_j(z) \tag{4.1}$$

for any $z \in \mathscr{D}$, where $\varpi_1, \dots, \varpi_\mu \in \mathbb{C}$ are called *connection constants*.

The following theorem[5] is an important ingredient in the proof of Theorems 4 and 5.

Theorem 7. *Let $\mathbb{K}$ denote an algebraic extension of $\mathbb{Q}$. Consider a minimal differential equation L of order μ, with coefficients in $\mathbb{K}[z]$ and admitting a solution at $z = 0$ which is a G-function $f \in \mathbb{K}[[z]]$. Let $\mathscr{D}$, ξ_1, ..., ξ_p be as above, $\zeta \in \mathbb{K} \cap (\mathscr{D} \cup \{\xi_1, \dots, \xi_p\})$ and $(g_1, \dots, g_\mu)$ be a basis of solutions given by Theorem 6. Then the connection constants $\varpi_1, \dots, \varpi_\mu$ defined by Equation (4.1) belong to $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$.*

The following corollary is a consequence of Theorem 7 and Lemma 5 (applied with $\mathbb{A} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$). It is used in the proof of Theorem 5.

Corollary 1. *Let $\mathbb{K}$, f, $\mathscr{D}$, ζ be as in Theorem 7. Then there exist $c \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$, $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Q}$ such that, as $z \to \zeta$ with $z \in \mathscr{D}$,*

$$f(z) = c\,\big(\log(\zeta - z)\big)^{\sigma} (\zeta - z)^{\tau}(1 + o(1)).$$

[5] As the proof shows, Theorem 7 holds under slightly weaker assumptions: it applies to any G-operator L such that $Lf = 0$, and also to $\zeta = \infty$.

4.3. Wronskian of fuchsian equations. Given a linear differential equation L with coefficients in $\overline{\mathbb{Q}}(z)$, of order μ and with a basis of solutions $f_1, f_2, \dots, f_\mu$, the wronskian $W = W(f_1, \dots, f_\mu)$ is the determinant

$$W(z) = \begin{vmatrix} f_1(z) & f_2(z) & \cdots & f_\mu(z) \\ f_1^{(1)}(z) & f_2^{(1)}(z) & \cdots & f_\mu^{(1)}(z) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(\mu-1)}(z) & f_2^{(\mu-1)}(z) & \cdots & f_\mu^{(\mu-1)}(z) \end{vmatrix}.$$

The wronskian can be defined in a more intrinsic way as follows. We write L as

$$y^{(\mu)}(z) + a_{\mu-1}(z)y^{(\mu-1)}(z) + \cdots + a_1(z)y(z) = 0$$

where $a_j(z) \in \overline{\mathbb{Q}}(z)$, $j = 1, \dots, \mu-1$. Then $W(z)$ is a solution of the linear equation

$$y'(z) = -a_{\mu-1}(z)y(z), \tag{4.2}$$

hence $W(z) = \nu_0 \exp\big(-\int a_{\mu-1}(z)dz\big)$. The value of the constant ν_0 is determined by the solutions $f_1, f_2, \dots, f_\mu$.

Lemma 9. *Let $\mathbb{K}$, f, L, $\mathscr{D}$, ζ, g_1, ..., g_μ be as in Theorem 7. Then the wronskian $W(z) = W(g_1, \dots, g_\mu)(z)$ is an algebraic function over $\overline{\mathbb{Q}}(z)$, and its zeros and singularities lie among the poles of $a_{\mu-1}(z)$.*

Proof. Since the differential equation (4.2) is fuchsian, Equation (5.1.16) in [24], p. 148, yields $W(z) = \nu \prod_{j=1}^{J}(z-p_j)^{-r_j}$ where $p_1, \dots, p_J \in \overline{\mathbb{Q}}$ are the poles of $a_{\mu-1}(z)$ (which are simple because L is fuschian), $r_1, \dots, r_j \in \mathbb{Q}$ (because L has rational exponents at its singularities), and $\nu \in \mathbb{C}^\star$. It remains to prove that ν is algebraic.

With this aim in view, we compute the determinant $W(z)$ for $z \in \mathscr{D}$ sufficiently close to ζ by means of the expansions of g_1,..., g_μ and their derivatives. This yields

$$W(z) = \sum_{s\in S}\sum_{t\in T}\big(\log(\zeta - z)\big)^s(\zeta-z)^t F_{s,t}(\zeta - z)$$

where $S \subset \mathbb{N}$ and $T \subset \mathbb{Q}$ are finite subsets, and the $F_{s,t}(z)$ are G-functions with coefficients in $\mathbb{K}$. Now Lemma 5 provides $c \in \mathbb{K}$, $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Q}$ such that, as $z \to \zeta$ with $z \in \mathscr{D}$:

$$W(z) = c\big(\log(\zeta - z)\big)^\sigma(\zeta - z)^\tau(1 + o(1)).$$

On the other hand we also have $\prod_{j=1}^{J}(z-p_j)^{-r_j} = \tilde{c}(\zeta - z)^{\tilde{\tau}}(1+o(1))$ for some $\tilde{c} \in \overline{\mathbb{Q}}^\star$ and $\tilde{\tau} \in \mathbb{Q}$. Since the quotient is a constant, namely ν, taking limits as $z \to \zeta$ yields $\sigma = 0$, $\tau = \tilde{\tau}$ and $\nu = c/\tilde{c} \in \overline{\mathbb{Q}}$. This concludes the proof of Lemma 9. □

4.4. Proof of Theorem 7. Let $R \geq 1$. For any $\xi \in (\mathscr{D} \setminus \{0, \zeta\}) \cap \mathbb{K}(i)$, let $r_\xi > 0$ be the distance of ξ to the border $\Delta_1 \cup \cdots \cup \Delta_p$ of $\mathscr{D}$ (with the notation of §4.1), and D_ξ be the open disk centered at ξ of radius r_ξ / R. Since ξ is not a singularity of L, there is a basis $g_{1,\xi}(z), \ldots, g_{\mu,\xi}(z)$ of solutions of $Ly = 0$ consisting in G-functions in the variable $\xi - z$ with coefficients in $\mathbb{K}(i)$ (by Theorem 6); these G-functions have radii of convergence $\geq r_\xi$, so that $g_{j,\xi}(z) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$ for any $z \in D_\xi \cap \mathbb{K}(i)$ and any j (see Lemma 4).

