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Embedding functors and their arithmetic properties

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Abstract. In this article, we focus on how to embed a torus T into a reductive group G with respect to a given root datum Ψ over a scheme S. This problem is also related to embedding an étale algebra with involution into a central simple algebra with involution (cf. [PR10]). We approach this problem by defining the embedding functor, which is representable and is a left homogeneous space over S under the automorphism group of G. In order to fix a connected component of the embedding functor, we define an orientation u of Ψ with respect to G. We show that the oriented embedding functor is also representable and is a homogeneous space under the adjoint action of G. Over a local field, the orientation u and the Tits index of G determine the existence of an embedding of T into G with respect to the given root datum Ψ. We also use the techniques developed in Borovoi's paper [Bo99] to prove that the local-global principle holds for oriented embedding functors in certain cases. Actually, the Brauer–Manin obstruction is the only obstruction to the local-global principle for the oriented embedding functor. Finally, we apply the results on oriented embedding functors to give an alternative proof of Prasad and Rapinchuk's Theorem, and to improve Theorem 7.3 in [PR10].

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Introduction

Let K be a field, A be a central simple algebra over K with involution τ, and E be an étale algebra over K with involution σ. Suppose that $\tau|_K = \sigma|_K$. Let k be the field of invariants K^τ, which is a global field. Motivated by the weak commensurability and length-commensurability between locally symmetric spaces ([PR09]), Prasad and Rapinchuk discuss in [PR10] the local-global principle for embeddings of (E, σ) into (A, τ) over K. This embedding problem is also related to studying the condition under which the isomorphism classes of simple groups are determined by their isomorphism classes of maximal tori over a number field (cf. [Ga12] and [PR09], Thm. 7.5).

Motivated by the work of Prasad and Rapinchuk, we consider the embedding problem of a twisted root datum. Loosely speaking, a twisted root datum is a torus equipped with some extra data related to the roots. In this article, we transform such embedding problem of algebras into a embedding problem of algebraic groups, in order to work in a more conceptual framework. Moreover, in this framework, our criteria can be applied not only to the classical groups but also to the exceptional groups. Instead of a global field, we work over an arbitrary scheme.

Let S be a scheme and G be a reductive group scheme over S. Given an S-torus T and a twisted root datum Ψ associated to T, we want to know when it is possible to embed T in G so that the corresponding twisted root datum $\Phi(\mathrm{G},\mathrm{T})$ is isomorphic to Ψ. To approach this problem, we first define the *embedding functor* $\mathfrak{E}(\mathrm{G},\Psi)$. Roughly speaking, each point of the embedding functor is a closed immersion f from T to G such that the twisted root datum $\Phi(\mathrm{G}, f(\mathrm{T}))$ is isomorphic to Ψ. For the formal definition, we refer to Section 1.1. Then our problem can be reformulated as: when is the set $\mathfrak{E}(\mathrm{G},\mathrm{T})(\mathrm{S})$ nonempty?

We first prove that the embedding functor is a sheaf for the étale topology (in the sense of big étale site). To be more precise, the embedding functor is a homogeneous sheaf under the action of the automorphism group $\underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G})$ over S, and it is a principal homogeneous space under the automorphism group $\underline{\mathrm{Aut}}(\Psi)$ over the scheme of maximal tori of G. Then by the result in [SGA3], Exp. X, 5.5, we conclude that the embedding functor $\mathfrak{E}(\mathrm{G},\Psi)$ is representable.

However, the embedding functor $\mathfrak{E}(\mathrm{G},\Psi)$ can be disconnected if $\underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G})$ is. Therefore, instead of dealing with $\mathfrak{E}(\mathrm{G},\Psi)$, we fix a particular connected component of $\mathfrak{E}(\mathrm{G},\Psi)$ which will be called an *oriented embedding functor*. The way we fix a connected component is to fix an *orientation* of Ψ with respect to G. An orientation between semisimple S-groups was previously defined by Petrov and Stavrova ([PS]). Here, we generalize it to an orientation between a twisted root datum Ψ and a reductive S-group G, which is an element u in $\underline{\mathrm{Isomext}}(\Psi,\mathrm{G})(\mathrm{S})$ (Section. 1.2.1). We show that the oriented embedding functor $\mathfrak{E}(\mathrm{G},\Psi,u)$ is homogeneous under the adjoint action of G over S and is principal homogeneous under the action of the Weyl group $\mathrm{W}(\Psi)$. Hence, $\mathfrak{E}(\mathrm{G},\Psi,u)$ is also representable (ref. [SGA3], Exp. X, 5.5). Moreover, in Theorem 3.12, we show that over a local field L, the orientation together with the Tits index of the given group determine the existence of L-points of the oriented embedding functor.

The main application of embedding functors is to the embedding problem of Azumaya algebras with involutions. Let $\widetilde{R}$ be a commutative ring, where 2 is invertible in $\widetilde{R}$. Let E be an étale algebra over $\widetilde{R}$ with involution σ and A be an Azumaya algebra with involution τ. Suppose $\sigma|_{\widetilde{R}} = \tau|_{\widetilde{R}}$. We ask when (E,σ) can be embedded into (A,τ). The case where $\widetilde{R}$ is a global field is discussed in Prasad and Rapinchuk's paper ([PR10]).

Over a commutative ring $\widetilde{R}$ in which 2 is an invertible element, we let R be the ring of invariants $\widetilde{R}^{\tau}$. If $\widetilde{R}$ is equal to R, then τ is said to be of the first kind. If $\widetilde{R}$ is a quadratic extension of R, then τ is said to be of the second kind. We consider the reductive group $\mathrm{G} = \mathrm{U}(\mathrm{A},\tau)^{\circ}$, the torus $\mathrm{T} = \mathrm{U}(\mathrm{E},\sigma)^{\circ}$ over R, and a twisted root datum Ψ attached to T. There is a nice correspondence between the R-points of the embedding functor $\mathfrak{E}(\mathrm{G},\Psi)$ and the embeddings $\iota\colon (\mathrm{E},\sigma)\to(\mathrm{A},\tau)$, namely:

Theorem. *Keep all the notation defined above. The set of k-embeddings from (E,σ) into (A,τ) is in one-to-one correspondence with the set of R-points of $\mathfrak{E}(\mathrm{G},\Psi)$, except for G of type D_4 or A of degree 2 with τ orthogonal.*

For G of type D_4, we have a finer treatment and we refer to Proposition 2.17. Moreover, for the involution τ of the second kind, we prove that there is an orientation u such that all R-points on the connected component $\mathfrak{E}(\mathrm{G},\Psi,u)$ are in one-to-one correspondence with R-embeddings from (E,σ) into (A,τ) (see Remark 2.16, Lemma 3.23).

The second part of this article is devoted to the arithmetic properties of the embedding functor. In particular, we want to know if the Hasse principle holds for the existence of k-points of the oriented embedding functor $\mathfrak{E}(\mathrm{G},\Psi,u)$, when k is a global field. Since $\mathfrak{E}(\mathrm{G},\Psi,u)$ is a homogeneous space under the group G whose stabilizer is a torus, we use the technique developed by Borovoi to solve this problem, cf. [Bo99]. Actually, in [Bo99], Borovoi proved that the Brauer–Manin obstruction to the Hasse principle is the only obstruction in this case. He also computed the obstruction using the Galois hypercohomology. We apply his result to show the following:

Theorem. *Let G, Ψ be as above, and T be the torus determined by Ψ. Let $u\in \underline{\mathrm{Isomext}}(\Psi,\mathrm{G})(k)$ be an orientation. Suppose that Ψ satisfies one of the following conditions:*

1. *all connected components of $\underline{\mathrm{Dyn}}(\Psi)(k^{\mathrm{s}})$ are of type C, where k^{s} is a separable closure of k.*
2. *T is anisotropic at some place $v\in\Omega_k$.*

Then the local-global principle holds for the existence of a k-point of the oriented embedding functor $\mathfrak{E}(\mathrm{G},\Psi,u)$. In particular, when Ψ is generic, the local-global principle holds.

Finally, for a global field k of characteristic different from 2, we combine these techniques and the correspondence established in Theorem 2.15 and Proposition 2.17 to give an alternative proof of Theorem A and Theorem 6.7 in [PR10]. Besides, we provide an example (3.22) to show that the Hasse principle fails in some cases when the involution τ is orthogonal and A is $\mathbf{M}_{2m}(\mathrm{D})$, where D is a division algebra over K. The main reason for the failure is that the embedding functor $\mathfrak{E}(\mathrm{G},\Psi)$ is disconnected in this case. Let $\mathfrak{E}(\mathrm{G},\Psi)=\mathrm{X}_1\coprod\mathrm{X}_2$. Then it may happen that the embedding functor has a k_v-point at each place v, but only X_1 has a k_{v_1}-point, at some place v_1, and only X_2 has a k_{v_2} point, at another place v_2. This explains the failure of the Hasse principle.

1. Some general facts and notation

In this section, we briefly recall the notation and definitions which will be used later. We also state some well-known theorems which are necessary for the development of the main results about the embedding functor. Most of the material here can be found in [SGA3], and in the Appendix A of the book by Conrad, Gabber, and Prasad [CGP].

1.1. Notation and conventions. Let S be a scheme and S' an S-scheme. For an S-scheme X, we let $X_{S'}$ be the scheme $X \underset{S}{\times} S'$ over S'. For a set Λ, we let Λ_S denote the disjoint union of the schemes S_i, where $i \in \Lambda$ and each S_i is isomorphic to S, i.e. $\Lambda_S = \coprod_{i\in\Lambda} S_i$. We call Λ_S the constant scheme over S of type Λ. (ref. [SGA3], Exp. I, 1.8).

Let Sch/S be the category of all S-schemes. Throughout this article, the étale site of S means the big étale site. Namely, we equip the category Sch/S with the following topology: for an S-scheme U, $\{U_i \xrightarrow{f_i} U\}_i$ is a covering of U if for each i, f_i is an étale morphism and $U = \bigcup_i f_i(U_i)$. For a detailed introduction to Grothendieck topology, we refer to the lecture notes by Brochard [Br].

1.2. Torsors and homogeneous spaces. Let G be an S-group sheaf for étale topology. Let $\mathcal{F}$ and X be S-sheaves. Let $p\colon \mathcal{F} \to X$ be a morphism between S-sheaves. Then $\mathcal{F}$ is called a right (resp. left) G-sheaf over X with respect to p if $\mathcal{F}$ is equipped with an G-action satisfying $p(fg) = p(f)$ (resp. $p(gf) = p(f)$) for all $(f, g) \in (\mathcal{F}\times G)(S')$ and for all S-schemes S'. Note that $X\times G$ can be equipped with a right G-action as $(x, g)\sigma = (x, g\sigma)$. Let p_X be the projection from $X\times G$ to X. Then $X\times G$ is a right G-sheaf over X with respect to p_X. A G-sheaf $\mathcal{F}$ over X with respect to p is *trivial* if it is isomorphic to $X\times G$ with respect to p_X as G-sheaves over X. A right G-sheaf $\mathcal{F}$ over X with respect to p is called a G-*torsor* over X if there is an epimorphism of S-sheaves $\pi\colon Y \to X$ such that $Y \underset{X}{\times} \mathcal{F}$ over Y is a trivial G-sheaf.

Proposition 1.1. *Let* G *be an* S*-group sheaf,* X *be a sheaf over* S*. Let* $\mathcal{F}$ *be a* G*-sheaf over* X *with respect to an* S*-sheaf morphism* $p\colon \mathcal{F} \to X$*. Then* $\mathcal{F}$ *is a torsor over* X *if and only if* p *is an epimorphism of* S*-sheaves and the morphism* $i\colon \mathcal{F}\times G \to \mathcal{F} \underset{X}{\times} \mathcal{F}$ *defined as* $i(x, h) = (x, xh)$ *is invertible.*

Proof. [DG70], Chap. III, §4, Corollary 1.7. □

Let G be an S-group sheaf. Let $\mathcal{F}$ and X be S-sheaves. Let $p\colon \mathcal{F} \to X$ be a morphism between S-sheaves. A G-sheaf $\mathcal{F}$ over X with respect to p is called a G-*homogeneous space* if p is an epimorphism of S-sheaves and the morphism $i\colon \mathcal{F}\times G \to \mathcal{F} \underset{X}{\times} \mathcal{F}$ defined as $i(x, h) = (x, xh)$ is an epimorphism between sheaves .

1.3. Root data and twisted root datum. Let $\psi = (\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$ be a root datum (ref. [SGA3], Exp. XXI, 1.1.1). Let $\Delta \subseteq \mathrm{R}$ be a system of simple roots of R. The root datum ψ plus a system of simple roots Δ of R is called a *pinning root datum*, and we denote it as $(\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee, \Delta)$.

The subgroup of the automorphism group of M generated by the reflections $\{s_\alpha\}_{\alpha\in\mathrm{R}}$ is called the Weyl group of ψ, and we denote it by $\mathrm{W}(\psi)$.

For a finite subset R (resp. $\mathrm{R}^\vee$)of M (resp. $\mathrm{M}^\vee$), we let $\Gamma_0(\mathrm{R})$ (resp. $\Gamma_0(\mathrm{R}^\vee)$) be the subgroup generated by R (resp. $\mathrm{R}^\vee$) and we let $\mathcal{V}(\mathrm{R})$ (resp. $\mathcal{V}(\mathrm{R}^\vee)$ be the vector space defined by $\Gamma_0(\mathrm{R}) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\Gamma_0(\mathrm{R}^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$).

A root datum is called *reduced* if for all $\alpha \in \mathrm{R}$, we have $2\alpha \notin \mathrm{R}$. A root datum is called *semisimple* if $\mathrm{rank}(\Gamma_0(\mathrm{R})) = \mathrm{rank}(\mathrm{M})$. A root datum $(\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$ is called *adjoint* (resp. *simply connected*) if $\mathrm{M} = \Gamma_0(\mathrm{R})$ (resp. $\mathrm{M}^\vee = \Gamma_0(\mathrm{R}^\vee)$).

We define the dual root datum of ψ to be $(\mathrm{M}^\vee, \mathrm{M}, \mathrm{R}^\vee, \mathrm{R})$, and denote it as $\psi^\vee$.

1.3.1. Radical and coradical of root data. Let

$$\mathrm{N} = \{x \in \mathrm{M} \mid \alpha^\vee(x) = 0 \text{ for all } \alpha^\vee \in \mathrm{R}^\vee\}.$$

Then the dual of N can be identified with $\mathrm{M}^\vee/\mathcal{V}(\mathrm{R}^\vee) \cap \mathrm{M}^\vee$ (ref. [SGA3], Exp. XXI, 6.3.1).

Define the coradical of ψ to be the root datum $(\mathrm{N}, \mathrm{N}^\vee, \emptyset, \emptyset)$ and denote it as $\mathrm{corad}(\psi)$. We define the radical of ψ to be $\mathrm{corad}(\psi^\vee)^\vee$, and denote it as $\mathrm{rad}(\psi)$.

1.3.2. Induced and coinduced root data. Given a root datum $\psi = (\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$, and a subgroup N of M which contains $\Gamma_0(\mathrm{R})$, let $i_{\mathrm{N}} : \mathrm{N} \to \mathrm{M}$ be the natural inclusion, and $i_{\mathrm{N}}^\vee : \mathrm{M}^\vee \to \mathrm{N}^\vee$ be the corresponding map on $\mathrm{M}^\vee$. Let $\mathrm{R}_{\mathrm{N}} = \mathrm{R}$ and $\mathrm{R}_{\mathrm{N}}^\vee = i_{\mathrm{N}}^\vee(\mathrm{R}^\vee)$. We define the root datum ψ_{N} as $(\mathrm{N}, \mathrm{N}^\vee, \mathrm{R}_{\mathrm{N}}, \mathrm{R}_{\mathrm{N}}^\vee)$, which is called the *induced root datum* of ψ respect to N. If $\mathrm{N} = \Gamma_0(\mathrm{R})$, then ψ_{N} is an adjoint root datum, and we denote it as $\mathrm{ad}(\psi)$. If $\mathrm{N} = \mathcal{V}(\mathrm{R}) \bigcap \mathrm{M}$, then ψ_{N} is a semisimple root datum, and we denote ψ_{N} as $\mathrm{ss}(\psi)$. We let $\mathrm{der}(\psi) = \mathrm{ss}(\psi^\vee)^\vee$, and $\mathrm{sc}(\psi) = \mathrm{ad}(\psi^\vee)^\vee$.

1.3.3. Morphisms between root data. Let

$$\psi_1 = (\mathrm{M}_1, \mathrm{M}_1^\vee, \mathrm{R}_1, \mathrm{R}_1^\vee) \quad \text{and} \quad \psi_2 = (\mathrm{M}_2, \mathrm{M}_2^\vee, \mathrm{R}_2, \mathrm{R}_2^\vee)$$

be two root data. A module morphism $f : \mathrm{M}_1 \to \mathrm{M}_2$ is a *morphism between* ψ_1 *and* ψ_2 if f induces a bijection between R_1 and R_2 and the transpose map ${}^t f : \mathrm{M}_2^\vee \to \mathrm{M}_1^\vee$ is a bijection between $\mathrm{R}_2^\vee$ and $\mathrm{R}_1^\vee$.

Proposition 1.2. *Keep all the notation above. If* $f : \mathrm{M}_1 \to \mathrm{M}_2$ *is a morphism between* ψ_1 *and* ψ_2*, then* ${}^t f(f(\alpha)^\vee) = \alpha^\vee$*, and the map* $s_\alpha \to s_{f(\alpha)}$ *for* $\alpha \in \mathrm{R}_1$ *extends to an isomorphism between* $\mathrm{W}(\psi_1)$ *and* $\mathrm{W}(\psi_2)$*,* *Proof.* [SGA3], Exp. XXI, 6.1.1 and 6.2.2. □

Let $\mathrm{Aut}(\psi)$ be the automorphism group of ψ, and fix a system of simple roots Δ of R. Define the abstract group

$$\mathrm{E}_\Delta(\psi) = \{u \in \mathrm{Aut}(\psi) \mid u(\Delta) = \Delta\}.$$

Then we have the following:

Proposition 1.3. $\mathrm{W}(\psi)$ *is a normal subgroup of* $\mathrm{Aut}(\psi)$*, and* $\mathrm{Aut}(\psi)$ *is a semi-direct product of* $\mathrm{W}(\psi)$ *by* $\mathrm{E}_\Delta(\psi)$.

Proof. [SGA3], Exp. XXI, 6.7.1 and 6.7.2. □

1.3.4. Twisted root data. Let T be an S-torus. Let $\mathcal{M}$ be the *character group scheme* associated to T, i.e. $\mathcal{M}(\mathrm{S}') = \mathrm{Hom}_{\mathrm{S}'\text{-gr}}(\mathrm{T}_{\mathrm{S}'}, \mathbb{G}_{m,\,\mathrm{S}'})$. Let $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ be a twisted root datum associated to T (ref. [SGA3], Exp. 22, Def. 1.9).

The root datum Ψ is *split* if T is split. A twisted root datum is called *reduced* if for all S-schemes S' and all $\alpha \in \mathcal{R}(\mathrm{S}')$, we have $2\alpha \notin \mathcal{R}(\mathrm{S}')$.

Let $\psi = (\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$ be a root datum. A twisted root datum Ψ is said to have *type* ψ at the point s of S if $\Psi_{\bar{s}} \simeq (\mathrm{M}_{\bar{s}}, \mathrm{M}^\vee_{\bar{s}}, \mathrm{R}_{\bar{s}}, \mathrm{R}^\vee_{\bar{s}})$.

Let $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ be a twisted root datum. Since at each $s \in \mathrm{S}$, there is an étale neighborhood such that T splits, we can define $\mathrm{ad}(\Psi)$, $\mathrm{sc}(\Psi)$, $\mathrm{ss}(\Psi)$, $\mathrm{der}(\Psi)$ étale locally, and by the functoriality of induced root data, define them over S by descent ([SGA3], Exp. XXI, 6.5).

Let $\Psi_1 = (\mathcal{M}_1, \mathcal{M}_1^\vee, \mathcal{R}_1, \mathcal{R}_1^\vee)$, $\Psi_2 = (\mathcal{M}_2, \mathcal{M}_2^\vee, \mathcal{R}_2, \mathcal{R}_2^\vee)$ be two twisted root data. Let T_1, T_2 be the tori determined by Ψ_1 and Ψ_2 respectively. An S-group morphism $f : \mathrm{T}_2 \to \mathrm{T}_1$ is a *morphism from* Ψ_1 *to* Ψ_2 if f induces an isomorphism from $\mathcal{R}_1$ to $\mathcal{R}_2$ and an isomorphism from $\mathcal{R}_2^\vee$ to $\mathcal{R}_1^\vee$. We can also define the induced twisted root data by étale descent, and define $\mathrm{ad}(\Psi)$, $\mathrm{ss}(\Psi)$, $\mathrm{der}(\Psi)$ and $\mathrm{sc}(\Psi)$ as we have done for the root data.

1.3.5. Weyl groups, Isom, Isomext and Isomint for twisted root data. Let Ψ be a twisted root datum. Suppose Ψ is split and $\Psi = (\mathrm{M}_\mathrm{S}, \mathrm{M}_\mathrm{S}^\vee, \mathrm{R}_\mathrm{S}, \mathrm{R}_\mathrm{S}^\vee)$. Let ψ be the root datum $(\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$. Let W be the Weyl group of ψ, and define $\mathrm{W}(\Psi) = \mathrm{W}_\mathrm{S}$. Suppose that Ψ is not split. Then we can find an étale covering $\{\mathrm{S}_i \to \mathrm{S}\}$ such that Ψ_{S_i} is split. By Proposition 1.6, the canonical isomorphism between $(\Psi_{\mathrm{S}_i})_{\mathrm{S}_j}$ and $(\Psi_{\mathrm{S}_j})_{\mathrm{S}_i}$ gives a canonical isomorphism between $\mathrm{W}(\Psi_{\mathrm{S}_i})_{\mathrm{S}_j}$ and $\mathrm{W}(\Psi_{\mathrm{S}_j})_{\mathrm{S}_i}$ and hence gives a descent data for $\{\mathrm{W}(\Psi_{\mathrm{S}_i})\}_i$, which allows us to define $\mathrm{W}(\Psi)$.

Let $\underline{\mathrm{Aut}}(\Psi)$ be the automorphism functor of Ψ. By descent, we can define the Weyl group $\mathrm{W}(\Psi)$. Then by Proposition 1.3, we can define the following exact sequence by étale descent:

$$1 \longrightarrow \mathrm{W}(\Psi) \longrightarrow \underline{\mathrm{Aut}}(\Psi) \longrightarrow \underline{\mathrm{Autext}}(\Psi) \longrightarrow 1.$$

Then we have the following proposition:

Proposition 1.4. *Keep the notation above. The automorphism group* $\underline{\mathrm{Aut}}(\Psi)$ *is representable by a twisted constant* S*-scheme, and* $\mathrm{W}(\Psi)$ *is normal in* $\underline{\mathrm{Aut}}(\Psi)$*. Let* $\underline{\mathrm{Autext}}(\Psi)$ *be the quotient group of* $\underline{\mathrm{Aut}}(\Psi)$ *by* $\mathrm{W}(\Psi)$*. Then* $\underline{\mathrm{Autext}}(\Psi)$ *is also representable by a twisted constant* S*-scheme.*

Proof. Note that we can find an étale covering $\{\mathrm{S}_i \to \mathrm{S}\}_i$ such that Ψ_{S_i} is split ([SGA3], Exp. X, 4.5). By Proposition 1.3 , $\underline{\mathrm{Aut}}(\Psi_{\mathrm{S}_i})$ and $\underline{\mathrm{Autext}}(\Psi_{\mathrm{S}_i})$ are constant group schemes over S_i. By [SGA3], Exp. X, 5.5, $\{\mathrm{S}_i \to \mathrm{S}\}_i$ gives an effective descent datum, so $\underline{\mathrm{Aut}}(\Psi_{\mathrm{S}_i})$ and $\underline{\mathrm{Autext}}(\Psi_{\mathrm{S}_i})$ are representable. □

Let Ψ_1, Ψ_2 be two twisted root data. Suppose Ψ_2 is a twisted form of Ψ_1. Let $\underline{\mathrm{Isom}}(\Psi_1, \Psi_2)$ be the isomorphism functor between Ψ_1 and Ψ_2. Then $\underline{\mathrm{Isom}}(\Psi_1, \Psi_2)$ is a right principal homogeneous space of $\underline{\mathrm{Aut}}(\Psi_1)$ and a left principle homogeneous of $\underline{\mathrm{Aut}}(\Psi_2)$. Since $\underline{\mathrm{Aut}}(\Psi_2)$ is representable, $\underline{\mathrm{Isom}}(\Psi_1, \Psi_2)$ is also representable.

