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On minimal spheres of area 4π and rigidity

Laurent Mazet and Harold Rosenberg*

Abstract. Let M be a complete Riemannian 3-manifold with sectional curvatures between 0 and 1. A minimal 2-sphere immersed in M has area at least 4π. If an embedded minimal sphere has area 4π, then M is isometric to the unit 3-sphere or to a quotient of the product of the unit 2-sphere with $\mathbb{R}$, with the product metric. We also obtain a rigidity theorem for the existence of hyperbolic cusps. Let M be a complete Riemannian 3-manifold with sectional curvatures bounded above by -1. Suppose there is a 2-torus T embedded in M with mean curvature one. Then the mean convex component of M bounded by T is a hyperbolic cusp, *i.e.*, it is isometric to $T \times \mathbb{R}$ with the constant curvature -1 metric: $e^{-2t} d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T.

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1. Introduction

Consider a smooth (C^∞) complete metric on the 2-sphere S whose curvature is between 0 and 1. It is well known that a simple closed geodesic in S has length at least 2π (see [4] or Klingenberg's theorem in higher dimension [3], [2]). It is less well known that when such an S has a simple closed geodesic of length exactly 2π, then S is isometric to the unit 2-sphere $\mathbb{S}^2_1$. This result is proved in [1], and the authors attribute the theorem to E. Calabi.

With this in mind, we consider what happens in a complete 3-manifold M with sectional curvatures between 0 and 1 (henceforth we suppose this curvature condition on M, unless stated otherwise).

Let Σ be an embedded minimal 2-sphere in M. Then the Gauss–Bonnet theorem and the Gauss equation tells us that the area of S is at least 4π: indeed we have

$$4\pi = \int_\Sigma \bar{K}_\Sigma = \int \det(A) + K_{T\Sigma} \le \int_\Sigma 1 = A(\Sigma) \qquad (1)$$

with $\det(A)$ the determinant of the shape operator which is non-positive. We prove in Theorem 1, that when the area of Σ equals 4π, then M is isometric to the unit

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3-sphere $\mathbb{S}^3_1$ or to a quotient of the product of the unit 2-sphere with $\mathbb{R}$, $\mathbb{S}^2_1 \times \mathbb{R}$, with the product metric.

We remark that Theorem 1 does not hold for embedded minimal tori. Given ε greater than zero, there are Berger spheres with curvatures between 0 and 1, which contain embedded minimal tori of area less than ε. But a minimal sphere always has area at least 4π.

It would be interesting to know what happens in higher dimensions. In the unit n-sphere $\mathbb{S}^n_1$, a compact minimal hyper-surface Σ always has volume at least the volume of the equatorial $n-1$ sphere $\mathbb{S}^{n-1}_1$. Is there a rigidity theorem when one allows metrics on $\mathbb{S}^n$ $(= M)$ of sectional curvatures between 0 and 1? Two questions arise. First, does an embedded minimal hyper-sphere Σ in M have volume at least the volume of $\mathbb{S}^{n-1}_1$. If this is so, and if Σ is an embedded minimal hyper-sphere with volume exactly the volume of $\mathbb{S}^{n-1}_1$, is M isometric to $\mathbb{S}^n_1$ or to $\mathbb{S}^{n-1}_1 \times \mathbb{R}$?

In the same spirit as Theorem 1, we prove a rigidity theorem for hyperbolic cusps. We recall that a 3-dimensional hyperbolic cusp is a manifold of the form $T \times \mathbb{R}$ with T a 2-torus and the hyperbolic metric $e^{-2t} d\sigma_0^2 + dt^2$ with $d\sigma_0^2$ a flat metric on T. In Theorem 2, we prove that if M is a complete Riemannian manifold with sectional curvatures bounded above by -1 and T is a constant mean curvature-1 torus embedded in M then the mean convex side of T in M is isometric to a hyperbolic cusp.

2. Minimal spheres of area 4π and rigidity of 3-manifolds

In this section, we prove a rigidity result for a Riemannian 3-manifold M whose sectional curvatures are between 0 and 1. As explained in the introduction, any minimal sphere in such a manifold has area at least 4π.

We denote by $\mathbb{S}^n_1$ the sphere of dimension n with constant sectional curvature 1. We then have the following result.

