On the number of finite subgroups of a lattice

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Abstract. We show that the number of conjugacy classes of maximal finite subgroups of a lattice in a semisimple Lie group is linearly bounded by the covolume of the lattice. Moreover, for higher rank groups, we show that this number grows sublinearly with covolume. We obtain similar results for isotropy subgroups in lattices. Geometrically, this yields volume bounds for the number of strata in the natural stratification of a finite-volume locally symmetric orbifold.

1. Introduction

Several families of infinite groups share the property that they have finitely many conjugacy classes of finite subgroups (henceforth, *the finiteness property*). In the realm of linear groups this was first proven by Jordan for $GL_n(\mathbb{Z})$. Using their reduction theory, Borel and Harish-Chandra generalized Jordan's theorem and proved the finiteness property holds for arithmetic groups of the form $\mathbb{G}(\mathbb{Z})$, where $\mathbb{G}$ is a linear algebraic group defined over $\mathbb{Q}$ [6]. This result was extended by Grunewald and Platonov to general arithmetic groups, as well as to their finite extensions [15]. Other families of groups known to enjoy this property are $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$ (cf. [10]), mapping class groups (cf. [8]), word hyperbolic groups, and CAT(0) groups (cf. [9]).

For a family of groups that has the finiteness property, it is natural to seek asymptotic bounds for the number of conjugacy classes of finite subgroups. For example, if Γ is a group with the finiteness property and $\Gamma_n < \Gamma$ is a sequence of finite-index subgroups, then by elementary group theory

$$F(\Gamma_n) \leq F(\Gamma) \cdot [\Gamma : \Gamma_n],$$

where $F(\cdot)$ is the number of conjugacy classes of finite subgroups. We generalize this "linear" bound to a family of lattices in a given semisimple Lie group. Naturally, the index of a group is replaced by its covolume. For reasons that will be made clear later, we only bound the number of conjugacy classes of *maximal* finite subgroups.

Theorem 1.1. *Let G be a connected semisimple Lie group with finite center and without compact factors. For a lattice $\Gamma < G$, denote by $f(\Gamma)$ the number of conjugacy classes of maximal finite subgroups in Γ. Then*

$$f(\Gamma) \leq c \cdot \text{vol}(G/\Gamma)$$

with a constant $c = c(G)$.

In particular, this theorem establishes the finiteness property for lattices in these semisimple Lie groups. We remark that if G is simple, mere finiteness already follows from the aforementioned results, unless Γ is a non-unform lattice in $\text{SO}(d, 1)$ ($d \geq 3$) or $\text{SU}(d, 1)$. Indeed, if G has higher rank, or is $\text{Sp}(d, 1)$ or F_4^{-20}, every lattice is arithmetic. If $G = \text{SO}(2, 1)$, or G has rank one and Γ is a uniform lattice, then Γ is word hyperbolic.

In some cases, we can make a stronger statement on the asymptotic growth of the number of conjugacy classes of maximal finite subgroups.

Theorem 1.2. *Let G be as in Theorem 1.1, and assume moreover that $\mathbb{R}$-rank$(G) \geq 2$ and G has Kazhdan's property (T). If Γ_n is a sequence of pairwise non-conjugate irreducible lattices in G, then*

$$\lim_n \frac{f(\Gamma_n)}{\text{vol}(G/\Gamma_n)} = 0.$$

The statement of the theorem does not hold for $G = \text{SO}(d, 1)$ ($d \geq 2$). Indeed, in Section 5 we exhibit a sequence of lattices $\Gamma_n < G$ such that $\text{vol}(G/\Gamma_n) \to \infty$ and

$$\liminf_n \frac{f(\Gamma_n)}{\text{vol}(G/\Gamma_n)} > 0.$$

The asymptotic behavior for lattices in other rank-one groups, as well as that for irreducible lattices in products of rank-one groups, remains unsettled.

The proof of Theorem 1.1 is based on an analysis of the action of Γ on the associated symmetric space $X = K \backslash G$. Each finite subgroup of Γ fixes a connected complete totally geodesic submanifold of X. This establishes a one-to-one correspondence between totally geodesic submanifolds fixed by some finite subgroup, and *isotropy* subgroups of Γ. In this correspondence, maximal finite subgroups correspond to minimal fixed submanifolds. The geometric equivalent of Theorem 1.1 is stated in Section 4 and is proved by non-positive curvature techniques.

From the geometric point of view, we can extend this result. The quotient orbifold $M = X/\Gamma$ has a natural stratification whose strata are the sets

$$M_{[H]} = \{x \in M \mid [\Gamma_{\tilde{x}}] = [H]\},$$

where $H < \Gamma$ is a finite subgroup, $\Gamma_{\tilde{x}}$ is the stabilizer in Γ of a lift of x, and brackets represent conjugacy classes. In this setting, a conjugacy class of a maximal finite subgroup corresponds to a stratum that does not contain any other stratum in its closure. The number of strata equals the number of conjugacy classes of isotropy subgroups in Γ. In Section 6 we prove the following extension of Theorems 1.1 and 1.2, which we state here in its geometric form.

Theorem 1.3. *Let X be a global symmetric space of non-compact type. Let $M = X/\Gamma$ be an X-orbifold. Denote by $s(M)$ the number of strata in the natural orbifold stratification of M. Then*

$$s(M) \le c \cdot \mathrm{vol}(M)$$

with a constant $c = c(X)$. If the rank of X is at least 2, and $\mathrm{Isom}(X)$ has property (T), then for any sequence M_n of irreducible X-orbifolds that are pairwise non-isometric we have

$$\lim_{n \to \infty} \frac{s(M_n)}{\mathrm{vol}(M_n)} = 0.$$

Lastly, we remark on the related problems of bounding the *size* of maximal finite subgroups, and the number of conjugacy classes of torsion elements in Γ. A bound on the former, combined with a bound on the number of conjugacy classes of maximal finite subgroups, would yield a bound (perhaps not optimal) on the latter. We were unable to achieve such bounds with the tools used in this work. In fact, in $\mathrm{SL}_2(\mathbb{R})$ the Gauss-Bonnet formula shows that there are arbitrarily large finite subgroups in lattices of bounded covolume. However, if $\Gamma < G$ is an arithmetic lattice and $H < \Gamma$ is a finite subgroup, then by [3]

$$|H| \le c_1 (\log \mathrm{vol}(G/\Gamma))^{c_2},$$

where c_1, c_2 are constants that depend on G alone[*].

Unfortunately, this does not prove that the number of conjugacy classes of torsion elements in a lattice in a higher rank simple Lie group grows sublinearly with volume, a statement that we conjecture to be true.

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[*]This follows by combining the inequalities 4.3, 4.5, and 4.6 of [3]. In [3], these inequalities are proved for standard lattices in $\mathrm{SO}(n, 1)$. In a private communication, Mikhail Belolipetsky reassured me that they hold in full generality, i.e. for arithmetic lattices in a semisimple algebraic group.

2. Generalities

We review some facts regarding isometries of Riemannian globally symmetric spaces. Let X be a Riemannian globally symmetric space with non-positive sectional curvature. We will always assume that the metric is rescaled so that $-1 \leq K \leq 0$.

2.1. The isometry group $\mathrm{Isom}(X)$ is a Lie group without center, with finitely many connected components. Its identity component can be realized as the connected component of the real points of a linear algebraic group.

For $g \in \mathrm{Isom}(X)$, we define the displacement function

$$d_g : X \to \mathbb{R}^{\geq 0}, \quad d_g(x) = d(gx, x).$$

The minimal displacement set of g is defined by

$$\mathrm{Min}(g) = \{x \in X \ : \ d_g(x) = \inf d_g\}.$$

An element $g \in \mathrm{Isom}(X)$ is *semisimple* if $\mathrm{Min}(g)$ is non-empty. If a semisimple element fixes a point in X it is *elliptic*. Otherwise, it is *hyperbolic*. In either case, $\mathrm{Min}(g)$ is a connected complete totally geodesic submanifold, and is thus a globally symmetric space [16], IV, §7. If $\mathrm{Min}(g)$ is empty, g is *parabolic*. If g is a hyperbolic isometry and d_g is constant, then g is called a Clifford isometry. We note that $g \in \mathrm{Isom}(X)^\circ$ is a semisimple isometry if and only if it is a semisimple element in the linear algebraic sense. If $g \neq 1$ is a unipotent element in the linear algebraic sense, then it is parabolic (for the converse to be true, it is necessary that $\inf d_g = 0$).

If A is a set of commuting semisimple isometries then $\bigcap_{\alpha \in A} \mathrm{Min}(\alpha)$ is non-empty.

We remark that if Γ is a group acting by isometries on X, we will call elements of Γ hyperbolic, elliptic, or parabolic according to the classification of the isometry by which they act.

2.2. The displacement function d_g is convex in the sense that $t \mapsto d_g(c(t))$ is a convex function for every geodesic c. Consequentially, the sub-level sets of d_g are convex sets.

