Comment. Math. Helv. 89 (2014), 783–818
DOI 10.4171/CMH/334

Commentarii Mathematici Helvetici

Knots in lattice homology

Peter Ozsváth, András I. Stipsicz and Zoltán Szabó

Abstract. Assume that Γ_{v_0} is a tree with vertex set $\mathrm{Vert}(\Gamma_{v_0}) = \{v_0, v_1, \dots, v_n\}$, and with an integral framing (weight) attached to each vertex except v_0. Assume furthermore that the intersection matrix of $G = \Gamma_{v_0} - \{v_0\}$ is negative definite. We define a filtration on the chain complex computing the lattice homology of G and show how to use this information in computing lattice homology groups of a negative definite graph we get by attaching some framing to v_0. As a simple application we produce new families of graphs which have arbitrarily many bad vertices for which the lattice homology groups are isomorphic to the corresponding Heegaard Floer homology groups.

Mathematics Subject Classification (2010). 57R, 57M.

Keywords. Lattice homology, Heegaard Floer homology, knot Floer homology.

1. Introduction

It is an eminent problem in low dimensional topology to find simple computational schemes for the recently defined invariants (e.g. Heegaard Floer and Monopole Floer homologies) of 3- and 4-manifolds. In particular, the minus-version HF^- of Heegaard Floer homology is of central importance. In [8] a computational scheme for the HF^- groups was presented, which is rather hard to implement in practice. This result was preceded by a more practical way of determining these invariants for those 3-manifolds which can be presented as boundary of a plumbing of spheres along a negative definite tree which has at most one "bad" vertex [21]. The idea of [21] was subsequently extended by Némethi [9], and in [10] a new invariant, *lattice homology* was proposed. It has been conjectured that lattice homology determines the Heegaard Floer groups when the underlying 3-manifold is given by a negative definite plumbing of spheres along a tree. Common features have been verified for both invariants. (For example, both theories satisfy a surgery exact triangle; see [19] for the Heegaard Floer setting, and [2], [12] for lattice homology.) Moreover, there is a spectral sequence which connects the two invariants. (See [17].) For further related results see [11], [13].

In the present work we extend these similarities by introducing filtrations on lattice homologies induced by vertices, mimicking the ideas of knot Floer homologies developed in the Heegaard Floer context in [22], [26]. This information then can be conveniently used to determine the lattice homology of the graph when the distinguished vertex is equipped with some framing; this is analogous to the surgery formulae in Heegaard Floer theory, cf. [24].

In more concrete terms, suppose that Γ_{v_0} is a given tree (or forest), with each vertex v in $\mathrm{Vert}(\Gamma_{v_0}) - v_0$ equipped with a framing (or weight) $m_v \in \mathbb{Z}$. Let G denote the tree (or forest) we get by deleting v_0 and the edges emanating from it. Suppose that G is negative definite. We will define the *master complex* $\mathbb{MCF}^\infty(\Gamma_{v_0})$ of Γ_{v_0}, which is a filtration on the chain complex defining the lattice homology of G equipped with a specific map, and will show

Theorem 1.1. *The master complex $\mathbb{MCF}^\infty(\Gamma_{v_0})$ determines the lattice homology of all negative definite framed trees (or forests) we get from Γ_{v_0} by attaching framings to v_0.*

By identifying the filtered chain homotopy type of the resulting master complex with the knot Floer homology of the corresponding knot in the plumbed 3-manifold, this method allows us to show that certain graphs have identical lattice and Heegaard Floer homologies. Recall that for a negative definite tree (or forest) G on the vertex set $\mathrm{Vert}(G)$, the vertex $v \in \mathrm{Vert}(G)$ is a *bad vertex* if $m_v + d_v > 0$, where m_v denotes the framing attached to v while d_v is the valency or degree of v (the number of edges emanating from v). A vertex is *good* if it is not bad, that is, $m_v + d_v \leq 0$. Now a connected sum formula for knot lattice homology (given in Subsection 4.1) enables us to extend the identification of lattice homology with Heegaard Floer homology to new families of graphs, including some with arbitrarily many bad vertices. As an example, we show

Theorem 1.2. *Consider the plumbing graph of Figure 1 on $3n + 1$ vertices, with the framing of v_0 an integer at most $-6n - 1$. Then the lattice homology of the graph is isomorphic to the Heegaard Floer homology HF^- of the 3-manifold defined by the plumbing.*

Remark 1.3. Notice that the graph of Figure 1 on $3n + 1$ vertices (after we attach a framing $-m \leq -6n - 1$ to the central vertex v_0) has n bad vertices. The case of $n = 2$ in the theorem was already proved by Némethi, cf. Example 4.4.1 of [10], see also [13] for related results. For a more general result along similar lines, see [18].

The paper is organized as follows. In Section 2 we review the basics of lattice homology for negative definite graphs. In Sections 3 and 4 we introduce the knot filtration on the lattice chain complex of the background graph, describe the master

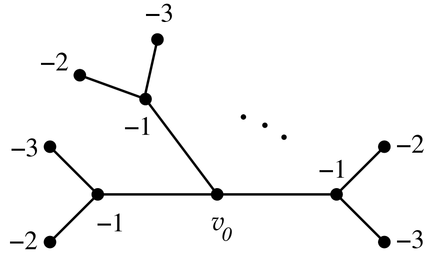

Figure 1. **The plumbing graph of the n-fold connected sum of the (right-handed) trefoil knot in S^3.** The valency of the central vertex v_0 is assumed to be $n \in \mathbb{N}$, and each edge emanating from v_0 connects it to a vertex with framing (-1). Furthermore these (-1)-vertices are connected to a (-2)- and a (-3)-framed leaf of the graph. Regarding v_0 as a circle in the plumbed 3-manifold defined by the rest of the graph, it can be identified with the n-fold connected sum of the trefoil knot in S^3.

complex and verify the connected sum formula. In Section 5 we show how to apply this information to determine the lattice homology of graphs we get by attaching various framings to the distinguished point v_0. In particular, we prove Theorem 1.1. In Section 6 we determine the knot filtration in one specific example, and verify Theorem 1.2.

Notation. Suppose that Γ is a tree (or forest), and G is the same graph equipped with framings, i.e., we attach integers to the vertices of Γ. The plumbing of disk bundles over spheres defined by G will be denoted by X_G, and its boundary 3-manifold is Y_G. Let M_G denote the incidence matrix associated to G (with framings in the diagonal). This matrix presents the intersection form of X_G in the basis provided by the vertices of the plumbing graph.

Suppose that Γ_{v_0} is a plumbing tree (or forest) with a distinguished vertex v_0 which is left unframed (but all other vertices of Γ_{v_0} are framed). Let G denote the plumbing graph we get by deleting the vertex v_0 (and all the edges adjacent to it). We will always assume that the plumbing trees/forests we work with are negative definite.

Remark 1.4. We can regard the unknot defined by v_0 in the plumbing picture as a (not necessarily trivial) knot in the plumbed 3-manifold Y_G.

Acknowledgements. PSO was supported by NSF grant number DMS-0804121. AS was supported by OTKA NK81203, by the ERC Grant LDTBud, by *Lendület program ADT* and by the Institute for Advanced Study. ZSz was supported by NSF grants DMS-0603940, DMS-0704053 and DMS-1006006. The present work is part of the authors' activities within CAST, a Research Network Program of the European Science Foundation.

2. Review of lattice homology

Lattice homology was introduced by Némethi in [10] (cf. also [11], [12], [13]). In this section we review the basic notions and concepts of this theory. Our main purpose is to set up notations which will be used in the rest of the paper.

Following [10], for a given negative definite plumbing tree G we define a $\mathbb{Z}$-graded combinatorial chain complex $(\mathbb{CF}^\infty(G), \partial)$ (and then a subcomplex $(\mathbb{CF}^-(G), \partial)$ of it), which is a module over the ring of Laurent polynomials $\mathbb{F}[U^{-1}, U]$ (and over the polynomial ring $\mathbb{F}[U]$, respectively), where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Define $\mathrm{Char}(G)$ as the set of characteristic cohomology elements of $H^2(X_G; \mathbb{Z})$, that is,

$$\mathrm{Char}(G) = \{K\colon H_2(X_G;\mathbb{Z}) \to \mathbb{Z} \mid \text{for all } x \in H_2(X_G;\mathbb{Z})\colon\ K(x) \equiv x\cdot x \bmod 2\}.$$

The *lattice chain complex* $\mathbb{CF}^\infty(G)$ is freely generated over $\mathbb{F}[U^{-1}, U]$ by the product $\mathrm{Char}(G) \times \mathbb{P}(\mathrm{Vert}(G))$, that is, by elements $[K, E]$ where $K \in \mathrm{Char}(G) \subset H^2(X_G; \mathbb{Z})$ and $E \subset \mathrm{Vert}(G)$. We introduce a $\mathbb{Z}$-grading on this complex, called the *δ-grading*, which is defined on the generator $[K, E]$ as the number of elements in E. To define the boundary map of the chain complex, we proceed as follows. Given a subset $I \subset E$, we define the *G-weight* $f([K, I])$ as the quantity

$$2f([K,I]) = \Big(\sum_{v\in I} K(v)\Big) + \Big(\sum_{v\in I} v\Big)\cdot\Big(\sum_{v\in I} v\Big). \tag{2.1}$$

Remark 2.1. Using the fact that G is negative definite, the integer $f([K, I])$ can be easily shown to be equal to

$$\frac{1}{8}\Big(\Big(K + \sum_{v\in I} 2v^*\Big)^2 - K^2\Big),$$

where $v^* \in H^2(X_G, Y_G; \mathbb{Z})$ denotes the Poincaré dual of the class $v \in H_2(X_G; \mathbb{Z})$ corresponding to the vertex $v \in \mathrm{Vert}(G)$. This form of $f(K, I)$ immediately implies, for example, the following useful identity: if $I \subset E$ then

$$f([K,I]) - f\Big(\Big[-K - \sum_{u\in E} 2u^*, E - I\Big]\Big) = f([K,E]). \tag{2.2}$$

We define the *minimal G-weight* $g([K, E])$ of $[K, E]$ by the formula

$$g([K,E]) = \min\{f([K,I]) \mid I \subset E\}.$$

The quantities $A_v([K, E])$ and $B_v([K, E])$ are defined as follows:

$$A_v([K,E]) = g([K, E - v]) \quad\text{and}\quad B_v([K,E]) = \min\{f([K,I]) \mid v \in I \subset E\}.$$

A simple argument shows that

$$B_v([K,E]) = \left(\frac{K(v)+v^2}{2}\right) + g([K+2v^*, E-v]). \tag{2.3}$$

It follows trivially from the definition that

$$\min\{A_v([K,E]), B_v([K,E])\} = g([K,E]).$$

Consider

$$a_v[K,E] = A_v([K,E]) - g([K,E]) \quad \text{and} \quad b_v[K,E] = B_v([K,E]) - g([K,E]).$$

and define the boundary map $\partial\colon \mathbb{CF}^\infty(G) \to \mathbb{CF}^\infty(G)$ by the formula

$$\partial[K,E] = \sum_{v\in E} U^{a_v[K,E]} \otimes [K, E-v] + \sum_{v\in E} U^{b_v[K,E]} \otimes [K+2v^*, E-v],$$

on a generator $[K,E]$ and extend this map U-equivariantly to the terms $U^j \otimes [K,E]$ and then linearly to $\mathbb{CF}^\infty(G)$. Notice that $a_v[K,E], b_v[K,E]$ are both nonnegative integers, and $\min\{a_v[K,E], b_v[K,E]\} = 0$ follows directly form the definitions. It is obvious that the boundary map decreases the δ-grading by one. Furthermore, it is a simple exercise to show that

Lemma 2.2. *The map ∂ is a boundary map, that is, $\partial^2 = 0$.*

Proof. The proof boils down to matching the exponents of the U-factors in front of various terms in $\partial^2[K,E]$ for a given generator $[K,E]$. This idea leads us to four equations to check. One of them, for example, relates the two U-powers in front of the two appearances $[K, E-v_1-v_2]$ in $\partial^2[K,E]$. We claim that

$$a_{v_1}[K,E] + a_{v_2}[K, E-v_1] = a_{v_2}[K,E] + a_{v_1}[K, E-v_2] \tag{2.4}$$

holds, therefore (over $\mathbb{F}$) the two terms cancel each other. Writing out the definitions of the terms in (2.4) we get

$$\begin{aligned} g([K,E-v_1]) - g([K,E]) + g([K,E-v_1-v_2]) - g([K,E-v_1]) \\ = g([K,E-v_2]) - g([K,E]) + g([K,E-v_1-v_2]) - g([K,E-v_2]), \end{aligned}$$

which trivially holds. The remaining three cases to check are:

$$\begin{aligned} a_{v_1}[K,E] + b_{v_2}[K,E-v_1] &= b_{v_2}[K,E] + a_{v_1}[K+2v_2^*, E-v_2], \\ b_{v_1}[K,E] + a_{v_2}[K+2v_1^*, E-v_1] &= a_{v_2}[K,E] + b_{v_1}[K,E-v_2], \end{aligned} \tag{2.5}$$

and finally

$$b_{v_1}[K,E] + b_{v_2}[K+2v_1^*, E-v_1] = b_{v_2}[K,E] + b_{v_1}[K+2v_2^*, E-v_2].$$

Using the definition of B_v given in (2.3), the equations reduce to similar equalities as in the first case. □

Remark 2.3. In [10] the theory is set up over $\mathbb{Z}$; for simplicity in the present paper we use the coefficients from the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ of two elements.

2.1. Connected sums. Suppose that the plumbing forest G is the union of G_1 and G_2, with no edges connecting any vertex of G_1 to any vertex of G_2. (In other words, G_1 and G_2 are both unions of components of G.) It is a simple topological fact that in this case Y_G decomposes as the connected sum of the two 3-manifolds Y_{G_1} and Y_{G_2}. Correspondingly, the $\mathbb{F}[U^{-1}, U]$-module $\mathbb{CF}^\infty(G)$ decomposes as the tensor product

$$\mathbb{CF}^\infty(G) \cong \mathbb{CF}^\infty(G_1) \otimes_{\mathbb{F}[U^{-1},U]} \mathbb{CF}^\infty(G_2), \tag{2.6}$$

and the definition of the boundary map ∂ shows that this decomposition holds on the chain complex level as well.