Theorem 1. *Let M be a complete Riemannian 3-manifold whose sectional curvatures satisfy $0 \le K \le 1$. Assume that there exists an embedded minimal sphere Σ in M with area 4π. Then the manifold M is isometric either to the sphere $\mathbb{S}^3_1$ or to a quotient of $\mathbb{S}^2_1 \times \mathbb{R}$.*

Proof. Let Φ be the map $\Sigma \times \mathbb{R} \to M$, $(p,t) \mapsto \exp_p(tN(q))$ where N is a unit normal vector field along Σ. In the following we focus on $\Sigma \times \mathbb{R}_+$; by symmetry of the configuration, the study is similar for $\Sigma \times \mathbb{R}_-$.

Σ is compact, so there is an ε such that Φ is an immersion and even an embedding on $\Sigma \times [0,\varepsilon)$. Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } \Sigma \times [0,\varepsilon)\};$$

ε_0 can be equal to $+\infty$. Using Φ, we pull back the Riemannian metric of M to $\Sigma \times [0, \varepsilon_0)$. This metric can be written $ds^2 = d\sigma_t^2 + dt^2$ where $d\sigma_t^2$ is a smooth family of metrics on Σ. With this metric, Φ becomes a local isometry from $\Sigma \times [0, \varepsilon_0)$ to M and $(\Sigma \times [0, \varepsilon_0), ds^2)$ has sectional curvatures between 0 and 1. Moreover, Σ_0 is minimal and has area 4π. Actually, we will prove the following facts.

Claim. *The metric $d\sigma_0^2$ has constant sectional curvature* 1 *so $(\Sigma, d\sigma_0^2)$ is isometric to $\mathbb{S}_1^2$. Moreover, we have two cases:*

(1) $\varepsilon_0 = \pi/2$ *and* $d\sigma_t^2 = \sin^2 t\, d\sigma_0^2$, *or*

(2) $\varepsilon_0 = +\infty$ *and* $d\sigma_t^2 = d\sigma_0^2$.

Let us denote by $\Sigma_t = \Sigma \times \{t\}$ the equidistant surfaces. We denote by $H(p,t)$ the mean curvature of Σ_t at the point (p,t) with respect to the unit normal vector ∂_t. We also define $\lambda(p,t) \geq 0$ such that $H + \lambda$ and $H - \lambda$ are the principal curvature of Σ_t at (p,t). We notice that $\lambda = 0$ if Σ_t is umbilical at (p,t).

The surfaces Σ_t are spheres, so, using the Gauss equation, the Gauss–Bonnet formula implies that

$$4\pi = \int_{\Sigma_t} \bar{K}_{\Sigma_t} = \int_{\Sigma_t} (H+\lambda)(H-\lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where $\bar{K}_{\Sigma_t}$ is the intrinsic curvature of Σ_t and K_t is the sectional curvature of the ambient manifold of the tangent space to Σ_t. Since $K_t \leq 1$, we obtain the following inequality:

$$\int_{\Sigma_t} \lambda^2 = \int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi \qquad (2)$$

where $A(\Sigma_t)$ is the area of Σ_t. In the following, we denote by $F(t)$ the right-hand side of this inequality.

Claim 1. *F is vanishing on* $[0, \varepsilon_0)$.

Since Σ_0 is minimal and has area 4π, we have $F(0) = 0$. We notice that this implies that $\lambda(p,0) = 0$, so Σ_0 is umbilical and $K_{T\Sigma_0} = 1$. Thus $(\Sigma_0, d\sigma_0)$ is isometric to $\mathbb{S}_1^2$.

We have the usual formulae:

$$\frac{\partial}{\partial t} A(\Sigma_t) = -\int_{\Sigma_t} 2H \quad \text{and} \quad \frac{\partial H}{\partial t} = \frac{1}{2}(\mathrm{Ric}(\partial_t) + |A_t|^2) \qquad (3)$$

where A_t is the shape operator of Σ_t and Ric is the Ricci tensor of $\Sigma \times [0, \varepsilon_0)$. Since the sectional curvatures of $M \times [0, \varepsilon_0)$ are non-negative, Ric is non-negative. So the second formula above implies that H is non-decreasing and thus $H \geq 0$ everywhere. Let us now compute and estimate the derivative of F:

$$\begin{aligned} F'(t) &= \int_{\Sigma_t} (2H\frac{\partial H}{\partial t} - 2H^3) - \int_{\Sigma_t} 2H \\ &= \int_{\Sigma_t} H(\mathrm{Ric}(\partial_t) + |A_t|^2 - 2H^2 - 2) \\ &= \int_{\Sigma_t} H\big((\mathrm{Ric}(\partial_t) - 2) + ((H+\lambda)^2 + (H-\lambda)^2 - 2H^2)\big) \\ &= \int_{\Sigma_t} H((\mathrm{Ric}(\partial_t) - 2) + 2\lambda^2) \\ &\leq 2\int_{\Sigma_t} H\lambda^2 \end{aligned}$$

