Comment. Math. Helv. 89 (2014), 929–936
DOI 10.4171/CMH/339

Commentarii Mathematici Helvetici

A Γ-structure on Lagrangian Grassmannians

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Abstract. For n odd the Lagrangian Grassmannian of $\mathbb{R}^{2n}$ is a Γ-manifold.

Mathematics Subject Classification (2010). 53Dxx, 55Mxx.

Keywords. Lagrangian Grassmannian, Gamma manifolds.

1. Introduction and statement of the result

We denote by $(\mathbb{R}^{2n}, \omega)$ the standard symplectic vector space. The (unoriented) Lagrangian Grassmannian $\mathscr{L}$ is the space of all Lagrangian subspaces of $\mathbb{R}^{2n}$. It is a homogeneous space

$$\mathscr{L} \cong \mathrm{U}(n)/\mathrm{O}(n),$$

see [AG01], [MS98]. Every Lagrangian subspace can be identified with the fixed point set of a linear orthogonal anti-symplectic involution. Using this identification, we define a smooth map

$$\Theta \colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$$

by

$$(R, S) \mapsto RSR,$$

which we think of as a product. On every space there are products such as constant maps and projections to one factor. In [Hop41] Hopf introduced the notion of Γ-manifolds which rules these trivial products out. The purpose of this paper is to prove that the above product gives the Lagrangian Grassmannian $\mathscr{L}$ the structure of a Γ-manifold for n odd.

Definition 1.1. A closed, connected, orientable manifold M carries the structure of a Γ-manifold if there exists a map

$$\Theta \colon M \times M \to M$$

such that the maps

$$x \mapsto \Theta(x, y_0) \quad \text{and} \quad y \mapsto \Theta(x_0, y)$$

have non-zero mapping degree for one and thus all pairs $(x_0, y_0) \in M \times M$.

It is well known that $\mathscr{L}$ is orientable if and only if n odd, see [Fuk68]. The main result of this article is the following theorem.

Theorem 1.2. *If n is odd, then $(\mathscr{L}, \Theta)$ is a Γ-manifold.*

Using Hopf's theorem ([Hop41], Satz 1), we get a new proof of the following corollary due to Fuks [Fuk68].

Corollary 1.3 ([Fuk68]). *For n odd, the rational cohomology ring of $\mathscr{L}$ is an exterior algebra on generators of odd degree.*

Remark 1.4. The cohomology ring of the oriented and unoriented Lagrangian Grassmannian was computed by Borel and Fuks for all n, see [Bor53a], [Bor53b], [Fuk68]. A nice summary of these results can be found in Chapter 22 of the book by Vassilyev [Vas88].

The above situation fits into the following general framework. It is well known that $\mathscr{L}$ embeds into $\mathrm{U}(n)$ as the set $\mathrm{U}(n) \cap \mathrm{Sym}(n)$, i.e. the symmetric unitary matrices. Indeed the image of a Lagrangian subspace $\Lambda \subset \mathbb{C}^n$ is the symmetric unitary matrix $A_\Lambda := uu^t \in \mathrm{U}(n) \cap \mathrm{Sym}(n)$ where $a \in \mathrm{U}(n)$ maps $\mathbb{R}^n$ onto Λ. The unique orthogonal anti-symplectic involution with fixed point set Λ is then the map $A_\Lambda \circ \tau$ where $\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ is complex conjugation. Thus, the Lagrangian Grassmannian $\mathscr{L}$ can be interpreted as the fixed point set of the involutive anti-isomorphism $A \mapsto A^T$ of $\mathrm{U}(n)$. On any Lie group G we can define a new product: $(g, h) \mapsto gh^{-1}g$. If $I \colon G \to G$ is an involutive anti-isomorphism then this new product restricts to a product on the fixed point set $\mathrm{Fix}(I)$. This is precisely the situation for the Lagrangian Grassmannian, namely the map Θ corresponds under the embedding of $\mathscr{L}$ into $\mathrm{U}(n)$ to $(g, h) \mapsto gh^{-1}g$.

For general Lie groups this new product does not always give rise to a Γ-structure for various reasons. For example, if we take $G = \mathrm{O}(n)$ resp. $G = \mathrm{U}(n)$ and $I(A) := A^{-1}$, then $\mathrm{Fix}(I)$ can be identified with $\bigcup_k G(k, n)$, the union of all real resp. complex Grassmannians, which is not connected. Another example is $G = \mathrm{SU}(n)$ with $I =$ transposition. Then for $n = 2$ we can identify $\mathrm{Fix}(I) \cong S^2$. But by Hopf's theorem ([Hop41], Satz 1) S^2 is not a Γ-manifold.