2.2. Spin^c structures and the J-map. Define an equivalence relation for the generators of the chain complex $\mathbb{CF}^\infty(G)$ as follows: we say that $[K, E]$ and $[K', E']$ are *equivalent* if $K - K' \in 2H^2(X_G, Y_G; \mathbb{Z})$. Since the boundary map respects this equivalence relation, the chain complex splits according to this relation.

It is easy to see that (since X_G is simply-connected) a characteristic cohomology class $K \in H^2(X_G; \mathbb{Z})$ uniquely determines a spin^c structure on X_G. By restricting this structure to the boundary 3-manifold Y_G we conclude that K naturally induces a spin^c structure $\mathbf{s}_K$ on Y_G. Two classes K, K' induce the same spin^c structure on Y_G if and only if they are equivalent in the above sense (that is, $K - K' \in 2H^2(X_G, Y_G; \mathbb{Z})$). Therefore the splitting of the chain complex $\mathbb{CF}^\infty(G)$ described above is parametrized by the spin^c structures of Y_G:

$$\mathbb{CF}^\infty(G) = \sum_{\mathbf{s} \in \mathrm{Spin}^c(Y_G)} \mathbb{CF}^\infty(G, \mathbf{s}),$$

where $\mathbb{CF}^\infty(G, \mathbf{s})$ is spanned by those pairs $[K, E]$ for which $\mathbf{s}_K = \mathbf{s}$.

Consider now the map

$$J[K, E] = \Big[-K - \sum_{v \in E} 2v^*, E\Big],$$

and extend it U-equivariantly (and linearly) to $\mathbb{CF}^\infty(G)$. Obviously J provides an involution on $\mathbb{CF}^\infty(G)$, and a simple calculation shows the following:

Lemma 2.4. *The J-map is a chain map, that is, $J \circ \partial = \partial \circ J$.*

Proof. The two compositions can be easily determined as

$$\begin{aligned}(J \circ \partial)[K, E] = \sum_{v \in E} \Big(U^{a_v[K,E]} \otimes \Big[-K - \sum_{u \in E - v} 2u^*, E - v\Big]\Big) \\ + \sum_{v \in E} \Big(U^{b_v[K,E]} \otimes \Big[-K - \sum_{u \in E} 2u^*, E - v\Big]\Big)\end{aligned}$$

and

$$(\partial \circ J)[K,E] = \sum_{v\in E} \Big(U^{a_v[-K-\sum_{u\in E} 2u^*,E]} \otimes [-K - \sum_{u\in E} 2u^*, E - v] \Big) + \sum_{v\in E} \Big(U^{b_v[-K-\sum_{u\in E} 2u^*,E]} \otimes \Big[-K - \sum_{u\in E-v} 2u^*, E - v \Big] \Big).$$

The fact that J is a chain map, then follows from the two identities

$$a_v[K,E] = b_v\Big[-K - \sum_{u\in E} 2u^*, E\Big] \quad \text{and} \quad a_v\Big[-K - \sum_{u\in E} 2u^*, E\Big] = b_v[K,E]. \tag{2.7}$$

In turn, these identities easily follow from the identity of (2.2), concluding the proof of the lemma. □

The J-map obviously respects the splitting of $\mathbb{CF}^\infty(G)$ according to spin^c structures. In fact, the spin^c structures represented by K and $-K$ are 'conjugate' to each other as spin^c structures on Y_G (cf. [19]), inducing the spin^c structures $\mathbf{s}, \bar{\mathbf{s}} \in \mathrm{Spin}^c(Y_G)$, respectively. The J-map therefore is just the manifestation of the conjugation involution of spin^c structures on the chain complex level. Indeed, J provides an isomorphism between the two subcomplexes $\mathbb{CF}^\infty(G,\mathbf{s})$ and $\mathbb{CF}^\infty(G,\bar{\mathbf{s}})$.

2.3. Gradings. The lattice chain complex $\mathbb{CF}^\infty(G)$ admits a *Maslov grading*: for a generator $[K,E]$ and $j \in \mathbb{Z}$ define $\mathrm{gr}(U^j \otimes [K,E])$ by the formula

$$\mathrm{gr}(U^j \otimes [K,E]) = -2j + 2g([K,E]) + |E| + \tfrac{1}{4}(K^2 + |\mathrm{Vert}(G)|).$$

Recall that K^2 is defined as the square of nK divided by n^2, where $nK \in H^2(X_G, Y_G; \mathbb{Z})$, hence it admits a cup square. (Here we use the fact that G is negative definite, hence $\det M_G \neq 0$, so the restriction of any cohomology class from X_G to its boundary Y_G is torsion.) The grading $\mathrm{gr}(U^j \otimes [K,E])$ is a rational number (rather than an integer).

Lemma 2.5. *The boundary map decreases the Maslov grading by one.*

Proof. We proceed separately for the two types of components of the boundary map. After obvious simplifications we get that

$$\mathrm{gr}(U^j \otimes [K,E]) - \mathrm{gr}(U^j \cdot U^{a_v[K,E]} \otimes [K, E-v]) = 2g([K,E]) + |E| + 2a_v[K,E] - 2g([K,E-v]) - |E-v|,$$

which, according to the definition of $a_v[K,E]$, is equal to 1. Similarly,

$$\mathrm{gr}(U^j \otimes [K,E]) - \mathrm{gr}(U^j \cdot U^{b_v[K,E]} \otimes [K + 2v^*, E - v]) = 1$$

follows from the same simplifications and Equation (2.3). □

It is not hard to see that the J-map preserves the Maslov grading. Indeed,

$$\begin{aligned}
&\mathrm{gr}([K,E]) - \mathrm{gr}(J[K,E]) \\
&\quad = \mathrm{gr}([K,E]) - \mathrm{gr}\Big(\Big[-K - \sum_{v\in E} 2v^*, E\Big]\Big) \\
&\quad = 2g([K,E]) - 2g\Big(\Big[-K - \sum_{v\in E} 2v^*, E\Big]\Big) + \tfrac{1}{4}\Big(K^2 - \Big(-K - \sum_{v\in E} 2v^*\Big)^2\Big).
\end{aligned}$$

Using the identity of (2.2) and the alternative definition of $f(K,E)$, it follows that the above difference is equal to zero.

Recall that the cardinality $|E|$ for a generator $[K,E]$ of $\mathbb{CF}^-(G)$ gives the δ-grading, which decomposes each $\mathbb{CF}^-(G,\mathbf{s})$ as

$$\mathbb{CF}^-(G,\mathbf{s}) = \bigoplus_{k=0}^n \mathbb{CF}^-_k(G,\mathbf{s}),$$

where $n = |\mathrm{Vert}(G)|$. It is easy to see that the differential ∂ decreases δ-grading by one.

2.4. Definition of the lattice homology. We define the lattice homology groups as follows. Consider $(\mathbb{CF}^\infty(G),\partial)$, and let $(\mathbb{CF}^-(G),\partial)$ denote the subcomplex generated by those generators $U^j \otimes [K,E]$ for which $j \geq 0$ (and equipped with the differential restricted to the subspace). Setting $U = 0$ in this subcomplex we get the complex $(\widehat{\mathbb{CF}}(G),\hat{\partial})$. Obviously all these chain complexes split according to spin^c structures and admit a Maslov grading, δ-grading and a J-map.

Definition 2.6. Define the *lattice homology* $\mathbb{HF}^\infty(G)$ as the homology of the chain complex $(\mathbb{CF}^\infty(G),\partial)$. The homology of the subcomplex $\mathbb{CF}^-(G)$ (with the boundary map ∂ restricted to it) will be denoted by $\mathbb{HF}^-(G)$, while the homology of $(\widehat{\mathbb{CF}}(G),\hat{\partial})$ is $\widehat{\mathbb{HF}}(G)$.

Since the chain complex $\mathbb{CF}^-(G)$ (and similarly, $\mathbb{CF}^\infty(G)$ and $\widehat{\mathbb{CF}}(G)$) splits according to spin^c structures, so does its homology, giving the decomposition

$$\mathbb{HF}^-(G) = \bigoplus_{\mathbf{s}\in \mathrm{Spin}^c(Y_G)} \mathbb{HF}^-(G,\mathbf{s}).$$

The δ-grading then decomposes $\mathbb{HF}^-(G,\mathbf{s})$ further as

$$\mathbb{HF}^-(G,\mathbf{s}) = \bigoplus_{k=0}^n \mathbb{HF}^-_k(G,\mathbf{s}),$$

where $n = |\mathrm{Vert}(G)|$. The Maslov grading provides an additional $\mathbb{Q}$-grading on $\mathbb{HF}^-(G,\mathbf{s})$, but we reserve the subscript $\mathbb{HF}^-_k(G,\mathbf{s})$ for the δ-grading.

Remark 2.7. The embedding $i\colon \mathbb{CF}^-(G) \to \mathbb{CF}^\infty(G)$ can be used to define a quotient complex $\mathbb{CF}^+(G)$ (with the differential inherited from this construction) which fits into the short exact sequence

$$0 \to \mathbb{CF}^-(G) \to \mathbb{CF}^\infty(G) \to \mathbb{CF}^+(G) \to 0.$$

The homology of this quotient complex will be denoted by $\mathbb{HF}^+(G)$. The same splittings as before (according to spin^c structures, the δ-grading and Maslov grading) apply to this theory is well. The short exact sequence above then induces a long exact sequence on the various homologies.

In a similar manner, $\mathbb{CF}^-(G)$ and $\widehat{\mathbb{CF}}(G)$ can be also connected by a short exact sequence:

$$0 \to \mathbb{CF}^-(G) \xrightarrow{U} \mathbb{CF}^-(G) \to \widehat{\mathbb{CF}}(G) \to 0,$$

where the first map is multiplication by U. This short exact sequence then induces a long exact sequence on homologies connecting $\mathbb{HF}^-(G)$ and $\widehat{\mathbb{HF}}(G)$:

$$\cdots \to \mathbb{HF}^-_q(G) \xrightarrow{U} \mathbb{HF}^-_q(G) \to \widehat{\mathbb{HF}}_q(G) \to \mathbb{HF}^-_{q-1} \to \cdots.$$

The homology group $\mathbb{HF}^-(G)$ is obviously an $\mathbb{F}[U]$-module. In the next result we describe an algebraic property these particular modules satisfy.

Theorem 2.8 (Némethi, [10]). *Suppose that G is a negative definite plumbing tree and $\mathbf{s}$ is a spin^c structure on Y_G. Then the homology $\mathbb{HF}^-(G,\mathbf{s})$ is a finitely generated $\mathbb{F}[U]$-module of the form*

$$\mathbb{HF}^-(G,\mathbf{s}) = \mathbb{F}[U] \oplus \bigoplus_i A_i,$$

where the modules A_i are cyclic modules of the form $\mathbb{F}[U]/(U^n)$. Furthermore the $\mathbb{F}[U]$-factor is in $\mathbb{HF}^-_0(G,\mathbf{s})$. □

Corollary 2.9. *The $\mathbb{F}[U^{-1},U]$-module $\mathbb{HF}^\infty(G,\mathbf{s}) = \mathbb{HF}^\infty_0(G,\mathbf{s})$ is isomorphic to $\mathbb{F}[U^{-1},U]$.*

Proof. By the Universal Coefficient Theorem we get that there is a short exact sequence

$$\begin{aligned}0 \to \mathbb{HF}^-_q(G,\mathbf{s}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1},U] \to \mathbb{HF}^\infty_q(G,\mathbf{s})\\ \to \operatorname{Tor}(\mathbb{HF}^-_{q-1}(G,\mathbf{s}), \mathbb{F}[U^{-1},U]) \to 0.\end{aligned}$$

Since

$$\operatorname{Tor}(\mathbb{F}[U], \mathbb{F}[U^{-1},U]) = \operatorname{Tor}(\mathbb{F}[U]/(U^n), \mathbb{F}[U^{-1},U]) = 0$$

and

$$(\mathbb{F}[U]/(U^n)) \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U] = 0,$$

while $\mathbb{F}[U] \otimes_{\mathbb{F}[U]} \mathbb{F}[U^{-1}, U] = \mathbb{F}[U^{-1}, U]$, the claim obviously follows. By Theorem 2.8 the single $\mathbb{F}[U]$-factor is in $\mathbb{HF}_0^-(G, \mathbf{s})$, hence we get that $\mathbb{HF}^\infty(G, \mathbf{s}) = \mathbb{HF}_0^\infty(G, \mathbf{s})$. □

Definition 2.10. Let

$$\mathbb{HF}^-_{\mathrm{red}}(G, \mathbf{s}) \subseteq \mathbb{HF}^-(G, \mathbf{s})$$

denote the kernel of the map $i_*\colon \mathbb{HF}^-(G, \mathbf{s}) \to \mathbb{HF}^\infty(G, \mathbf{s})$ induced by the embedding $i\colon \mathbb{CF}^-(G, \mathbf{s}) \to \mathbb{CF}^\infty(G, \mathbf{s})$. This group is finite dimensional as a vector space over $\mathbb{F}$ and is called the *reduced lattice homology* of $(G, \mathbf{s})$.

