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Constructing equivalences with some extensions to the divisor and topological invariance of projective holonomy

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Abstract. Given topologically equivalent germs of holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$, under some hypothesis, we construct topological equivalences extending to some regions of the divisor after resolution of singularities. As an application we study the topological invariance of the projective holonomy representation.

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1. Introduction

Let $h\colon (\mathbb{C}^2, 0) \mapsto (\mathbb{C}^2, 0)$ denote a topological equivalence between two germs $\mathcal{F}$ and $\mathcal{F}'$ of holomorphic foliations with isolated singularity at $0 \in \mathbb{C}^2$, i.e., h is an orientation preserving homeomorphism mapping leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}'$. Cerveau and Sad in [24] pose the following problem: Assuming $\mathcal{F}$ is a non-dicritical generalized curve, it is true that the projective holonomy groups of $\mathcal{F}$ and $\mathcal{F}'$ are topologically conjugated? Also in [24] the authors give a positive answer for a generic class of foliations $\mathcal{F}$ and assuming that h is a topologically trivial deformation. A stronger result is obtained by Marín in [12] under the assumption of complex hyperbolicity of the singularities of $\mathcal{F}$ after a single blow up and removing the topological triviality of h. In [16], by using a notion of extended holonomy, the authors give a positive answer under the assumption that all singularities of $\mathcal{F}$ after a single blow up are non-degenerate and have exactly two separatrices. In a recent work ([15]), D. Marín and J.-F. Mattei give a global monodromy notion which allows to solve the problem for Generic General Type foliations. Following [15], a non-dicritical generalized curve $\mathcal{F}$ is of General Type if after resolution all singularities in the strict transform of $\mathcal{F}$ are linearizable or resonant. Such $\mathcal{F}$ is of Generic General Type if "some" irreducible components of the exceptional divisor have a non-solvable holonomy group (see genericity condition (G) in [15]). When

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the resolution of $\mathcal{F}$ does not have nodal singularities the genericity condition (G) is equivalent to the existence of a single divisor component having a non-solvable holonomy group, in this case the topological equivalence h is transversely conformal ([18]) and the principal result of [15] shows that the projective holonomy of each irreducible component of the exceptional divisor is a topological invariant. In the present work, given topologically equivalent germs of foliations $\mathcal{F}$ and $\mathcal{F}'$ and under some additional hypothesis, we construct topological equivalences extending to some regions of the divisor after the resolution of singularities of $\mathcal{F}$ and $\mathcal{F}'$. We give the precise statement of this construction in Theorem 7. When $\mathcal{F}$ is a non-dicritical generalized curve, it is known that $\mathcal{F}'$ is also a generalized curve and the resolutions of $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic ([3]), although h does not extend necessarily to the divisor after resolution. In this case Theorem 7 gives the following result.

Theorem 1. *Let h be a topological equivalence between two non-dicritical generalized curves $\mathcal{F}$ and $\mathcal{F}'$ with isolated singularity at $0 \in \mathbb{C}^2$. Then we may construct a topological equivalence $\bar{h}$ between $\mathcal{F}$ and $\mathcal{F}'$ which, after resolution, extends as a homeomorphism to a neighborhood of each linearizable or resonant singularity of $\mathcal{F}$ which is not a corner*[1].

As a direct application we obtain:

Corollary 2. *Let $\mathcal{F}$ be a non-dicritical generalized curve whose reduction of singularities is achieved after a single blow up. Assume that after resolution the strict transform of $\mathcal{F}$ has a linearizable or resonant singularity. Then the projective holonomy representation of $\mathcal{F}$ is a topological invariant.*

If $\mathcal{F}$ is of general type, as was pointed out to me by the referee, we can combine Theorem 1 with the results of [13] and [14] to prove the topological invariance of the projective holonomy of some exceptional divisor components without using the transverse rigidity hypothesis assumed in [15].

Corollary 3. *Let $\mathcal{F}$ be singularity of general type. Let D be an irreducible component of the exceptional divisor in the resolution of $\mathcal{F}$ such that D meets the strict transform of the separatrix curve of $\mathcal{F}$. Then the projective holonomy representation of D is a topological invariant.*

Also as a corollary of Theorem 1 we obtain the following extension result.

Corollary 4. *Let $\mathcal{F}$ be a singularity of general type whose reduction of singularities is achieved after a single blow up. Then, if $\mathcal{F}$ and $\mathcal{F}'$ are topologically equivalent, the strict transforms of $\mathcal{F}$ and $\mathcal{F}'$ after resolution are also topologically equivalent.*

[1] A corner is a singular point of the exceptional divisor.

Following [15], a nodal separatrix of $\mathcal{F}$ is an irreducible separatrix whose strict transform in the resolution of $\mathcal{F}$ meets the exceptional divisor at a nodal singular point. Is worth to notice as a corollary of Proposition 13 that the nodal separatrices of general type foliations are topological invariants:

Corollary 5. *Let $\mathcal{F}$ be a singularity of general type and let h be a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$. Let S be a nodal separatrix of $\mathcal{F}$. Then $h(S)$ is a nodal separatrix of $\mathcal{F}'$ and the Camacho–Sad indices along the strict transforms of S and $h(S)$ coincide.*

This corollary allows us to remove the $\mathcal{N}$-conjugacy hypothesis assumed in [15].

As a corollary of the proof of Theorem 7, we may replace the linearizing-resonant hypothesis by the assumption that the holonomy group of $\mathcal{F}$ is non-solvable to prove the following result, which is a particular case of the results obtained in [15].

Corollary 6. *Let $\mathcal{F}$ be a non-dicritical generalized curve whose reduction of singularities is achieved after a single blow up. Assume that the holonomy group of $\mathcal{F}$ is non-solvable. Suppose that $\mathcal{F}'$ is topologically equivalent to $\mathcal{F}$ by a homeomorphism which preserves the orientation of the leaves. Then we may construct a topological equivalence $\bar{h}$ between $\mathcal{F}$ and $\mathcal{F}'$ such that, after resolution, we have that*

1. *$\bar{h}$ extends to the divisor as a homeomorphism,*
2. *$\bar{h}$ preserves the Hopf fibration,*
3. *$\bar{h}$ is holomorphic close to each singularity whose eigenvalue is not a real positive number, and*
4. *if p is a singularity of the strict transform of $\mathcal{F}$ with eigenvalue $\lambda \in \mathbb{R}^+ \backslash \mathbb{Q}^+$, then the eigenvalue λ remains invariant by $\bar{h}$.*

In particular, the analytic type of all the singularities after resolution are invariants.[2]

The paper is organized as follows. In Section 2 we state Theorem 7 and prove Theorem 1. In Section 3 we prove Corollaries 2, 3 and 4. In Section 4 we make a first construction in order to prove Theorem 7. In Section 5 we proof a topological lemma. In Section 6 we divide the proof of Theorem 7 in two cases and in next section we prove the theorem in the first case: when the singularity is a node. In the remaining sections we prove Theorem 7 in the non-nodal case.

2. The extension theorem

Let $\mathcal{F}$ be a holomorphic foliation on the open set $U \subset \mathbb{C}^2$ with isolated singularity at $0 \in \mathbb{C}^2$. Let $\pi : M \mapsto \mathbb{C}^2$ be the composition of a finite sequence of blow ups.

[2] Remember that the eigenvalue λ determines the analytic type of a singularity, provided that $\lambda \in \mathbb{R}^+ \backslash \mathbb{Q}^+$.

We only consider blow ups at singular points of $\mathcal{F}$ or some strict transform of $\mathcal{F}$. The divisor $E = \pi^{-1}(0)$ is an union of projective lines with normal crossings such that $\pi\colon M\backslash E \mapsto \mathbb{C}^2\backslash\{0\}$ is an isomorphism. Let S be an irreducible separatrix of $\mathcal{F}$ through $0 \in \mathbb{C}^2$. It is possible to order the sequence of blow ups composing π and realize first all the blow ups involving points of S or some strict transform of S, that is, we may write π as composition of blow ups $\pi = \pi_1 \circ \cdots \circ \pi_n$ such that for some $k \in \{1, \ldots, n\}$ we have the following:

(1) π_1 is the projection associated to the blow up at $0 \in \mathbb{C}^2$.

(2) For all $j \in \{2, \ldots, k\}$ the map π_j is the projection associated to the blow up at the point p_j with $\pi_1 \circ \cdots \circ \pi_{j-1}(p_j) = 0$ and such that p_j is contained in the strict transform of S by $\pi_1 \circ \cdots \circ \pi_{j-1}$.

(3) If $j > k$, then π_j is the projection associated to a blow up in a point outside the strict transform of S by $\pi_1 \circ \cdots \circ \pi_{j-1}$.

It is easy to see that the number k depends only on π and S. Let us denote $k = k_\pi(S)$. Consider another holomorphic foliation $\mathcal{F}'$ with isolated singularity at $0 \in \mathbb{C}^2$. Let $\pi'\colon M' \mapsto \mathbb{C}^2$ be finite a composition of blow ups and let $E' = \pi^{-1}(0)$. Let $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ denote the strict transforms of $\mathcal{F}$ and $\mathcal{F}'$ by π and π' respectively. Consider a topological equivalence $h\colon U \mapsto U'$ between $\mathcal{F}$ and $\mathcal{F}'$. We know that h lifts to a homeomorphism

$$\tilde{h} = \pi'^{-1} h \pi\colon \pi^{-1}(U)\backslash E \mapsto \pi'^{-1}(U')\backslash E'$$

which takes leaves of $\tilde{\mathcal{F}}$ to leaves of $\tilde{\mathcal{F}}'$ and such that $\tilde{h}(w) \to E'$ as $w \to E$. Conversely, if W and W' are neighborhoods of E and E' respectively and $\bar{h}\colon W\backslash E \mapsto W'\backslash E'$ is a homeomorphism taking leaves of $\tilde{\mathcal{F}}$ to leaves of $\tilde{\mathcal{F}}'$ and such that $\bar{h}(w) \to E'$ as $w \to E$, then $\bar{h}$ induces a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$. Thus, by simplicity, we will say that any such $\bar{h}$ is a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$. Moreover, when no confusion arises we will often denote $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ simply by $\mathcal{F}$ and $\mathcal{F}'$ respectively.

We recall that a singularity p of a holomorphic foliation is called reduced if it is generated in local coordinates by a vector field of the form

$$\lambda_1 x(1 + \cdots)\frac{\partial}{\partial x} + \lambda_2 y(1 + \cdots)\frac{\partial}{\partial y},$$

where $\lambda_2 \neq 0$ and $\lambda = \lambda_1/\lambda_2$ is not a rational positive number. The singularity is non-degenerate when $\lambda_1 \cdot \lambda_2 \neq 0$ and is called resonant if λ_1/λ_2 is a rational (non-positive) number. The number $\lambda = \lambda_1/\lambda_2$ (or λ^{-1}) is called the eigenvalue of the singularity.

Theorem 7. *Let h be a topological equivalence between two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ with isolated singularity at $0 \in \mathbb{C}^2$. Let $\pi\colon M \mapsto \mathbb{C}^2$ and $\pi'\colon M' \mapsto \mathbb{C}^2$*

be finite compositions of blow ups. Let S be an irreducible separatrix of $\mathcal{F}$. Set $S' = h(S)$ and let $\widetilde{S}$ and $\widetilde{S}'$ denote the strict transforms of S and S' by π and π' respectively. Let p and p' be the intersections of $\widetilde{S}$ and $\widetilde{S}'$ with its respective divisors. Let (t, x) and (t', x') be holomorphic coordinates at p and p' respectively. Suppose that the following conditions hold:

(1) *The foliations are not degenerate at p and p'.*

(2) *The exceptional divisors are given by $\{x = 0\}$ and $\{x' = 0\}$ and they are invariant by the strict transforms of $\mathcal{F}$ and $\mathcal{F}'$ respectively.*

(3) *$\widetilde{S}$ and $\widetilde{S}'$are given by $\{t = 0\}$ and $\{t' = 0\}$ respectively.*

(4) $k_\pi(S) = k_{\pi'}(S')$.

Then, given $\varepsilon > 0$ we may construct a topological equivalence $\bar{h}$ between $\mathcal{F}$ and $\mathcal{F}'$ such that, for some numbers $a, b, a', b' \in (0, \varepsilon)$, we have

(1) *$\bar{h}$ maps $\{|t| \le a, 0 < |x| \le b\}$ into $\{|t'| \le a', 0 < |x'| \le b'\}$,*

(2) *$\bar{h}$ maps $\{|t| = a, 0 < |x| \le b\}$ into $\{|t'| = a', 0 < |x'| \le b'\}$,*

(3) *close to the divisor and outside*

$$\{|t| \le \varepsilon, |x| < \varepsilon\} \cup h^{-1}(|t'| \le \varepsilon, |x'| < \varepsilon)$$

we have $\bar{h} = h$.

Moreover, if p is linearizable or resonant, the following additional properties hold:

(4) *$\bar{h}$ extends as a topological equivalence to $\{|t| \le a, |x| \le b\}$,*

(5) *$\bar{h}(\{|t| \le a, x = 0\}) = \{|t'| \le a', x' = 0\}$ and $\bar{h}(0,0) = (0,0)$,*

(6) *$\bar{h}$ maps each disc $\Sigma_u = \{t = u, |x| \le b\}$, $|u| = a$, into a disc $\Sigma_{u'} = \{t = u', |x| \le b\}$, $|u'| = a'$.*

Given a germ of holomorphic singular foliation $\mathcal{F}$, we know by Seidenberg's desingularization Theorem that after a suitable finite sequence of blow ups, all the singularities of the strict transform of $\mathcal{F}$ are reduced. If $\mathcal{F}$ is dicritical (infinitely many separatrix), after some suitable additional blow ups we arrive to the following situation:

(1) The separatrices of $\mathcal{F}$ have became smooth, disjoint and transverse to the divisor.

(2) No separatrix passes through a corner.

(3) The singularities appearing in the blow-up are reduced an lie in invariant projective lines.

In this case the foliation $\mathcal{F}$ is said to be desingularized.

Definition 8 ([3]). A germ of holomorphic foliation $\mathcal{F}$ with isolated singularity at $0 \in \mathbb{C}^2$ is called a generalized curve if after resolution all its singularities are non-degenerate.

Theorem. ([3]) *If $\mathcal{F}$ is a generalized curve and $\mathcal{F}'$ is topologically equivalent to $\mathcal{F}$ at $0 \in \mathbb{C}^2$, then $\mathcal{F}'$ is also a generalized curve and both $\mathcal{F}$ and $\mathcal{F}'$ have isomorphic desingularizations.*

Proof of Theorem 1. Let $\pi: M \mapsto \mathbb{C}^2$ and $\pi': M' \mapsto \mathbb{C}^2$ be the minimal resolutions of $\mathcal{F}$ and $\mathcal{F}'$. Let $p_1, \dots, p_n$ be the linearizable or resonant singularities of the strict transform of $\mathcal{F}$ which are not corners. There are holomorphic coordinates (t, x) in a neighborhood of $p_1 \simeq (0, 0)$ such that

(1) the exceptional divisor is given by $\{x = 0\}$,

(2) $\widetilde{S} = \{t = 0\}$ is the strict transform of an irreducible separatrix S of $\mathcal{F}$.

The set $S' = h(S)$ is a separatrix (irreducible) of $\mathcal{F}'$ and its strict transform $\widetilde{S}'$ by π' intersects the exceptional divisor at a singularity p_1'. It is easily verified, since the resolutions of $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic, that $k_\pi(S) = k_{\pi'}(S')$. Let us apply Theorem 7 to construct a topological equivalence h_1 between $\mathcal{F}$ and $\mathcal{F}'$ which extends as a homeomorphism to a neighborhood V_1 of p_1. In the same way, we have a singularity p_2' in the desingularization of $\mathcal{F}'$ associated to p_2. We apply again Theorem 7 to obtain a topological equivalence h_2 between $\mathcal{F}$ and $\mathcal{F}'$ which extends to a neighborhood V_2 of p_2 and such that close to the divisor and out of

$$\{|t| \leq \varepsilon, |x| < \varepsilon\} \cup h^{-1}(|t'| \leq \varepsilon, |x'| < \varepsilon)$$

we have $h_2 = h_1$, where (t, x) and (t', x') are holomorphic coordinates at p_2 and p_2' respectively. If $\varepsilon > 0$ is taken small enough such that V_1 and $h_1(V_1)$ are disjoint of $\{|t| \leq \varepsilon, |x| < \varepsilon\}$ and $\{|t'| \leq \varepsilon, |x'| < \varepsilon\}$ respectively, we have $h_2 = h_1$ on V_1. Then h_2 actually extends as a homeomorphism to neighborhoods of both p_1 and p_2. Repeating this argument a finite number of times we finish the proof. □

3. Projective holonomy representation

Consider now a foliation $\mathcal{F}$ such that after a single blow up $\pi: \widehat{\mathbb{C}^2} \mapsto \mathbb{C}^2$ of the origin the exceptional divisor $D = \pi^{-1}(0)$ is invariant by the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ by π, that is, $D^* = D \backslash \mathrm{Sing}(\widetilde{\mathcal{F}})$ is a leaf of $\widetilde{\mathcal{F}}$. Let q be a point in D^* and Σ a small complex disc passing through q and transverse to $\widetilde{\mathcal{F}}$. For any loop γ in D^* based on q there is an holonomy map $H_{\mathcal{F}}(\gamma): (\Sigma, q) \mapsto (\Sigma, q)$ which only depends on the homotopy class of γ in the fundamental group $\Gamma = \pi_1(D^*)$. The map $H_{\mathcal{F}}: \Gamma \mapsto \mathrm{Diff}(\Sigma, q)$ is known as the projective holonomy representation of $\mathcal{F}$. Identifying $(\Sigma, q) \simeq (\mathbb{C}, 0)$ the image of $H_{\mathcal{F}}$ defines up to conjugation a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ which is known as the holonomy group of $\mathcal{F}$.

Definition 9. The representations $H: \Gamma \mapsto \mathrm{Diff}(\mathbb{C}, 0)$ and $H': \Gamma' \mapsto \mathrm{Diff}(\mathbb{C}, 0)$ are topologically conjugated if there exist an isomorphism $\varphi: \Gamma \mapsto \Gamma'$ and a germ of homeomorphism $h\colon (\mathbb{C},0) \mapsto (\mathbb{C},0)$ such that $H' \circ \varphi(\gamma) = h \circ H(\gamma) \circ h^{-1}$ for all $\gamma \in \Gamma$.