If C is a convex subset of X and $x \in X$, there is a unique point $\pi_C(x) \in C$ – the projection of x to C – which is closest to x. If C is invariant under an isometry g, then $d_g(\pi_C(x)) \leq d_g(x)$. This follows from the fact that the projection does not increase distances. Consequentially, if $x \notin C$ and $c : [0, \infty) \to X$ is a geodesic ray with $c(0) = \pi_C(x)$ and $c(t_0) = x$, then $d_g(c(t))$ is non-decreasing.

2.3. Let Γ be a countable group acting properly by isometries on X. By [23], a non-trivial Clifford isometry acts by a translation on the Euclidean factor of X, and trivially on the complementary factor. From this, it follows that the set T of Clifford isometries in Γ forms a normal, finitely generated, free abelian subgroup.

If $k = \mathrm{rk}_{\mathbb{R}} \, T$, then there is an isometric splitting $X \simeq X_1 \times X_2$ with $X_2 \simeq \mathbb{R}^k$, such that every non-trivial element of T acts trivially on X_1 and as a translation on X_2. Also, the action of T on X_2 is cocompact. We shall call this decomposition the *Clifford splitting*[†] of X with respect to Γ. Note that this need not coincide with the de Rham decomposition, because X_1 may contain a Euclidean factor.

Every element $\gamma \in \Gamma$ preserves the Clifford splitting, and can be written as (γ_1, γ_2) with $\gamma_i \in \mathrm{Isom}(X_i)$. We denote by p_i the projection of Γ to $\mathrm{Isom}(X_i)$, and set $\Gamma_i = p_i(\Gamma)$ $(i = 1, 2)$.

Proposition 2.1. Γ_1 *acts properly on* X_1.

Proof. Suppose in contradiction that $\gamma_1^{(n)} \in \Gamma_1$ is a sequence such that $\gamma_1^{(n)} \to \infty$ but $\gamma_1^{(n)} x_1$ converges for some $x_1 \in X_1$.

There are elements $\gamma_2^{(n)} \in \Gamma_2$ such that $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)}) \in \Gamma$. Fix any $x_2 \in X_2$. Since T acts trivially on X_1 and cocompactly on X_2, by multiplying the $\gamma^{(n)}$ by elements of T, we may assume $d(\gamma_2^{(n)} x_2, x_2) < K$, where K is some constant that depends on T. Passing to a subsequence we have that $\gamma_2^{(n)} x_2$ converges, and thus $\gamma^{(n)}(x_1, x_2)$ converges. This contradicts to properness of the action of Γ. $\square$

The kernel of the projection p_1 projects to a subgroup Γ_2' of Γ_2 that acts properly on X_2. Note that T injects into Γ_2', and thus Γ_2' acts cocompactly on X_2.

As we have remarked, the Euclidean factor of the Clifford decomposition may be smaller than the Euclidean factor of the de Rham decomposition. However, if X/Γ has finite volume, then the decompositions coincide [12], Theorem I. In this case, the statement of Proposition 2.1 follows from [12], Corollary F.

2.4. For a group of isometries Γ, $x \in X$, and $\varepsilon > 0$, we denote

$$\Gamma_\varepsilon(x) = \langle \gamma \in \Gamma : d_\gamma(x) < \varepsilon \rangle.$$

Recall the classical Margulis Lemma:

Theorem 2.2 ([21], Chapter 4). *Let X be a globally symmetric space of non-positive curvature. There are constants $\varepsilon > 0$ and $m \in \mathbb{N}$ (depending on X) such that for every discrete group $\Gamma < \mathrm{Isom}(X)$ and every $x \in X$, $\Gamma_\varepsilon(x)$ contains a normal nilpotent subgroup N of index at most m. Moreover, N is the intersection of $\Gamma_\varepsilon(x)$ with a connected nilpotent group in $\mathrm{Isom}(X)$.*

We will need a slightly stronger version of this theorem. Namely, we want the constants ε and m to depend only on the *dimension* of X (with our standing assumption

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[†]This splitting can be regarded as a special case of Gromov's ess^a-vol decomposition, with $a = \infty$, cf. [5], §12.

on curvature bounds). To this end, we first observe that if the metric is rescaled by some factor, then the constant ε can be rescaled by the same factor, and m is can be left unaltered. Therefore, if the constants of the Margulis Lemma are given for a space with minimal curvature exactly -1, then they will be adequate for any rescaling of this space with curvature $-1 \leq K \leq 0$. Next, recall that a complete simply connected symmetric space of non-positive curvature admits a decomposition into a product of a Euclidean factor and irreducible symmetric spaces of non-compact type. By the classification of É. Cartan, there are – up to isometry and rescaling of the metric – only finitely many symmetric spaces of non-compact type of a given dimension [16], Chapter X. Therefore, in each dimension we need only consider finitely many spaces with metric rescaled so that the minimal curvature is -1. It follows that for $d \in \mathbb{N}$, we may choose ε and m such that the statement of the theorem holds with these constants for every space of dimension $\leq d$.

We state two important consequences of the fact that the nilpotent subgroup N stipulated in Theorem 2.2 is contained in a connected nilpotent linear group. By a theorem of Lie, a connected linear solvable Lie group can be conjugated to a subgroup of upper triangular matrices. Hence, the commutator subgroup of such group consists of unipotent elements. We claim that if N contains a parabolic element, then it contains a central parabolic element. If N is abelian there is nothing to prove. Otherwise, the elements of the $k - 1$-iterated commutator, where k is the nilpotency rank of N, are central and parabolic. Similar considerations are made in [13].

Moreover, the semisimple elements in N consist of an abelian subgroup [7], §10. In particular, if H is a finite subgroup of $\mathrm{Isom}(X)$, then it has a normal abelian subgroup of index no more than m.

2.5. If γ is an element in a discrete group for which the Margulis Lemma holds, then γ^i is in the corresponding normal nilpotent subgroup, for some $i \leq m$. With the purpose of relating properties of γ^i to those of γ, we make the following definition.

Definition 2.3. A semisimple isometry $g \in \mathrm{Isom}(X)$ is *stable* if $C(g^i) = C(g)$ for every $i = 1, \ldots, m$, where $m = m(X)$ is the constant of the Margulis Lemma. Here, $C(g)$ is the centralizer of g in $\mathrm{Isom}(X)$.

We remark that dependence on the constant of the Margulis Lemma introduces ambiguity to this definition. However, since it is an auxiliary notion, we will tolerate this. Moreover, we will later fix the space X, and consider subspaces of it. The constant of the Margulis Lemma will be set once and for all, and stability of isometries of a subspace Y of X will be defined using this constant.

Since $C(g)$ acts transitively on $\mathrm{Min}(g)$, stability of g also implies that $\mathrm{Min}(g^i) = \mathrm{Min}(g)$ for $i = 1, \ldots, m$. A slightly weaker notion of stability was introduced by Gromov [5], §12; in his definition, the latter equality is the defining property.

It is easy to see that translations on a Euclidean space are stable (in fact, if $g \in$ Isom($\mathbb{R}^n$) is a translation then $C(g^i) = C(g)$ for every $i \geq 1$). This implies that in the general setting Clifford isometries are stable. Indeed, an isometry commutes with a Clifford isometry if and only if their actions on the Euclidean factor commute.

For other semisimple isometries, we have the following "stabilization" lemma, proved in Lemma 1.4 of [13] (cf. [5], §12.5 and [20], Proposition 2.5).

Lemma 2.4. *There is a constant $M = M(X)$ such that for every $g \in$ Isom(X) there exists $i \leq M$ such that g^i is stable.*

As we have observed, for a given $d \in \mathbb{N}$, it is possible to choose the constants of the Margulis Lemma so that they will be adequate for all symmetric spaces of dimension $\leq d$. The same reasoning shows that in the previous lemma, it is possible to choose a constant M that will suit all spaces of dimension $\leq d$.

Lemma 2.5. *Let $\alpha \in \Gamma$ be a stable isometry. If $\min d_\alpha < \varepsilon$ (the constant of the Margulis Lemma) then α commutes with every Clifford isometry in Γ.*

Proof. Let $X = X_1 \times X_2$ be the Clifford splitting with respect to Γ. Let $R_\lambda \colon X \to X$ be the map that acts trivially on X_1, and by the homothety $x \mapsto \lambda x$ on X_2. Note that R_λ conjugates Γ to a group of isometries of X. Moreover, if β is a Clifford translation then the displacement of $R_\lambda \beta R_\lambda^{-1}$ is λ times the displacement of β.