2.5. Examples. We conclude this section by working out a simple example which will be useful in our later discussions.

Example 2.11. Suppose that the tree G has a single vertex v with framing -1. The chain complex $\mathbb{CF}^\infty(G)$ is generated over $\mathbb{F}[U^{-1}, U]$ by the elements

$$\{[2n+1,\ \{v\}],\ [2n+1,\ \emptyset] \mid n \in \mathbb{Z}\},$$

where a characteristic vector on G is denoted by its value $2n+1$ on v. The boundary map on $[2n+1, \emptyset] = [2n+1]$ is given by $\partial[2n+1] = 0$ and by

$$\partial[2n+1, \{v\}] = \begin{cases} [2n+1] + U^n \otimes [2n-1] & \text{if } n \geq 0, \\ U^{-n} \otimes [2n+1] + [2n-1] & \text{if } n < 0. \end{cases}$$

These formulae also describe the chain complexes $\mathbb{CF}^-(G)$ and $\widehat{\mathbb{CF}}(G)$ (generated over $\mathbb{F}[U]$ and over $\mathbb{F}$). Let us consider the map F from $\mathbb{CF}^\infty(G)$ to the subcomplex $\mathbb{F}[U^{-1}, U]\langle[-1]\rangle \subseteq \mathbb{CF}^\infty(G)$ generated by the element $[-1]$, defined as

$$F([2n+1, E]) = \begin{cases} 0 & \text{if } E = \{v\}, \\ U^{\frac{1}{2}n(n+1)} \otimes [-1] & \text{if } E = \emptyset. \end{cases}$$

This map provides a chain homotopy equivalence between $\mathbb{CF}^\infty(G)$ and $\mathbb{F}[U^{-1}, U]$ (the latter equipped with the differential $\partial = 0$), as shown by the chain homotopy

$$H([2n+1, E]) = \begin{cases} 0 & \text{if } E = \{v\} \text{ or } n = -1, \\ \sum_{i=0}^{n} U^{s_i} \otimes [2(n-i)+1, v] & \text{if } E = \emptyset \text{ and } n \geq 0, \\ \sum_{i=0}^{-n-2} U^{r_i} \otimes [2(n+i+1)+1, v] & \text{if } E = \emptyset \text{ and } n < -1, \end{cases}$$

where $s_0 = 0$ and $s_i = s_{i-1} + b_v[2(n-i-1)-1, v] = \frac{1}{2}i(2n+1-i)$, $r_0 = 0$ and $r_i = r_{i-1} + a_v[2(n+i)+1, v] = -\frac{1}{2}i(2n+1+i)$. In conclusion, the homology

$\mathbb{HF}^\infty(G)$ (and similarly $\mathbb{HF}^-(G)$ and $\widehat{\mathbb{HF}}(G)$) is generated by the class of $[-1]$ over $\mathbb{F}[U^{-1}, U]$ (and over $\mathbb{F}[U]$ and $\mathbb{F}$, respectively). In particular, $\mathbb{HF}^-_i(G) = 0$ for $i > 0$.

Recall that for the disjoint union $G = G_1 \cup G_2$ of two trees/forests the chain complex of G (and therefore the lattice homology of G) splits as the tensor product of the lattice homologies of G_1 and G_2 (over the coefficient ring of the chosen theory). As a quick corollary we get

Corollary 2.12. *Suppose that $G = G_1 \cup G_2$ where G_2 is the graph encountered in Example 2.11. Then $\mathbb{HF}^-(G) \cong \mathbb{HF}^-(G_1)$. (Similar statements hold for the other versions of the theory.)*

Proof. By the connected sum formula (Equation (2.6)), and by the computation in Example 2.11 we get that

$$\mathbb{HF}^-(G) \cong \mathbb{HF}^-(G_1) \otimes_{\mathbb{F}[U]} \mathbb{HF}^-(G_2) \cong \mathbb{HF}^-(G_1) \otimes_{\mathbb{F}[U]} \mathbb{F}[U] \cong \mathbb{HF}^-(G_1),$$

verifying the statement. □

3. The knot filtration on lattice homology

Denote the vertices of the tree Γ_{v_0} by $V = \mathrm{Vert}(\Gamma_{v_0}) = \{v_0, v_1, \ldots, v_n\}$. Assume that each v_j with $j > 0$ is equipped with a framing $m_j \in \mathbb{Z}$, but leave the vertex v_0 unframed. In the following we will assume that $G = \Gamma_{v_0} - v_0$ is negative definite. The reason for this assumption is that for more general graphs lattice homology provides groups isomorphic to the corresponding Heegaard Floer homology groups only after completion; in particular after allowing infinite sums in the chain complex. For such elements, however, the definition of any filtration requires more care. To avoid these technical difficulties, here we restrict ourselves to the negative definite case.

For a framing $m_0 \in \mathbb{Z}$ on v_0 denote the framed graph we get from Γ_{v_0} by $G_{v_0} = G_{v_0}(m_0)$. (We will always assume that m_0 is chosen in such a way that $G_{v_0}(m_0)$ is also negative definite.) Let $\Sigma \in H_2(X_{G_{v_0}}; \mathbb{Q})$ be a homology class satisfying

$$\Sigma = v_0 + \sum_{j=1}^{n} a_j \cdot v_j \ \ (\text{where } a_j \in \mathbb{Q}), \quad \text{and} \quad v_j \cdot \Sigma = 0 \ \ (\text{for all } j > 0). \tag{3.1}$$

Notice that since $G = \Gamma_{v_0} - v_0$ is assumed to be negative definite, the class Σ exists and is unique. In the next two sections we will follow the convention that characteristic classes on G and subsets of $V - \{v_0\}$ will be denoted by K and E respectively, while the characteristic classes on G_{v_0} and subsets of V will be denoted by L and H, respectively.

Lemma 3.1. *Let us fix a generator* $[K,E] \in \mathrm{Char}(G) \times \mathbb{P}(V - v_0)$ *of the lattice chain complex* $\mathbb{CF}^\infty(G)$ *of* G*. There is a unique element* $L = L_{[K,E]} \in \mathrm{Char}(G_{v_0})$ *with the properties that for* $H_E = E \cup \{v_0\}$*,*

- $L|_G = K$*, and*
- $a_{v_0}[L, H_E] = b_{v_0}[L, H_E] = 0$.

Proof. The equality $a_{v_0}[L, H_E] = b_{v_0}[L, H_E]$ is, by definition, equivalent to $A_{v_0}([L, H_E]) = B_{v_0}([L, H_E])$. By its definition $A_{v_0}([L, H_E]) = g([K, E])$ is independent of $L(v_0)$ (and of the framing $m_0 = v_0^2$ of v_0), while since $K(v_j) = L(v_j)$ for $j > 0$, by Equation (2.3)

$$2B_{v_0}([L, H_E]) = L(v_0) + v_0^2 + 2g([K + 2v_0^*, E]).$$

The identity $2A_{v_0}([L, H_E]) = 2B_{v_0}([L, H_E])$ then uniquely specifies $L(v_0)$:

$$\begin{aligned} L(v_0) &= -v_0^2 + 2g([K, E]) - 2g([K + 2v_0^*, E]) \\ &= -v_0^2 + \min_{I \subset E} \Big(\sum_{v \in I} K(v) + \Big(\sum_{v \in I} v\Big)^2 \Big) \\ &\qquad - \min_{I \subset E} \Big(\sum_{v \in I} K(v) + \Big(\sum_{v \in I} v\Big)^2 + 2v_0 \cdot \Big(\sum_{v \in I} v\Big) \Big). \end{aligned}$$

Since K is characteristic, both minima are even, and therefore $L(v_0) \equiv v_0^2 \pmod 2$, implying that L is also characteristic. □

Definition 3.2. We define the *Alexander grading* $A([K,E])$ of a generator $[K,E]$ of $\mathbb{CF}^\infty(G)$ by the formula

$$A([K, E]) = \tfrac{1}{2}(L(\Sigma) + \Sigma^2) \in \mathbb{Q},$$

where $L = L_{[K,E]}$ is the extension of K found in Lemma 3.1 and Σ is the (rational) homology element in $H_*(X_{G_{v_0}}; \mathbb{Q})$ associated to v_0 in Equation (3.1). (In the above formula we regard $L \in H^2(X_{G_{v_0}}; \mathbb{Z})$ as a cohomology class with rational coefficients.) Notice that since $v_j \cdot \Sigma = 0$ for all $j > 0$, the above expression is equal to $\frac{1}{2}(L(\Sigma) + v_0 \cdot \Sigma)$. We extend this grading to expressions of the form $U^j \otimes [K, E]$ with $j \in \mathbb{Z}$ by

$$A(U^j \otimes [K, E]) = -j + A([K, E]).$$

In the definition above we fixed a framing m_0 on v_0, and it is easy to see that both the values of $L(v_0)$ and of $\Sigma^2 = v_0 \cdot \Sigma$ depend on this choice.

Lemma 3.3. *The value* $A([K,E])$ *is independent of the choice of the framing* $m_0 = v_0^2$ *of* v_0.

Proof. By the identities of Lemma 3.1 it is readily visible that $L(v_0)$ (and hence $L(\Sigma)$) changes by -1 if v_0^2 is replaced by v_0^2+1. Since Σ^2 changes exactly as v_0^2 does, the sum $L(v_0)+\Sigma^2$ (and hence $\frac{1}{2}(L(\Sigma)+\Sigma^2)$) does not depend on the chosen framing v_0^2 on v_0. □

Since Σ is not an integral homology class, there is no reason to expect that $A([K,E])$ is an integer in general. On the other hand, it is easy to see that if K, K' represent the same spin^c structure then $A([K,E]) - A([K',E'])$ is an integer: if $K' = K + 2y^*$ (with $y \in H_2(X_G;\mathbb{Z})$) then

$$A([K,E]) - A([K',E']) = \tfrac{1}{2}(L_{[K,E]} - L_{[K',E']})(v_0) \in \mathbb{Z}$$

since $y \cdot \Sigma = 0$ and both $L_{[K,E]}$ and $L_{[K',E']}$ are characteristic cohomology classes.

Definition 3.4. For each spin^c structure $\mathbf{s}$ of G there is a rational number $i_{\mathbf{s}} \in [0,1)$ with the property that mod 1 the Alexander grading $A([K,E])$ for a pair $[K,E]$ with $\mathbf{s}_K = \mathbf{s}$ is congruent to $i_{\mathbf{s}}$.

Remark 3.5. For a rational homology sphere Y and a knot $K \subset Y$ the Alexander grading defined in Heegaard Floer homology is generally not an integer. On the other hand, for a fixed spin^c structure $\mathbf{s}$ all generators representing $\mathbf{s}$ have Alexander gradings which differ by integers. Therefore the mod 1 residue of the Alexander grading of a generator is an invariant of the spin^c structure, giving rise to a similar rational number in $[0,1)$ in Heegaard Floer homology as $i_{\mathbf{s}}$ defined above in the lattice homology context.

Definition 3.6. The Alexander grading A of generators naturally defines a filtration $\{\mathcal{F}_i\}$ on the chain complex $\mathbb{CF}^\infty(G)$ (which we will still denote by A and will call the *Alexander filtration*) as follows: an element $x \in \mathbb{CF}^\infty(G)$ is in $\mathcal{F}_i$ if every component of x (when written in the $\mathbb{F}$-basis $U^j \otimes [K,E]$) has Alexander grading at most i. Intersecting the above filtration with the subcomplex $\mathbb{CF}^-(G)$ we get the Alexander filtration A on $\mathbb{CF}^-(G)$. Similarly, the definition provides Alexander filtrations on the chain complexes $\widehat{\mathbb{CF}}(G)$ and $\mathbb{CF}^+(G)$.

Equipped with the Alexander filtration, now $(\mathbb{CF}^\infty(G), \partial)$ is a filtered chain complex, as the next lemma shows.

Lemma 3.7. *The chain complex $\mathbb{CF}^\infty(G)$ (and similarly, $\mathbb{CF}^-(G)$ and $\widehat{\mathbb{CF}}(G)$) equipped with the Alexander filtration A is a filtered chain complex, that is, if $x \in \mathcal{F}_i$ then $\partial x \in \mathcal{F}_i$.*

Proof. We need to show that for a generator $[K,E]$ the inequality $A(\partial[K,E]) \leq A([K,E])$ holds. Recall that $\partial[K,E]$ is the sum of two types of elements. In the following we will deal with these two types separately, and verify a slightly stronger statement for these components.

Let us first consider the component of the boundary of the shape of $U^{a_v[K,E]} \otimes [K, E-v]$ for some $v \in E$. We claim that in this case

$$A([K,E]) - A(U^{a_v[K,E]} \otimes [K, E-v]) = a_v[K + 2v_0^*, E] \tag{3.2}$$

holds, obviously implying that the Alexander grading of this boundary component is not greater than that of $[K, E]$. To verify the identity of (3.2), write Σ as $v_0 + \sum_{j=1}^n a_j \cdot v_j$, and note that twice the left-hand side of Equation (3.2) is equal to

$$K\Big(\sum_{j=1}^n a_j \cdot v_j\Big) + L_{[K,E]}(v_0) + \Sigma^2 + 2g([K, E-v]) - 2g([K,E])$$
$$- K\Big(\sum_{j=1}^n a_j \cdot v_j\Big) - L_{[K,E-v]}(v_0) - \Sigma^2,$$

which, after the simple cancellations and the extensions found in Lemma 3.1, is equal to

$$2g([K,E]) - 2g([K+2v_0^*, E]) + 2g([K, E-v])$$
$$- 2g([K,E]) - 2g([K, E-v]) + 2g([K+2v_0^*, E-v].$$

After further cancellations, this expression gives $2a_v[K + 2v_0^*, E]$, verifying Equation (3.2). Since $a_v \geq 0$, Equation (3.2) concludes the argument in this case.

Next we compare the Alexander grading of the term $U^{b_v[K,E]} \otimes [K+2v^*, E-v]$ to $A([K,E])$. Now we claim that

$$A([K,E]) - A(U^{b_v[K,E]} \otimes [K + 2v^*, E-v]) = b_v[K + 2v_0^*, E]. \tag{3.3}$$

As before, after substituting the defining formulae into the terms of twice the left-hand side of (3.3) we get

$$K\Big(\sum_{j=1}^n a_j \cdot v_j\Big) + L_{[K,E]}(v_0) + \Sigma^2 + 2B_v[K,E] - 2g([K,E])$$
$$- (K + 2v^*)\Big(\sum_{j=1}^n a_j \cdot v_j\Big) - L_{[K+2v^*, E-v]}(v_0) - \Sigma^2.$$

From the fact that $v^*(\Sigma) = 0$ we get that $2v^*(\sum_{j=1}^n a_j \cdot v_j) = -2v \cdot v_0$, hence by considering the form of B_v given in (2.3) we get that this term is equal to

$$2g([K,E]) - 2g([K+2v_0^*, E]) + 2g([K+2v^*, E-v]) + K(v) + v^2 + 2v \cdot v_0$$
$$- 2g([K,E]) - 2g([K+2v^*, E-v]) + 2g([K+2v^*+2v_0^*, E-v]),$$

and this expression is obviously equal to $2b_v[K+2v_0^*, E]$. Once again, since $b_v \geq 0$, the statement of the lemma follows. $\square$

Definition 3.8. We define the filtered chain complex $(\mathbb{CF}^\infty(G), \partial, A)$ (and similarly $(\mathbb{CF}^-(G), \partial, A)$ and $(\widehat{\mathbb{CF}}(G), \partial, A)$) the *filtered lattice chain complex* of the vertex v_0 in the graph Γ_{v_0}.