Proof of Corollary 2. Let p be a linearizable or resonant singularity of $\mathcal{F}$ after resolution. By Theorem 1 we have a topological equivalence h between $\mathcal{F}$ and $\mathcal{F}'$ extending to a neighborhood of p. Moreover, by the last property given by Theorem 7 there is a regular point q in the divisor and a disc Σ through q transverse to $\mathcal{F}$ such that $h(\Sigma)$ is contained in a disc Σ' through $q' = h(q)$ transverse to $\mathcal{F}'$. At this point we can follow the proof given in [12]. □

Proof of Corollary 3. Let $\mathcal{F}'$ be a foliation topologically equivalent to $\mathcal{F}$ and let D' be the irreducible component of the exceptional divisor corresponding to D in the resolution of $\mathcal{F}'$. By Theorem 1 there is a topological equivalence h between $\mathcal{F}$ and $\mathcal{F}'$ conjugating transverse sections Σ and Σ' to D and D' respectively. We can apply Theorem A of [14] to obtain a new homeomorphism g (not longer foliated) conjugating the separatrices S and S' of $\mathcal{F}$ and $\mathcal{F}'$ extending to the exceptional divisor and inducing the same action that f on $\pi_1(U \backslash S) \to \pi_1(U' \backslash S')$, where U and U' are suitable neighborhoods of the singularities constructed by foliated assembly according to Definition 2.1.1 of [13]. Moreover, there are no topological obstructions to have $g = f$ on Σ. Consider $\phi\colon = g|_D\colon D \to D'$ and its action in homotopy level $\phi_*\colon \pi_1(D^*) \to \pi_1(D'^*)$, where D^* and D'^* are obtained from D and D' by removing the singularities. Consider a loop $\gamma \in \pi_1(D^*)$ and its corresponding holonomy map h. For $p \in \Sigma^* := \Sigma \backslash D$ we consider a path β contained in the leaf L of $\mathcal{F}$ passing through $p = \beta(0)$ and realizing the holonomy map h, that is, β is mapped onto γ by the Hopf fibration associated to D and $\beta(1) = h(p)$. Consider a path θ contained in Σ^* joining $h(p)$ and p. Then the loop $f(\beta\theta)$ is homotopic to $g(\beta\theta)$ which is contained in a tubular neighborhood W' of D'. According to [13] we can choose W' such that it is 1-$\mathcal{F}'$-connected in U'^* (see Theorems 2.1.2 and 3.2.1 of [13]). Since $f = g$ on Σ we deduce that $f(\beta) \subset L' := f(L)$ is homotopic to $g(\beta) \subset W'$ with fixed endpoints. By the foliated 1-connexity of W' in U'^* we obtain a path $\beta' \subset L' \cap W'$ which is homotopic to $f(\beta)$ in L' and to $g(\beta)$ in U'^*. Let $\pi'\colon W' \to D'$ be the Hopf fibration associated to D'. Then we see that $\pi'(\beta')$ is homotopic to $\phi_*(\gamma)$ in D'^*. Hence $f(h(p)) = f(\beta)(1) = \beta'(1) = h'(f(p))$ where h' is the holonomy map associated to the loop $\phi_*(\gamma) \in \pi_1(D'^*)$. Consequently $f \circ h \circ f^{-1} = h'$. □

Proof of Corollary 4. By Theorem 1 we have a topological equivalence h between $\mathcal{F}$ and $\mathcal{F}'$ which extends as a homeomorphism and preserves the Hopf fibration near the singularities.The holonomy representations are topologically conjugated by an isomorphism induced by a homeomorphism $\phi\colon D \to D$ which coincides with h near the singularities. By a lifting path argument using ϕ we can redefine h outside some neighborhoods of the singularities to obtain a topological equivalence $\bar{h}$ extending to the divisor. □

4. A preliminary isotopy

As a first step in order to prove Theorem 7, we will prove the following:

Theorem 10. *Let h be a topological equivalence between two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ with isolated singularity at $0 \in \mathbb{C}^2$. Let $\pi: M \mapsto \mathbb{C}^2$ and $\pi': M' \mapsto \mathbb{C}^2$ be finite compositions of blow ups. Let S be an irreducible separatrix of $\mathcal{F}$. Denote $S' = h(S)$ and let $\tilde{S}$ and $\tilde{S}'$ be the strict transforms of S and S' by π and π' respectively. Let p and p' the intersections of $\tilde{S}$ and $\tilde{S}'$ with the respective exceptional divisors. Let (t,x) and (t',x') be holomorphic coordinates in the neighborhoods V of $p \simeq (0,0)$ and V' of $p' \simeq (0,0)$, respectively. Suppose that the following conditions hold:*

(1) *The exceptional divisors are given by $\{x = 0\}$ and $\{x' = 0\}$ and are invariant by (the strict transforms) $\mathcal{F}$ and $\mathcal{F}'$ respectively.*

(2) *$\tilde{S}$ and $\tilde{S}'$ are given by $\{t = 0\}$ and $\{t' = 0\}$.*

Then given $\varepsilon > 0$ there is $b \in (0,\varepsilon)$ and a topological equivalence $\bar{h}$ between $\mathcal{F}$ and $\mathcal{F}'$ with the following properties:

(1) *$\bar{h}$ is defined in a neighborhood of the set $\{(0,x) : 0 < |x| \le b\}$, which is mapped into $\{(0,x') : 0 < |x'| < \varepsilon\}$.*

(2) *There exists $\delta > 0$ such that for all r in a neighborhood of b, the set $\{|t| < \delta, |x| = r\}$ is mapped by $\bar{h}$ into a set of type $\{|x'| = r'\}$ with $r' \in (0,\varepsilon)$.*

(3) *For $|z|$ close to b the set $\{|t| < \delta,\ x = z\}$ is mapped into a set of type $\{x' = cte\}$.*

(4) *Close to the divisor we have $\bar{h} = h$.*

Proof. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be the circles $\{(0,x) : |x| = r_0\}$ and $\{(0,x) : |x| = r_1\}$ in V, where $0 < r_0 < r_1 < \varepsilon$. The curves $\mathcal{C}_0$ and $\mathcal{C}_1$ are contained in the separatrix $\{t = 0\} \subset V$. Fix a_0 and b_0 in $\mathcal{C}_0$, with $a_0 \neq b_0$. It is possible to modify the homeomorphism h in such way, on some neighborhoods of a_0 and b_0, the new homeomorphism, still denoted by h, maps the sets $\{x = cte\}$ into the sets $\{x' = cte\}$. Take a circle $\mathcal{C}_1'$ in the separatrix $\{t' = 0\} \subset V'$ containing $h(\mathcal{C}_0)$ in its interior, that is, $\mathcal{C}_1' = \{(0,x') : |x'| = r_1'\}$ with $|x'| < r_1'$ whenever $(0,x') \in h(\mathcal{C}_0)$. By taking r_1 small enough we may assume $r_1' < \varepsilon$. Let A be a segment of ratio with endpoints a_0 and $a_1 \in \mathcal{C}_1$. Thus A connect $\mathcal{C}_0$ and $\mathcal{C}_1$ and $A \backslash \{a_0, a_1\}$ is contained in the annulus bounded by $\mathcal{C}_0$ and $\mathcal{C}_1$. In the same way, let B be a segment of ratio, disjoint of A, with endpoints b_0 and $b_1 \in \mathcal{C}_1$. Consider the usual orientations of $\mathcal{C}_0$ and $\mathcal{C}_1$ an take a diffeomorphism $\theta: \mathcal{C}_1 \mapsto \mathcal{C}_1'$ such that $h(\mathcal{C}_0)$ and $\theta(\mathcal{C}_1)$ are homologous in $\{(0,x') : x' \neq 0\}$. Take injective maps $\alpha: A \mapsto \{t' = 0\}$ and $\beta: B \mapsto \{t' = 0\}$ such that

(1) $\alpha(a_0) = h(a_0), \alpha(a_1) = \theta(a_1)$,

(2) $\beta(b_0) = h(b_0), \beta(b_1) = \theta(b_1)$,

(3) $\alpha(A) \cap \beta(B) = \emptyset$,

(4) $\alpha(A \backslash \{a_0, a_1\})$ and $\beta(B \backslash \{b_0, b_1\})$ are contained in the annulus bounded by $h(\mathcal{C}_0)$ and $\mathcal{C}_1'$.

Let $\mathcal{I}$ be the annulus bounded by $\mathcal{C}_0$ and $\mathcal{C}_1$ in $\{t = 0\}$. We have $\mathcal{I} \backslash \{A, B\} = D \cup \widetilde{D}$, where D and $\widetilde{D}$ are simply connected domains. The boundary of D is a Jordan curve which is the union of the curves A, C_0, B and C_1, where C_0 and C_1 are segments of $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. Let π be the projection $(t, x) \to (0, x)$. Let L_x denote the leaf of $\mathcal{F}$ passing through $x \in \pi^{-1}(a_0)$. If Σ is a small enough neighborhood (a disc) of a_0 in $\pi^{-1}(a_0)$, for all $x \in \Sigma$ there is a domain D_x in L_x such that $x \in D_x$ and $\pi : \overline{D}_x \mapsto \overline{D}$ is a diffeomorphism (D is a lifting). The domain D_x in L_x is bounded by a Jordan curve, which is the union of the paths $A_x = \pi^{-1}(A)$, $C_{0x} = \pi^{-1}(C_0)$, $B_x = \pi^{-1}(B)$ and $C_{1x} = \pi^{-1}(C_1)$. Define $g : \partial D \mapsto \{t' = 0\}$ as

$$g = \begin{cases} \alpha & \text{on } A, \\ \beta & \text{on } B, \\ h & \text{on } C_0, \\ \theta & \text{on } C_1. \end{cases}$$

It is easy to see that g is continuous and injective. Denote also by π the projection $(t', x') \to (0, x')$. Observe that the Jordan curve $g(\partial D)$ in $\{t' = 0\}$ does not link the point $(0, 0)$. Therefore there is a lifting of $g(\partial D)$ to any leaf close enough to the separatrix. Then, if Σ is small enough, there is a Jordan curve $\tilde{J}_x$ in the leaf passing through $h(x)$ such that $\pi : \tilde{J}_x \mapsto g(\partial D)$ is a homeomorphism. Therefore we have that the map $f_x : \partial D_x \mapsto \tilde{J}_x$ defined by $\pi \circ f_x = g \circ \pi$ is a homeomorphism. Observe that, on C_{0x}, we have that $\pi \circ h$ is arbitrarily close to $h \circ \pi$ when $x \in \Sigma$ is close to a_0. Then, since $\pi \circ f_x = g \circ \pi = h \circ \pi$ on C_{0x}, we have that $\pi \circ h$ is arbitrarily close to $\pi \circ f_x$. Hence, since $f_x(x) = h(x)$ when x is close to a_0, we have that $f_x(y)$ is arbitrarily close to $h(y)$ for all $y \in C_{0x}$. Recall that, on neighborhoods U_a and U_b of a_0 and b_0 respectively, we have that h takes fibres $x = cte$ to fibres $x' = cte$, that is, $h \circ \pi = \pi \circ h$. Then, on $(U_a \cup U_b) \cap C_{0x}$, we have that $\pi \circ f_x = \pi \circ h$. Thus, since $f_x(y)$ is close to $h(y)$ and they are in the same leaf, we conclude that $f_x(y) = h(y)$ for all $y \in (U_a \cup U_b) \cap C_{0x}$ (whenever x is close to a_0). Then the function $h_x : \partial D_x \mapsto V'$, defined as $h_x = h$ on C_{0x} and $h_x = f_x$ on $\partial D_x \backslash C_{0x}$, is continuous and its image is contained in a leaf.

Assertion. If x is close enough to a_0, the map h_x is injective.

Proof. Clearly h_x is injective on C_{0x} and $\partial D_x \backslash C_{0x}$ separately. Then it is sufficient to prove that $h_x(C_{0x})$ and $h_x(\partial D_x \backslash C_{0x})$ are disjoint. Let $I_x = C_{0x} \backslash (U_a \cap U_b)$ and $I = C_0 \backslash (U_a \cap U_b)$. If x is close to a_0, we have that $\pi \circ h$ is arbitrarily close to $h \circ \pi = g \circ \pi$, on $I_x \subset C_{0x}$. Then the set $\pi \circ h(I_x)$ is arbitrarily close to $g \circ \pi(I_x)$. On the other hand, observe that, when x is close to a_0, the set $g \circ \pi(I_x)$ is arbitrarily close to the set $g(I)$. Then, when x is close to a_0, the set $\pi \circ h(I_x)$ is arbitrarily close to $g(I)$. Thus, since I is compact and disjoint of the closure of $\partial D \backslash C_0$, we have that $\pi \circ h(I_x)$ is disjoint of $g(\partial D \backslash C_0) = \pi \circ f_x(\partial D_x \backslash C_{0x})$. Therefore $h(I_x) = h_x(I_x)$ is disjoint of $f_x(\partial D_x \backslash C_{0x})$. On the other hand, $h_x(C_{0x} \cap (U_a \cup U_b)) = h(C_{0x} \cap (U_a \cup U_b)) = f_x(C_{0x} \cap (U_a \cup U_b))$ and is therefore disjoint of $f_x(\partial D_x \backslash C_{0x})$.

The Jordan curve $h_x(\partial D_x)$ is the boundary of a simply connected domain D'_x in the leave passing through x. It follows from the construction that h_x depends continuously on x. Then, by Lemma 11 below we have that h_x extends to a homeomorphism $h_x : \bar{D}_x \mapsto \bar{D}'_x$, which depends continuously on x. The homeomorphism h_x has the following properties:

(1) $h_x = h$ on C_{0x},

(2) $\pi \circ h_x = \alpha \circ \pi$ on A_x,

(3) $\pi \circ h_x = \beta \circ \pi$ on B_x,

(4) $\pi \circ h_x = \theta \circ \pi$ on C_{1x}.

The domain $\widetilde{D}$ is bounded by the union of the paths A, $\widetilde{C}_0$, B and $\widetilde{C}_1$, where $\widetilde{C}_0$ and $\widetilde{C}_1$ are segments of $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. For $x \in \Sigma$, let $\widetilde{D}_x$ be the lifting of $\widetilde{D}$ to the leaf passing through x, that is, $\pi : \bar{\widetilde{D}}_x \mapsto \bar{\widetilde{D}}$ is a diffeomorphism. Let $\tilde{A}_x = \pi^{-1}(A)$, $\widetilde{C}_{0x} = \pi^{-1}(\widetilde{C}_0)$, $\widetilde{B}_x = \pi^{-1}(B)$ and $\widetilde{C}_{1x} = \pi^{-1}(\widetilde{C}_1)$. Analogously, reducing Σ if necessary, for all $x \in \Sigma$ we construct the map $\tilde{h}_x : \bar{\widetilde{D}}_x \mapsto V'$ such that

(1) $\tilde{h}_x$ is a homeomorphism onto its image,

(2) $\widetilde{D}'_x = \tilde{h}_x(\widetilde{D}_x)$ is contained in the leaf passing through $h(x)$,

(3) $\tilde{h}_x = h$ on $\widetilde{C}_{0x}$,

(4) $\pi \circ \tilde{h}_x = \alpha \circ \pi$ on $\tilde{A}_x$,

(5) $\pi \circ \tilde{h}_x = \beta \circ \pi$ on $\widetilde{B}_x$,

(6) $\pi \circ \tilde{h}_x = \theta \circ \pi$ on $\widetilde{C}_{1x}$.

By reducing Σ we may assume that D_x and $\widetilde{D}_x$ are contained in V and that h_x and $\tilde{h}_x$ are defined for all $x \in \bar{\Sigma}$. Let $\mathcal{D} = \bigcup_{x \in \bar{\Sigma}} D_x$ and $\widetilde{\mathcal{D}} = \bigcup_{x \in \bar{\Sigma}} \widetilde{D}_x$. Let $f : \mathcal{D} \mapsto V'$ and $\tilde{f} : \widetilde{\mathcal{D}} \mapsto V'$ be defined by $f = h_x$ on D_x and $\tilde{f} = \tilde{h}_x$ on $\widetilde{D}_x$. Clearly f and $\tilde{f}$ are continuous and it is easy to see that

$$f = \tilde{f} \text{ on } \mathcal{D} \cap \widetilde{\mathcal{D}}. \tag{4.1}$$

In fact, if $z \in \mathcal{D} \cap \widetilde{\mathcal{D}}$, then $\pi(z)$ is contained in A or B. Suppose that $\pi(z) \in B$. Then $z \in B_x = \widetilde{B}_{\tilde{x}}$ for some $x, \tilde{x}$. Then it suffices to show that $f(w) = \tilde{f}(w)$ for all $w \in B_x = \widetilde{B}_{\tilde{x}}$, that is, $h_x(w) = \tilde{h}_{\tilde{x}}(w)$ for all $w \in B_x = \widetilde{B}_{\tilde{x}}$. But $\pi \circ h_x(w) = \beta \circ \pi(w) = \pi \circ \tilde{h}_{\tilde{x}}(w)$ for all $w \in B_x$, then, since B is an interval, it suffices to show that $h_x(w) = \tilde{h}_{\tilde{x}}(w)$ for some $w \in B_x = \widetilde{B}_{\tilde{x}}$. Let $w_0 \in B_x$ be the point such that $\pi(w_0) \in C_0$. Then $w_0 \in C_{0x} \cap \widetilde{C}_{0\tilde{x}}$ and we have by definition that $h_x(w_0) = \tilde{h}_{\tilde{x}}(w_0) = h(w_0)$.

It is easy to see that $\mathcal{D} \cup \widetilde{\mathcal{D}}$ contains a set of type $\{(t, x) : |t| \leq \bar{\delta}, r_0 \leq |x| \leq r_1\}$ for $\bar{\delta} > 0$. Let W be a neighborhood of the divisor E such that

(1) h is defined on $\overline{W} \backslash E$,

(2) $W \cap V = \{(t, x) \in V : |x| < r_0\}$,

(3) $h(\overline{W} \backslash E)$ does not intersect the set $\{(t', x') \in V' : |x'| \geq r_1'\}$.

Define the map $\bar{h}$ on $(W \cup \mathcal{D} \cup \widetilde{\mathcal{D}}) \backslash E$ as $\bar{h} = h$ on $W \backslash E$, $\bar{h} = f$ on $\mathcal{D}$ and $\bar{h} = \tilde{f}$ on $\widetilde{\mathcal{D}}$. It follows from (4.1) and conditions (1), (2), (3) above that $\bar{h}$ is a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ and maps the set $\{0 < |t| \leq \bar{\delta}, |x| = r_1\}$ into $\{|x'| = r_1'\}$ in V'. Moreover $\bar{h}$ maps the subsets $\{x = cte\}$ of $\{0 < |t| \leq \delta, |x| = r_1\}$ into the subsets $\{x' = cte\}$ of $\{|x'| = r_1'\}$. Finally, by a lifting path argument we finish the proof of Theorem 10. □

Lemma 11. *Let $f_t : \partial\mathbb{D} \mapsto \mathbb{C}$ be an injective map for all t and suppose that f_t depends continuously on t. Let U_t be the interior domain of $f_t(\partial\mathbb{D})$. Then there exists a continuous family of homeomorphisms $\bar{f}_t : \bar{\mathbb{D}} \mapsto \bar{U}_x$ extending f_t, that is, such that $\bar{f}_t = f_t$ on $\partial\mathbb{D}$ for all t.*

5. Homological compatibility

In this section we prove Theorem 12, which shows that some homological data is preserved by the equivalence h. This result has been previously obtained in the case of an orientation preserving homeomorphism in [15] (Theorem 6.2.1) using Theorem 3.16 of [14].

Theorem 12. *Let S and S' be irreducible analytic curves with isolated singularity at $0 \in \mathbb{C}^2$. Let $h : U \mapsto U'$ be a topological equivalence between S and S', that is, h is an orientation preserving homeomorphism such that $h(S \cap U) = S' \cap U'$, $h(0) = 0$. Let $\pi : M \mapsto \mathbb{C}^2$ and $\pi' : M' \mapsto \mathbb{C}^2$ be finite compositions of blow ups such that $k_\pi(S) = k_{\pi'}(S')$. Let $\widetilde{S}$ and $\widetilde{S}'$ be the strict transforms of S and S' by π and π' respectively. Let p and p' be the intersections of $\widetilde{S}$ and $\widetilde{S}'$ with its respective divisors and take holomorphic coordinates (t, x) and (t', x') at p and p' respectively such that*

(1) *$\widetilde{S}$ and $\widetilde{S}'$ are given by $\{t = 0\}$ and $\{t' = 0\}$,*

(2) *the exceptional divisors are given by $\{x = 0\}$ and $\{x' = 0\}$ respectively.*

Take $a, b, a', b' > 0$ such that

(1) *the set $\{|t| \leq a, 0 < |x| \leq b\}$ is contained in the domain of definition of h,*

(2) $h(\{(0,x) : 0 < |x| \le b\}) \subset \{(0,x') : 0 < |x'| < b'\}$,

(3) $h(\{|t| \le a, |x| = b\}) \subset \{|t'| < a', 0 < |x'| < b'\}$.