where the last inequality comes from $\mathrm{Ric}(\partial_t) - 2 \leq 0$ because of the hypothesis on the sectional curvatures. If we choose $\varepsilon < \varepsilon_0$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times [0, \varepsilon]$. So for $t \in [0, \varepsilon]$, using the inequality (2), we get $F'(t) \leq 2CF(t)$. Then $F(t) \leq F(0)e^{2Ct} = 0$ on $[0, \varepsilon]$. So $F \leq 0$ on $[0, \varepsilon_0)$ and, because of (2), $F = 0$ on $[0, \varepsilon_0)$; this finishes the proof of Claim 1.

The first consequence of Claim 1 is that all the equidistant surfaces Σ_t are umbilical (see inequality (2)); so $\lambda \equiv 0$. In the computation of the derivative of F, this implies that

$$\int_{\Sigma_t} H(\mathrm{Ric}(\partial_t) - 2) = 0.$$

Since $H(\mathrm{Ric}(\partial_t) - 2) \leq 0$ everywhere, we obtain

$$H(\mathrm{Ric}(\partial_t) - 2) = 0 \quad \text{everywhere.} \tag{4}$$

Moreover the umbilicity and (3) imply that $\dfrac{\partial H}{\partial t} = \frac{1}{2}\mathrm{Ric}(\partial_t) + H^2$. We now prove the following claim.

Claim 2. *Let $(p,t) \in \Sigma \times [0, \varepsilon_0)$ ($t > 0$) be such that $H(p,t) > 0$ then $H(q,t) > 0$ for any $q \in \Sigma$*

In other words, when the mean curvature is positive at a point of an equidistant, it is positive at any point of this equidistant. We recall that H is increasing in the t variable, so when it becomes positive it stays positive.

So assume that $H(p,t) > 0$ and consider $\Omega = \{q \in \Sigma \mid H(q,t) > 0\}$ which is a nonempty open subset of Σ. Let $q \in \Omega$. Since $H(q,t) > 0$, $\mathrm{Ric}(\partial_t)(q,t) = 2$ by (4). Thus $\mathrm{Ric}(\partial_t)(r,t) = 2$ for any $r \in \bar{\Omega}$. So if $r \in \bar{\Omega}$, then, for $s < t$, $\mathrm{Ric}(\partial_t)(r,s) > 0$ for s close to t and, by (3), this implies that $H(r,t) > 0$ and $r \in \Omega$. So Ω is closed and $\Omega = \Sigma$. This finishes the proof of Claim 2.

Let us assume that there is an $\varepsilon_1 > 0$ such that $H(p,t) = 0$ for $(p,t) \in \Sigma \times [0, \varepsilon_1]$ and $H(p,t) > 0$ for any $(p,t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$. Because of the evolution equation of H, this implies that $\mathrm{Ric}(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_1]$. On $\Sigma \times (\varepsilon_1, \varepsilon_0)$, we have $\mathrm{Ric}(\partial_t) = 2$ because of (4). So by continuity of $\mathrm{Ric}(\partial_t)$, we get a contradiction and then we have two possibilities:

(1) $H = 0$ on $\Sigma \times [0, \varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = 0$ on $\Sigma \times [0, \varepsilon_0)$;

(2) $H > 0$ on $\Sigma \times (0, \varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = 2$ on $\Sigma \times [0, \varepsilon_0)$.

In the first case, this implies that the sectional curvature of any 2-plane orthogonal to Σ_t is zero. Thus $d\sigma_t^2 = d\sigma_0^2$. Since the map Φ ceases to be an immersion only if $d\sigma_t^2$ becomes singular this implies that $\varepsilon_0 = +\infty$. Thus $\Sigma \times \mathbb{R}_+$ with the induced metric is isometric to $\mathbb{S}_1^2 \times \mathbb{R}_+$ and Φ is a local isometry from $\mathbb{S}_1^2 \times \mathbb{R}_+$ to M.

In the second case, the sectional curvature of any 2-plane orthogonal to Σ_t is equal to 1. The sectional curvature of Σ_t is also 1, since the inequality in (2) is an equality by Claim 1. Thus $d\sigma_t^2 = \sin^2 t\, d\sigma_0$ and $\varepsilon_0 = \pi/2$. This also implies that $\Phi(p, \pi/2)$ is a point. So $\Sigma \times [0, \pi/2]$ with the metric ds^2 is isometric to a hemisphere of $\mathbb{S}_1^3$ and the map Φ is a local isometry from that hemisphere to M.