Remark 3.9. Recall that the chain complex $\mathbb{CF}^-(G)$ splits according to the spin^c structures of the 3-manifold Y_G. By intersecting the Alexander filtration with the subcomplexes $\mathbb{CF}^-(G, \mathbf{s})$ for every spin^c structure $\mathbf{s}$, we get a splitting of the filtered chain complex according to spin^c structures as well. The same remark applies to the $\mathbb{CF}^\infty$ and $\widehat{\mathbb{CF}}$ theories.

Definition 3.10. The *knot lattice homology* $\mathbb{HFK}^-(\Gamma_{v_0})$ (respectively $\mathbb{HFK}^\infty(\Gamma_{v_0})$, $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$) of v_0 in the graph Γ_{v_0} is defined as the homology of the graded object associated to the filtered chain complex $(\mathbb{CF}^-(G), \partial, A)$ (and of $(\mathbb{CF}^\infty(G), \partial, A)$, $(\widehat{\mathbb{CF}}(G), \hat{\partial}, A)$, respectively). As before, the groups $\mathbb{HFK}^-(\Gamma_{v_0})$ (and similarly $\mathbb{HFK}^\infty(\Gamma_{v_0})$ and $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$) split according to the spin^c structures of Y_G, giving rise to the groups $\mathbb{HFK}^-(\Gamma_{v_0}, \mathbf{s})$ for $\mathbf{s} \in \mathrm{Spin}^c(Y_G)$.

Let us fix a spin^c structure $\mathbf{s}$ on Y_G. The group $\mathbb{HFK}^-(\Gamma_{v_0}, \mathbf{s})$ then splits according to the Alexander gradings as

$$\bigoplus\nolimits_a \mathbb{HFK}^-(\Gamma_{v_0}, \mathbf{s}, a),$$

and the components $\mathbb{HFK}^-(\Gamma_{v_0}, \mathbf{s}, a)$ are further graded by the absolute δ-grading (originating from the cardinality of the set E for a generator $[K, E]$) and by the Maslov grading.

The relation between the Alexander filtration and the J-map is given by the following formula:

Lemma 3.11. $A(J[K, E])) = -A([K - 2v_0^*, E])$.

Proof. Recall that $J[K, E] = [-K - \sum_{v \in E} 2v^*, E]$. With the extension L of $-K - \sum_{v \in E} 2v^*$ given by Lemma 3.1, and with the choice $v_0^2 = 0$ we have that

$$2A(J[K, E]) = \Big(-K - \sum_{v \in E} 2v^*\Big)(\Sigma - v_0) + L(v_0) + \Sigma^2.$$

Since $v^*(\Sigma) = 0$, by the definition of $L(v_0)$ and the identity of Remark 2.1 this expression is equal to

$$\begin{aligned} &-K(\Sigma - v_0) + 2v_0 \cdot \Big(\sum_{v \in E} v\Big) + \Sigma^2 + 2g[K, E] \\ &\qquad - 2f[K, E] - 2g[K - 2v_0^*, E] + 2f[K - 2v_0^*, E]. \end{aligned}$$

With the same argument the identity

$$\begin{aligned}2A([K-2v_0^*,E]) &= K(\Sigma-v_0)-2v_0^*(\Sigma-v_0)+L'(v_0)+\Sigma^2\\ &= K(\Sigma-v_0)-\Sigma^2+2g[K-2v_0^*,E]-2g[K,E]\end{aligned}$$

follows (since $v_0\cdot\Sigma=\Sigma^2$ and $v_0^2=0$). Now the identity of the lemma follows from the observation that $f[K,E]-f[K-2v_0^*,E]-v_0\cdot(\sum_{v\in E}v)=0$. □

A variant of the J-map, adapted to the distinguished vertex $v_0\in\Gamma_{v_0}$ (and to the filtration given by v_0) is given as follows. Define $J_{v_0}\colon\mathbb{CF}^\infty(G)\to\mathbb{CF}^\infty(G)$ by the formula

$$[K,E]\mapsto\Big[-K-\sum_{u\in E}2u^*-2v_0^*,E\Big],$$

on a generator $[K,E]$ and extend U-equivariantly and linearly to $\mathbb{CF}^\infty(G)$. It is easy to see that $J_{v_0}^2=\mathrm{Id}$. The result of the previous lemma can be restated as

$$A(J_{v_0}[K,E])=-A[K,E].$$

For the next statement recall from Definition 3.4 the quantity $i_{\mathbf{s}}$ associated to a spin^c structure $\mathbf{s}$ on G.

Lemma 3.12. *The map sending the generator* $[K,E]\in\mathbb{CF}^\infty(G,\mathbf{s})$ *to*

$$U^{i_{\mathbf{s}}-A([K,E])}J_{v_0}[K,E]$$

is a chain map.

Proof. We show first that the application of the above map to $U^{a_v[K,E]}\otimes[K,E-v]$ for some $v\in E$ is equal to

$$U^{i_{\mathbf{s}}-A([K,E])}\cdot U^{b_v[-K-\sum_{u\in E}2u^*-2v_0^*,E]}\otimes\Big[-K-\sum_{u\in E}2u^*-2v_0^*+2v^*,E-v\Big].$$

The identification of $J_{v_0}(U^{a_v[K,E]}\otimes[K,E])$ with the above term easily follows from the observation that

$$a_v[K,E]+i_{\mathbf{s}}-A([K,E-v])=i_{\mathbf{s}}-A([K,E])+b_v\Big[-K-\sum_{u\in E}2u^*-2v_0^*,E\Big].\tag{3.4}$$

Equation (3.4), however, is a direct consequence of the equality

$$b_v\Big[-K-\sum_{u\in E}2u^*-2v_0^*,E\Big]=a_v[K+2v_0^*,E]$$

and the definitions of the terms describing the Alexander gradings. A similar computation shows the identity for the other type of boundary components (involving the terms of the shape $U^{b_v[K,E]}\otimes[K+2v^*,E-v]$), concluding the proof. □

Examples 3.13. Two examples of the filtered chain complexes associated to certain graphs can be determined as follows. Since both examples describe the unknot in S^3, it is not surprising that the filtered chain complexes are filtered chain homotopy equivalent. (These examples will be used in later arguments.)

- Consider first the graph Γ_{v_0} with two vertices $\{v_0, v\}$, connected by a single edge, and with (-1) as the framing of v. The chain complex of $G = \Gamma_{v_0} - v_0$ has been determined in Example 2.11. A straightforward calculation shows that $A([2n+1]) = n+1$ and
$$A([2n+1, \{v\}]) = \begin{cases} n+1 & \text{if } n \geq 0, \\ n & \text{if } n < 0. \end{cases}$$
This formula then describes the Alexander filtration on $\mathbb{CF}^-(G)$. (Recall that $A(U^i \otimes [K, E]) = -j + A([K, E])$.) It is easy to see that the chain homotopy encountered in Example 2.11 respects the Alexander filtration, hence the filtered lattice chain complex $(\mathbb{CF}^\infty(G), A)$ is filtered chain homotopic to $\mathbb{F}[U^{-1}, U]$, generated by the element g in filtration level 0. In conclusion, $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$ and $\mathbb{HFK}^-(\Gamma_{v_0})$ are both generated by the element $[-1]$ (over $\mathbb{F}$ and $\mathbb{F}[U]$, respectively), and the Alexander and Maslov gradings of the generator are both equal to 0.
- In the second example consider the graph Γ'_{v_0} on the same two vertices $\{v_0, v\}$, now with no edges at all. (That is, Γ'_{v_0} is given from Γ_{v_0} by erasing the single edge of Γ_{v_0}.) The background graph G (and hence the chain complex $\mathbb{CF}^-(G)$) is obviously the same as in the first example, but the Alexander grading A' is much simpler now: $A'([2n+1]) = A'([2n+1, \{v\}]) = 0$ for all $n \in \mathbb{Z}$. Once again, the chain homotopy of Example 2.11 is a filtered chain homotopy, hence we can apply it to determine the filtered lattice chain complex of Γ'_{v_0}, concluding that $(\mathbb{CF}^\infty(G), A')$ is filtered chain homotopic to $\mathbb{F}[U^{-1}, U]$ with the generator in Alexander grading 0. Once again $\mathbb{HFK}^-(\Gamma'_{v_0})$ is generated by $[-1]$.

In conclusion, the filtered chain complexes of the two examples are filtered chain homotopic to each other. The filtered homotopy between the two examples is not a surprise: the two filtered chain complexes are associated to the unknot U in S^3 and both constructions are motivated by the construction of $\mathrm{CFK}^-(U)$.

4. The master complex and the connected sum formula

As we will see in the next section, the filtered chain complexes defined in the previous section (together with certain maps, to be discussed below) contain all the relevant information we need for calculating the lattice homologies of graphs we get by attaching various framings to v_0. The Alexander filtration A on $\mathbb{CF}^\infty(G)$ can be enhanced to a double filtration by considering the double grading

$$U^j \otimes [K,E] \mapsto (-j, A(U^j \otimes [K,E])). \tag{4.1}$$

In fact, this doubly filtered chain complex determines (and is determined by) the filtered chain complex $(\mathbb{CF}^-(G), A)$. Notice that multiplication by U decreases Maslov grading by 2, $-j$ by 1 and Alexander grading by 1.

In describing the further structures we need, it is slightly more convenient to work with $\mathbb{CF}^\infty(G)$, and therefore we will consider the doubly filtered chain complex above. In the following we will find it convenient to equip $\mathbb{CF}^\infty(G)$ with the following map.

Definition 4.1. The map $N\colon \mathbb{CF}^\infty(G) \to \mathbb{CF}^\infty(G)$ is defined by the formula

$$N(U^j \otimes [K,E]) = U^{i_{\mathbf{s}_K} - A[K,E]+j} \otimes [K + 2v_0^*, E]. \tag{4.2}$$

Notice that N does not preserve the spin^c structure of a given element. Indeed, if $\mathbf{s}_{v_0}$ denotes the spin^c structure we get by twisting $\mathbf{s}$ with v_0^* (and hence we get $c_1(\mathbf{s}_{v_0^*}) = c_1(\mathbf{s}) + 2v_0^*$), then N maps $\mathbb{CF}^\infty(G,\mathbf{s})$ to $\mathbb{CF}^\infty(G,\mathbf{s}_{v_0})$. By choosing another rational number r (with $r \equiv i_{\mathbf{s}_K} \mod 1$) instead of $i_{\mathbf{s}_K}$ in the above formula, we get only multiples of N (multiplied by appropriate monomials of U).

Lemma 4.2. *The map N is a chain map, and provides an isomorphism between the chain complex $\mathbb{CF}^\infty(G,\mathbf{s})$ and $\mathbb{CF}^\infty(G,\mathbf{s}_{v_0^*})$.*

Proof. The fact that N is a chain map follows from the identities

$$a_v[K,E] - A([K,E-v]) = a_v[K+2v_0^*,E] - A([K,E]) \tag{4.3}$$

and

$$b_v[K,E] - A([K+2v^*,E-v]) = b_v[K+2v_0^*,E] - A([K,E]). \tag{4.4}$$

These identities follow easily from the definitions of the terms. To show that N is an isomorphism, let the spin^c structure $\mathbf{s}_{-v_0^*}$ be denoted by $\mathbf{t}$ and consider the map

$$M(U^j \otimes [K,E]) = U^{A([K-2v_0^*,E])+j-i_{\mathbf{t}}} \otimes [K-2v_0^*,E].$$

M is also a chain map (as the identities similar to (4.3) and (4.4) show), and M and N are inverse maps. It follows therefore that N is an isomorphism between chain complexes. □

Notice that N can be written as the composition of the J-map with the map $U^{i_{\mathbf{s}}-A([K,E])}J_{v_0}[K,E]$ considered in Lemma 3.12.

Definition 4.3. Suppose that for $i = 1, 2$ the triples (C_i, A_i, j_i) are doubly filtered chain complexes and $N_i : C_i \to C_i$ are given maps. Then the map $f : C_1 \to C_2$ is an *equivalence* of these structures if f is a (doubly) filtered chain homotopy equivalence commuting with N_i, that is, $f \circ N_1 = N_2 \circ f$.

With this definition at hand, now we can define the *master complex* of Γ_{v_0} as follows.

Definition 4.4. Suppose that Γ_{v_0} is given. Consider $\mathbb{CF}^\infty(G)$ with the double filtration $(-j, A)$ as above, together with the map N defined in Definition 4.1. The equivalence class of the resulting structure is the *master complex* of Γ_{v_0}.

As a simple example, a model for the master complex for each of the two cases in Example 3.13 can be easily determined: regarding the map $U^j \otimes [K, E] \mapsto (-j, A(U^j \otimes [K, E]))$ as a map into the plane, (a representative of) the master complex will have a $\mathbb{Z}_2$ term for each coordinate (i, i), and all other terms (and all differentials) are zero. In addition, the map N in this model is equal to the identity. (Note that in this case the background 3-manifold is diffeomorphic to S^3, hence admits a unique spin^c structure.) In short, the master complex for both cases in Example 3.13 is $\mathbb{F}[U^{-1}, U]$, with the Alexander grading of U^j being equal to j and with $N = id$.