Let $r_0 > 0$ be the radius of convergence of the G-function $f(z)$, and D_0 denote the open disk centered at 0 with radius r_0/R. Finally, for any $j \in \{1, \ldots, \mu\}$ we let $g_{j,\zeta}(z) = g_j(z)$; by assumption there exists $r_\zeta > 0$ such that

$$g_{j,\zeta}(z) = \sum_{s \in S_j} \sum_{t \in T_j} \big(\log(\zeta - z)\big)^s (\zeta - z)^t F_{s,t,j}(\zeta - z)$$

for any $z \in \mathscr{D}$ such that $|z - \zeta| < r_\zeta$, where $S_j \subset \mathbb{N}$ and $T_j \subset \mathbb{Q}$ are finite subsets and the $F_{s,t,j}$ are G-functions with coefficients in $\mathbb{K}$ and radii of convergence $\geq r_\zeta$. Then we let D_ζ be the open disk centered at ζ with radius r_ζ/R, so that for any $z \in D_\zeta \cap \mathbb{K}(i)$ and any j we have $g_{j,\zeta}(z) \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$ by Lemmas 4, 7 and 8.

Following a smooth injective compact path from 0 to ζ inside $\mathscr{D} \cup \{0, \zeta\}$, we can find $s - 2$ points $\xi_2, \ldots, \xi_{s-1} \in (\mathscr{D} \setminus \{0, \zeta\}) \cap \mathbb{K}(i)$ (with $s \geq 3$) such that $D_{k-1} \cap D_k \neq \emptyset$ for any $k \in \{2, \ldots, s\}$, where we let $D_k = D_{\xi_k}$ and $\xi_1 = 0$, $\xi_s = \zeta$.

As in the beginning of §4.2, we have connection constants $\varpi_{j,2} \in \mathbb{C}$ such that

$$f(z) = \sum_{j=1}^{\mu} \varpi_{j,2}\, g_{j,\xi_2}(z) \tag{4.3}$$

for any $z \in \mathscr{D}$. In the same way, for any $z \in \mathscr{D}$, any $k \in \{3, \ldots, s\}$ and any $j \in \{1, \ldots, \mu\}$ we have

$$g_{j,\xi_{k-1}}(z) = \sum_{\ell=1}^{\mu} \varpi_{j,k,\ell}\, g_{\ell,\xi_k}(z). \tag{4.4}$$

Obviously the connection constants $\varpi_j \in \mathbb{C}$ in Theorem 7 are obtained by making products of the vector $(\varpi_{j,2})_{1 \leq j \leq \mu}$ and the matrices $(\varpi_{j,k,\ell})_{1 \leq j,\ell \leq \mu}$ (for $k \in \{3, \ldots, s\}$), because $g_{j,\xi_s}(z) = g_j(z)$. Since $\mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$ is a ring and $R \geq 1$ can be any real number, Theorem 7 follows from the fact that all constants $\varpi_{j,2}$ and $\varpi_{j,k,\ell}$ in (4.3) and (4.4) belong to $\mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$. We will prove it now for (4.4); the proof is similar for (4.3).

Let $k \in \{3, \ldots, s\}$ and $j \in \{1, \ldots, \mu\}$. We differentiate $\mu - 1$ times Equation (4.4), so that we get the μ equations

$$g^{(s)}_{j,\xi_{k-1}}(z) = \sum_{\ell=1}^{\mu} \varpi_{j,k,\ell}\, g^{(s)}_{\ell,\xi_k}(z), \quad s = 0, \ldots, \mu - 1.$$

We choose $z = \rho_k \in D_{k-1} \cap D_k \cap \mathbb{K}(i)$ outside the poles of $a_{\mu-1}(z)$ (with the notation of §4.3). Doing so yields a system of μ linear equations in the μ unknowns $\varpi_{j,k,\ell}$, $\ell = 1, \dots, \mu$, which can be solved using Cramer's rule because the determinant of the system (namely $W(\rho_k)$, where $W(z)$ is the wronskian of L built on the basis of solutions $g_{1,\xi_k}(z), \dots, g_{\mu,\xi_k}(z)$) does not vanish, by Lemma 9. Using again Lemma 9, we have $W(\rho_k) \in \overline{\mathbb{Q}}^\star$ and therefore $\frac{1}{W(\rho_k)} \in \overline{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$ by Lemma 7. Now Cramer's rule yields the following expression for $\varpi_{j,k,\ell}$:

$$\frac{1}{W(\rho_k)}\begin{vmatrix} g_{1,\xi_k}(\rho_k) & \cdots & g_{\ell-1,\xi_k}(\rho_k) & g_{j,\xi_{k-1}}(\rho_k) & g_{\ell+1,\xi_k}(\rho_k) & \cdots & g_{\mu,\xi_k}(\rho_k) \\ g^{(1)}_{1,\xi_k}(\rho_k) & \cdots & g^{(1)}_{\ell-1,\xi_k}(\rho_k) & g^{(1)}_{j,\xi_{k-1}}(\rho_k) & g^{(1)}_{\ell+1,\xi_k}(\rho_k) & \cdots & g^{(1)}_{\mu,\xi_k}(\rho_k) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ g^{(\mu-1)}_{1,\xi_k}(\rho_k) & \cdots & g^{(\mu-1)}_{\ell-1,\xi_k}(\rho_k) & g^{(\mu-1)}_{j,\xi_{k-1}}(\rho_k) & g^{(\mu-1)}_{\ell+1,\xi_k}(\rho_k) & \cdots & g^{(\mu-1)}_{\mu,\xi_k}(\rho_k) \end{vmatrix}.$$

Since $\rho_k \in D_{k-1} \cap D_k$, the entries in this determinant belong to the ring $\mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$ (as noticed above), so that $\varpi_{j,k,\ell} \in \mathbf{G}^{\mathrm{cv}}_{R,\mathbb{K}(i)}$. This concludes the proof of Theorem 7.

