On Schoen surfaces

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Abstract. We give a new construction of the irregular surfaces of general type with $p_g = 5$, $\chi = 2$, $K^2 = 8$, recently discovered by C. Schoen in [24]. Our approach proves that, if S is a general Schoen surface, its canonical map is a finite morphism of degree 2 onto a canonical surface with invariants $p_g = 5$, $\chi = 6$, $K^2 = 8$, a complete intersection of a quadric and a quartic hypersurface in $\mathbb{P}^4$, with 40 even nodes.

1. Introduction

Let S be a smooth projective irregular surface. Let

$$\varphi_S : \wedge^2 H^0(S, \Omega_S) \to H^0(S, K_S) \tag{1.1}$$

be the natural map. We call the vectors of the space $\wedge^2 H^0(S, \Omega_S)$ the *formal 2-forms* of S. The *rank* of a formal 2-form ω is the minimum dimension of a subspace $V \subseteq H^0(S, \Omega_S)$ such that $\omega \in \wedge^2 V$; it is an even integer.

A famous theorem by Castelnuovo and De Franchis says that there is a non–zero formal 2–form ω of rank 2 in $\ker(\varphi_S)$ if and only if there exists an *irrational pencil* of genus $b \geq 2$ on S, i.e. a surjective morphism $f : S \to B$, where B is a smooth genus b curve and there exist $\omega_1, \omega_2 \in H^0(B, \omega_B)$ such that $\omega = f^*(\omega_1) \wedge f^*(\omega_2)$.

Existence of higher rank formal 2-forms in $\ker(\varphi_S)$ are more rare and their geometric interpretation more difficult (see [3]). E.g., the existence of such forms is relevant in the study of the fundamental group of S (see [1, 2]).

With a completely different viewpoint in mind (i.e., Tate and Hodge conjectures), C. Schoen discovered in [24] remarkable minimal irregular surfaces of general type

with invariants $p_g = 5$, $\chi = 2$, $K^2 = 16$ (from now on called *Schoen surfaces*). They enjoy the property that $\ker(\varphi_S)$ is generated by a formal 2–form of rank 4 (hence they are *Lagrangian surfaces* in the sense of [3], i.e. they are Lagrangian subvarieties of their Albanese variety A, for an appropriate symplectic structure on A). Furthermore they also enjoy the property that $p_g = 2q - 3$, i.e. p_g is the minimum possible with respect to q for surfaces with no irrational pencils of genus $b \geq 2$ (see [17, 19, 20] for the existence of surfaces with $p_g = 2q - 3$).

Other interesting topological properties of Schoen surfaces are pointed out in §3 below.

Schoen's construction is as follows. He finds a reducible surface V, the transverse union of two irreducible components, inside the principally polarized abelian variety $A \times A$, where $A = J(C)$ and C is a general smooth irreducible genus 2 curve. He shows that V smooths to the required surface. The smoothing relies on two main tools: Bloch's semiregularity (enjoyed by V in $A \times A$) and an explicit deformation of varieties of type $A \times A$ to simple principally polarized abelian varieties in which the class of V stays of Hodge type $(2, 2)$.

The aim of this paper is to give a different, slightly more geometric, approach to Schoen's construction. It will give us some more information, namely (see Theorem 4.1 below):

Theorem 1.1. *Let S be a general Schoen surface. Its canonical map $\varphi_K : S \to \mathbb{P}^4$ is a finite morphism of degree 2 onto a canonical surface with invariants $p_g = 5$, $\chi = 6$, $K^2 = 8$ and 40 even nodes, which is a complete intersection of a quadric and a quartic hypersurface in $\mathbb{P}^4$. The ramification of φ_K takes place at the nodes.*

When the canonical map of a surface of general type has degree $n > 1$ onto a surface, that surface either has $p_g = 0$ or is itself canonically embedded (see [4, Th. 3.1]). Schoen surfaces provide one more example of the latter, rather rare, case (see [11]; see also the recent preprint [5]).

Our construction, described in §4, starts from the same reducible surface V considered by Schoen. It turns out that (a slight modification of) V is the double cover of a surface Z, which has 40 nodes and otherwise has normal crossing singularities. The surface Z sits in the closure of the moduli space of complete intersections of a quadric and a quartic hypersurface in $\mathbb{P}^4$ and an easy count of parameters shows that it deforms to a surface Y which is still a complete intersection of a quadric and a quartic, and has 40 nodes and no other singularity. Then we show that the 40 nodes are even. The double cover of Y branched at the 40 nodes are Schoen surfaces, and counting parameters one sees that in this way one gets them all.

These ideas can be applied to other similar situations in order to find more examples of irregular surfaces, but we do not dwell on this here.

The paper is organized as follows. In §2 we recall a few useful know facts. In §3 we recall Schoen's main result and, using of [1, 2], we discuss some properties of

the fundamental group of Schoen surfaces. Finally, §4 is devoted to our alternative construction.

Notation. We use standard notation in algebraic geometry. Specifically, if X is a surface with locally Gorenstein singularities (so that the *dualizing* or *canonical* sheaf ω_X is a line bundle), we denote by K_X the divisor class of $|\omega_X|$ and we set $p_g(X) := h^0(X, \omega_X)$, $\chi(X) := \chi(\mathcal{O}_X) = \chi(\omega_X)$, $K_X^2 := \omega_X^2$, $q(X) = h^1(X, \mathcal{O}_X)$. We may drop the indication of X when this is clear from the context. We note that if X has only Du Val singularities, then the above invariants for X and for a minimal desingularization of X coincide.