Obviously, by fixing a spin^c structure $\mathbf{s} \in \mathrm{Spin}^c(Y_G)$ we can consider the part $\mathbb{MCF}^\infty(\Gamma_{v_0}, \mathbf{s})$ of the master complex generated by those elements $U^j \otimes [K, E]$ which satisfy the constraint $\mathbf{s}_K = \mathbf{s}$. As we noted earlier, N maps components of the master complex corresponding to various spin^c structures into each other.

4.1. The connected sum formula.

Suppose that Γ_{v_0} and Γ'_{w_0} are two graphs with distinguished vertices v_0, w_0. Their connected sum is defined in the following:

Definition 4.5. Let Γ_{v_0} and Γ'_{w_0} be two graphs with distinguished vertices v_0 and w_0. Their *connected sum* is the graph obtained by taking the disjoint union of Γ_{v_0} and Γ'_{w_0}, and then identifying the distinguished vertices $v_0 = w_0$. The resulting graph

$$\Delta_{(v_0 = w_0)} = \Gamma_{v_0} \#_{(v_0 = w_0)} \Gamma'_{w_0}$$

(which will be a tree/forest provided both Γ_{v_0} and Γ'_{w_0} were trees/forests) has a distinguished vertex $v_0 = w_0$.

Remark 4.6. Notice that this construction gives the connected sum of the two knots specified by v_0 and w_0 in the two 3-manifolds Y_G and $Y_{G'}$.

Recall that for the disjoint graphs $G = \Gamma_{v_0} - v_0$ and $G' = \Gamma'_{w_0} - w_0$ the chain complex $\mathbb{CF}^\infty(G \cup G')$ of their connected sum is simply the tensor product of $\mathbb{CF}^\infty(G)$ and $\mathbb{CF}^\infty(G')$ (over $\mathbb{F}[U^{-1}, U]$). We will denote the Alexander grading/filtration on $\mathbb{CF}^\infty(G)$ by A_{v_0} and on $\mathbb{CF}^\infty(G')$ by A_{w_0}.

Theorem 4.7. *For the Alexander grading $A_\#$ of the generator $[K_1, E_1] \otimes [K_2, E_2] \in \mathbb{CF}^\infty(G \cup G')$ induced by the distinguished vertex $v_0 = w_0$ in $\Delta_{(v_0=w_0)}$ we have that*

$$A_\#([K_1, E_1] \otimes [K_2, E_2]) = A_{v_0}([K_1, E_1]) + A_{w_0}([K_2, E_2]).$$

Proof. For simplicity fix $v_0^2 = w_0^2 = 0$ and consider Σ_{v_0} and Σ_{w_0} on the respective sides of the connected sum. By the calculation from Lemma 3.1 it follows that for the extensions L_i of K_i over the distinguished points v_0, w_0, and extension L over $v_0 = w_0$ we have

$$L_{E_1 \cup E_2}(v_0 = w_0) = (L_1)_{E_1}(v_0) + (L_2)_{E_2}(w_0).$$

Since $\Sigma^2_{v_0=w_0} = (\Sigma_{v_0} + \Sigma_{w_0})^2 = \Sigma^2_{v_0} + \Sigma^2_{w_0}$, the above equality shows that both terms of the defining equation of the Alexander grading are additive, concluding the result. □

As a corollary, we can now show that

Theorem 4.8. *The master complexes of Γ_{v_0} and Γ'_{w_0} determine the master complex of the connected sum $\Delta_{(v_0=w_0)}$.*

Proof. As we saw above, the chain complexes for Γ_{v_0} and Γ'_{w_0} determine the chain complex of $\Delta_{(v_0=w_0)}$ by taking their tensor product. This identity immediately shows that the j-filtration on the result is determined by the j-filtrations on the components. The content of Theorem 4.7 is that the Alexander filtration on the connected sum is also determined by the Alexander filtrations of the pieces. Finally, the map N is built from the maps J and J_{v_0}, which simply add for the connected sum, implying the result. A minor adjustment is needed in the last step: if $i_{\mathbf{s}}$ and $i_{\mathbf{s}'}$ are the rational numbers determined by Definition 3.4 for the spin^c structures $\mathbf{s}$ and $\mathbf{s}'$, then for $\mathbf{s} \# \mathbf{s}'$ we take either their sum (if it is in $[0, 1)$) or $i_{\mathbf{s}} + i_{\mathbf{s}'} - 1$. □

As a simple application of this formula, consider a graph Γ_{v_0} and associate to it two further graphs as follows. Both graphs are obtained by adding a further element e to $\mathrm{Vert}(\Gamma_{v_0})$, equipped with the framing (-1). We can proceed in the following two ways:

(1) Construct $\Gamma^+_{v_0}$ by adding an edge connecting e and v_0 to Γ_{v_0}.

(2) Define $\Gamma^d_{v_0}$ by simply adding e (with the fixed framing (-1)) without adding any extra edge.

For a pictorial presentation of the two graphs, see Figure 2. It is easy to see that $\Gamma^+_{v_0}$ is the connected sum of Γ_{v_0} and the first example in 3.13, while $\Gamma^d_{v_0}$ is the connected sum of Γ_{v_0} and the second example of 3.13. Since the master complexes of the two graphs of Example 3.13 coincide, we conclude that

Corollary 4.9. *The master complexes* $\mathbb{MCF}^\infty(\Gamma^+_{v_0})$ *and* $\mathbb{MCF}^\infty(\Gamma^d_{v_0})$ *are equal. In fact, both master complexes are equal to* $\mathbb{MCF}^\infty(\Gamma_{v_0})$.

Proof. Both master complexes are the tensor product (over $\mathbb{F}[U^{-1}, U]$) of the master complex of Γ_{v_0} and of $\mathbb{F}[U^{-1}, U]$, concluding the argument. □

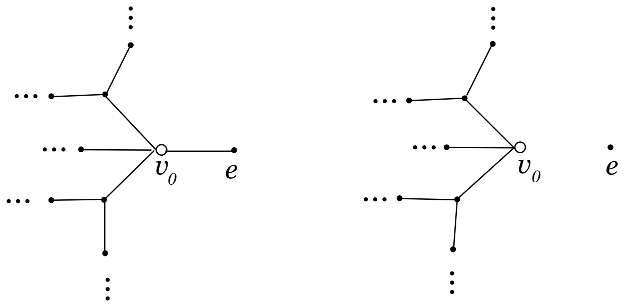

Figure 2. **The two graphs $\Gamma^+_{v_0}$ (on the left) and $\Gamma^d_{v_0}$ (on the right) derived from a given graph Γ_{v_0}.** The framing of e is (-1) in both cases, and v_0 is the distinguished vertex (hence admits no framing and is denoted by a hollow circle) in both graphs.

5. Surgery along knots

A formula for computing the lattice homology for the graph G_{v_0} (we get from Γ_{v_0} by attaching appropriate framing to v_0) can be derived from the knowledge of the master complex of Γ_{v_0}, according to the following result:

Theorem 5.1. *The master complex* $\mathbb{MCF}^\infty(\Gamma_{v_0})$ *of* Γ_{v_0} *determines the lattice homology of the result of the graph obtained by marking* v_0 *with any integer* $n \in \mathbb{Z}$, *for which the resulting graph is negative definite.*

In order to verify this result, first we describe the chain complex computing lattice homology as a mapping cone of related objects. As before, consider the tree Γ_{v_0} in which each vertex except v_0 is equipped with a framing. The plumbing graph G is then given by deleting v_0 from Γ_{v_0}. Let $G_{v_0} = G_{v_0}(n)$ denote the plumbing graph we get from Γ_{v_0} by attaching the framing $n \in \mathbb{Z}$ to v_0. Suppose that for the chosen n the graph G_{v_0} is negative definite. Our immediate aim is to present the chain complex $\mathbb{CF}^-(G_{v_0})$ as a mapping cone of related objects. These related objects then will be reinterpreted in terms of the master complex $\mathbb{MCF}^\infty(\Gamma_{v_0})$.

Consider the two-step filtration on $\mathbb{CF}^-(G_{v_0})$ where the filtration level of $U^j \otimes [L, H]$ is 1 or 0 according to whether v_0 is in H or v_0 is not in H. Denoting the elements with filtration at most 0 by $\mathbb{B}$, we get a short exact sequence

$$0 \longrightarrow \mathbb{B} \longrightarrow \mathbb{CF}^-(G_{v_0}) \longrightarrow \mathbb{D} \longrightarrow 0.$$

Explicitly, $\mathbb{B}$ is generated (over $\mathbb{F}[U]$) by pairs $[L,H]$ with $v_0 \notin H$, while a nontrivial element in $\mathbb{D}$ can be represented by (linear combinations of) terms $U^j \otimes [L,H]$ where $v_0 \in H$. Indeed, the quotient complex $\mathbb{D}$ can be identified with the complex $(\mathbb{T}, \partial_{\mathbb{T}})$, where $\mathbb{T}$ is generated over $\mathbb{F}[U]$ by those elements $[L,H]$ of $\mathrm{Char}(G) \times \mathbb{P}(V)$ for which $v_0 \in H$, and

$$\partial_{\mathbb{T}}[L,H] = \sum_{v \in H - v_0} U^{a_v[L,H]} \otimes [L, H-v] + \sum_{v \in H - v_0} U^{b_v[L,H]} \otimes [L + 2v^*, H - v].$$

Notice that there are two obvious maps $\partial_1, \partial_2 \colon \mathbb{T} \to \mathbb{B}$: For a generator $[L,H]$ of $\mathbb{T}$ (with $v_0 \in H$) consider

$$\partial_1[L,H] = U^{a_{v_0}[L,H]} \otimes [L, H - v_0], \quad \partial_2[L,H] = U^{b_{v_0}[L,H]} \otimes [L + 2v_0^*, H - v_0]. \tag{5.1}$$

It follows from $\partial^2 = 0$ that both maps $\partial_1, \partial_2 \colon \mathbb{T} \to \mathbb{B}$ are chain maps. It is easy to see that

Lemma 5.2. *The mapping cone of* $(\mathbb{T}, \mathbb{B}, \partial_1 + \partial_2)$ *is chain homotopy equivalent to the chain complex* $\mathbb{CF}^-(G_{v_0}(n)))$ *computing the lattice homology* $\mathbb{HF}^-(G_{v_0}(n))$ *of the result of* n*-surgery on* v_0. ☐

Next we identify the above terms using the Alexander filtration on $\mathbb{CF}^\infty(G)$ induced by v_0. We will use the class Σ characterized in Equation (3.1).

Definition 5.3. Consider the subcomplex $B_i \subset \mathbb{B} \subset \mathbb{CF}^-(G_{v_0})$ generated by $[L,H]$ where $\frac{1}{2}(L(\Sigma) + \Sigma^2) = i \in \mathbb{Q}$. (Recall that since $[L,H]$ is in $\mathbb{B}$, the set H does not contain v_0. Also, as before, we regard $L \in H^2(X_{G_{v_0}}; \mathbb{Z})$ as a cohomology class with rational coefficients.) Since $v_j^*(\Sigma) = v_j \cdot \Sigma = 0$ for all $j \neq 0$, it follows that B_i is, indeed, a subcomplex of $\mathbb{B}$ for any rational i, and obviously $\bigoplus_{i \in \mathbb{Q}} B_i = \mathbb{B}$.

Proposition 5.4. *There is an isomorphism* $\varphi \colon B_i \to B_{i+1}$.

Proof. Define the map φ by sending a generator $[L,H]$ of B_i to $[L',H]$ where

$$L'(v_j) = \begin{cases} L(v_0) + 2 & \text{if } j = 0, \\ L(v_j) & \text{if } j \neq 0. \end{cases}$$

Since $v_0 \notin H$, it follows that $f([L,H]) = f([L',H])$ (where f is defined in Equation (2.1)), hence the resulting map is an isomorphism between the chain complexes B_i and B_{i+1}. ☐

Proposition 5.5. *The sum* $B = \bigoplus_{0 \le i < 1} B_i$ *is isomorphic to* $\mathbb{CF}^-(G)$.

Proof. Consider the map $F' \colon B \to \mathbb{CF}^-(G)$ induced by the forgetful map F' defined as $[L, H] \mapsto [L|_G, H]$. It is easy to see that (since H does not contain v_0) the map F' is a chain map. Indeed, F' is an isomorphism: one needs to check only that every element $[L|_G, H]$ admits a unique lift to $[L, H] \in B_i$ with $0 \le i < 1$. The condition $\frac{1}{2}(L(\Sigma) + \Sigma^2) = \frac{1}{2}L(v_0) + \frac{1}{2}(L|_G)(\Sigma - v_0) + \frac{1}{2}\Sigma^2 \in [0, 1)$ uniquely characterizes the value of $\frac{1}{2}L(v_0)$ by the fact that $L(v_0) \equiv v_0^2 \mod 2$. □

Remark 5.6. Obviously, the same argument shows that for any $r \in \mathbb{Q}$ the sum $\bigoplus_{r \le i < r+1} B_i$ is isomorphic to $\mathbb{CF}^-(G)$.

The above statement admits a spin^c-refined version as follows. Notice first that if we fix a spin^c structure $\mathbf{t}$ on the 3-manifold $Y_{G_{v_0}}$ we get after the surgery, and also fix i, then there is a unique spin^c structure $\mathbf{s}$ on Y_G induced by $(\mathbf{t}, i)$. Indeed, if the cohomology class L satisfies $\mathbf{s}_L = \mathbf{t}$ and $\frac{1}{2}(L(\Sigma) + \Sigma^2) = i$, and L' is another representative of $\mathbf{t}$, then

$$L' = L + \sum_{i=0}^{n} 2n_i v_i^*.$$

In order for L' to be also in B_i, however, the coefficient n_0 of v_0^* in the above sum must be equal to zero, hence $L|_G$ and $L'|_G$ represent the same spin^c structure on Y_G. We will denote this restriction by $(\mathbf{t}, i)|_G$. Then the above isomorphism F' provides

Lemma 5.7. *Let* $B_i(\mathbf{t})$ *be the subcomplex of* B_i *generated by those pairs for which* L *represents the spin*^{c} *structure* $\mathbf{t}$. *The map* F' *provides an isomorphism between* $B_i(\mathbf{t})$ *and* $\mathbb{CF}^-(G, (\mathbf{t}, i)|_G)$.

