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Flexible bundles over rigid affine surfaces

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Abstract. We construct a smooth rational affine surface S with finite automorphism group but with the property that the group of automorphisms of the cylinder $S \times \mathbb{A}^2$ acts infinitely transitively on the complement of a closed subset of codimension at least two. Such a surface S is in particular rigid but not stably rigid with respect to the Makar-Limanov invariant.

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1. Introduction

A complex affine variety X is called *rigid* if it does not admit non trivial algebraic actions of the additive group $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{C}}$. This is the case for "most" affine varieties, for instance for every affine curve different from the affine line $\mathbb{A}^1$ and for every affine variety whose normalization has non negative logarithmic Kodaira dimension. The notion was actually introduced by Crachiola and Makar-Limanov in [4] under the more algebraic equivalent formulation that the *Makar-Limanov invariant* $\mathrm{ML}(X)$ of X, which is defined as the algebra consisting of regular functions on X invariant under all algebraic $\mathbb{G}_a$-actions, is equal to the coordinate ring $\Gamma(X, \mathcal{O}_X)$ of X.

Among many important questions concerning this invariant, the understanding of its behavior under the operation consisting of taking cylinders $X \times \mathbb{A}^n$, $n \geq 1$, over a given affine variety X has focused a lot of attention during the last decade, in connexion with the Zariski Cancellation Problem. Of course, rigidity is lost even when passing to the cylinder $X \times \mathbb{A}^1$ since these admit non trivial $\mathbb{G}_a$-actions by translations on the second factor. But one could expect that such actions are essentially the unique possible ones in the sense that the projection $\mathrm{pr}_X : X \times \mathbb{A}^1 \to X$ is invariant for every $\mathbb{G}_a$-action on $X \times \mathbb{A}^1$, a property which translates algebraically to the fact that $\mathrm{ML}(X \times \mathbb{A}^1) = \Gamma(X, \mathcal{O}_X)$. This property was indeed established by Makar-Limanov [14] and this led to wonder

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more generally whether a rigid variety is *stably rigid* in the sense that the equality $\mathrm{ML}(X \times \mathbb{A}^n) = \Gamma(X, \mathcal{O}_X)$ holds for arbitrary $n \geq 1$. Stable rigidity of smooth affine curves is easily confirmed as a consequence of the fact that a smooth rigid curve does not admit any dominant morphism from the affine line, and more generally every rigid affine curve is in fact stably rigid [4]. Stable rigidity is also known to hold for smooth factorial rigid surfaces by virtue of a result of Crachiola [5], and without any indication of a potential counter-example, it seems that the implicit working conjecture so far has been that every rigid affine variety should be stably rigid.

In this article, we construct a smooth rigid surface S which fails stable rigidity very badly, the cylinder $S \times \mathbb{A}^2$ being essentially as remote as possible from a rigid variety in terms of richness of $\mathbb{G}_a$-actions on it. Here "richness" has to be interpreted in the sense of a slight weakening of the notion of flexibility introduced recently in [1, 2] that we call *flexibility in codimension one*: a normal affine variety X is said to be flexible in codimension one if for every closed point x outside a possibly empty closed subset of codimension at least two in X, the tangent space $T_x X$ of X at x is spanned by tangent vectors to orbits of $\mathbb{G}_a$-actions on X. Clearly, the Makar-Limanov invariant of a variety with this property is trivial, consisting of constant functions only. Now our main result can be stated as follows:

Theorem 1.1. *Let $V \subset \mathbb{P}^3$ be smooth cubic surface and let $D = V \cap H$ be a hyperplane section of V consisting of the union of a smooth conic and its tangent line. Then $S = V \setminus D$ is a smooth rigid affine surface whose cylinder $S \times \mathbb{A}^2$ is flexible in codimension one.*

A noteworthy by-product is that while the automorphism group $\mathrm{Aut}(S)$ of S is finite, actually isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if the cubic surface V is chosen general, Theorem 0.1 in [1] implies that $\mathrm{Aut}(S \times \mathbb{A}^2)$ acts infinitely transitively on the complement of a closed subset of codimension at least two in $S \times \mathbb{A}^2$.