Define $\underline{\mathrm{Isomext}}(\Psi_1, \Psi_2) = \mathrm{W}(\Psi_2) \setminus \underline{\mathrm{Isom}}(\Psi_1, \Psi_2)$.

Note that for $f \in \underline{\mathrm{Isom}}(\Psi_1, \Psi_2)(\mathrm{S})$, we have $f^{-1} \circ \mathrm{W}(\Psi_2) \circ f = \mathrm{W}(\Psi_1)$ by Proposition 1.2. Therefore we have a natural isomorphism from $\underline{\mathrm{Isomext}}(\Psi_1, \Psi_2)$ to $\underline{\mathrm{Isom}}(\Psi_1, \Psi_2)/\mathrm{W}(\Psi_1)$. Then $\underline{\mathrm{Isomext}}(\Psi_1, \Psi_2)$ is a left $\underline{\mathrm{Autext}}(\Psi_2)$-principal homogeneous space and a right $\underline{\mathrm{Autext}}(\Psi_1)$-principal homogeneous space. An *orientation* of Ψ_1 with respect to Ψ_2 is an S-point of $\underline{\mathrm{Isomext}}(\Psi_1, \Psi_2)$.

Suppose that there is $u \in \underline{\mathrm{Isomext}}(\Psi_1, \Psi_2)(\mathrm{S})$. Then we can regard S as an $\underline{\mathrm{Isomext}}(\Psi_1, \Psi_2)$-scheme through u and define

$$\underline{\mathrm{Isomint}}_u(\Psi_1, \Psi_2) := \mathrm{S} \underset{\underline{\mathrm{Isomext}}(\Psi_1,\Psi_2)}{\times} \underline{\mathrm{Isom}}(\Psi_1, \Psi_2).$$

1.4. Reductive groups. An S-group scheme G is called *reductive (resp. semisimple)* if it is affine and smooth over S, and all the geometrical fibers are connected and reductive (resp. semisimple) (ref. [SGA3], Exp. XIX, Def. 2.7).

Let G be a reductive S-group scheme and suppose that T is a maximal torus in G. We let $\Phi(\mathrm{G}, \mathrm{T})$ be the twisted root datum of G with respect to T (ref. [SGA3], Exp. XXII, 1.10).

For a point $s \in \mathrm{S}$, let $\kappa(s)$ be the residue field of s and $\overline{\kappa(s)}$ be the algebraic closure of $\kappa(s)$. Let $\bar{s}$ be the scheme $\mathrm{Spec}(\overline{\kappa(s)})$. The *type* of G at s is the type of $\Phi(\mathrm{G}_{\bar{s}}, \mathrm{T}_0)$, where T_0 is a maximal torus of $\mathrm{G}_{\bar{s}}$ (ref. [SGA3], Exp. 22, Def. 2.6.1, 2.7). Note that the type of G is locally constant over S (ref [SGA3], Exp. 22, Prop. 2.8).

A reductive S-group G is *split* if there is a maximal torus T of G and a root datum $(\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$ such that $\Phi(\mathrm{G}, \mathrm{T}) \simeq (\mathrm{M}_\mathrm{S}, \mathrm{M}_\mathrm{S}^\vee, \mathrm{R}_\mathrm{S}, \mathrm{R}_\mathrm{S}^\vee)$ and satisfying the following:

1. S is nonempty and each root $\alpha \in \mathrm{S}$ (resp. $\alpha^\vee \in \mathrm{R}^\vee$) can be identified as a constant map from S to M (resp. $\mathrm{M}^\vee$).

2. Let $\mathfrak{g} = \mathrm{Lie}(G/S)$ and $\mathfrak{t} = \mathrm{Lie}(T/S)$. Under the adjoint action of T, $\mathfrak{g} = \mathfrak{t} \oplus \coprod_{\alpha \in R} \mathfrak{g}^{\alpha}$, where the $\mathfrak{g}^{\alpha}$'s are free $\mathcal{O}_S$-modules.

In this case, we say that G is *split relatively to* T (ref. [SGA3], Exp. XXII, 1.13 and 2.7).

Let us endow S with the étale topology. Let S′ be an S-scheme, and G be an S-group scheme. Let $\underline{\mathrm{Aut}}_{\text{S-gr}}(G)$ be the sheaf of group automorphisms of G. Then we can define the group homomorphism

$$\mathrm{ad}\colon G \to \underline{\mathrm{Aut}}_{\text{S-gr}}(G)$$

which maps an element g of $G(S')$ to an automorphism of $G_{S'}$ defined by the conjugation by g. Let $\underline{\mathrm{Centr}}(G)$ be the center of G. Then the image sheaf of ad is isomorphic to $G/\underline{\mathrm{Centr}}(G)$ and $\mathrm{ad}(G)$ is normal in $\underline{\mathrm{Aut}}_{\text{S-gr}}(G)$. So we have the exact sequence of S-group sheaves:

$$1 \longrightarrow \mathrm{ad}(G) \longrightarrow \underline{\mathrm{Aut}}_{\text{S-gr}}(G) \longrightarrow \underline{\mathrm{Autext}}(G) \longrightarrow 1.$$

Theorem 1.5. *Let* S *be a scheme and* G *be a reductive* S*-group scheme. For the exact sequence of* S*-sheaves:*

$$1 \longrightarrow \mathrm{ad}(G) \longrightarrow \underline{\mathrm{Aut}}_{S\text{-}gr}(G) \xrightarrow{p} \underline{\mathrm{Autext}}(G) \longrightarrow 1,$$

we have the following:

(i) $\underline{\mathrm{Aut}}_{S\text{-}gr}(G)$ *is represented by a separated, smooth* S*-scheme.*

(ii) $\underline{\mathrm{Autext}}(G)$ *is represented by a twisted finitely generated constant scheme.*

(iii) *Suppose that* G *splits relatively to* T*, and* $\Phi(G,T) \simeq (M_S, M_S^{\vee}, R_S, R_S^{\vee})$. *Let* $(\psi, \Delta) = (M, M^{\vee}, R, R^{\vee}, \Delta)$ *be a pinning root datum. Then there is a monomorphism between sheaves* $a\colon E_{\Delta}(\psi)_S \to \underline{\mathrm{Aut}}_{S\text{-}gr}(G)$ *such that*

$$p \circ a\colon E_{\Delta}(\psi)_S \to \underline{\mathrm{Autext}}(G)$$

is an isomorphism.

Proof. [SGA3], Exp. XXIV, Theorem 1.3. □

For a subgroup scheme H of G, let $\underline{\mathrm{Aut}}_{\text{S-gr}}(G,H)$ be the subsheaf of $\underline{\mathrm{Aut}}_{\text{S-gr}}(G)$ which normalizes H, i.e. $\underline{\mathrm{Aut}}_{\text{S-gr}}(G,H) = \underline{\mathrm{Norm}}_{\underline{\mathrm{Aut}}_{\text{S-gr}}(G)}(H)$ (cf. [SGA3], Exp. $\mathrm{VI_B}$, Def. 6.1 (iii)). We let $\underline{\mathrm{Autext}}(G,H)$ be the quotient sheaf

$$\underline{\mathrm{Aut}}_{\text{S-gr}}(G,H)/\underline{\mathrm{Norm}}_{\mathrm{ad}(G)}(H).$$

1.5. Dynkin diagrams. For each reductive S-group G, we can associate a *Dynkin diagram scheme* $\underline{\mathrm{Dyn}}(\mathrm{G})$ to G (ref. [SGA3], Exp. XXIV, 3.2 and 3.3). Moreover we have the following:

Proposition 1.6. *If* G *is semisimple* (*resp. adjoint or simply connected*)*, then the morphism*

$$\underline{\mathrm{Autext}}(\mathrm{G}) \to \underline{\mathrm{Aut}}_{\mathrm{Dyn}}(\underline{\mathrm{Dyn}}(\mathrm{G}))$$

is a monomorphism (resp. isomorphism).

Proof. [SGA3], Exp. XXIV, 3.6. □

Given a twisted root datum Ψ over S, we can also define the Dynkin scheme of Ψ in a similar way and denote it by $\underline{\mathrm{Dyn}}(\Psi)$. We also have a natural morphism from $\underline{\mathrm{Autext}}(\Psi)$ to $\underline{\mathrm{Aut}}_{\mathrm{Dyn}}(\underline{\mathrm{Dyn}}(\Psi))$, which will be a monomorphism (resp. isomorphism) if Ψ is reduced semisimple (resp. reduced adjoint or reduced simply connected).

For a root datum ψ, we can associate to each connected component of its Dynkin diagram $\mathrm{Dyn}(\psi)$ a type according to the classification of Dynkin diagrams (ref. [SGA3], Exp. XXI, 7.4.6). Let $\mathbf{T}$ be the set of all types of Dynkin diagram. Similarly, for each Dynkin scheme $\mathbf{D}$ over S, we can associate the scheme of connected components $\mathbf{D}_0$ to D (ref. [SGA3], Exp. XXIV, 5.2). We can also define a morphism

$$a\colon \mathbf{D}_0 \to \mathbf{T}_{\mathrm{S}}.$$

Let $\mathbf{v} \in \mathbf{T}$. If $\mathbf{D}_0 = a^{-1}(\mathbf{v})$, then we say $\mathbf{D}$ is *isotypical of type* $\mathbf{v}$. If the Dynkin scheme $\underline{\mathrm{Dyn}}(\Psi)$ is connected at each fiber over S and is of constant type $\mathbf{v}$, then we say that Ψ is *simple* of type $\mathbf{v}$.

1.6. Parabolic subgroups. Let S be a scheme and G be a reductive S-group. A subgroup scheme P of G over S is called parabolic if

1. P is smooth over S.
2. For each $s \in \mathrm{S}$, the quotient $\mathrm{G}_{\bar{s}}/\mathrm{P}_{\bar{s}}$ is proper.

Let us keep the notation in Section 1.4. Let $\mathcal{E} = \{\mathrm{G}, \mathrm{T}, \mathrm{R}, \Delta, \{\mathrm{X}_\alpha\}_{\alpha\in\Delta}\}$ be a pinning of G and P be a parabolic subgroup. The pinning E is said to be adapted to P if P contains T and $\mathrm{Lie}(\mathrm{P}/\mathrm{S}) = \mathfrak{t} \oplus \coprod_{\alpha\in\mathrm{R}'} \mathfrak{g}^\alpha$, where R' is a subset of R which contains all the positive roots. In this case, we denote $\Delta(\mathrm{P}) = \Delta \cap -\mathrm{R}'$.

Let $\mathrm{Of}(\underline{\mathrm{Dyn}}(\mathrm{G}))$ be the functor defined as the following: for each S-scheme S', $\mathrm{Of}(\underline{\mathrm{Dyn}}(\mathrm{G}))(\mathrm{S}')$ is the set of all subschemes of $\underline{\mathrm{Dyn}}(\mathrm{G})_{\mathrm{S}'}$ which are open and closed. Then $\mathrm{Of}(\underline{\mathrm{Dyn}}(\mathrm{G}))$ is a twisted finite constant scheme. Let $\underline{\mathrm{Par}}(\mathrm{G})$ be the functor defined by $\underline{\mathrm{Par}}(\mathrm{G})(\mathrm{S}')$ is the set of all parabolic subgroups of G'_{S}, for each S-scheme S'. One can define a morphism

$$\mathbf{t}\colon \underline{\mathrm{Par}}(\mathrm{G}) \to \mathrm{Of}(\underline{\mathrm{Dyn}}(\mathrm{G}))$$

satisfying the following:

1. $\mathbf{t}$ is functorial in G.
2. If $\mathcal{E}$ is a pinning of G adapted to the parabolic subgroup P, then $\mathbf{t}(\mathrm{P}) = \Delta(\mathrm{P})_{\mathrm{S}}$.

For a parabolic subgroup P of G, we call $\mathbf{t}(\mathrm{P})$ the type of P.

Proposition 1.7. *Let* S, G *be as above. Let* P *be a parabolic subgroup of* G. *Let* t' *be a section of* $\mathrm{Of}(\underline{\mathrm{Dyn}}(\mathrm{G}))$ *over* S *and* $t' \supseteq \mathbf{t}(\mathrm{P})$. *Then there is a unique parabolic subgroup* P' *of* G *which contains* P *and the type of* P' *is* t'.

Proof. [SGA3], Exp. XXVI, Lemme 3.8. □

2. Embedding functors

Let S be a scheme and G be a reductive group over S. Let T be an S-torus and Ψ be a root datum associated to T. We would like to know if we can embed T in G as a maximal torus such that the twisted root datum $\Phi(\mathrm{G},\mathrm{T})$ is isomorphic to Ψ. To answer this question, we first define the embedding functor $\mathfrak{E}(\mathrm{G},\Psi)$. The embedding functor is representable and is a left $\underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G})$-homogeneous space. Briefly speaking, each S-point of $\mathrm{E}(\mathrm{G},\Psi)$ corresponds to an embedding from T to G with respect to Ψ.

In the second part, we first define an orientation v of Ψ with respect to G. Once we can fix an orientation, we can fix a connected component of $\mathrm{E}(\mathrm{G},\Psi)$, which is called an oriented embedding functor. The oriented embedding functor $\mathfrak{E}(\mathrm{G},\Psi,v)$ is also representable and is a left G-homogeneous space.

In the end of this section, we show that the embedding functor has an interpretation in the embedding problem of Azumaya algebras with involution. Moreover, we show that there is a one-to-one correspondence between the k-points of the embedding functor and the k-embeddings from an étale k-algebra with involution into an Azumaya algebra with involution.

2.1. Embedding functors. Let S be a scheme, G be a reductive S-group scheme. Let T be an S-torus. Let $\mathcal{M}$ be the character group scheme associated to T, and $\Psi = (\mathcal{M},\mathcal{M}^\vee,\mathcal{R},\mathcal{R}^\vee)$ be a root datum associated to T. We define the *embedding functor* by

$$\mathfrak{E}(\mathrm{G},\Psi)(\mathrm{S}') \left\{ f : \mathrm{T}_{\mathrm{S}'} \hookrightarrow \mathrm{G}_{\mathrm{S}'} \;\middle|\; \begin{array}{l} f \text{ is both a closed immersion and a group} \\ \text{homomorphism which induces an} \\ \text{isomorphism } f^{\Psi}\colon \Psi_{\mathrm{S}'} \xrightarrow{\sim} \Phi(\mathrm{G}_{\mathrm{S}'}, f(\mathrm{T}_{\mathrm{S}'})) \\ \text{such that } f^{\Psi}(\alpha) = \alpha \circ f^{-1}|_{f(\mathrm{T}_{\mathrm{S}'})} \text{ for all} \\ \alpha \in \mathcal{M}(\mathrm{S}''), \text{ for each } \mathrm{S}'\text{-scheme } \mathrm{S}'' \end{array} \right\}$$

for S' a scheme over S. In this article, we always assume that at each geometric point $\bar{s} \in \mathrm{S}$, the root datum $\Psi_{\bar{s}}$ is isomorphic to the root datum of $\mathrm{G}_{\bar{s}}$. Therefore, $\mathfrak{E}(\mathrm{G},\Psi)$ is not empty in our case.

The embedding functor $\mathfrak{E}(G, \Psi)$ is naturally equipped with a left $\underline{\mathrm{Aut}}_{S\text{-gr}}(G)$-action defined as compositions of functions. Namely define

$$l : \underline{\mathrm{Aut}}_{S\text{-gr}}(G) \times \mathfrak{E}(G, \Psi) \to \mathfrak{E}(G, \Psi)$$

as $l(\sigma, f) = \sigma \circ f$ for all $\sigma \in \underline{\mathrm{Aut}}_{S\text{-gr}}(G)(S')$, $f \in \mathfrak{E}(G, \Psi)(S')$ and S' an S-scheme.

Since $\underline{\mathrm{Aut}}(\Psi) \subseteq \underline{\mathrm{Aut}}_{S\text{-gr}}(\mathcal{M})$ and $\underline{\mathrm{Aut}}_{S\text{-gr}}(T) = \underline{\mathrm{Aut}}_{S\text{-gr}}(\mathcal{M})^{op}$, we can regard $\underline{\mathrm{Aut}}(\Psi)$ as a subgroup of $\underline{\mathrm{Aut}}_{S\text{-gr}}(T)$ through the inverse map between $\underline{\mathrm{Aut}}_{S\text{-gr}}(\mathcal{M})$ and $\underline{\mathrm{Aut}}_{S\text{-gr}}(\mathcal{M})^{op}$. We define a right $\underline{\mathrm{Aut}}(\Psi)$-action on $\mathfrak{E}(G, \Psi)$ as a composition of an automorphism of T followed by a closed embedding from $\mathfrak{E}(G, \Psi)$.

Now, let $\mathcal{T}$ be the scheme of maximal tori of G (cf. [SGA3], XII, 1.10). We think about the morphism $\pi : \mathfrak{E}(G, \Psi) \to \mathcal{T}$ defined as $\pi(f) = f(T_{S'})$, where $f \in \mathfrak{E}(G, \Psi)(S')$, and S' is a scheme over S. Then we have the following:

Theorem 2.1. *In the sense of the étale topology, $\mathfrak{E}(G, \Psi)$ is a homogeneous space over* S *under the left $\underline{\mathrm{Aut}}_{S\text{-}gr}(G)$-action, and a torsor over $\mathcal{T}$ under the right $\underline{\mathrm{Aut}}(\Psi)$-action. Moreover, $\mathfrak{E}(G, \Psi)$ is representable by an* S*-scheme.*

Proof. We divide the argument into the following three parts:

Claim. $\mathfrak{E}(G, \Psi)$ *is a sheaf for the étale topology.*

Proof. Let $\{S_i \to S\}$ be an étale covering. Since $\mathfrak{E}(G, \Psi)$ is a subfunctor of $\underline{\mathrm{Hom}}_{S-gr}(T, G)$ and $\underline{\mathrm{Hom}}_{S-gr}(T, G)$ is a sheaf, we only need to prove that for $f \in \underline{\mathrm{Hom}}_{S-gr}(T, G)(S)$ if $f_{S_i} \in \mathfrak{E}(G, \Psi)(S_i)$, then $f \in \mathfrak{E}(G, \Psi)(S)$.

We note that to verify that f is a closed immersion and $f(T)$ is a maximal torus, it is enough to verify it étale locally. Since f_{S_i} is in $\mathfrak{E}(G, \Psi)(S_i)$, by the definition of $\mathfrak{E}(G, \Psi)$, f_{S_i} is a closed immersion and $f(T)_{S_i}$ is a maximal torus. Hence f is a closed immersion and $f(T)$ is a maximal torus. Finally, f^{Ψ} is an isomorphism étale locally, so f^{Ψ} is an isomorphism. We conclude that $\mathfrak{E}(G, \Psi)$ is a sheaf. □

Claim. $\mathfrak{E}(G, \Psi)$ *is homogeneous under the left $\underline{\mathrm{Aut}}_{S\text{-}gr}(G)$-action, which is defined as composition of functions.*

Proof. Let S' be a scheme over S, and f_1, f_2 be two elements in $\mathfrak{E}(G, \Psi)(S')$. Let $F_i = f_i(T_{S'})$, i =1, 2 respectively. Then there exists an étale neighborhood U of S' where F_1 and F_2 are conjugated (ref. [SGA3], Exp. 12, Theorem 1.7), so we can assume $F_{1,U} = F_{2,U}$. Moreover, we can even assume G_U is split relatively to $F_{1,U}$ (ref. [SGA3], Exp. 22, 2.3). By abuse of notation, we still use $f_2 \circ f_1^{-1}$ to denote the morphism from $F_{1,U}$ to $F_{2,U}$. Then by the definition of the $\mathfrak{E}(G, \Psi)$-functor, we know that $f_2 \circ f_1^{-1}$ induces an automorphism on $\Phi(G_U, f(F_{1,U}))$. According to Theorem 1.5, we can find σ, which is an automorphism of G_U, such that $\sigma \circ f_2 = f_1$, which proves the claim. □

Claim. $\mathfrak{S}(\mathrm{G},\Psi)$ *is a right* $\underline{\mathrm{Aut}}(\Psi)$*-torsor over* $\mathcal{T}$ *for étale topology.*

Proof. We first prove that $\pi : \mathfrak{S}(\mathrm{G},\Psi) \to \mathcal{T}$ is surjective as an S-sheaf morphism for the étale topology. For an S-scheme S' and an element F in $\mathcal{T}(\mathrm{S}')$, which means that F is a maximal torus in $\mathrm{G}_{\mathrm{S}'}$, for each $s' \in \mathrm{S}'$, we can find an étale open neighborhood $\mathrm{U}' \to \mathrm{S}'$ such that $\Psi_{\mathrm{U}'}$ splits and $\mathrm{G}_{\mathrm{U}'}$ splits relatively to $\mathrm{F}_{\mathrm{U}'}$. Therefore, $\Psi_{\mathrm{U}'}$ and $\Phi(\mathrm{G},\mathrm{F})_{\mathrm{U}'}$ are isomorphic as we assume that both of them are with the same type at each geometric point. Hence, there is $f \in \mathfrak{S}(\mathrm{G},\Psi)(\mathrm{U}')$ such that $\pi(f) = \mathrm{F} \underset{\mathrm{S}'}{\times} \mathrm{U}'$.

Next, let us show that $\mathfrak{S}(\mathrm{G},\Psi) \underset{\mathrm{S}}{\times} \underline{\mathrm{Aut}}(\Psi) \simeq \mathfrak{S}(\mathrm{G},\Psi) \underset{\mathcal{T}}{\times} \mathfrak{S}(\mathrm{G},\Psi)$ as $\underline{\mathrm{Aut}}(\Psi)$-space. By identifying $\underline{\mathrm{Aut}}(\Psi)$ with a subgroup of $\underline{\mathrm{Aut}}(\mathrm{T})$, we regard $\sigma \in \underline{\mathrm{Aut}}(\Psi)(\mathrm{S}')$ as an element of $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{T})(\mathrm{S}')$. Define

$$m\colon \mathfrak{S}(\mathrm{G},\Psi) \underset{\mathrm{S}}{\times} \underline{\mathrm{Aut}}(\Psi) \to \mathfrak{S}(\mathrm{G},\Psi) \underset{\mathcal{T}}{\times} \mathfrak{S}(\mathrm{G},\Psi)$$

as $m(f,\sigma) = (f, f \circ \sigma)$ for all S' a scheme over S.