Acknowledgements. Parts of this article were written during a visit of the first two authors at the Forschungsinstitut für Mathematik (FIM), ETH Zürich. The authors thank the FIM for its stimulating working atmosphere.

This work is supported by the SFB 878 – Groups, Geometry and Actions (PA), by the Alexander von Humboldt Foundation (UF), by Israel Science Foundation grant 1321/2009 and by Marie Curie grant No. 239381 (JPS).

2. Proof of Theorem 1.2

We recall that the (unoriented) Lagrangian Grassmannian $\mathscr{L}$ is the homogeneous space

$$\mathscr{L} \cong \mathrm{U}(n)/\mathrm{O}(n),$$

see [AG01], [MS98]. Since n is odd, $\mathscr{L}$ is a closed connected orientable manifold [Fuk68]. The space $\mathscr{L}$ is naturally identified with the space of linear orthogonal anti-symplectic involutions of $\mathbb{R}^{2n}$ with the standard symplectic structure. Using this identification, we define the map

$$\Theta\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$$

by $(R, S) \mapsto RSR$. In order to prove Theorem 1.2, it suffices to show for one choice of basepoint R_0 that the mapping degrees of

$$S \mapsto \Theta(R_0, S) \quad \text{and} \quad S \mapsto \Theta(S, R_0)$$

are non-zero. Since

$$S \mapsto \Theta(R_0, S) = R_0 S R_0 \mapsto \Theta(R_0, R_0 S R_0) = R_0 R_0 S R_0 R_0 = S,$$

the first map is an involution and therefore has mapping degree ± 1. The non-trivial case is to compute the mapping degree of

$$\Theta_0(S) := \Theta(S, R_0) = S R_0 S.$$

Theorem 1.2 follows immediately from the following proposition.

Proposition 2.1. *The mapping degree of Θ_0 equals*

$$\deg \Theta_0 = 2^{m+1}$$

where $n = 2m + 1$.

Proof. Identify $\mathbb{R}^{2n} = \mathbb{C}^n$ in the standard way. Denote by $\tau\colon \mathbb{C}^n \to \mathbb{C}^n$ the map given by complex conjugation of all coordinates simultaneously. It is a standard fact, see for instance [MS98], that an orthogonal symplectic map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to a unitary map $\mathbb{C}^n \to \mathbb{C}^n$. It follows that an orthogonal anti-symplectic map

$R\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ can be written as the composition $A \circ \tau\colon \mathbb{C}^n \to \mathbb{C}^n$ for A a unitary linear map. The condition $R^2 = \mathrm{Id}$ then translates to $A\bar{A} = \mathrm{Id}$. So, we identify

$$\mathscr{L} = \{A \in \mathrm{U}(n) \mid A\bar{A} = \mathrm{Id}\}.$$

Under this identification, the map Θ is given by

$$\Theta(A, B) = A\bar{B}A.$$

Let B_0 be the unitary matrix corresponding to R_0. Then the map Θ_0 is given by

$$\Theta_0(A) = \Theta(A, B_0) = A\bar{B}_0A.$$

In the following, we take $B_0 = B$, the diagonal matrix with entries $b_{jk} = e^{i\theta_j}\delta_{jk}$ where

$$0 < \theta_j < 2\pi, \quad \theta_1 < \theta_2 < \cdots < \theta_n.$$

For this choice of B_0, we show that Id is a regular value of Θ_0 and compute the signed cardinality of $\Theta_0^{-1}(\mathrm{Id})$.

Indeed, if $\Theta_0(A) = \mathrm{Id}$, then $A\bar{B}A = \mathrm{Id}$, and therefore $\bar{A}B = A$. Throughout this paper, we do *not* use the Einstein summation convention. Letting a_{jk} denote the matrix entries of A, we have

$$\bar{a}_{jk}e^{i\theta_k} = a_{jk}.$$

Write $a_{jk} = r_{jk}e^{i\psi_{jk}}$, where $r_{jk} \in \mathbb{R}$ and $0 \le \psi_{jk} < \pi$. So,

$$e^{i2\psi_{jk}} = a_{jk}/\bar{a}_{jk} = e^{i\theta_k},$$

and therefore $\psi_{jk} = \theta_k/2$. Writing the unitary condition for A in terms of r_{jk} and ψ_{jk}, we have

$$\delta_{jl} = \sum_k a_{jk}\bar{a}_{lk} = \sum_k r_{jk}r_{lk}e^{i(\psi_{jk}-\psi_{lk})} = \sum_k r_{jk}r_{lk}.$$