Let $t'_0, x'_0 \in \mathbb{C}$ *with* $0 < |t'_0| \le a'$ *and* $0 < |x'_0| \le b'$ *and define the paths* $\alpha, \beta \colon [0,1] \mapsto M$, $\alpha', \beta' \colon [0,1] \mapsto M'$ *by* $\alpha(s) = (ae^{2\pi is}, b)$, $\beta(s) = (a, be^{2\pi is})$, $\alpha'(s) = (t'_0 e^{2\pi is}, x'_0)$, $\beta'(s) = (t'_0, x'_0 e^{2\pi is})$. *Then, in the first homology group of* $T' = \{0 < |t'| \le a', 0 < |x'| \le b'\}$ *we have*

$$[h(\alpha)] = \xi[\alpha'] \quad \textit{and} \quad [h(\beta)] = \xi[\beta'],$$

where $\xi = 1$ *or* -1 *according to* h *preserves or reverses the natural orientations of* S *and* S'.

Proof. For some integers m, n we have

$$h(\beta) = m\beta' + n\alpha' \quad \text{in } H_1(T'). \tag{5.1}$$

Let β_0 and β'_0 be the paths defined by $\beta_0 = (0, be^{2\pi is})$, $\beta'_0 = (0, b'e^{2\pi is})$, $s \in [0,1]$. If $Q = \{|t| \le a, 0 < |x| \le b\}$ and $Q' = \{|t'| \le a', 0 < |x'| \le b'\}$ it is easy to see that $\beta = \beta_0$ in $H_1(Q)$, $\beta' = \beta'_0$ in $H_1(Q')$ and $h(\beta_0) = \xi\beta'_0$ in $H_1(Q')$. Then $h(\beta) = \xi\beta'$ in $H_1(Q')$. On the other hand it follows from equation (5.1) that $h(\beta) = m\beta'$ in $H_1(Q')$, hence $m = \xi$. Then we have

$$h(\beta) = \xi\beta' + n\alpha' \quad \text{in } H_1(T'). \tag{5.2}$$

Take neighborhoods W and W' of the divisors $E = \pi^{-1}(0)$ and $E' = \pi'^{-1}(0)$ respectively, with the following properties:

(1) W contains the set $\{|t| \le a, |x| \le b\}$,

(2) $W \cap \{t = 0\}$ is homeomorphic to a disc,

(3) $h(W \cap \{t = 0\}) \subset \{t' = 0\}$,

(4) $h(W \backslash E) = W' \backslash E$,

(5) $\pi(W)$ and $\pi'(W')$ are homeomorphic to balls.

Let $S_0 = \pi(W \cap \{t = 0\})$ and $S'_0 = \pi \circ h(W \cap \{t = 0\})$. Since $\pi(W)$ is homeomorphic to $\mathbb{C}^2$ and S_0 is closed in $\pi(W)$ and homeomorphic to $\mathbb{C}$, we have by Alexander's duality that $H_1(\pi(W) \backslash S_0) \simeq \mathbb{Z}$. Then, since $W_0 = W \backslash (E \cup \{t = 0\})$ is homeomorphic to $\pi(W) \backslash S_0$, we have $H_1(W_0) \simeq \mathbb{Z}$ and it is easy to see that α is a generator of this group. In the same way, if $W'_0 = W' \backslash (E' \cup \{t' = 0\})$ and we assume x'_0 small enough[3] we have that α' is a generator of the group $H_1(W'_0) \simeq \mathbb{Z}$. Since h preserves orientation it follows from the topological invariance of the intersection number (see [7] p. 200) that

$$h(\alpha) = \xi\alpha' \quad \text{in } H_1(W'_0). \tag{5.3}$$

[3] Without loss of generality we may suppose x'_0 arbitrarily small.

Then, if $\beta = k\alpha$ ($k \in \mathbb{Z}$) in $H_1(W_0)$, we obtain:

$$h(\beta) = k\xi\alpha' \quad \text{in } H_1(W_0'). \tag{5.4}$$

Since S and S' have isomorphic reductions and $k_\pi(S) = k_{\pi'}(S')$ we also have

$$\beta' = k\alpha' \quad \text{in } H_1(W_0'). \tag{5.5}$$

We may assume x_0' small such that β' and α' are contained in a set of type $T_\epsilon' = \{0 < |t'| \leq a', 0 < |x'| \leq \varepsilon\}$ with $T_\epsilon' \subset W_0'$. Then it is easy to see that we may write equation (5.2) in $H_1(T_\epsilon')$ and therefore in $H_1(W_0')$, that is,

$$h(\beta) = \xi\beta' + n\alpha' \quad \text{in } H_1(W_0').$$

Then, by using equations (5.4) and (5.5) we obtain $n = 0$. On the other hand, let

$$h(\alpha) = q\alpha' + r\beta' \quad \text{in } H_1(T')$$

with $q, r \in \mathbb{Z}$. Then, since α' is homologous to zero in Q' we obtain

$$h(\alpha) = r\beta' \quad \text{in } H_1(Q'). \tag{5.6}$$

Clearly α is homologous to zero in $\{|t| \leq a, |x| = b\}$ and hence, since $h(\{|t| \leq a, |x| = b\})$ is contained in Q', we deduce that $h(\alpha) = 0$ in $H_1(Q')$. It follows from equation (5.6) that $r = 0$ and thus $h(\alpha) = q\alpha'$ in $H_1(T')$. As before, we may write this equation in $H_1(W_0')$, that is, $h(\alpha) = q\alpha'$ in $H_1(W_0')$. Finally, it follows from equation (5.3) that $q = \xi$. □

6. Topological invariance of nodal separatrices

The following proposition allows us to divide the proof of Theorem 7 in two cases:

(1) The singularities p and p' are nodes with equal (positive irrational) eigenvalue.

(2) The singularities p and p' are non-nodal.

Proposition 13. *Under the conditions of Theorem 7, we have that p is a nodal singularity if and only p' is a nodal singularity. In this case the eigenvalues of p and p' are equal.*

Proof. Suppose that p has a real irrational positive eigenvalue. We know that in this case p is linearizable. Then the holonomy associated to $\widetilde{S}$ is linearizable. Let $q \in \widetilde{S}\backslash\{p\}$ and Σ a disc through q transverse to $\mathcal{F}$. Let $\gamma \subset \widetilde{S}\backslash\{p\}$ be a simply loop based on q and let $g\colon (\Sigma, q) \mapsto (\Sigma, q)$ its holonomy map. We know that $h(\Sigma)$ is a continuous disc transverse to $\mathcal{F}'$ through the point $q' = h(q)$. By a local deformation of h we may assume that $\Sigma' = h(\Sigma)$ is a complex disc transverse to $\mathcal{F}'$ and clearly $h\colon (\Sigma, q) \mapsto (\Sigma', q')$ is a topological conjugation between g and the holonomy $g'\colon (\Sigma', q') \mapsto (\Sigma', q')$ associated to the loop $h(\gamma)$ in $\widetilde{S}'\backslash\{p'\}$. But g is linearizable and this property is a topological invariant in $\mathrm{Diff}(\mathbb{C}, 0)$, then the holonomy associated to $\widetilde{S}'$ is also linearizable, hence the singularity p' is linearizable. Consider holomorphic coordinates (t, x) and (t', x') at p and p' respectively such that

(1) p and p' are generated by the holomorphic vector fields $\lambda t \frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ and $\lambda' t' \frac{\partial}{\partial t'} + x' \frac{\partial}{\partial x'}$ respectively,

(2) the exceptional divisors are given by $\{x = 0\}$ and $\{x' = 0\}$ respectively,

(3) $\widetilde{S}$ and $\widetilde{S}'$ are given by $\{t = 0\}$ and $\{t' = 0\}$ respectively.

By Theorem 10 we may assume that

(1) there are numbers $r, r', \delta, \delta' > 0$ such that the set $\{|t| < \delta, |x| = r\}$ is mapped by h into $\{|t'| < \delta', |x'| = r'\}$, and

(2) if $|z| = r$, the set $\{|t| < \delta, x = z\}$ is mapped by h into a set of type $\{x' = cte\}$.

Take $(a, b), (a', b') \in \mathbb{C}^2$ such that $|b| = r$, $|a| < \delta$ and $h(a, b) = (a', b')$. Define the paths $\beta_0(s) = (0, be^{2\pi i s}), s \in [0, 1]$ and $\beta_0'(s) = (0, b'e^{2\pi i s}), s \in [0, 1]$ in $\widetilde{S}$ and $\widetilde{S}'$ respectively. The holonomy maps associated to β_0 and β_0' computed in $\Sigma = \{(t, b) : |t| < \delta\}$ and $\Sigma' = \{(t', b') : |t'| < \delta'\}$ are given by $t \mapsto e^{2\pi i\lambda}t$ and $t' \mapsto e^{2\pi i\lambda'}t'$ respectively. Suppose first that h preserves the orientation of the leaves. Then $h(\beta_0)$ is homotopic to β_0' and therefore $h\colon \Sigma \mapsto \Sigma'$ is a topological conjugation between the maps $t \mapsto e^{2\pi i\lambda}t$ and $t' \mapsto e^{2\pi i\lambda'}t'$. Then, since $|e^{2\pi i\lambda}| = 1$ we have by the topological invariance of the rotation number that

$$e^{2\pi i\lambda'} = e^{2\pi i\lambda}. \tag{6.1}$$

Since the holonomy maps are irrational rotations, the orbits of the points (a, b) and (a', b') are dense in the circles $C = \{(t, b) : |t| = |a|\} \subset \Sigma$ and $C = \{(t', b') : |t'| = |a'|\} \subset \Sigma'$ respectively. Therefore h maps C onto C' and it is easy to prove that $h|_C : C \to C'$ is a given by

$$h|_C(t, b) = ((a'/a)t, b') \quad \text{for all } (t, b) \in C. \tag{6.2}$$

Let $\{\lambda\} = \lambda - [\lambda]$ and define the paths $\theta(s) = (ae^{2\pi i\{\lambda\}(1-s)}, b), s \in [0, 1]$ and $\theta'(s) = (a'e^{2\pi i\{\lambda'\}(1-s)}, b'), s \in [0, 1]$ in Σ and Σ' respectively. From (6.1) we have $\{\lambda\} = \{\lambda'\}$ and from (6.2) we obtain $h(\theta(s)) = \theta'(s)$ for all $s \in [0, 1]$. Define in M the path $\gamma(s) = (ae^{2\pi i\lambda s}, be^{2\pi i s})$, $s \in [0, 1]$. This path is a segment of orbit of the 1-foliation induced by $\mathcal{F}$ in $\{|t| < \delta, |x| = r\}$. The orbits of this foliation are mapped by h into orbits of the 1-foliation induced by $\mathcal{F}'$ in $\{|t'| < \delta', |x'| = r'\}$. It is easy to see that $h(\gamma)$ is a positive reparametrization of the path $\gamma'(s) = (a'e^{2\pi i\lambda' s}, b'e^{2\pi i s})$,

$s \in [0,1]$ in M'. It follows that $h(\gamma * \theta)$ is a positive reparametrization of $\gamma' * \theta'$ and therefore

$$h(\gamma * \theta) = \gamma' * \theta' \quad \text{in } H_1(T'), \tag{6.3}$$

where $T' = \{0 < |t'| \leq |a'|, 0 < |x'| \leq |b'|\}$. Define the paths $\alpha, \beta\colon [0,1] \mapsto M$, $\alpha', \beta'\colon [0,1] \mapsto M'$ by $\alpha(s) = (ae^{2\pi i s}, b)$, $\beta(s) = (a, be^{2\pi i s})$, $\alpha'(s) = (a'e^{2\pi i s}, b')$, $\beta'(s) = (a', b'e^{2\pi i s})$. If $T = \{0 < |t| \leq |a|, 0 < |x| \leq |b|\}$, it is easy to see that $\gamma * \theta = [\lambda]\alpha + \beta$ and $\gamma' * \theta' = [\lambda']\alpha' + \beta'$ in the groups $H_1(T)$ and $H_1(T')$ respectively. Then from equation (6.3) we obtain

$$h(\gamma * \theta) = [\lambda']\alpha' + \beta' \quad \text{in } H_1(T'). \tag{6.4}$$

On the other hand, it follows from Theorem 12 that $h(\gamma * \theta) = [\lambda]\alpha' + \beta'$ in $H_1(T')$, so equation (6.4) gives $[\lambda]\alpha' + \beta' = [\lambda']\alpha' + \beta'$. Thus $[\lambda] = [\lambda']$ and therefore $\lambda = \lambda'$. Suppose now that h reverses the orientation of the leaves. In this case $h\colon \Sigma \mapsto \Sigma'$ reverses orientation and is a topological conjugation between the holonomy map associated to β_0 and the inverse of the holonomy map associated to β_0'. Therefore $h(t,b) = ((a'/\bar{a})\bar{t}, b')$ for all $(t,b) \in C$ and we obtain as before that $\{\lambda\} = \{\lambda'\}$. By redefining now $\theta'(s) = (a'e^{-2\pi i\{\lambda'\}(1-s)}, b'), s \in [0,1]$ and $\theta'(s) = (a'e^{-2\pi i\{\lambda'\}(1-s)}, b'), s \in [0,1]$ we obtain again that $h(\gamma * \theta)$ is a positive reparametrization of $\gamma' * \theta'$ and we may also write equation (6.3). As before $\gamma * \theta = [\lambda]\alpha + \beta$ in $H_1(T)$ but in this case we have $\gamma' * \theta' = -[\lambda']\alpha' - \beta'$ in $H_1(T')$. It follows from Theorem 12 that $h(\gamma * \theta) = -[\lambda]\alpha' - \beta'$, so we have $-[\lambda]\alpha' - \beta' = -[\lambda']\alpha' - \beta'$ and we obtain again $\lambda = \lambda'$. □

7. Proof of Theorem 7 in the nodal case

This section is completely devoted to prove Theorem 7 when p and p' are nodal singularities. Since the proof is slightly too long, the proof contains a series of intermediary propositions (14 to 25). We also use some lemmas which are enounced at the end of the section.

Let λ be the eigenvalue of p and p'. There are coordinates (t,x) at p and (t',x') at p' such that the following holds:

(1) The foliations are locally generated by the vector fields $t\frac{\partial}{\partial t} + \lambda x \frac{\partial}{\partial x}$ and $t'\frac{\partial}{\partial t'} + \lambda x' \frac{\partial}{\partial x'}$ respectively.

(2) The exceptional divisors E and E' are given by $\{x = 0\}$ and $\{x' = 0\}$ respectively.

Let $\mathcal{B}$ and $\mathcal{B}'$ be closed balls in the coordinates (t,x) and (t',x') centered at p and p' respectively. Each leaf of $\mathcal{F}|_{\mathcal{B}}$ other than the separatrices $\{t = 0\}$ and $\{x = 0\}$ is dense in a 3-submanifold which separates the ball $\mathcal{B}$ in two connected components. Each of those connected components contains a separatrix. Let $\mathcal{B}_* = \{(t,x) \in \mathcal{B} :$

$x \neq 0\}$ and $\mathcal{B}'_* = \{(t', x') \in \mathcal{B}' : x' \neq 0\}$ and let $\mathcal{H}$ and $\mathcal{H}'$ be denote the space of leaves of $\mathcal{F}|_{\mathcal{B}_*}$ and $\mathcal{F}'|_{\mathcal{B}'_*}$ respectively.

Proposition 14. *Consider $L \in \mathcal{H}$ and assume that there is an open ball B centered at p such that $h(L \cap B)$ is contained in a leaf $L' \in \mathcal{H}'$. Then there is a ball B' centered at p' such that $h^{-1}(L' \cap B')$ is contained in L.*

Proof. Since the set $\partial B \cap \bar{L}$ is compact and disjoint of the divisor, we may take a ball B' centered at p' such that $h^{-1}(B')$ is disjoint of $\partial B \cap \bar{L}$. If w is contained in $L \cap B$ we have $h(w) \in L'$. Thus, if $w \to p$, then $h(w)$ tends to the divisor an we have necessarily that $h(w) \mapsto p'$. Therefore we may take $w \in L \cap B$ with $h(w) \in B'$. Consider any point $z \in L' \cap B'$. Let $C \subset L' \cap B'$ be a set connecting $h(w)$ to z. Since $h^{-1}(B')$ is disjoint of $\partial B \cap \bar{L}$, the set $h^{-1}(C)$ is contained in $\tilde{L} \backslash \partial B$, where $\tilde{L}$ is the leaf of $\mathcal{F}$ containing L. Observe that $L \cap B$ is a connected component of $\tilde{L} \backslash \partial B$. Then, since the connected set $h^{-1}(C) \subset \tilde{L} \backslash \partial B$ contains the point $w \in L \cap B$ we have $h^{-1}(C) \subset L \cap B$, hence $h^{-1}(z) \in L$. □

Define $\mathcal{A}$ as the set of the leaves $L \in \mathcal{H}$ for which there is an open ball B centered at p such that $h(L \cap B)$ is contained in a leaf $L' \in \mathcal{H}'$ denoted by $h_*(L)$. By Proposition 14, $h_*(\mathcal{A})$ is contained in the set $\mathcal{A}'$ of the leaves $L' \in \mathcal{H}'$ for which there is an open ball B' centered at p' such that $h^{-1}(L' \cap B')$ is contained in a leaf in $\mathcal{H}$. By applying the proposition in the other direction we conclude that $h_*(\mathcal{A}) = \mathcal{A}'$ and h_* is a bijection. Clearly $\mathcal{A}$ and $\mathcal{A}'$ are non-empty since they contain the separatrices $\{t = 0\}$ and $\{t' = 0\}$ respectively.

Proposition 15. *If $L \in \mathcal{H}$ is close to the separatrix $\{t = 0\}$ then $L \in \mathcal{A}$.*

Proof. Denote $\{t = 0\}$ and $\{t' = 0\}$ by S and S' respectively. Let B' be a ball centered at p' with $\bar{B}' \subset \mathcal{B}'$ and take a ball B centered at p such that $h(B)$[4] does not meet some neighborhood Ω of the compact set $S' \cap \partial B'$. Fix $z_0 \in S \cap B$ with $h(z_0) \in B'$ and assume z close enough to z_0 such that $h(z) \in B'$. Let $L_z \in \mathcal{H}$ and $L'_z \in \mathcal{H}'$ be the leaves through z and $h(z)$ respectively. By assuming $h(z)$ close enough to $h(z_0) \in S'$ we have that $\overline{L'}_z \cap \partial B' \subset \Omega$. Then $h(L_z \cap B) \subset h(B)$ is disjoint of $\overline{L'}_z \cap \partial B'$ and we have $h(L_z \cap B) \subset F' \backslash \partial B'$, where F' is the leaf of $\mathcal{F}'$ through $h(z)$. Observe that $L'_z \cap B'$ is a connected component of $F' \backslash \partial B'$. Then $h(L_z \cap B)$ is connected and intersects (at least in $h(z)$) the connected component $L'_z \cap B'$ of $F' \backslash \partial B'$, hence $h(L_z \cap B) \subset L'_z \cap B'$ and therefore $L_z \in \mathcal{A}$. □

Proposition 16. *If $L \in \mathcal{A}$, there is an open ball B such that $h(\bar{L} \cap B)$ is contained in $\overline{h_*(L)}$. Therefore any leaf contained in $\bar{L}$ is an element of $\mathcal{A}$.*

[4]We denote by $h(A)$ the set $h(A \cap \operatorname{dom}(h))$.