Given $(f_1, f_2) \in (\mathfrak{S}(\mathrm{G},\Psi) \underset{\mathcal{T}}{\times} \mathfrak{S}(\mathrm{G},\Psi))(\mathrm{S}')$, we let $\mathrm{F} = f_1(\mathrm{T}_{\mathrm{S}'}) = f_2(\mathrm{T}_{\mathrm{S}'})$ and $\Phi = \Phi(\mathrm{G}_{\mathrm{S}'},\mathrm{F})$. Then both f_1^{Ψ}, f_2^{Ψ} induce isomorphisms from $\Psi_{\mathrm{S}'}$ to Φ, so $(f_1^{\Psi})^{-1} \circ f_2^{\Psi}$ is an automorphism of $\Psi_{\mathrm{S}'}$. So we can define

$$i : \mathfrak{S}(\mathrm{G},\Psi) \underset{\mathcal{T}}{\times} \mathfrak{S}(\mathrm{G},\Psi) \to \mathfrak{S}(\mathrm{G},\Psi) \underset{\mathrm{S}}{\times} \underline{\mathrm{Aut}}(\Psi)$$

as $i(f_1, f_2) = (f_1, f_1^{-1} \circ f_2)$. Then we have

$$i \circ m(f,\sigma) = i(f, f \circ \sigma) = (f, f^{-1} \circ f \circ \sigma) = (f,\sigma);$$
$$m \circ i(f_1, f_2) = m(f_1, f_1^{-1} \circ f_2) = (f_1, f_2).$$

Therefore i is the inverse map of m and the claim follows from Proposition 1.1. □

Now we want to show that $\mathfrak{S}(\mathrm{G},\Psi)$ is a scheme. As we have mentioned in Proposition 1.2, the group scheme $\underline{\mathrm{Aut}}(\Psi)$ is étale locally constant. Therefore, the $\underline{\mathrm{Aut}}(\Psi)$-torsor $\mathfrak{S}(\mathrm{G},\Psi)$ is representable by [SGA3], Exp. X, 5.5. □

For a maximal torus X of G, we let X^{ad} be the corresponding torus in $\mathrm{ad}(\mathrm{G})$. Note that $\mathrm{X}/\underline{\mathrm{Centr}}(\mathrm{G}) \xrightarrow{\sim} \mathrm{X}^{\mathrm{ad}}$ (ref. [SGA3], Exp. 24, Prop. 2.1). For $f \in \mathfrak{S}(\mathrm{G},\Psi)(\mathrm{S}')$, we define the stabilizer of f under the $\underline{\mathrm{Aut}}_{\mathrm{S}'\text{-gr}}(\mathrm{G}_{\mathrm{S}'})$ as

$$\underline{\mathrm{Stab}}(f)(\mathrm{S}'') = \left\{x \in \mathrm{Aut}_{\mathrm{S}''\text{-gr}}(\mathrm{G}_{\mathrm{S}''}) \mid x \circ f_{\mathrm{S}''} = f_{\mathrm{S}''}\right\}.$$

Proposition 2.2. *Let* $f \in \mathfrak{S}(\mathrm{G},\Psi)(\mathrm{S}')$ *and* $\mathrm{X} = f(\mathrm{T}_{\mathrm{S}'})$*. Then* $\underline{\mathrm{Stab}}(f)$ *is isomorphic to* X^{ad}*.*

Proof. Let $\sigma \in \mathrm{Aut}_{\mathrm{S}''\text{-gr}}(\mathrm{G}_{\mathrm{S}''})$. Then $\sigma \in \underline{\mathrm{Stab}}(f)(\mathrm{S}'')$ if and only if $\sigma|_{\mathrm{X}}$ is the identity map on X, which means $\underline{\mathrm{Stab}}(f) = \underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}, \mathrm{id}_{\mathrm{X}})$. Since $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}, \mathrm{id}_{\mathrm{X}}) = \mathrm{X}^{\mathrm{ad}}$ (ref. [SGA3], Exp. 24, Prop. 2.11), $\underline{\mathrm{Stab}}(f) = \mathrm{X}^{\mathrm{ad}}$. □

2.2. Oriented embedding functors

2.2.1. The definition of an orientation. Let $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ be a twisted reduced root datum over S, and G be a reductive group over S. Suppose that Ψ and G have the same type at each $s \in \mathrm{S}$. From Theorem 2.1, we know that $\mathfrak{E}(\mathrm{G}, \Psi)$ is a homogeneous space under the action of $\underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G})$. However, $\underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G})$ may be disconnected, so we would like to fix an extra datum "v" to make our embedding functor together with "v" to be a homogeneous space under the adjoint action of G. The "v" will be called an orientation of Ψ with respect to G.

First, we suppose that G has a maximal torus T. Let $\Phi(\mathrm{G},\mathrm{T})$ be the twisted root datum of G with respect to T.

For an S-scheme S′, and for $\sigma \in \underline{\mathrm{Aut}}_{\text{S-gr}}(\mathrm{G},\mathrm{T})(\mathrm{S}')$, σ induces an automorphism on $\Phi(\mathrm{G},\mathrm{T})$, and induces a left action on $f \in \underline{\mathrm{Isom}}(\Psi, \Phi(\mathrm{G},\mathrm{T}))(\mathrm{S}')$ which is defined as

$$(\sigma \cdot f)(x) = f(x) \circ \sigma^{-1},$$

for all $x \in \mathcal{M}_{\mathrm{S}'}(\mathrm{S}'')$, where S″ is an S′-scheme.

Let T′ be another maximal torus of G, and $\underline{\mathrm{Transt}}_{\mathrm{G}}(\mathrm{T},\mathrm{T}')$ be the strict transporter from T to T′ (cf. [SGA3], Exp. $\mathrm{VI_B}$, Def. 6.1 (ii)). Then we have a natural morphism (for the convention, we refer to [Gir], Chap. III, Def. 1.3.1.):

$$\underline{\mathrm{Transt}}_{\mathrm{G}}(\mathrm{T},\mathrm{T}') \overset{\underline{\mathrm{Norm}}_{\mathrm{G}}(\mathrm{T})}{\wedge} \underline{\mathrm{Isom}}(\Psi, \Phi(\mathrm{G},\mathrm{T})) \to \underline{\mathrm{Isom}}(\Psi, \Phi(\mathrm{G},\mathrm{T}')).$$

Since $\underline{\mathrm{Transt}}_{\mathrm{G}}(\mathrm{T},\mathrm{T}')$ is a right principal homogeneous space under $\underline{\mathrm{Norm}}_{\mathrm{G}}(\mathrm{T})$ and $\underline{\mathrm{Norm}}_{\mathrm{G}}(\mathrm{T})$ acts on the left of $\underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G},\mathrm{T}))$ trivially, we have the following canonical morphism:

$$\begin{aligned}\underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G},\mathrm{T})) &\simeq \underline{\mathrm{Transt}}_{\mathrm{G}}(\mathrm{T},\mathrm{T}') \overset{\underline{\mathrm{Norm}}_{\mathrm{G}}(\mathrm{T})}{\wedge} \underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G},\mathrm{T})) \\ &\simeq \underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G},\mathrm{T}')).\end{aligned}$$

Therefore, for G with a maximal torus T, we can define

$$\underline{\mathrm{Isomext}}(\Psi, \mathrm{G}) := \underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G},\mathrm{T})).$$

In general, since G has a maximal torus étale locally, we can find an étale covering $\{\mathrm{S}_i \to \mathrm{S}\}_i$ such that $\mathrm{G}_{\mathrm{S}_i}$ has a maximal torus, and we can define $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ by the descent data of $\underline{\mathrm{Isomext}}(\Psi_{\mathrm{S}_i}, \mathrm{G}_{\mathrm{S}_i})$.

An *orientation* of Ψ with respect to G is an S-point of $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$. A twisted root datum Ψ together with an orientation $v \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(\mathrm{S})$ is called an oriented root datum and we denote it as (Ψ, v).

One can also define the functor $\underline{\mathrm{Isomext}}(\mathrm{G}, \Psi)$ in the same way. Suppose that G is with a maximal torus T. Then there is a natural isomorphism ι between $\underline{\mathrm{Isom}}(\Psi, \Phi(\mathrm{G},\mathrm{T}))$ and $\underline{\mathrm{Isom}}(\Phi(\mathrm{G},\mathrm{T}), \Psi)$ sending u to u^{-1}. This isomorphism also induces an isomorphism between $\underline{\mathrm{Isomext}}(\Psi, \Phi(G,T))$ and $\underline{\mathrm{Isomext}}(\Phi(G,T), \Psi)$. Let T' be another maximal torus of G. We have the following commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Transt}}_G(T,T') \overset{\underline{\mathrm{Norm}}_G(T)}{\wedge} \underline{\mathrm{Isom}}(\Psi, \Phi(G,T)) & \longrightarrow & \underline{\mathrm{Isom}}(\Phi(G,T), \Psi) \overset{\underline{\mathrm{Norm}}_G(T)}{\wedge} \underline{\mathrm{Transt}}_G(T',T) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Isom}}(\Psi, \Phi(G,T')) & \longrightarrow & \underline{\mathrm{Isom}}(\Phi(G,T'), \Psi) \end{array}$$

Therefore, the morphism ι defines an isomorphism between $\underline{\mathrm{Isomext}}(\Psi, G)$ and $\underline{\mathrm{Isomext}}(G, \Psi)$ and we can define ι for an arbitrary reductive group G by descent.

Remark 2.3. Actually, in our case, there is no difference between the transporter $\underline{\mathrm{Trans}}_G(T, T')$ and the strict transporter $\underline{\mathrm{Transt}}_G(T, T')$ since both T and T' are maximal tori.

Proposition 2.4. *Let G' be another reductive group over S. Suppose that G' and Ψ have the same type at each fibre over S. Then we will have the following map:*

$$\underline{\mathrm{Isomext}}(\Psi, G') \times \underline{\mathrm{Isomext}}(G, \Psi) \to \underline{\mathrm{Isomext}}(G, G').$$

Proof. To see this, we first suppose that both G and G' have maximal tori. Let T and T' be the maximal tori of G and G' respectively. Then the natural map from $\underline{\mathrm{Isom}}(\Psi, \Phi(G', T')) \times \underline{\mathrm{Isom}}(\Phi(G,T), \Psi)$ to $\underline{\mathrm{Isom}}(\Phi(G,T), \Phi(G', T'))$, induces the map

$$\underline{\mathrm{Isomext}}(\Psi, \Phi(G', T')) \times \underline{\mathrm{Isomext}}(\Phi(G,T), \Psi) \to \underline{\mathrm{Isomext}}(\Phi(G,T), \Phi(G', T')).$$

We now want to show that $\underline{\mathrm{Isomext}}(\Phi(G,T), \Phi(G', T')) \simeq \underline{\mathrm{Isomext}}(G, G')$. Note that we have natural morphisms from $\underline{\mathrm{Isom}}_{S\text{-gr}}(G, T; G', T')/\underline{\mathrm{Norm}}_{\mathrm{ad}(G)}(\mathrm{ad}(T))$ to $\underline{\mathrm{Isomext}}(\Phi(G,T), \Phi(G', T'))$. By [SGA3], Exp. XXIV, 2.2, we have

$$\underline{\mathrm{Isom}}_{S\text{-gr}}(G, T; G', T')/\underline{\mathrm{Norm}}_{\mathrm{ad}(G)}(\mathrm{ad}(T)) \xrightarrow{\sim} \underline{\mathrm{Isomext}}(G, G')$$

So we have a map

$$\iota_1 : \underline{\mathrm{Isomext}}(G, G') \to \underline{\mathrm{Isomext}}(\Phi(G,T), \Phi(G', T')).$$

Note that $\underline{\mathrm{Isomext}}(G; G')$ and $\underline{\mathrm{Isomext}}(\Phi(G,T), \Phi(G', T'))$ are principal homogeneous spaces under $\underline{\mathrm{Autext}}(G)$ and $\underline{\mathrm{Autext}}(\Phi(G,T))$ respectively.

By [SGA3], Exp. XXIV, 2.1, we have that $\underline{\mathrm{Autext}}(G,T) \simeq \underline{\mathrm{Autext}}(G)$. Moreover, by Theorem 1.5 and Proposition 1.3, the natural map between $\underline{\mathrm{Autext}}(G,T)$ and $\underline{\mathrm{Autext}}(\Phi(G,T))$ is an isomorphism on each geometric fiber, so

$$\underline{\mathrm{Autext}}(G) \simeq \underline{\mathrm{Autext}}(G,T) \simeq \underline{\mathrm{Autext}}(\Phi(G,T)).$$

Under these identifications, the map ι_1 is a morphism between $\underline{\mathrm{Autext}}(G)$-principal homogeneous spaces. So it is an isomorphism.

For general reductive groups G and G', we can define this map by descent. □

Remark 2.5. For a semisimple group G, there is also a definition of an orientation of G (ref. [PS]). Let G^{qs} be a quasi-split form of G, and T' be a maximal torus of G^{qs}. If we replace Ψ above by $\Phi(G^{qs}, T')$, then an orientation of Ψ with respect to G is called an orientation of G in [PS], §2.

2.2.2. Oriented embedding functors. Given an oriented twisted root datum (Ψ, v) of Ψ with respect to G, we define the *oriented embedding functor* as:

$$\mathfrak{E}(G, \Psi, v)(S') = \left\{ f : T_{S'} \hookrightarrow G_{S'} \;\middle|\; \begin{array}{l} f \in \mathfrak{E}(G, \Psi)(S'), \text{ and the image} \\ \text{of } f^{\Psi} \text{ in } \underline{\mathrm{Isomext}}(\Psi, G)(S') \text{ is } v. \end{array} \right\}$$

With all the notation defined above, we have the following result similar to Theorem 2.1:

Theorem 2.6. *Suppose that* G *is reductive. Then in the sense of the étale topology,* $\mathfrak{E}(G, \Psi, v)$ *is a left homogeneous space under the adjoint action of* G*, and a torsor over* $\mathcal{T}$ *under the right* $W(\Psi)$*-action. Moreover,* $\mathfrak{E}(G, \Psi, v)$ *is representable by an affine* S*-scheme.*

Proof. Since $\mathfrak{E}(G, \Psi)$ and $\underline{\mathrm{Isomext}}(\Psi, G)$ are sheaves, $\mathfrak{E}(G, \Psi, v)$ is an S-sheaf.

Let S' be an S-scheme, $f_1, f_2 \in \mathfrak{E}(G, \Psi, v)(S')$ and $T_i = f_i(T_{S'})$, for $i = 1, 2$ respectively. There is an étale neighborhood U of S' such that G splits relatively to $T_{i,U}$'s and hence there is $g \in \underline{\mathrm{Transt}}(T_1, T_2)(U)$ (ref. [SGA3], Exp. XXIV, 1.5). By the definition of $\mathfrak{E}(G, \Psi, v)$, we know that f_1^{Ψ} and f_2^{Ψ} have the same image in $\underline{\mathrm{Isomext}}(\Psi, G)(S')$, and therefore $g \cdot f_1^{\Psi}$ and f_2^{Ψ} have the same image in $\underline{\mathrm{Isomext}}(\Psi, \Phi(G, T_2))(S')$. Since G_U splits relatively to $T_{2,U}$, we can find $n \in \underline{\mathrm{Norm}}_G(T)(U)$ such that $n \cdot g \cdot f_1^{\Psi} = f_2^{\Psi}$, which proves that $\mathfrak{E}(G, \Psi, v)$ is a homogeneous space under the adjoint action of G.

Next, we show that $\pi : \mathfrak{E}(G, \Psi, v) \to \mathcal{T}$ is surjective as a morphism of sheaves. As we have seen in the proof of Theorem 2.1, $\pi : \mathfrak{E}(G, \Psi) \to \mathcal{T}$ is surjective, so for an S-scheme S' and $X \in \mathcal{T}(S')$, there is an étale covering $\{S'_i \to S'\}$ such that for each i, there is $f_i \in \mathfrak{E}(G, \Psi)(S'_i)$ with $\pi(f) = X_{S'_i}$. Moreover, we can assume $G_{S'_i}$ is split relatively to $X_{S'_i}$. Then $\underline{\mathrm{Aut}}_{S\text{-gr}}(G, T)(S'_i)$ is mapped surjectively to $\underline{\mathrm{Autext}}(G)(S'_i)$ (ref. [SGA3], Exp. XXIV, 2.1), which allows us to find $\sigma_i \in \underline{\mathrm{Aut}}_{S\text{-gr}}(G, T)(S'_i)$ such that $\sigma_i \circ f_i \in \mathfrak{E}(G, \Psi, v)(S'_i)$. Therefore, $\pi : \mathfrak{E}(G, \Psi, v) \to \mathcal{T}$ is surjective.

Finally, we want to prove that $\mathfrak{E}(G, \Psi, v)$ is a right $W(\Psi)$-torsor over $\mathcal{T}$. We identify $W(\Psi)$ with a subgroup of $\underline{\mathrm{Aut}}_{S\text{-gr}}(T)$. So for $w \in W(\Psi)(S')$, we can regard it as an element in $\underline{\mathrm{Aut}}_{S\text{-gr}}(T)(S')$. By the definition of $\underline{\mathrm{Isomext}}(\Psi, G)$, $W(\Psi)$ acts trivially on $\underline{\mathrm{Isomext}}(\Psi, G)$. Therefore, we can consider the map

$$m_v : \mathfrak{E}(G, \Psi, v) \underset{S}{\times} W(\Psi) \to \mathfrak{E}(G, \Psi, v) \underset{\mathcal{T}}{\times} \mathfrak{E}(G, \Psi, v)$$

defined as $m_v(f, w) = (f, f \circ w)$, for $f \in \mathfrak{E}(G, \Psi, v)(S')$, $w \in W(\Psi)(S')$, where S' is an S-scheme.

On the other hand, given $f_1, f_2 \in \mathfrak{E}(\mathrm{G}, \Psi, v)(\mathrm{S}')$ with $f_1(\mathrm{T}) = f_2(\mathrm{T})$, f_1^{Ψ} and f_2^{Ψ} have the same image in $\underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G}, f_1(\mathrm{T})))$, so $f_1^{-1} \circ f_2$ defines an element in $\mathrm{W}(\Psi)(\mathrm{S}')$.

Then we can define the map

$$i_v : \mathfrak{E}(\mathrm{G}, \Psi, v) \underset{\mathcal{T}}{\times} \mathfrak{E}(\mathrm{G}, \Psi, v) \to \mathfrak{E}(\mathrm{G}, \Psi, v) \underset{\mathrm{S}}{\times} \mathrm{W}(\mathrm{T})$$

as $i_v(f_1, f_2) = (f_1, f_1^{-1} \circ f_2)$ for $(f_1, f_2) \in \mathfrak{E}(\mathrm{G}, \Psi, v)(\mathrm{S}') \underset{\mathcal{T}}{\times} \mathfrak{E}(\mathrm{G}, \Psi, v)(\mathrm{S}')$, S' an S-scheme. As what we have verified in the proof of Theorem 2.1, we have that i_v, m_v are the inverse maps of each other. Again, by Proposition 1.1, we conclude that $\mathfrak{E}(\mathrm{G}, \Psi, v)$ is a $\mathrm{W}(\Psi)$-torsor over $\mathcal{T}$ and by [SGA3], Exp. X, 5.5, $\mathfrak{E}(\mathrm{G}, \Psi, v)$ is representable. Since $\mathcal{T}$ is affine and $\mathrm{W}(\Psi)$ is finite, $\mathfrak{E}(\mathrm{G}, \Psi, v)$ is represented by an affine S-scheme. □

For a reductive group G, we let der(G) be the derived group of G and ss(G) be the semisimple group associated to G. Let sc(G) be the simply connected group associated to der(G). The following corollary allows us to reduce the oriented embedding problem of reductive groups to that of semisimple simply connected groups, which is useful for arithmetic purposes.

Corollary 2.7. *Let* $v \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(\mathrm{S})$. *Then* v *induces an orientation* $v_{\mathrm{der}} \in \underline{\mathrm{Isomext}}(\mathrm{der}(\Psi), \mathrm{der}(\mathrm{G}))(\mathrm{S})$. *Moreover, we have a natural isomorphism*

$$\mathfrak{E}(\mathrm{G}, \Psi, v) \xrightarrow{\sim} \mathfrak{E}(\mathrm{der}(\mathrm{G}), \mathrm{der}(\Psi), v_{\mathrm{der}}).$$

One can also replace der(Ψ) *and* der(G) *by* ad(Ψ) *and* ad(G)*,* ss(Ψ) *and* ss(G)*,* sc(Ψ) *and* sc(G) *respectively.*

Proof. The key point lies in the functoriality of the induced and coinduced operation on the root data and the one-to-one correspondence between the maximal tori of G and the maximal tori of der(G) (resp. ad(G), sc(G)), which gives us a natural isomorphism from $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ to $\underline{\mathrm{Isomext}}(\mathrm{der}(\Psi), \mathrm{der}(\mathrm{G}))$ (resp. $\underline{\mathrm{Isomext}}(\mathrm{ad}(\Psi), \mathrm{ad}(\mathrm{G}))$, $\underline{\mathrm{Isomext}}(\mathrm{ss}(\Psi), \mathrm{ss}(\mathrm{G}))$, $\underline{\mathrm{Isomext}}(\mathrm{sc}(\Psi), \mathrm{sc}(\mathrm{G}))$). Hence, we only prove the case for der(Ψ) and der(G) in detail, all the other cases can be proved similarly.

Suppose that G has a maximal torus T. Let $\mathrm{T}' = \mathrm{T} \cap \mathrm{der}(\mathrm{G})$. Then T' is a maximal torus of der(G) and $\mathrm{der}(\Phi(\mathrm{G}, \mathrm{T})) = \Phi(\mathrm{der}(\mathrm{G}), \mathrm{T}')$. Moreover, the scheme of maximal tori of G is isomorphic to the scheme of maximal tori of der(G) (ref. [SGA3], Exp. XXII, 6.2.7, 6.2.8). Therefore, there is a natural morphism i_{der} from $\underline{\mathrm{Isom}}(\Psi, \Phi(\mathrm{G}, \mathrm{T}))$ to $\underline{\mathrm{Isom}}(\mathrm{der}(\Psi), \Phi(\mathrm{der}(\mathrm{G}), \mathrm{T}'))$. Moreover, by Proposition 1.6, the natural morphism from Ψ to der(Ψ) induces an isomorphism from $\mathrm{W}(\Psi)$ to $\mathrm{W}(\mathrm{der}(\Psi))$, so i_{der} induces a natural morphism from $\underline{\mathrm{Isomext}}(\Psi, \Phi(\mathrm{G}, \mathrm{T}))$ to $\underline{\mathrm{Isomext}}(\mathrm{der}(\Psi), \Phi(\mathrm{der}(\mathrm{G}), \mathrm{T}'))$, and hence a natural morphism from $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$

to $\underline{\mathrm{Isomext}}(\mathrm{der}(\Psi), \mathrm{der}(G))$ by descent. Therefore, given $v \in \underline{\mathrm{Isomext}}(\Psi, G)(S)$, we can have $v_{\mathrm{der}} \in \underline{\mathrm{Isomext}}(\mathrm{der}(\Psi), \mathrm{der}(G))$ induced by v.

Since $W(\Psi)$ is isomorphic to $W(\mathrm{der}(\Psi))$, by Theorem 2.6, both $\mathfrak{E}(G, \Psi, v)$ and $\mathfrak{E}(\mathrm{der}(G), \mathrm{der}(\Psi), v_{\mathrm{der}})$ are $W(\Psi)$-torsors over the scheme of maximal tori. Thus, the natural morphism

$$\mathfrak{E}(G, \Psi, v) \xrightarrow{\sim} \mathfrak{E}(\mathrm{der}(G), \mathrm{der}(\Psi), v_{\mathrm{der}})$$

is an isomorphism. □

2.3. Examples–embedding functors and embedding problems of Azumaya algebras with involution. In this section, we want to show the relations between the embedding functors and embedding problems of Azumaya algebras with involution. For the background of Azumaya algebras, we refer to the book by Knus [KN], Chap. III, §5, and the paper [KPS90].

Let K be a commutative ring and suppose that 2 is invertible in K. Let A be an Azumaya algebra over K of degree n equipped with an involution τ. Let $k = K^\tau$ be the elements in K fixed by τ. If $k = K$, then τ is said to be of the first kind. If K is an étale quadratic extension over k, then τ is said to be of the second kind. Let E be a commutative étale algebra over K of rank n equipped with an involution σ. Assume $\sigma|K = \tau|K$.

Let $U(E, \sigma)$ and $U(A, \tau)$ be two algebraic k-groups defined as follows: for any commutative k-algebra C,

$$U(E, \sigma)(C) = \{x \in E \otimes_k C \mid x\sigma(x) = 1\},$$

and

$$U(A, \tau)(C) = \{x \in A \otimes_k C \mid x\tau(x) = 1\}.$$

Let $T = U(E, \sigma)^\circ$, the identity component of $U(E, \sigma)$, and $G = U(A, \tau)^\circ$. Since 2 is invertible in K, G is smooth at each fiber.

Then we associate a root datum Ψ to T. The idea is to associate a “split form” (A_0, τ_0) (resp. (E_0, σ_0)) to each (A, τ) (resp. (E, σ)). From the split form (A_0, τ_0), we get a group G_0 with a split maximal torus T_0. Let $\Phi(G_0, T_0)$ be the root datum of G_0 with respect to T_0. Then we use the isomorphism between $\underline{\mathrm{Aut}}(E_0, \sigma_0)$ and $\underline{\mathrm{Aut}}(\Psi_0)$ to associate a twisted root datum Ψ to (E, σ). This allows us to transfer a k-embedding from (E, σ) to (A, τ) to a k-point of the embedding functor $\mathfrak{E}(G, \Psi)$. Moreover, we will show that the k-points of $\mathfrak{E}(G, \Psi)$ are in one-to-one correspondence with the k-embeddings from (E, σ) to (A, τ). To simplify things, we always assume that A and E have constant rank over K.

2.3.1. The root datum associated to T

Notation for the case where the involution τ is of the second kind. If τ is an involution of the second kind, then we let A_0 be $\mathbf{M}_{n,k} \times \mathbf{M}^{\mathrm{op}}_{n,k}$, where $\mathbf{M}_{n,k}$ stands for the $n \times n$-matrix algebra defined over k, and let E_0 be $k^n \times k^n$, which is viewed as an étale algebra over $k \times k$. In this case, we let τ_0 be the exchange involution of A_0 defined by $\tau_0(M, N) = (N, M)$. Let $\iota_0 \colon E_0 \to A_0$ be defined as $\iota_0(x_1, \dots, x_n, y_1, \dots, y_n) = (\mathrm{diag}(x_1, \dots, x_n), \mathrm{diag}(y_1, \dots, y_n))$, where $\mathrm{diag}(x_1, \dots, x_n)$ stands for the diagonal matrix with the (i,i)-th entry x_i. Clearly it is a $k \times k$-homomorphism and the image of ι_0 is invariant under τ_0. Let σ_0 be the exchange involution on E_0 induced by τ_0. Let $T_0 = U(E_0, \sigma_0)$ and $G_0 = U(A_0, \tau_0)$ and $f_0 \colon T_0 \to G_0$ be the embedding induced by ι_0. Let Ψ_0 be the root datum associated to T_0 defined as

$$\Psi_0(C) = \Phi(G_0, f_0(T_0))(C) \circ f_0$$

for any k-algebra C.