Thus r_{jk} is an orthogonal matrix. Furthermore, the condition $A\bar{A} = \mathrm{Id}$ translates to

$$\delta_{jl} = \sum_k a_{jk}\bar{a}_{kl} = \sum_k r_{jk}r_{kl}e^{i(\theta_k-\theta_l)/2}.$$

In particular, taking $j = l$, we obtain

$$1 = \sum_k r_{jk}r_{kj}\cos((\theta_k - \theta_j)/2).$$

Writing $r'_{jk} = \cos((\theta_k - \theta_j)/2)r_{jk}$, we can reformulate the preceding equation in terms of the inner product of the row and column vectors $r'_{j\cdot}$ and $r_{\cdot j}$. Namely,

$$r'_{j\cdot} \cdot r_{\cdot j} = 1. \tag{2.1}$$

On the other hand, since r_{jk} is unitary, $|r_{\cdot j}| = 1$ and

$$|r'_{j\cdot}|^2 = \sum_k r_{jk}^2 \cos^2((\theta_k - \theta_j)/2) \le \sum_k r_{jk}^2 = |r_{j\cdot}|^2 = 1,$$

with equality only if $r_{jk} = 0$ when $k \neq j$. Applying Cauchy–Schwartz to equation (2.1), we have

$$1 \le |r'_{j\cdot}||r_{\cdot j}| = |r'_{j\cdot}| \le 1.$$

Thus equality must hold, and the matrix r_{jk} is diagonal. Moreover, orthogonality implies that $r_{jk} = \pm\delta_{jk}$. Summing up, $A \in \Theta_0^{-1}(\mathrm{Id})$ if and only if we have $A = A^\epsilon$, where

$$\epsilon = (\epsilon_1, \dots, \epsilon_n), \quad \epsilon_k \in \{0, 1\},$$

and A^ϵ is the matrix with elements

$$a_{jk}^\epsilon = e^{i(\theta_k/2 + \epsilon_k \pi)}.$$

In particular, $\Theta_0^{-1}(\mathrm{Id})$ has unsigned cardinality 2^n.

It remains to show that Id is a regular value and compute the signs. Let $\mathrm{Sym}(n)$ denote the space of real $n \times n$ symmetric matrices. It is easy to see that the tangent space to $\mathscr{L}$ at $A = \mathrm{Id}$ is given by

$$T_{\mathrm{Id}}\mathscr{L} = \{T \in \mathfrak{u}(n) \mid T + \overline{T} = 0\} = \{iQ \mid Q \in \mathrm{Sym}(n)\}.$$

Recall that $\mathrm{U}(n)$ acts on $\mathscr{L}$ by $A \mapsto UA\bar{U}^{-1}$. Thus, if $A = U\bar{U}^{-1}$, we have an isomorphism

$$\kappa_U : T_{\mathrm{Id}}\mathscr{L} \xrightarrow{\sim} T_A\mathscr{L}$$

given by $T \mapsto UT\bar{U}^{-1}$. Since $\mathscr{L}$ is a $\mathrm{U}(n)$ homogeneous space, the isomorphism κ_U preserves orientation. Moreover, for $T \in T_{A^\epsilon}\mathscr{L}$ we have

$$d\Theta_0|_{A^\epsilon}(T) = T\overline{B}A^\epsilon + A^\epsilon\overline{B}T = T\overline{A}^\epsilon + \overline{A}^\epsilon T.$$

If $U^\epsilon \in \mathrm{U}(n)$ satisfies

$$A^\epsilon = U^\epsilon(\bar{U}^\epsilon)^{-1},$$

then A^ϵ is a regular point of Θ_0 if the linear map

$$\alpha^\epsilon = d\Theta_0|_{A^\epsilon} \circ \kappa_U : T_{\mathrm{Id}}\mathscr{L} \to T_{\mathrm{Id}}\mathscr{L}$$

is invertible, and in that case the sign of A^ϵ is $\operatorname{sign}\det(\alpha^\epsilon)$. Explicitly,

$$\begin{aligned}
\alpha^\epsilon(T) &= U^\epsilon T(\bar{U}^\epsilon)^{-1}\overline{A}^\epsilon + \overline{A}^\epsilon U^\epsilon T(\bar{U}^\epsilon)^{-1} \\
&= U^\epsilon T(U^\epsilon)^{-1} + \bar{U}^\epsilon T(\bar{U}^\epsilon)^{-1} \\
&= U^\epsilon T(U^\epsilon)^{-1} - \bar{U}^\epsilon \overline{T}(\bar{U}^\epsilon)^{-1} \\
&= 2i\,\mathrm{Im}(U^\epsilon T(U^\epsilon)^{-1}).
\end{aligned}$$