Proof. Let B be an open ball centered at p such that $h(L \cap B)$ is contained in a leaf $L' \in \mathcal{H}'$. Let $z \in \bar{L} \cap B$ and $z_n \in L$ with $z_n \to z$. Since B is open we may assume that $z_n \in B$ for all $n \in \mathbb{N}$. Then $h(z_n) \in L'$ for all $n \in \mathbb{N}$ and we have $h(z) = \lim h(z_n) \in \bar{L}'$. Thus, if $L_1 \in \mathcal{H}$ is contained in $\bar{L} \backslash \{p\}$ we have $h(L_1 \cap B) \subset h(\bar{L} \cap B) \subset \bar{L}' \subset T'$ and, since $h(L_1)$ is a connected subset of a leaf of $\mathcal{F}$, we conclude that $h(L_1)$ is contained in a leaf in $\mathcal{H}'$. □

Proposition 17. *Let $L \in \mathcal{A}$ and $L' = h_*(L)$. There is a ball B centered at p such that the connected component of $B \backslash \bar{L}$ intersecting $\{t = 0\}$ is contained in the connected component of $\mathcal{B}' \backslash \bar{L}'$ intersecting $\{t' = 0\}$.*

Proof. As in Proposition 16 we may find a ball B' centered at p such that

$$h^{-1}(B' \cap \bar{L}') \subset \bar{L}. \tag{7.1}$$

Let V' be the connected component of $\mathcal{B}' \backslash \bar{L}'$ intersecting $\{t' = 0\}$. Take a neighborhood W' of the divisor E' such that

(1) $W' \cap \bar{L}' \subset B'$,

(2) if Ω is the connected component of $W' \backslash \bar{L}'$ intersecting $\{t' = 0\}$, then $\Omega \subset V'$.

It follows from (7.1) and (1) above that

$$h^{-1}(W' \cap \bar{L}') \subset \bar{L}. \tag{7.2}$$

Let B be a ball centered at p such that $h(B) \subset W'$. Let V be the connected component of $B \backslash \bar{L}$ intersecting $\{t = 0\}$. Then $h(V) \subset W$ is connected and it follows from (7.2) that $h(V) \subset W' \backslash \bar{L}'$. Then, since $h(V)$ is connected and intersects $\{t' = 0\}$, we have m $h(V) \subset \Omega$. Thus, it follows from (2) that $h(V) \subset V'$. □

If $F, L \in \mathcal{H}$ are not separatrices, we will write $F > L$ or $L < F$ to means that F and the separatrix $\{t = 0\}$ are contained in the same connected component of $\mathcal{B} \backslash \bar{L}$.

Proposition 18. *If $F > L$ and $L \in \mathcal{A}$, then $F \in \mathcal{A}$ and $h_*(F) > h_*(L)$.*

Proof. Let B a ball centered at p given by Proposition 17 and let V and V' be as in the proof of this proposition. Since $F > L$, then $F \cap B \subset V$ and by Proposition 17 we have $h(F \cap B) \subset V' \subset \mathcal{B}'$. It is easy to see that this implies $F \in \mathcal{A}$ and $h_*(F) > h_*(L)$. □

Proposition 19. *At least one of the equalities $\mathcal{A} = \mathcal{H}$ or $\mathcal{A}' = \mathcal{H}'$ holds.*

Proof. Assume by contradiction that $\mathcal{A} \neq \mathcal{H}$ and $\mathcal{A}' \neq \mathcal{H}'$. As a first step we will prove that there exists $L \in \mathcal{H}$ (not a separatrix) such that

$$F > L \Rightarrow F \in \mathcal{A} \quad \text{and} \quad F < L \Rightarrow F \notin \mathcal{A}. \tag{7.3}$$

The closure of a leaf $L \in \mathcal{H}$ is contained in a set of type $|x|/|t|^{\lambda} = r \in (0, +\infty]$. We denote $r = r(L)$. It is easy to see that $F > L$ is equivalent to $r(F) > r(L)$. Then $(r(F) > r(L), L \in \mathcal{A})$ implies $F \in \mathcal{A}$ and therefore we deduce that $r(\mathcal{A}) \subset (0, +\infty]$ is an interval. Since $\mathcal{A} \neq \mathcal{H}$ we see that $\rho := \inf(r(\mathcal{A})) > 0$. Now, if we take $L \in \mathcal{H}$ such that $r(L) = \rho$ it is easy to see that (7.3) holds.

Now we continue with the proof of Proposition 19. Suppose first that $L \notin \mathcal{A}$ and take $L' \in \mathcal{H}'\backslash\mathcal{A}'$. Let B' be a ball centered at p' with $\bar{B}' \subset \mathcal{B}'$. Since L is not the separatrix $\{t = 0\}$, there is a neighborhood W' of the divisor E' such that

(1) $W' \cap \bar{L}' \subset B'$,

(2) if $V'$is the connected component of $W'\backslash\bar{L}'$ intersecting $\{t' = 0\}$, then $V' \subset B'$.

Let B be a ball centered at p such that $h(\bar{B}) \subset W'$ and let V be the connected component of $B\backslash\bar{L}$ intersecting $\{t = 0\}$. Let $z \in V$ and $F \in \mathcal{H}$ be the leaf containing z. Clearly F is contained in the connected component of $\mathcal{B}\backslash\bar{L}$ intersecting $\{t = 0\}$, hence $F > L$ and therefore $F \in \mathcal{A}$, by (7.3). Then $h_*(F) \in \mathcal{A}'$ and we have $h_*(F) \cap \bar{L}' = \emptyset$, otherwise $L' \subset \overline{h_*(F)}$ and Proposition 16 implies $L' \in \mathcal{A}'$, which is a contradiction. Therefore $z \notin \bar{L}'$ and it follows that $h(V) \cap \bar{L}' = \emptyset$, that is $h(V) \subset W'\backslash\bar{L}'$. Therefore, since $h(V)$ is connected and intersects $\{t' = 0\}$, we deduce from (2) that $h(V) \subset V' \subset B'$. Thus, since $L \cap B \subset \bar{V}$, we have $h(L \cap B) \subset h(\bar{V}) \subset \overline{h(V)} \subset \bar{B}' \subset \mathcal{B}'$, hence $L \in \mathcal{A}$, which is a contradiction. Suppose now that $L \in \mathcal{A}$ and let $L' = h_*(L)$. Let B' a ball centered at p' with $\bar{B}' \subset \mathcal{B}'$ and take a ball B centered at p such that $h(L \cap B) \subset L' \cap B'$. Since $\partial B' \cap \bar{L}'$ is compact and far of the divisor, we may assume B small enough such that $h(B)$ is disjoint of a neighborhood Ω of $\bar{L}' \cap \partial B'$. Choose a point $z_0 \in L \cap B$. Thus, since $h(L \cap B) \subset L' \cap B'$, we have $h(z_0) \in L' \cap B'$. It is easy to see that we may find a point z arbitrarily close to z_0 such that the leaf $F \in \mathcal{H}$ through z satisfies $F < L$ and therefore $F \notin \mathcal{A}$. Since $h(z_0) \in B'$ we may assume $h(z) \in B'$. Let $F' \in \mathcal{H}'$ be the leaf through $h(z)$. Again by taking $h(z)$ close enough to $h(z_0) \in L'$ we may also assume that and $\bar{F}' \cap \partial B' \subset \Omega$. Let $\tilde{F}'$ be the leaf of $\mathcal{F}'$ containing F' and observe that $F' \cap B'$ is a connected component of $\tilde{F}'\backslash\partial B'$. Since $h(F \cap B) \subset h(B)$ and $\bar{F}' \cap \partial B' \subset \Omega$ we have that $h(F \cap B)$ is disjoint of $\bar{F}' \cap \partial B'$. Then $h(F \cap B) \subset \tilde{F}'\backslash\partial B'$. Thus, since $h(F \cap B)$ is connected and intersect (at least in $h(z)$) the connected component $F' \cap B'$ of $\tilde{F}'\backslash\partial B'$, we deduce that $h(F \cap B) \subset F' \cap B'$. But this means that $F \in \mathcal{A}$, which is a contradiction. □

Given $L'_0 \in \mathcal{H}'$ we will find a neighborhood $W' = W'(L'_0)$ of the divisor E' with the following property:

$$\text{If } L' > L'_0 \text{ and } F' \text{ is a leaf of } \mathcal{F}'|_{W'} \text{ intersecting } L', \text{ then } F' \subset L'. \tag{7.4}$$

Suppose first W' is any neighborhood of E' and let F' be a leaf of $\mathcal{F}|'_{W'}$ intersecting $L' > L'_0$. If F' is not contained in L', then F' intersects the boundary $\partial L' = L' \cap \partial\mathcal{B}$

of L'. But it is easy to see that the union of the sets $\{\partial L'\}_{L'>L'_0}$ is contained in a compact set K disjoint of the divisor E'. Then it suffices to take W' disjoint of K.

Proposition 20. *Let B be a ball centered at p and W a neighborhood of the divisor E such that $B \subset W$ and $h(W) \subset W'$. Given $z \in B \backslash E$, let $L_z \in \mathcal{A}$ be the leaf through z and let F_z be the leaf of $\mathcal{F}|_W$ containing $L_z \cap B$. Then, if $h_*(L_z) > L'_0$, we have $h(F_z) \subset h_*(L_z)$.*

Proof. It follows from the definition of $h(F_z)$ that $h(F_z) \cap h_*(L_z) \neq \emptyset$. Then, since $h_*(L_z) > L'_0$, the property 7.4 implies that the leaf F'_z of $\mathcal{F}'|_{W'}$ containing $h(F_z)$ is a subset of $h_*(L_z)$. Therefore $h(F_z) \subset h_*(L_z)$. □

Now, by global considerations we prove the following.

Proposition 21. *Both equalities $\mathcal{A} = \mathcal{H}$ and $\mathcal{A}' = \mathcal{H}'$ hold. Thus h_* is a bijection between $\mathcal{H}$ and $\mathcal{H}'$.*

Proof. By Proposition 19 we may assume that $\mathcal{A} = \mathcal{H}$. Suppose by contradiction that $\mathcal{A}' \neq \mathcal{H}'$. Fix $L'_0 \in \mathcal{H}' \backslash \mathcal{A}'$ and let W', W, B, L_z and F_z as in Proposition 20.

Claim 22. For all $z \in B \backslash E$ the set $\overline{F_z}$ intersects the divisor only at p.

Let $z \in B \backslash E$. Since $h_*(L_z) \in \mathcal{A}$ and $L'_0 \notin \mathcal{A}'$ we deduce from Propositions 16 and 18 that $h_*(L_z) > L'_0$. Then Proposition 20 implies that $h(F_z) \subset h_*(L_z)$. Now, suppose that $w_n \in F_z$ tends to the divisor as $n \to \infty$. Then $h(w_n) \in h_*(L_z)$ tends to the divisor and therefore $h(w_n)$ tends to p'. Since $h_*(L_z) \in \mathcal{A}'$, if $h(w_n)$ is close enough to p' we have necessarily $w_n \in L_z$, hence w_n tends to p. Thus Claim 22 is proved.

By a suitable finite composition of blow ups we construct a map $\tilde{\pi}\colon \widetilde{M} \to M$ such that the strict transform of $\mathcal{F}$ by $\tilde{\pi}$ has only reduced singularities. Since p is yet a reduced singularity we may assume that $\tilde{\pi}$ does not involve any blow up at p. Thus we may locally identify the spaces $\widetilde{M}$ and M at the points $\tilde{\pi}^{-1}(p) \simeq p$. Let $\widetilde{\mathcal{F}}$ denote the strict transform of $\mathcal{F}$ restricted to the set $\widetilde{W} = \tilde{\pi}^{-1}(W)$. For all $z \in B \backslash E$ the leaf F_z of $\mathcal{F}|_W$ defines a leaf $\widetilde{F}_z$ of $\widetilde{\mathcal{F}}$. Let $\widetilde{E} = \tilde{\pi}^{-1}(E)$ and $D \subset \widetilde{E}$ be the projective line containing p.

Claim 23. Any singularity $q \neq p$ of $\widetilde{\mathcal{F}}$ in D has a real negative eigenvalue.[5]

Let (t, x) be holomorphic coordinates at q and $a, b > 0$ such that

(1) $q \simeq (0, 0)$ and D is given by $\{x = 0\}$,

[5]This a consequence of the contradiction hypothesis $\mathcal{A}' \neq \mathcal{H}'$.

(2) the set $T = \{|t| \leq a, |x| \leq b\}$ is contained in $\widetilde{W}$ and q is the unique singularity in T,

(3) any point in $R = \{|t| = a, 0 < |x| < b\}$ belongs to $\widetilde{F}_z$ for some $z \in B \backslash E$.

Let w be any point in R, let L be the leaf of $\widetilde{\mathcal{F}}|_T$ through w, and $z \in B$ such that $\widetilde{F}_z$ contains w. We have by (2) that $L \subset \widetilde{F}_z$. Then $\bar{L} \subset \bar{\widetilde{F}}_z$ and, since $p \notin T$, it follows from Claim 22 that $\bar{L} \cap D = \emptyset$. Thus Claim 23 is a direct consequence of Lemma 28 below.

Suppose that D_1 is a projective line in $\widetilde{E}$ intersecting D. Observe that the union of the $\widetilde{F}_z$ contains a neighborhood of any regular point in D. Then, since by Claim 23 the singularity at $D \cap D_1$ has a real negative eigenvalue, there is a neighborhood U of this singularity such that

$$U \backslash (D \cup D_1) \subset \bigcup_{z \in B \backslash E} \widetilde{F}_z.$$

Let $\Sigma_1 \subset U \backslash D$ be a disc transverse to D_1. Then, if $q_1 \neq q$ is a singularity in D_1, there are coordinates at q_1 satisfying the conditions (1), (2) and (3) in the proof of Claim 23 with q_1 and D_1 instead of q and D. Thus we may prove that all singularities in D_1 have eigenvalue in $\mathbb{R}_{<0}$. If we continue with this argument along the divisor $\widetilde{E}$ we conclude that all the singularities of $\widetilde{\mathcal{F}}$ other than p have eigenvalue in $\mathbb{R}_{<0}$.

Let $\widetilde{S} \subset \widetilde{M}$ be the strict transform of the union of the separatrices of $\mathcal{F}$ in $(\mathbb{C}^2, 0)$. Since all singularity other than p has eigenvalue in $\mathbb{R}_{<0}$, there exists a neighborhood $\widetilde{\Omega}$ of $\widetilde{E}$ such that the union of the $\widetilde{F}_z$ contains the set $\widetilde{\Omega} \backslash (\widetilde{E} \cup \widetilde{S})$. Then

$$\widetilde{S} \cup \bigcup_{z \in B \backslash E} \widetilde{F}_z$$

contains the set $\widetilde{\Omega} \backslash \widetilde{E}$ and therefore

$$G = h(\widetilde{S}) \cup h\Big(\bigcup \widetilde{F}_z\Big) \cup E'$$

is a neighborhood of the divisor E'. But this is a contradiction because it follows from Proposition 20 that $h(\bigcup \widetilde{F}_z)$ is contained in $\bigcup h_*(L_z) \subset \mathcal{B}'$ and clearly $h(\widetilde{S}) \cup \mathcal{B}' \cup E'$ is not a neighborhood of E'. Proposition 21 is proved. □

At this point we have a correspondence between the leaves in $\mathcal{H}$ with the leaves in $\mathcal{H}'$. Moreover, given corresponding leaves $L \in \mathcal{H}$ and $L' \in \mathcal{H}'$ we have $h(L \cap B) \subset L'$ for a small enough ball B centered at p. Let F and F' be the leaves of $\mathcal{F}$ and $\mathcal{F}'$ containing L and L' respectively. The map $h|_F$ maps the pair $(F, L \cap B)$ onto the pair $(F', h(L \cap B))$. From the topological structure of nodal singularities and using the fact $h(L \cap B) \subset L'$ we can prove that the pairs $(F, L \cap B)$ and $(F', h(L \cap B))$ are homeomorphic to (F, L) and (F', L'). This allows us to construct a new homeomorphism $h_F : F \mapsto F'$ mapping (F, L) onto (F', L'). In the remainder of the proof we construct the maps h_F depending continuously on F and such that $h_F = h$ outside $\mathcal{B}$. We make this construction in such way the total homeomorphisms obtained extends to the divisor in a neighborhood of the nodal singularity.

Naturally we may assume that the sets $\{|t| \le 1, |x| \le 1\}$ and $\{|t'| \le 1, |x'| \le 1\}$ are contained in the balls $\mathcal{B}$ and $\mathcal{B}'$ respectively. Take $b \in (0,1)$ and consider $w = (1, b) \in \mathcal{B}$. Let $L_w \in \mathcal{H}$ be the leaf through w. If b is taken small enough, $\overline{h_*(L_w)}$ intersects $\{|t'| = 1, |x'| \le 1\}$ in a set of type $\{|t'| = 1, |x'| = b'\}$ for some $b' \in (0,1)$. Set $T = \{0 < |t| \le 1, 0 < |x| \le b\}, T' = \{0 < |t'| \le 1, 0 < |x'| \le b'\}$, $R = \{|t| = 1, |x| \le b\}$, $R' = \{|t'| = 1, |x'| \le b'\}$, $R_* = \{|t| = 1, 0 < |x| \le b\}$ and $R'_* = \{|t'| = 1, 0 < |x'| \le b'\}$.

Proposition 24. *There exists a homeomorphism onto its image $f : R_* \mapsto T$ such that the following holds:*

(1) *If $z \in L \in \mathcal{H}$, then $f(z) \in L$ and $h(f(z)) \in h_*(L)$.*

(2) $h(f(R_*)) \subset T'$.

Proof. Consider the real flow (tangent to the foliation) $\phi^s(t,x) = (te^{-s}, xe^{-\lambda s})$. Given $z = (t,x) \in R_*$, we have $\phi^s(z) \to p$ as $s \to +\infty$ and clearly $\phi^s(z)$, $s \ge 0$ is contained in a leaf $L \in \mathcal{H}$. By Proposition 21 we have $L \in \mathcal{A}$ and therefore for s big enough we have that $h(\phi^s(z))$ is contained in a leaf $L' \in \mathcal{H}'$. Since $h(\phi^s(z))$ tends to the divisor and $\overline{L}'$ meets the divisor only at p we deduce that $h(\phi^s(z)) \to p'$ when $s \to +\infty$. Then we may define

$$\tau_0(z) = \inf\{\tau \ge 0 : h(\phi^s(z)) \in T' \text{ for all } s > \tau\}.$$

Let us prove that $\tau_0 : R_* \mapsto [0, +\infty]$ is upper semi-continuous. Suppose on the contrary that there is a sequence $(z_n)_{n\in\mathbb{N}}$ of points in R_* with $z_n \to z \in R_*$ and such that $\tau_0(z_n) \ge \tau_0(z) + 2\varepsilon$ for some $\varepsilon > 0$. Then for all $n \in \mathbb{N}$ we find $s_n > \tau_0(z) + \varepsilon$ such that $h(\phi^{s_n}(z_n)) \notin T'$. Suppose first that $\{s_n\}$ is bounded. Then by passing to a subsequence if necessary we may assume $s_n \to s \ge \tau_0(z) + \varepsilon$, so that $h(\phi^{s_n}(z_n)) \to h(\phi^s(z))$, but this is a contradiction because $h(\phi^{s_n}(z_n)) \notin T'$ for all $n \in \mathbb{N}$ and $s > \tau_0(z)$ implies $h(\phi^s(z)) \in T'$. Otherwise, again by passing a subsequence we may suppose $s_n \to +\infty$. Then $\phi^{s_n}(z_n) \to p$ and therefore $h(\phi^{s_n}(z_n))$ tends to the divisor. Let $L_n \in \mathcal{H}$ be the leaf through z_n. Since $z_n \to z$ there is $L \in \mathcal{H}$ such that $L_n > L$ for all $n \in \mathbb{N}$. Let V and V' be as in the proof of Proposition 17. Thus, for n big enough we have $\phi^{s_n}(z_n) \in V$ and, by Proposition 17, $h(\phi^{s_n}(z_n)) \in V'$. Then, since $h(\phi^{s_n}(z_n))$ tends to the divisor, we conclude that $h(\phi^{s_n}(z_n)) \to p'$, a contradiction since $h(\phi^{s_n}(z_n)) \notin T'$ for all $n \in \mathbb{N}$. Now, by Lemma 29 below there exists a continuous function $\tau : R_* \mapsto \mathbb{R}^+$ such that $\tau > \tau_0$. Then $h(\phi^{\tau(z)}(z)) \in T$ for all $z \in R_*$ and we finally define $f(z) = \phi^{\tau(z)}(z)$. □

Proposition 25. *There exists a homeomorphism* $H\colon R \mapsto R'$ *with the following properties:*

(1) $H(\{(t,x)\in R : t = u\}) = \{(t',x')\in R' : t' = u\}$ *for all* u.