5. Proof of Theorem 4

The main part in the proof of Theorem 4 is to prove that $\mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$; this will be done below. We deduce Theorem 4 from this inclusion as follows, by Lemmas 2 and 3. If $\mathbb{K} \not\subset \mathbb{R}$, we have

$$\mathbf{G}^{\mathrm{a.c.}}_{\mathbb{K}} \subset \mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} \subset \mathbf{G}^{\mathrm{a.c.}}_{\mathbb{K}}$$

and Theorem 4 follows. If $\mathbb{K} \subset \mathbb{R}$, we have:

$$\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} \subset \mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}} \cap \mathbb{R} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)} \cap \mathbb{R} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}}$$

so that $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$. The inclusion $\mathbf{G}^{\mathrm{a.c.}}_{\mathbb{K}} \subset \mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$ is trivial; let us prove that $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} \subset \mathbf{G}^{\mathrm{a.c.}}_{\mathbb{K}}$. Let $\xi_1, \xi_2 \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$, and f, g, h be G-functions with rational coefficients and radii of convergence > 2 such that $f(1) = \xi_1$, $g(1) = \xi_2$, and $h(1) = \sqrt[4]{2}$. Then $k(z) = f(z) + g(z)h(z)\sqrt[4]{1 - \frac{z}{2}}$ is a G-function with coefficients in $\mathbb{Q} \subset \mathbb{K}$, and $\xi_1 + i\xi_2$ is the value at 1 of an analytic continuation of k (obtained after a small loop around $z = 2$). This concludes the proof that $\mathbf{G}^{\mathrm{a.c.}}_{\mathbb{K}} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}} + i\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}}$ if $\mathbb{K} \subset \mathbb{R}$.

The rest of the section is devoted to the proof that $\mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}} \subset \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$. Let $\xi \in \mathbf{G}^{\mathrm{a.c.}}_{\overline{\mathbb{Q}}}$; we may assume $\xi \neq 0$. There exists a G-function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with coefficients $a_n \in \overline{\mathbb{Q}}$, and $z_0 \in \overline{\mathbb{Q}}$, such that ξ is one of the values at z_0 of the multivalued analytic continuation of f. Replacing $f(z)$ with $f(z_0 z)$, we may assume $z_0 = 1$.

Let L denote the minimal differential equation satisfied by f, and $\xi_1, \dots, \xi_p$ be the singularities of L. To keep the notation simple (and because the general case can be proved along the same lines), we shall assume that there is an open subset $\mathscr{D} \subset \mathbb{C}$ (as in §4.1) such that $1 \in \mathscr{D} \cup \{\xi_1, \dots, \xi_p\}$ and $\xi = f(1)$, where f denotes the analytic continuation of the G-function $\sum a_n z^n$ to $\mathscr{D}$. If 1 is a singularity of L then $f(1)$ is the (necessarily finite) limit of $f(z)$ as $z \to 1$, $z \in \mathscr{D}$.

The coefficients a_n ($n \geq 0$) belong to a number field $\mathbb{K} = \mathbb{Q}(\beta)$ for some primitive element β of degree d say. We can assume without loss of generality that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$, i.e, that all Galois conjugates of β are in $\mathbb{K}$. There exist d sequences of rational numbers $(u_{j,n})_{n\geq 0}$, $j = 0, \dots, d-1$, such that, for all $n \geq 0$, $a_n = \sum_{j=0}^{d-1} u_{j,n}\beta^j$ and thus (at least formally)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{j=0}^{d-1} \beta^j \sum_{n=0}^{\infty} u_{j,n} z^n. \tag{5.1}$$

The power series $U_j(z) = \sum_{n=0}^{\infty} u_{j,n} z^n$ are G-functions (see [17], Proposition VIII.1.4, p. 266), so that Equation (5.1) holds as soon as $|z|$ is sufficiently small. Moreover U_j has rational coefficients, so that it satisfies a differential equation with coefficients in $\mathbb{Q}[z]$ (see for instance [17], Proposition VIII.2.1 (iv), p. 268). We let L_j denote a minimal one, of order μ_j. Let $\mathscr{S}_j$ denote the set of singularities of L_j, and $\mathscr{S} = \mathscr{S}_0 \cup \cdots \cup \mathscr{S}_{d-1}$. Let Γ denote a compact broken line without multiple points from 0 to 1 inside $\mathscr{D} \cup \{0, 1\}$. Since $\mathscr{S}$ is a finite set, we may assume that $\Gamma \cap \mathscr{S} \subset \{0, 1\}$ and find a (small) simply connected open subset $\Omega \subset \mathbb{C}$ such that $\Gamma \setminus \{0, 1\} \subset \Omega \subset \mathscr{D} \setminus \{1\}$ and $\Omega \cap \mathscr{S} = \emptyset$. If Γ and Ω are chosen appropriately, it is possible to construct $\mathscr{D}_0$, ..., $\mathscr{D}_{d-1}$ as in §4.1 (with respect to L_0, ..., L_{d-1}) such that $\Omega \subset \mathscr{D}_0 \cap \cdots \cap \mathscr{D}_{d-1}$. Since Ω is simply connected and $1 \notin \Omega$, we choose a continuous determination of $\log(1-z)$ for $z \in \Omega$. Now Equation (5.1) holds in a neighborhood of 0, and 0 lies in the closure of Ω so that, by analytic continuation,

$$f(z) = \sum_{j=0}^{d-1} \beta^j U_j(z) \text{ for any } z \in \Omega. \tag{5.2}$$