2. Preliminaries

2.1. Surfaces with normal crossing singularities. We recall a few known facts (see [7, 8] and references therein). Let $V = V_1 \cup \cdots \cup V_n$ be a reducible, projective surface such that:

(i) the irreducible components $V_1, \ldots, V_n$ are smooth;

(ii) the *double curves* $C_{ij} := V_i \cap V_j$ are smooth and irreducible, and V_i, V_j intersect transversally along C_{ij}, for $1 \leqslant i < j \leqslant n$;

(iii) V has a finite number of *triple points* $T_{ijk} := V_i \cap V_j \cap V_k$ and V around T_{ijk} is analytically isomorphic to the surface of equation $xyz = 0$ in $\mathbb{A}^3$ around the origin, for $1 \leqslant i < j < k \leqslant n$. We set $T_{ij} := \sum_{k \neq i,j} T_{ijk}$ for the *triple point divisor* on C_{ij}, for $1 \leqslant i < j \leqslant n$;

(iv) V has no other singularity.

Given V as above, one forms the graph G_V:

▷ with *vertices* $v_1, \ldots, v_n$ corresponding to the components $V_1, \ldots, V_n$;

▷ with *edges* c_{ij} corresponding to the double curves C_{ij}, with $1 \leqslant i < j \leqslant n$;

▷ with *faces* t_{ijk} corresponding to the triple points T_{ijk}, with $1 \leqslant i < j < k \leqslant n$.

In the above setting the dualizing sheaf ω_V is invertible and one has

$$\omega_V|_{V_i} \cong \omega_{V_i} \otimes \mathcal{O}_{V_i}(\sum_{j \neq i} C_{ij}), \text{ for } 1 \leqslant i \leqslant n, \tag{2.1}$$

hence

$$K_V^2 = \sum_{i=1}^{n}(K_{V_i} + \sum_{j \neq i} C_{ij})^2. \tag{2.2}$$

Moreover

$$\chi(\mathcal{O}_V) = \sum_{i=1}^{n} \chi(\mathcal{O}_{V_i}) - \sum_{1 \leqslant i < j \leqslant n} \chi(\mathcal{O}_{C_{ij}}) + t(V) \tag{2.3}$$

where $t(V)$ is the number of triple points of V, i.e. the number of faces of G_V.

Let

$$\Phi_V : \bigoplus_{i=1}^{n} H^1(V_i, \mathcal{O}_{V_i}) \to \bigoplus_{1 \leqslant i < j \leqslant n} H^1(C_{ij}, \mathcal{O}_{C_{ij}})$$

be the natural map and let $p_g(V) = h^0(V, \omega_V)$. Then

$$p_g(V) = b_2(G_V) + \sum_{i=1}^{n} p_g(V_i) + \dim(\mathrm{coker}(\Phi_V)). \tag{2.4}$$

If $V = X_0$ is the central fiber of a projective, flat family of surfaces $f : \mathcal{X} \to \mathbb{D}$, over a disc $\mathbb{D}$, with smooth total space $\mathcal{X}$ and smooth fibers $X_t = f^{-1}(t)$, for $t \in \mathbb{D} - \{0\}$, then these smooth fibres have invariants $p_g(X_t) = p_g(V)$, $K_{X_t}^2 = K_V^2$ and $\chi(\mathcal{O}_{X_t}) = \chi(\mathcal{O}_V)$.

If V sits in a flat family $f : \mathcal{X} \to \mathbb{D}$ as above, one says that V is *smoothable* and that $f : \mathcal{X} \to \mathbb{D}$ is a *smoothing* of V. Then the *triple point formula* holds

$$N_{C_{ij}|V_i} \otimes N_{C_{ij}|V_j} \otimes \mathcal{O}_{C_{ij}}(T_{ij}) \cong \mathcal{O}_{C_{ij}}, \text{ for } 1 \leqslant i < j \leqslant n. \tag{2.5}$$

We recall the following definition from [14]: V is said to be *d–semistable* if

$$\mathcal{O}_C(-V) := \bigotimes_{i=1}^{n} \frac{\mathcal{I}_{V_i|V}}{\mathcal{I}_{V_i|V}\mathcal{I}_{C|V}} \cong \mathcal{O}_C$$

where $C = \cup_{1 \leqslant i < j \leqslant n} C_{ij}$ is the singular locus of V and the tensor product is taken as $\mathcal{O}_C$–modules. If V is smoothable, then V is d–semistable (see [14, Proposition (1.12)]), but the converse is in general false. In any event, $\mathcal{O}_C(-V)$ is a line bundle on C (see [14, Proposition (1.10)]). One defines $\mathcal{O}_C(V) := \mathcal{O}_C(-V)^*$, and $\mathcal{O}_{C_{ij}}(V) := \mathcal{O}_C(V)|_{C_{ij}}$. Note that $\mathcal{O}_C(V) = \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$ is the T_V^1 *sheaf* of V (see [14, Proposition (2.3)]).

Lemma 2.1. *In the above setting, one has*

$$\mathcal{O}_{C_{ij}}(V) = N_{C_{ij}|V_i} \otimes N_{C_{ij}|V_j} \otimes \mathcal{O}_{C_{ij}}(T_{ij}). \tag{2.6}$$

Hence (2.5) *is necessary for d–semistability. If the dual graph of the singular locus C of V is a tree, i.e.*

$$p_a(C) = \sum_{1 \leqslant i < j \leqslant n} p_a(C_{ij}),$$

then (2.5) *is also sufficient for d–semistability.*

Proof. Formula (2.6) is an immediate consequence of the definition of $\mathcal{O}_C(V)$. If the dual graph of C is a tree, then $\mathcal{O}_C$ is the unique line bundle on C whose restriction to each component of C is trivial. □