(2) *If* $z \in L \in \mathcal{H}$, *then* $H(z) \in h_*(L)$.

Proof. Let $\Sigma_* = \{(1,x) : 0 < |x| \le b\}$ and $\Sigma'_* = \{(1,x') : 0 < |x'| \le b'\}$. By Proposition 24 the set $\mathcal{D} = h \circ f(\Sigma_*)$ is contained in T'. Fix $\bar{z} \in \mathcal{D}$ and $\gamma_{\bar{z}} \subset T'$ a path in the leaf through $\bar{z}$ with $\gamma_{\bar{z}}(0) = \bar{z}$ and $\gamma_{\bar{z}}(1) \in \Sigma'_*$. Given $z \in \mathcal{D}$, choose a path $\alpha_z \subset \mathcal{D}$ joining z with $\bar{z}$. Denote by π the projection $(t',x') \mapsto t'$ and let $\gamma_z\colon [0,1] \mapsto T'$ be the path in the leaf through z which is the lifting by the fibration $t' = cte$ of the curve $\pi(\alpha_z * \gamma_{\bar{z}})$. Then $\gamma_z(1)$ is a point in Σ'_*. Suppose that $\tilde{\alpha}_z \subset \mathcal{D}$ is another path joining z with z_0. Then $\tilde{\alpha}_z * \alpha_z^{-1} \subset \mathcal{D}$ is the image by h of a closed path θ in $f(\Sigma_*)$. Since $f(\Sigma_*)$ is homotopic to Σ_* in T we have that θ does not link the separatrix $\{t = 0\}$. Thus, it follows from Theorem 12 that $\tilde{\alpha}_z * \alpha_z^{-1}$ does not link $\{t' = 0\}$. Then the paths $\pi(\alpha_z * \gamma_{\bar{z}})$ and $\pi(\tilde{\alpha}_z * \gamma_{\bar{z}})$ are homotopic in $\{(t',0) : t' \neq 0\}$ and therefore the point $\gamma_z(1) \in \Sigma'_*$ does not depend on the path α_z. Thus $g(z) = \gamma_z(1)$ defines a map $g\colon \mathcal{D} \mapsto \Sigma'_*$. It is not difficult to prove that g is injective[6]. Define $H\colon \Sigma_* \mapsto \Sigma'_*$ by $H = g \circ h \circ f$. Then H is injective and it follows from Proposition 24 that $H(w) \in h_*(L) \in \mathcal{H}'$ whenever $w \in L \in \mathcal{H}$. Let $w \in \Sigma$ and $L_w \in \mathcal{H}$ the leaf through w. If w is close to $(1,0) \in \Sigma$, then L_w is close to $\{x = 0\}$. In this case, we know that $h_*(L_w)$ is close to $\{x' = 0\}$. Therefore, since $H(w) \in h_*(L_w)$, we have that $H(w) \to (1,0) \in \Sigma'$ as $w \to (1,0) \in \Sigma$. Then by setting $H(1,0) = (1,0)$ we extend H as a homeomorphism of $\Sigma = \{(1,x) : |x| \le b\}$ onto its image in $\Sigma' = \{(1,x') : |x'| \le b'\}$. Let $r\colon \Sigma \mapsto \Sigma$ and $r'\colon \Sigma' \mapsto \Sigma'$ be the holonomy maps associated to positively oriented circles around $(0,0)$ in $\{x = 0\}$ and $\{x' = 0\}$ respectively. Let us prove that H conjugates the maps r and r'^{ξ}, where $\xi = 1$ or -1 according to h preserves or reverses the orientation of the leaves. Let $w \in \Sigma$ and $\theta \subset R$ be the path in the leaf through w joining it with $r(w)$. Take any path $\eta \subset \Sigma$ joining $r(w)$ with w. Let α and α' as in Theorem 12. Then $\theta * \eta$ is homologous to α in $\{t \neq 0\}$ and therefore $f(\theta * \eta)$ is homologous to α in $\{t \neq 0\}$. Suppose first that h preserves the orientation of the leaves. Then by Theorem 12 we have that $h \circ f(\theta * \eta)$ is homologous to α' in $\{t' \neq 0\}$. Parametrize the path $h \circ f(\eta) \subset \mathcal{D}$ by z_t, $t \in [0,1]$, $z_0 = h \circ f(r(w))$, $z_1 = h \circ f(w)$. For all $t \in [0,1]$ we may construct the path γ_{z_t} as above, depending continuously on $t \in [0,1]$. The path γ_{z_t} is contained in a leaf and $\gamma_{z_t}(1) = g(z_t)$. The map $G\colon [0,1] \times [0,1] \mapsto T'$ defined by $G(t,s) = \gamma_{z_t}(s)$ is continuous and maps the boundary of the square onto

$$h \circ f(\eta) * \gamma_{z_1} * (g \circ h \circ f(\eta))^{-1} * \gamma_{z_0}^{-1}.$$

Then this path is homotopically trivial in $\{t' \neq 0\}$, so that $\gamma_{z_0} * h \circ f(\theta * \eta)\gamma_{z_0}^{-1}$

[6] We make a complete proof in a similar situation in Subsection 10.1.

is homologous to

$$\vartheta = \gamma_{z_0} * h \circ f(\theta) * \gamma_{z_1}^{-1} * g \circ h \circ f(\eta)$$

in $\{t' \neq 0\}$. Then ϑ is homologous to α' in $\{t' \neq 0\}$. Observe that ϑ has the part $\gamma_{z_0} * h \circ f(\theta) * \gamma_{z_1}^{-1}$ contained in a leaf and the part $g \circ h \circ f(\eta)$ contained in Σ'. Then, since path $\gamma_{z_0} * h \circ f(\theta) * \gamma_{z_1}^{-1}$ joins the point $H(w)$ with $H(r(w))$, we conclude that $H(r(w)) = r' \circ H(w)$. If h reverses the orientation of the leaves the proof follows as above but in this case we have that ϑ is homologous to $-\alpha'$ in $\{t' \neq 0\}$, so that $H(r(w)) = r'^{-1} \circ H(w)$.

Recall that $w \in \{(1,x) : |x| = b\}$ implies that $h_*(L_w) \cap \Sigma'$ is contained in $\{(1,x') : |x'| = b'\}$. Then $H(\overline{\Sigma})$ intersects $\{(1,x') : |x'| = b'\}$ and, since $H(\overline{\Sigma})$ is invariant by the irrational rotation r', we deduce that $H(\overline{\Sigma}) = \overline{\Sigma}'$. Now, since the 1-foliations induced in R and R' are suspensions of r and r' respectively, it is easy to extend H to a homeomorphism $H\colon R \mapsto R'$ satisfying the assertions of the proposition. □

Define the function $g\colon R \mapsto T$ by $g(t,x) = \phi^1(t,x) = (te^{-1}, xe^{-\lambda})$. This map is a homeomorphism between R and $\widetilde{R} = \{(t,x) : |t| = e^{-1}, |x| \leq be^{-\lambda}\}$.

Lemma 26. *Let $f, g\colon R_* \to T$ be homeomorphisms onto is image. Suppose that $f(z)$ and $g(z)$ are contained in the leaf trough z for all $z \in R_*$. Let V_f and V_g be the closures in T of the connected components of $\mathcal{T} \backslash f(R_*)$ and $\mathcal{T} \backslash g(R_*)$ containing R_*, where $\mathcal{T}$ is the union of leaves $L \in \mathcal{H}$ meeting R_*. Then there exists a leaf preserving homeomorphism $\Phi\colon V_f \mapsto V_g$ such that $\Phi|_{R_*} = id$ and $\Phi(f(z)) = g(z)$ for all $z \in R_*$.*

Proof. Given $z \in R_*$, let L_z^f and L_z^g be the leaves of $\mathcal{F}|_{V_f}$ and $\mathcal{F}|_{V_g}$ through z. The interiors of L_z^f and L_z^g are conformally equivalent to the unit disc and we may consider the Poincaré metric on L_z^f and L_z^g. Let $\gamma_z^f : \mathbb{R} \mapsto L_z^f$ be the geodesic in L_z^f with $\gamma_z^f(-\infty) = z$ and $\gamma_z^f(+\infty) = f(z)$ and set $I_z^f = \gamma_z^f(\mathbb{R}_{\pm\infty})$. Define analogously $\gamma_z^g : \mathbb{R} \mapsto L_z^g$ and I_z^g. Let $\Phi_z : I_z^f \mapsto I_z^g$ be the homeomorphism such that $\Phi_z(\gamma_z^f(s)) = \gamma_z^g(s)$ for all $s \in \mathbb{R}$. Define $\Phi\colon V_f \mapsto V_g$ by $\Phi|_{I_z^f} = \Phi_z$ for all $z \in R_*$. It is not difficult to see that Φ is a leaf preserving homeomorphism. □

If f is given by Proposition 24 and g is the map defined above, Lemma 26 gives us a leaf preserving homeomorphism $\Phi\colon V_f \mapsto V_g$ such that $\Phi|_{R_*} = \mathrm{id}$ and $\Phi(f(z)) = g(z)$ for all $z \in R_*$. Take a neighborhood W of the divisor E containing $\{|t| \leq 1, |x| \leq b\}$ and set $W_* = W \backslash (\{|t| \leq 1\} \cup E)$, $W_f = W_* \cup V_f$ and $W_g = W_* \cup V_g$. Since $\Phi|_{R_*} = \mathrm{id}$ we may continuously extend Φ to W_f by setting $\Phi|_{W_*} = \mathrm{id}$. Then $\Phi\colon W_f \mapsto W_g$ is a leaf preserving homeomorphism. Define $f'\colon R'_* \mapsto T'$ by $f' = h \circ f \circ H^{-1}$ and $g'\colon R' \mapsto T'$ by $g'(t',x') = \phi^1(t',x') = (t'e^{-1}, x'e^{-\lambda})$. By

Proposition 24 we may apply Lemma 26 to f' and g' to obtain a homeomorphism $\Phi'\colon V_{f'} \mapsto V_{g'}$ such that $\Phi'|_{R'_*} = \mathrm{id}$ and $\Phi'(f'(z')) = g'(z')$ for all $z' \in R'_*$. Set $W' = h(W)$, $W'_* = W' \backslash \{|t'| \le 1\}$, $W'_{f'} = W'_* \cup V_{f'}$ and $W'_{g'} = W'_* \cup V_{g'}$ and extend Φ' to a leaf preserving homeomorphism $\Phi'\colon W'_{f'} \mapsto W'_{g'}$. Then it is easy to see that the homeomorphism $\bar{h} = \Phi' \circ h \circ \Phi^{-1}$ is a topological equivalence between $\mathcal{F}|_{W_g}$ and $\mathcal{F}'|_{W'_{g'}}$. Set $\widetilde{R}_* = g(R_*) = \{(t,x) : |t| = e^{-1}, 0 < |x| \le be^{-\lambda}\}$ and observe that $\bar{h}|_{\widetilde{R}_*} = g' \circ H \circ g^{-1}$. Then $\bar{h}$ extends to $\widetilde{R}$ and maps this set homeomorphically onto $\widetilde{R}' = \{(t,' x') : |t'| = e^{-1}, |x'| \le b'e^{-\lambda}\}$. Now we apply Lemma 27 below to extend $\bar{h}$ to $\{|t| \le e^{-1}, |x| \le be^{-\lambda}\}$ as a topological equivalence and this finishes the proof of Theorem 7 in the nodal case.

Lemma 27. *Let $\mathcal{F}$ be the foliation in $\mathbb{C}^2$ generated by the holomorphic vector field $t\frac{\partial}{\partial t} + \lambda x \frac{\partial}{\partial x}$, where λ is an irrational positive number. Let $a, b, a', b' > 0$ and $h\colon \{|t| = a, |x| \le b\} \mapsto \{|t| = a', |x| \le b'\}$ a homeomorphism such that*

(1) *h is a topological equivalence between the 1-foliations induced by $\mathcal{F}$ in $\{|t| = a, |x| \le b\}$ and $\{|t| = a', |x| \le b'\}$,*

(2) *h is expressed as $h(t,x) = (h_1(t), h_2(t,x))$.*

Then h extends as a topological equivalence between $\{|t| \le a, |x| \le b\}$ and $\{|t| \le a', |x| \le b'\}$.

Proof.[7] Clearly h maps the disc $\{(a,x) : |x| \le b\}$ onto the disc $\{(h_1(a), x) : |x| \le b'\}$ and h conjugates the holonomies $(a,x) \mapsto (a, e^{2\pi i\lambda}x)$ and $(h_1(a), x) \mapsto (h_1(a), e^{2\pi i \lambda}x)$ defined on these discs. Since λ is irrational it is easy to see that h maps the circle $\Gamma = \{(a,x) : |x| = b\}$ onto the circle $\Gamma' = \{(h_1(a), x) : |x| = b'\}$ and there is $\nu \in \mathbb{C}^*$ such that $h(a,x) = (h_1(a), \nu x)$ for all $x \in \mathbb{C}$ with $|x| = b$. Since for any $\alpha, \beta \in \mathbb{C}^*$ the map $(t,x) \mapsto (\alpha t, \beta x)$ is a global auto-conjugation of $\mathcal{F}$, by composing h with a suitable such map if necessary we may assume that $a = b = a' = b' = h_1(a) = \nu = 1$. Then $h(1,x) = (1,x)$ for all $x \in \mathbb{C}$ with $|x| = 1$. Clearly the map h_1 is a homeomorphism of the circle $\{|t| = 1\}$ onto itself. Since the map $(t,x) \mapsto (\bar{t}, x)$ is a global auto-conjugation of $\mathcal{F}$, we may assume that h_1 preserves orientation. Then there is an increasing homeomorphism $\phi\colon [0,1] \to [0,1]$ such that $h_1(e^{2\pi i s}) = e^{2\pi i \phi(s)}$ for all $s \in [0,1]$. The orbits of the 1-foliation induced by $\mathcal{F}$ on $\{(t,x) : |t| = |x| = 1\}$ are parametrized by $(e^{2\pi i s}, e^{2\pi i \lambda s} z)$, $s \in \mathbb{R}$, $|z| = 1$. Observe that h maps each circle $\{(e^{2\pi i s}, x) : |x| = 1\}$ onto the circle $\{(e^{2\pi i \phi(s)}, x) : |x| = 1\}$. Moreover h conjugates the 1-foliation on $\{(t,x) : |t| = |x| = 1\}$ with itself and $h(1,z) = (1,z)$ if $|z| = 1$. Then it is easy to see that

$$h(e^{2\pi i s}, e^{2\pi i \lambda s} z) = (e^{2\pi i \phi(s)}, e^{2\pi i \lambda \phi(s)} z),$$

[7] We may also find a proof of this lemma in [5].

for $s \in [0,1]$, $|z| = 1$. Let $\phi_t \colon [0,1] \to [0,1]$, $t \in [1/2, 1]$ be a continuous family of homeomorphism such that $\phi_{1/2} = \mathrm{id}$ and $\phi_1 = \phi$. For $1/2 \leq r \leq 1$, $|z| = 1$ and $s \in [0,1)$ define

$$h(re^{2\pi i s}, e^{2\pi i \lambda s} z) = (re^{2\pi i \phi_r(s)}, e^{2\pi i \lambda \phi_r(s)} z).$$

It is not difficult to see that this extends de conjugation h to the set $\{(t,x) : 1 \geq |t| \geq 1/2,\ |x| = 1\}$. Moreover, if $|t| = 1/2$ and $|x| = 1$ we have $h(t,x) = (t,x)$ and we can extend h to the set $\{(t,x) : |t| \leq 1/2,\ |x| = 1\}$ as the identity map. Then the extended h is an auto-conjugation of the 1-foliation defined by $\mathcal{F}$ on $\partial(\mathbb{D} \times \mathbb{D})$. Finally, since the singularity at $0 \in \mathbb{C}^2$ is in the Poincaré domain, topologically the foliation $\mathcal{F}$ on the bidisc $\mathbb{D} \times \mathbb{D}$ is a "cone" generated by the 1-foliation on $\partial(\mathbb{D} \times \mathbb{D})$. Then it is easy to extend h to the interior of the bidisc. □

Lemma 28. *Let $\mathcal{F}$ be a holomorphic foliation on a neighborhood of the set $T = \{|t| \leq a, |x| \leq b\}$ with an isolated singularity at $0 \in \mathbb{C}^2$. Suppose that*

(1) *the singularity at $0 \in \mathbb{C}^2$ is reduced and $D = \{x = 0\}$ is a separatrix, and*

(2) *if L is the leaf of $\mathcal{F}|_T$ passing through a point in $R = \{|t| = a,\ 0 < |x| < b\}$, then $\bar{L} \cap D = \emptyset$.*

Then the singularity at $0 \in \mathbb{C}^2$ has a real negative eigenvalue.

Proof. By condition (2) we see that $0 \in \mathbb{C}^2$ could not be neither a hyperbolic neither a nodal singularity. It remains to prove that $0 \in \mathbb{C}^2$ is not a saddle node. Suppose that $0 \in \mathbb{C}^2$ is a saddle node and assume first that D is the strong separatrix. By the Flower Theorem is easy to see that a leaf L through a point $p \in R$ close enough to D is such that $\bar{L}$ contains D, which contradicts property (2). Suppose now that D is the weak separatrix. By the topological structure (see for example [9]) of the saddle node we may find a leaf L through a point in R such that L intersects the set $\{0 < |t| < a, |x| = b\}$ at a point q close enough to the strong separatrix $\{t = 0\}$ in such way (as above) $\bar{L}$ contains the strong separatrix. Then $\bar{L}$ contains $0 \in \mathbb{C}^2$, which contradicts property (2). □

Lemma 29. *If $\tau_0 \colon R \mapsto \mathbb{R}$ is upper semi-continuous, there exists a continuous function $\tau \colon R \mapsto \mathbb{R}$ such that $\tau > \tau_0$.*

Proof. It is easy to prove. □

8. Topological structure of a non-nodal simply singularity

Let $\mathcal{F}$ be a holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$ of eigenvalue $\lambda \notin \mathbb{R}_0^+$. Let (x,y) be coordinates such that $\{x = 0\}$ and $\{y = 0\}$ are the separatrices of the singularity. We may find a holomorphic vector field Z generating $\mathcal{F}$ such that

$$Z = \lambda_1 x(1+\cdots)\frac{\partial}{\partial x} + \lambda_2 y(1+\cdots)\frac{\partial}{\partial y},$$

where $re(\lambda_1) > 0 > re(\lambda_2)$. Thus, in a neighborhood U of $0 \in \mathbb{C}^2$ we have $Z = xA\frac{\partial}{\partial x} + yB\frac{\partial}{\partial y}$ with $re(A) > 0 > re(B)$. Let ϕ be the real flow associated to Z and let $a, b > 0$ be such that $P = \{|x| \le a, |y| \le b\} \subset U$. Let z be any point in $T = P\backslash\{xy = 0\}$. Write $\phi(t, z) = (x(t), y(t))$ and put $g(t) = |x(t)|^2$. A straightforward computation shows that

$$g'(t) = 2|x(t)|^2 re\{A(t)\} > 0,$$

hence the function $|x(t)|$ is strictly increasing. Analogously we may prove that the function $|y(t)|$ is strictly decreasing. Thus, since $z = (x_0, y_0)$ with $|x_0| \le a$ and $|y_0| \le b$ we have that the orbit of z intersects the set $\{|x| \le a, |y| = b\}$ at exactly one point w. Therefore we have $z = \phi(s, w)$ with $0 \le s \le \tau(w)$, where $\tau(w) \ge 0$ is the unique real number such that $\phi(\tau(w), w)$ is contained in the set $\{|x| = a, |y| \le b\}$. Since Z is transverse to $\{|x| = a, |y| \le b\}$, we have that τ depends continuously on w. Moreover observe that Z is transverse to the sets $\{|x| = cte \ne 0\}$ and $\{|y| = cte \ne 0\}$.