We shall now expand this equality around the point 1, which lies also in the closure of Ω. For any $j \in \{0, \dots, d-1\}$, let $(g_{j,1}, \dots, g_{j,\mu_j})$ denote a basis of solutions of the differential equation $L_j y = 0$ provided by Theorem 6 with $\zeta = 1$. Then Theorem 7 gives $\varpi_{j,1}, \dots, \varpi_{j,\mu_j} \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$ such that $U_j(z) = \varpi_{j,1} g_{j,1}(z) + \cdots + \varpi_{j,\mu_j} g_{j,\mu_j}(z)$ for any $z \in \Omega$. Since $\beta^j \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$ by Lemma 7, Equation (5.2) yields finite subsets $S \subset \mathbb{N}$ and $T \subset \mathbb{Q}$ such that, for $z \in \Omega$ sufficiently close to 1,

$$f(z) = \sum_{s\in S} \sum_{t\in T} \big(\log(1-z)\big)^s (1-z)^t F_{s,t}(1-z)$$

where the functions $F_{s,t}(z)$ are holomorphic at 0 and have Taylor coefficients at 0 in $\mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$. Then Lemma 5 gives $c \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$, $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Q}$ such that $f(z) = c\big(\log(1-z)\big)^{\sigma}(1-z)^{\tau}(1+o(1))$ as $z \to 1$ with $z \in \Omega$. Since $\lim_{z\to 1} f(z) = \xi \neq 0$, we have $\sigma = \tau = 0$ and $\xi = c \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{Q}(i)}$. This concludes the proof of Theorem 4.

6. Rational approximations to quotients of values of G-functions

This section is devoted to the proof of Theorem 5: in §6.1 we prove that (i) $\Rightarrow$ (iii), and in §6.2 that (ii) $\Rightarrow$ (i). Since (iii) obviously implies (ii), this will conclude the proof.

6.1. Construction of rational approximants. Assume that assertion (i) holds. Let $\xi_1, \xi_2 \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} \setminus \{0\}$ be such that $\xi = \xi_1/\xi_2$. Let $R \geq 1$, and $U(z) = \sum_{n=0}^{\infty} u_n z^n$, $V(z) = \sum_{n=0}^{\infty} v_n z^n$ be G-functions with coefficients in $\mathbb{K}$ and radii of convergence $> R$, such that $U(1) = \sum_{n=0}^{\infty} u_n = \xi_1$ and $V(1) = \sum_{n=0}^{\infty} v_n = \xi_2$.

For any $n \geq 0$, let $a_n = \sum_{k=0}^{n} u_k$ and $b_n = \sum_{k=0}^{n} v_k$, $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$. Then $A(z) = U(z)\sum_{n=0}^{\infty} z^n = \frac{U(z)}{1-z}$ and $B(z) = \frac{V(z)}{1-z}$ are G-functions with coefficients in $\mathbb{K}$ and radii of convergence $= 1$. Moreover $\lim_{n\to+\infty} a_n = \xi_1$ and $\lim_{n\to+\infty} b_n = \xi_2$ so that $a_n, b_n \neq 0$ for any n sufficiently large, and

$$\big|a_n - \xi b_n\big| = \big|(a_n - \xi_1) - \xi(b_n - \xi_2)\big| \leq \sum_{k=n+1}^{\infty} |u_k| + |\xi| \sum_{k=n+1}^{\infty} |v_k| = \mathcal{O}\big(R^{-n}\big)$$

because $u_n, v_n = \mathcal{O}(R^{-n})$ as $n \to +\infty$ and we may assume $R \geq 2$. Therefore $A(z) - \xi B(z)$ has radius of convergence $\geq R$, thereby concluding the proof that (i) $\Rightarrow$ (iii).

6.2. Application of Singularity Analysis. Let us prove that (ii) $\Rightarrow$ (i) in Theorem 5.

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be G-functions with coefficients in $\mathbb{K}$, such that $b_n \neq 0$ for infinitely many n and $a_n - \xi b_n = o(b_n)$. Since $\xi \neq 0$, we have $a_n \neq 0$ for infinitely many n: none of $A(z)$ and $B(z)$ is a polynomial. Therefore these G-functions have finite positive radii of convergence, say ρ and $\tilde{\rho}$ respectively.

Let us denote by L the minimal differential equation over $\mathbb{K}[z]$ satisfied by $A(z)$, and by $\rho\zeta_1, \ldots, \rho\zeta_q$ the pairwise distinct singularities of $A(z)$ of modulus ρ (so that $|\zeta_1| = \cdots = |\zeta_q| = 1$). Then we have $q \geq 1$, and all $\rho\zeta_i$ are singularities of L and are algebraic numbers.