2.2. Double covers. The contents of this section are well known. We recall them to fix notation and terminology.

Let X be a projective scheme (over an algebraically closed field k of characteristic $p \neq 2$, though we work over $\mathbb{C}$ in this paper). A *double cover* of X is a scheme Y and a finite morphism $f : X \to Y$ of degree 2. The datum of such a double cover is equivalent to give two line bundles $\mathcal{L}, \mathcal{M}$ on X such that $\mathcal{M}^{\otimes 2} = \mathcal{L}$ plus a section $s \in H^0(X, \mathcal{L})$. Let $(U_i)_{i \in I}$ be a finite covering of X over which both $\mathcal{L}$ and $\mathcal{M}$ trivialize, let $(\xi_{ij})_{i,j \in I}$ be the corresponding cocycle for $\mathcal{M}$, let z_i be the coordinate in the fibre of $\mathcal{M}$ over U_i and let $(s_i)_{i \in I}$ be the local functions defining s. Then we have

$$z_i = \xi_{ij} z_j, \text{ and } s_i = \xi_{ij}^2 s_j \text{ for all } i, j \in I$$

and the locus Y

$$z_i^2 = s_i, \text{ for all } i \in I$$

in the total space of $\mathcal{M}$ is well defined and, via the natural projection to X, is a double cover $f : Y \to X$. The zero locus B of s is the *branch locus* of the covering and $R := f^{-1}(B)$ is the *ramification locus*. As schematic counter image of B, R has a non–reduced scheme structure. Note that B is not necessarily a Cartier divisor on X: e.g. if s is the zero section, then Y is a double structure on X. Similarly, if X is reducible, s could be zero on some component of X.

Lemma 2.2. *Let $f : \mathcal{X} \to \mathbb{D}$ be a flat projective family over a disc. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ and set $\mathcal{L}_0 = \mathcal{L}|_{X_0}$. Assume there is a line bundle $\mathcal{M}_0$ on X_0 such that $\mathcal{M}_0^{\otimes 2} = \mathcal{L}_0$. Then, up to shrinking $\mathbb{D}$, there is a line bundle $\mathcal{M}$ on $\mathcal{X}$ such that $\mathcal{M}^{\otimes 2} = \mathcal{L}$ and $\mathcal{M}_0 = \mathcal{M}|_{X_0}$.*

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite covering of $\mathcal{X}$ over which $\mathcal{L}$ trivializes and $\mathcal{M}_0$ trivializes on $\mathcal{V} = (V_i)_{i \in I}$ with $V_i = U_i \cap X_0$ for all $i \in I$. Let $(\xi_{ij})_{i,j \in I}$ be the cocycle for $\mathcal{L}$ on $\mathcal{U}$ and let $(\eta_{ij})_{i,j \in I}$ be the cocycle for $\mathcal{M}_0$ on $\mathcal{V}$. Then

$$\eta_{ij} = \sqrt{\xi_{ij}\big|_{V_{ij}}}, \text{ for all } i, j \in I$$

which encodes the choice of a suitable determination of the square root. Then we may choose the same determination of the square root defining

$$\zeta_{ij} = \sqrt{\xi_{ij}}, \text{ for all } i, j \in I$$

on U_{ij} for all $i, j \in I$, and this gives the cocycle $(\zeta_{ij})_{i,j \in I}$ defining $\mathcal{M}$ on $\mathcal{X}$. $\square$

2.3. Hypernodes. An *hypernode* of an n dimensional variety X, with $n \geq 2$, is a point p such that the analytic germ of (X, p) is isomorphic to the quotient singularity $(\mathbb{C}^n/\sigma, \mathbf{0})$, where

$$\sigma : \mathbf{x} \in \mathbb{C}^n \to -\mathbf{x} \in \mathbb{C}^n.$$

If $n = 2$ this is called a *node*, and it is an A_1–singularity. A minimal resolution $\tilde{X}$ of X at p is gotten by a single blow–up. The exceptional divisor E is isomorphic to $\mathbb{P}^{n-1}$ and $N_{E|\tilde{X}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-2)$.

If X is a projective variety with hypernodes $p_1, \ldots, p_h$, and no other singularity, we can consider its minimal desingularization $f : \tilde{X} \to X$. Then $\tilde{X}$ has the exceptional divisors $N_1, \ldots, N_h$ contracted by f to the hypernodes $p_1, \ldots, p_h$. Set $N := \sum_{i=1}^{h} N_i$. One says that $p_1, \ldots, p_h$ are *even*, if $\mathcal{O}_{\tilde{X}}(N)$ is divisible by 2 in $\text{Pic}(\tilde{X})$. This happens if and only if there is a commutative diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\ g\ } & Y \\
{\scriptstyle \tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\tilde{X} & \xrightarrow[\ f\ ]{} & X
\end{array}
$$

where $Y, \tilde{Y}$ are smooth varieties, $\pi, \tilde{\pi}$ are finite morphisms of degree 2, and $\tilde{\pi}$ is branched at N, whereas π is branched at $p_1, \ldots, p_h$. The counter images of $p_1, \ldots, p_h$ are points $q_1, \ldots, q_h \in Y$ and g is the blow–up of Y at $q_1, \ldots, q_h$.

3. Schoen surfaces

Let $V_1 = A$ be an abelian surface with $C \subset A$ a smooth curve of genus $g \geq 2$. One has $N_{C|A} \cong \omega_C$. Let $V_2 = C \times C$ and let $\Delta \subset V_2$ be the diagonal. Then $\Delta \simeq C$ and $N_{\Delta|V_2} \cong \omega_C^*$. Let V be the reducible surface consisting of $V_1 \cup V_2$ glued along $C \subset V_1$ and the diagonal $\Delta \subset V_2$.

Proposition 3.1. *The invariants of V are*

$$p_g = 1 + g^2, \ \chi = g(g - 1), \ K^2 = 8g(g - 1).$$

Proof. This follows from (2.2), (2.3) and (2.4). The details can be left to the reader. Only note that the map Φ_V is surjective, since $h^1(A, \mathcal{O}_A(-C)) = 0$ because C is ample on A. $\qquad\square$

Schoen proves in [24] that:

Theorem 3.2. *If $g = 2$, then V is smoothable to surfaces with a 4–dimensional generically smooth moduli space.*

Remark 3.3. In [24, Proposition 10.1, (ii)], it is stated that for the general Schoen surface S one has $\text{rk}(NS(S)) = 2$. As one can directly see with an argument as in [15], the right statement is instead that $\text{rk}(NS(S)) = 1$ (that was also pointed to us in [25]).