Writing $T = iQ$, we can think of α^ϵ as the map $\mathrm{Sym}(n) \to \mathrm{Sym}(n)$ given by

$$\alpha^\epsilon(Q) = 2\,\mathrm{Re}(U^\epsilon Q (U^\epsilon)^{-1}).$$

For convenience, we take U^ϵ to be the unitary linear map given by

$$u^\epsilon_{jk} = e^{i(\theta_k/4 + \epsilon_k \pi/2)} \delta_{jk}.$$

Then, denoting by q_{jk} the matrix elements of Q, we have

$$\begin{aligned}\alpha^\epsilon(Q)_{jk} &= 2\,\mathrm{Re}\left(e^{i\left((\theta_j-\theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\right)}\right) q_{jk} \\ &= 2\cos\big((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\big) q_{jk}.\end{aligned}$$

Since Q is a symmetric matrix, it is determined by q_{jk} for $j \le k$. Thus

$$\det(\alpha^\epsilon) = \prod_{j \le k} 2\cos\big((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\big) q_{jk}.$$

We need to show that this determinant does not vanish and compute its sign. For $j = k$, clearly $\cos\big((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\big) = 1$. For $j < k$, by assumption, $0 < \theta_j < \theta_k < 2\pi$, so

$$-\frac{\pi}{2} < \frac{\theta_j - \theta_k}{4} < 0.$$

It follows that for all $j \le k$, we have $\cos\big((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\big) \neq 0$. Therefore $\det(\alpha^\epsilon) \neq 0$ for all ϵ and Id is a regular value. Moreover,

$$\cos\big((\theta_j - \theta_k)/4 + (\epsilon_j - \epsilon_k)\pi/2\big) < 0 \quad \text{if and only if} \quad \epsilon_j = 0,\ \epsilon_k = 1.$$

Let Υ_n be the set of all binary sequences $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. For $\epsilon \in \Upsilon_n$ define $\mathrm{sign}(\epsilon)$ to be the number modulo 2 of pairs $j < k$ such that $\epsilon_j = 0$ and $\epsilon_k = 1$. The upshot of the preceding calculations is that

$$\mathrm{sign}\det(\alpha^\epsilon) = \mathrm{sign}(\epsilon),$$

therefore

$$\deg \Theta_0 = \sum_{\epsilon \in \Upsilon_n} (-1)^{\mathrm{sign}(\epsilon)}.$$

A combinatorial argument given below in Lemma 2.2 then implies the theorem. □

Lemma 2.2. For $n = 2m + 1$, we have

$$d_n := \sum_{\epsilon \in \Upsilon_n} (-1)^{\mathrm{sign}(\epsilon)} = 2^{m+1}.$$

Proof. Let M_n denote the number of $\epsilon \in \Upsilon_n$ such that $\mathrm{sign}(\epsilon) = 0$. Then

$$d_n = M_n - (2^n - M_n) = 2M_n - 2^n.$$

For $\epsilon \in \Upsilon_n$ denote by $\mathrm{par}(\epsilon)$ the parity of ϵ, or in other words the number modulo 2 of j such that $\epsilon_j = 1$. Let P_n denote the number of $\epsilon \in \Upsilon_n$ such that $\mathrm{sign}(\epsilon) + \mathrm{par}(\epsilon) = 0$. By analyzing what happens when we adjoin either 1 or 0 to the beginning of a sequence $\epsilon \in \Upsilon_{n-1}$, we find that

$$M_n = M_{n-1} + P_{n-1}, \quad P_n = (2^{n-1} - P_{n-1}) + M_{n-1}.$$

Iterating these recursions twice, we obtain

$$M_n = M_{n-2} + P_{n-2} + 2^{n-2} - P_{n-2} + M_{n-2} = 2M_{n-2} + 2^{n-2}.$$

Clearly $M_1 = 2$, so $d_1 = 2$. Using the preceding recursion for M_n, we obtain

$$d_n = 2(2M_{n-2} + 2^{n-2}) - 2^n = 2(2M_{n-2} - 2^{n-2}) = 2d_{n-2}.$$

The lemma follows by induction. □

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Received December 13, 2012

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