We let $i_{T_0} \colon \mathbb{G}_{m,k} \to T_0$ (resp. $i_{G_0} \colon \mathbb{G}_{m,K} \to G_0$) denote the embedding defined by the $k \times k$-structure morphism of E_0 (resp. A_0), and let $i_T \colon R^{(1)}_{K/k}(\mathbb{G}_{m,K}) \to T$ (resp. $i_G \colon R^{(1)}_{K/k}(\mathbb{G}_{m,K}) \to G$) denote the embedding defined by the K-structure morphism of E (resp. A).

An isomorphism between $(E_0, \sigma_0, k \times k)$ and (E, σ, K) is a k-isomorphism between E_0 and E commuting with the involutions, and sends $k \times k$ to K. Let $\mathfrak{X} = \underline{\mathrm{Isom}}((E_0, \sigma_0, k \times k), (E, \sigma, K))$ be the isomorphism functor between $(E_0, \sigma_0, k \times k)$ and (E, σ, K). Note that $\mathfrak{X}(\bar{s})$ is not empty for each geometric point $\bar{s}$ of $\mathrm{Spec}(k)$, if and only if $\mathrm{rank}_k E^\sigma = \mathrm{rank}_K E$. Throughout this article, we assume that $\mathfrak{X}$ is non-empty. Then $\mathfrak{X}$ is a right $\underline{\mathrm{Aut}}(E_0, \sigma_0, k \times k)$-torsor.

Notation for the case where the involution τ is of the first kind. For τ an involution of the first kind, we let $A_0 = \mathbf{M}_{n,k}$, and $E_0 = k^n$. Let $\iota_0 \colon E_0 \to A_0$ be defined as $\iota(x_1, \dots, x_n) = \mathrm{diag}(x_1, \dots, x_n)$.

If τ is an orthogonal involution and n is odd, we let $n = 2m + 1$ and $B = (b_{ij})_{0 \le i,j \le 2m}$, where

$$b_{i,j} = \begin{cases} 1 & \text{if } i = j = 0, \\ 1 & \text{if } i = j \pm m, \text{ with } i, j \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

For τ an orthogonal involution and n even, we let $n = 2m$ and $B = (b_{i,j})_{1 \le i,j \le 2m}$, where

$$b_{i,j} = \begin{cases} 1 & \text{if } i = j \pm m, \text{ with } i, j \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

For τ a symplectic involution, we let $n = 2m$ and $\mathrm{B} = (b_{i,j})_{1\le i,j\le 2m}$, where

$$b_{i,j} = \begin{cases} 1 & \text{if } j = i+m, \\ -1 & \text{if } j = i-m, \\ 0 & \text{otherwise.} \end{cases}$$

Let τ_0 be the involution on A_0 defined by $\tau_0(\mathrm{M}) = \mathrm{B}\mathrm{M}^t\mathrm{B}^{-1}$, and let σ_0 be the involution on E_0 induced by τ_0. Let $\mathrm{T}_0 = \mathrm{U}(\mathrm{E}_0, \sigma_0)^\circ$ and $\mathrm{G}_0 = \mathrm{U}(\mathrm{A}_0, \tau_0)^\circ$. Let $f_0\colon \mathrm{T}_0 \to \mathrm{G}_0$ be the embedding induced by ι_0.

Let Ψ_0 be the root datum associated to T_0 defined as

$$\Psi_0(\mathrm{C}) = \Phi(\mathrm{G}_0, f_0(\mathrm{T}_0))(\mathrm{C}) \circ f_0$$

for any k-algebra C. For τ of the first kind, let $\mathfrak{X} = \underline{\mathrm{Isom}}((\mathrm{E}_0, \sigma_0), (\mathrm{E}, \sigma))$.

Note that $\mathfrak{X}(\bar{s})$ is non-empty for each geometric point $\bar{s}$ of $\mathrm{Spec}(k)$ if and only if $\mathrm{rank}_K \mathrm{E}^\sigma = \lceil \frac{1}{2} \mathrm{rank}_K \mathrm{E} \rceil$. Throughout this article, we assume that $\mathfrak{X}$ is non-empty.

The definition of the twisted root datum Ψ

Definition 2.8. 1. Suppose that τ is of the first kind. A *k-embedding* $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$ is an injective k-homomorphism commuting with the involutions.

2. A *k-embedding* $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$ is an injective k-homomorphism commuting with the involutions and sending K to K.

3. Let ι be a k-embedding from (E, σ) to (A, τ). Define the isomorphisms between $(\mathrm{E}_0, \mathrm{A}_0, \iota_0)$ and $(\mathrm{E}, \mathrm{A}, \iota)$ to be pairs (α, β), where α is an isomorphism between (E_0, σ_0) and (E, σ), β is an isomorphism between (A_0, τ_0) and (A, τ), and α, β satisfy $\iota \circ \alpha = \beta \circ \iota_0$. Let $\underline{\mathrm{Isom}}((\mathrm{E}_0, \mathrm{A}_0, \iota_0), (\mathrm{E}, \mathrm{A}, \iota))$ be the isomorphism functor between $(\mathrm{E}_0, \mathrm{A}_0, \iota_0)$ and $(\mathrm{E}, \mathrm{A}, \iota)$.

4. A morphism $f: \mathrm{T} \to \mathrm{G}$ is called an embedding if it is a closed immersion and a group homomorphism. Let $f: \mathrm{T} \to \mathrm{G}$ be an embedding. Define the isomorphisms between $(\mathrm{G}_0, \mathrm{T}_0, f_0)$ and $(\mathrm{G}, \mathrm{T}, f)$ to be pairs (h, g), where h is an isomorphism from T_0 to T and g is an isomorphism from G_0 to G, and h, g satisfy $f \circ h = g \circ f_0$. Let $\underline{\mathrm{Isom}}((\mathrm{G}_0, \mathrm{T}_0, f_0), (\mathrm{G}, \mathrm{T}, f))$ be the isomorphism functor between $(\mathrm{G}_0, \mathrm{T}_0, f_0)$ and $(\mathrm{G}, \mathrm{T}, f)$.

Remark 2.9. Suppose that τ is of the second kind, and ι is an embedding from (E, σ) to (A, τ). For a k-algebra C and $(\alpha, \beta) \in \underline{\mathrm{Isom}}((\mathrm{E}_0, \mathrm{A}_0, \iota_0), (\mathrm{E}, \mathrm{A}, \iota))(\mathrm{C})$, α will automatically be in $\underline{\mathrm{Isom}}((\mathrm{E}_0, \sigma_0, k \times k), (\mathrm{E}, \sigma, K))(\mathrm{C})$, because β sends the center of A_0 to the center of A and $\iota \circ \alpha = \beta \circ \iota_0$.

Remark 2.10. Let f be a k-point of $\mathfrak{E}(\mathrm{G}, \Psi)$. Since f induces an isomorphism between Ψ and $\Phi(\mathrm{G}, f(\mathrm{T}))$, f induces an isomorphism between $\mathrm{rad}(\Psi)$ and

$\mathrm{rad}(\Phi(\mathrm{G}, f(\mathrm{T})))$. As $i_{\mathrm{T}}(\mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}))$ (resp. $i_{\mathrm{G}}(\mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K})))$ is the torus associated to $\mathrm{rad}(\Psi)$ (resp. $\mathrm{rad}(\Phi(\mathrm{G}, f(\mathrm{T})))$), f maps $i_{\mathrm{T}}(\mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}))$ to $i_{\mathrm{G}}(\mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}))$.

The following lemma enables us to attach a twisted root datum to the torus T.

Lemma 2.11. *Let* $\mathrm{S} = \mathrm{Spec}(k)$. *Then we have the following:*

(1) *The canonical homomorphism from* $\underline{\mathrm{Aut}}(\mathrm{E}_0, \mathrm{A}_0, \iota_0)$ *to* $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-}gr}(\mathrm{G}_0, f_0(\mathrm{T}_0))$ *is an isomorphism except for* G_0 *of type* D_4 *or* A *of degree* 2 *with* τ *orthogonal.*

(2) *If the involution* τ_0 *is of the first kind, then there is a canonical monomorphism* j_{E_0} *from* $\underline{\mathrm{Aut}}(\mathrm{E}_0, \sigma_0)$ *to* $\underline{\mathrm{Aut}}(\Psi_0)$. *In particular, if* Ψ_0 *is not of type* D_4, *then the homomorphism* j_{E_0} *is an isomorphism.*

(3) *If the involution* τ_0 *is of the second kind, then there is a canonical isomorphism from* $\underline{\mathrm{Aut}}(\mathrm{E}_0, \sigma_0, k \times k)$ *to* $\underline{\mathrm{Aut}}(\Psi)$.

Proof. To verify that the canonical homomorphism j_{A_0} from $\underline{\mathrm{Aut}}(\mathrm{E}_0, \mathrm{A}_0, \iota_0)$ to $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0, f_0(\mathrm{T}_0))$ is an isomorphism, it suffices to verify that the natural morphism from $\underline{\mathrm{Aut}}(\mathrm{A}_0, \tau_0)$ to $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0)$ is an isomorphism, since the automorphism preserves $\iota_0(\mathrm{E}_0)$ if and only if it preserves $f_0(\mathrm{T}_0)$. To see that the natural morphism from $\underline{\mathrm{Aut}}(\mathrm{A}_0, \tau_0)$ to $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0)$ is an isomorphism, we check it case by case. For G_0 of type A_n, let σ be the automorphism of (A_0, τ_0) which maps $(\mathrm{M}, \mathrm{N}) \in \mathrm{A}_0$ to $(\mathrm{N}^t, \mathrm{M}^t)$, where N^t denotes the transpose of N. Then $\underline{\mathrm{Aut}}(\mathrm{A}_0, \tau_0)$ is $\mathrm{PGL}_{n+1} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{Z}/2\mathbb{Z}$ is generated by σ. Note that σ induces the outer automorphism of $\mathrm{G}_0 = \mathrm{GL}_{n+1}$, and we have $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0) = \underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{GL}_{n+1}) = \mathrm{PGL}_{n+1} \rtimes \mathbb{Z}/2\mathbb{Z}$. For G_0 of type B_n, we have

$$\underline{\mathrm{Aut}}(\mathrm{A}_0, \tau_0) = \mathrm{PGO}(\mathrm{A}_0, \tau_0) \cong \mathrm{G}_0$$

(cf. [KMRT98], Thm. 12.15 and Prop. 12.4). Since G_0 is adjoint of type B_n in this case, we have $\mathrm{G}_0 = \underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0)$ and hence

$$\underline{\mathrm{Aut}}(\mathrm{A}_0, \tau_0) \cong \mathrm{G}_0 = \underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{G}_0).$$

Similar calculation can be done for G_0 of type C_n or G_0 of type D_n with $n \geq 2$ and $n \neq 4$, and we refer to [KMRT98], Theorem 26.14 and Theorem 26.15.

To prove (2), we first note that there is a natural isomorphism j_{E_0} from $\underline{\mathrm{Aut}}(\mathrm{E}_0, \sigma_0)$ to $\underline{\mathrm{Aut}}_{\mathrm{S}\text{-gr}}(\mathrm{T}_0)$. To see that the image of j_{E_0} is contained in $\underline{\mathrm{Aut}}(\Psi)$, we verify it case by case. For example, for σ_0 orthogonal and E of degree $2m$, the automorphism group $\underline{\mathrm{Aut}}(\mathrm{E}_0, \sigma_0)$ is isomorphic to the constant group scheme $((\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathrm{S}_m)_{\mathrm{S}}$. We can check that the corresponding action of $((\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathrm{S}_m)_{\mathrm{S}}$ on T_0 actually preserves the root datum Ψ_0. Moreover, by [Bou], Plan. IV, we know that $((\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathrm{S}_m)_{\mathrm{S}}$ is exactly the automorphism group of Ψ_0 for $m \neq 4$ and is a subgroup of $\underline{\mathrm{Aut}}(\Psi_0)$ for $m = 4$. From this, we conclude that j_{E_0} maps $\underline{\mathrm{Aut}}(\mathrm{E}_0, \sigma_0)$ isomorphically to $\underline{\mathrm{Aut}}(\Psi)$ for Ψ_0 not of type D_4. One can check the other cases in the same way, which allows us to conclude the statement (2). One can prove (3) in the same way. □

Since T_0 is a maximal torus of G_0, we have the following exact sequence:

$$0 \longrightarrow \mathrm{ad}(T_0) \longrightarrow \underline{\mathrm{Aut}}_{\text{S-gr}}(G_0, f_0(T_0)) \longrightarrow \underline{\mathrm{Aut}}(\Psi_0) \longrightarrow 0$$

by Proposition 1.3 and [SGA3], Exp. XXIV, Proposition 2.6. Therefore, we can summarize the above lemma as the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(T_0) & \longrightarrow & \underline{\mathrm{Aut}}(A_0, E_0, \iota_0) & \longrightarrow & \underline{\mathrm{Aut}}(E_0, \sigma) & \longrightarrow & 0\,, \\ & & \downarrow & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & \mathrm{ad}(T_0) & \longrightarrow & \underline{\mathrm{Aut}}_{\text{S-gr}}(G_0, f_0(T_0)) & \longrightarrow & \underline{\mathrm{Aut}}(\Psi_0) & \longrightarrow & 0 \end{array}$$

where α, β are isomorphisms if τ_0 is of the first kind and G_0 is not of type D_4. For τ_0 of the second kind, we just replace $\underline{\mathrm{Aut}}(E_0, \sigma)$ by $\underline{\mathrm{Aut}}(E_0, \sigma, k \times k)$ and α, β are isomorphisms. Now if the involution τ_0 is of the first kind, we define the twisted root datum Ψ related to T as

$$\Psi := \mathfrak{X} \overset{\mathrm{Aut}(E_0,\sigma_0)}{\wedge} \Psi_0.$$

If the involution τ_0 is of the second kind, we define the twisted root datum Ψ related to T as

$$\Psi := \mathfrak{X} \overset{\mathrm{Aut}(E_0,\sigma_0,k\times k)}{\wedge} \Psi_0.$$

Remark 2.12. If we regard Ψ_0 as a set of combinatorial data satisfying the axioms of root data, the canonical morphism β between $\underline{\mathrm{Aut}}(E_0, \sigma_0)$ and $\underline{\mathrm{Aut}}(\Psi_0)$ is defined over any arbitrary base. However, for the involution τ_0 of the first kind, the group G_0 is not reductive over arbitrary base. Hence, we ask 2 to be invertible over the base so that Ψ_0 can be regarded as a root datum of G_0.

Remark 2.13. We have a canonical morphism from $\mathfrak{X}$ to $\underline{\mathrm{Isom}}(\Psi_0, \Psi)$ which is an isomorphism except if Ψ_0 is of type D_4 by Lemma 2.11. The natural morphism from $\underline{\mathrm{Isom}}((A_0, \tau_0), (A, \tau))$ to $\underline{\mathrm{Isom}}(G_0, G)$ over k is a canonical monomorphism which is an isomorphism except if G is of type D_4, since $\underline{\mathrm{Aut}}(A_0, \tau_0) = \underline{\mathrm{Aut}}(G_0)$ except for G_0 of type D_4 ([KMRT98], Chap. IV, §23 and §26).

Remark 2.14. For A of degree 2 with τ orthogonal, the corresponding split group G_0 is actually the one dimensional split torus. Therefore G_0 acts trivially on itself but non-trivially on A_0. However, the conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ induces a nontrivial isomorphism of G_0. Hence the natural morphism from $\underline{\mathrm{Aut}}(A_0, E_0, \iota_0)$ to $\underline{\mathrm{Aut}}_{\text{S-gr}}(G_0, f_0(T_0))$ is surjective but not injective. However, in this case, we have

$$\underline{\mathrm{Aut}}(E_0, \sigma_0) \simeq (\mathbb{Z}/2\mathbb{Z})_K \simeq \underline{\mathrm{Aut}}(\Psi_0),$$

so the natural map from $\underline{\mathrm{Isom}}(E_0, E)$ to $\underline{\mathrm{Isom}}(\Psi_0, \Psi)$ is still an isomorphism.

2.3.2. Embedding functors and embedding problems for algebras with involution

Theorem 2.15. *Keep the notation defined above. Then:*

(1) *The set of k-embeddings from* (E, σ) *into* (A, τ) *is in one-to-one correspondence with the set of k-points of* $\mathfrak{E}(\mathrm{G}, \Psi)$, *except for* G *of type D_4 or* A *of degree* 2 *with τ orthogonal.*

(2) *If τ is of the second kind, then the set of K-algebra embeddings from* (E, σ) *into* (A, τ) *is in one-to-one correspondence with the set of k-points f of* $\mathfrak{E}(\mathrm{G}, \Psi)$ *which satisfy $f \circ i_{\mathrm{T}} = i_{\mathrm{G}}$.*

Proof. The crucial ingredient of the proof is Lemma 2.11. We prove (1) first. Let ι be a k-embedding from (E, σ) to (A, τ). Clearly, ι induces an embedding $f : \mathrm{T} \to \mathrm{G}$. To see that f is a k-point of $\mathfrak{E}(\mathrm{G}, \Psi)$, we need to verify that f induces an isomorphism between Ψ and $\Phi(\mathrm{G}, f(\mathrm{T}))$.

Let $\mathfrak{Y} = \underline{\mathrm{Isom}}((\mathrm{E}_0, \mathrm{A}_0, \iota_0), (\mathrm{E}, \mathrm{A}, \iota))$. By Lemma 2.11, we have

$$\mathfrak{Y} \simeq \underline{\mathrm{Isom}}((\mathrm{G}_0, f_0(\mathrm{T}_0)), (\mathrm{G}, f(\mathrm{T}))).$$

This allows us to define an isomorphism from $\mathfrak{Y} \overset{\underline{\mathrm{Aut}}(\mathrm{E}_0, \mathrm{A}_0, \iota_0)}{\wedge} (\mathrm{G}_0, f_0(\mathrm{T}_0))$ to $(\mathrm{G}, f(\mathrm{T}))$, which induces an isomorphism from $\mathfrak{Y} \overset{\underline{\mathrm{Aut}}(\mathrm{E}_0, \mathrm{A}_0, \iota_0)}{\wedge} \Phi(\mathrm{G}_0, f_0(\mathrm{T}_0))$ to $\Phi(\mathrm{G}, f(\mathrm{T}))$.

Given a k-algebra C and $(\alpha, \beta) \in \mathfrak{Y}(\mathrm{C})$, we have a natural map from $\mathfrak{Y}$ to $\mathfrak{X}$ which maps (α, β) to α. By the definition of Ψ_0, f_0 induces an isomorphism between $\Phi(\mathrm{G}_0, f_0(\mathrm{T}_0))$ and Ψ_0. Therefore we have the following natural map

$$\mathfrak{Y} \overset{\underline{\mathrm{Aut}}(\mathrm{G}_0, f_0(\mathrm{T}_0))}{\wedge} \Phi(\mathrm{G}_0, f_0(\mathrm{T}_0)) \simeq \mathfrak{Y} \overset{\underline{\mathrm{Aut}}(\mathrm{G}_0, f_0(\mathrm{T}_0))}{\wedge} \Psi_0 \to \mathfrak{X} \overset{\underline{\mathrm{Aut}}(\Psi_0)}{\wedge} \Psi_0 = \Psi.$$

Since $\mathfrak{Y} \overset{\underline{\mathrm{Aut}}(\mathrm{G}_0, f_0(\mathrm{T}_0))}{\wedge} \Phi(\mathrm{G}_0, f_0(\mathrm{T}_0))$ is isomorphic to $\Phi(\mathrm{G}, f(\mathrm{T}))$, we conclude that f induces a natural map from $\Phi(\mathrm{G}, f(\mathrm{T}))$ to Ψ. Since this natural map becomes an isomorphism at each geometric fiber, it is an isomorphism and hence $f \in \mathfrak{E}(\mathrm{G}, \Psi)(k)$, and we denote it as $\mathrm{I}_{\mathrm{A}}(\iota)$.

Given $f \in \mathfrak{E}(\mathrm{G}, \Psi)(k)$, now we want to define a k-embedding ι associated to f. Note that f induces an isomorphism from $\Phi(\mathrm{G}, f(\mathrm{T}))$ to Ψ by definition.

Let $\mathfrak{Z} = \underline{\mathrm{Isom}}((\mathrm{G}_0, \mathrm{T}_0, f_0), (\mathrm{G}, \mathrm{T}, f))$. Note that for a k-algebra C and $(h, g) \in \mathfrak{Z}(\mathrm{C})$, g induces an isomorphism between $\Phi(\mathrm{G}_0, f_0(\mathrm{T}_0))$ and $\Phi(\mathrm{G}, f(\mathrm{T}))$. Hence h is an element of $\underline{\mathrm{Isom}}(\Psi_0, \Psi)(\mathrm{C})$ and we have a natural morphism from $\mathfrak{Z}$ to $\underline{\mathrm{Isom}}(\Psi_0, \Psi)$.

By Lemma 2.11, there is a canonical isomorphism from $\underline{\mathrm{Isom}}((\mathrm{E}_0, \sigma_0), (\mathrm{E}, \sigma))$ (resp. $\underline{\mathrm{Isom}}((\mathrm{E}_0, \sigma_0, k \times k), (\mathrm{E}, \sigma, K))$, if τ is of the second kind) to $\underline{\mathrm{Isom}}(\Psi_0, \Psi)$, so we have a canonical morphism from $\mathfrak{Z}$ to $\underline{\mathrm{Isom}}((\mathrm{E}_0, \sigma_0), (\mathrm{E}, \sigma))$, and hence a canonical map from $\mathfrak{Z} \wedge (\mathrm{E}_0, \sigma_0)$ to (E, σ). Similarly, by Remark 2.13, we have a canonical map from $\mathfrak{Z}$ to $\underline{\mathrm{Isom}}((\mathrm{A}_0, \tau_0), (\mathrm{A}, \tau))$, and hence a canonical map from $\mathfrak{Z} \wedge (\mathrm{A}_0, \tau_0)$ to (A, τ). Therefore, we get a k-embedding $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$ from the map $\mathfrak{Z} \wedge (\mathrm{E}_0, \sigma_0)$ to $\mathfrak{Z} \wedge (\mathrm{A}_0, \tau_0)$ induced by ι_0, and we denote ι as $\mathrm{J}_\mathrm{G}(f)$.

Clearly, $\mathrm{I}_\mathrm{A}\mathrm{J}_\mathrm{G}(f) = f$ since we construct ι from $\mathfrak{Z}$ and $\mathfrak{Z} \bigwedge (\mathrm{G}_0, \mathrm{T}_0, f_0)$ is canonically isomorphic to $(\mathrm{G}, \mathrm{T}, f)$. On the other hand, we have

$$\mathrm{J}_\mathrm{G} \circ \mathrm{I}_\mathrm{A}(\iota) = \iota$$

because of the canonical isomorphism from

$$\underline{\mathrm{Isom}}((\mathrm{E}_0, \mathrm{A}_0, \iota_0), (\mathrm{E}, \mathrm{A}, \iota))$$

to

$$\underline{\mathrm{Isom}}((\mathrm{G}_0, \mathrm{T}_0, f_0), (\mathrm{G}, \mathrm{T}, f)),$$

where f is induced by ι. Hence, the first assertion follows.

We now prove (2). Clearly, if $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$ is a K-embedding, then the corresponding k-embedding f will be a k-point of $\mathfrak{E}(\mathrm{G}, \Psi)$ and f satisfies $f \circ i_\mathrm{T} = i_\mathrm{G}$.