Let $\theta_0 \in (-\pi/2, \pi/2)$ and $\Delta_0 = \{z \in \mathbb{C},\ z = 1 \text{ or } \arg(z-1) \equiv \theta_0 \bmod 2\pi\}$. For any $i \in \{1, \ldots, q\}$, let $\Delta_i = \rho\zeta_i \Delta_0 = \{\rho\zeta_i z,\ z \in \Delta_0\}$. Denoting by $\xi_1 = \rho\zeta_1$,

$\dots$, $\xi_q = \rho\zeta_q$, ξ_{q+1}, $\dots$, ξ_p the singularities of L, we may assume (by choosing θ_0 properly) that Δ_1, $\dots$, Δ_q and some appropriate half-lines Δ_{q+1}, $\dots$, Δ_p satisfy the assumptions made at the beginning of §4.1, so that we can take $\mathscr{D} = \mathbb{C} \setminus (\Delta_1 \cup \dots \cup \Delta_p)$. Choosing arbitrary determinations for $\log(\rho\zeta_i)$ $(i = 1, \dots, q)$, and also a continuous one for $\log z$ when $z \in \mathbb{C} \setminus \Delta_0$, we may define $\log(\rho\zeta_i - z)$ to be $\log(\rho\zeta_i) + \log\left(1 - \frac{z}{\rho\zeta_i}\right)$ for $z \in \mathscr{D}$ sufficiently close to $\rho\zeta_i$ (because $\frac{1}{\rho\zeta_i}\Delta_i = \Delta_0$). For any $i \in \{1, \dots, q\}$, Corollary 1 yields $c_i \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)} \setminus \{0\}$, $\sigma_i \in \mathbb{N}$ and $\tau_i \in \mathbb{Q}$ such that

$$
\begin{aligned}
A(z) &= c_i \big(\log(\rho\zeta_i - z)\big)^{\sigma_i} \big(\rho\zeta_i - z\big)^{\tau_i} (1 + o(1)) \\
&= c_i (\rho\zeta_i)^{\tau_i} \left(\log\left(1 - \frac{z}{\rho\zeta_i}\right)\right)^{\sigma_i} \left(1 - \frac{z}{\rho\zeta_i}\right)^{\tau_i} (1 + o(1))
\end{aligned}
$$

as $z \to \rho\zeta_i$ with $z \in \mathscr{D}$. Replacing $A(z)$ and $B(z)$ with their ℓ-th derivatives from the beginning, where ℓ is a sufficiently large integer, we may assume $\tau_1 < 0$ (because $\rho\zeta_1$ is a singularity of $A(z)$). Let $\tau = \min(\tau_1, \dots, \tau_q) < 0$, and σ denote the maximal value of σ_i among those indices i such that $\tau_i = \tau$. Let $g(z) = (\log(1-z))^{\sigma}(1-z)^{\tau}$ for $z \in \mathbb{C} \setminus \Delta_0$, and $d_i = c_i(\rho\zeta_i)^{\tau_i}$ if $(\sigma_i, \tau_i) = (\sigma, \tau)$, $d_i = 0$ otherwise. Then $(d_1, \dots, d_q) \neq (0, \dots, 0)$ and, for any $i \in \{1, \dots, q\}$, we have $d_i \in \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$ (by Lemma 7, because $\rho\zeta_i \in \overline{\mathbb{Q}}$). Finally,

$$
A(z) = d_i g\left(\frac{z}{\rho\zeta_i}\right) + o\left(g\left(\frac{z}{\rho\zeta_i}\right)\right) \tag{6.1}
$$

as $z \to \rho\zeta_i$ with $z \in \mathscr{D}$. We have checked all assumptions of Theorem VI.5 (§VI.5, p. 398) of [20] (see also [21]). This result enables one to transfer this estimate (6.1) around the singularities on the circle of convergence into an asymptotic estimate for the coefficients of $A(z)$, namely

$$
a_n = \frac{(-1)^{\sigma}}{\Gamma(-\tau)} \cdot \frac{(\log n)^{\sigma}}{\rho^n n^{\tau+1}} \cdot \big(\chi_n + o(1)\big), \text{ with } \chi_n = \sum_{i=1}^{q} d_i \zeta_i^{-n}. \tag{6.2}
$$

Remark. Equation (6.2), the proof of which is based on Singularity Analysis, seems to be interesting for itself (and not only as a step in the proof of Theorem 5).

The same arguments with $B(z)$ provide $\tilde{\rho}$, $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{\zeta}_1$, $\dots$, $\tilde{\zeta}_{\tilde{q}}$, $\tilde{d}_1$, $\dots$, $\tilde{d}_{\tilde{q}}$ such that

$$
b_n = \frac{(-1)^{\tilde{\sigma}}}{\Gamma(-\tilde{\tau})} \cdot \frac{(\log n)^{\tilde{\sigma}}}{\tilde{\rho}^n n^{\tilde{\tau}+1}} \cdot \big(\tilde{\chi}_n + o(1)\big), \quad \text{with } \tilde{\chi}_n = \sum_{i=1}^{\tilde{q}} \tilde{d}_i \tilde{\zeta}_i^{-n}. \tag{6.3}
$$

Let $\mathscr{N}_0 = \{n \in \mathbb{N}, b_n = 0\}$ and $\mathscr{N} = \mathbb{N} \setminus \mathscr{N}_0$. By assumption $\mathscr{N}$ is infinite, and $a_n = 0$ for any $n \in \mathscr{N}_0$ sufficiently large. In what follows, we assume implicitly that $\mathcal{N}_0$ is infinite (otherwise the proof is the same, and even easier since everything works as if $\mathcal{N}_0 = \emptyset$ and $\mathcal{N} = \mathbb{N}$).