Doing the same study for $\Sigma \times \mathbb{R}_-$, we get in the first case a local isometry $\Phi\colon \mathbb{S}_1^2 \times \mathbb{R} \to M$ and in the second case a local isometry $\Phi\colon \mathbb{S}_1^3 \to M$. Since $\mathbb{S}_1^2 \times \mathbb{R}$ and $\mathbb{S}_1^3$ are simply connected, Φ is then the universal cover of M and M is then isometric to a quotient of $\mathbb{S}_1^2 \times \mathbb{R}$ or $\mathbb{S}_1^3$. Since Φ is injective on Σ this implies that in the second case, Φ is actually injective and then a global isometry. □

Remark 1. In the proof, since Φ is injective on Σ, the possible quotients of $\mathbb{S}_1^2 \times \mathbb{R}$ are either $\mathbb{S}_1^2 \times \mathbb{R}$ or its quotient by the subgroup generated by an isometry of the form $\mathbb{S}_1^2 \times \mathbb{R} \to \mathbb{S}_1^2 \times \mathbb{R}$, $(p,t) \mapsto (\alpha(p), t + t_0)$ with α an isometry of $\mathbb{S}_1^2$ and $t_0 \neq 0$.

Remark 2. Something can be said about constant mean curvature H_0 spheres in a Riemannian 3-manifold with sectional curvatures between 0 and 1. Indeed, the computation (1) implies that the area of Σ is larger than $\frac{4\pi}{1+H_0^2}$, which is the area of a geodesic sphere in $\mathbb{S}_1^3$ of mean curvature H_0. Moreover, if Σ has area $\frac{4\pi}{1+H^2}$, the above proof can be adapted to prove that the mean convex side of Σ is isometric to a spherical cap of $\mathbb{S}_1^3$ with constant mean curvature H_0 (see Theorem 2 below, for a similar result in the hyperbolic case).

Remark 3. Let M be a Riemannian n-manifold whose sectional curvatures are between 0 and 1 and let Σ be a minimal 2-sphere in M. A computation similar to (1) proves also that the area of Σ is larger than 4π. It also implies that, if Σ has area 4π, Σ is totally geodesic and isometric to $\mathbb{S}_1^2$.

3. Existence of hyperbolic cusps

Let $(\mathbb{T}^2, g)$ be a flat 2 torus, the manifold $\mathbb{T}^2 \times \mathbb{R}_+$ with the complete Riemannian metric $e^{-2t} g + dt^2$ is a hyperbolic 3-dimensional cusp. $\mathbb{T}^2 \times \mathbb{R}$ is actually isometric to the quotient of a horoball of $\mathbb{H}^3$ by a $\mathbb{Z}^2$ subgroup of isometries of $\mathbb{H}^2$ leaving the horoball invariant. Any $\mathbb{T}^2 \times \{t\}$ has constant mean curvature 1. The following theorem says that, in certain 3-manifolds, a constant mean curvature 1 torus is necessarily the boundary of a hyperbolic cusp.

Theorem 2. *Let M be a complete Riemannian 3-manifold with its sectional curvatures satisfying $K \leq -1$. Assume that there exists a constant mean curvature 1 torus T embedded in M. Then T separates M and its mean convex side is isometric to a hyperbolic cusp.*

As a consequence, the existence of this torus implies that M can not be compact. The proof uses the same ideas as in Theorem 1

Proof. Let us consider the map $\Phi\colon T \times \mathbb{R}_+ \to M$, $(p,t) \mapsto \exp_p(tN(p))$ where N is the unit normal vector field normal to T such that N is the mean curvature vector of T. Let us define

$$\varepsilon_0 = \sup\{\varepsilon > 0 \mid \Phi \text{ is an immersion on } T \times [0,\varepsilon)\}.$$

Using Φ, we pull back the Riemannian metric of M to $T \times [0,\varepsilon_0)$; it can be written $ds^2 = dt^2 + d\sigma_t^2$. We define $T_t = T \times \{t\}$ the equidistant surfaces to T_0. We also denote by $H(p,t)$ the mean curvature of the equidistant surfaces at (p,t) with respect to ∂_t. We finally define $\lambda(p,t)$ such that $H + \lambda$ and $H - \lambda$ are the principal curvatures of T_t at (p,t).