Now suppose that $f \in \mathfrak{E}(\mathrm{G}, \Psi)(k)$ and $f \circ i_\mathrm{T} = i_\mathrm{G}$. Then we need to verify that the map $\mathrm{J}_\mathrm{G}(f)$ from $\mathfrak{Z} \wedge (\mathrm{E}_0, \sigma_0)$ to $\mathfrak{Z} \wedge (\mathrm{A}_0, \tau_0)$ is a K-morphism. From the construction of $\mathrm{J}_\mathrm{G}(f)$ it is clear that it suffices to prove that the two maps from $\underline{\mathrm{Isom}}((\mathrm{G}_0, \mathrm{T}_0, f_0), (\mathrm{G}, \mathrm{T}, f))(\mathrm{R})$ to $\underline{\mathrm{Isom}}(\mathbb{G}_{m,k}, \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}))(\mathrm{R})$, which map (h, g) to $i_\mathrm{T}^{-1} \circ h \circ i_{0,\mathrm{T}}$ and $i_\mathrm{G}^{-1} \circ g \circ i_{0,\mathrm{G}}$ respectively, coincide. However, it is a direct consequence from the fact that $f \circ i_\mathrm{T} = i_\mathrm{G}$, since

$$\begin{aligned} i_\mathrm{T}^{-1} \circ h \circ i_{0,\mathrm{T}} &= i_\mathrm{G}^{-1} \circ f \circ h \circ i_{0,\mathrm{T}} \\ &= i_\mathrm{G}^{-1} \circ g \circ f_0 \circ i_{0,\mathrm{T}} \\ &= i_\mathrm{G}^{-1} \circ g \circ i_{0,\mathrm{G}}. \end{aligned}$$

Therefore, $\mathrm{J}_\mathrm{G}(f)$ is a K-algebra morphism. □

Remark 2.16. Let τ be of the second kind. Suppose that $\mathfrak{E}(\mathrm{G}, \Psi)(k)$ is nonempty and fix $f \in \mathfrak{E}(\mathrm{G}, \Psi)(k)$. If $f \circ i_\mathrm{T} \neq i_\mathrm{G}$, then $f \circ \sigma$ will satisfy $(f \circ \sigma) \circ i_\mathrm{T} = i_\mathrm{G}$ since σ acts on $\mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K})$ as -1. Therefore, the existence of a k-embedding will imply the existence of a K-embedding. Moreover, we will see that the condition $f \circ i_\mathrm{T} = i_\mathrm{G}$ gives a particular orientation $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(k)$.

Now we want to consider the case where G_0 is of type D_4. Note that since there is a natural monomorphism from $\underline{\mathrm{Aut}}(\mathrm{A}_0, \mathrm{E}_0, \iota_0)$ to $\underline{\mathrm{Aut}}_{k\text{-gr}}(\mathrm{G}_0, f_0(\mathrm{T}_0))$, we can still get a k-point of the embedding functor $\mathfrak{E}(\mathrm{G}, \Psi)$ from a k-embedding $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$. The problem is that given a k-point f of the embedding functor $\mathfrak{E}(\mathrm{G}, \Psi)$, we can not get a k-embedding from f as we have done in the proof of Theorem 2.15, because the canonical map from $\underline{\mathrm{Aut}}(\mathrm{A}_0, \mathrm{E}_0, \iota_0)$ to $\underline{\mathrm{Aut}}_{k\text{-gr}}(\mathrm{G}_0, f_0(\mathrm{T}_0))$ is not an isomorphism.

To fix the problem, we first observe that $\mathrm{ad}(G_0)$ (resp. $W(\Psi_0)$) is in the image of the canonical morphism from $\underline{\mathrm{Aut}}(A_0, \tau_0)$ (resp. $\underline{\mathrm{Aut}}(E_0, \sigma_0)$) to $\underline{\mathrm{Aut}}_{k\text{-gr}}(G_0)$ (resp. $\underline{\mathrm{Aut}}(\Psi_0)$). So instead of associating a split form (A_0, τ_0) to (A, τ), we consider all quasi-split forms of (A, τ). Note that $\underline{\mathrm{Aut}}(A_0, \tau_0)/\mathrm{ad}(G_0)$ is the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_k$ and we can find a section from $(\mathbb{Z}/2\mathbb{Z})_k$ to $\underline{\mathrm{Aut}}(A_0, E_0, \iota_0)$. For example we can send 1 in $\mathbb{Z}/2\mathbb{Z}$ to the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us fix the section from $(\mathbb{Z}/2\mathbb{Z})_k$ to $\underline{\mathrm{Aut}}(A_0, E_0, \iota_0)$ as above. Let

$$\underline{\mathrm{Isomext}}(A_0, \tau_0; A, \tau) := \underline{\mathrm{Isom}}(A_0, \tau_0; A, \tau)/\mathrm{ad}(G_0).$$

Then for each (A, τ) we can associate a quasi-split form

$$(A_q, \tau_q) = \underline{\mathrm{Isomext}}(A_0, \tau_0; A, \tau) \overset{(\mathbb{Z}/2\mathbb{Z})_k}{\wedge} (A_0, \tau_0).$$

Moreover, since the section has image in $\underline{\mathrm{Aut}}(A_0, E_0, \iota_0)$, we get an étale algebra with involution (E_q, σ_q) and a embedding $\iota_q\colon (E_q, \sigma_q) \to (A_q, \tau_q)$ from the datum (A_0, E_0, ι_0). Let $G_q = U(A_q, \tau_q)^\circ$ and $T_q = U(E_q, \sigma_q)$ and $f_q\colon T_q \to G_q$ be the morphism induced by ι_q.

From our construction, the group G in an inner form of G_q, so we can always fix an orientation v in $\underline{\mathrm{Isomext}}(G_q, G)(k)$. Then we have the following result:

Proposition 2.17. *Let u be a k-point of* $\underline{\mathrm{Isomext}}(\Psi, G)$. *Then each k-point of the oriented embedding functor* $\mathfrak{E}(G, \Psi, u)$ *corresponds to a k-embedding ι from* (E, σ) *to* (A, τ).

Proof. The way to prove it is exactly the same as in Theorem 2.15. The only different thing is that we stay in the inner case. First we fix an orientation v in $\underline{\mathrm{Isomext}}(G_q, G)(k)$. Let u_q be the orientation in $\underline{\mathrm{Isomext}}(\Psi_q, G_q)(k)$ which comes from the map f_q. Then there is an orientation $v' = u^{-1} \circ v \circ u_q$ in $\underline{\mathrm{Isomext}}(\Psi_q, \Psi)(k)$ by Proposition 2.4.

For $f \in \mathfrak{E}(G, \Psi, u)(k)$, we consider the $\underline{\mathrm{Norm}}_{\mathrm{ad}(G_q)}(\mathrm{ad}(f_q(T_q)))$-torsor

$$\underline{\mathrm{Isomint}}_v(G_q, f_q(T_q); G, f(T)).$$

Let $\mathfrak{Z}' = \underline{\mathrm{Isomint}}_v(\mathrm{G}_q, f_q(\mathrm{T}_q); \mathrm{G}, f(\mathrm{T}))$. Clearly, we have a canonical morphism from $\mathfrak{Z}'$ to $\underline{\mathrm{Isomint}}_v(\mathrm{G}_q, \mathrm{G})$ which is an $\mathrm{ad}(\mathrm{G}_q)$-torsor. We also have a canonical morphism from $\mathfrak{Z}'$ to $\underline{\mathrm{Isomint}}_{v'}(\Psi_q, \Psi)$ which is a $\mathrm{W}(\Psi_q)$-torsor. Note that $\mathrm{ad}(\mathrm{G}_q)$ (resp. $\mathrm{W}(\Psi_q)$) are in the image of the canonical morphism from $\underline{\mathrm{Aut}}(\mathrm{A}_q, \tau_q)$ (resp. $\underline{\mathrm{Aut}}(\mathrm{E}_q, \sigma_q)$) to $\underline{\mathrm{Aut}}_{k\text{-gr}}(\mathrm{G}_q)$ (resp. $\underline{\mathrm{Aut}}(\Psi_q)$) as they do in the split case. So we get a k-embedding $\iota\colon (\mathrm{E}, \sigma) \to (\mathrm{A}, \tau)$ from the map $\mathfrak{Z}' \overset{\mathrm{W}(\Psi_q)}{\wedge} (\mathrm{E}_q, \sigma_q)$ to $\mathfrak{Z}' \overset{\mathrm{ad}(\mathrm{G}_q)}{\wedge} (\mathrm{A}_q, \tau_q)$ induced by ι_q. □

3. Arithmetic properties of embedding functors

In this section, we focus on the arithmetic properties of the embedding functor. The main arithmetic technique which we use here has been developed by Borovoi [Bo99].

In the first part, we recall the main result in [Bo99]. In the second part, we give a criterion for an oriented embedding functor to satisfy the local-global principle. Besides, over a local field L, we use the Tits index to give a necessary and sufficient condition for an oriented embedding functor to have an L-point.

For a field k with characteristic different from 2, embedding an étale algebra over k into a central simple algebra over k commuting with involutions is equivalent to finding a k-point of the corresponding embedding functor. In Section 2.4, we use the arithmetic properties of oriented embedding functors to give an alternative proof of Theorem A, Theorem 6.7 and Theorem 7.3 in the work of Prasad and Rapinchuk [PR10].

Throughout this section, we let k be a global field and k^{s} be a separable closure. Let Γ be the absolute Galois group of k, and Ω_k be the set of all places of k.

We start this section with some general facts of the local-global principle of homogeneous spaces established in Borovoi's papers.

3.1. The local-global principle for homogeneous spaces. First, we let k be a number field. For a k-linear algebraic group G, we let G° to be the connected component containing the neutral element of G. Let G^{u} be the unipotent radical of G°; $\mathrm{G}^{\mathrm{red}} = \mathrm{G}^\circ/\mathrm{G}^{\mathrm{u}}$; G^{ss} be the derived group of $\mathrm{G}^{\mathrm{red}}$; $\mathrm{G}^{\mathrm{tor}}=\mathrm{G}^{\mathrm{red}}/\mathrm{G}^{\mathrm{ss}}$. Let $\mathrm{G}^{\mathrm{ssu}} = \ker[\mathrm{G}^\circ \to \mathrm{G}^{\mathrm{tor}}]$. If $\mathrm{G}/\mathrm{G}^{\mathrm{ssu}}$ is abelian, we let $\mathrm{G}^{\mathrm{mult}} = \mathrm{G}/\mathrm{G}^{\mathrm{ssu}}$ which is a multiplicative group.

Let X be a left homogeneous space under a connected linear algebraic group G over k. Let $x \in \mathrm{X}(k^{\mathrm{s}})$ and $\bar{\mathrm{H}} = \mathrm{Stab}_{\mathrm{G}_{k^{\mathrm{s}}}}(x)$ be the stabilizer of x.

Throughout this section, we will assume that G^{ss} is simply connected, and $\bar{\mathrm{H}}/\bar{\mathrm{H}}^{\mathrm{ssu}}$ is abelian.

Since Γ has a natural action on $\mathrm{G}(k^{\mathrm{s}})$, we can define $\mathfrak{G}$ to be the semidirect product $\mathrm{G}(k^{\mathrm{s}}) \rtimes \Gamma$. We have a $\mathfrak{G}$-action on $\mathrm{X}(k^{\mathrm{s}})$ defined as $(g, \sigma)x = g \cdot \sigma(x)$. Let

$\mathfrak{H} = \mathrm{Stab}_{\mathfrak{G}}(x)$. Then we have the following exact sequence:

$$(*) \quad 1 \longrightarrow \bar{\mathrm{H}}(k^{\mathrm{s}}) \xrightarrow{\ i\ } \mathfrak{H} \xrightarrow{\ p\ } \Gamma \longrightarrow 1,$$

where $i(\bar{h}) = (\bar{h}, 1)$, and p is the projection to Γ.

Since $\bar{\mathrm{H}}^{\mathrm{mult}}$ is commutative, we can define a Γ-action on $\bar{\mathrm{H}}^{\mathrm{mult}}$ by conjugation. To be precise, for $\sigma \in \Gamma$, choose $(g_\sigma, \sigma) \in p^{-1}(\sigma)$. Then since $g_\sigma \cdot \sigma(x) = x$, we have $\mathrm{int}(g_\sigma)^\sigma \bar{\mathrm{H}}^{\mathrm{mult}} = \bar{\mathrm{H}}^{\mathrm{mult}}$. Note that the above definition does not depend on the lifting of σ in $\mathfrak{H}$ because $\bar{\mathrm{H}}^{\mathrm{mult}}$ is commutative. Hence we get a Γ-action on $\bar{\mathrm{H}}^{\mathrm{mult}}$. Since the point x is defined over some finite extension L of k, $\bar{\mathrm{H}}^{\mathrm{mult}}$ is defined over L. Moreover, for each $\sigma \in \mathrm{Gal}(k^{\mathrm{s}}/L)$, we can choose $g_\sigma = 1$. Hence, there is a k-form H^m of $\bar{\mathrm{H}}^{\mathrm{mult}}$ defined by this Γ-action (cf. [BS64], 2.12, [Se], Chap. V, §4, n°. 20, and [FSS98], 1.15). One can verify that the isomorphism class of H^m is independent of the choice of the geometric point x. Therefore, given G and X, the isomorphism class of H^m is well-defined ([Bo99], 1.2).

Let $\bar{j}: \bar{\mathrm{H}} \to \mathrm{G}_{k^{\mathrm{s}}}$ be the natural inclusion. Then $\bar{j}$ induces a group morphism from $\bar{\mathrm{H}}^{\mathrm{mult}}$ to $\mathrm{G}^{\mathrm{tor}}_{k^{\mathrm{s}}}$, which descends to a group morphism $j: \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}}$ over k.

Consider the complex

$$0 \longrightarrow \mathrm{H}^m \xrightarrow{\ j\ } \mathrm{G}^{\mathrm{tor}} \longrightarrow 0,$$

where H^m is in degree -1 and $\mathrm{G}^{\mathrm{tor}}$ is in degree 0. Let $\mathrm{H}^1(k, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}})$ be the first Galois hypercohomology group of the above complex, and $Ш^1(k, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}})$ be the kernel of the localization map $\mathrm{H}^1(k, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}}) \to \prod_{v \in \Omega_k} \mathrm{H}^1(k_v, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}})$.

For $\sigma \in \Gamma$, let $(g_\sigma, \sigma) \in \mathfrak{H}$. Let $u_{\sigma,\tau} = g_{\sigma\tau}(g_\sigma {}^\sigma g_\tau)^{-1}$, $\hat{u}_{\sigma,\tau}$ be the image of $u_{\sigma,\tau}$ in $\bar{\mathrm{H}}^{\mathrm{mult}}(k^{\mathrm{s}})$, and $\hat{g}_\sigma$ be the image of g_σ in $\mathrm{G}^{\mathrm{tor}}(k^{\mathrm{s}})$. Then $(\hat{u}, \hat{g})$ is a hypercocycle, and we let $\eta(\mathrm{X}) = \mathrm{Cl}(\hat{u}, \hat{g}) \in \mathrm{H}^1(k, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}})$. Note that $\eta(\mathrm{X})$ is well defined (see [Bo99], 1.4).

We will make use of the following two theorems later.

Theorem 3.1. *Let k_v be a nonarchimedean local field of characteristic* 0. *Let* G, X *be as above. If* $\eta(\mathrm{X}) = 0$, *then* X *has a k_v-point.*

Proof. [Bo99], Thm. 2.1. □

Theorem 3.2. *Let k be a number field, and let* G, X *be as above. Assume that* $\mathrm{X}(k_v)$ *is nonempty for every place v of k and* $\eta(\mathrm{X}) = 0$. *Then* X *has a rational point.*

Proof. [Bo99], Thm. 2.2. □

Remark 3.3. Note that if X has a k_v-point at all places $v \in \Omega_k$, then $\eta(\mathrm{X})$ lies in $Ш^1(k, \mathrm{H}^m \to \mathrm{G}^{\mathrm{tor}})$.

Remark 3.4. In fact, up to a sign, $\eta(\mathrm{X})$ is the Brauer–Manin obstruction of X (ref. [Bo99], Thm. 4.5).

Now, we let k be a global function field, for example, $k = \mathbb{F}_q(t)$, where $\mathbb{F}_q$ is a finite field with q elements. One may ask if Theorem 3.1 holds over k. Indeed, we have similar results when G is a connected reductive group over k and X is a G-homogeneous space for the étale topology. Since X is a G-homogeneous space for the étale topology, the set $\mathrm{X}(k^{\mathrm{s}})$ is nonempty. Let $x \in \mathrm{X}(k^{\mathrm{s}})$ and $\overline{\mathrm{H}} = \mathrm{Stab}_{\mathrm{G}_{k^{\mathrm{s}}}}(x)$. Suppose that $\overline{\mathrm{H}}$ is connected reductive. Then we can define an k-torus H^m, which is an k-form of $\overline{\mathrm{H}}^{\mathrm{mult}}$ as above. Let $\mathfrak{H}$ and $\eta(\mathrm{X})$ be defined as above. Then we have the following:

Proposition 3.5. *Let k be a global function field. Let* G *be a connected reductive group over k and* X *be a* G*-homogeneous space for the étale topology. Let $x \in \mathrm{X}(k^{\mathrm{s}})$. Define $\overline{\mathrm{H}}$ as above. Suppose that G^{ss} is simply connected and $\overline{\mathrm{H}}$ is a torus. If $\eta(\mathrm{X}) = 0$, then* X *has a k-point. The same result also holds over k_v for $v \in \Omega_k$.*

Proof. The key point of the proof is that $\mathrm{H}^1(k, \mathrm{G}^{\mathrm{ss}}) = 0$, for G^{ss} semisimple simply connected ([H75], Satz A and [Th08], Thm. A).

For $\sigma \in \Gamma$, let $(g_\sigma, \sigma) \in \mathfrak{H}$. Let $u_{\sigma,\tau} = g_{\sigma\tau}(g_\sigma {}^\sigma g_\tau)^{-1}$. As above, we have

$$\eta(\mathrm{X}) = \mathrm{Cl}(\hat{u}, \hat{g}) \in \mathrm{H}^1(k, \mathrm{H}^m \xrightarrow{j} \mathrm{G}^{\mathrm{tor}}).$$

Since $\overline{\mathrm{H}}$ is a torus, we have $\overline{\mathrm{H}}^{\mathrm{mult}} = \overline{\mathrm{H}}$ and H^m is a k-form of $\overline{\mathrm{H}}$. Suppose that $\eta(\mathrm{X}) = 0$. Then we have $a_\sigma \in \mathrm{H}^m(k^{\mathrm{s}})$ and $s \in \mathrm{G}(k^{\mathrm{s}})$ such that

$$(\hat{u}_{\sigma,\tau}, \hat{g}_\sigma) = (-\partial(a_\sigma), j(a_\sigma)\partial s),$$

i.e. $u_{\sigma,\tau} = a_{\sigma\tau}(a_\sigma {}^\sigma a_\tau)^{-1}$ and $g_\sigma = s^{-1} \cdot j(a_\sigma) \cdot {}^\sigma s \pmod{\mathrm{G}^{\mathrm{ss}}}$. After replacing g_σ by $a_\sigma^{-1} g_\sigma$, we can assume $u_{\sigma,\tau} = 0$. We also replace x by $s \cdot x$, and we get $g_\sigma \in \mathrm{G}^{\mathrm{ss}}(k^{\mathrm{s}})$ and $u_{\sigma,\tau} = 0$. Therefore, (g_σ) is a cocycle of Γ with values in G^{ss}. Since $\mathrm{H}^1(k, \mathrm{G}^{\mathrm{ss}}) = 0$ when G^{ss} is semisimple simply connected, there is $t \in \mathrm{G}^{\mathrm{ss}}(k^{\mathrm{s}})$ such that $s_\sigma = t^{-1}\,{}^\sigma t$. Then $t \cdot x$ is a k-point of X. □

Remark 3.6. For k with positive characteristic, we ask G to be reductive because we want to ensure that $\mathrm{G}^{\mathrm{red}}$, G^{ss} and $\mathrm{G}^{\mathrm{tor}}$ are properly defined. Otherwise, it may happen that the k-unipotent radical of G is trivial but G is not reductive (see [CGP], Example 1.1.3).

Remark 3.7. The above proposition is also true over a totally imaginary number field k, because in this case, $\mathrm{H}^1(k, \mathrm{G}^{\mathrm{ss}}) = 0$ by Kneser's Theorem ([K], Chap. IV, Thm. 1 and Chap. V, Thm. 1).

3.2. Local-global principle for oriented embedding functors. Let k be a global field. Unless otherwise specified, G is a reductive k-group, Ψ is a twisted root datum, and T is the k-torus determined by Ψ. In the following, we always assume that G and Ψ have the same type.

Let sc(T) be the torus determined by the simply connected root datum sc(Ψ). We will show that the only obstruction to the local-global principal for the oriented functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ lies in the Shafarevich group $Ш^2(k, \mathrm{sc}(\mathrm{T}))$. Moreover, the local-global principle holds for the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ if $\underline{\mathrm{Dyn}}(\Psi)$ is of type C or T is anisotropic at some place v.

Note that since G is reductive, we have $\mathrm{G}^{\mathrm{ss}} = \mathrm{der}(\mathrm{G})$. A direct application of Theorem 3.2 is the following:

Proposition 3.8. *Let u be an orientation of Ψ with respect to* G*. Then the only obstruction for $\mathfrak{E}(\mathrm{G}, \Psi, u)$ to satisfy the local-global principle lies in the group $Ш^2(k, \mathrm{sc}(\mathrm{T}))$. In particular, if* G *has no outer automorphisms and $Ш^2(k, \mathrm{sc}(\mathrm{T}))$ vanishes, then $\mathfrak{E}(\mathrm{G}, \Psi)$ satisfies the local-global principle.*

Before proving the above proposition, we prove the following lemma:

Lemma 3.9. *Suppose further that* G *is a semisimple simply connected group over k, and that Ψ is a twisted simply connected root datum. Let $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(k)$. As we have shown in Theorem 2.6, the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ is a left homogeneous space under the adjoint* G*-action. Then under this* G*-action, the corresponding H^m (which is defined in Section 3.1) is isomorphic to* T.

Proof. Given $f \in \mathfrak{E}(\mathrm{G}, \Psi, u)(k^{\mathrm{s}})$, the stabilizer of f in $\mathrm{G}_{k^{\mathrm{s}}}$ is $f(\mathrm{T}_{k^{\mathrm{s}}})$. Since $f(\mathrm{T}_{k^{\mathrm{s}}})$ is a torus, we have $f(\mathrm{T}_{k^{\mathrm{s}}})^{\mathrm{mult}} = f(\mathrm{T}_{k^{\mathrm{s}}})$. For $\sigma \in \Gamma$, σ acts on f as ${}^{\sigma}f = \sigma \circ f \circ \sigma^{-1}$. Let $\mathfrak{G} = \mathrm{G}(k^{\mathrm{s}}) \rtimes \Gamma$, and $\mathfrak{H} = \mathrm{Stab}_{\mathfrak{G}}(f)$. For $g \in \mathrm{G}(k^{\mathrm{s}})$, we let $\mathrm{int}(g)$ denote the conjugation action of g on G. Then for $(g_\sigma, \sigma) \in \mathfrak{H}$, we have $\mathrm{int}(g_\sigma) \circ {}^{\sigma}f = f$, which means $g_\sigma \cdot {}^{\sigma}f(t) \cdot g_\sigma^{-1} = f(t)$ for all $t \in \mathrm{T}(k^{\mathrm{s}})$. Therefore, we have

$$g_\sigma \cdot \sigma(f(t)) \cdot g_\sigma^{-1} = f(\sigma(t)),$$

which means f is a k-isomorphism between T and $f(\mathrm{T}_{k^{\mathrm{s}}})^m$. Therefore, the H^m defined in Section 2.1 is isomorphic to T. □

Now, we are ready to prove Proposition 3.8.

Proof of Proposition 3.8. By Corollary 2.7, it suffices to prove this proposition for $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})$. Since $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})$ is a homogeneous space under sc(G), by Lemma 3.9 we know that the H^m corresponding to this sc(G)-action is isomorphic to sc(T).