By Equations (6.2) and (6.3), we have as $n \to +\infty$ with $n \in \mathcal{N}$,

$$\frac{a_n}{b_n} = (-1)^{\sigma-\tilde{\sigma}} \frac{\Gamma(-\tilde{\tau})}{\Gamma(-\tau)} \cdot \frac{\chi_n + o(1)}{\tilde{\chi}_n + o(1)} \cdot \left(\frac{\tilde{\rho}}{\rho}\right)^n n^{\tilde{\tau}-\tau} (\log n)^{\sigma-\tilde{\sigma}}. \tag{6.4}$$

Now the left-hand side tends to $\xi \neq 0$ as $n \to +\infty$ with $n \in \mathcal{N}$. If $(\rho, \sigma, \tau) \neq (\tilde{\rho}, \tilde{\sigma}, \tilde{\tau})$ then $\left|\frac{\chi_n+o(1)}{\tilde{\chi}_n+o(1)}\right|$ tends to 0 or $+\infty$ as $n \to +\infty$ with $n \in \mathcal{N}$. Since both χ_n and $\tilde{\chi}_n$ are bounded, this implies that χ_n or $\tilde{\chi}_n$ tends to 0 as $n \to +\infty$ with $n \in \mathcal{N}$. Since $\chi_n = o(1)$ and $\tilde{\chi}_n = o(1)$ as $n \to \infty$ with $n \in \mathcal{N}_0$ (using (6.2) and (6.3), because $a_n = b_n = 0$ for $n \in \mathcal{N}_0$ sufficiently large), we have $\lim_{n\to+\infty} \chi_n = 0$ or $\lim_{n\to+\infty} \tilde{\chi}_n = 0$. By Lemma 6 this implies $d_1 = \cdots = d_q = 0$ or $\tilde{d}_1 = \cdots = \tilde{d}_{\tilde{q}} = 0$, which is a contradiction.

Therefore we have $(\rho, \sigma, \tau) = (\tilde{\rho}, \tilde{\sigma}, \tilde{\tau})$ in Equation (6.4), so that $\frac{a_n}{b_n} = \frac{\chi_n+o(1)}{\tilde{\chi}_n+o(1)}$ as $n \to +\infty$ with $n \in \mathcal{N}$. Therefore $\frac{\chi_n - \xi\tilde{\chi}_n + o(1)}{\tilde{\chi}_n+o(1)} = \frac{a_n}{b_n} - \xi$ tends to 0 as $n \to +\infty$ with $n \in \mathcal{N}$. Since $\tilde{\chi}_n$ is bounded, we deduce $\lim_{n\to+\infty} \chi_n - \xi\tilde{\chi}_n = 0$ (using the fact that $\chi_n = o(1)$ and $\tilde{\chi}_n = o(1)$ as $n \to \infty$ with $n \in \mathcal{N}_0$). Writing $\chi_n - \xi\tilde{\chi}_n = \sum_{j=1}^t \kappa_j \omega_j^n$ where $\{\omega_1, \ldots, \omega_t\} = \{\zeta_1^{-1}, \ldots, \zeta_q^{-1}, \tilde{\zeta}_1^{-1}, \ldots, \tilde{\zeta}_{\tilde{q}}^{-1}\}$ with $\omega_1, \ldots, \omega_t$ pairwise distinct, Lemma 6 yields $\kappa_1 = \cdots = \kappa_t = 0$. Reordering the ζ_j's and the ω_k's if necessary, we may assume that $d_1 \neq 0$ and $\omega_1 = \zeta_1^{-1}$. Then $\kappa_1 = d_1 - \xi\tilde{d}_i$ if there is a (necessarily unique) i such that $\omega_1 = \tilde{\zeta}_i^{-1}$, and $\kappa_1 = d_1$ otherwise. Since $\kappa_1 = 0 \neq d_1$, there is such an i and it satisfies $\tilde{d}_i \neq 0$ and $\xi = d_1/\tilde{d}_i \in \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)})$. If $\mathbb{K} \not\subset \mathbb{R}$ then $\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}} = \mathbf{G}^{\mathrm{cv}}_{\mathbb{K}(i)}$ by Theorem 4; otherwise we have $\xi \in \mathbb{R} \cap \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\overline{\mathbb{Q}}}) = \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\overline{\mathbb{Q}}\cap\mathbb{R}}) = \mathrm{Frac}(\mathbf{G}^{\mathrm{cv}}_{\mathbb{K}})$ by Theorem 4 and Lemma 2. In both cases, this concludes the proof of Theorem 5.

7. Perspectives

7.1. Other classes of arithmetic power series. It is natural to wonder if the results presented in this paper can be adapted to other classes of arithmetic power series. The most natural class is that of E-functions, also introduced by Siegel in [30]. The definition of these functions (see the Introduction) is formally similar to that of G-functions, but of course the presence of $n!$ at the denominator of the Taylor coefficients changes drastically the properties of E-functions. An E-function is entire and André proved in Theorem 4.3 of [4] that any E-function is solution of a linear differential equation with polynomial coefficients (not necessarily minimal) whose singularities are 0 (a regular singularity with rational exponents) and infinity (an irregular singularity in general). Like the set of G-functions, the set of E-functions enjoys certain stability properties; for instance, it is a ring.

Let us denote by $\mathbf{E}$ as the set of all values of E-functions at algebraic points. This is the analogue of $\mathbf{G}$ and it is a ring; it would be interesting to prove a result on $\mathbf{E}$ analogous to Theorem 1. However we do not even know what a reasonable conjecture would be in this respect; what is clear is that the situation is really different, as the following result shows (we are indebted to the referee for suggesting its proof to us).