Lemma 30. *Let $b_1 \in (0, b)$ and let I and J be open intervals such that $\bar{I} \subset (0, a)$ and $\bar{J} \subset (0, b_1)$. Then there exists $\delta > 0$ and a map g such that*

(1) *g is a homeomorphism between $Q = \{(x, y) : |x| \le a, 0 < |y| \le b\}$ and $Q\backslash\{(0, y) : |y| \le b_1\}$,*

(2) *g preserve the leaves of $\mathcal{F}$,*

(3) *$g = \mathrm{id}$ on $\{(x, y) : (|x| - a)(|y| - b) = 0\}$,*

(4) *for all $r \in \bar{I}$ we have that g maps $\{|x| = r, 0 < |y| < \delta\}$ into a set of type $\{|y| = r'\}$ with $r' \in \bar{J}$.*

Proof. Let $R = \{(x, y) : 0 < |x| \le \delta, |y| = b\}$ with $0 < \delta < a$. Take functions $\alpha\colon [5, 6] \mapsto \mathbb{R}$ and $\beta\colon [0, 3] \mapsto \mathbb{R}$ such that

(1) α is strictly increasing with $\alpha([5, 6]) = \bar{I}$,

(2) β is strictly decreasing with $\beta(0) = b$, $\beta(1) = b_1$ and $\beta([2, 3]) = \bar{J}$.

It is easy to see that for δ small enough the orbit of any $z \in \overline{R}$ intersects each set $\{|y| = \beta(s)\}$. Since the flow is transverse to the sets $\{|y| = \beta(s)\}$, we have continuous functions $\tau_s\colon \overline{R} \mapsto \mathbb{R}^+$ such that $\phi(\tau_s(z), z) \in \{|y| = \beta(s)\}$ for all $z \in R$, $s \in [0, 3]$. Make $\phi(t, z) = (x(t), y(t))$ and observe that

(1) $|y(\tau_3(z))| = \beta(3) > 0$ and $|x(\tau(z))| = a > 0$,

(2) $|x(\tau_3(z))| \to 0$ and $|y(\tau(z))| \to 0$ as $|z| \to 0$.

Therefore by reducing δ we may assume that

$$|y(\tau_3(z))| - |x(\tau_3(z))| > 0 > |y(\tau(z))| - |x(\tau(z))|.$$

Then, since $|y(t)| - |x(t)|$ is strictly decreasing we have a continuous function $\tau_4\colon R \mapsto \mathbb{R}^+$ defined by $\phi(\tau_4(z), z) \in \{|x| = |y|\}$. By reducing δ if necessary we have $|x(\tau_4(z))| < \alpha(5)$ and we also obtain continuous functions $\tau_s\colon R \mapsto \mathbb{R}^+$ such that $\phi(\tau_s(z), z) \in \{|x| = \alpha(s)\}$ for all $z \in R$, $s \in [5,6]$. Observe that $\tau_3 < \tau_4$ and $\tau_4(z) \to \infty$ as $z \to \{x = 0\}$. We define $\tau_4(z) = \infty$ if $z \in \bar{R} \cap \{x = 0\}$ and construct a continuous family of functions $\tau_s\colon \bar{R} \mapsto \mathbb{R}^+$, $s \in (3,4)$ such that

(1) $\tau_s < \tau_{s'}$ for all $s, s' \in [3,4]$, $s < s'$,

(2) $\tau_s(z) \to \tau_3(z)$ as $s \to 3$ for all $z \in \bar{R}$,

(3) $\tau_s(z) \to \tau_4(z)$ as $s \to 4$ for all $z \in \bar{R}$.

We extend the family τ_s by making

$$\begin{aligned} \tau_s &= (5-s)\tau_4 + (s-4)\tau_5 \quad \text{if } s \in [4,5],\\ \tau_s &= (7-s)\tau_6 + (s-6)\tau_7 \quad \text{if } s \in [6,7], \end{aligned}$$

where $\tau_7 = \tau$. It is easy to see that $\tau_s < \tau_{s'}$ for all $s, s' \in [0,7]$, $s < s'$. Take an increasing homeomorphism $f\colon [0,7] \mapsto [0,7]$ such $f([5,6]) = [2,3]$, $f([0,4]) = [0,1]$. We write $w = \phi(\tau_s(z), z)$, $z \in \bar{R}$, and define $\Delta(w) = \tau_{f(s)}(z) - \tau_s(z)$. Take a continuous function $\rho\colon [0,\delta] \mapsto [0,1]$ such that $\rho = 1$ on $[0, \delta/2]$ and $\rho = 0$ near of δ. Define now $g(w) = \phi(\rho(|z|)\Delta(w), w)$. The map g is defined on $V = \{\phi(\tau_s(z), z) : z \in \bar{R},\ z \in \operatorname{dom}(\tau_s)\}$ and may be extended to Q by making $g = \mathrm{id}$ on $Q \backslash \bar{V}$. It is not difficult to see that g satisfies the assertions of the lemma.

Lemma 31. *Given a_1 with $a > a_1 > 0$, there exists a map g such that*

(1) *g is a homeomorphism between $P \backslash \{(x,0) : |x| \le a_1\}$ and $P \backslash \{0\}$,*

(2) *g preserve the leaves of $\mathcal{F}$,*

(3) *g maps $\{(x,0) : a_1 < |x| \le a\}$ onto $\{(x,0) : 0 < |x| \le a\}$ with $g(x,0) \to (0,0)$ as $|x| \to a_1$,*

(4) *$g = \mathrm{id}$ on $\{(x,y) : |x| = a$ or $|y| = b\}$,*

Proof. Let $R = \{(x,y) : 0 < |y| \le \delta,\ |x| = a\}$ with $0 < \delta < b$. Now, we denote by ϕ the real flow associated to $-Z$. As in the proof of Lemma 30, for δ small enough we may construct a continuous family of functions $\tau_s\colon \bar{R} \mapsto \mathbb{R} \cup \{+\infty\}$, $s \in [0,3]$ such that

(1) $\tau_0 = 0$,

(2) $\tau_s < \tau_{s'}$ for all $s, s' \in [0,3]$, $s < s'$,

(3) for all $s \in (0,2)$ the function τ_s take values in $\mathbb{R}^+$,

(4) $\tau_2(z) \in \{|x| = |y|\}$ for all $z \in R$,

(5) $\tau_3(z) \in \{2|x| = |y|\}$ for all $z \in R$.

Take an increasing homeomorphism $f : [0,3] \mapsto [0,3]$ such $f([0,1]) = [0,2]$. As before, we write $w = \phi(\tau_s(z), z)$, $z \in \overline{R}$ and define $g(w) = \phi(\rho(|z|)\Delta(w), w)$, where $\Delta(w) = \tau_{f(s)}(z) - \tau_s(z)$ and $\rho\colon [0,\delta] \mapsto [0,1]$ is such that $\rho = 1$ on $[0, \delta/2]$ and $\rho = 0$ near of δ. The map g is defined on $V = \{\phi(\tau_s(z), z) : z \in \overline{R},\ z \in \operatorname{dom}(\tau_s)\}$ and may be extended to $P \backslash \{(x,0) : |x| \leq a_1\}$ by making $g = \mathrm{id}$ on $P \backslash \overline{V}$. Then g satisfies the assertions of the lemma.

9. Proof of first part of Theorem 7 in the non-nodal case

In this section we prove the first part of Theorem 7, that is: Given $\varepsilon > 0$ we construct a topological equivalence $\bar{h}$ between $\mathcal{F}$ and $\mathcal{F}'$ such that, for some numbers $a, b, a', b' \in (0, \varepsilon)$, we have

(1) $\bar{h}$ maps $\{|t| \leq a, 0 < |x| \leq b\}$ into $\{|t'| \leq a', 0 < |x'| \leq b'\}$,

(2) $\bar{h}$ maps $\{|t| = a, 0 < |x| \leq b\}$ into $\{|t'| = a', 0 < |x'| \leq b'\}$,

(3) close to the divisor and outside

$$\{|t| \leq \varepsilon, |x| < \varepsilon\} \cup h^{-1}(|t'| \leq \varepsilon, |x'| < \varepsilon)$$

we have $\bar{h} = h$.

Actually we will prove the following stronger version of item (2) above:

(2′) For some $a_1 \in (0,a), a'_1 \in (0,a')$, the sets $\{|t| = r, 0 < |x| \leq b\}_{r \in [a_1, a]}$ are mapped by $\bar{h}$ into the sets $\{|t'| = r',\ 0 < |x'| \leq b'\}_{r' \in [a'_1, a']}$

It follows from Theorem 10 that there is a topological equivalence $\tilde{h}$ such that for some $\mathsf{a}, \mathsf{a}', \mathsf{b} \in (0,\varepsilon)$ we have the following:

(1) For all s in a neighborhood of b, the set $\{|t| < \mathsf{a}, |x| = s\}$ is mapped by $\tilde{h}$ into the set $\{|t'| < \mathsf{a}', |x'| = \beta(s)\}$, where β is an increasing continuous function.

(2) Close to the divisor we have $\tilde{h} = h$.

Take $\mathsf{b}_1 < \mathsf{b}$ and an open interval J in the domain of definition of β such that $\bar{J} \subset (0, \mathsf{b}_1)$. Let $\mathsf{b}' = \beta(\mathsf{b})$, $\mathsf{b}'_1 = \beta(\mathsf{b}_1)$, $J' = \beta(J)$ and take open intervals I and I' such that $I \subset (0,\mathsf{a})$, $\bar{I}' \subset (0, \mathsf{a}')$. Clearly we may assume a, a', b, b' be small enough such that $\{|t| \leq \mathsf{a}, |x| \leq \mathsf{b}\}$ and $\{|t'| \leq \mathsf{a}', |x'| \leq \mathsf{b}'\}$ are contained in neighborhoods as in Section 8. Thus, by Lemma 30 there exist homeomorphisms g and g' and numbers $\delta, \delta' > 0$ such that

(1) g maps $Q = \{|t| \le \mathsf{a}, 0 < |x| \le \mathsf{b}\}$ onto $Q\backslash\{0\} \times [0, \mathsf{b}_1]$,

(2) g' maps $Q' = \{|t'| \le \mathsf{a}', 0 < |x'| \le \mathsf{b}'\}$ onto $Q'\backslash\{0\} \times [0, \mathsf{b}_1']$,

(3) g and g' are leaf preserving and are equal to the identity on $\{(t,x) \in Q : (|t| - \mathsf{a})(|x| - \mathsf{b}) = 0\}$ and $\{(t',x') \in Q' : (|t'| - \mathsf{a}')(|x'| - \mathsf{b}') = 0\}$ respectively,

(4) g maps the sets $\{|t| = s, 0 < |x| \le \delta\}_{s \in \bar{I}}$, into the sets $\{|x| = s\}_{s \in \bar{J}}$,

(5) g' maps the sets $\{|t'| = s, 0 < |x'| \le \delta'\}_{s \in \bar{I}'}$ into the sets $\{|x'| = s\}_{s \in \bar{J}'}$.

Outside the exceptional divisor we may extend g and g' as the identity map. Clearly g and g' are topological equivalences of $\mathcal{F}$ with itself and $\mathcal{F}'$ with itself respectively. Then $\bar{h} = g'^{-1} \circ \tilde{h} \circ g$ is a topological equivalence between $\mathcal{F}$ and $\mathcal{F}$ and it is not difficult to see that, if δ is taken small enough, the following properties hold:

(1) The sets $\{|t| = s,\ 0 < |x| < \delta\}_{s \in \bar{I}}$ are mapped by $\bar{h}$ into the sets $\{|t'| = s,\ 0 < |x'| < \mathsf{b}'\}_{s' \in \bar{I}'}$.

(2) Close to the divisor and out of

$$\{|t| \le \varepsilon, |x| < \varepsilon\} \cup h^{-1}(|t'| \le \varepsilon, |x'| < \varepsilon)$$

we have $\bar{h} = h$.

Let $b' = \mathsf{b}'$ and take $a \in I$, $a' \in I$ be such that $\{|t'| = a',\ 0 < |x'| < b'\}$ contains $\bar{h}(|t| = a,\ 0 < |x| < \delta)$.

Assertion. There exists $\bar{\delta} > 0$ such that $\{|t'| = a',\ 0 < |x'| < \bar{\delta}\}$ is contained in $\bar{h}(|t| = a, 0 < |x| < \delta)$.

Take $\bar{\delta} > 0$ such that for all $(t', x') \in \bar{h}(|t| = a,\ |x| = \delta)$ we have $|x'| > \bar{\delta}$. Since $\bar{h}$ is a homeomorphisms, the set $X = \bar{h}(|t| = a,\ 0 < |x| < \delta) \cap \{|t'| = a', 0 < |x'| < \bar{\delta}\}$ is open in $\{|t'| = a',\ 0 < |x'| < \bar{\delta}\}$. Obviously the set X is non-empty, then it suffices to show that X is closed in $\{|t'| = a',\ 0 < |x'| < \bar{\delta}\}$. Let $(t_k, x_k) \in \{|t| = a, 0 < |x| < \delta\}$ be such that $\bar{h}(t_k, x_k)$ tends to a point q in $\{|t'| = a', 0 < |x'| < \bar{\delta}\}$. We may assume that $(t_k, x_k) \to (t_0, x_0)$. Clearly $x_0 \ne 0$ because q is not a point in the divisor $\{x' = 0\}$. Then $(t_0, x_0) \in \{|t| = a, 0 < |x| \le \delta\}$. By the choice of $\bar{\delta}$ and the injectivity of $\bar{h}$ we have that $(t_0, x_0) \in \{|t| = a, 0 < |x| < \delta\}$. Then $q = \bar{h}(t_0, x_0) \in X$ and X is therefore closed in $\{|t'| = a', 0 < |x'| < \bar{\delta}\}$. Assertion is proved.

Take $b \in (0, \delta)$ small enough such that

$$A = \bar{h}(\{|t| < a, 0 < |x| \le b\})$$

intersects $B = \{|t'| \le a', |x| \le b'\}$ in a set contained in $\{|t'| \le a', |x| < \bar{\delta}\}$. Then

$$A \cap \partial B \subset \{|t'| = a', 0 < |x| < \bar{\delta}\}.$$

But $\{|t'| = a', 0 < |x| < \bar{\delta}\}$ is contained in the set $\bar{h}(|t| = a, 0 < |x| < \delta)$, which is disjoint of A, since $\bar{h}$ is injective. Then $A \cap \partial B = \emptyset$. Finally, for complete the proof we show that A is contained in B. The set A is connected and it intersects the separatrix $\{t' = 0, |x'| < b'\} \subset B$. Then $A \not\subset B$ implies $A \cap \partial B \neq \emptyset$, which is a contradiction.

10. The-linearizing/resonant case

Let $\bar{h}$ be the homeomorphism constructed in Section 9. By simplicity we denote $\bar{h}$ also by h. Let $\mathcal{G}$ be the foliation of dimension 1 induced in $R' = \{|t'| = a', 0 < |x'| < b'\}$ by $\mathcal{F}'$. Let ϕ be the flow associated to $\mathcal{G}$ such that if $z = (t', *) \in R'$, then $\phi(s,z) = (e^{2\pi i s}t', *)$. For $\delta \in (0,b)$ let $D = D(\delta) = \{(a,x) : 0 < |x| < \delta\}$ and $\mathcal{D} = \mathcal{D}(\delta) = h(D)$. Let $\Sigma = \{(a', x') : |x'| < b'\}$. It is in the proof of the following Proposition where the linearizing-resonant hypothesis is used. This proposition is the key to redressing the transverse sections $\Sigma_u = \{t = u, |x| \leq b\}$ in the proof of the second part of Theorem 7. By $[c,d]$ we denote the closed interval with endpoints c and d, even if $c > d$.

Proposition 32. *If δ is small enough, there exists a continuous function $\tau\colon \mathcal{D} \mapsto \mathbb{R}$ such that*

(1) *$\phi(t,z) \in R'$ and $f(z) = \phi(\tau(z), z) \in \Sigma$ for all $z \in \mathcal{D}$, $t \in [0, \tau(z)]$,*

(2) *$f : \mathcal{D} \mapsto \Sigma$ is a homeomorphism onto its image,*

(3) *$f(\mathcal{D}) = \Omega\backslash\{o\}$, where $o = (a', 0) \in \Sigma$ and $\Omega \subset \Sigma$ is a topological disc containing o,*

(4) *$f(z) \to o$ as $z \in \mathcal{D}$ tends to the divisor $\{x' = 0\}$.*

It is easy to see that there exists $z_0 \in \mathcal{D}$ and $s_0 \in \mathbb{R}$ such that $\phi(s_0, z_0) \in \Sigma$ and $\phi(s, z_0) \in R'$ for all $s \in [0, s_0]$. Let z be any point in $\mathcal{D}$. Take any path $\gamma\colon [0,1] \mapsto \mathcal{D}$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. If $z_0 = (t_0, *)$, we may write $\gamma(s) = (e^{2\pi i \theta(s)} t_0, *)$, where $\theta\colon [0,1] \mapsto \mathbb{R}$ is continuous and $\theta(0) = 0$. We define $\tau(z) = s_0 - \theta(1)$. Let $\gamma'\colon [0,1] \mapsto \mathcal{D}$ be another path joining z_0 and z and let $\theta'\colon [0,1] \mapsto \mathbb{R}$ be the corresponding function. It is easy to see that $\theta'(1) - \theta(1)$ is the linking number between the path $\gamma^{-1} \circ \gamma'$ and the vertical $\{t = 0\}$ and therefore equal to zero, by Theorem 12. Thus τ is well defined and it is easy to see that it is a continuous function. Take $\tilde{\delta} > 0$ be such that $T' = \{|t'| = a', 0 < |x'| < \tilde{\delta}\}$ is contained in $h(\{|t| = a, 0 < |x| < b\})$. We divide the proof of Proposition 32 in three cases.