Let $x \in \text{Min}(\alpha)$ and let $\beta \in \Gamma$ be a Clifford isometry. By conjugating by R_λ with $\lambda > 0$ sufficiently small, we may assume that $d_\beta(x) < \varepsilon$. Let N be the normal nilpotent subgroup of $\Gamma_\varepsilon(x)$, with index $i \leq m$. Since $\alpha^i, \beta^i \in N$ are semisimple, they commute. Thus $\beta^i \in C(\alpha^i) = C(\alpha)$ by stability of α, and therefore $\alpha \in C(\beta^i) = C(\beta)$ by stability of β. $\qquad\square$

2.6. Let Γ be a group acting properly (but possibly not faithfully) on X. For $x \in X$, we denote $d_\Gamma(x) = \inf d_\gamma(x)$, where γ ranges over all elements of Γ that act non-trivially on X. The δ-thick part of X/Γ is defined as

$$(X/\Gamma)_{\geq \delta} = \{x \in X/\Gamma \ : \ d_\Gamma(\tilde{x}) \geq \delta\}$$

where $\tilde{x}$ is any lift of x to X. We denote $\text{vol}_{\geq \delta}(X/\Gamma) = \text{vol}((X/\Gamma)_{\geq \delta})$. If X is a point, we set $\text{vol}(X) = 1$ and $\text{vol}_{\geq \delta}(X/\Gamma) = 1$ for every $\delta > 0$.

In the presence of a Euclidean factor, one cannot hope to relate any property of Γ to the volume of X/Γ; indeed, by conjugating Γ by a homothety of the Euclidean factor, the volume of the quotient can be made arbitrarily small. The following definition aims at overcoming this difficulty.

Definition 2.6. Let $X \simeq X_1 \times X_2$ ($X_2 \simeq \mathbb{R}^k$) be the Clifford decomposition with respect to Γ. Let Γ_1 be the projection of Γ to Isom(X_1). We define $\text{vol}^{\text{nc}}(X/\Gamma) = \text{vol}(X_1/\Gamma_1)$ (for "no Clifford volume") and $\text{vol}^{\text{nc}}_{\geq \delta}(X/\Gamma) = \text{vol}_{\geq \delta}(X_1/\Gamma_1)$.

The following proposition lets us estimate the volume of the thick part of a space by a discrete set.

Proposition 2.7. *Let Y be a totally geodesic submanifold of X, and let Γ be a group acting properly on Y. There exists a constant $c = c(X, \delta)$ such that the following holds. If $\mathcal{N}$ is a δ-discrete set in $(Y/\Gamma)_{\geq \delta}$ (i.e. every two points in $\mathcal{N}$ are at distance $\geq \delta$) then*

$$|\mathcal{N}| \leq c \cdot \mathrm{vol}_{\geq \frac{\delta}{2}}(Y/\Gamma).$$

Furthermore, if $\mathcal{N}$ is a maximal δ-discrete set then

$$|\mathcal{N}| \geq c^{-1} \cdot \mathrm{vol}_{\geq \delta}(Y/\Gamma).$$

The proof is standard and uses two ideas. First, that balls of radius $\delta/4$ around points of $\mathcal{N}$ are disjoint in $(Y/\Gamma)_{\geq \delta/2}$, and are injective images of balls in Y. Second, that the volume of injected balls can be bounded from below by the volume of Euclidean balls with the same radius. Similarly, if $\mathcal{N}$ is maximal then the (metric) balls of radius 2δ around points of $\mathcal{N}$ cover $(Y/\Gamma)_{\geq \delta}$, and the volume of these balls can be bounded from above by the Bishop–Gromov theorem.

3. Fixed submanifolds

In this section, X is a Riemannian globally symmetric space of non-positive sectional curvature, with $-1 \leq K \leq 0$. The constants ε and m are the constant of the Margulis Lemma.

3.1. Submanifolds fixed by maximal finite subgroups.
By a theorem of Kazhdan and Margulis [17], a finite volume locally symmetric manifold of non-compact type has a point where the injectivity radius is greater than some constant that depends only on the universal cover. We develop a variation on this result.

Let $M = X/\Gamma$ be an orbifold, and let $H < \Gamma$ be a finite group. If we restrict ourselves to points on $\mathrm{Fix}(H)$, it does not make any sense to seek a point with $d_\Gamma > 0$. The best one can hope for is that the only elements that translate by less than some constant are those of H. Of course, even this cannot be expected if there happens to be a hyperbolic element that acts by a small translation on $\mathrm{Fix}(H)$. Assuming this does not happen, we have the following.

Proposition 3.1. *Assume Γ is a group acting properly by isometries on X, such that $\mathrm{vol}_{\geq \mu}(X/\Gamma) < \infty$ for every $\mu > 0$. Let $\delta < \varepsilon$. Then for every maximal finite subgroup $H < \Gamma$, either there is a hyperbolic element $\gamma \in \Gamma$ with $\mathrm{Min}(\gamma) \supseteq \mathrm{Fix}(H)$ and $\min d_\gamma \leq \delta$, or there exists a point $x \in \mathrm{Fix}(H)$ such that $\Gamma_{\delta/2m}(x) = H$.*

Several lemmata will proceed the proof of this proposition. In what follows we set $S = \text{Fix}(H)$ and denote by Γ_S the stabilizer of S in Γ.

Lemma 3.2. S/Γ_S *is a manifold and the map* $S/\Gamma_S \to X/\Gamma$ *is an embedding.*

Proof. First, we show that every elliptic element in Γ_S acts trivially on S, proving that S/Γ_S is a manifold. If $\gamma \in \Gamma_S$ is elliptic then it fixes some point $x \in S$. The stabilizer of x is a finite group containing H and γ, hence by maximality $\gamma \in H$. Therefore, γ acts trivially on S.

Next, we prove that the map $S/\Gamma_S \to X/\Gamma$ is an embedding. To prove that it is injective, let $x \in S$ and assume $\gamma x \in S$ for $\gamma \in \Gamma$. On one hand, $H \leq \Gamma_{\gamma x}$ because $\gamma x \in S$. On the other hand, $\gamma H \gamma^{-1} \leq \gamma \Gamma_x \gamma^{-1} = \Gamma_{\gamma x}$. Since H is maximal, there is equality in both cases. Thus γ normalizes H and it keeps $S = \text{Fix}(H)$ invariant, hence $\gamma \in \Gamma_S$.

To complete the proof, let x_n be a sequence in S and assume that $\gamma_n x_n$ converges to $y \in S$ for some sequence $\gamma_n \in \Gamma$. We show that $\gamma_n \in \Gamma_S$ for sufficiently large n.

Fix $\delta > 0$ such that $\Gamma_\delta(y) = H$ (it exists because $y \in S$, and Γ acts properly). For sufficiently large n, $d(\gamma_n x_n, y) < \delta/2$, hence $\gamma_n H \gamma_n^{-1} \subseteq \Gamma_\delta(y) = H$, and maximality implies equality. Then γ_n normalizes H and hence stabilizes S. $\square$

We will need the following elementary lemma from §12 of [5]:

Lemma 3.3. *For every* $\delta > 0$ *and* $k \in \mathbb{N}$ *there exists* $0 < \mu < \delta$, *such that following holds. For every* $x \in X$, *if* $\Gamma_\delta(x)$ *is finite of size at most* k, *then there exists a point* $y \in X$ *such that* $d(x, y) < \delta/4$ *and* $d_\Gamma(y) > \mu$.

The constant μ depends on δ, k, and the space X. But here again, it can be chosen to accommodate for all spaces of dimension $\leq \dim(X)$.

For a point $x \in X$, let $\rho(x) = \inf_\gamma d_\gamma(x)$ where γ ranges over all non-elliptic elements in Γ. Note that if $x \in S$ then $\rho(x)$ is at most $d_{\Gamma_S}(x)$ (w.r.t. the action of Γ_S on S).

To study the behavior of ρ at points of S, it is useful to define the notion of quasi-thickness which was introduced and studied in [20].

Definition 3.4. Let $M = X/\Gamma$ be an orbifold. For $\delta > 0$ and $k \in \mathbb{N}$, the (δ, k)-*quasi-thick part* of M is defined as

$$M_{\geq \delta, k} = \{x \in X : |\Gamma_\delta(x)| \leq k\}/\Gamma.$$

We will need the following simple fact.

Claim. *If* $\text{vol}_{\geq \mu}(M) < \infty$ *for every* $\mu > 0$, *then* $M_{\geq \delta, k}$ *is compact for every* $\delta > 0$ *and* $k \in \mathbb{N}$.

Proof. Let $\delta > 0$ and $k \in \mathbb{N}$. Let $x \in M_{\geq \delta, k}$ and fix a lift $\tilde{x} \in X$. By Lemma 3.3, there is some $\mu < \delta$ such that the ball of radius $\delta/2$ around $\tilde{x}$ contains a ball of radius $\mu/4$ that injects to the $\frac{\mu}{2}$-thick part of M (the displacement function is 2-Lipschitz).