Since $\mathrm{sc}(\mathrm{G})^{\mathrm{tor}}$ is trivial, by Theorem 3.2 and Proposition 3.5, the only obstruction for $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})$ to satisfy the local-global principle lies in the group $Ш^2(k, \mathrm{sc}(\mathrm{T}))$. The rest of the proposition then follows. □

Let k_v be a nonarchimedean local field. Then by combining previous results, we get the following corollary:

Corollary 3.10. *If the group* $\mathrm{H}^2(k_v, \mathrm{sc}(\mathrm{T}))$ *is trivial, then the oriented embedding functor* $\mathfrak{E}(\mathrm{G}, \Psi, u)$ *has a* k_v*-point.*

Proof. By Corollary 2.7, it is enough to prove that $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})(k_v)$ is nonempty. As a result of Lemma 3.9, the group H^m for the sc(G)-homogeneous space $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})$ is isomorphic to sc(T). Since $\mathrm{sc}(\mathrm{G})^{\mathrm{tor}}$ is trivial, we have

$$\mathrm{H}^1(k_v, \mathrm{H}^m \to \mathrm{sc}(\mathrm{G})^{\mathrm{tor}}) = \mathrm{H}^2(k_v, \mathrm{sc}(\mathrm{T})).$$

By Theorem 3.1 and Proposition 3.5, the set $\mathfrak{E}(\mathrm{sc}(\mathrm{G}), \mathrm{sc}(\Psi), u_{\mathrm{sc}})(k_v)$ is nonempty if $\mathrm{H}^2(k_v, \mathrm{sc}(\mathrm{T}))$ is trivial. □

For a twisted root datum Ψ, the Galois group Γ has a natural action on Ψ_{k^s}. Therefore, we have a group homomorphism from Γ to $\underline{\mathrm{Aut}}(\Psi)(k^s)$. Recall that Ψ is said to be *generic* if the image of Γ in $\underline{\mathrm{Aut}}(\Psi)(k^s)$ contains the Weyl group $\mathrm{W}(\Psi)(k^s)$.

Theorem 3.11. *Let* G *be a reductive group over* k*,* Ψ *be a twisted root datum over* k*, and* T *be the torus determined by* Ψ*. Let* $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(k)$*. Suppose that* Ψ *satisfies one of the following conditions:*

(1) *All connected components of* $\underline{\mathrm{Dyn}}(\Psi)(k^s)$ *are of type* C*.*

(2) T *is anisotropic at one place* $v \in \Omega_k$*.*

Then the local-global principle holds for the existence of a k*-point of the oriented embedding functor* $\mathfrak{E}(\mathrm{G}, \Psi, u)$*. In particular, when* Ψ *is generic, the local-global principle holds.*

Proof. If Ψ satisfies one of the above conditions, then $\mathrm{sc}(\Psi)$ also satisfies one of them. Therefore, we can assume that Ψ and G are semisimple simply connected.

By Proposition 3.8, the local-global principle holds for the existence of k-points of the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ if $Ш^2(k, \mathrm{T})$ vanishes. Therefore, it is enough to prove $Ш^2(k, \mathrm{T}) = 0$ for Ψ satisfying either condition.

Suppose that Ψ satisfies condition (1). Let Ψ_0 be the split simple, simply connected root datum of type C_n (ref. [Bou], Plan. III). Let E_0 be the étale algebra $k^n \times k^n$ and σ_0 be the involution which exchanges the two copies of k^n. When Ψ is simple simply connected of type C_n, Ψ corresponds to some twisted form (E, σ) of (E_0, σ_0) by Lemma 2.11, and the torus T determined by Ψ is $\mathrm{U}(\mathrm{E}, \sigma) = \mathrm{R}_{\mathrm{E}^\sigma/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m))$.

Consider the exact sequence:

$$1 \longrightarrow \mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m) \longrightarrow \mathrm{R}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m) \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

By Hilbert Theorem 90, we have

$$0 \longrightarrow H^2(E^\sigma, R^{(1)}_{E/E^\sigma}(\mathbb{G}_m)) \longrightarrow H^2(E^\sigma, R_{E/E^\sigma}(\mathbb{G}_m)).$$

By Shapiro's Lemma, $Ш^2(E^\sigma, R_{E/E^\sigma}(\mathbb{G}_m)) = Ш^2(E, \mathbb{G}_m)$. By the Brauer–Hasse–Noether Theorem, $Ш^2(E, \mathbb{G}_m) = 0$. Therefore, we have

$$Ш^2(k, T) = Ш^2(E^\sigma, R^{(1)}_{E/E^\sigma}(\mathbb{G}_m)) = 0.$$

For Ψ not simple, since we can decompose Ψ_{k^s} into a product of isotypic root data ([SGA3], Exp. XXI, 6.4.1 and 7.1.6), we can also decompose Ψ into a product of isotypic twisted root data Ψ_i by descent. By the same reasoning as in [SGA3], Exp. XXIV, 5.8, or in [CGP], Theorem A.5.14, we know that there exists some étale algebra F_i over k such that Ψ_{i,F_i} is a product of copies of the twisted simple root datum $\Psi_{i,0}$ over F_i, and the automorphism group $\mathrm{Aut}(F_i/k)$ acts on Ψ_{i,F_i} by permuting $\Psi_{i,0}$'s. So we have $\Psi_i = R_{F_i/k}(\Psi_{i,0})$ and the torus T will take the form $\prod_i R_{F_i/k}(T_{i,0})$, where $T_{i,0}$ is the torus determined by the twisted root datum $\Psi_{i,0}$. Then we know that $\Psi_{i,0}$ is a twisted root datum defined by an étale algebra with involution over F_i (Section 2.3.1). As in the above discussion, we will have $T_{i,0} = U(E_i, \sigma_i)$ where E_i is an étale algebra over F_i. By Shapiro's Lemma,

$$Ш^2(k, R_{F_i/k}(T_{i,0})) = Ш^2(F_i, T_{i,0}) = Ш^2(E_i^{\sigma_i}, R^{(1)}_{E_i/E_i^{\sigma_i}}(\mathbb{G}_m)) = 0.$$

By Proposition 3.8, the theorem holds when Ψ satisfies the first condition.

Now, suppose that T is anisotropic at some place $v \in \Omega_k$. Then by Kneser's Theorem (ref. [San81], Lemma 1.9), we have $Ш^2(k, T) = 0$.

To complete the proof, we will show that if Ψ is generic, then T is anisotropic at some place v. Suppose that Ψ is generic. Let L be a finite Galois extension of k which splits T. Then there exists an element $\sigma \in \mathrm{Gal}(L/k)$ such that σ acts on Ψ_L as the Coxeter element $\omega \in W(\Psi)(L)$. Let M be the character group of T_L. Then the set $M^\omega = 0$ by Theorem 1 in [Bou], Chap. V, §6, and hence $M^\sigma = 0$. By Čebotarev Density Theorem, there exists a place v such that σ generates the Frobenius map at v, so T is anisotropic at v. □

3.3. Oriented embedding functors over local fields. Let G be a reductive group over a local field L, and Ψ be a twisted root datum over L. Suppose that G and Ψ have the same type and $\underline{\mathrm{Isomext}}(\Psi, G)(L)$ is not empty. Let $u \in \underline{\mathrm{Isomext}}(\Psi, G)(L)$. In the following, we are going to show that the existence of an L-point of the oriented embedding functor is actually determined by the Tits indices of Ψ and G. Note that the existence of an orientation u is important here, because it gives a map between the Dynkin schemes $\underline{\mathrm{Dyn}}(G)$ and $\underline{\mathrm{Dyn}}(\Psi)$, which allows us to compare the Tits indices of G and Ψ. An orientation also allows us to replace the reductive group G by the adjoint group ad(G) or simply connected group sc(G) as we have shown in Corollary 2.7.

3.3.1. Tits indices. We recall briefly the definition of the Tits index. For a detailed introduction on Tits indices, we refer to Tits's paper [T66]. For the Tits indices of reductive groups over connected semilocal rings, one can refer to [SGA3], Exp. XXVI, §5, §6, and §7. One can also look at Petrov and Stavrova's paper [PS], §5. For the Tits indices of a twisted root datum, we refer to Gille's paper [Gi], §7.

Let S be the spectrum of a semilocal ring. Let G be an S-reductive group. For each G, there exists a minimal parabolic subgroup $\mathrm{P}_{\min}$ of G. Let $t_{\min}$ be the type of $\mathrm{P}_{\min}$. Note that given G, the type $t_{\min}$ is well defined (ref. [SGA3], Exp. XXVI, 5.7). Moreover, if S is connected, then we call the type $t_{\min}$ the Tits index of G, and denote it by $\Delta^\circ(\mathrm{G})$.

For a reduced root datum $\psi = (\mathrm{M}, \mathrm{M}^\vee, \mathrm{R}, \mathrm{R}^\vee)$, a parabolic subset P is a closed subset of R which contains a system of simple roots. For a reduced twisted root datum $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$ over S, a parabolic subsheaf (fpqc) $\mathcal{P}$ is a subsheaf of R which is locally isomorphic to a parabolic subset. Let $\underline{\mathrm{Par}}(\Psi)$ be the functor such that for each S-scheme S′, $\underline{\mathrm{Par}}(\Psi)(\mathrm{S}')$ is the set of all the parabolic subsheaves (fpqc) of Ψ over S′. Similarly, we can define a type map $\mathbf{t}_\Psi$ from $\underline{\mathrm{Par}}(\Psi)$ to $\underline{\mathrm{Dyn}}(\Psi)$.

Let $t_{\min}$ be the type of a minimal parabolic subsheaf of Ψ ([Gi], Prop. 7.1). If S is connected, then we call $t_{\min}$ the Tits index of Ψ, and denote it by $\Delta^\circ(\Psi)$. Note that $\Delta^\circ(\mathrm{G})$ and $\Delta^\circ(\Psi)$ only depend on the roots, so they are invariant under the operations sc, ad,….

3.3.2. A criterion for the existence of points of the oriented embedding functor over a local field L. Let L be a local field of arbitrary characteristic. We have the following criterion for the existence of an L-point of the oriented embedding functor:

Theorem 3.12. *Let* G *be a reductive group over a local field L, and Ψ be a twisted root datum over L. Suppose that* G *and Ψ have the same type and* $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(L)$ *is not empty. Let $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(L)$. Then $\mathfrak{E}(\mathrm{G}, \Psi, u)(L) \neq \emptyset$ if and only if $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(\mathrm{G})$.*

Proof. First, we suppose that $\mathfrak{E}(\mathrm{G}, \Psi, u)(L) \neq \emptyset$ and let $f \in \mathfrak{E}(\mathrm{G}, \Psi, u)(L)$. Since $\Psi \simeq \Phi(\mathrm{G}, f(\mathrm{T}))$, from Prop. 7.3.2 in [Gi], $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(\mathrm{G})$.

Now, suppose $u(\Delta^\circ(\Psi)) \supseteq \Delta^\circ(\mathrm{G})$, and we want to show that $\mathfrak{E}(\mathrm{G}, \Psi, u)(L)$ is nonempty. Again, by Corollary 2.7, we only need to consider the problem for sc(G) and sc(Ψ). Therefore, we can assume G is simply connected, and Ψ is reduced simply connected.

Let T be the torus determined by Ψ and $\mathrm{I} = \Delta^\circ(\Psi)$. We start with the case where T is anisotropic, i.e. $\mathrm{I} = \underline{\mathrm{Dyn}}(\Psi)(L)$.

Case 1. L is non-archimedean. Since T is anisotropic, by Tate–Nakayama Theorem, we have $\mathrm{H}^2(L, \mathrm{T}) = 0$ (cf. [K], 3.2, Thm. 5). Since G and Ψ are simply connected, by Corollary 3.10, the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ has an L-point.

Case 2. $L = \mathbb{R}$. In this case, we consider the oriented embedding functor $\mathfrak{E}(\mathrm{ad(G)}, \mathrm{ad}(\Psi), u_{\mathrm{ad}})$. Let σ be the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let ad(T) be the torus associated to the root datum $\mathrm{ad}(\Psi)$. Since T is anisotropic, the torus ad(T) is also anisotropic and $\mathrm{ad(T)} \simeq (\mathrm{R}^{(1)}_{\mathbb{R}/\mathbb{C}}(\mathbb{G}_m))^r$.

Suppose that ad(G) is anisotropic and pick a maximal torus S of ad(G). Since $\mathrm{ad}(\Psi)$ and ad(G) have the same type, there is a $\mathbb{C}$-point f of $\mathfrak{E}(\mathrm{ad(G)}, \mathrm{ad}(\Psi), u_{\mathrm{ad}})$ which maps ad(T) to S. Since σ acts on the character group of the anisotropic torus by -1, σ commutes with f. Therefore, f is an $\mathbb{R}$-point of the oriented embedding functor $\mathfrak{E}(\mathrm{ad(G)}, \mathrm{ad}(\Psi), u_{\mathrm{ad}})$.

Suppose that ad(G) is not anisotropic. Then we can find an anisotropic form $\widetilde{\mathrm{G}}$ of ad(G) by [Ge91], Corollary 7. Since $\widetilde{\mathrm{G}}$ has the same type with $\mathrm{ad}(\Psi)$, by the above argument, we have an $\mathbb{R}$-point f of $\mathfrak{E}(\widetilde{\mathrm{G}}, \mathrm{ad}(\Psi))$. Then f defines an $\mathbb{R}$-point $\tilde{u}$ of $\underline{\mathrm{Isomext}}(\mathrm{ad}(\Psi), \widetilde{\mathrm{G}})$. The orientation u_{ad} together with $\tilde{u}$ gives an orientation $u_{\mathrm{ad}} \circ \tilde{u}^{-1} \in \underline{\mathrm{Isomext}}(\widetilde{\mathrm{G}}, \mathrm{ad(G)})(\mathbb{R})$. Hence ad(G) is an inner form of $\widetilde{\mathrm{G}}$. However, the natural inclusion from $\mathrm{H}^1(\mathbb{R}, f(\mathrm{ad(T)}))$ to $\mathrm{H}^1(\mathbb{R}, \widetilde{\mathrm{G}})$ is surjective ([Ge91], Thm. 3), so ad(G) has an anisotropic torus S. Let h belong to $\mathfrak{E}(\mathrm{ad(G)}, \mathrm{ad}(\Psi), u_{\mathrm{ad}})(\mathbb{C})$ and suppose that h maps $\mathrm{ad}(T)$ to S. Again, since σ acts on the character group of the anisotropic torus by -1, σ commutes with h and h is an $\mathbb{R}$-point of the oriented embedding functor $\mathfrak{E}(\mathrm{ad(G)}, \mathrm{ad}(\Psi), u_{\mathrm{ad}})$. By Corollary 2.7, the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ has an $\mathbb{R}$-point.

Therefore, the proposition is true when T is anisotropic.

Now, suppose that T is arbitrary. Since $u(\mathrm{I}) \supseteq \Delta^\circ(\mathrm{G})$, we can find a parabolic subgroup $\mathrm{P_I}$ of G such that the type of P is $u(\mathrm{I})$ by Proposition 1.7. Let $\mathrm{L_I}$ be a Levi subgroup of $\mathrm{P_I}$ and T' be a maximal torus of $\mathrm{L_I}$. Let $\Psi' = \Phi(\mathrm{G}, \mathrm{T}')$, and $\Psi'_{\mathrm{I}} = \Phi(\mathrm{L_I}, \mathrm{T}')$. Let $\mathcal{P}_{\mathrm{I}}$ be the subsheaf of roots of Ψ' which is determined by $\mathrm{P_I}$. Note that u corresponds to an element in $\underline{\mathrm{Isomext}}(\Psi, \Psi')(L)$, which we still denote as u.

Let $\Psi = (\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}, \mathcal{R}^\vee)$. Let $\mathcal{P}$ be a minimal parabolic subsheaf of $\mathcal{R}$. Then by definition, type $\mathcal{P} = \mathrm{I}$. Let $\mathcal{R}_{\mathrm{I}}$ be the subsheaf of $\mathcal{P}$ defined by the property: for any L-scheme X,

$$x \in \mathcal{R}_{\mathrm{I}}(\mathrm{X}), \text{ if and only if both } x \text{ and } -x \text{ are in } \mathcal{P}_{\mathrm{I}}(\mathrm{X}).$$

Let Ψ_{I} be the root system given by $(\mathcal{M}, \mathcal{M}^\vee, \mathcal{R}_{\mathrm{I}}, \mathcal{R}_{\mathrm{I}}^\vee)$. Define

$$\mathrm{Q} = \underline{\mathrm{Isomint}}_u(\Psi, \mathcal{P}; \Psi', \mathcal{P}_{\mathrm{I}}) = \underline{\mathrm{Isom}}(\Psi, \mathcal{P}; \Psi', \mathcal{P}_{\mathrm{I}}) \bigcap \underline{\mathrm{Isomint}}_u(\Psi, \Psi').$$

Note that Q is a right $\mathrm{W}(\Psi_{\mathrm{I}})$-torsor over $\mathrm{Spec}(L)$ (for the étale topology), so Q is representable. By the definition of Q, each $h \in \mathrm{Q(X)}$ will send the sheaf $\mathcal{R}_{\mathrm{I}}$ to the sheaf of roots of $\mathrm{L_I}$, because $\mathrm{L_I}$ is the unique Levi subgroup of $\mathrm{P_I}$ which contains T'. Therefore, we have a natural map

$$i_{\mathrm{I}} \colon \mathrm{Q} \to \underline{\mathrm{Isom}}(\Psi_{\mathrm{I}}, \Psi'_{\mathrm{I}}).$$

Let L^{s} be a separable closure of L. Let $x \in \mathrm{Q}(L^{\mathrm{s}})$. By the definition of Q, the image of x in $\underline{\mathrm{Isomext}}(\Psi, \Psi')(L^{\mathrm{s}})$ is u. Moreover, since Q is a right $\mathrm{W}(\Psi_{\mathrm{I}})$-torsor and $\mathrm{W}(\Psi_{\mathrm{I}})$ acts trivially on $\underline{\mathrm{Isomext}}(\Psi_{\mathrm{I}}, \Psi'_{\mathrm{I}})$, $i_{\mathrm{I}}(x)$ defines an L-point of $\underline{\mathrm{Isomext}}(\Psi_{\mathrm{I}}, \Psi'_{\mathrm{I}})$ and hence an L-point of $\underline{\mathrm{Isomext}}(\Psi_{\mathrm{I}}, \mathrm{Ł}_{\mathrm{I}})$. We denote it by u_{I}. Note that the definition of u_{I} is independent of the choice of T'.

Now we consider the functor $\mathfrak{E}(\mathrm{L}_{\mathrm{I}}, \Psi_{\mathrm{I}}, u_{\mathrm{I}})$. We claim that if $\mathfrak{E}(\mathrm{L}_{\mathrm{I}}, \Psi_{\mathrm{I}}, u_{\mathrm{I}})$ has an L-point, then $\mathfrak{E}(\mathrm{G}, \Psi, u)$ has an L-point.

Suppose that $\mathfrak{E}(\mathrm{L}_{\mathrm{I}}, \Psi_{\mathrm{I}}, u_{\mathrm{I}})$ has an L-point. Let $f \in \mathfrak{E}(\mathrm{L}_{\mathrm{I}}, \Psi_{\mathrm{I}}, u_{\mathrm{I}})(L)$. Then we replace the torus T' above by $f(\mathrm{T})$. By the definition of Q and u_{I}, we have a natural morphism

$$j : \mathrm{Q} \to \underline{\mathrm{Isomint}}_{u_{\mathrm{I}}}(\Psi_{\mathrm{I}}, \Psi'_{\mathrm{I}}).$$

Since both of them are $\mathrm{W}(\Psi_{\mathrm{I}})$-torsors, j is an isomorphism. As $\mathfrak{E}(\mathrm{L}_{\mathrm{I}}, \Psi_{\mathrm{I}}, u_{\mathrm{I}})$ has an L-point, $\underline{\mathrm{Isomint}}_{u_{\mathrm{I}}}(\Psi_{\mathrm{I}}, \Psi'_{\mathrm{I}})(L)$ is not empty, so Q has an L-point as well, which means $\underline{\mathrm{Isomint}}_{u}(\Psi, \Psi')(L) \neq \emptyset$. Hence, $\mathfrak{E}(\mathrm{G}, \Psi, u)$ has an L-point.

Now, by Corollary 2.7 it is enough to prove that $\mathfrak{E}(\mathrm{der}(\mathrm{L}_{\mathrm{I}}), \mathrm{der}(\Psi_{\mathrm{I}}), u_{\mathrm{I,der}})$ has an L-point. Note that $\mathrm{der}(\Psi)$ is reduced simply connected as Ψ is (ref. [SGA3], Exp. XXI, 6.5.11). Since the torus $\mathrm{der}(\mathrm{T})$ determined by $\mathrm{der}(\Psi)$ is anisotropic, it follows that $\mathfrak{E}(\mathrm{der}(\mathrm{L}_{\mathrm{I}}), \mathrm{der}(\Psi_{\mathrm{I}}), u_{\mathrm{I,der}})$ has an L-point as we have seen above. This finishes the proof. □

Example 3.13. The above theorem does not hold over arbitrary fields. Here is an example. Let $K = \mathbb{Q}(\sqrt{-1})$ and σ be the conjugation on K, and $k = \mathbb{Q}$. Let T be the torus $\mathrm{T} = \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_m)$. Since T is of dimension 1, there is only one semisimple simply connected root datum with respect to T. Let Ψ be this root datum. Let v_1, v_2 be two places of $\mathbb{Q}$ of the form $4n + 1$. Then Ψ splits at v_1 and v_2. Let D be a quaternion algebra over $\mathbb{Q}$ corresponding to $1/2$ in $\mathbb{Q}/\mathbb{Z} \simeq \mathrm{H}^2(\mathbb{Q}_{v_i}, \mathbb{G}_m)$ for $i = 1$, 2, and corresponding to 0 in the other places. Note that such a quaternion exists by the Brauer–Hasse–Noether's Theorem. Let G be $\mathrm{SL}_1(\mathrm{D})$. Since G has no outer form, there is an orientation u between Ψ and G. Since both Ψ and G are anisotropic over $\mathbb{Q}$, we have $u(\Delta^{\circ}(\Psi)) \supseteq \Delta^{\circ}(\mathrm{G})$. However, at place v_1 and v_2, the root datum Ψ splits but G is anisotropic, so $\mathfrak{E}(\mathrm{G}, \Psi, u)(\mathbb{Q}_{v_1}) = \emptyset$. Therefore, $\mathfrak{E}(\mathrm{G}, \Psi, u)(\mathbb{Q}) = \emptyset$ and Theorem 3.12 does not hold over $\mathbb{Q}$.

3.4. Applications-the problem of embedding an étale algebra in a central simple algebra with respect to involutions.

Let K be a field, (E, σ) be an étale K-algebra with involution σ, and (A, τ) be a central simple algebra over K with involution τ. Assume $\sigma|_K = \tau|_K$. Let $k = K^{\sigma}$. From now on, we assume that k is a global field of characteristic different from 2. Let Ω_k be the set of all places of k. Fix a separable closure k^{s} of k. Let $\mathcal{G} = \mathrm{Gal}(k^{\mathrm{s}}/k)$ and $\mathcal{G}_v = \mathrm{Gal}(k^{\mathrm{s}}_v/k_v)$ for $v \in \Omega_k$. Let $\mathrm{T} = \mathrm{U}(\mathrm{E}, \sigma)^{\circ}$, and $\mathrm{G} = \mathrm{U}(\mathrm{A}, \tau)^{\circ}$. Note that by the definition of $\mathrm{U}(\mathrm{E}, \sigma)^{\circ}$, $\mathrm{T} = \mathrm{R}_{\mathrm{E}^{\sigma}/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^{\sigma}}(\mathbb{G}_m))$. We keep the notation defined in Section 1.3.

In the paper [PR10], Prasad and Rapinchuk consider the local-global principle for the K-embedding from (E,σ) into (A,τ). As we have mentioned in Theorem 2.15, the local-global principle for the existence of k-embeddings from (E,σ) into (A,τ) is equivalent to the local-global principle for the existence of k-points of $\mathfrak{E}(\mathrm{G},\Psi)$. Here, we will reduce the original problem to the existence of k-points of oriented embedding functors, and prove that the local-global principle holds in certain cases by computing the Shafarevich group $\text{Ш}(k,\mathrm{sc}(\mathrm{T}))$. In the special case where G is an orthogonal group, Bayer-Fluckiger gives necessary and sufficient conditions for the local-global principle to hold ([B]).

3.4.1. Symplectic involutions. For τ symplectic, Ψ and G are semisimple simply connected of type C_n, which is the first case in Theorem 3.11, so we just restate the result as the following:

Proposition 3.14. *If τ is symplectic, then the local-global principle holds for the existence of K-embeddings of* (E,σ) *into* (A,τ).

3.4.2. Orthogonal involutions. Throughout this subsection, for an étale algebra F over K, we let $\mathrm{M_F}$ be the character group of the torus $\mathrm{R}_{\mathrm{F}/K}(\mathbb{G}_m)$ and let $\mathrm{J_F}$ be the character group of the torus $\mathrm{R}^{(1)}_{\mathrm{F}/K}(\mathbb{G}_m)$. Note that $K = k$ when τ is an orthogonal involution.