Proof. By the above discussion it is clear that F' maps $B_i(\mathbf{t})$ to $\mathbb{CF}^-(G, (\mathbf{t}, i)|_G)$. The map is injective, hence to show the isomorphism we only need to verify that F' is onto. Obviously $L(\Sigma) + \Sigma^2 = 2i$ and $L|_G = K$ determines $L(v_0)$, and it is not hard to see that for the resulting cohomology class $\mathbf{s}_L = \mathbf{t}$. □

In conclusion, the complexes $\mathbb{B}$, $B_i(\mathbf{t})$ and $B = \bigoplus_{i \in [0,1)} B_i$ can be recovered from $\mathbb{CF}^-(G)$, and hence from the master complex.

The complex $\mathbb{T}$ also admits a decomposition into $\bigoplus_{i \in \mathbb{Q}} T_i$ where the generator $[L, H]$ with $v_0 \in H$ belongs to T_i if $\frac{1}{2}(L(\Sigma) + \Sigma^2) = i \in \mathbb{Q}$. Notice that the map ∂_1 defined in (5.1) maps T_i into $B_i \subset \mathbb{B}$, while when we apply ∂_2 to T_i, we get a map pointing to $B_{i+v_0^*(\Sigma)} \subset \mathbb{B}$.

Recall that in the definitions of B_i and T_i we used the fixed framing attached to the vertex v_0. In the following we show that the result will be actually independent of this choice. To formulate the result, suppose that for the fixed framing $v_0^2 = n$ the complex $\mathbb{B} = \mathbb{B}(n)$ splits as $\bigoplus_i B_i(n)$ (and similarly, $\mathbb{T} = \mathbb{T}(n)$ splits as $\bigoplus_i T_i(n)$).

Lemma 5.8. *The chain complexes $B_i(n)$ and $B_i(n+1)$ (and similarly $T_i(n)$ and $T_i(n+1)$) are isomorphic.*

Proof. Consider the map $t\colon B_i(n) \to B_i(n+1)$ which sends the generator $[L,H]$ to $[L',H]$ where $L'(v_j) = L(v_j)$ for all $j > 0$ and $L'(v_0) = L(v_0) - 1$. Notice that by changing the framing on v_0 from n to $n+1$ we increase Σ^2 by 1. Since $L'(\Sigma) = L(\Sigma) - 1$, and the above map t is invertible, the claim follows. Since the function f we used in the definition of the boundary map takes the same value for $[L,H]$ as for $[L',H]$, the map t is, indeed, a chain map between the chain complexes. The reasoning for the map $t'\colon T_i(n) \to T_i(n+1)$ is similar. □

Our next goal is to reformulate $\mathbb{T}$ (and its splitting as $\bigoplus_{i\in\mathbb{Q}} T_i$) in terms of the master complex $\mathbb{MCF}^\infty(\Gamma_{v_0})$. As before, recall that for a spin^c structure $\mathbf{t}$ on $Y_{G_{v_0}}$ and i we have a restricted spin^c structure $\mathbf{s} = (\mathbf{t},i)|_G$ on Y_G. Consider the subcomplex $S_i(\mathbf{s}) = S_i((\mathbf{t},i)|_G) \subset \mathbb{CF}^\infty(G,\mathbf{s})$ generated by the elements

$$\{U^j \otimes [K,E] \in \mathbb{CF}^-(G,\mathbf{s}) \mid -j \le 0, A(U^j \otimes [K,E]) \le i\}.$$

Lemma 5.9. *For a spin^c structure $\mathbf{t}$ the chain complex $T_i(\mathbf{t})$ and the subcomplex $S_i((\mathbf{t},i)|_G)$ are isomorphic as chain complexes.*

Proof. Define the map $F = F_i^{\mathbf{t}}\colon T_i(\mathbf{t}) \to S_i((\mathbf{t},i)|_G)$ on the generator $[L,H]$ by the formula

$$F([L,H]) = U^{a_{v_0}[L,H]} \otimes [L|_G, H - v_0].$$

The exponent of U in this expression is obviously nonnegative and the spin^c structure of the image is equal to $(\mathbf{t},i)|_G$. Therefore, in order to show that $F([L,H]) \in S_i((\mathbf{t},i)|_G)$, we need only to verify that

$$A(F([L,H])) \le i = \tfrac{1}{2}(L(\Sigma) + \Sigma^2). \tag{5.2}$$

In fact, we claim that

$$\frac{1}{2}(L(\Sigma) + \Sigma^2) - A(U^{a_{v_0}([L,H])} \otimes [L|_G, H - v_0]) = b_{v_0}[L,H]. \tag{5.3}$$

By substituting the definitions of the various terms in the left hand side of this equation (after multiplying it by 2), and applying the obvious simplifications we get

$$\begin{aligned} &L(v_0) + 2g([L,H-v_0] - 2g([L,H]) + v_0^2 \\ &\quad - 2g([L|_G, H - v_0]) + 2g([L|_G + 2v_0^*, H - v_0]). \end{aligned}$$

Since $g([L|_G, H - v_0]) = g([L, H - v_0])$, this expression is clearly equal to $2b_{v_0}[L,H]$, concluding the argument. Since $b_{v_0}[L,H]$ is nonnegative, Equation (5.3) immediately implies Inequality (5.2).

Finally, a simple argument shows that F is a chain map: The two necessary identities

$$a_{v_0}[L,H] + a_v[L|_G, H - v_0] = a_v[L,H] + a_{v_0}[L, H - v]$$

and

$$a_{v_0}[L,H] + b_v[L|_G, H - v_0] = b_v[L,H] + a_{v_0}[L + 2v^*, H - v]$$

are reformulations of Equations (2.4) and (2.5) (together with the observation that $f(L|_G, I) = f(L, I)$ once $v_0 \notin I$).

Next we show that F is an isomorphism. For $[K,E]$ on G there is a unique extension $[L,H]$ on G_{v_0} with $[L|_G, H - v_0] = [K,E]$ and $\frac{1}{2}(L(\Sigma) + \Sigma^2) = i$, hence the injectivity of F easily follows. To show that F is onto, fix an element $U^j \otimes [K,E] \in S_i((\mathbf{t}, i)|_G)$ and consider $[L,H] \in T_i(\mathbf{t})$ with $F([L,H]) = U^{a_{v_0}[L,H]} \otimes [K,E]$. If $a_{v_0}[L,H] = 0$ then $U^j \otimes [L,H]$ maps to $U^j \otimes [K,E]$ under F. In case $a_{v_0}[L,H] > 0$ then $b_{v_0}[L,H] = 0$ and so by the identity of (5.3) we get that $A(U^{a_{v_0}[L,H]} \otimes [K,E]) = i$. Therefore $A(U^j \otimes [K,E]) \leq i$ implies that $j \geq a_{v_0}[L,H]$, hence $U^{j - a_{v_0}[L,H]} \otimes [L,H]$ is in $T_i(\mathbf{t})$ and maps under F to $U^j \otimes [K,E]$, concluding the proof. □

The subcomplexes of $\mathbb{T}$ admit a certain symmetry, induced by the J-map.

Lemma 5.10. *The J-map induces an isomorphism J_i between the chain complexes T_i and T_{-i}. This isomorphism intertwines the maps ∂_1 and ∂_2; more precisely ∂_2 on T_i is equal to $J_i^{-1} \circ \partial_1 \circ J_i$ (and ∂_1 on T_i is equal to $J_i^{-1} \circ \partial_2 \circ J_i$).*

Proof. Recall the definition $J[L,H] = [-L - \sum_{v \in H} 2v^*, H]$ of the J-map on the chain complex $\mathbb{CF}^-(G_{v_0})$. Applying it to the complex T_i, we claim that we get a chain complex isomorphism $J_i \colon T_i \to T_{-i}$: from the fact $(-L - \sum_{v \in H} 2v^*)(\Sigma) = -L(\Sigma) - 2v_0 \cdot \Sigma$ (since $v_0 \in H$ and for all other v_i we have that $v_i \cdot \Sigma = 0$) together with the observation that $\Sigma^2 = v_0 \cdot \Sigma$, it follows that

$$\frac{1}{2}((-L - \sum_{v \in H} 2v^*)(\Sigma) + \Sigma^2) = \frac{1}{2}(-L(\Sigma) - \Sigma^2) = -\frac{1}{2}(L(\Sigma) + \Sigma^2).$$

This equation shows that J_i maps T_i to T_{-i}. The claim $\partial_2 = J_i^{-1} \circ \partial_1 \circ J_i$ (where ∂_2 is taken on T_i while ∂_1 on T_{-i}) then simply follows from the identities of (2.7) in Lemma 2.4. □

The same idea as above shows that

Lemma 5.11. *The restriction of J to B_i provides an isomorphism $B_i \to B_{-i+v_0^*(\Sigma)}$ of chain complexes.*

Proof. Indeed, if $v_0 \notin H$, then $(-L - \sum_{v\in H} 2v^*)(\Sigma) = -L(\Sigma)$, hence

$$\frac{1}{2}\Big((-L - \sum_{v\in H} 2v^*\Big)(\Sigma) + \Sigma^2\Big) = \tfrac{1}{2}(-L(\Sigma) + \Sigma^2) = -\tfrac{1}{2}(L(\Sigma) + \Sigma^2) + \Sigma^2,$$

and $\Sigma^2 = v_0^*(\Sigma)$. □

Next we identify the two maps ∂_1 and ∂_2 of the mapping cone $(\mathbb{T}, \mathbb{B}, \partial_1 + \partial_2)$ in the filtered lattice chain complex context. Notice that $S_i(\mathbf{s})$ is naturally a subcomplex of $\mathbb{CF}^-(G, \mathbf{s})$; let the inclusion $S_i(\mathbf{s}) \subset \mathbb{CF}^-(G, \mathbf{s})$ be denoted by η_1. It is obvious from the definitions that for the maps F', F of Proposition 5.5 and Lemma 5.9

$$F'(\partial_1[L, H]) = \eta_1(F([L, H])).$$

The subcomplex $S_i(\mathbf{s})$ admits a further natural embedding into the complex $V_i(\mathbf{s})$ which is generated by the elements $\{U^j \otimes [K, E] \mid A(U^j \otimes [K, E]) \le i\}$ in $\mathbb{CF}^\infty(G, \mathbf{s})$. ($V_i(\mathbf{s})$ is the subcomplex of $\mathbb{CF}^\infty(G, \mathbf{s})$ when we regard this latter as an $\mathbb{F}[U]$-module.) Recall that $\mathbf{s}_{v_0}$ denotes the spin^c structure we get from $\mathbf{s}$ by twisting it with v_0^*.

Proposition 5.12. *The subcomplex $V_i(\mathbf{s})$ is isomorphic to $\mathbb{CF}^-(G, \mathbf{s}_{v_0})$.*

Proof. Consider the map $U^{i-i_{\mathbf{s}}}N$ from Definition 4.1 mapping from $\mathbb{CF}^\infty(G, \mathbf{s})$ to $\mathbb{CF}^\infty(G, \mathbf{s}_{v_0})$. It is easy to see that this map provides an isomorphism between $V_i(\mathbf{s})$ and $\mathbb{CF}^-(G, \mathbf{s}_{v_0})$, since

$$j(U^{i-i_{\mathbf{s}}} \otimes N(U^k \otimes [K, E])) = i + k - A([K, E])$$

is nonnegative if and only if $i \ge -k + A([K, E]) = A(U^k \otimes [K, E])$. □

Define now $\eta_2\colon S_i(\mathbf{s}) \to \mathbb{CF}^-(G, \mathbf{s}_{v_0})$ as the composition of the embedding $S_i(\mathbf{s}) \to V_i(\mathbf{s})$ with the map $U^{i-i_{\mathbf{s}}}N$. With this definition in place the identity

$$\eta_2 \circ F = F' \circ \partial_2$$

easily follows:

$$(\eta_2 \circ F)[L, H] = U^{a_{v_0}[L,H]+i-A([L|_G, H-v_0])} \otimes [L|_G + 2v_0^*, H - v_0],$$

$$(F' \circ \partial_2)[L, H] = U^{b_{v_0}[L,H]}[L + 2v_0^*|_G, H - v_0],$$

and the two right-hand side terms are equal by the identity of (5.3). Now we are in the position to turn to the proof of the main result of this section, Theorem 5.1.

Proof of Theorem 5.1. Fix the framing n of v_0 in such a way that $G_{v_0} = G_{v_0}(n)$ is a negative definite plumbing graph. Fix a spin^c structure $\mathbf{t}$ on $Y_{G_{v_o}}$. Our goal is now to determine the chain complex $\mathbb{CF}^-(G_{v_0}, \mathbf{t})$ from the master complex of Γ_{v_0}. As we discussed earlier in this section, it is sufficient to recover the subcomplexes $T_i(\mathbf{t})$, $B_i(\mathbf{t})$ (for $i \in \{q + n \cdot \Sigma^2 \mid n \in \mathbb{N}\}$ for an appropriate $q \in \mathbb{Q}$) and the maps $\partial_1 \colon T_i(\mathbf{t}) \to B_i(\mathbf{t})$ and $\partial_2 \colon T_i(\mathbf{t}) \to B_{i+v_0^*(\Sigma)}(\mathbf{t})$.