10.1. Proof of Proposition 32 when the holonomy is a rotation. In this case we may take δ small enough such that for all $z \in \mathcal{D}$, all the orbit of $\mathcal{G}$ passing through z is contained in T'. Therefore $\phi(t,z) \in R'$ for all $z \in \mathcal{D}$ and for all $t \in [0, \tau(z)]$. It follows from the construction of τ that $f(z) = \phi(\tau(z), z) \in \Sigma$. We shall prove that

f is injective. Suppose that $f(z) = f(z')$. Let $\gamma\colon [0,1] \mapsto \mathcal{D}$ be a curve joining z and z'. Let $s \in [0,1]$ and let α and β be the paths $\phi((1-s)\tau(z), z)$ and $\phi(s\tau(z'), z')$ respectively. Let θ be the closed path $\gamma * \beta * \alpha$. For $t \in [0,1]$ we define γ_t, α_t and β_t by the expressions $\phi(t\tau \circ \gamma(s), \gamma(s))$, $\phi((1-s+ts)\tau(z), z)$ and $\phi((s+t(1-s))\tau(z'), z')$ respectively. It is easy to see that $\gamma_t * \beta_t * \alpha_t$ define a homotopy between θ and a path contained in Σ. Then θ does not link the separatrix $\{t' = 0\}$ and therefore, by Theorem 12, the path $h^{-1}(\theta)$ does not link $\{t' = 0\}$. Observe that the path $h^{-1}(\theta)$ has the part $h^{-1}(\gamma)$ contained in D. On the other hand, $h^{-1}(\beta * \alpha)$ is a path contained in a leaf of the foliation $\mathcal{F}$ restricted to $\{0 < |t| \leq a, 0 < |x| \leq b\}$. Since $h^{-1}(\beta * \alpha)$ joins $h^{-1}(z)$ and $h^{-1}(z')$ (points in D) we have that $h^{-1}(z) = g(h^{-1}(z'))$, where g is the holonomy map associated to the projection of $h^{-1}(\theta)$ in $\{x = 0\}$. Then, since $h^{-1}(\theta)$ does not link $\{t = 0\}$, we have that $g = \mathrm{id}$, hence $z = z'$. Let $O(z)$ be the orbit of $\mathcal{G}$ passing through z. We know that $O(z)$ tends to $\{x' = 0\}$ as z tends to $\{x' = 0\}$. It follows that $f(z) \to o$ as z tends to $\{x' = 0\}$. Topologically, we may identify $\mathcal{D}$ with $\mathbb{D} \backslash \{0\}$. Then we extend the function f to $\mathbb{D}$ by making $f(0) = o$. This extension is a homeomorphism and $\Omega = f(\mathbb{D})$ is therefore homeomorphic to a disc. This finishes the proof in this case.

10.2. Proof of Proposition 32 when the holonomy is hyperbolic. Given $z \in \mathcal{D}$ take a complex disc Σ_z passing through z and transverse to $\mathcal{F}'$. In a neighborhood U_z of z is well defined a leaf preserving projection $\pi_z\colon U_z \mapsto \Sigma_z$. It is not difficult to prove, since $\mathcal{D}$ is a continuous transversal to $\mathcal{F}$, that in a small neighborhood Δ_z of z in $\mathcal{D}$ the restriction $\pi_z\colon \Delta_z \mapsto \Sigma_z$ is a homeomorphism onto its image. The charts $\{\pi_z\}_{z \in \mathcal{D}}$ define a natural complex structure on $\mathcal{D}$. Then $\mathcal{D}$, since it is homeomorphic to an annulus, it is analytically equivalent to an annulus $\{z \in \mathbb{C} : 0 \leq r < |z| \leq 1\}$ for some $r \geq 0$. The holonomy map of the separatrix $x = 0$ is a contractive function $g\colon D \mapsto D$. Consider the map $g' = h \circ g \circ h^{-1}\colon \mathcal{D} \mapsto \mathcal{D}$. Clearly $g'\colon \mathcal{D} \mapsto \mathcal{D}$ is not trivial at homology level and is holomorphic, because it is continuous and leaf preserving. Then, since g' is not an isomorphism, it follows from the annulus theorem (see [19], p. 211) that $r = 0$ and $\mathcal{D}$ is therefore analytically equivalent to a punctured disc.

By using linearizing coordinates we may assume that the foliation $\mathcal{G}$ extends to the set $\{(t', x') : |t'| = a', x' \in \mathbb{C}\}$ and is the suspension of a hyperbolic automorphism of $\mathbb{C}$. Then we have a map $f\colon \mathcal{D} \mapsto \{(a', x) : x \in \mathbb{C}\}$ defined by $f(z) = \phi(\tau(z), z)$. Observe that f is holomorphic, because it is a continuous leaf preserving map. Identifying $\mathcal{D}$ with $\mathbb{D} \backslash \{0\}$, we have by the Riemann Extension Theorem that f extends to a holomorphic map $f : \mathbb{D} \mapsto \mathbb{C}$, $f(0) = 0$. Since $\mathcal{G}$ is the suspension of an hyperbolic automorphism of $\mathbb{C}$, there exists a set $\widetilde{R} \subset T'$ such that

(1) $\widetilde{R}$ contains all segment of orbit with endpoints in $\widetilde{R}$,

(2) $\widetilde{R}$ contains the set $\{(t', x') : |t'| = a', |x| < \epsilon\}$ for some $\epsilon > 0$.

Since $f(0) = 0$, by reducing $\mathcal{D}$ if necessary we may assume that $\mathcal{D}$ and $f(\mathcal{D})$ are contained in $\widetilde{R}$. It is not difficult to see that the proof of the injectivity of f given in Case 1 also works in this case. Then f maps $\mathbb{D}$ homeomorphically into $\mathbb{C}$ and therefore $\Omega = f(\mathbb{D})$ is a topological disc. This finishes the proof in the hyperbolic case.

10.3. Proof of Proposition 32 when the holonomy is resonant non-linearizable. In this case the foliations near the singularities p and p' are generated by vector fields of the form $t\frac{\partial}{\partial t} + \lambda x(1+\cdots)\frac{\partial}{\partial x}$ and $t'\frac{\partial}{\partial t'} + \lambda' x'(1+\cdots)\frac{\partial}{\partial x'}$ with $\lambda, \lambda' \in \mathbb{Q}_{<0}$. Let ψ and ψ' be the real flows associated to these vector fields respectively. Given $z = (a, x) \in D$, there is a unique $s(z) \in \mathbb{R}$ such that $\psi(s(z), z) \in \{|x| = b\}$. Let γ_z be the path $\psi(s, z)$, $s \in [0, s(z)]$ and define $\rho(z) = \psi(s(z), z)$. For all $w \in \{0 < |t'| < a', 0 < |x'| \le b'\}$ define $\pi(w)$ as the intersection of the orbit of w by the flow ψ' with R'. As in Section 4 we may construct a topological equivalence $\bar{h}$ such that

(1) $\bar{h}$ is defined in a neighborhood of the set $\{(0, x) : 0 < |x| \le b\}$,

(2) $\{|t| \le a, |x| \le b\} \cap \operatorname{dom}(\bar{h})\}$ is mapped by $\bar{h}$ into $\{|t'| \le a', |x'| \le b'\}$,

(3) For $\epsilon > 0$ small enough and for all $\mu \in \mathbb{S}^1 \subset \mathbb{C}$, $\bar{h}$ maps the set $\{|t| \le \epsilon, x = \mu b\}$ into the set $\{|t'| < a, x' = \mu b'\}$,

(4) close to the divisor we have $\bar{h} = h$.

If δ is small enough we have $\gamma_z \subset \operatorname{dom}(\bar{h})$ and $\bar{h}(\gamma_z) \subset \{|t'| \le a', |x'| \le b'\}$. The path $\pi(\bar{h}(\gamma_z))$ is contained in a orbit of the flow ϕ and is homotopic in this orbit to a path of the form $\phi(s, \bar{h}(z))$, $s \in [0, \tau_z]$ for some $\tau_z \in \mathbb{R}$ such that $\phi(\tau_z, \bar{h}(z)) = \pi(\bar{h}(\rho(z))$. By (4) we may assume that $\bar{h}(z) = h(z)$ for all $z \in D$. Then $\phi(s, w) \in R'$ for all $w \in \mathcal{D}$, $s \in [0, \tau_1(w)]$, where $\tau_1(w) = \tau_{h^{-1}(w)}$. Let $\mathcal{D}_1 = \{\phi(\tau_1(w), w) : w \in \mathcal{D}\}$. We will prove that there is a continuous function $\tau_2 \colon \mathcal{D}_1 \mapsto \mathbb{R}$ such that $\phi(s, w) \in R'$ and $\phi(\tau_2(w), w) \in \Sigma$ for all $w \in \mathcal{D}_1$, $s \in [0, \tau_2(w)]$. Since $\mathcal{D}_1$ does not link the vertical $\{t' = 0\}$ there exists a continuous function $\theta \colon \mathcal{D}_1 \mapsto \mathbb{R}$ such that $w = (a' e^{2\pi i \theta(w)}, *)$ for all $w \in \mathcal{D}_1$.

Assertion. The function θ is bounded.

Given $\mu \in \mathbb{S}^1$ let $I_\mu = \{(t, \mu b) : t \in (0, \epsilon_1]\}$, where $\epsilon_1 \in (0, \epsilon)$ and ϵ is as in item (3) above. Let $U_\mu = \{(t, \mu b) : |t| < \epsilon\}$ and $U'_\mu = \{(t', \mu b') : |t'| < a\}$ and observe that $\bar{h}|_{U_\mu} : U_\mu \mapsto U'_\mu$ conjugates the holonomies of the separatrices $\{t = 0\}$ and $\{t' = 0\}$ computed on U_μ and U'_μ respectively. Therefore, if $r_\mu > 0$ and θ_μ are continuous real functions such that

$$\bar{h}(\zeta) = (r_\mu(\zeta) e^{2\pi i \theta_\mu(\zeta)}, \mu b') \tag{10.1}$$

for all $\zeta \in I_\mu$, it follows from Lemma 33 that $\theta_\mu(I_\mu)$ has finite diameter $M_\mu \in \mathbb{R}$. Observe that, since the orbits of the flow ψ' are contained in the sets $\{t'/|t'| = cte\}$, we have

$$\pi \bar{h}(\zeta) = \pi(r_\mu(\zeta) e^{2\pi i \theta_\mu(\zeta)}, \mu b') = (a' e^{2\pi i \theta_\mu(\zeta)}, *). \tag{10.2}$$

Moreover, the orbits of ψ passing through a point of $\bigcup I_\mu$ are all contained in $\{(t,x) : t \in \mathbb{R}_{>0}\}$, then these orbits intersects $\{(t,x) : |t| = a, |x| < b\}$ at points in D. Thus, by taking ϵ_1 small enough we may assume that $\bigcup I_\mu$ is contained in $\rho(D)$. Then $\pi\bar{h}(\zeta) \in \mathcal{D}_1$ for all $\zeta \in I_\mu$ and therefore $\pi\bar{h}(\zeta) = (a' e^{2\pi i\theta(\pi\bar{h}(\zeta))}, *)$. It follows from equation (10.2) that there is some integer n_μ such that $\theta(\pi\bar{h}(\zeta)) = \theta_\mu(\zeta) + n_\mu$ for all $\zeta \in I_\mu$. This implies that the diameter of $\theta(\pi\bar{h}(I_\mu))$ is equal to M_μ. We may take $\delta_1 \in (0,\delta)$ small enough such that

(1) I_μ intersects the set $K = \rho(\{(a,x) : \delta_1 \leq |x| \leq \delta\})$ for all $\mu \in \mathbb{S}^1$,

(2) $\rho(\{(a,x) : 0 < |x| \leq \delta_1\})$ is contained in $\bigcup I_\mu$.

Then $\theta(\mathcal{D}_1) \subset \theta\pi\bar{h}(K) \cup \bigcup \theta\pi\bar{h}(I_\mu)$ and each $\theta\pi\bar{h}(I_\mu)$ intersects the compact set $\theta\pi\bar{h}(K)$. Thus, it suffices to show that $\{M_\mu : \mu \in \mathbb{S}^1\}$ is bounded. Suppose by contradiction that there is a sequence $\{\mu_k\} \subset \mathbb{S}^1$ with $M_{\mu_k} \to \infty$ and $\mu_k \to \bar{\mu} \in \mathbb{S}^1$. Since $\bar{h}$ is a topological equivalence, for large k there are holonomy maps $f_k : \{(t, \mu_k b) : |t| \leq \epsilon_1\} \mapsto U_{\bar{\mu}}$ and $g_k : \bar{h}(U_{\bar{\mu}}) \mapsto U'_{\mu_k}$ such that

(1) $\bar{h}(z) = g_k \circ \bar{h} \circ f_k(z)$ for all $z \in \{(t, \mu_k b) : |t| \leq \epsilon_1\}$,

(2) f_k and g_k tends to the identity as $k \to \infty$.

We can parametrize $\bar{h}(f_k(I_{\mu_k}))$ by $(r_k(\zeta)e^{2\pi i\theta_k(\zeta)}, \bar{\mu}b')$, $\zeta \in I_{\mu_k}$, where $r_k > 0$ and θ_k are real continuous functions. If k is large enough we have that $f_k(I_{\mu_k})$ is C^1-close to $I_{\bar{\mu}}$ and Lemma 33 below implies that the image of θ_k has diameter bounded by some constant C independent of k. For k large we may write $g_k(w, \bar{\mu}b') = (wc_k(w)e^{2\pi i\vartheta_k(w)}, \mu_k b')$, where $c_k > 0$ and ϑ_k are real continuous functions with $||\vartheta_k|| < 1$. Then for all $\zeta \in I_{\mu_k}$,

$$\begin{aligned}
\bar{h}(\zeta) &= g_k \circ \bar{h} \circ f_k(\zeta) \\
&= g_k(r_k(\zeta)e^{2\pi i\theta_k(\zeta)}, \bar{\mu}b') \\
&= (r_k(\zeta)e^{2\pi i\theta_k(\zeta)}c_k(*)e^{2\pi i\vartheta_k(*)}, \mu_k b') \\
&= (r_k c_k e^{2\pi i(\theta_k(\zeta)+\vartheta_k(*))}, \mu_k b').
\end{aligned}$$

On the other hand, we have from equation (10.1) that $\bar{h}(\zeta) = (r_{\mu_k}(\zeta)e^{2\pi i\theta_{\mu_k}(\zeta)}, \mu_k b')$ for all $\zeta \in I_{\mu_k}$. Therefore we have $\theta_{\mu_k}(\zeta) = \theta_k(\zeta) + \vartheta_k(*) + n_k$ for all $\zeta \in I_{\mu_k}$ for some $n_k \in \mathbb{Z}$. It follows that $M_{\mu_k} \leq C + 2$ for all k big enough, which is a contradiction. Assertion is proved.

Define $\tau_2(w) = -\theta(w)$ for all $w \in \mathcal{D}_1$ and let $M > 0$ be such that $||\theta|| \leq M$. Now, keeping θ invariable we can reduce δ in order to have $\phi(s,w) \in T'$ for all $w \in \mathcal{D}_1$, $s \in [0, \tau_{2(w)}]$. Clearly we have $\phi(\tau_2(w), w) \in \Sigma$ for all $w \in \mathcal{D}_1$. The injectivity of f follows as before, so Proposition 32 is proved.

Lemma 33. *Let h map $D = \{z \in \mathbb{C} : |z| \leq r\}$ homeomorphically into $\mathbb{C}$ with $h(0) = 0$. Suppose further that h is a topologically conjugation between two germs*

$f, g\colon (\mathbb{C}, 0) \mapsto (\mathbb{C}, 0)$ of biholomorphism with resonant fixed point at $0 \in \mathbb{C}$. Given a simply path $\gamma\colon [0,1] \mapsto D$ with $\gamma(0) = 0$, take a real continuous function θ such that $h(\gamma(t)) = |h(\gamma(t))|e^{2\pi i\theta(t)}$ and define $d(\gamma) \in \mathbb{R} \cup \{\infty\}$ as the diameter of $\theta([0,1]) \subset \mathbb{R}$. Then there is a constant $C > 0$ such that $d(\gamma) \leq C$ for all γ whose image is contained in the complement of $\{tu : t > 0\}$ for some $u \in \mathbb{C}^*$.

Proof. Let $D^* = D \backslash \{0\}$, $B = \exp^{-1}(D^*)$ and $B' = \exp^{-1}(h(D^*)$. The homeomorphism h may be lifted to a homeomorphism $H\colon B \mapsto B'$ such that $h \circ \exp = \exp \circ H$. It is easy to see that any γ satisfying the hypothesis of the lemma may be lifted by exp into the set $T = B \cap \{0 < \operatorname{im}(z) < 4\pi\}$ Then it is sufficient to show that there is some constant $k > 0$ such that $H(T)$ is contained in $\{|\operatorname{im}(z)| \leq k\}$. Suppose that there is some path Γ satisfying the hypothesis of the lemma and such that $d(\Gamma) < \infty$. Then we may find two lifting Γ_1 and Γ_2 of Γ in B such that the set T is contained in the closed region K bounded by Γ_1 and Γ_2 in B. Since $d(\Gamma) < \infty$ there is $k > 0$ such that $H(\Gamma_1)$ and $H(\Gamma_2)$ are contained in $\{|\operatorname{im}(z)| \leq k\}$. In this case it is easy to see that $H(K) \subset \{|\operatorname{im}(z)| \leq k\}$ and therefore $H(T)$ is contained in $\{|\operatorname{im}(z)| \leq k\}$. Now we prove the existence of Γ. By the Flower Theorem (Leau–Fatou), considering a repelling petal of f, we may find a simply curve $\Gamma\colon [0,1] \mapsto D$, $\Gamma(0) = 0$ and a disc $D_0 \subset D$ centered at $0 \in \mathbb{C}$ such that the following holds:

(1) The path $\Gamma((0,1])$ is contained in the complement of $\{tu : t > 0\}$ for some $u \in \mathbb{C}^*$.

(2) For all $z \in \Gamma((0,1])$ there is some $n \in \mathbb{Z}_{\geq 0}$ with $f^{\circ n}(z) \notin D_0$.

Again by the Flower Theorem, considering a attracting petal of g, we may find $u_0 \in \mathbb{C}$, $|u_0| = 1$ and $\epsilon > 0$ such that for all $z \in \{tu_0 : 0 < t \leq \epsilon\}$ we have $g^{\circ n}(z) \in f(D_0)$ for all $n \in \mathbb{Z}_{\geq 0}$. Then, since h conjugates f and g, we deduce that $h(\Gamma)$ does not intersect $\{tu_0 : 0 < t \leq \epsilon\}$. Thus $h(\Gamma)$ intersects the ray $\{tu_0 : t > 0\}$ only finitely many times and therefore $d(\Gamma) < \infty$. □

Remark 34. We conjecture that Lemma 33 is true, in general, when the germs f and g are non-linearizable. If this would be the case, the theorems of the paper would be true without the linearizing/resonant hypothesis. The construction of an extension to a neighborhood of p depends only on the boundedness of the function θ (Subsection 10.3). In particular, the function θ is bounded if the homeomorphism in Lemma 33 is a conformal map, we have this situation for example if the topological equivalence between the foliations is transversely conformal. In [18] the author shows some general situations where the topological equivalence is necessarily transversely conformal, for example if the resolution of $\mathcal{F}$ is non-dicritical, has no nodes or saddle-nodes and has some component of the divisor with non-solvable holonomy group.

11. Proof of the second part of Theorem 7 in the non-nodal case

In this section, under the linearizing/resonant hypothesis, we prove the second part of Theorem 7. We continue with the notation established in Section 10. Denote also $C = \{|t| = a, x = 0\}$, $C' = \{|t'| = a', x' = 0\}$, $R = \{|t| = a, 0 < |x| \leq b\}$ and $\zeta_0 = (a, 0)$. We have in $\bar{R}$ a foliation of dimension 1 induced by $\mathcal{F}$. Recall the real flow ϕ on $\bar{R}'$ defined in last section. We also denote by ϕ the real flow on $\bar{R}$ such that $\phi(s, z) = (e^{2\pi i s}t, *)$ for $z = (t, *) \in \bar{R}$. Choose the orientation of C given by the flow ϕ. Let θ be a oriented circle in R homotopic to C in $\bar{R}$ and take a diffeomorphism $g : C \mapsto C'$, $g(\zeta_0) = (a', 0)$ such that $g(C)$ is homotopic to $h(\theta)$ in $\bar{R}' = \{|t'| = a', |x'| \leq b'\}$. Let $R_{\delta'} = \{(t', x') \in R' : |t'| = a', 0 < |x'| < \delta'\}$ and assume $\delta' > 0$ be such that

(1) $\phi(s, z) \in R'$ for all $z \in R_{\delta'}$, $s \in [-1, 1]$,

(2) $\phi(s, z) \in \{|t| = a, |x| < \delta\}$ for all $z \in h^{-1}(R_{\delta'})$, $s \in [-1, 1]$.