If $M_{\geq \delta, k}$ is not compact, then there is a sequence of points x_i in $M_{\geq \delta, k}$ such that $d(x_i, x_j) > \delta$ for $i \neq j$. Then the balls of radius $\delta/2$ around these points are disjoint, and the intersection of each of these balls with $M_{\geq \frac{\mu}{2}}$ contains an injected ball of radius $\mu/4$. Since these disjoint injected balls all have the same volume, this contradicts the assumption that $M_{\geq \frac{\mu}{2}}$ has finite volume. $\square$

Lemma 3.5. *For every $a > 0$, the image of the set $C = \{x \in S \mid \rho(x) \geq a\}$ in X/Γ is compact.*

Proof. We may assume that a is smaller than ε of the Margulis Lemma. Let $x \in X/\Gamma$ be a point with preimage $\tilde{x} \in C$. Let N be the normal nilpotent subgroup of $\Gamma_{a/2m}(\tilde{x})$. Since its index is bounded by m, it can be generated by words of length at most $2m$ in the generators of $\Gamma_{a/2m}(\tilde{x})$[‡]. Hence, N is generated by elements γ with $d_\gamma(x) < a$. Since $\rho(\tilde{x}) \geq a$, these generators must be elliptic, and therefore they commute. It follows that N is finite, hence so is $\Gamma_{a/2m}(\tilde{x})$. Therefore, $\Gamma_{a/2m}(\tilde{x}) = H$ by the maximality of the latter. It follows that x is in the $(\frac{a}{2m}, |H|)$-quasi-thick part of M, which is compact.

Since C is closed and Γ_S-invariant, its image in S/Γ_S is closed. Hence, by Lemma 3.2 its image in X/Γ is closed, and the claim follows. $\square$

Proof of Proposition 3.1. Suppose that there are no hyperbolic elements $\gamma \in \Gamma$ with $\min d_\gamma \leq \delta$ and $\text{Min}(\gamma) \supseteq S$.

It follows from Lemma 3.5 that there exists a point $y \in S$ where ρ attains its maximum on S. Let us denote by $\#\rho(x)$ the number of elements $\gamma \in \Gamma$ for which $d_\gamma(x) = \rho(x)$. We can assume y is chosen such that $\#\rho(y)$ is minimal (among points in S where ρ is maximal).

We claim that $\rho(y) \geq \delta$. Let us first show that this will prove the proposition. Indeed, we have observed in the proof of Lemma 3.5 that if $\rho(y) \geq \delta$ then $\Gamma_{\delta/2m}(y)$ is finite, hence equal to H by maximality.

Now suppose contrarily that $\rho(y) < \delta$. Let $\mu = \rho(y)$, and let Σ be the set of non-elliptic elements γ with $d_\gamma(y) = \mu$. Let $\Delta = \langle H, \Sigma \rangle$ and let N be the normal nilpotent subgroup of Δ.

There are two cases. First, let us assume that Σ contains a parabolic element. Then N contains parabolic elements, and therefore contains a central parabolic element $\gamma_1 \in N$. Since N has finite index in Δ, γ_1 has finitely many conjugates, say, $\gamma_1, \ldots, \gamma_k$. Define $D(x) = \sum_i d_{\gamma_i}(x)$. Then for every $a > 0$, $C = D^{-1}([0, a])$ is closed, convex, and Δ-invariant. For sufficiently small a, $y \notin C$. Let $z \in C$ be the projection of y onto C, and let $c \colon [0, \infty) \to X$ be a geodesic ray with $c(0) = z$ and

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[‡]For a proof of this, see the proof of Lemma 2.2 in [20].

$c(t_0) = y$. Every $\gamma \in \Delta$ keeps C invariant, hence d_γ is non-decreasing along c. Also, since H fixes y and keeps C invariant, it also fixes z, and hence fixes c pointwise. We claim that for some $\gamma \in \Sigma$, d_γ is strictly increasing along c. If $d_\gamma \circ c$ is not strictly increasing, then it is locally constant at some point, and since $d_\gamma \circ c$ is analytic, this implies that d_γ is constant along c. Since this is true for the generators of Δ, $d_\gamma \circ c$ is bounded and thus constant for every $\gamma \in \Delta$. This would imply that D is constant along c and thus $y \in C$, which is a contradiction.

We have shown that for $t > t_0$, $d_\gamma(c(t)) \geq \mu$ for all $\gamma \in \Sigma$, and $d_\gamma(c(t)) > \mu$ for at least one $\gamma \in \Sigma$. We may choose $t_1 > t_0$ sufficiently small, such that for every $\gamma \in \Gamma$, $d_\gamma(c(t_1)) > \mu$ whenever $d_\gamma(c(t_0)) > \mu$.

Therefore, $\rho(c(t_1)) \geq \mu$, and if $\rho(c(t_1)) = \mu$ then $\#\rho(c(t_1)) \leq \#\rho(y) - 1$. This is a contradiction to the choice of y.

Now, suppose Σ does not contain a parabolic element, hence it consists of hyperbolic elements. Let i be the index of N in Δ. Then the elements γ^i, $\gamma \in \Sigma$, are hyperbolic and thus commute. It follows that $C = \bigcap_{\gamma \in \Delta} \mathrm{Min}(\gamma^i)$ is non-empty, and it is closed, convex and Δ-invariant. Let $z \in C$ be the projection of y to C. Then, as before, $z \in S$. By our assumption, there are no hyperbolic elements γ with $\mathrm{Min}(\gamma) \supseteq S$ and $\min d_\gamma \leq \delta$. Therefore, $C \cap S$ is properly contained in S. As in the previous case, we take a geodesic ray in S which starts at z and is perpendicular to C at z. Along this geodesic, d_γ is non-decreasing for every $\gamma \in \Delta$, and is strictly increasing for at least one $\gamma \in \Sigma$. This leads to contradiction, as in the previous case. $\qquad\square$

3.2. Stable singular submanifolds.

We now formulate a proposition which is key to the inductive reasoning needed for the proof of the main theorem. This proposition is an analogy to Gromov's Theorem 12.11 in [5], and parts of its proof follow along the lines of his proof.

Let $A \subset \Gamma$ be a set of stable semisimple isometries. If $Y = \bigcap_{\alpha \in A} \mathrm{Min}(\alpha)$ is non-empty then it is a connected complete totally geodesic submanifold of X. If, furthermore, $\min d_\alpha < \varepsilon$ for every $\alpha \in A$, we call Y a *stable singular submanifold*. Two submanifolds Y_1, Y_2 are called *non-conjugate* if $Y_2 \neq \gamma Y_1$ for all $1 \neq \gamma \in \Gamma$.

Note that if A defines a stable singular submanifold Y then the elements of A commute. Indeed, if $\alpha_1, \alpha_2 \in A$ then $\alpha_1, \alpha_2 \in \Gamma_\varepsilon(y)$ for some $y \in Y$. If N is the normal nilpotent subgroup of $\Gamma_\varepsilon(y)$, then $\alpha_1^i, \alpha_2^i \in N$ for some $i \leq m$. It follows that α_1^i and α_2^i commute, hence by stability so do α_1 and α_2.

For a submanifold Y we denote by Γ_Y the stabilizer of Y in Γ.

Proposition 3.6. *Let W be a connected complete totally geodesic submanifold of X, and let Γ be a group acting properly by isometries on W. Let Σ be a set of non-conjugate stable singular proper submanifolds in W. For every $\varepsilon_1 > 0$ there are constants $\varepsilon_2 = \varepsilon_2(\varepsilon_1, X)$ and $c = c(\varepsilon_1, X)$ such that*

$$\sum_{Y \in \Sigma} \mathrm{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Y/\Gamma_Y) \leq c \cdot \mathrm{vol}^{\mathrm{nc}}_{\geq \varepsilon_2}(W/\Gamma). \tag{1}$$

Proof. The proof is by induction on the dimension of W. Clearly, the claim is true for submanifolds of dimension 0 because there are no proper submanifolds.

For $Y \in \Sigma$ we denote $\Sigma_Y = \{Z \in \Sigma \ : \ Z \subsetneq Y\}$. Let $Z \in \Sigma_Y$ and denote by A_Y, A_Z the (maximal) set of stable elements such that $Y = \bigcap_{\alpha \in A_Y} \operatorname{Min}(\alpha)$ and $Z = \bigcap_{\alpha \in A_Z} \operatorname{Min}(\alpha)$. Then $A_Y \subsetneq A_Z$. Fix some $z \in Z$. The elements of A_Z commute, from which we deduce that every element of A_Z keeps Y invariant. Thus, Z can be considered a stable submanifold of Y with respect to Γ_Y.

By induction, we may assume that there are constants ε_2' and c' such that (1) holds for submanifolds with dimension lower than $\dim(W)$. Hence, considering the action of Γ_Y on Y,

$$\sum_{Z \in \Sigma_Y} \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Z / \Gamma_{Z,Y}) \leq c' \cdot \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_2'}(Y / \Gamma_Y),$$

where $\Gamma_{Z,Y} = \Gamma_Y \cap \Gamma_Z$.

Note that from the fact that $\Gamma_{Z,Y} < \Gamma_Z$ it is not hard to deduce that

$$\operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Z / \Gamma_Z) \leq \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Z / \Gamma_{Z,Y}).$$

The important point is that every element of A_Y keeps Z invariant, so it commutes with every Clifford isometry in Γ_Z (by Lemma 2.5). Therefore, every Clifford isometry in Γ_Z keeps Y invariant. It follows that Z has the same Clifford splitting w.r.t Γ_Z or $\Gamma_{Z,Y}$.