It is not known if V is smoothable for $g \geq 3$. This is an intriguing question, especially for $g = 3$ (see Remark 3.7 below).

Schoen surfaces verify $K^2 = 8\chi$. Surfaces whose universal cover is $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C}/\Im m(z) > 0\}$ is the *Siegel upper-half plane*, also verify $K^2 = 8\chi$ and have infinite fundamental group. Teicher and Moishezon constructed in [21, 22] finitely many families of surfaces with $K^2 = 8\chi$ and finite (even trivial) fundamental group. The following proposition shows a remarkable property of Schoen surfaces:

Proposition 3.4. *The universal cover of a Schoen surface S is not $\mathbb{H} \times \mathbb{H}$. Since $q(S) = 4$, $\pi_1(S)$ is not finite and finite étale covers of Schoen surfaces give an infinite number of families of surfaces with $K^2 = 8\chi$ whose universal cover is not $\mathbb{H} \times \mathbb{H}$.*

Proof. If S has universal cover $\mathbb{H} \times \mathbb{H}$, then it is the quotient of $\mathbb{H} \times \mathbb{H}$ by a discrete cocompact subgroup Γ of $\text{Aut}(\mathbb{H} \times \mathbb{H})$ acting freely. By [16], either Γ is *reducible*, and S is *isogenous* to the product of two curves (i.e. it is a quotient of a product of two curves by a fixed–point free group action), or Γ is *irreducible* and S is regular.

The latter case cannot happen, because $q(S) = 4$. Also the former case cannot happen. Indeed, in [23, Proposition 6.1] Schoen proved that a surface dominated by a map from a product of curves is *Albanese standard*, i.e. the class of its image into its Albanese variety A sits in the subring of $H^{\bullet}(A, \mathbb{Q})$ generated by the divisor classes. By contrast, by [24, Theorem 1.1, (iii)] Schoen surfaces are *Albanese exotic*, i.e. not Albanese standard. $\square$

Note that, according to [13], Schoen surfaces do not possess any *semi special tensor*.

If S is a Schoen surface, set $G := \pi_1(S)$. We denote by $\{G_n\}_{n \in \mathbb{N}}$ the *lower central series* of G, defined as

$$G_1 = G, \quad G_{n+1} = [G_n, G], \quad \text{for} \quad n \geq 1,$$

where $[\cdot, \cdot]$ denotes the *commutator subgroup*. The group $G_{\text{ab}} = G_1/G_2$ is the *abelianization* of G, and in the present case $G_{\text{ab}} \cong H^1(S, \mathbb{Z}) \cong \mathbb{Z}^8$.

By [1, Corollary 1.44], [2], the group $G_2/G_3 \otimes \mathbb{C}$ is isomorphic to the (dual of the) kernel of the natural map

$$\psi_S : \wedge^2 H^1(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C}).$$

The Betti numbers of Schoen surfaces S are $b_1 = 8$ and $b_2 = 22$, moreover $h^{1,1}(S) = b_2 - 2p_g = 12$. The space $\wedge^2 H^1(S, \mathbb{C})$ is 28-dimensional. The map ψ_S respects the Hodge decomposition, hence it is the direct sum of the map φ_S in (1.1) and of its conjugate, and of the map

$$\phi_S : H^{1,0}(S) \otimes H^{0,1}(S) \rightarrow H^{1,1}(S).$$

We see that $\dim(\ker \phi_S) \geq 4$. The general Schoen surface S has no irrational pencil. This follows from Remark 3.3; alternatively, if $f : S \to B$ is an irrational pencil, then the genus of B is at most 3 (because $q(S) = 4$ and S is of Albanese general type) and this would imply that $\text{Alb}(S)$ is non–simple, contradicting [24, Theorem 6.1, (i)]. In particular S has no morphism $f : S \to B$ to a curve B of genus $b \geq 2$. Since this is a deformation invariant property (see [9]), the same holds for any Schoen surface. Hence, by Castelnuovo–De Franchis' Theorem, the map φ_S cannot have a kernel of dimension bigger than 1, hence it is surjective with a 1–dimensional kernel. Moreover $\dim(\ker \phi_S) \leq 5$ by [10, Proposition 2.2.5]. We have (see [24, Proposition 9.1]):

Corollary 3.5. *Let S be a Schoen surface. Then $6 \leq \dim(\ker(\psi_S)) \leq 7$, hence $6 \leq \dim(G_2/G_3 \otimes \mathbb{C}) \leq 7$ and G is not abelian.*

Remark 3.6. Schoen surfaces are such that both φ_S and ϕ_S have a non–trivial kernel. As far as we know the only other surfaces with no irrational pencils enjoying the same property and discovered so far are the *LG–surfaces* studied in [6].

Remark 3.7. Consider again the reducible surface V for $g \geq 3$. Suppose V is smoothable and that S is a general surface in a smoothing of V. Since

$$\dim(\wedge^2 H^0(S, \Omega_S)) = \frac{1}{2}(g+2)(g+1) < p_g, \text{ for } g \geq 4,$$

we cannot conclude directly that φ_S has a non–trivial kernel if $g \geq 4$. Similarly, one computes $h^{1,1} = 2(g^2 + 2)$, hence we cannot conclude that ϕ_S has a non–trivial kernel if $g \geq 4$. The borderline case $g = 3$ is attractive. If V is smoothable to a surface S, then ϕ_S has a non–trivial kernel of dimension at least 3, hence, as in the Schoen surface case, the fundamental group $\pi_1(S)$ is not abelian and it would be interesting to understand it. Moreover, either φ_S is an isomorphism, or φ_S would have a non–trivial kernel. In the former case S would contradict a conjecture to the effect that the Fano surface of lines of a smooth cubic threefold and the symmetric product of curves are the only surfaces S with $q \geq 3$ such that φ_S is an isomorphism (see [18] and also [12]). In the latter case, S would again be a generalized Lagrangian surface in the sense of [3], and these surfaces are quite rare and interesting on their own.