Our construction is inspired by earlier work of Bandman and Makar-Limanov [3] which actually already contained the basic ingredients to construct a counter-example to stable rigidity, in the form of a lifting lemma for $\mathbb{G}_a$-actions which asserts that if $\mathrm{q} : Z \to Y$ is a line bundle over a normal affine variety Y then $\mathrm{ML}(Z) \subseteq \mathrm{ML}(Y)$, and an example of a non trivial line bundle $\mathrm{p} : L \to \tilde{S}$ over a smooth rational rigid affine surface $\tilde{S}$ for which $\mathrm{ML}(L) \subsetneqq \mathrm{ML}(\tilde{S})$. Indeed, with these informations, the property that $\mathrm{ML}(\tilde{S} \times \mathbb{A}^2)$ is a proper sub-algebra of $\mathrm{ML}(\tilde{S})$ could have been already deduced as follows: letting $\mathrm{p}' : L' \to \tilde{S}$ be a line bundle representing the class of the inverse of L in the Picard group of $\tilde{S}$, the lifting lemma applied to the rank 2 vector bundle $E = L \oplus L' = L \times_{\tilde{S}} L' \to \tilde{S}$ considered as a line bundle over L via the first projection implies that $\mathrm{ML}(E) \subseteq \mathrm{ML}(L) \subsetneqq \mathrm{ML}(\tilde{S})$. But combined with a result of Pavaman Murthy [15] which asserts in particular that every vector bundle on such a surface $\tilde{S}$ is isomorphic to the direct sum of its determinant and a trivial bundle, the construction of E guarantees that it is isomorphic to the trivial bundle $\tilde{S} \times \mathbb{A}^2$ and hence that $\mathrm{ML}(\tilde{S} \times \mathbb{A}^2) \subsetneqq \mathrm{ML}(\tilde{S})$.

Noting that the aforementioned result of Pavaman Murthy also applies to surfaces S as in Theorem 1.1 enables a similar construction as in the previous paragraph, provided that such an S admits a line bundle $\mathrm{p} : L \to S$ whose total space is flexible in codimension one, and that flexibility in codimension one lifts to total spaces of line bundles. The lifting property follows easily from the fact that every line bundle admits $\mathbb{G}_a$-linearizations, but the existence of a line bundle $\mathrm{p} : L \to S$ with the desired property is trickier to establish. To construct such a bundle, we exploit the fact that S admits an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{P}^1$, i.e. a faithfully flat morphism with generic fiber isomorphic to affine line. The strategy then consists in constructing a suitable $\mathbb{A}^1$-fibered affine surface $\pi_F : S_F \to \mathbb{P}^1$ flexible in codimension one and to which a variant of the famous Danielewski fiber product trick [6] can be applied to derive the existence of an affine threefold flexible in codimension one and carrying simultaneously the structure of a line bundle over S and S_F.

The article is organized as follows. In the first section we review basic results about rigid and flexible affine varieties, with a particular focus on the case of affine surfaces, and we establish that flexibility in codimension one does indeed lift to total spaces of line bundles (see Lemma 2.3). Section two is devoted to the study of the class of affine surfaces S considered in Theorem 1.1 and the construction of their aforementioned flexible mates S_F. The appropriate variant of the Danielewski fiber product trick needed to achieve the proof of Theorem 1.1 is discussed in the last section.