Let $\Sigma' \subseteq \Sigma$ be the set of stable submanifolds which are maximal in Σ with respect to inclusion. Then we have

$$\sum_{Y \in \Sigma} \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Y / \Gamma_Y) \leq \sum_{Z \in \Sigma'} \sum_{Z \supseteq Y \in \Sigma} \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Y / \Gamma_Y)$$

$$\leq \sum_{Z \in \Sigma'} \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_1}(Z / \Gamma_Z) + \sum_{Z \in \Sigma'} c' \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_2'}(Z / \Gamma_Z) \leq (c' + 1) \sum_{Z \in \Sigma'} \operatorname{vol}^{\mathrm{nc}}_{\geq \varepsilon_2'}(Z / \Gamma_Z)$$

(here we assume $\varepsilon_2' < \varepsilon_1$). Our problem is thus reduced to the case of maximal stable submanifolds.

We will henceforth assume that Σ is a set of maximal stable proper submanifolds, and prove that there exist ε_2 and c for which (1) holds. We may assume that $\varepsilon_1 < \varepsilon$.

Let $W \simeq W_1 \times \mathbb{R}^d$ be the Clifford splitting of W with respect to Γ. Suppose Y is a maximal stable submanifold, and A_Y is the set of stable elements α with $\min d_\alpha < \varepsilon$ such that $\bigcap_{\alpha \in A_Y} \operatorname{Min}(\alpha) = Y$. By maximality, $\operatorname{Min} \alpha = Y$ for every $\alpha \in A_Y$.

By Lemma 2.5, every element in A_Y commutes with the Clifford transformations in Γ. Hence, every Clifford isometry of W keeps Y invariant, and therefore restricts to a Clifford isometry of Y. Therefore, the Clifford splitting of Y (with respect to Γ_Y) can be made compatible with that of W; it can be written as $Y = Y_1 \times \mathbb{R}^k \times \mathbb{R}^d$, with $Y_1 \times \mathbb{R}^k \subset W_1$. Since we are only interested in the projections of isometries to W_1 and to Y_1, we may simply assume $d = 0$.

Every $\beta \in \Gamma_Y$ can be written in the form $\beta = (\beta_1, \beta_2)$ with $\beta_1 \in \text{Isom}(Y_1)$ and $\beta_2 \in \text{Isom}(\mathbb{R}^k)$. Denote by p_1 the projection of Γ_Y to $\text{Isom}(Y_1)$. Recall that $p_1(\Gamma_Y)$ acts properly on Y_1.

We will now describe a procedure that associates to a point in the ε_1-thick part of $Y_1/p_1(\Gamma_Y)$ a point in the ε_2-thick part of W/Γ (ε_2 will be defined in this process).

Let $y' \in Y_1$ be a point with $d_{p_1(\Gamma_Y)}(y') \geq \varepsilon_1$. Let $y = (y', 0) \in Y_1 \times \mathbb{R}^k = Y$. Let $c: [0, \infty) \to W$ be a geodesic ray with $c(0) = y$ and $c'(0) \perp Y$. For every $\alpha \in A_Y$, α keeps Y invariant, and $d_\alpha \circ c$ has a unique minimum at 0. Hence, by convexity $d_\alpha \circ c$ is monotone increasing and unbounded. Fix $z = c(t_0)$, $t_0 > 0$ such that $d_\alpha(z) \geq \varepsilon/2$ for all $\alpha \in A_Y$, and $d_\alpha(z) = \varepsilon/2$ for some $\alpha \in A_Y$. We henceforth fix α to be the latter.

We proceed to find a constant ε_4 for which $\Gamma_{\varepsilon_4}(z)$ is finite and bounded. Fix $\varepsilon_3 = \varepsilon_1/2mM$. Suppose that $d_\beta(z) < \varepsilon_3$ for some $\beta \in \Gamma$. By Lemma 2.4, β^j is m-stable for some $j < M$. Since $d_{\beta^j}(y) < \varepsilon$, β^j and α commute. It follows that β^j keeps $Y = \text{Min}(\alpha)$ invariant. Since y is the projection of z to Y, $d_{\beta^j}(y) \leq d_{\beta^j}(z) \leq j d_\beta(z) < M\varepsilon_3 < \varepsilon_1$.

Let us turn our attention to the projections of β^j. On one hand, $d_{\beta_1^j}(y') < \varepsilon_1$, so by our hypotheses β_1^j must fix Y_1 pointwise. Therefore, β^j is in the kernel of the projection $\Gamma \to \Gamma_2$. Recall that the action of this kernel on Y_2 is proper and cocompact. Hence, by the Bieberbach theorem ([21], §4), β_2^{jk} is either trivial or a translation for some $k \leq m$. Therefore, β^{jk} is either trivial or acts as a Clifford translation on Y. The latter is impossible; indeed, it would imply that $\text{Min}(\beta^{jk}) \supseteq Y$, and by maximality of Y, $\text{Min}(\beta^{jk}) = Y$ (recall that we are assuming there are no Clifford translations on W). But this would imply that $\beta^{jk} \in A_Y$ and contradicts the choice of z, because $d_{\beta^{jk}}(z) < mM\varepsilon_3 < \varepsilon/2$. Thus, the order of β is bounded by mM.

Now let $\varepsilon_4 = \varepsilon_3/2m$. Let N be the normal nilpotent subgroup of $\Gamma_{\varepsilon_4}(z)$. Then N is generated by elements that translate z by less than $2m\varepsilon_4 = \varepsilon_3$. It follows that N is generated by elliptic elements whose order is bounded by mM. Therefore, N is abelian, and its exponent is bounded by $(mM)!$. Since N is an abelian subgroup of $SO(n)$, it can be generated by a set of n elements, hence $|\Gamma_{\varepsilon_4}(z)| \leq m(mM!)^n$.

Now, by Lemma 3.3 there is a positive constant $\varepsilon_2 < \varepsilon_4$ such that if $|\Gamma_{\varepsilon_4}(z)| \leq m(mM!)^n$ then there exists a point $w \in W$ with $d(z, w) < \varepsilon_4/4$ such that $d_\Gamma(w) > \varepsilon_2$. The point w is the point associated to y'. Note that

$$d_\alpha(w) \leq d_\alpha(z) + 2 \cdot \frac{\varepsilon_4}{4} < \varepsilon. \tag{2}$$

Now let $\Sigma = \{Y^1, \ldots, Y^s\}$ be a set of maximal stable proper submanifolds of W. For each i, choose a maximal set of points $\{y'_{i,1}, \ldots, y'_{i,t_i}\}$ in Y^i that projects to an ε_1-discrete set in the ε_1-thick part of $Y_1^i/p_1(\Gamma_{Y^i})$. To each point $y'_{i,j}$ we attach

a point $w_{i,j} \in W$ by the procedure described above. We have seen that these points project to the ε_2-thick part of W/Γ. Let us denote the projection of $w_{i,j}$ by $\bar{w}_{i,j}$. The set of $\bar{w}_{i,j}$'s is not necessarily ε_2-discrete. However, we claim that for every point $\bar{w}_{i,j}$, there are at most $2^n - 1$ points $\bar{w}_{k,l}$ for which $d(\bar{w}_{i,j}, \bar{w}_{k,l}) < \varepsilon_2$. If this is true, we can find in $\{\bar{w}_{i,j}\}$ a subset of size $\lfloor |\{\bar{w}_{i,j}\}|/2^n \rfloor$ which is ε_2-discrete. Using the volume estimates of Proposition 2.7, this will complete the proof. We turn to prove this final claim.

First, we show that if that if $d(\bar{w}_{i,j}, \bar{w}_{i,k}) < 2\varepsilon_2$ then $j = k$. Indeed, if this inequality holds, we may assume that the points are chosen such that $d(w_{i,j}, w_{i,k}) < 2\varepsilon_2$. By the construction, it follows that

$$ d(y'_{i,1}, y'_{i,2}) \leq d(y_{i,1}, y_{i,2}) < 2\varepsilon_2 + 2 \cdot \frac{\varepsilon_4}{4} < \varepsilon_1, $$

which is only possible if $j = k$.

Next, suppose there is a point $\bar{w}_{i,j}$ for which there are 2^n other points of distance $< \varepsilon_2$. Then each two such points are at distance $< 2\varepsilon_2$. By the preceding paragraph, the points correspond to different submanifolds, and we may thus renumber the indices so that $d(\bar{w}_{1,1}, \bar{w}_{i,1}) < \varepsilon_2$ for $i = 2, \ldots, 2^n + 1$. Also, we may assume that $d(w_{1,1}, w_{i,1}) < \varepsilon_2$.