4. Different construction of Schoen surfaces

Here we propose an approach to the construction of Schoen surfaces different from the original one. It provides the following additional bit of information:

Theorem 4.1. *Let S be a general Schoen surface. The canonical map $\varphi_K : S \to \mathbb{P}^4$ of S is a finite morphism of degree 2 onto a canonical surface with invariants $p_g = 5$, $\chi = 6$, $K^2 = 8$ and 40 even nodes, a complete intersection of a quadric and a quartic hypersurface in $\mathbb{P}^4$. The ramification of φ_K takes place at the nodes.*

We start by looking at the dualizing sheaf ω_V, which, by (2.1), is the bundle obtained by gluing $\mathcal{O}_A(C)$ on $A = V_1$ and $\omega_{V_2}(\Delta)$ on $V_2 = C \times C$ along C: this is possible since the two bundles both restrict to ω_C on C. Then we modify ω_V by *twisting by* V_2, which means considering the line bundle $\mathcal{L}$ on V obtained by gluing $\mathcal{O}_A(2C)$ on A and ω_{V_2} on V_2, the two bundles both restricting to $\omega_C^{\otimes 2}$ on C.

Remark 4.2. Suppose $V = X_0$ is the central fiber of a projective family of surfaces $f : \mathcal{X} \to \mathbb{D}$, as in §2.1. Then $\omega_V = \omega_{\mathcal{X}}|_{X_0}$. Twisting by V_2, as we did, is the same as considering the line bundle $\mathcal{L} = \omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(V_2)|_{X_0}$. Note that both $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{X}} \otimes \mathcal{O}_{\mathcal{X}}(V_2)$ restrict to the canonical bundle on the general surface of the family. Hence $\mathcal{L}$, as well as ω_V, is a *limit* of the canonical bundle of X_t for $t \in \mathbb{D} - \{0\}$.

Lemma 4.3. *We have $h^0(V, \mathcal{L}) = p_g(V) = 5$ and the map $\phi_{\mathcal{L}} : V \to \mathbb{P}^4$ is a morphism.*

Proof. One has a cartesian diagram

$$
\begin{array}{ccc}
H^0(V, \mathcal{L}) & \xrightarrow{\quad s_1 \quad} & H^0(A, \mathcal{O}_A(2C)) \\
\downarrow{\scriptstyle s_2} & & \downarrow{\scriptstyle r_1} \\
H^0(V_2, \omega_{V_2}) \cong H^0(C, \omega_C)^{\otimes 2} & \xrightarrow{\quad r_2 \quad} & H^0(C, \omega_C^{\otimes 2})
\end{array}
$$

where r_1, r_2 are restriction maps. One has $h^0(A, \mathcal{O}_A(2C)) = 4$, $h^0(V_2, \omega_{V_2}) = 4$, r_1 is surjective since $h^1(A, \mathcal{O}_A(C)) = 0$ and r_2 is surjective by Noether's theorem. Since $h^0(C, \omega_C^{\otimes 2}) = 3$, we have $h^0(V, \mathcal{L}) = 5$. Moreover $p_g(V) = 5$ follows from (2.4), because $h^1(A, \mathcal{O}_A(-C)) = 0$ implies Φ_V is surjective. Finally, the surjectivity of r_1 and r_2 implies the surjectivity of both s_1, s_2, and since $|2C|$ and $|\omega_{V_2}|$ are base point free, also $|\mathcal{L}|$ is base point free. $\qquad\square$

We note that $\phi_{\mathcal{L}} : V \to \mathbb{P}^4$ is composed with an involution ι of V, which restricts to the involution $\pm$ on A and to $\mathfrak{i} \times \mathfrak{i}$ on V_2, where $\mathfrak{i}$ is the hyperelliptic involution on C. Note that the canonical map of $V_2 = C \times C$ is a $\mathbb{Z}_2^2$–cover of $\mathbb{P}^1 \times \mathbb{P}^1$ given by the action of $\mathfrak{i}$ separately on each coordinate. The involution ι has 46 isolated fixed points on V:

▷ the 16 points of order two on A, 6 of which lie on C and coincide with its Weierstrass points (we assume C is stable by multiplication by the $\pm$ involution on A);

▷ 36 points on V_2, the ones having as coordinates the Weierstrass points on C, 6 of them lie on $\Delta \cong C$ and coincide with the 6 Weierstrass points on $C \subset A$;

▷ in conclusion 40 isolated fixed points are in the smooth locus of V, the remaining 6 are on the double curve $C \cong \Delta$.

Accordingly, $W = V/\iota$, is the union of two components:

▷ $\Sigma = A/\pm$, the Kummer surface of A, with 16 nodes, 6 on $\Gamma := C/\mathfrak{i} \cong \mathbb{P}^1$;

▷ $T = V_2/\mathfrak{i} \times \mathfrak{i}$, with 36 nodes, 6 on $\Gamma' = \Delta/\mathfrak{i} \times \mathfrak{i} = C/\mathfrak{i} \cong \mathbb{P}^1$;

▷ Σ and T are glued along the double curve R, which is Γ on Σ and Γ' on T, in

such a way that the 6 nodes located there coincide in the obvious way and the tangent cones to two coinciding nodes have in common only the tangent line to R;

$\triangleright$ W has 40 more nodes off R.