The case where the degree of A is odd. Let us consider the case where τ is orthogonal, and $\mathrm{A} = \mathbf{M}_{2n+1}(K)$. In this case, the corresponding group G is adjoint of type B_n, so there is no outer automorphisms. By Theorem 2.15 and Proposition 3.8, to prove the local-global principle for the K-embeddings here, it suffices to prove that $\text{Ш}^2(K,\mathrm{sc}(\mathrm{T}))$ vanishes. Note that in this case, $\mathrm{E} = K\times \mathrm{E}'$ and σ acts trivially on the component K, so $\mathrm{T} = \mathrm{R}_{\mathrm{E}'^\sigma/K}(\mathrm{R}^{(1)}_{\mathrm{E}'/\mathrm{E}'^\sigma}(\mathbb{G}_m))$.

Let $\mathrm{E}'^\sigma = \prod_{i=1}^r \mathrm{F}_i$, where F_i is a field over K for all i. Let $d = (d_1,\dots,d_r)$ be an element in E'^σ such that $\mathrm{E}' = \mathrm{E}'^\sigma[x]/(x^2-d) = \prod_{i=1}^r \mathrm{F}_i[x]/(x^2-d_i)$. Let $\mathrm{E}_i = \mathrm{F}_i[x]/(x^2-d_i)$ for all i and $\mathrm{E}_{i,v}$ (resp. $\mathrm{F}_{i,v}$) be $\mathrm{E}_i\otimes_K K_v$ (resp. $\mathrm{F}_i\otimes_K K_v$) for all $v\in\Omega_K$.

Theorem 3.15. *Suppose τ is orthogonal, and $\mathrm{A} = \mathbf{M}_{2n+1}(K)$. If there is a place $v\in\Omega_K$ such that the following condition holds:*

$$\textit{for all } i,\ d_i \in \mathrm{F}_i^{\times 2} \textit{ if and only if } d_i \in (\mathrm{F}_{i,v})^{\times 2},$$

then the local-global principle for the existence of K-embeddings from (E,σ) *into* (A,τ) *holds.*

We start with some calculations:

Lemma 3.16. $\mathrm{sc}(\mathrm{T}) = \mathrm{R}^{(1)}_{\mathrm{E}'/K}(\mathbb{G}_m)/\mathrm{R}^{(1)}_{\mathrm{E}'^{\sigma}/K}(\mathbb{G}_m)$.

Proof. Consider the exact sequence over E'^{σ}:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{R}_{\mathrm{E}'/\mathrm{E}'^{\sigma}}(\mathbb{G}_m) \longrightarrow \mathrm{R}^{(1)}_{\mathrm{E}'/\mathrm{E}'^{\sigma}}(\mathbb{G}_m) \longrightarrow 1\ ,$$

where the map from $\mathrm{R}_{\mathrm{E}'/\mathrm{E}'^{\sigma}}(\mathbb{G}_m)$ to $\mathrm{R}^{(1)}_{\mathrm{E}'/\mathrm{E}'^{\sigma}}(\mathbb{G}_m)$ sends x in $\mathrm{R}_{\mathrm{E}'/\mathrm{E}'^{\sigma}}(\mathbb{G}_m)(R)$ to $x/\sigma(x)$, for any K-algebra R. Let us take the Weil restriction of the above sequence over K. Then we get the exact sequence:

$$(1)\quad 1 \longrightarrow \mathrm{R}_{\mathrm{E}'^{\sigma}/K}(\mathbb{G}_m) \longrightarrow \mathrm{R}_{\mathrm{E}'/K}(\mathbb{G}_m) \longrightarrow \mathrm{T} \longrightarrow 1\ .$$

Let M (resp. P) be the character group of T (resp. sc(T)).

First, suppose that (E',σ) is split. Then $\mathrm{Aut}(\mathrm{E}',\sigma) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathrm{S}_n$ and there is a basis $\{e_i\}_i$ of M such that the S_n-part of $\mathrm{Aut}(\mathrm{E}',\sigma)$ acts on $\{e_i\}_i$ by permuting the indices and $(\mathbb{Z}/2\mathbb{Z})^n$ acts on $\{e_i\}_i$ by change the sign of e_i. In this case, $\mathrm{P} = \mathrm{M} + \frac{1}{2}(e_1 + \cdots + e_n)$. We choose a basis $\{\varepsilon_i, \epsilon_i\}_{i=1}^n$ (resp. $\{h_i\}_{i=1}^n$) of $\mathrm{M}_{\mathrm{E}'}$ (resp. $\mathrm{M}_{\mathrm{E}'^{\sigma}}$) on which $\mathrm{Aut}(\mathrm{E}',\sigma)$ acts as the following: S_n permutes the indices i and $(\mathbb{Z}/2\mathbb{Z})^n$ exchanges $\varepsilon_i, \epsilon_i$ (resp. $(\mathbb{Z}/2\mathbb{Z})^n$ acts trivially on h_i).

The we have the following exact sequence corresponding to (1):

$$0 \longrightarrow \mathrm{M} \xrightarrow{\ \imath\ } \mathrm{M}_{\mathrm{E}'} \xrightarrow{\ \jmath\ } \mathrm{M}_{\mathrm{E}'^{\sigma}} \longrightarrow 0\ ,$$

where $\imath$ maps e_i to $\varepsilon_i - \epsilon_i$ and $\jmath$ maps ε_i, ϵ_i to h_i. Consider the map $\bar{\imath}$ from M to $\mathrm{J}_{\mathrm{E}'}$ induced by $\imath$. Then $\bar{\imath}(e_1 + \cdots + e_n) = 2(\bar{\varepsilon}_1 + \cdots + \bar{\varepsilon}_n)$, where $\bar{\varepsilon}_i$ is the image of ε_1 in $\mathrm{J}_{\mathrm{E}'}$. Hence $\bar{\imath}$ induces a map from P to $\mathrm{J}_{\mathrm{E}'}$ and we have the following exact sequence:

$$(2)\quad 0 \longrightarrow \mathrm{P} \longrightarrow \mathrm{J}_{\mathrm{E}'} \longrightarrow \mathrm{J}_{\mathrm{E}'^{\sigma}} \longrightarrow 0.$$

Since all the maps constructed are equivariant under $\mathrm{Aut}(\mathrm{E}',\sigma)$, we conclude $\mathrm{sc}(\mathrm{T}) = \mathrm{R}^{(1)}_{\mathrm{E}'/K}(\mathbb{G}_m)/\mathrm{R}^{(1)}_{\mathrm{E}'^{\sigma}/K}(\mathbb{G}_m)$. □

Now we use the above lemma to compute $Ш^2(K, \mathrm{sc}(\mathrm{T}))$.

Proof of Theorem 3.15. Keep the notation of Lemma 3.16. By the Poitou–Tate duality (ref. [NSW], Chap. VIII, Thm. 8.6.9), we have

$$Ш^2(K, \mathrm{sc}(\mathrm{T})) \simeq Ш^1(K, \mathrm{P})^*.$$

Hence, it is enough to show that $Ш^1(K, \mathrm{P}) = 0$.

From the exact sequence (2) in the proof of Lemma 3.16, we derive the commutative exact diagram:

$$
\begin{array}{ccccccccccc}
 & & & & 0 & \longrightarrow & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{M}^{\mathcal{G}} & \longrightarrow & \mathrm{M}^{\mathcal{G}}_{\mathrm{E}'} & \longrightarrow & \mathrm{M}^{\mathcal{G}}_{\mathrm{E}'\sigma} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{P}^{\mathcal{G}} & \longrightarrow & \mathrm{J}^{\mathcal{G}}_{\mathrm{E}'} & \longrightarrow & \mathrm{J}^{\mathcal{G}}_{\mathrm{E}'\sigma} & \longrightarrow & \mathrm{H}^1(K,\mathrm{P}) & \longrightarrow \mathrm{H}^1(K,\mathrm{J}_{\mathrm{E}'}) & \longrightarrow \cdots \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \mathrm{H}^1(K,\mathbb{Z}) = 0 & & \mathrm{H}^1(K,\mathbb{Z}) = 0 & & & &
\end{array}
$$

Again, $Ш^1(K, \mathrm{J}_{\mathrm{E}'}) = Ш^2(K, \mathrm{R}^{(1)}_{\mathrm{E}'/K}(\mathbb{G}_m))^*$. By Hilbert Theorem 90, we have that $\mathrm{H}^2(K, \mathrm{R}^{(1)}_{\mathrm{E}'/K}(\mathbb{G}_m))$ injects into $\mathrm{H}^2(K, \mathrm{R}_{\mathrm{E}'/K}(\mathbb{G}_m))$. However, $Ш^2(K, \mathrm{R}_{\mathrm{E}'/K}(\mathbb{G}_m))$ vanishes, so does $Ш^1(K, \mathrm{J}_{\mathrm{E}'})$. Let $x \in Ш^1(K, \mathrm{P})$. Since $Ш^1(K, \mathrm{J}_{\mathrm{E}'}) = 0$, we have $y \in \mathrm{J}^{\mathcal{G}}_{\mathrm{E}'\sigma}$ mapped to x.

Let $\mathrm{I} = \{1, 2, \ldots, r\}$. Let I_1 be the subset of I such that $i \in \mathrm{I}_1$ if and only if $d_i \in \mathrm{F}_i^{\times 2}$. Let $\mathrm{I}_2 = \mathrm{I} \setminus \mathrm{I}_1$. Note that $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}'\sigma} = \bigoplus_{i=1}^r \mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$ and $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}'} = \bigoplus_{i=1}^r \mathrm{M}^{\mathcal{G}}_{\mathrm{E}_i}$. For $i \in \mathrm{I}_1$, $\mathrm{E}_i \simeq \mathrm{F}_i \times \mathrm{F}_i$, so $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}_i} \simeq \mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i} \bigoplus \mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$ and $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}_i}$ is mapped surjectively onto $\mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$.

Let γ_i be a basis of $\mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$. For $i \in \mathrm{I}_2$, we have the following observation:

Lemma 3.17. *For $i \in \mathrm{I}_2$ and $y \in \mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$, y is in the image of $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}_i}$ if and only if the coefficient of γ_i in y is even.*

Proof of Lemma 3.17. Since E_i is a field over F_i with degree 2 for $i \in \mathrm{I}_2$, the module $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}_i}$ is of rank 1 and is generated by $\sum_{\epsilon_j, \varepsilon_j \in \mathrm{M}_{\mathrm{E}_i}} (\epsilon_j + \varepsilon_j)$. Since the element $\sum_{\epsilon_j, \varepsilon_j \in \mathrm{M}_{\mathrm{E}_i}} (\epsilon_j + \varepsilon_j)$ is mapped to $2\gamma_i$ in $\mathrm{M}^{\mathcal{G}}_{\mathrm{F}_i}$, the lemma then follows. □

We return to the proof of Theorem 3.15. Since $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}'\sigma}$ is mapped surjectively onto $\mathrm{J}^{\mathcal{G}}_{\mathrm{E}'\sigma}$, $\mathrm{J}^{\mathcal{G}}_{\mathrm{E}'\sigma}$ is generated by γ_i's. Let $\bar{\gamma}_i$ be the image of γ_i in $\mathrm{J}_{\mathrm{E}'\sigma}$. Let $y = \sum_{i=1}^r a_i \bar{\gamma}_i$. If for all $i \in \mathrm{I}_2$, the a_i's have the same parity, then we can find $z = \sum_{i=1}^r b_i \gamma_i$, which is a lifting of y in $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}'\sigma}$, such that b_i is even for any $i \in \mathrm{I}_2$. Then by Lemma 3.17, z is in the image of $\mathrm{M}^{\mathcal{G}}_{\mathrm{E}'}$ and hence y is in the image of $\mathrm{J}^{\mathcal{G}}_{\mathrm{E}'}$. So it is enough to prove that for all $i \in \mathrm{I}_2$, the a_i's have the same parity.

Now, let v be a place of K such that for all i, $d_i \in \mathrm{F}_i^{\times 2}$ if and only if $d_i \in (\mathrm{F}_{i,v})^{\times 2}$. Since x is in $Ш^1(K, \mathrm{P})$, y is in the image $\mathrm{J}^{\mathcal{G}_v}_{\mathrm{E}'_v/K_v}$. For each $i \in \mathrm{I}_2$, since d_i is not a square in $\mathrm{F}_{i,v}$, there is some $h_{j(i)} \in \mathrm{M}_{\mathrm{F}_i}$ such that there exists $\tau_{j(i)} \in \mathcal{G}_v$ which exchanges $\epsilon_{j(i)}$ and $\varepsilon_{j(i)}$. Therefore, for all $i \in \mathrm{I}_2$, the coefficients of $\bar{h}_{j(i)}$'s in the expression of y have the same parity. Since the coefficient of $\bar{h}_{j(i)}$ in y is a_i, we know that all a_i's have the same parity for $i \in \mathrm{I}_2$. By Lemma 3.17, y is in the image of $\mathrm{J}^{\mathscr{G}}_{\mathrm{E}'}$, which means $\text{Ш}^1(K, \mathrm{P}) = 0$. □

Remark 3.18. A special case of the above theorem is when there is a place v such that $\mathrm{sc}(\mathrm{T})$ is anisotropic over K_v. We now show that $\mathrm{sc}(\mathrm{T})$ is anisotropic over K_v implies all $d_i \notin (\mathrm{F}_{i,v})^{\times 2}$. To see this, we note that in our case here, $\mathrm{sc}(\mathrm{T})$ is anisotropic if and only if T is anisotropic. If there is $d_i \in (\mathrm{F}_{i,v})^{\times 2}$, then $\mathrm{M}_{E_{i,v}/K_v} = \mathrm{M}_{F_{i,v}/K_v} \oplus \mathrm{M}_{F_{i,v}/K_v}$. Let α be a nontrivial element in $\mathrm{M}^{\mathscr{G}_v}_{F_{i,v}/K_v}$. Then $(\alpha, -\alpha) \in \mathrm{M}^{\mathscr{G}_v}_{E_{i,v}/K_v}$ and it is in the image of M, which means $\mathrm{M}^{\mathscr{G}_v}$ is nontrivial and contradicts to the condition that T is anisotropic over K_v. Therefore, $d_i \notin (\mathrm{F}_{i,v})^{\times 2}$ for all i.

The case where the degree of A is even. Throughout this paragraph, we let A be $\mathbf{M}_{2n}(K)$, or $\mathbf{M}_n(\mathrm{D})$ with orthogonal involution τ, where D is a quaternion division algebra over K. In this case, the corresponding group G is semisimple of type D_n, and $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ satisfies the local-global principle.

For A satisfying one of the conditions in Theorem 3.19, we first show that $\mathfrak{E}(\mathrm{G}, \Psi)(K_v)$ is nonempty implies that $\mathfrak{E}(\mathrm{G}, \Psi, u)(K_v)$ is nonempty for any orientation u. (See Lemma 3.20.) Then we prove that the local global principle holds for the oriented embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$. By Theorem 2.15 and Proposition 2.17, we get the local-global principle for the existence of K-embeddings from (E, σ) into (A, τ).

We first fix some notation. Let $\mathrm{E}^\sigma = \prod_{i=1}^r \mathrm{F}_i$, where the F_i's are fields over K. Let $d = (d_1, \ldots, d_r)$ be in E^σ and $\mathrm{E} = \mathrm{E}^\sigma[x]/(x^2 - d) = \prod_{i=1}^r \mathrm{F}_i[x]/(x^2 - d_i)$. Let $\mathrm{E}_i = \mathrm{F}_i[x]/(x^2 - d_i)$, and $\mathrm{E}_{i,v}$ (resp. $\mathrm{F}_{i,v}$) be $\mathrm{E}_i \otimes_K K_v$ (resp. $\mathrm{F}_i \otimes_K K_v$) for all $v \in \Omega_K$.

Theorem 3.19. *Suppose that* A *is equal to one of the following:*

(1) $\mathbf{M}_{2n}(K)$, $n > 1$.

(2) $\mathbf{M}_{2m+1}(\mathrm{D})$, *where* D *is a quaternion division algebra over* K.

(3) $\mathbf{M}_{2m}(\mathrm{D})$, *where* D *is a quaternion division algebra over* K, *and at each place* $v \in \Omega_K$, *if* A *is not split and the discriminant splits, then* E_v *is not split over* E^σ_v, *i.e.* $\mathrm{E}_v \neq \mathrm{E}^\sigma_v \times \mathrm{E}^\sigma_v$.

If there is a place $v \in \Omega_K$ *such that for all* i, $d_i \in \mathrm{F}_i^{\times 2}$ *if and only if* $d_i \in (\mathrm{F}_{i,v})^{\times 2}$, *then the local-global principle for the* K*-embedding of* (E, σ) *into* (A, τ) *holds.*

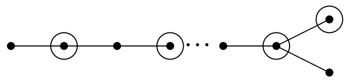
First we prove the following lemma:

Lemma 3.20. *For* A *satisfying one of the three conditions in Theorem 3.19, the existence of a* K_v*-point of* $\mathfrak{S}(\mathrm{G}, \Psi)$ *implies the existence of a* K_v*-point of* $\mathfrak{S}(\mathrm{G}, \Psi, u)$ *for any* $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(K_v)$.

Proof. Suppose that there is a K_v-point f of $\mathfrak{S}(\mathrm{G}, \Psi)$. By Theorem 3.12, there is an orientation u' induced by f such that $u'(\Delta^\circ(\Psi_{K_v})) \supseteq \Delta^\circ(\mathrm{G}_{K_v})$.

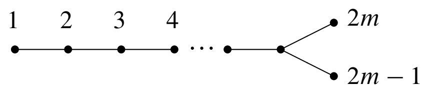
According to the list of all possible Tits indices (ref. [T66]), if A satisfies (1) or (2) in Theorem 3.19, then the Tits index of G_{K_v} will be symmetric under $\underline{\mathrm{Autext}}(\mathrm{G}_{K_v})$. Therefore, for any $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(K_v)$, we have that $u(\Delta^\circ(\Psi_{K_v}))$ contains $\Delta^\circ(\mathrm{G}_{K_v})$, and again, by Theorem 3.12, we have $\mathfrak{S}(\mathrm{G}, \Psi, u)(K_v) \neq \emptyset$.

Now assume that A satisfies (3) in Theorem 3.19. If over K_v, A is not split and the discriminant splits, then G is a non-split inner form over K_v. In this case, the possible Tits indices of G are symmetric except the following case:



Suppose that $\Delta^\circ(\mathrm{G}_{K_v})$ takes the above nonsymmetric form. We will show that condition (3) in Theorem 3.19 forces $\Delta^\circ(\Psi_{K_v})$ to be symmetric under $\underline{\mathrm{Autext}}(\Psi_{K_v})$ in this case.

Consider the Dynkin diagram of Ψ:



Suppose that $\mathrm{I} = \Delta^\circ(\Psi_{K_v})$ is not symmetric under $\underline{\mathrm{Autext}}(\Psi)$. Without loss of generality, we suppose that the vertex $2m$ is not in I. Let I' be the Dynkin subdiagram with vertices $1, \dots, 2m-1$ which is of type A_{2m-1}. So $\mathrm{I} \subseteq \mathrm{I}'$.

Since there is $f \in \mathfrak{S}(\mathrm{G}, \Psi)(K_v)$, G_{K_v} has a parabolic subgroup P_I with the type I and P_I contains $f(\mathrm{T}_{K_v})$ by Proposition 1.7. Let G_0, Ψ_0 be the split form of G and Ψ respectively (Section. 2.3.1), and let T_0 be the split torus determined by Ψ_0. Let $\mathrm{P}_{0,\mathrm{I}}$ be a parabolic subgroup of G_{0,K_v} with type I and contains T_{0,K_v}.

Let $\mathscr{P}_\mathrm{I}$ (resp. $\mathscr{P}_{0,\mathrm{I}}$) be the subsheaf of the sheaf of roots of Ψ (resp. Ψ_0) determined by P_I (resp. $\mathrm{P}_{0,\mathrm{I}}$). Define $\mathrm{W}_{0,\mathrm{I}} = \mathrm{W}(\Psi_{0,\mathrm{I}})$ and $\mathrm{W}_\mathrm{I} = \mathrm{W}(\Psi_\mathrm{I})$ as we have done in the proof of Theorem 3.12. Define $\mathrm{W}_{0,\mathrm{I}'} = \mathrm{W}(\Psi_{0,\mathrm{I}'})$ in the same way.

Let Ψ_0 be $(\mathrm{M}_0, \mathrm{M}_0^\vee, \mathrm{R}_0, \mathrm{R}_0^\vee)$, and $\{e_i\}_{i=1}^{2m}$ be a basis of M_0 such that R_0 is the set $\{\pm e_i \pm e_j\}_{i<j}$, where the vertex i corresponds to $e_i - e_{i+1}$ for $i = 1, \dots, 2m-1$, and the vertex $2m$ corresponds to $e_{2m-1} + e_{2m}$. Let S_n be the permutation group of n elements. Then we have

$$\underline{\mathrm{Aut}}(\Psi_{0,K_v})(K_v) = (\mathbb{Z}/2\mathbb{Z})^{2m} \rtimes \mathrm{S}_{2m},$$

where S_{2m} acts on R_0 by permuting the indices of $\{e_i\}_{i=1}^{2m}$, and $(\mathbb{Z}/2\mathbb{Z})^{2m}$ acts on R_0 by exchanging the sign of e_i's ([Bou], Plan. IV). Under this basis, $\mathrm{W}_{0,\mathrm{I}'}$

is just the permutation group of the set $\{e_i\}_{i=1}^{2m}$. Therefore, the natural inclusion $\iota_{\mathrm{W}}\colon \mathrm{W}_{0,\mathrm{I}'} \to \underline{\mathrm{Aut}}(\Psi_{0,K_v})$ sends $w \in \mathrm{W}_{0,\mathrm{I}'} \simeq \mathrm{S}_{2m}$ to $(1, w) \in (\mathbb{Z}/2\mathbb{Z})^{2m} \rtimes \mathrm{S}_{2m}$.

Since G_{K_v} is an inner form of G_{0,K_v}, there is an orientation

$$\mu \in \underline{\mathrm{Isomext}}(\Psi_{0,K_v}, \Phi(\mathrm{G}_{K_v}, f(\mathrm{T}_{K_v})))(K_v).$$

The orientation μ together with u'^{-1} gives an orientation

$$\nu \in \underline{\mathrm{Isomext}}(\Psi_{0,K_v}, \Psi_{K_v})(K_v).$$

We then define

$$\mathrm{Q} = \underline{\mathrm{Isomint}}_{\nu}(\Psi_{0,K_v}, \mathscr{P}_{0,\mathrm{I}}; \Psi_{K_v}, \mathscr{P}_{\mathrm{I}})$$

as we have done in the proof of Proposition 3.12. Since $\mathrm{W}_{0,\mathrm{I}} \subseteq \mathrm{W}_{0,\mathrm{I}'}$, we can regard $\mathrm{W}_{0,\mathrm{I}}$ as a subgroup of $\{1\} \rtimes \mathrm{S}_{2m} \subseteq \underline{\mathrm{Aut}}(\Psi_0)$ through ι_{W}. Since $\Psi_{K_v} = \mathrm{Q} \overset{\mathrm{W}_{0,\mathrm{I}}}{\wedge} \Psi_{0,K_v}$, by Remark 2.13, $(\mathrm{E}_v, \sigma) \simeq \mathrm{Q} \overset{\mathrm{W}_{0,\mathrm{I}}}{\wedge} (\mathrm{E}_{0,v}, \sigma_0)$. Therefore, $\mathrm{E}_v \simeq \mathrm{E}'_v \times \mathrm{E}'_v$ with σ acts on E_v as the exchange of the two copies of E'_v, which contradicts to the assumption (3) in Theorem 3.19! Therefore, I is symmetric under $\underline{\mathrm{Autext}}(\Psi_{K_v})$ and we conclude that $u(\Delta^\circ(\Psi_{K_v})) \supseteq \Delta^\circ(\mathrm{G}_{K_v})$ for any orientation u. Again, by Theorem 3.12, we have $\mathfrak{E}(\mathrm{G}, \Psi, u)(K_v) \neq \emptyset$, for any $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(K_v)$. ☐

Next, we prove that the $\underline{\mathrm{Autext}}(\mathrm{G})$-torsor $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ satisfies the local-global principle. Namely,

Lemma 3.21. *Let* G *(resp.* Ψ*) be the corresponding semisimple group (resp. root datum) defined by* A *(resp.* E*). If the* $\underline{\mathrm{Autext}}(\mathrm{G})$*-torsor* $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ *has a* K_v*-point at each place* $v \in \Omega_K$*, then* $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ *has a* K*-point.*

Proof. If A is not equal to $\mathbf{M}_8(K)$ or $\mathrm{A} = \mathbf{M}_4(\mathrm{D})$, then $\underline{\mathrm{Autext}}(\mathrm{G})$ is $(\mathbb{Z}/2\mathbb{Z})_K$, so the local-global principle for $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ holds in this case.