By our construction and (2), there are stable elements α_i ($1 \leq i \leq 2^n + 1$) such that $d_{\alpha_i}(w_{1,1}) < \varepsilon$. Hence, by their stability, these elements commute, and the submanifolds $Y_i = \text{Min}(\alpha_i)$ have a non-empty intersection. Moreover, at the point of intersection the submanifolds intersect orthogonally (see proof in [5] for additional details). There is a one-to-one correspondence between these subspaces and their tangent space at the point of intersection, and the tangent subspaces are orthogonal, as well. Hence, there is an orthogonal basis of the tangent space, such that the tangent space to each of the Y_i's is spanned by vectors of this basis. Hence, there are at most 2^n such spaces (= the number of subsets of the basis). This contradicts our assumption that $Y_1, \ldots, Y_{n+1}$ are distinct subspaces. $\square$

4. Proof of Theorems 1.1 and 1.2

First, we make a reduction of Theorem 1.1 to the case where G has trivial center. Suppose the theorem holds for groups without center, and let G be a connected semisimple Lie group with finite center Z. Denote by π the projection of G to the adjoint group $G' = G/Z$.

Let $\Gamma < G$ be a lattice. Since Z is finite, every maximal finite subgroup in Γ contains $\Gamma \cap Z$. It is easy to check that this implies that $f(\Gamma) = f(\pi(\Gamma))$. Therefore,

$$ f(\Gamma) = f(\pi(\Gamma)) \leq c \cdot \text{vol}(G'/\pi(\Gamma)) \leq c \cdot \text{vol}(G/\Gamma), $$

where $c = c(G')$.

Assume now that G has trivial center. Let K be a maximal compact subgroup. The associated symmetric space $X = K \backslash G$ is of non-compact type, and we may assume that the metric is normalized to meet the curvature bounds. There exists a constant $D = D(G)$ such that $\mathrm{vol}(X/\Gamma) = D \cdot \mathrm{vol}(G/\Gamma)$. Since X does not have Euclidean factors, $\mathrm{vol}^{\mathrm{nc}}_{\geq \mu}(X/\Gamma) \leq \mathrm{vol}(X/\Gamma)$. Therefore, Theorem 1.1 follows directly from the following.

Theorem 4.1. *Let X be a global symmetric space of non-positive curvature. For a group Γ acting properly by isometries on X, denote by $f(\Gamma)$ the number of conjugacy classes of maximal finite subgroups of Γ. Then there exist constants $\mu = \mu(X)$ and $c = c(X)$ such that $f(\Gamma) \leq c \cdot \mathrm{vol}^{\mathrm{nc}}_{\geq \mu}(X/\Gamma)$.*

Proof. Throughout the proof, ε and m are the constants of the Margulis Lemma, and M is the constant of Lemma 2.4. We choose all constants so that these lemmas hold for any symmetric space of dimension $\leq \dim(X)$.

We will prove by induction that for every $d \leq \dim(X)$ there are constants μ_d, c_d such that if Y is a connected complete totally geodesic submanifold of X of dimension $\leq d$, and Γ is a group acting properly on Y then $f(\Gamma) \leq c_d \cdot \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_d}(Y/\Gamma)$.

If $\dim(Y) = 0$, and Γ acts properly on Y, then Γ is finite, and therefore $f(\Gamma) = 1$. We may fix $c_0 = 1$ and μ_0 to be any positive number.

Suppose now that μ_i exist for $i = 0, \ldots, d-1$, and let us prove that μ_d exists. Let Y be a connected complete totally geodesic submanifold of X of dimension d, and let Γ be a group of isometries acting properly on Y.

Step 1. Let us first assume that Γ contains Clifford isometries. Let $T \leq \Gamma$ be the subgroup of Clifford isometries, and let $Y = Y_1 \times Y_2$, $Y_2 \simeq \mathbb{R}^k$ ($k \geq 1$) be the Clifford splitting with respect to Γ. Denote by Γ_i the projection of Γ to $\mathrm{Isom}(Y_i)$, $i = 1, 2$. Recall that Γ_1 acts properly on Y_1, and that the subgroup T is contained in the kernel of the projection $\Gamma \to \Gamma_1$ and acts cocompactly on Y_2.

Our goal is to show that a maximal finite subgroup of Γ has a bounded number of possible projections to Γ_1 and Γ_2, up to conjugation in Γ. Then we will show that this yields the appropriate bound on the number of conjugacy classes of maximal finite subgroups in Γ.

For a finite subgroup H_1 in Γ_1, define $\Gamma(H_1) = p_1^{-1}(H_1)$ and $\Gamma_2(H_1) = p_2(\Gamma(H_1))$. Since the restriction of p_2 to $\Gamma(H_1)$ has finite kernel, $\Gamma_2(H_1)$ acts properly on Y_2. Moreover, T is contained in $\Gamma(H_1)$ and injects into $\Gamma_2(H_1)$, and therefore $\Gamma_2(H_1)$ acts cocompactly on Y_2.

We claim that $f(\Gamma_2(H_1))$ is bounded by a constant D that depends only on $\dim(X)$. This follows immediately from the fact that, up to isometry, there are finitely many crystallographic groups of every dimension [4].

Now let $\{H_1^{(1)}, \ldots, H_r^{(1)}\}$ be a maximal collection of non-conjugate maximal finite subgroups in Γ_1. For every $1 \leq i \leq r$, let $\{H_{i,1}^{(2)}, \ldots, H_{i,s_i}^{(2)}\}$ be a maximal

collection of non-conjugate maximal finite subgroups of $\Gamma_2(H_i^{(1)})$. By the induction hypothesis, $r = f(\Gamma_1) \leq \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_{d-1}}(Y_1/\Gamma_1) = \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_{d-1}}(Y/\Gamma)$ (the latter equation being the definition of $\mathrm{vol}^{\mathrm{nc}}$). Also, $s_i \leq D$ for every i.

Let $H < \Gamma$ be a maximal finite subgroup. By conjugating in Γ we may assume that the projection of H to Γ_1 is contained in $H_i^{(1)}$ for some i. Furthermore, by conjugating by an element of $\Gamma(H_i^{(1)})$ we may assume that H projects into $H_{i,j}^{(2)}$ for some $1 \leq j \leq s_i$. Note that this conjugation does not change the projection of H to Γ_1. Our claim will be proved if we show that there is a single maximal finite subgroup with such projections.

To this end, let $y_1 \in Y_1$, $y_2 \in Y_2$ be points fixed by $H_i^{(1)}$, $H_{i,j}^{(2)}$, respectively. If K_1, K_2 are two maximal finite subgroups of Γ which both project into $H_i^{(1)}$, $H_{i,j}^{(2)}$, then they both fix (y_1, y_2). But by maximality, they are both equal to the stabilizer of this point in Γ.

We conclude that

$$f(\Gamma) \leq D \cdot \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_{d-1}}(Y/\Gamma). \tag{3}$$

Step 2. We henceforth assume Γ does not contain Clifford isometries.

Let Σ be a maximal set of non-conjugate stable singular proper submanifolds of Y. For $Z \in \Sigma$, denote by Γ_Z the stabilizer of Z in Γ, by Γ_Z^1 the fixator (pointwise stabilizer) of Z in Γ, and by f_Z the number of non-conjugate maximal finite subgroups of Γ that are contained in Γ_Z.

Observe that if H is a maximal finite subgroup of Γ that keeps Z invariant, then it fixes some point $z \in Z$. By maximality, $H = \Gamma_z$, and therefore H contains Γ_Z^1. Therefore, maximal finite subgroups that are contained in Γ_Z are in a natural one-to-one correspondence with maximal finite subgroups of Γ_Z/Γ_Z^1. Moreover, this correspondence respects conjugation by elements of Γ_Z.

It follows that $f_Z \leq f(\Gamma_Z/\Gamma_Z^1) \leq \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_{d-1}}(Z/\Gamma_Z)$, by the induction hypothesis. By Proposition 3.6, we have

$$\sum_{Z \in \Sigma} f_Z \leq \sum_{Z \in \Sigma} \mathrm{vol}^{\mathrm{nc}}_{\geq \mu_{d-1}}(Z/\Gamma_Z) \leq E_1 \cdot \mathrm{vol}^{\mathrm{nc}}_{\geq \delta_1}(Z/\Gamma_Z), \tag{4}$$

with $\delta_1 = \delta_1(\mu_{d-1})$, $E_1 = E_1(\mu_{d-1})$.

It remains to bound the number of non-conjugate maximal finite subgroups that do not stabilize any stable singular proper submanifold.

Step 3. Let $\mathcal{H}$ be a set of non-conjugate maximal finite subgroup of Γ that do not stabilize any stable singular proper submanifold. Let $H \in \mathcal{H}$, and let $y \in \mathrm{Fix}(H)$. By maximality, $H = \Gamma_y$.

We claim that there is no hyperbolic element $\alpha \in \Gamma$ with $\mathrm{Min}(\alpha) \supseteq \mathrm{Fix}(H)$ and $d_\alpha(y) < \varepsilon/M$ (recall, M is the constant of Lemma 2.4). Suppose α is such an element. Then for some $j \leq M$, α^j is stable, and $d_{\alpha^j}(y) < \varepsilon$. The elements in $\{\gamma \alpha^j \gamma^{-1} \mid \gamma \in H\}$ are all stable and contained in $\Gamma_\varepsilon(y)$, hence they commute. It

follows that $Z = \bigcap_{\gamma \in H} \text{Min}(\gamma \alpha^j \gamma^{-1})$ is non-empty, and it is an H-invariant stable singular submanifold. By our assumption, there are no Clifford isometries in Γ, so Z is a proper submanifold. But this contradicts our assumption on H.