Next we modify W in order to make it with normal crossing singularities. To do this, we minimally resolve the singularities of both Σ and T. This produces two surfaces Σ', T'. We abuse notation and still denote by Γ and Γ' the proper transforms of these curves on Σ', T'. Then we glue Σ', T' along Γ and Γ', and call again R the double curve of the reducible surface $W' = \Sigma' \cup T'$ thus obtained. Note that:

$\triangleright$ Σ' has (-2)–curves $N_1, \ldots, N_{16}$ and we may assume that $N_{11}, \ldots, N_{16}$ intersect Γ;

$\triangleright$ T' has (-2)–curves $M_1, \ldots, M_{36}$ and we may assume that $M_{31}, \ldots, M_{36}$ intersect Γ';

$\triangleright$ in conclusion W' has the (-2)–curves $N_1, \ldots, N_{10}, M_1, \ldots, M_{30}$, whereas the curves N_{10+i}, M_{30+i} meet each other and the double curve R at a point x_i, for $1 \leqslant i \leqslant 6$.

Finally, in order to create a (potentially) smoothable surface, we need the triple point formula (2.5) to be verified (which is not the case for W'). In order to obtain this, we form a new surface Z' by sticking 6 planes $P_i \cong \mathbb{P}^2$ in W' in the following way: P_i contains the two curves N_{10+i}, M_{30+i} as lines meeting at x_i, for $1 \leqslant i \leqslant 6$. The surface Z' has normal crossing singularities and it respects the triple point formula (2.5). We will also consider the surface Z with 40 nodes obtained from Z' by contracting the (-2)–curves $N_1, \ldots, N_{10}, M_1, \ldots, M_{30}$ to nodes $n_1, \ldots, n_{10}, m_1, \ldots, m_{30}$.

Lemma 4.4. *The surfaces Z, Z' have invariants*

$$p_g = 5, \ \chi = 6, \ K^2 = 8.$$

Proof. It suffices to compute the invariants for Z'. The surface Σ' is a $K3$. Moreover $H^0(T, \omega_T)$ is the space of invariants of $H^0(V_2, \omega_{V_2}) \cong H^0(C, \omega_C)^{\otimes 2}$ under the hyperelliptic involution $\mathfrak{i}$ on C. Since $\mathfrak{i}$ changes the sign of holomorphic 1–forms on C, we have

$$H^0(T, \omega_T) \cong H^0(C, K_C)^{\otimes 2},$$

hence T (and also T') has $p_g = 4$. The same argument shows that T has $q = 0$. The assertion $p_g(Z') = 5$ follows from (2.4), by noticing that $b_2(G_{Z'}) = 0$ and $\mathrm{coker}(\Phi_{Z'}) = 0$ because the double curves of Z' are all rational.

The computations of K^2 and χ follow in a similar way by (2.2) and (2.3). $\qquad\square$

We could consider $\omega_{Z'}$, but as above this is not quite the right thing to do, because, among other things, this sheaf is negative on the planes P_i, for $1 \leqslant i \leqslant 6$.

Rather we consider its twist $\mathcal{N}$ by T', which restricts to:

▷ the line bundle $\mathcal{O}_{\Sigma'}(2\Gamma + \sum_{i=1}^{6} N_{10+i})$ on Σ';

▷ the canonical bundle $\omega_{T'}$ on T';

▷ the trivial bundle on each of the planes P_i, for $1 \leqslant i \leqslant 6$.

One has:

▷ the linear system $|2\Gamma + \sum_{i=1}^{6} N_{10+i}|$ on Σ' is base point free and birationally maps Σ' to the quartic Kummer surface $\Sigma \subset \mathbb{P}^3$, by contracting $N_1, \ldots, N_{16}$ to the nodes of Σ;

▷ the canonical system $|\omega_{T'}|$ is base point free and we have a commutative diagram of morphisms

$$
\begin{array}{ccccc}
T' & \xrightarrow{\;\;f\;\;} & T & \xleftarrow{\;\;g\;\;} & V_2 = C \times C \\
& \searrow_{h_{T'}} & \downarrow{h_T} & \swarrow_{h_{V_2}} & \\
& & Q \subset \mathbb{P}^3 & &
\end{array}
$$

where $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric, f is birational, h_T, $h_{T'}$ and g have degree 2, and h_{V_2} has degree 4.

Lemma 4.5. *We have $h^0(Z', \mathcal{N}) = p_g(Z') = 5$ and the map $\phi_{\mathcal{N}} : Z' \to \mathbb{P}^4$ is a morphism factoring through a morphism $\phi : Z \to \mathbb{P}^4$, whose image $\bar{Z}$ is the union of a Kummer surface Σ lying in a hyperplane Π and of a (double) quadric Q lying in another hyperplane Π', and Σ and Q meet along a conic Γ which is a plane section of Q and passes through 6 nodes of Σ.*

Proof. We have a cartesian diagram

$$
\begin{array}{ccc}
H^0(Z, \mathcal{N}) & \xrightarrow{\;\;s_1\;\;} & H^0(\Sigma', \mathcal{O}_{\Sigma'}(2\Gamma + \sum_{i=1}^{6} N_{10+i})) \\
\downarrow{s_2} & & \downarrow{r_1} \\
H^0(T', \omega_{T'}) & \xrightarrow{\;\;r_2\;\;} & H^0(\Gamma, \mathcal{O}_\Gamma \otimes \mathcal{N}) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))
\end{array}
$$

where r_1, r_2 are restriction maps, both surjective. Note that the planes P_i, with $1 \leqslant i \leqslant 6$, do not play any role in computing $H^0(Z, \mathcal{N})$ since the restriction of $\mathcal{N}$ on each of them is trivial, hence the corresponding constant value of the sections of $\mathcal{N}$ on the P_i's is determined by the values of the sections on Σ' and T' agreeing on Γ. The proof goes as the one of Lemma 4.3. $\qquad\square$