Identify $T_i(\mathbf{t})$ with the subcomplex $S_i((\mathbf{t}, i)|_G)$ and $B_i(\mathbf{t})$ with $\mathbb{CF}^-(G, (\mathbf{t}, i)|_G)$ (both as subcomplexes of $\mathbb{CF}^\infty(G, (\mathbf{t}, i)|_G)$) by the maps F and F'. As we showed earlier, the natural embedding of $S_i((\mathbf{t}, i)|_G) \subset \mathbb{CF}^-(G, (\mathbf{t}, i)|_G)$ can play the role of ∂_1, while the embedding $S_i((\mathbf{t}, i)|_G) \to V_i((\mathbf{t}, i)|_G)$ composed with $U^{i-i_{(\mathbf{t},i)|_G}} N$ plays the role of ∂_2 in this model. These subcomplexes and maps are all determined by $\mathbb{CF}^\infty(G)$, the two filtrations and the map N on it. Since by its definition the master complex of Γ_{v_0} equals this collection of data, the theorem is proved. □

5.1. Computation of the master complex. When computing the homology $\mathbb{HF}^-(G_{v_0}(n))$ from $(\oplus S_i, \bigoplus_{k\in\mathbb{Z}} \mathbb{CF}^-(G), \eta_1, \eta_2)$ we can first take the homologies $H_*(S_i)$ and $\mathbb{HF}^-(G)$ and consider the maps $H_*(\eta_1)$ and $H_*(\eta_2)$ induced by η_1, η_2 on these smaller complexes. This method provides more manageable chain complexes to work with, but it also loses some information: the resulting homology will be isomorphic to the homology of the original mapping cone only as a vector space over $\mathbb{F}$, and not necessarily as a module over the ring $\mathbb{F}[U]$. Nevertheless, sometimes this partial information can be applied very conveniently.

As an example, we show how to recover (in favorable situations, like the one considered in Section 6 or in [18]) the knot lattice homology $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$ from the homologies of S_i. Let us consider the following iterated mapping cone. First consider the mapping cones C_i of (S_i, S_{i+1}, ψ_i) for $i = n, n-1$, and then consider the mapping cone $D(n)$ of $(C_n, C_{n-1}, (\phi_{i+1}, \phi_i))$. (For a schematic picture of the chain complex, see Figure 3.) In the next lemma we will still need to use the complexes S_i rather than their homologies.

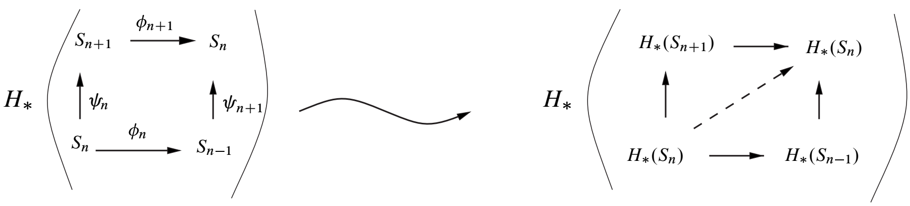

Figure 3. **The iterated mapping cone $D(n)$ on the S_i's.** The maps are defined as ϕ_i, ψ_i with appropriate choices of i on the left, and the homomorphisms induced by these maps on the right. When taking homologies first, we might need to encounter a nontrivial map indicated by the dashed arrow.

Lemma 5.13. *The homology $H_*(D(n))$ is isomorphic to $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$.*

Proof. Factoring S_{n+1} with the image of $\psi_n \colon S_n \to S_{n+1}$ we compute the homology of the horizontal strip in the master complex with $A = n + 1$ and nonnegative U-power (i.e., $j \geq 0$). Similarly, with the help of $\psi_{n-1} \colon S_{n-1} \to S_n$ we get the homology of the horizontal strip with $A = n$ and nonnegative U-power. The iterated mapping cone in the statement maps the upper strip into the lower one by multiplying it by U, localizing the computation to one coordinate with $A = n$ and vanishing U-power. The homology of this complex is by definition the knot lattice homology $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$. □

Unfortunately, if we first take the homologies of the complexes S_i and then form the mapping cones in the above discussion, we might get different homology. The reason is that when taking homologies of the S_i we might need to consider a diagonal map, as indicated by the dashed arrow of Figure 3. Under favorable circumstances (eg. in Section 6 and in [18]), however, the diagonal map can be determined to be zero, and in those cases $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$ can be computed from the homologies of S_i (and the maps induced by ϕ_i, ψ_i on these homologies). From the knowledge of $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$ we can recover the nontrivial groups in the master complex: multiplication by U^n simply translates $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$ (located on the y-axis) with the vectors (n, n) $(n \in \mathbb{Z})$. In some special cases appropriate *ad hoc* arguments help us to reconstruct the differentials and the map N on the master complex (which do not follow from the computation of $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$), getting $\mathbb{MCF}^\infty(\Gamma_{v_0})$ back from $H_*(S_i)$ and the maps $H_*(\Psi_i)$ and $H_*(\Phi_i)$. Such simple calculations are carried out in detail in [18].

Remember also that first taking the homology and then the mapping cone causes some information loss: the result will coincide with the homology of the mapping cone as a vector space over $\mathbb{F}$, but not necessarily as an $\mathbb{F}[U]$-module. The vector space underlying the $\mathbb{F}[U]$-module $\mathbb{HF}^-$ is already an interesting invariant of the graph. The module structure can be reconstructed by considering the mapping cones with coefficient rings $\mathbb{F}[U]/(U^n)$ for every $n \in \mathbb{N}$, cf. [17], Lemma 4.12.

6. An example: the right-handed trefoil knot

In this section we give an explicit computation of the filtered lattice chain complex (introduced in Section 3) for the right-handed trefoil knot in S^3. It is a standard fact that this knot can be given by the plumbing diagram Γ_{v_0} of Figure 4. Notice that in this example the background manifold is diffeomorphic to S^3, hence admits a unique spin^c structure, and therefore we do not need to record it. (Related explicit computations can be found in [13].)

Using the results of [9], [10] first we will determine $H_*(T_i)$ and $H_*(B)$ when the framing $v_0^2 = -7$ is fixed on v_0.

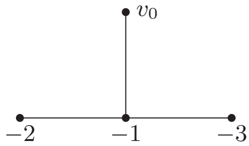

Figure 4. **The plumbing tree Γ_{v_0} describing the right-handed trefoil knot in S^3.** Interpreting the graph as a plumbing tree, the repeated blow-down of the (−1)-, (−2)- and (−3)-framed vertices turn the circle corresponding to v_0 into the right-handed trefoil knot.

Proposition 6.1. *Suppose that Γ_{v_0} is given by the diagram of Figure 4. Then $H_*(B) \cong \mathbb{F}[U]$.*

Proof. The graph $G = \Gamma_{v_0} - v_0$ is negative definite with one bad vertex, hence the result of [10] (cf. also [9]) applies and shows that the lattice homology of it is isomorphic to the Heegaard Floer homology of the 3-manifold Y_G defined by the plumbing. Since G presents S^3 as a 3-manifold and $H_*(B) \cong \mathbb{HF}^-(G)$, the claim follows. □

Consequently the lattice homology group $\mathbb{HF}^-(G) = \mathbb{HF}^-_0(G) \cong H_*(B)$ is generated by a single element, and it has to be a linear combination of elements of the form $[K, E]$ with $E = \emptyset$ (since the entire homology of a negative definite graph with at most one bad vertex is supported in this level). The generator has Maslov grading 0, which by the definition of the grading means that $\frac{1}{4}(K^2 + 3) = 0$, i.e., $K^2 = -3$. There are exactly 8 such cohomology classes on G, and it is easy to verify that these are all homologous to each other (when thought of as cycles in lattice homology), so any one of them can represent the generator of $\mathbb{HF}^-(G) = \mathbb{F}[U]$. By denoting the vertex of G with framing $-i$ by v_i ($i = 1, 2, 3$), we define the vector K as

$$(K(v_1), K(v_2), K(v_3)) = (-1, 0, 1). \tag{6.1}$$

Simple calculation shows that $K^2 = -3$, hence $[K, \emptyset]$ generates $\mathbb{HF}^-(G)$. We will need one further computational fact for the group $\mathbb{HF}^-(G)$:

Lemma 6.2. *The element $[K', \emptyset] \in \mathbb{CF}^-(G)$ given by $(K'(v_1), K'(v_2), K'(v_3)) = (1, 0, 1)$ is homologous to $U \otimes [K, \emptyset]$, where K is given by (6.1) above.*

Proof. Consider the element

$$x = [(1, 0, 1), \{v_1\}] + [(-1, 2, 3), \{v_3\}] + [(1, 2, -3), \{v_1\}] + [(-1, 4, -1), \{v_2\}].$$

It is an easy computation to show that $\partial x = [(1, 0, 1), \emptyset] + U \otimes [(1, 0, -1), \emptyset]$. Since both $[K, \emptyset]$ and $[(1, 0, -1), \emptyset]$ generate $\mathbb{HF}^-(G)$, the proof is complete. □

Before calculating $H_*(T_i)$, we determine the maps $H_*(\partial_1), H_*(\partial_2)\colon H_*(T_i) \to H_*(B)$ on certain elements. To this end, for $j \in \mathbb{Z}$ consider the elements $L_j \in H^2(X_{G_{v_0}};\mathbb{Z})$ (with framing $v_0^2 = -7$ attached to v_0) defined as

$$(L_j(v_1), L_j(v_2), L_j(v_3), L_j(v_0)) = (-1, 0, 1, 2j+1).$$

Since $\Sigma = v_0 + 6v_1 + 3v_2 + 2v_3$, by the choice $v_0^2 = -7$ we get $\Sigma^2 = -1$. This implies that $\frac{1}{2}(L_j(\Sigma) + \Sigma^2) = j-2$, hence the element $[L_j, \{v_0\}]$ is in T_{j-2}. Simple calculation shows that

$$a_{v_0}[L_j, \{v_0\}] = \begin{cases} 0 & \text{if } j-3 \geq 0, \\ -(j-3) & \text{if } j-3 < 0 \end{cases}$$

and

$$b_{v_0}[L_j, \{v_0\}] = \begin{cases} j-3 & \text{if } j-3 \geq 0, \\ 0 & \text{if } j-3 < 0. \end{cases}$$

With notations $a_j = a_{v_0}([L_j, \{v_0\}])$ and $b_j = b_{v_0}([L_j, \{v_0\}])$ we conclude that (with the conventions for K and K' above, and with the identification of B with $\mathbb{CF}^-(G)$)

$$\partial_1[L_j, \{v_0\}] = U^{a_j} \otimes K \quad \text{and} \quad \partial_2[L_j, \{v_0\}] = U^{b_j} \otimes K',$$

and the latter element (according to Lemma 6.2) is homologous to $U^{b_j+1} \otimes K$. This shows that for $j \geq 3$ the homology class of $H_*(T_{j-2})$ represented by the element $[L_j, \{v_0\}]$ maps under (∂_1, ∂_2) to $((-1,0,1), U^{j-2} \otimes (-1,0,1)) \in \mathbb{HF}^-(G) \times \mathbb{HF}^-(G)$. Applying the J-symmetry we can then determine the (∂_1, ∂_2)-image of $J[L_j, \{v_0\}] \in T_{2-j}$ $(j \geq 3)$ as well. (Notice that although $J[L_j, \{v_0\}]$ and $[L_{-j+4}, \{v_0\}]$ are both elements of $T_{-(j-2)}$, they are not necessarily homologous.) For $j = 2$ the class $[L_2, \{v_0\}] \in T_0$ maps to $(U \otimes (-1,0,1), U \otimes (-1,0,1))$. Now we are in the position to determine the homologies $H_*(T_i)$, as well as the maps on them. Notice first that since G represents S^3, the Alexander gradings are all integer valued, hence we have a nontrivial complex T_i for each $i \in \mathbb{Z}$.

Proposition 6.3. *The homology $H_*(T_i)$ is isomorphic to $\mathbb{F}[U]$.*

Proof. Notice first that $H_*(T_i)$ cannot have any nontrivial U-torsion: since ∂_1, ∂_2 map to $H_*(B) = \mathbb{F}[U]$, such part of the homology stays in the kernel of ∂_1 and ∂_2, hence would give nontrivial homology in $\mathbb{HF}^-_1(G_{v_0})$ (supported in $|E| = 1$). This, however, contradicts the fact that for negative definite graphs with at most one bad vertex we have that $\mathbb{HF}^-_1(G_{v_0}) = 0$ [10], [21]. If $i > 0$ and $H_*(T_i)$ is not cyclic, then (by the J-symmetry) the same applies to $H_*(T_{-i})$. Consider the surgery coefficient n with the property that ∂_2 on T_i and ∂_1 on T_{-i} point to the same B. Then $H_*(T_i) \oplus H_*(T_{-i}) \to H_*(B) \oplus H_*(B) \oplus H_*(B)$ will have nontrivial kernel, once again producing nontrivial elements in $\mathbb{HF}^-_1(G_{v_0}(n))$, a group which vanishes for any (negative enough) surgery on v_0. Therefore if $i \neq 0$, the group $H_*(T_i)$ is cyclic with trivial U-torsion, consequently isomorphic to $\mathbb{F}[U]$. For the same reason, $H_*(T_0)$ can have at most two generators, and if it has two generators, then the two maps ∂_1 and ∂_2 have different elements in their kernel. Suppose that $H_*(T_0)$ is not cyclic. In this case (for the choice $v_0^2 = -7$) the $U = 1$ homology can be easily computed and shown to be zero, contradicting the fact that in the single spin^c structure on $Y_{G_{v_0}(-7)}$ this homology is equal to $\mathbb{F}$. This last argument then implies that $H_*(T_0) = \mathbb{F}[U]$ and concludes the proof of the proposition. □

Now our earlier computations of the maps show that for $i > 0$ the map ∂_1 maps $[L_{i+2}, \{v_0\}] \in T_i$ into the generator of $\mathbb{HF}^-(G)$, hence $[L_{i+2}, \{v_0\}]$ generates $H_*(T_i)$. Furthermore, this reasoning shows that ∂_1 is an isomorphism and the map $\partial_2 \colon H_*(T_i) \to \mathbb{HF}^-(G)$ is multiplication by U^i. By the J-symmetry this computation also determines the maps ∂_1, ∂_2 on all $H_*(T_i)$ with $i \neq 0$. On T_0 the situation is slightly more complicated: both maps ∂_1, ∂_2 take $[L_2, \{v_0\}]$ to U-times the generator of $\mathbb{HF}^-(G)$. This can happen in two ways. Either $[L_2, \{v_0\}]$ generates $H_*(T_0)$ (and the maps ∂_1, ∂_2 are both multiplications by U), or the cycle $[L_2, \{v_0\}]$ is homologous to one of the form $U \otimes g$, where g can be represented by a sum of generators (of the form $[L', \{v_0\}]$), each of Maslov grading two greater than the Maslov grading of $[L_2, \{v_0\}]$. Thus, our aim is to show that there are no generators in the requisite Maslov grading.