Given $\zeta \in C$, define $\vartheta(\zeta) \in [0, 1)$ by $\zeta = \phi(\zeta_0, \vartheta(\zeta))$ and let $\vartheta'(\zeta) \in \mathbb{R}$ be such that $\phi(s\vartheta'(\zeta), g(\zeta_0))$, $s \in [0, 1]$ is a positive reparametrization of the path $g(\phi(s\vartheta(\zeta), \zeta_0))$, $s \in [0, 1]$. Clearly ϑ and ϑ' are continuous on $C \backslash \{\zeta\}$ and they have a simply discontinuity at ζ_0. Let π be the projection $(t, x) \to t$ in R. Given $z \in R_{\delta'}$, make $\zeta(z) = \pi \circ h^{-1}(z)$ and let $\theta(z) \in \mathbb{R}$ be such that $\phi(-s\theta(z), z)$, $s \in [0, 1]$ is a positive reparametrization of $h \circ \phi(-s\vartheta(\zeta(z)), h^{-1}(z))$, $s \in [0, 1]$. From (2) and the definition of θ it is easy to see that $\phi(-\theta(z), z) \in \mathcal{D}$ for all $z \in R_{\delta'}$. In Section 10 we found the function τ defined on $\mathcal{D}$. Now, we extend τ to $R_{\delta'}$ by making:

$$\tau(z) = -\theta(z) + \tau \circ \phi(-\theta(z), z) + \vartheta'(\zeta(z)). \tag{11.1}$$

Assertion. τ is continuous and $\phi(s\tau(z), z) \in R'$ for all $z \in R_{\delta'}$, $s \in [0, 1)$.

Let $z_0 \in \mathcal{D}$. It is sufficient to show that $\tau(z) \to \tau(z_0)$ whenever $z \to z_0 \in \mathcal{D}$ with $1/2 < \vartheta(\zeta(z)) < 1$. If $\vartheta(\zeta(z)) \to 1$ we have that $\theta(z) \to \theta_0$, where θ_0 is such that $\phi(-s\theta_0, z_0)$, $s \in [0, 1]$ is a positive reparametrization of $h \circ \phi(-s, h^{-1}(z_0))$, $s \in [0, 1]$. Then $z_1 := \phi(-\theta_0, z_0) = h \circ \phi(-1, h^{-1}(z_0)) \in \mathcal{D}$. Let $\gamma : [0, 1] \mapsto \mathcal{D}$ be any path such that $\gamma(0) = z_1$ and $\gamma(1) = z_0$. For all $t \in [0, 1]$ define the paths γ_t and α_t by $\gamma_t(s) = \phi(t\tau \circ \gamma(s), \gamma(s))$ and

$$\alpha_t(s) = \phi((1 - s)t\tau(z_0) + s(t\tau(z_1) - \theta_0), z_0)$$

for $s \in [0, 1]$. The paths $\alpha_t * \gamma_t$ are closed and give a homotopy between $\alpha_0 * \gamma$ and $\alpha_1 * \gamma_1$. By the definition of θ_0, the path α_0 is homotopic in R' to the path $h \circ \phi(-s, h^{-1}(z_0))$, $s \in [0, 1]$. Then $\alpha_0 * \gamma$ is homotopic to the path $h(\tilde{\alpha} * \tilde{\gamma})$, where $\tilde{\alpha}$ is the path $\phi(-s, h^{-1}(z_0))$, $s \in [0, 1]$ and $\tilde{\gamma} = h^{-1} \circ \gamma$. But the path $\tilde{\alpha} * \tilde{\gamma}$ is homotopic to $-C$ in $\bar{R}$. Then, it follows from the definition of g that $\alpha_0 * \gamma$

is homotopic to $g(-C)$ in $\bar{R}'$. Therefore $\alpha_1 * \gamma_1$ is homotopic to $g(-C)$ in $\bar{R}'$. Observe that, since $\gamma_1 \subset \Sigma$, the path $\alpha_1 * \gamma_1$ is homotopic in $\bar{R}'$ to the closed path $\phi((1-s)\tau(z_0) + s(\tau(z_1) - \theta_0), g(\zeta_0))$, $s \in [0,1]$. Then $g(-C)$ is homotopic to $\phi(s(\tau(z_1) - \tau(z_0) - \theta_0), q)$, $s \in [0,1]$, where $q = \phi(\tau(z_0), g(\zeta_0))$. On the other hand, since $\vartheta(\zeta(z)) \to 1$ as $z \to z_0$ with $1/2 < \vartheta(\zeta(z)) < 1$, it follows from the definition of ϑ' that $\vartheta'(\zeta(z)) \to \xi$, where ξ (equal to 1 or -1) is such that $\phi(-s\xi, g(\zeta_0))$, $s \in [0,1]$ is a positive reparametrization of $g(-C) = g \circ \phi(-s, \zeta_0)$, $s \in [0,1]$. Then $g(-C)$ is homotopic to $\phi(-s\xi, g(\zeta_0)) = \phi(-s\xi, q)$, $s \in [0,1]$. It follows that the paths $\phi(s(\tau(z_1) - \tau(z_0) - \theta_0), q)$ and $\phi(-s\xi, q)$ are homotopic in $\bar{R}'$ and this implies that

$$\xi = -\tau(z_1) + \tau(z_0) + \theta_0.$$

Thus, if $z \to z_0$ with $1/2 < \vartheta(\zeta(z)) < 1$, we have that $\theta(z) \to \theta_0$, $\tau \circ \phi(-\theta(z), z) \to \tau \circ \phi(-\theta_0, z_0) = \tau(z_1)$, $\vartheta'(\zeta(z)) \to \xi = -\tau(z_1) + \tau(z_0) + \theta_0$ and by replacing in (11.1) we obtain that $\tau(z) \to \tau(z_0)$. Therefore τ is continuous. On the other hand it is easy to see that $\phi(s\tau(z), z) \in R'$ for all $z \in R_\delta$, $s \in [0,1]$. The assertion is proved.

Define the map

$$f : R_{\delta'} \mapsto R', \quad f(z) = \phi(\tau(z), z).$$

This map f is an extension of the map $f : \mathcal{D} \to \Sigma$ given by Proposition 32. Given $\zeta = (t_\zeta, 0) \in C$, let $g(\zeta) = (t'_\zeta, 0)$ and define the sets

$$\mathcal{D}_\zeta = h(\{(t_\zeta, x) : 0 < |x| < \delta\}),$$
$$\Sigma_\zeta = \{(t'_\zeta, x') : |x'| < b'\}.$$

Observe that $f(z) \in \Sigma_\zeta$ for all $z \in \mathcal{D}_\zeta \cap R_{\delta'}$. Moreover, the map $f_\zeta = f|_{\mathcal{D}_\zeta \cap R_{\delta'}} : \mathcal{D}_\zeta \cap R_{\delta'} \mapsto \Sigma_\zeta$ may be expressed as $f_\zeta = g' f_0 h g h^{-1}$, where $g(w) = \phi(-\vartheta(\zeta), w)$, $g'(w) = \phi(\vartheta'(\zeta), w)$ and $f_0 = f|_{\mathcal{D} \cap R_{\delta'}}$. Clearly g and g' are diffeomorphisms and by Proposition 32 the map f_0 is a homeomorphism. Then f_ζ is a homeomorphism onto its image and $f_\zeta(z)$ tends to the divisor as z tends to the divisor. Then we conclude that

(1) f is a homeomorphism onto its image,

(2) $f(z)$ tends to the divisor as z tends to the divisor,

(3) f maps $\mathcal{D}_\zeta \cap R_{\delta'}$ into the vertical Σ_ζ.

Observe that, for some $\delta_1 > 0$, $f \circ h$ maps each vertical $\{(t_\zeta, x) : 0 < |x| < \delta_1\}$ into the vertical $(t'_\zeta, x') : 0 < |x'| < b'\}$.

Now, for some $\varepsilon > 0$, $\delta'' > 0$, we will extend f to the set $V = \{(t', x') : a' - \varepsilon \leq |t'| \leq a' + \varepsilon, 0 < |x'| < \delta''\}$. Take first any $\delta'' \in (0, \delta')$. For $\varepsilon > 0$ small enough we may extend the flow ϕ in the natural way:

(1) ϕ is defined on V,

(2) $\mathcal{F}'$ is invariant by ϕ,

(3) $\phi(s,z) = (e^{2\pi i s}t', *)$ whenever $z = (t', *)$.

By reducing ε if necessary we have the following property: given $z \in V$, there is a path $\alpha_z : [0,1] \mapsto \{(t,x) : 0 < |x| < b\}$ such that

(1) α_z is contained in the leaf of $\mathcal{F}$ and $\alpha_z(0) = h^{-1}(z)$,

(2) $\alpha_z(s) = (t_z(s), x_z(z))$ with $t(s) = (1-s)t_z(0) + sa\frac{t_z(0)}{|t_z(0)|}$,

that is, α_z is the lifting to a leaf of a radial segment in $\{x = 0\}$ such that $\alpha_z(0) = h^{-1}(z)$ and $\alpha_z(1) \in R$. Let $\gamma_z(s) = h \circ \alpha_z(s) = (t'_z(s), x'_z(s))$. There is a continuous function $\theta_z : [0,1] \mapsto \mathbb{R}$ with $\theta_z(0) = 0$ and such that

$$t'_z(s) = \frac{t'_z(0)}{|t'_z(0)|}|t'_z(s)|e^{2\pi i \theta_z(s)}.$$

Observe that $\gamma_z(1) \in R'$ for all $z \in V$ and we may assume $\gamma_z(1) \in R_\delta$ if δ'' is taken small enough. Then we extend τ and f by the expressions

$$\tau(z) = \theta_z(1) + \tau(\gamma_z(1))$$

and

$$f(z) = \phi(\tau(z), z).$$

It is easy to see that these functions are continuous. Let $R_{\delta''}(r) = \{|t'| = r, 0 < |x'| < \delta''\}$ and $R'(r) = \{|t'| = r, 0 < |x'| < b'\}$. Let $t_0 \in \mathbb{C}$ be such that $h(\{(t_0, x) : 0 < |x| < \delta_0\})$ is contained in $R'(r)$. We may write $t_0 = ku_0$ with $k > 0$ and $|u_0| = a$. We know $h(\{(u_0, x) : 0 < |x| < \delta_0\})$ is mapped by f homeomorphically into a set $\{(u'_0, x') : 0 < |x'| < b'\}$ with $|u'_0| = a'$. It follows from the construction that, if $\mathcal{D}(t_{0,\epsilon}) = h(\{(t_0, x) : 0 < |x| < \epsilon\})$ is contained in $R'(r)$, then $\mathcal{D}(t_{0,\epsilon})$ is mapped by f homeomorphically into $\Sigma(t_0) = \{((r \backslash a')u'_0, x') : 0 < |x'| < b'\}$. Then f maps each $R_{\delta'}(r)$ homeomorphically into $R'(r)$. Moreover, it is not difficult to see that

(1) $\phi(s\tau(z), z) \in R'(r)$ for all $z \in R_{\delta'}$, $s \in [0,1]$,

(2) for all $\rho \in [0,1]$ we have that $g_\rho(z) = \phi(\rho\tau(z), z)$, maps $R_{\delta'}(r)$ homeomorphically into $R'(r)$,

(3) g_ρ tends to the divisor as z tends to the divisor.

Now, take $\rho : [a' - \varepsilon, a' + \varepsilon] \mapsto [0,1]$ such that $\rho(a' - \varepsilon) = \rho(a' + \varepsilon) = 0$ and $\rho = 1$ on a neighborhood of a' and define

$$F(z) = \phi(\rho(r)\tau(z), z) \quad \text{if } z \in R_{\delta'}(r).$$

It is easy to see that

(1) F preserves the leaves of $\mathcal{F}$,

(2) F maps V homeomorphically onto its image,

(3) $F = \text{id}$ on $R_{\delta'}(a' - \varepsilon) \cup R'_{\delta'}(a' + \varepsilon)$,

(4) if $\epsilon > 0$ is small and $|t_0|$ is close to a, then F maps each set $\mathcal{D}(t_0, \epsilon) = h(\{(t_0, x) : 0 < |x| < \epsilon\})$ homeomorphically into a vertical $\{t' = cte\}$,

(5) $F(z)$ tends to the divisor as z tends to the divisor.

We may extend F to a topological equivalence of $\mathcal{F}'$ with itself.

From above we have that $\tilde{h} = F \circ h$ is a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$. By reducing b if necessary we may assume that

(1) $\tilde{h}$ maps $\{|t| \le a, 0 < |x| \le b\}$ into $\{|t'| \le a', 0 < |x'| \le b'\}$,

(2) there are numbers $a_1 \in (0, a)$, $a'_1 \in (0, a')$ such that $\tilde{h}$ extends as a homeomorphism to the set $\{(t, 0) : a_1 \le |t| \le a\}$ which is mapped onto $\{(t', 0) : a'_1 \le |t| \le a'\}$.

Let $P = \{|t| \le a, < |x| \le b\}$ and $P' = \{|t'| \le a', 0 < |x'| \le b'\}$. By Lemma 31 there are homeomorphisms g and g' such that

(1) g maps $P \backslash \{(t, 0) : |t| \le a_1\}$ onto $P \backslash \{(0, 0)\}$,

(2) g' maps $P' \backslash \{(t', 0) : |t'| \le a'_1\}$ onto $P' \backslash \{(0, 0)\}$,

(3) g and g' preserve the leaves of $\mathcal{F}$ and $\mathcal{F}'$ respectively,

(4) g maps $\{(t, 0) : a_1 < |t| \le a\}$ onto $\{(t, 0) : 0 < |t| \le a\}$ with $g(t, 0) \to (0, 0)$ as $|t| \to a_1$,

(5) g' maps $\{(t', 0) : a'_1 < |t'| \le a'\}$ onto $\{(t', 0) : 0 < |t'| \le a'\}$ with $g(t', 0) \to (0, 0)$ as $|t'| \to a'_1$,

(6) $g = \text{id}$ and $g' = \text{id}$ on $\{|t| = a, |x| \le b\}$ and $\{|t'| = a', |x'| \le b'\}$ respectively.

We may extend g and g' to topological equivalences of $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then $\bar{h} = g' \circ \tilde{h} \circ g^{-1}$ is a topological equivalence between $\mathcal{F}$ and $\mathcal{F}'$ and it is easy to see that $\bar{h}$ extends to P as a leaf preserving homeomorphism.

Proof of Corollary 6. If the projective holonomy is non-solvable, we can construct a topologically equivalence extending after resolution (see Remark 34). Since the equivalence is transversely holomorphic, by a well known lifting path argument we can modify this equivalence near each non-nodal singularity to obtain a topologically equivalence $\bar{h}$ which is holomorphic near each such singularity. The last statement of the corollary follows from Proposition 13. □

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References

[1] L. Bers, *Riemann surfaces*. Lectures given at the Institute of Mathematical Sciences, New York University, New York 1957–1958.

[2] W. Burau, Kennzeichnung der Schlauchknoten. *Abh. Math. Sem. Ham. Univ.* **9** (1932), 125–133. Zbl 0006.03402 MR 3069587

[3] C. Camacho, A. Lins and P. Sad, Topological invariants and equidesingularization for holomorphic vector fields. *J. Differential Geom.* **20** (1984), 143–174. Zbl 0576.32020 MR 0772129

[4] C. Camacho and P. Sad, Pontos singulares de equações diferenciais analíticas. 16^o Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro 1987. MR 0953780

[5] L. Camara, Non-linear analytic differential equations and their invariants. Tese, IMPA 2001.

[6] J. Dieudonné, *Foundations of modern analysis*. Pure and Applied Mathematics 10, Academic Press, New York 1960. Zbl 0100.04201 MR 0120319

[7] A. Dold, *Lectures on algebraic topology*. Grundlehren Math. Wiss. 200, Springer-Verlag, Berlin 1972. Zbl 0234.55001 MR 0415602

[8] X. Gómez Mont, J. Seade and A. Verjovsky, The index of a holomorphic flow with an isolated singularity. *Math. Ann.* **291** (1991), no. 4, 737–751. Zbl 0725.32012 MR 1135541

[9] F. Loray, *5 leçons sur la structure transverse d'une singularité de feuilletage holomorphe en dimension 2 complexe.* Monographies Red TMR Europea Sing. Ec. Dif. Fol. 1 (1999), p. 1–92.

[10] J. F. Mattei and D. Cerveau, Formes intégrables holomorphes singulières. *Astérisque* **97** (1982), Soc. Math. France, Paris. Zbl 0545.32006 MR 0704017

[11] J. F. Mattei and R. Moussu, Holonomie et intégrales premiéres. *Ann. Sci. Éc. Norm. Supér.* (4) **13** (1980), no. 4, 469–523. Zbl 0458.32005 MR 0608290

[12] D. Marín, Moduli spaces of germs of holomorphic foliations in the plane. *Comment. Math. Helv.* **78** (2003), 518–539. Zbl 1054.32018 MR 1998392

[13] D. Marín and J. F. Mattei, Incompressibilité des feuilles de germes de feuilletages holomorphes singuliers. *Ann. Sci. Éc. Norm. Supér.* (4) **41** (2008), no. 6, 855–903. Zbl 1207.32028 MR 2504107

[14] D. Marín and J. F. Mattei, Mapping class group of a plane curve germ. *Topology Appl.* **158** (2011), 1271–1295. Zbl 1273.32034 MR 2806361

[15] D. Marín and J. F. Mattei, Monodromy and topological classification of germs of holomorphic foliations. *Ann. Sci. Éc. Norm. Supér.* (4) **3** (2012), 405–445. Zbl 06109719 MR 3014482

[16] L. Ortíz-Bobadilla, E. Rosales-Gonzáles and S. M. Voronin, Extended holonomy and topological invariance of vanishing holonomy group. *J. Dyn. Control Syst.* **14** (2008), no. 3, 299–358. Zbl 1203.32012 MR 2425303

[17] Ch. Pommerenke, *Boundary behaviour of conformal maps.* Grundlehren Math. Wiss. 299, Springer-Verlag, Berlin 1992. Zbl 0762.30001 MR 1217706

[18] J. Rebelo, On transverse rigidity for singular foliations in $(\mathbb{C}^2, 0)$. *Ergodic Theory Dynam. Systems* **31** (2011), 935–950. Zbl 1232.37030 MR 2794955

[19] R. Remmert, *Classical topics in complex function theory*. Grad. Texts in Math. 172, Springer-Verlag, New York 1998. Zbl 0895.30001 MR 1483074

[20] W. Rudin, *Real and complex analysis*. McGraw-Hill, New York 1987. Zbl 0925.00005 MR 0924157

[21] R. Rosas, The differentiable invariance of the algebraic multiplicity of a holomorphic vector field. *J. Differential Geom.* **83** (2009), 337–376. Zbl 1188.32007 MR 2577472

[22] M. E. Taylor, *Partial differential equations I*. Appl. Math. Sci. 115, Springer-Verlag, New York 1996. Zbl 0869.35002 MR 1395148

[23] O. Zariski, On the topology of algebroid singularities. *Amer. J. Math.* **54** (1932), 453–465. JFM 58.0614.02 MR 1507926

[24] D. Cerveau and P. Sad, Problémes de modules pour les formes différentielles singulières dans le plan complexe. *Comment. Math. Helv.* **61** (1986), no. 2, 222–253. Zbl 0604.58004 MR 0856088

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