Lemma 4.6. *Notation as in Lemma 4.5. Then $\bar{Z}$ is the complete intersection of the quadric $\Pi \cup \Pi'$ and of a quartic hypersurface.*

Proof. We may choose homogeneous coordinates $(x_0 : \ldots : x_4)$ in $\mathbb{P}^4$ so that Π has equation $x_0 = 0$ and Π' equation $x_1 = 0$. Suppose that the equation of Σ in Π is $F(x_1, \ldots, x_4) = 0$ and the equation of Q in Π' is $G(x_0, x_2, \ldots, x_4) = 0$. We may write

$$G(x_0, x_2, \ldots, x_4) = x_0^2 + x_0 q_1(x_2, x_3, x_4) + q_2(x_2, x_3, x_4)$$

where q_1, q_2 are homogeneous polynomials of degree given by the index. We may assume that

$$F(0, x_2, x_3, x_4) = G^2(0, x_2, x_3, x_4) = q_2^2(x_2, x_3, x_4).$$

Consider the homogeneous polynomial of degree 4

$$H(x_0, \ldots, x_4) = \sum_{i=0}^{4} x_0^{4-i} f_i(x_1, \ldots, x_4)$$

where

$$f_0 = 1, \ \ f_1 = 2q_1, \ \ f_2 = q_1^2 + 2q_2, \ \ f_3 = 2q_1q_2, \ \ f_4 = F.$$

The quartic $H = 0$ intersects Π in Σ and Π' in the quartic with equation

$$x_0^4 + 2x_0^3 q_1 + x_0^2(q_1^2 + 2q_2) + 2x_0 q_1 q_2 + q_2^2 = 0$$

which is the double quadric $G^2 = 0$. The assertion follows. $\square$

Lemma 4.6 shows that the 40–nodal (and otherwise normal crossings) surface Z sits on the boundary of a partial compactification $\mathfrak{M}$ of the moduli space of complete intersections of a quadric and a quartic in $\mathbb{P}^4$, which are canonical surfaces with invariants $p_g = 5$, $\chi = 6$, $K^2 = 8$. One has $\dim(\mathfrak{M}) = 10\chi - 2K^2 = 44$. In $\mathfrak{M}$ each node imposes, as well known, one condition at most, and therefore Z is contained in some irreducible, locally closed subset $\mathcal{Z} \subset \mathfrak{M}$ of dimension $\dim(\mathcal{Z}) \geq 4$ of 40–nodal surfaces.

Lemma 4.7. *The general surface in $\mathcal{Z}$ has 40 nodes and no other singularity.*

Proof. The reducible surfaces Z depend on 3 moduli (i.e. the moduli of C). So they fill up a proper subvariety $\mathcal{Z}'$ of $\mathcal{Z}$. The local to global Ext spectral sequence gives the exact sequence

$$0 \to H^1(Z, \Theta_Z) \to \mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega_Z^1, \mathcal{O}_Z) \to H^0(Z, \mathcal{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)) \cong \mathbb{C} \oplus \mathbb{C}^{40}.$$
$$(4.1)$$

To explain the last isomorphism, note that $\mathcal{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z)$ is supported at the singular locus of Z, which consists of the double curve $D := \Gamma + \sum_{i=1}^{6}(N_{10+i} + M_{10+i})$ plus the 40 nodes $n_1, \ldots, n_{10}, m_1, \ldots, m_{30}$. By Lemma 2.1 (which clearly applies to this case, though Z is singular off the double curve), one has

$$\mathcal{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z) \otimes \mathcal{O}_D \cong \mathcal{O}_D.$$

Moreover

$$\mathcal{E}xt^1_{\mathcal{O}_Z}(\Omega_Z, \mathcal{O}_Z) \otimes \mathcal{O}_z \cong \mathcal{O}_z, \ \text{ for } z = n_1, \ldots, n_{10}, m_1, \ldots, m_{30}.$$

Recall that the vector spaces in (4.1) have the following meaning:

$\triangleright$ $H^1(Z, \Theta_Z)$ is the tangent space to *locally trivial* deformations of Z;

$\triangleright$ $\mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z)$ is the tangent space of all deformations of Z. Consider the kernel $\mathbf{K}$ of the projection

$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z) \to H^0(Z, \bigoplus_{i=1}^{10} \mathcal{O}_{n_i} \oplus \bigoplus_{i=1}^{30} \mathcal{O}_{m_i}) \cong \mathbb{C}^{40}$$

which is the tangent space to deformations of Z keeping the 40 nodes $n_1, \ldots, n_{10}$, $m_1, \ldots, m_{30}$, i.e. it is the tangent space to Z in $\mathcal{Z}$. The sequence (4.1) can be replaced by

$$0 \to H^1(Z, \Theta_Z) \to \mathbf{K} \to H^0(D, \mathcal{O}_D) \cong \mathbb{C}.$$

Let us take now a deformation $f : \mathcal{X} \to \mathbb{D}$ of Z inside $\mathcal{Z}$ parametrized by a disc $\mathbb{D}$, which is not tangent to $\mathcal{Z}'$, in particular it is not a locally trivial deformation of Z. Then the tangent vector to this deformation is an element in $\mathbf{K}$ not in $H^1(Z, \Theta_Z)$, hence it maps to a non–zero element in $H^0(D, \mathcal{O}_D)$. By a (suitable version of) [14, Proposition (2.5)], one may assume (up to shrinking $\mathbb{D}$) that $\mathcal{X}$ is smooth off the curve A described by the deformations of the 40 nodes. The assertion follows. $\square$

Let us consider the desingularization $\mathcal{Y} \to \mathcal{X}$, which is obtained by blowing-up $\mathcal{X}$ along the singular curve A (see proof of Lemma 4.7). By composing with $f : \mathcal{X} \to \mathbb{D}$ we have a new family $g : \mathcal{Y} \to \mathbb{D}$ which is a smoothing of Z'. We denote by E the exceptional divisor over A. It intersects the general surface Y_t of the family, for $t \neq 0$, in the (-2)–curves deforming $N_1, \ldots, N_{10}, M_1, \ldots, M_{30}$ on Z'.