The surfaces T_t are tori so, by the Gauss equation and the Gauss–Bonnet formula, we have

$$0 = \int_{T_t} \bar{K}_{T_t} = \int_{T_t} H^2 - \lambda^2 + K_t$$

where K_t is the sectional curvature of the ambient manifold of the tangent space to T_t. Since $K_t \leq -1$, we obtain the inequality

$$\int_{T_t} \lambda^2 = \int_{T_t} H^2 + K_t \leq \int_{T_t} H^2 - A(T_t).$$

We denote by $F(t)$ the right-hand term of the above inequality. By hypothesis, $H(p,0) = 1$ so $F(0) = 0$ and $F(t) \geq 0$ for any $t \geq 0$. Let us compute the derivative of F:

$$\begin{aligned} F'(t) &= \int_{T_t} \Big(2H\frac{\partial H}{\partial t} - 2H^3\Big) + \int_{T_t} 2H \\ &= \int_{T_t} H\big(\mathrm{Ric}(\partial_t) + |A_t|^2 - 2H^2 + 2\big) = \int_{T_t} H\big((\mathrm{Ric}(\partial_t) + 2) + 2\lambda^2\big). \end{aligned}$$

Since $H(p,0) = 1$, we can consider $\varepsilon \in (0,\varepsilon_0)$ such that $0 < H \le C$ on $T \times [0,\varepsilon]$. Since $\mathrm{Ric}(\partial_t) + 2 \le 0$ we get

$$F'(t) \le \int_{T_t} 2H\lambda^2 \le 2CF(t).$$

Thus $F(t) \le F(0)e^{2Ct}$ for $t \in [0,\varepsilon]$; this implies $F(t) = 0$ on that segment. We then obtain $\lambda = 0$ on $T \times [0,\varepsilon]$ (the equidistant surfaces are umbilical) and $\mathrm{Ric}(\partial_t) = -2$ since $H > 0$. Thus H satisfies the differential equation $\dfrac{\partial H}{\partial t} = -2 + 2H^2$. This gives that $H = 1$ on $T \times [0,\varepsilon]$ since $H = 1$ on T_0. Thus we can let ε tend to ε_0 to obtain that $F(t) = 0$ on $[0,\varepsilon_0)$ and $\mathrm{Ric}(\partial_t) = -2$ and $H = 1$ on $T \times [0,\varepsilon_0)$. Since $0 = \int_{T_t} H^2 + K_t$ and $K_t \le -1$, it follows that $K_t = -1$ for all t in the interval. We then have proved that the sectional curvature of $T \times [0,\varepsilon_0)$ with the metric ds^2 is equal to -1 for any 2-plane. Moreover, we get that $d\sigma_0^2$ is flat and that $d\sigma_t^2 = e^{-2t} d\sigma_0^2$. This implies that Φ is actually an immersion on $T \times \mathbb{R}_+$ ($\varepsilon_0 = +\infty$) and $T \times \mathbb{R}_+$ is isometric to a hyperbolic cusp. Φ is then a local isometry from this hyperbolic cusp to M.

To finish the proof, let us prove that Φ is in fact injective. If this is not the case, let $\varepsilon_1 > 0$ be the smallest ε such that Φ is not injective on $T \times [0,\varepsilon]$. This implies that there exist p and q in T such that either

- $\Phi(p,0) = \Phi(q,\varepsilon_1)$, or
- $\Phi(p,\varepsilon_1) = \Phi(q,\varepsilon_1)$ (with $p \ne q$ in this case).

Let U and V be respective neighborhoods of $(p,0)$ (or (p,ε_1)) in T_0 (or T_{ε_1}) and (q,ε_1) in T_{ε_1} such that Φ is injective on them. Since ε_1 is the smallest one, $\Phi(U)$ and $\Phi(V)$ are two constant mean curvature 1 surfaces in M that are tangent at $\Phi(q,\varepsilon_1)$. Moreover, in the first case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ so by the maximum principle $\Phi(U) = \Phi(V)$. Thus $\Phi(T_0)$ would be equal to $\Phi(T_{\varepsilon_1})$ which is impossible since these two surfaces do not have the same area. In the second case, $\Phi(U)$ is included in the mean convex side of $\Phi(V)$ and then Φ is not injective on T_s for s near t, $s < t$, which is a contradiction. □

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Laurent Mazet, Université Paris-Est, Laboratoire d'Analyse et Mathématiques Appliquées, CNRS UMR8050, UFR des Sciences et Technologie, 61 avenue du Général de Gaulle, 94010 Créteil cedex, France

E-mail: laurent.mazet@math.cnrs.fr

Harold Rosenberg, Instituto de Matematica Pura y Aplicada, 110 Estrada Dona Castorina, Rio de Janeiro 22460-320, Brazil

E-mail: hrosen@free.fr