Now, by Proposition 3.1 we may replace y with a point $y \in \text{Fix}(H)$ such that $\Gamma_{\varepsilon/2mM}(y) = H$.

Following the same reasoning, every element $\alpha \in H$ has order bounded by M; otherwise, $\bigcap_{\gamma \in H} \text{Min}(\gamma \alpha^j \gamma^{-1})$ would be a non-empty H-invariant stable singular proper submanifold, for some $j \leq M$. By identifying H with a subgroup of $\text{SO}(n)$, we see that its normal abelian subgroup is generated by at most n elements. Since the index of this subgroup is a most m, we deduce that $|H| \leq M^n m$.

By Lemma 3.3 there is a positive constant $\delta_2 < \varepsilon/8$ such that if

$$|\Gamma_{\varepsilon/2mM}(y)| \leq M^n m$$

then there is a point $z \in Y$ with $d(x, y) < \varepsilon/8$ such that $d_\Gamma(y) > \delta_2$. We denote this point z_H.

We claim that the set of points $\{z_H : H \in \mathcal{H}\}$ projects to a δ_2-discrete set in the δ_2-thick part of Y/Γ. By the choice of z_H, the projection is indeed in the δ_2-thick part. It remains to show discreteness. To this end, let $H_1, H_2 \in \mathcal{H}$, let y_1, y_2 be the corresponding fixed points, and let z_1, z_2 be the points obtained for each group. Suppose, in contradiction, that the projections of z_1, z_2 are not δ_2-separated. By replacing H_2 by a conjugate, we may assume $d(z_1, z_2) < \delta_2$. This implies that

$$d(y_1, y_2) \leq \delta_2 + 2 \cdot \varepsilon/8 < \varepsilon/2.$$

Consequentially, $H_1 = \Gamma_{y_1} \leq \Gamma_\varepsilon(y_2) = H_2$, hence $H_1 = H_2$, a contradiction. By the volume estimate of 2.7,

$$|\mathcal{H}| \leq E_2 \cdot \text{vol}_{\geq \delta_2/2}(Y/\Gamma) = E_2 \cdot \text{vol}^{nc}_{\geq \delta_2/2}(Y/\Gamma), \tag{5}$$

where E_2 is a constant depending on X.

Step 4. Take $c_d = \max(D, E_1 + E_2, c_{d-1})$ and $\mu_d = \min(\delta_1, \delta_2/2, \mu_{d-1})$. It follows easily from (3), (4), and (5) that

$$f(\Gamma) \leq c_d \cdot \text{vol}^{nc}_{\geq \mu_d}(Y/\Gamma). \qquad \square$$

Proof of Theorem 1.2. Assume G has $\mathbb{R}$-rank at least 2, and Kazhdan's property (T). In this case, we have the following result

Theorem 4.2. *Suppsose that Γ_n is a sequence of irreducible lattices in G such that* $\text{vol}(G/\Gamma_n) \to \infty$. *Then for every $R > 0$,*

$$\lim_n \frac{\text{vol}_{\leq R}(X/\Gamma_n)}{\text{vol}(X/\Gamma_n)} = 0.$$

This is Corollary 4.8 of [1] (a version for simple groups appears as Corollary 2.5 of [2]). We note that by Wang's finiteness theorem (see [22], 8.1), if the Γ_n are pairwise non-conjugate then indeed $\mathrm{vol}(G/\Gamma_n) \to \infty$.

We now reflect on the proof of Theorem 4.1. The basic technique of the proof is, essentially, to bound the size of certain sets by assigning to each element of the set a point in the δ-thick part of X/Γ (for some $\delta > 0$ which is smaller than ε), and making sure these points are δ-discrete. This is done in two places. The first, is in step 2 of the proof, and relies on Proposition 3.6. The second is done directly in step 3. (Note that we are restricting our attention to the proof of the last step of the induction, i.e. $d = \dim(X)$.) If we look closer into the proofs of the proposition and of step 3, we see that in both cases the points in the δ-thick part are contained in the $\frac{\varepsilon}{2}$-thin part of X/Γ. In fact, this is crucial to the proof of δ-discreteness. Therefore, we also have a bound

$$f(\Gamma) \le c' \cdot \mathrm{vol}_{\le \varepsilon}(X/\Gamma)$$

(of course, this bound does not lend itself to induction, and is therefore not highlighted in the proof of the theorem). The proof is completed by appealing to Theorem 4.2. $\qquad\square$

5. A construction in $SO(d, 1)$

5.1. We first describe a general setting that assures that a group contains a sequence of subgroups of finite index for which the number of non-conjugate maximal finite subgroups grows linearly with index. Later, we will construct lattices in $SO(d, 1)^\circ$ that realize this setting.

Proposition 5.1. *Let Φ be a countable group with a finite subgroup H. Assume that there is an epimorphism $\varphi\colon \Phi \to \mathbb{Z}$ such that the normalizer of H is contained in $\ker \varphi$. Let Φ_n be the kernel of the composition*

$$\Phi \xrightarrow{\varphi} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}.$$

Then there are n non-conjugate subgroups of Φ_n that are all conjugate to H in Φ.

Proof. Fix some $t \in \varphi^{-1}(1)$. Consider the subgroups $H_i = t^i H t^{-i}$, $i \ge 0$. Since they are finite, they are all contained in $\ker \varphi$, hence in Φ_n, for all n.

We claim that $H_0, \ldots, H_{n-1}$ are non-conjugate in Φ_n. Indeed, if H_i, H_j ($0 \le i, j < n$) are conjugate then $t^{-j} g t^i$ normalizes H for some $g \in \Phi_n$. By our assumption on H and the definition of Φ_n, $j - i = \varphi(g) \equiv 0 \pmod{n}$, and therefore $i = j$. $\qquad\square$

In our application, Φ will be a lattice. As the covolume of Φ_n is proportional to $n = [\Phi : \Phi_n]$, the number of non-conjugate finite subgroups constructed in the

previous proposition grows linearly with the covolume of Φ_n. We also remark that if we start with a *maximal* finite subgroup H, the construction yields non-conjugate maximal subgroups.

5.2.   Now, we construct an appropriate lattice in $G = \mathrm{SO}(d,1)^{\circ}$, the connected component of $\mathrm{SO}(d,1)$. The starting point of our construction is the result of Millson [18] concerning the positivity of the first Betti number of certain congruence subgroups in standard arithmetic lattices of G.

Let $X = \{(x_1, \ldots, x_{d+1}) \mid x_1^2 + \cdots + x_d^2 - x_{d+1}^2 = -1, \ x_{d+1} > 0\}$ be the upper sheet of the hyperboloid, induced with a metric from the Minkowski space. This is a model for the d-dimensional real hyperbolic space. The linear action of G on $\mathbb{R}^{d+1}$ induces an isometric action on X.

Let q be a prime, and let $Q(X_1, X_2, \ldots, X_{d+1}) = X_1^2 + \cdots + X_d^2 - \sqrt{q} X_{d+1}^2$ be a quadratic form. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}[\sqrt{q}]$. Clearly, $\mathrm{SO}(Q, \mathbb{R})$ is conjugate to $\mathrm{SO}(d,1)$, and we denote by Γ the corresponding conjugate of $\mathrm{SO}(Q, \mathcal{O})$ in $\mathrm{SO}(d,1)$, intersected with $\mathrm{SO}(d,1)^{\circ}$. Then Γ is a lattice in G. For an ideal $\mathfrak{p}$ in $\mathcal{O}$, let $\Gamma(\mathfrak{p}) = \{\gamma \in \Gamma \mid \gamma \equiv I \mod \mathfrak{p}\}$. We will fix a certain $\mathfrak{p}$ shortly, and denote $\Delta = \Gamma(\mathfrak{p})$.

Denote by Y the hyperplane $\{(x_1, \ldots, x_{d+1}) \in X \mid x_1 = 0\}$. Let Δ_Y the subgroup of isometries in Δ stabilizing Y (in Millson's work, this is described as the group of isometries commuting with the reflection around Y). For almost all $\mathfrak{p}$, Δ is torsion-free, and it is shown that for $\mathfrak{p}$ deep enough, Y/Δ_Y embeds as a non-separating oriented hyperplane in X/Δ. We fix such an ideal $\mathfrak{p}$. Note that Δ is normal in Γ.

As explained in [18], the homology class carried by Y/Δ_Y is non-trivial, and its Poincaré dual yields an epimorphism $\phi \colon \Delta \to \mathbb{Z}$.

Denote by $H < \Gamma$ the group of isometries stabilizing Y and fixing the point $(0, \ldots, 0, 1) \in Y$. Note that H is not trivial; it contains, for example, the diagonal matrices with -1 in two of the first d entries, and 1 on the rest. Moreover, by considering all the matrices of this form, we see that $(0, \ldots, 0, 1)$ is the *unique* fixed point of H.