Lemma 4.8. *The 40 nodes on the general surface of $\mathcal{Z}$ are even.*

Proof. Consider the divisor $E + P$ on $\mathcal{Y}$, where $P = \sum_{i=1}^6 P_i$ (we abuse notation here and denote by P_i its strict transform on $\mathcal{Y}$, for $1 \leqslant i \leqslant 6$). We note that $\mathcal{O}_{Z'}(E + P)$ is divisible by 2 in $\mathrm{Pic}(Z')$. Indeed:

(i) $\mathcal{O}_{P_i}(E + P) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$, for $1 \leqslant i \leqslant 6$;

(ii) $\mathcal{O}_{\Sigma'}(E + P) \cong \mathcal{O}_{\Sigma'}(N_1 + \ldots + N_{16})$, which is divisible by two, because the 16 nodes of the Kummer surface are even;

(iii) $\mathcal{O}_{T'}(E + P) \cong \mathcal{O}_{T'}(M_1 + \ldots + M_{36})$, which is also divisible by two, because the 36 nodes of T' are even.

Moreover the halves of the bundles appearing in (i), (ii) and (iii) above naturally glue to give a line bundle $\mathcal{M}_0$ on Z' such that $\mathcal{M}_0^{\otimes 2} = \mathcal{O}_{Z'}(E + P)$. Then, by Lemma 2.2, up to shrinking $\mathbb{D}$, we may assume that there is a line bundle $\mathcal{M}$ on $\mathcal{Y}$ such that $\mathcal{M}|_{Z'} = \mathcal{M}_0$ and $\mathcal{M}^{\otimes 2} = \mathcal{O}_{\mathcal{Y}}(E + P)$. Since $\mathcal{O}_{Y_t}(E + P) = \mathcal{O}_{Y_t}(E)$ for $t \neq 0$, the assertion follows. $\square$

We are now in position to finish the:

Proof of Theorem 4.1. If $Y \in \mathcal{Z}$ is the general surface, we can consider the double cover $\pi : S \to Y$ branched at the 40 nodes of Y. The surface S is smooth and one computes its invariants to be the same as for Schoen surfaces. Moreover $\pi^*(\omega_Y) = \omega_S$. Next we have to show that these surfaces are indeed Schoen surfaces, i.e. they come from smoothings of surfaces of type V.

The proof of Lemma 4.8 shows that there is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{S}' & \xrightarrow{\ \pi'\ } & \mathcal{Y} \\
& \searrow & \downarrow g \\
& & \mathbb{D}
\end{array}
$$

where π' is a double cover branched along $E + P$. One has that $\mathcal{S}'$ is smooth, because so is $E + P$. Let $E' + P'$ be the ramification divisor on $\mathcal{S}'$. Note also that the central fibre of $\mathcal{S}'$, which is a double cover of Z', is nothing but V plus 6 double planes P_i' whose sum is P', each covering one of the planes P_i, for $1 \leqslant i \leqslant 6$.

Next we simultaneously contract $E + P$ and $E' + P'$, thus getting a new commutative diagram

$$
\begin{array}{ccccc}
& & \mathcal{S}' & & \\
& \swarrow & & \searrow{\pi'} & \\
\mathcal{S} & \xrightarrow{\ \pi\ } & \mathcal{X}' & \longleftarrow & \mathcal{Y} \\
& \searrow{h} & \downarrow h & \swarrow{g} & \\
& & \mathbb{D} & &
\end{array}
$$

where:
▷ $\mathcal{S}' \to \mathcal{S}$ is the contraction of $E + P$ and $\mathcal{S}$ is smooth;
▷ $\mathcal{Y} \to \mathcal{X}'$ is the contraction of $E' + P'$ and $\mathcal{X}'$ has 6 hypernodes arising from the contraction of the six components of P and a curve A of double points coming from the contraction of E;
▷ $\pi : \mathcal{S} \to \mathcal{X}'$ is ramified along A and along the 6 hypernodes;
▷ the family $h : \mathcal{S} \to \mathbb{D}$ is a smoothing of the reducible surface V as dictated by 3.2.

To finish our proof we have to show that in this way we do get all Schoen surfaces. By Theorem 3.2, Schoen surfaces depend on 4 moduli. On the other hand, the double covers we found here depend on $\dim(\mathcal{Z}) \geq 4$ moduli. This proves our assertion. $\qquad\square$

Remark 4.9. It is worth stressing that our approach does give an alternative proof of the existence of Schoen surfaces and of the computation of their number of moduli, since we do not need to rely on Theorem 3.2. Indeed, the argument of the proof of Theorem 4.1, shows that there are smoothings of V, depending on $\dim(\mathcal{Z}) \geq 4$

moduli. It takes a few lines in [24, §2] to compute the cohomology of Θ_V and one has $h^1(V, \Theta_V) = 3$. Then we have the exact sequence

$$0 \to H^1(V, \Theta_V) \to \text{Ext}^1_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)$$
$$\to H^0(V, \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)) \cong H^0(C, \mathcal{O}_C) \cong \mathbb{C}$$

and we prove here that the rightmost map is non–zero. This shows that $\dim(\text{Ext}^1_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)) = 4$ and that the deformations in $\text{Ext}^1_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)$ are unobstructed. In addition we have $\dim(\text{Ext}^1_{\mathcal{O}_V}(\Omega^1_V, \mathcal{O}_V)) \geq \dim(\mathcal{Z}) \geq 4$, which proves that $\dim(\mathcal{Z}) = 4$.

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