Proposition 2. *Let f be an E-function with coefficients in $\mathbb{Q}(i)$, and $\alpha \in \overline{\mathbb{Q}}$ be such that $f(1) = \alpha$ or $f(1) = e^{\alpha}$. Then $\alpha \in \mathbb{Q}(i)$.*

Proof. Let $\phi(z)$ denote either α or $e^{\alpha z}$, with $\alpha \in \overline{\mathbb{Q}}$; assume there exists an E-function f with coefficients in $\mathbb{Q}(i)$ such that $f(1) = \phi(1)$. Replacing $f(z)$ with $f(z) - \beta$ or $f(z)e^{-\beta z}$ for a suitable $\beta \in \mathbb{Q}(i)$, we may assume that α has zero trace over $\mathbb{Q}(i)$. Now there exist $\overline{\mathbb{Q}}(z)$-linearly independent E-functions $f_1, \ldots, f_n$ with coefficients in $\mathbb{Q}(i)$ such that $f_1(1) = \phi(1)$ and the vector $\underline{f} = {}^t(f_1, \ldots, f_n)$ is a solution of the differential system $\underline{y}' = A\underline{y}$ where A is an $n \times n$ matrix with entries in $\mathbb{Q}(i)(z)$. Modifying $f_1, \ldots, f_n$ if necessary as in the proof of Theorem 1.5 of [11], we may assume that 1 is not a pole of an entry of A. Using Beukers' version of Siegel–Shidlovskii's theorem (namely Theorem 1.3 of [11]), the relation $f_1(1) = \phi(1)$ can be lifted to $P_1(z)f_1(z) + \cdots + P_n(z)f_n(z) = P_0(z)\phi(z)$ with $P_0, \ldots, P_n \in \overline{\mathbb{Q}}[z]$ such that $P_0(1) = P_1(1) = 1$ and $P_2(1) = \cdots = P_n(1) = 0$.

If $\phi(z) = \alpha$, taking the trace over $\mathbb{Q}(i)$ yields $Q_0, \ldots, Q_n \in \mathbb{Q}(i)[z]$ such that $Q_1(z)f_1(z) + \cdots + Q_n(z)f_n(z) = Q_0(z)$ with $Q_1(1) = 1$, $Q_2(1) = \cdots = Q_n(1) = 0$, and $Q_0(1) = 0$ since α has zero trace. Therefore $f_1(1) = 0$, and $\alpha = 0$.

If $\phi(z) = e^{\alpha z}$, we take the norm over $\mathbb{Q}(i)$ of the relation $P_1(z)f_1(z) + \cdots + P_n(z)f_n(z) = P_0(z)e^{\alpha z}$. Letting d denote the degree of a finite Galois extension of $\mathbb{Q}(i)$ which contains α and all coefficients of $P_0, \ldots, P_n$, this provides (since α has zero trace) a relation $\sum_{\underline{\kappa}} Q_{\underline{\kappa}}(z) f_{\underline{\kappa}}(z) = Q_0(z)$ where $Q_0 \in \mathbb{Q}(i)[z]$, $\underline{\kappa} = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ is such that $\kappa_1 + \cdots + \kappa_n = d$, $f_{\underline{\kappa}}(z) = f_1(z)^{\kappa_1} \ldots f_n(z)^{\kappa_n}$, and $Q_{\underline{\kappa}}(z) \in \overline{\mathbb{Q}}[z]$ is such that $Q_{\underline{\kappa}}(1) = 0$ for $\underline{\kappa} \neq (d, 0, \ldots, 0)$ and $Q_{(d,0,\ldots,0)}(1) = 1$. Taking $z = 1$ yields $f_1(1)^d = Q_0(1) \in \mathbb{Q}(i)$ hence $e^{\alpha} \in \overline{\mathbb{Q}}$, so that $\alpha = 0$.

This concludes the proof of Proposition 2. $\square$

The possibility of a result analogous to Theorem 3 is also uncertain. It is easy to describe the limits of sequences A_n/B_n where $A_n, B_n \in \overline{\mathbb{Q}}$, $B_n \neq 0$ for all large enough n and $\sum_{n=0}^{\infty} A_n z^n$ and $\sum_{n=0}^{\infty} B_n z^n$ are E-functions. This is simply $\mathrm{Frac}(\mathbf{G})$, because the series $\sum_{n=0}^{\infty} n! A_n z^n$ and $\sum_{n=0}^{\infty} n! B_n z^n$ are G-functions, and conversely if $\sum_{n=0}^{\infty} a_n z^n$ is a G-function, then $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. This can hardly be the analogue we seek. We now observe that given an E-function $f(z) = \sum_{n=0}^{\infty} A_n z^n$, the sequence p_n/q_n, with $p_n = \sum_{k=0}^{n} A_k$ and $q_n = 1$, tends to $f(1)$, but $\sum_{n=0}^{\infty} p_n z^n = \frac{f(z)}{1-z}$ is not an E-function and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ is a G-function. Hence a result analogous to Theorem 3 and involving $\mathbf{E}$ might be achieved by considering simultaneously E and G-functions. It is also possible that similar questions might be easier to answer in the larger class of *arithmetic Gevrey series* introduced by André in [4], [5].

7.2. Possible applications to irrationality questions. The Diophantine theory of E-functions is well understood after the works of many authors, among which we may cite Siegel [30] and Shidlovskii [29], and more recently André [5] and Beukers [11]. An E-function essentially takes transcendental values at all non-zero algebraic points, and the algebraic points where it may take an algebraic value are fully controlled *a priori*.

This is far from being true for a non-algebraic G-function. There are many examples in the literature of G-functions taking algebraic values at some algebraic points without an obvious reason, see for example [10]. After the pioneering works of Galochkin [22] and Bombieri [12], it is known that, given a transcendental G-function f, if α is a non-zero algebraic number of modulus $\leq c$, then $f(\alpha)$ cannot be an algebraic number of degree $\leq d$. Here, $c > 0$ and $d \geq 1$ are explicit quantities that depend on f and on the degree and height of α. A typical example is that if $\alpha = 1/q$ is the inverse of an integer, then $f(\alpha)$ is an irrational number provided that $|q| \geq Q$ is sufficiently large in terms of f. An important issue is that the constant c is usually much smaller than the radius of convergence of f: the point where the value is taken has to be very close to 0.