For G an inner form, the outer automorphism group $\underline{\mathrm{Autext}}(\mathrm{G})$ is the symmetric group S_3. Therefore, to prove the local-global principal for the S_3-torsor $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$, we only need to prove $\text{Ш}^1(K, \mathrm{S}_3) = 0$. Consider the exact sequence:

$$(1)\ \ 0 \to \mathbb{Z}/3\mathbb{Z} \to \mathrm{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

From the above exact sequence, we get the following exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to \mathrm{S}_3 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{H}^1(K, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1(K, \mathrm{S}_3) \to \mathrm{H}^1(K, \mathbb{Z}/2\mathbb{Z}).$$

Since the map from S_3 ro $\mathbb{Z}/2\mathbb{Z}$ is surjective, we have

$$0 \to \mathrm{H}^1(K, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1(K, \mathrm{S}_3) \to \mathrm{H}^1(K, \mathbb{Z}/2\mathbb{Z}).$$

However, the group $\text{Ш}^1(K, \mathbb{Z}/2\mathbb{Z}) = 0$, so the set $\text{Ш}^1(K, \mathrm{S}_3)$ is in the image of $\mathrm{H}^1(K, \mathbb{Z}/3\mathbb{Z})$.

Recall that $\mathscr{G}$ is the absolute Galois group of K. Note that $\mathrm{H}^1(K, \mathbb{Z}/3\mathbb{Z}) = \mathrm{Hom}_{\mathrm{gr}}(\mathscr{G}, \mathbb{Z}/3\mathbb{Z})$, where $\mathrm{Hom}_{\mathrm{gr}}(\mathscr{G}, \mathbb{Z}/3\mathbb{Z})$ is the set of continuous homomorphisms from $\mathscr{G}$ to $\mathbb{Z}/3\mathbb{Z}$. Suppose $\alpha \in \mathrm{Hom}_{\mathrm{gr}}(K, \mathbb{Z}/3\mathbb{Z})$ is mapped into $\text{Ш}^1(K, \mathrm{S}_3)$. Then since the symmetric group S_n is surjective to the group $\mathbb{Z}/2\mathbb{Z}$ for each place $v \in \Omega_K$, we have

$$0 \to \mathrm{H}^1(K_v, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1(K_v, \mathrm{S}_3) \to \mathrm{H}^1(K_v, \mathbb{Z}/2\mathbb{Z}).$$

Hence, the homomorphism α is in $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$. Now we claim that $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$ is trivial, i.e. for each $\alpha \in \mathrm{Hom}_{\mathrm{gr}}(K, \mathbb{Z}/3\mathbb{Z})$, if α is in $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$, then α is the trivial homomorphism. Suppose that α is not the trivial homomorphism. Let $\mathscr{H}$ be the kernel of α. Let $L = (K^{\mathrm{s}})^{\mathscr{H}}$. Then L is a Galois extension of K with Galois group $\mathbb{Z}/3\mathbb{Z}$ and we can regard it as a $\mathbb{Z}/3\mathbb{Z}$-torsor. Since the homomorphism α is in $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$, L_v is split completely over K_v for each place $v \in \Omega_K$. This contradicts Chebotarev's density Theorem! Therefore, α is the trivial homomorphism and $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$ is trivial. Since $\text{Ш}^1(K, \mathrm{S}_n)$ is in the image of $\text{Ш}^1(K, \mathbb{Z}/3\mathbb{Z})$, $\text{Ш}^1(K, \mathrm{S}_n)$ is also trivial.

For $\mathrm{A} = \mathbf{M}_4(\mathrm{D})$ and G an outer form, let L be the splitting field of the discriminant of A. We choose a splitting z from $\mathbb{Z}/2\mathbb{Z}$ to S_3 and we twist the sequence (1) by z. Since z acts on $\mathbb{Z}/3\mathbb{Z}$ as -1, we have the exact sequence

$$(2)\ \ 0 \to \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}) \to {}_z(\mathrm{S}_3) \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

where we regard $\mathbb{Z}/3\mathbb{Z}$ as a constant group scheme. Note that z is invariant under the twisting because $\mathbb{Z}/2\mathbb{Z}$ is commutative. Therefore, the sequence still splits. Consider the exact sequence derived from (2):

$$\begin{aligned} 0 &\to \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z})(K) \to {}_z(\mathrm{S}_3)(K) \to (\mathbb{Z}/2\mathbb{Z})(K) \\ &\to \mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z})) \to \mathrm{H}^1(K, {}_z(\mathrm{S}_3)) \to \mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/2\mathbb{Z})). \end{aligned}$$

Since the sequence (2) splits, ${}_z(\mathrm{S}_3)(K)$ is mapped onto $(\mathbb{Z}/2\mathbb{Z})(K)$. Hence we have the exact sequence

$$0 \to \mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z})) \to \mathrm{H}^1(K, {}_z(\mathrm{S}_3)) \to \mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/2\mathbb{Z})).$$

Since $\text{Ш}^1(K, \mathbb{Z}/2\mathbb{Z}) = 0$, the set $\text{Ш}^1(K, {}_z(\mathrm{S}_3)(K))$ is contained in the image of $\mathrm{H}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}))$. Again, because the exact sequence (2) splits, for each place $v \in \Omega_K$, we have

$$0 \to \mathrm{H}^1(K_v, \mathrm{R}^{(1)}_{L_v/K_v}(\mathbb{Z}/3\mathbb{Z})) \to \mathrm{H}^1(K_v, {}_z(\mathrm{S}_3)) \to \mathrm{H}^1(K_v, \mathrm{R}^{(1)}_{L_v/K_v}(\mathbb{Z}/2\mathbb{Z})).$$

Therefore, $\text{Ш}^1(K, {}_z(\mathrm{S}_3)(K))$ is in the image of $\text{Ш}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}))$. Now, we only need to prove $\text{Ш}^1(K, \mathrm{R}^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z})) = 0$. By Shapiro's Lemma, we have that $Ш^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z})) = Ш^1(L, \mathbb{Z}/3\mathbb{Z})$. As we have proved above, the group $Ш^1(L, \mathbb{Z}/3\mathbb{Z}) = 0$, so $Ш^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z})) = 0$. Consider the following exact sequence:

$$0 \to R^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}) \to R_{L/K}(\mathbb{Z}/3\mathbb{Z}) \xrightarrow{\mathrm{Nr}} \mathbb{Z}/3\mathbb{Z} \to 0.$$

In our case, the norm map Nr from $R_{L/K}(\mathbb{Z}/3\mathbb{Z})(K) = \mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}/3\mathbb{Z}$ is just the multiplication by 2. Hence the map Nr is a surjective map from $R_{L/K}(\mathbb{Z}/3\mathbb{Z})(K)$ to $(\mathbb{Z}/3\mathbb{Z})(K)$, and thus the map from $H^1(K, R^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}))$ to $H^1(K, R_{L/K}(\mathbb{Z}/3\mathbb{Z}))$ is injective. Hence, $Ш^1(K, R^{(1)}_{L/K}(\mathbb{Z}/3\mathbb{Z}))$ is also trivial. Therefore, in both cases, the local-global principle for $\underline{\mathrm{Isomext}}(\Psi, G)$ holds. □

Now we have all the ingredients to prove Theorem 3.19.

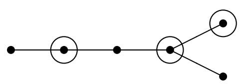
Proof of Theorem 3.19. Suppose that $(E \otimes K_v, \sigma \otimes \mathrm{id}_{K_v})$ can be embedded into $(A \otimes K_v, \tau \otimes \mathrm{id}_{K_v})$ over K_v for each $v \in \Omega_K$, i.e., $E(G, \Psi)(K_v) \neq \emptyset$ for each $v \in \Omega_K$. Then we have $\underline{\mathrm{Isomext}}(\Psi, G)(K_v) \neq \emptyset$ for each place v. By Lemma 3.21, we can fix an orientation u.

By Lemma 3.20, the oriented embedding functor $\mathfrak{E}(G, \Psi, u)$ has a K_v-point for each $v \in \Omega_K$. By Proposition 3.8, the only obstruction for $\mathfrak{E}(G, \Psi, u)$ to satisfy the local-global principle lies in $Ш^2(K, \mathrm{sc}(T))$. As the proof of Theorem 3.15 shows, $Ш^2(K, \mathrm{sc}(T))$ vanishes if there is a place $v \in \Omega_K$ such that for all i, $d_i \in F_i^{\times 2}$ if and only if $d_i \in (F_i \otimes_K K_v)^{\times 2}$. Therefore, the oriented embedding functor $E(G, \Psi, u)$ satisfies the local-global principle in this case. By Theorem 2.15 and Proposition 2.17, the local-global principle for the existence of K-embeddings from (E, σ) into (A, τ) holds. □

In the following, we provide an example when the local-global principle for the embedding functor fails.

Example 3.22. Let K be $\mathbb{Q}(\sqrt{-1})$, and $F = K[x]/(x^2-3)$. Let $E' = F \times F \times F$ and $E = E' \times E'$. Let σ be the K-automorphism of E which exchanges the two copies of E'. Then σ is an involution and $E^\sigma \simeq E'$. With the notation defined in Section 2.3, we know that the right $\underline{\mathrm{Aut}}(E_0, \sigma_0)$-torsor $\underline{\mathrm{Isom}}((E_0, \sigma_0), (E, \sigma))$ defines a class in $H^1(K, S_6)$, where S_6 is contained in the Weyl group of Ψ_0 ([Bou], Plan. IV). Let Ψ be the corresponding root datum. Since Ψ comes from a class of $H^1(K, S_6)$, Ψ is an inner form of Ψ_0.

Let us fix four places of K such that F is not split over K_v. For example, we can take a place v which corresponds to a prime number of the form $7 + 12l$, where l is a positive integer. By Gauss reciprocity, $x^2 - 3$ is not split at v. Let $v_1, \ldots, v_4$ be the four places mentioned above. At these places, the corresponding Tits index of Ψ is the following:



Consider the following central isogeny:

$$1 \longrightarrow \mu_2 \times \mu_2 \longrightarrow \mathrm{Spin}_6 \longrightarrow \mathrm{PSO}_6 \longrightarrow 1.$$

Since we have no real places, by [San81], Corollary 4.5, we have

$$\mathrm{H}^1(K, \mathrm{PSO}_6) \xrightarrow{\sim} \mathrm{H}^2(K, \mu_2 \times \mu_2).$$

Also at each finite place v, we have

$$\mathrm{H}^1(K_v, \mathrm{PSO}_6) \xrightarrow{\sim} \mathrm{H}^2(K_v, \mu_2 \times \mu_2) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

(ref. [K], Chap. IV, Thm. 1 and Thm. 2). Let $[\xi_i]$ be the class in $\mathrm{H}^1(K_{v_i}, \mathrm{PSO}_6)$ corresponding to $(1,0)$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, for i=1, 2. Let $[\xi_i]$ be the class in $\mathrm{H}^1(K_{v_i}, \mathrm{PSO}_6)$ corresponding to $(0,1)$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, for i=3, 4. For the other places $v \in \Omega_K \setminus \{v_1,\ v_2,\ v_3,\ v_4\}$, we let $[\xi_v]$ in $\mathrm{H}^1(K_v, \mathrm{PSO}_6)$ correspond to $(0,0) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By the Brauer–Hasse–Noether Theorem, we know that there exists a class $[\xi]$ in $\mathrm{H}^1(K, \mathrm{PSO}_6)$ such that the image of $[\xi]$ in $\mathrm{H}^1(K_v, \mathrm{PSO}_6)$ is $[\xi_v]$ for each $v \in \Omega_K$.

Choose a cocycle ζ which represents the class $[\xi]$. Let G be the K-form of G_0 twisted by ζ. Since G and Ψ are inner forms of G_0 and Ψ_0 respectively, we can fix an orientation u of Ψ with respect to G. Without loss of generality, we can choose the orientation u such that $u(\Delta^\circ(\Psi_{K_{v_1}})) \supseteq (\Delta^\circ(\mathrm{G}_{K_{v_1}}))$. Note that there is no orientation u' such that $u'(\Delta^\circ(\Psi_{K_v}))$ contains $\Delta^\circ(\mathrm{G}_{K_v})$ for both $v = v_1$ and $v = v_3$.

For each place $v \in \Omega_K \setminus \{v_3,\ v_4\}$, we have $u(\Delta^\circ(\Psi_{K_v})) \supseteq \Delta^\circ(\mathrm{G}_{K_v})$, so by Theorem 3.12, there is a K_v point of $\mathfrak{E}(\mathrm{G}, \Psi, u)$. On the one hand, for the place $v \in \{v_3,\ v_4\}$, by Theorem 3.12, $\mathfrak{E}(\mathrm{G}, \Psi, u)(K_v)$ is empty. Therefore, the embedding functor $\mathfrak{E}(\mathrm{G}, \Psi, u)$ has no K-points. For the same reason, we conclude $\mathfrak{E}(\mathrm{G}, \Psi, u')(K) = \emptyset$ for the other orientation u'. Hence $\mathfrak{E}(\mathrm{G}, \Psi)$ has no K-point. However, at each place v, we can always find an orientation $u_v \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(K_v)$ such that $u_v(\Delta^\circ(\Psi_{K_v})) \supseteq \Delta^\circ(\mathrm{G}_{K_v})$, so the embedding functor $\mathfrak{E}(\mathrm{G}, \Psi)$ has a K_v-point for each place v. Therefore, the local-global principle fails in this case.

3.4.3. Involutions of the second kind. In this section, A is of degree n over K and τ is of the second kind. The corresponding reductive group G is of type A_{n-1}. In this case, K and k are no longer the same.

Recall that $i_{\mathrm{T}}\colon \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}) \to \mathrm{T}$ (resp. $i_{\mathrm{G}}\colon \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_{m,K}) \to \mathrm{G}$) denote the embedding defined by the K-structure morphism of E (resp. A). We first interpret the K-morphism condition into an orientation. Namely, we show that the following are equivalent:

1. A k-embedding f is a K-embedding.
2. $f \circ i_{\mathrm{T}} = i_{\mathrm{G}}$.
3. f is a k-point of $\mathfrak{E}(\mathrm{G}, \Psi, u)$ for some particular orientation u.

Using the following lemma, we can concretely define the orientation u mentioned above.

Lemma 3.23. *Let* $\mathrm{Z} = \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_m)$ *and* $\Psi_{\mathrm{Z}} = \Phi(\mathrm{Z}, \mathrm{Z})$. *Then*

(1) *The natural homomorphism from* $\underline{\mathrm{Aut}}(\Psi)$ *to* $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi))$ *induces an isomorphism from* $\underline{\mathrm{Autext}}(\Psi)$ *to* $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$.

(2) $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$ *is a trivial* $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$*-torsor.*

Proof. Let j_{T} be the homomorphism from the character group of T to the character group of Z induced by i_{T}. Then j_{T} induces an isomorphism between $\mathrm{rad}(\Psi)$ and Ψ_{Z} and we have a canonical way to identify $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi))$ and $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$. Consider the natural morphism from $\underline{\mathrm{Aut}}(\Psi)$ to $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi))$. Since the Weyl group acts trivially on $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi))$, we have a natural morphism η from $\underline{\mathrm{Autext}}(\Psi)$ to $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi))$. Note that since Z is a torus of dimension one, $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}}) \simeq \mathbb{Z}/2\mathbb{Z}$. Hence, $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi)) \simeq \mathbb{Z}/2\mathbb{Z}$.

To prove that η is an isomorphism, we only need to check it over k^{s}, so we can assume that Ψ is split and of type $(\mathrm{M}, \mathrm{M}^{\vee}, \mathrm{R}, \mathrm{R}^{\vee})$.

We first prove the injectivity of η. By the definition of Ψ, we can find a basis $\{e_i\}_{i=1,\dots,n}$ of M such that $\Delta = \{e_i - e_{i+1}\}_{i=1,\dots,n-1}$ is a system of simple roots of R (ref. [Bou], Plan. I). By Proposition 1.3, $\mathrm{Autext}(\Psi) \simeq \mathrm{E}_{\Delta}(\Psi)$. Let $h \in \mathrm{E}_{\Delta}(\Psi)$ and suppose that h acts on $\mathrm{rad}(\Psi)$ trivially. We claim that h acts on Ψ trivially.

To see this, we note that h induces an isomorphism on the Dynkin diagram, so h can only act on Δ trivially or exchange $e_i - e_{i+1}$ with $e_{n-i} - e_{n-i+1}$.

Let $h(e_1) = \sum_{i=1}^{n} a_i e_i$.

Suppose that h exchanges $e_i - e_{i+1}$ with $e_{n-i} - e_{n-i+1}$. Since

$$\alpha^{\vee}(e_1) = h(\alpha)^{\vee}(h(e_1)) \quad \text{for all } \alpha^{\vee} \in \mathrm{R}^{\vee},$$

we have

$$\begin{aligned} a_{n-1} &= a_n + 1, \\ a_i &= a_{n-1}, \quad i = 1, \dots, n-1. \end{aligned}$$

Besides, h acts on $\mathrm{rad}(\Psi)$ trivially, so $e_1 - h(e_1) = \sum_{i=1}^{n} b_i(e_i - e_{i+1})$. By summing up the coefficients, we have $\sum_{i=1}^{n} a_i = 1$ and hence $n a_{n-1} = 2$. Since a_i's are integers, the only possibility is $n = 2$ and $a_1 = 1$. In this case, $h(e_1) = e_1$ and $h(e_1 - e_2) = e_1 - e_2$, so h is identity.

Now suppose that h acts on Δ trivially. Then by the same reasoning, we have

$$\begin{aligned} a_1 &= a_2 + 1, \\ a_2 &= a_i, \quad i = 2, \dots, n. \end{aligned}$$

Besides, h acts on $\mathrm{rad}(\Psi)$ trivially, so $\sum_{i=1}^{n} a_i = 1$. Therefore, $a_1 = 1$ and $a_i = 0$ for $i \neq 1$, which means h is the identity.

This proves that η is injective.

On the other hand, since the -1 map on Ψ induces the -1 map on $\mathrm{rad}(\Psi)$ and $\underline{\mathrm{Aut}}(\mathrm{rad}(\Psi)) \simeq \mathbb{Z}/2\mathbb{Z}$, we have η is surjective and hence an isomorphism.

To prove (2), we choose a maximal torus T' of G. Note that since K is in the center of A, $i_{\mathrm{G}}(\mathrm{Z})$ is in the center of G and hence $i_{\mathrm{G}}(\mathrm{Z})$ is in T'. Let j_{G} be the map between character groups of T' and Z induced by i_{G}. Let $\Psi_{\mathrm{G}} = \Phi(\mathrm{G}, \mathrm{T}')$. Then we have a natural morphism from $\underline{\mathrm{Isom}}(\Psi, \Psi_{\mathrm{G}})$ to $\underline{\mathrm{Isom}}(\mathrm{rad}(\Psi), \mathrm{rad}(\Psi_{\mathrm{G}}))$. Since the Weyl group $\mathrm{W}(\Psi)$ acts trivially on $\mathrm{rad}(\Psi)$, the above morphism induces an morphism from $\underline{\mathrm{Isomext}}(\Psi, \Psi_{\mathrm{G}})$ to $\underline{\mathrm{Isom}}(\mathrm{rad}(\Psi), \mathrm{rad}(\Psi_{\mathrm{G}}))$, which is an isomorphism. Through j_{T} and j_{G}, we have an isomorphism from $\underline{\mathrm{Isom}}(\mathrm{rad}(\Psi), \mathrm{rad}(\Psi_{\mathrm{G}}))$ to $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$, which sends f^{Ψ} to $j_{\mathrm{G}} \circ f^{\Psi} \circ j_{\mathrm{T}}^{-1}$. Therefore, we have

$$\zeta \colon \underline{\mathrm{Isomext}}(\Psi, \Psi_{\mathrm{G}}) \to \underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}}).$$

Since $\underline{\mathrm{Autext}}(\Psi) \simeq \underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$ and ζ is compatible with the $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$-action, ζ is an isomorphism between $\underline{\mathrm{Aut}}(\Psi_{\mathrm{Z}})$-principal homogeneous spaces. Since there is a canonical isomorphism from $\underline{\mathrm{Isomext}}(\Psi, \Psi_{\mathrm{G}})$ to $\underline{\mathrm{Isomext}}(\Psi, \mathrm{G})$, the result then follows. □

With the notation defined in the above lemma, we let $u \in \underline{\mathrm{Isomext}}(\Psi, \mathrm{G})(k)$ be $\zeta^{-1}(1)$. Then for a k-embedding f, we see that $f \circ i_{\mathrm{T}} = i_{\mathrm{G}}$ if and only if f is a k-point of $\mathfrak{E}(\mathrm{G}, \Psi, u)$. Hence again, we can reduce the embedding problem to the existence of rational points of $f \in \mathfrak{E}(\mathrm{G}, \Psi, u)$ and reformulate Prasad–Rapinchuk's Theorem as follows ([PR10], Thm. 4.1):

Theorem 3.24. *Suppose that τ is an involution of the second type. If* E *is a field, then the local-global principle for the K-embeddings from* (E, σ) *to* (A, τ) *holds.*

Proof. By Lemma 3.23, we can fix an orientation u such that f is a k-point of $\mathfrak{E}(\mathrm{G}, \Psi, u)$ if and only if f is a K-embedding. By Remark 2.16, $\mathfrak{E}(\mathrm{G}, \Psi, u)(k_v)$ is nonempty if and only if $\mathfrak{E}(\mathrm{G}, \Psi)(k_v)$ is nonempty. Hence, it suffices to show that the local-global principal for $\mathfrak{E}(\mathrm{G}, \Psi, u)$ holds. By Theorem 3.2, we only need to show that $\text{Ш}^2(k, \mathrm{sc}(\mathrm{T}))$ vanishes. Consider the exact sequence

$$0 \longrightarrow \mathrm{sc}(\mathrm{T}) \longrightarrow \mathrm{R}_{\mathrm{E}^{\sigma}/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^{\sigma}}(\mathbb{G}_m)) \longrightarrow \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_m) \longrightarrow 0,$$

from which we derive the long exact sequence

$$\begin{aligned}\longrightarrow \mathrm{H}^1(k, \mathrm{sc}(\mathrm{T})) &\longrightarrow \mathrm{H}^1(k, \mathrm{R}_{\mathrm{E}^{\sigma}/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^{\sigma}}(\mathbb{G}_m))) \longrightarrow \mathrm{H}^1(k, \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_m)) \\ \longrightarrow \mathrm{H}^2(k, \mathrm{sc}(\mathrm{T})) &\longrightarrow \mathrm{H}^2(k, \mathrm{R}_{\mathrm{E}^{\sigma}/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^{\sigma}}(\mathbb{G}_m))) \longrightarrow \cdots .\end{aligned}$$

Since $Ш^2(k, \mathrm{R}_{\mathrm{E}^\sigma/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m))) = Ш^2(\mathrm{E}^\sigma, \mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m)) = 0$, we know that $Ш^2(k, \mathrm{sc}(\mathrm{T}))$ is in the image of $\mathrm{H}^1(k, \mathrm{R}^{(1)}_{K/k}(\mathbb{G}_m)) = k^\times/\mathrm{Nr}_{K/k}(K^\times)$, where $\mathrm{Nr}_{K/k}$ denotes the norm map from K to k. Let x be an element of $k^\times$ and suppose that x is mapped to $Ш^2(k, \mathrm{sc}(\mathrm{T}))$. At each place $v \in \Omega_k$, let x_v be the image of x in k_v. Since x is mapped to $Ш^2(k, \mathrm{sc}(\mathrm{T}))$, x belongs to $k^\times \bigcap \mathrm{Nr}_{\mathrm{E}^\sigma/k}(\mathrm{I}_{\mathrm{E}^\sigma})\mathrm{Nr}_{K/k}(\mathrm{I}_K)$, where $\mathrm{I}_{\mathrm{E}^\sigma}$ and I_K are idèle groups of E^σ and K respectively. By Hasse multinorm principle (ref. [PlR], Prop. 6.11), x belongs to $\mathrm{Nr}_{\mathrm{E}^\sigma/k}(\mathrm{E}^\sigma)\mathrm{Nr}_{K/k}(K)$, so x is in the image of $\mathrm{H}^1(k, \mathrm{R}_{\mathrm{E}^\sigma/k}(\mathrm{R}^{(1)}_{\mathrm{E}/\mathrm{E}^\sigma}(\mathbb{G}_m)))$. Hence x is mapped to 0 in $Ш^2(k, \mathrm{sc}(\mathrm{T}))$, which implies $Ш^2(k, \mathrm{sc}(\mathrm{T})) = 0$. The theorem then follows. □

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