Specifically, we have that

$$\mathrm{gr}[L_2, \{v_0\}] = -1,$$

while

$$\mathrm{gr}[K, \{v_0\}] = 2g[K, \{v_0\}] + 1 + \tfrac{1}{4}(K^2 + 4),$$

which in turn can be 1 only if $K^2 = -4$ and $g[K, \{v_0\}] = 0$; $K^2 = -4$ implies that $K(v_0) \leq 5$, while $g[K, \{v_0\}] = 0$ implies that $K(v_0) \geq 7$, a contradiction.

We have therefore identified the mapping cone $(\bigoplus_i H_*(T_i), \bigoplus_{k\in\mathbb{Z}} H_*(B), H_*(\partial_1 + \partial_2))$. For a schematic picture of the maps, see Figure 5.

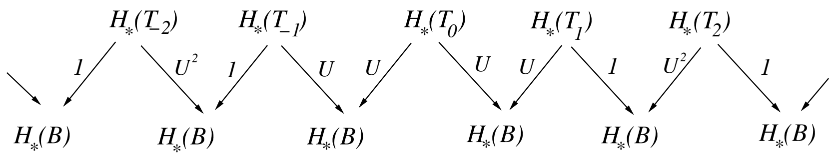

Figure 5. **The schematic diagram of the homology groups of $H_*(T_i)$, of $H_*(B)$ and the maps between them.** All homologies are isomorphic to $\mathbb{F}[U]$, and the maps are all multiplication by some power of U (as indicated in the diagram). The sequence of homologies continue in both directions to $\pm\infty$.

We are now ready to describe the master complex of Γ_{v_0}. We start by determining the groups on the line $j = 0$ — equivalently, we compute $\widehat{\mathbb{HFK}}(\Gamma_{v_0})$. For this computation, the formula of Lemma 5.13 turns out to be rather useful. Indeed, since $H_*(T_i) = \mathbb{F}[U]$, there is no diagonal map in the mapping cone of Figure 3.

The map $H_*(\Psi_i)\colon H_*(T_i) \to H_*(T_{i+1})$ can be determined from the fact that composing it with the map $H_*(T_{i+1}) \to H_*(B)$ we get $H_*(T_i) \to H_*(B)$. Since $\partial_1\colon H_*(T_i) \to H_*(B)$ is an isomorphism for $i \geq 1$, so are all the maps $H_*(\Psi_i)$. Using the same principle for $i = 0$ (and noticing that $H_*(T_0) \to H_*(B)$ is multiplication by U) we get that $H_*(\Psi_0)$ is also multiplication by U. Repeating the same argument it follows that $H_*(\Psi_{-1})$ is an isomorphism, while $H_*(\Psi_i)$ is multiplication by U for all $i \leq -2$. The iterated mapping cone construction of Lemma 5.13 shows that the group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$ vanishes if the two maps $H_*(\Psi_n)$ and $H_*(\Psi_{n-1})$ are the same, and the group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$ is isomorphic to $\mathbb{F}$ is the two maps above differ. (For similar computations see [18].) The computation of the maps $H_*(\Psi_i)$ above shows that

Lemma 6.4. *For Γ_{v_0} given by Figure 4 the knot lattice group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, n)$ is isomorphic to $\mathbb{F}$ for $n = -1, 0, 1$ and vanishes otherwise.* □

Indeed, with the convention used in Equation 6.1, the group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, 1)$ can be represented by

$$x_1 = [(-1, 0, 1), \emptyset],$$

while the group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, -1)$ by

$$x_{-1} = [(-1, 0, -1), \emptyset].$$

It is straightforward to determine the Alexander gradings of these elements, and requires only a little more work to show that these two generators are not boundaries of elements of the same Alexander grading. A quick computation gives that the Maslov grading of x_1 is 0, while the Maslov grading of x_{-1} is -2. Since the homology of the elements with $j = 0$ gives $\mathbb{F}$ in Maslov grading 0 (as the $\widehat{\mathbb{HF}}$-invariant of S^3), we conclude that the generator x_0 of the group $\widehat{\mathbb{HFK}}(\Gamma_{v_0}, 0) = \mathbb{F}$ must be of Maslov grading -1. Furthermore, x_{-1} is one of the components of ∂x_0.

Similarly, since the homology along the line $A = 0$ is also $\mathbb{F}$ (supported in Maslov grading 0), it is generated by $U^{-1} \otimes x_{-1}$ and therefore there is a nontrivial map from x_0 to $U \otimes x_1$. Furthermore, this picture is translated by multiplications by all powers of U, providing nontrivial maps on the master complex. There is no more nontrivial map by simple Maslov grading argument. The filtered chain complex $\mathbb{CF}^\infty(\Gamma_{v_0})$ is then described by Figure 6. (By convention, a solid dot symbolizes $\mathbb{F}$, while an arrow stands for a nontrivial map between the two 1-dimensional vector spaces.) Furthermore, as the map N is U-equivariant, it is equal to the identity. Comparing this result with [24] we get that

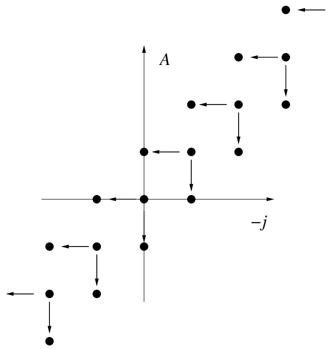

Figure 6. **The schematic diagram of the master complex $\mathbb{MCF}^{\infty}(\Gamma_{v_0})$.** As usual, nontrivial groups are denoted by dots, while nontrivial maps between them are symbolized by arrows.

Proposition 6.5. *The master complex of Γ_{v_0} determined above is filtered chain homotopic to the master complex of the right-handed trefoil knot in Heegaard Floer homology* (*as it is given in* [22]). *Consequently the filtered lattice chain complex of the right-handed trefoil* (*given by Figure* 4) *is filtered chain homotopy equivalent to the filtered knot Floer chain complex of the same knot.* □

Remark 6.6. Essentially the same argument extends to the family of graphs $\{\Gamma_{v_0}(n) \mid n \in \mathbb{N}\}$ we get by modifying the graph Γ_{v_0} of Figure 4 by attaching a string of $(n-1)$ vertices, each with framing (-2) to the (-3)-framed vertex of Γ_{v_0}. The resulting knot can be easily shown to be the $(2, 2n+1)$ torus knot. A straightforward adaptation of the argument above provides an identifications of the filtered chain homotopy types of the master complexes (in lattice homology) of these knots with the master complexes in knot Floer homology.

As an application, consider the connected sum of n trefoil knots. (For a plumbing diagram, see Figure 1.)

Proof of Theorem 1.2. According to Proposition 6.5, together with the connected sum formula for lattice homology and the Künneth formula for knot Floer homology, we get that the two filtered chain complexes for v_0 in Figure 1 (the filtered lattice chain complex and the knot Floer chain complex) are filtered chain homotopic to each other. (See Figure 7 for the master complex we get in the $n = 2$ case.) Equip the vertex v_0 of Figure 1 with framing $m_0 \leq -6n - 1$. Then the corresponding 3-manifold is $(m_0 + 6n)$-surgery on the n-fold connected sum of trefoil knots in S^3. Since the master complex determines the chain complex of the surgery in the same manner in the two theories, the lattice homology of this graph is isomorphic to the Heegaard Floer homology of the corresponding 3-manifold. □

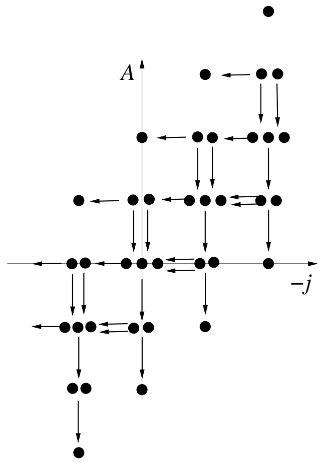

Figure 7. The master complex for the knot $T \# T$ (where T is the right-handed trefoil knot).

Remark 6.7. Notice that this graph has exactly n bad vertices, therefore the above result provides further evidence to the conjectured isomorphism of lattice and Heegaard Floer homologies. (For related results also see [13].) More generally, the identification of the master complexes of knots in S^3 (in fact in any Y_G which is an L-space) is given in [18].

References

[1] V. Colin, P. Ghiggini and K. Honda, HF = ECH via open book decompositions: a summary. Preprint, arXiv:1103.1290.

[2] J. Greene, A surgery triangle for lattice cohomology. *Algebr. Geom. Topol.* **13** (2013), 441–451. Zbl 1280.57028 MR 3031647

[3] P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*. New Math. Monogr. 10, Cambridge University Press, Cambridge 2007. Zbl 1158.57002 MR 2388043

[4] C. Kutluhan, Y.-J. Lee and C. Taubes, HF = HM: Heegaard Floer homology and Seiberg–Witten Floer homology. Preprint, arXiv:1007.1979.

[5] C. Manolescu and P. Ozsváth, Heegaard Floer homology and integer surgeries on links. Preprint, arXiv:1011.1317.

[6] C. Manolescu, P. Ozsváth and S. Sarkar, A combinatorial description of knot Floer homology. *Ann. of Math.* **169** (2009), 633–660. Zbl 1179.57022 MR 2480614

[7] C. Manolescu, P. Ozsváth, Z. Szabó and D. Thurston, On combinatorial link Floer homology. *Geom. Topol.* **11** (2007), 2339–2412. Zbl 1155.57030 MR 2372850

[8] C. Manolescu, P. Ozsváth and D. Thurston, Grid diagrams and Heegaard Floer invariants. Preprint, arXiv:0910.0078.

[9] A. Némethi, On the Ozsváth-Szabó invariant of negative definite plumbed 3-manifolds. *Geom. Topol.* **9** (2005) 991–1042. Zbl 1138.57301 MR 2140997

[10] A. Némethi, Lattice cohomology of normal surface singularities. *Publ. RIMS. Kyoto Univ.* **44** (2008), 507–543. Zbl 1149.14029 MR 2426357

[11] A. Némethi, Some properties of the lattice cohomology. In *Proceedings of the Geometry Conference meeting*, Yamagata University, Japan, September 2010, to appear.

[12] A. Némethi, Two exact sequences for lattice cohomology. In *Noncommutative geometry and global analysis*, Proceedings of the conference organized to honor H. Moscovici's 65th birthday, Contemp. Math. 546, Amer. Math. Soc., Providence, RI, 2011, 249–269. Zbl 1244.32016 MR 2815139

[13] A. Némethi and F. Román, The lattice cohomology of $S^3_{-d}(K)$. In *Zeta functions in algebra and geometry*, Contemp. Math. 566, Amer. Math. Soc., Providence, RI, 2012, 261–292. Zbl 1270.57046 MR 2858927

[14] W. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. Amer. Math. Soc.* **268** (1981), 299–344. Zbl 0546.57002 MR 0632532

[15] P. Ozsváth, A. Stipsicz and Z. Szabó, A combinatorial description of the $U^2 = 0$ version of Heegaard Floer homology. Internat. Math. Res. Notices 2011 **2011**, 5412–5448. Zbl 1248.57010 MR 2855074

[16] P. Ozsváth, A. Stipsicz and Z. Szabó, Combinatorial Heegaard Floer homology and nice Heegaard diagrams. *Adv. in Math.* **231** (2012), 102–171. Zbl 1260.57051 MR 2935385

[17] P. Ozsváth, A. Stipsicz and Z. Szabó, A spectral sequence on lattice homology. *Quantum Topol.*, to appear; preprint, arXiv:1206.1654.

[18] P. Ozsváth, A. Stipsicz and Z. Szabó, Knot lattice homology in L-spaces. Preprint, arXiv:1207.3889.

[19] P. Ozsváth and Z. Szabó, Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math.* **159** (2004), 1027–1158. Zbl 1081.57013 MR 2113020

[20] P. Ozsváth and Z. Szabó, Holomorphic disks and three–manifold invariants: properties and applications. *Ann. of Math.* **159** (2004), 1159–1245. Zbl 1081.57013 MR 2113020

[21] P. Ozsváth and Z. Szabó, On the Floer homology of plumbed three-manifolds. *Geom. Topol.* **7** (2003), 185–224. Zbl 1130.57302 MR 1988284

[22] P. Ozsváth and Z. Szabó, Holomorphic disks and knot invariants. *Adv. Math.* **186** (2004), 58–116. Zbl 1062.57019 MR 2065507

[23] P. Ozsváth and Z. Szabó, Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebr. Geom. Topol.* **8** (2008), 615–692. Zbl 1144.57011 MR 2443092

[24] P. Ozsváth and Z. Szabó, Knot Floer homology and integer surgeries. *Algebr. Geom. Topol.* **8** (2008), 101–153. Zbl 1181.57018 MR 2377279

[25] P. Ozsváth and Z. Szabó, Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.* **11** (2011), 1–68. Zbl 1226.57044 MR 2764036

[26] J. Rasmussen, Floer homology and knot complements. PhD Thesis, Harvard University, 2003, arXiv:math.0306378.

[27] S. Sarkar and J. Wang, An algorithm for computing some Heegaard Floer homologies. *Ann. Math.* **171** (2010), 1213–1236. Zbl 1228.57017 MR 2630063

[28] C. Taubes, Embedded contact homology and Seiberg-Witten Floer cohomology I–V. *Geom. Topol.* **14** (2010), 2497–3000. Zbl 1275.57037 Zbl 1276.57024 Zbl 1276.57025 Zbl 1276.57026 Zbl 1276.57027 MR 2746723 MR 2746724 MR 2746725 MR 2746726 MR 2746727

Received August 14, 2012

Peter Ozsváth, Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.

E-mail: petero@math.princeton.edu

András I. Stipsicz, Rényi Institute of Mathematics, Budapest, Hungary, and Institute for Advanced Study, Princeton, NJ 08540, U.S.A.

E-mail: stipsicz@math-inst.hu

Zoltán Szabó, Department of Mathematics, Princeton University, Princeton, NJ 08544, U.S.A.

E-mail: szabo@math.princeton.edu