On the contrary, a few results are known in which such a restriction is not necessary. One of them is Wolfart's theorem [33] on transcendence of values of Gauss' hypergeometric function at algebraic points. Another, more related to the present paper, is Apéry's proof of the irrationality of $\zeta(3)$; it involves evaluating a G-function on the border of its disk of convergence. The starting point of his method is given by Theorem 5: he constructs two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of rational numbers, whose generating functions are G-functions [6], such that a_n/b_n tends to $\zeta(3)$. To prove irrationality, more is needed, i.e., one also has to find a suitable common denominator D_n of a_n and b_n, and then prove that the linear form $D_n a_n + D_n b_n \zeta(3) \in \mathbb{Z} + \mathbb{Z}\zeta(3)$ tends to 0 without being equal to 0. (In this case, $D_n = \mathrm{lcm}(1, 2, \ldots, n)^3$.) The growth of D_n is usually the main problem in attempts at proving irrationality in Apéry's style. Indeed, there exist many examples of values $f(\alpha)$ of a G-function f at an algebraic point α having approximations in the sense of Theorem 3 (iii) (see [28] for references), but the growth of the relevant denominators D_n prevents one to prove irrationality when the modulus of α is too close to the radius of convergence of f. For instance, this approach has failed so far to establish the irrationality of $\zeta(5)$ or of Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$.

In the following proposition, we explain in details how the growth of D_n, the radii of convergence and the irrationality exponent $\mu(\xi)$ of ξ are connected. Recall that $\mu(\xi)$ is the supremum of the set of real numbers μ such that, for infinitely many

[6] This was apparently first observed by Dwork in [16]; see also [18], §1.10, for references.

fractions p/q, $|\xi - p/q| < q^{-\mu}$. In particular ξ is said to be a Liouville number if $\mu(\xi) = +\infty$.

Proposition 3. *Let $\xi \in \mathbf{G} \cap \mathbb{R}$. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ be G-functions, with rational coefficients and radii of convergence $= r > 0$, such that $A(z) - \xi B(z)$ has a finite radius of convergence, which is $\geq R > r$. Let $C \geq 1$ be such that a_n and b_n have a common denominator $\leq C^{n(1+o(1))}$ (as $n \to +\infty$). Then:*

- *If $C < R$ then $\xi \notin \mathbb{Q}$ and $\mu(\xi) \leq 1 - \frac{\log(C/r)}{\log(C/R)}$.*
- *Necessarily $C \geq \sqrt{Rr}$.*

This proposition is analogous to the other ones used to bound $\mu(\xi)$ from above when small linear forms $a_n \xi - b_n$ are available; the main difference here is that we do not assume $\lim_{n\to\infty} |a_n\xi - b_n|^{1/n}$ to exist. We hope this proposition can be used to make some progress towards Conjecture 1 stated in the introduction; of course the difficult point is to construct the G-functions with a control upon the denominators of a_n and b_n (so that C is not too large).

We have considered here only the case of one number ξ, but G-functions also arise in proofs of linear independence, in the same way as in Apéry's, for instance concerning the irrationality [8], [27] of $\zeta(s)$ for infinitely many odd $s \geq 3$.

Proof of Proposition 3. The second assertion follows from the first one because $\mu(\xi) \geq 2$ for any $\xi \in \mathbb{R} \setminus \mathbb{Q}$. Let us prove the first one.

Let $p_n = D_n a_n \in \mathbb{Z}$ and $q_n = D_n b_n \in \mathbb{Z}$, where n is sufficiently large and $D_n \in \mathbb{Z}$ is such that $1 \leq D_n \leq C^n$ (increasing C slightly if necessary). Decreasing R slightly if necessary, we may assume that the radius of convergence of $A(z) - \xi B(z)$ is $> R$, so that $|q_n \xi - p_n| \leq (C/R)^n$ for any n sufficiently large. Since $C < R$ and $q_n\xi - p_n \neq 0$ for infinitely many n (because $A(z) - \xi B(z)$ has a finite radius of convergence), this implies $\xi \notin \mathbb{Q}$. Moreover there exists a non-trivial linear recurrence relation $P_0(n)u_n + P_1(n)u_{n+1} + \cdots + P_r(n)u_{n+r} = 0$, with coefficients $P_j(n) \in \mathbb{Z}[n]$, satisfied by both sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. We claim that for any n sufficiently large, the vectors (p_n, q_n), (p_{n+1}, q_{n+1}), ..., (p_{n+r}, q_{n+r}) span the $\mathbb{Q}$-vector space $\mathbb{Q}^2$. Using Lemma 3.2 in [23], this implies $\mu(\xi) \leq 1 - \frac{\log(C/r')}{\log(C/R)}$ for any $r' < r$, because $|p_n|, |q_n| \leq (C/r')^n$ for any n sufficiently large. To prove the claim we argue by contradiction, and assume (permuting $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ if necessary) that for some $\lambda \in \mathbb{Q}$ we have $q_k = \lambda p_k$ for any $k \in \{n, n+1, \ldots, n+r\}$. Then the sequence $(b_i - \lambda a_i)_{i\geq n}$ satisfies the above-mentioned recurrence relation, and its first $r+1$ terms vanish. If n is sufficiently large then $P_r(i) \neq 0$ for any $i \geq n+r+1$ (because we may assume P_r to be non-zero), so that $q_i - \lambda p_i = b_i - \lambda a_i = 0$ for any $i \geq n$. Since $\lim_{i\to+\infty} q_i\xi - p_i = 0$ and $p_i \neq 0$ for infinitely many n, we deduce $\lambda\xi = 1$, in contradiction with the fact that $\xi \notin \mathbb{Q}$. $\square$

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