Since H normalizes Δ, it acts by isometries on X/Δ. Furthermore, since H stabilizes Y it normalizes Δ_Y and stabilizes Y/Δ_Y. It follows that H fixes the homology class of Y/Δ_Y and its Poincaré dual. In other words, $\phi(h\gamma h^{-1}) = \phi(\gamma)$ for all $\gamma \in \Delta$ and $h \in H$. By the following lemma, whose proof is left to the reader, we can extend ϕ to $H \ltimes \Delta$.

Lemma 5.2. *Let $\Delta < \Gamma$ be a subgroup, and $H < \Gamma$ a finite subgroup contained in the normalizer of Δ. A homomorphism $\phi \colon \Delta \to \mathbb{Z}$ can be extended to $H\Delta$ iff $\phi(h\gamma h^{-1}) = \phi(\gamma)$ for all $\gamma \in \Delta$ and $h \in H$.*[§]

[§]$H\Delta$ is a subgroup because H normalizes Δ. If $H \cap \Delta = \{1\}$ then it is equal to $H \ltimes \Delta$.

We will denote the extension of ϕ to $H \ltimes \Delta$ by $\tilde{\phi}$. To use Proposition 5.1 (with $H \ltimes \Delta$ as the group Φ), we must show that the normalizer of H in $H \ltimes \Delta$ is contained in $\ker \tilde{\phi}$. Since $H \leq \ker \tilde{\phi}$, it suffices to show that H is self-normalizing in $H \ltimes \Delta$. To this end, suppose $\gamma \in \Delta$ normalizes H. Then γ must fix the unique fixed point of H, but since Δ is torsion-free this forces γ to be 1.

Finally, observe that H is a maximal finite subgroup in $H \ltimes \Delta$. Indeed, if $g \in H \ltimes \Delta$, $g \notin H$, then there exists $1 \neq \gamma \in \Delta$ such that $g\gamma^{-1} \in H$. Therefore, the group generated by H and g contains γ, and is therefore infinite.

6. Isotropy subgroups

Theorem 1.1 does not provide a bound on the number of conjugacy classes of non-maximal finite subgroups (or torsion elements) in Γ, because we have no effective bound on the size of finite subgroups of Γ. However, we will now show that there is an effective bound on the number of non-conjugate *isotropy* subgroups in Γ, i.e. subgroups that are stabilizers of points in X. It is easy to see that a stabilizer of the submanifold Y of X is a stabilizer of some (but not every) point in Y. Hence there is a one-to-one correspondence between isotropy subgroups and fixed submanifolds.

Let us begin by proving a lemma regarding linear groups.

Definition 6.1. Let T be a subset of $\mathrm{GL}_n(\mathbb{C})$. A set of vectors $\mathcal{B}$ in $\mathbb{C}^n$ *splits* T, if $\mathcal{B}$ contains a basis of eigenvectors for every $t \in T$.

If $H < \mathrm{GL}_n(\mathbb{C})$ is an abelian group consisting of semisimple elements then there is a set vectors of size n that splits H. This is a restatement of the fact that the elements of H can be simultaneously diagonalized. If H is finite but not abelian, there is generally no uniform bound (depending only on n) on the size of a splitting set. Indeed, the minimal splitting set for a dihedral group D_{2m} realized in the standard way as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ is $2m$. Nonetheless, in this example, there is a cyclic subgroup of index 2, and all other elements (of order 2) constitute one or two conjugacy classes. Thus, if one considers splitting "up-to-conjugacy", the size of a minimal splitting set is 4 or 6. We make generalize this phenomenon:

Lemma 6.2. *There exists a function $f(n)$ such that for every finite group G of $\mathrm{GL}_n(\mathbb{C})$ there exists a set $\mathcal{B}$ such that $G\mathcal{B}$ splits G and $|\mathcal{B}| < f(n)$.*

Remark. Saying that $G\mathcal{B}$ splits G is equivalent to saying that $\mathcal{B}$ splits a set of representatives of the conjugacy classes in G.

Proof. By a theorem of Jordan [11], 36.13, there is a normal abelian subgroup $A \lhd G$ such that $[G : A] < j(n)$, where j is a function of n alone. It therefore suffices to

prove that for any coset gA there is a set of vectors $\mathcal{C}$ such that $G\mathcal{C}$ splits gA, and $|\mathcal{C}|$ is bounded by a constant that depends only on n.

Let gA be a coset of A in G. Set $H = \langle A, g \rangle$. Assume first that H acts irreducibly on $\mathbb{C}^n$. Since A is abelian, $C := C_A(g) \leq Z(H)$, and by Schur's lemma, $Z(H) < Z(\mathrm{GL}_n(\mathbb{C}))$.

Let

$$K = [A, g^{-1}] = \{[a, g^{-1}] \; : \; a \in A\}.$$

It is straightforward to check that K is a subgroup of A because A is normal and abelian. Moreover, the map $a \mapsto [a, g^{-1}]$ is a homomorphism of A onto K with kernel C. Note that $K \leq \mathrm{SL}_n(\mathbb{C})$, and hence $K \cap C \leq Z(\mathrm{SL}_n(\mathbb{C}))$. Thus, $|K \cap C| \leq n$. We conclude that

$$|KC| = \frac{|K||C|}{|K \cap C|} \geq \frac{|K||C|}{n} = \frac{|A|}{n},$$

hence the index of KC in A is at most n.

Choose representatives $a_1, \ldots, a_s$ ($s \leq n$) for the cosets of KC in A. For each i, choose a basis of eigenvectors for ga_i, and let $\mathcal{C}$ be the union of these bases. Clearly, $|\mathcal{C}| \leq n^2$. We claim that $H\mathcal{C}$, and moreover $A\mathcal{C}$, splits gA. To this end, let $y \in gA$. For some i, $y \in ga_i KC$ and write $y = ga_i[a, g^{-1}]c$ ($[a, g^{-1}] \in K, c \in C$). Then

$$y = cga_i a(g^{-1}a^{-1}g) = cg(g^{-1}a^{-1}g)a_i a = ca^{-1}ga_i a.$$

Since $\mathcal{C}$ splits ga_i, $A\mathcal{C}$ splits $a^{-1}ga_i a$, and since c is a scalar matrix, it also splits y.

In the general case, decompose $\mathbb{C}^n$ to H-irreducible subspaces $V_1, \ldots, V_r$. In each subspace V_i, we take a set $\mathcal{C}_i$ such that $A\mathcal{C}_i$ splits the restriction of gA to V_i, and that $|\mathcal{C}_i| \leq \dim(V_i)^2$. Now $\mathcal{C} = \bigcup \mathcal{C}_i$ has the required properties: $A\mathcal{C}$ splits gA and $|\mathcal{C}| \leq \sum \dim(V_i)^2 \leq n^2$. $\qquad\square$

Remark. The proof shows in fact that we may take a set $\mathcal{B}$ such that $|\mathcal{B}| < f(n)$ and $A\mathcal{B}$ already (rather than $G\mathcal{B}$) is a splitting set.

Corollary 6.3. *Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then the number of G-orbits in the set of fixed subspaces $\{\mathrm{Fix}(g) : g \in G\}$ is bounded by a function of n.*

Corollary 6.4. *Let X be a global symmetric space of non-compact type, and Γ a discrete subgroup of $\mathrm{Isom}(X)$. Denote by $i(\Gamma)$ the number of conjugacy classes of isotropy subgroups in Γ. There is a constant $c' = c'(X)$ such that*

$$i(\Gamma) \leq c' \cdot f(\Gamma).$$

Proof. The claim will follow if we show that there is a uniform bound on the number of isotropy subgroups contained in a maximal finite subgroup of Γ.

Let $H < \Gamma$ be a maximal finite subgroup, and let $x \in \mathrm{Fix}(H)$. We may identify H as a subgroup of $\mathrm{SO}(n)$ ($n = \dim(X)$) through the action of H on $T_x(X)$, the

tangent bundle at x. We note that by replacing the action of H on X by an action on $T_x(X)$, the set of isotropy subgroups in H is unchanged. Since conjugacy classes of isotropy subgroups correspond to H-orbits of fixed subspaces, the assertion follows at once from the previous corollary. $\qquad\square$

Corollary 6.5 (Theorem 1.3 of the introduction). *Let X be a symmetric space X of non-compact type. Let $M = X/\Gamma$ be an X-orbifold. Denote by $s(M)$ the number of strata in the natural orbifold stratification of M. Then*

$$s(M) \leq c \cdot \mathrm{vol}(M),$$

with a constant $c = c(X)$. If the rank of X is at least 2 and $\mathrm{Isom}(X)$ has property (T), then for any sequence M_n of irreducible X-orbifolds that are pairwise non-isometric we have

$$\lim_{n\to\infty} \frac{s(M_n)}{\mathrm{vol}(M_n)} = 0.$$

Proof. Strata correspond to conjugacy classes of isotropy subgroups. Hence, this follows immediately from Corollary 6.4, and Theorems 1.1 and 1.2. $\qquad\square$

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