2. Preliminaries on (stable) rigidity and flexibility

2.1. Rigid and flexible affine varieties.

Given a normal complex affine variety $X = \mathrm{Spec}(A)$, we denote by $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X) \simeq \mathcal{H}om_X(\Omega^1_{X/\mathbb{C}}, \mathcal{O}_X)$ the sheaf of germs of $\mathbb{C}$-derivations from $\mathcal{O}_X$ to itself. It is a coherent sheaf of $\mathcal{O}_X$-modules whose global sections coincide with elements of the A-module $\mathrm{Der}_{\mathbb{C}}(A)$ of $\mathbb{C}$-derivations of A. We denote by $\mathrm{MLND}_{\mathbb{C}}(A)$ the sub-A-module of $\mathrm{Der}_{\mathbb{C}}(A)$ generated by locally nilpotent $\mathbb{C}$-derivations, i.e. $\mathbb{C}$-derivations $\partial : A \to A$ for which every element of A is annihilated by a suitable power of ∂. Recall that such derivations coincide precisely with velocity vector fields of $\mathbb{G}_a$-actions on X (see e.g. [12]).

Definition 2.1. A normal affine variety $X = \mathrm{Spec}(A)$ is called:

a) Rigid if $\mathrm{MLND}_{\mathbb{C}}(A) = \{0\}$, equivalently X does not admit non trivial $\mathbb{G}_a$-actions,

b) Flexible in codimension 1, or 1-flexible for short, if the support of the co-kernel of the natural homomorphism $\mathrm{MLND}_{\mathbb{C}}(A) \otimes_A \mathcal{O}_X \to \mathcal{D}er_X(\mathcal{O}_X)$ has codimension at least 2 in X.

2.1.1. The above definition of 1-flexibility says equivalently that there exists a closed subset $Z \subset X$ of codimension at least 2 such that the restriction of $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)$ over $X \setminus Z$ is generated by elements of $\mathrm{MLND}_{\mathbb{C}}(A)$. A closed point $x \in X$ at which the natural homomorphism $\mathrm{MLND}_{\mathbb{C}}(A) \otimes_A \mathcal{O}_{X,x} \to \mathcal{D}er_X(\mathcal{O}_X)_x$ is surjective is called a flexible point of X, this property being equivalent by virtue of Nakayama Lemma to the fact that the Zariski tangent space $T_x X$ of X at x is spanned by the tangent vectors to orbits of $\mathbb{G}_a$-actions on X. The set X_{flex} of flexible points is contained in the regular locus X_{reg} of X and is invariant under the action of the automorphism group $\mathrm{Aut}(X)$ of X. In particular, if there exists a flexible point $x \in X$ such that the complement of the $\mathrm{Aut}(X)$-orbit of x is contained in a closed subset of codimension at least two, then X is flexible in codimension 1.

2.1.2. We warn the reader that our definition of flexibility for a normal affine variety X is weaker than the one introduced earlier in [1, 2] which asks in addition that $X_{\mathrm{flex}} = X_{\mathrm{reg}}$. Since for a 1-flexible variety the set $X \setminus X_{\mathrm{flex}}$ has codimension at least two in X, this makes essentially no difference for global properties of X depending on regular functions, for instance the Makar-Limanov invariant of a 1-flexible affine variety is trivial. Furthermore, all the properties of the regular locus of a flexible variety in the sense of *loc. cit.* hold for the open subset X_{flex} of a 1-flexible variety X, for instance the sub-group of $\mathrm{Aut}(X)$ generated by its one-parameter unipotent sub-groups acts infinitely transitively on X_{flex}.

Clearly, the only 1-flexible affine curve is the affine line $\mathbb{A}^1$. While the classification of flexible affine surfaces in the stronger sense of [1, 2] is not known and most probably quite intricate, 1-flexible surfaces coincide with the so-called *Gizatullin surfaces* [13] with no non constant invertible functions. More precisely, we have the following characterization (see also [1, Example 2.3]).

Theorem 2.2. *For a normal affine surface S, the following are equivalent:*

a) *S is 1-flexible,*

b) *S admits two $\mathbb{A}^1$-fibrations over $\mathbb{A}^1$ with distinct general fibers,*

c) *$\Gamma(S, \mathcal{O}_S^*) = \mathbb{C}^*$ and S admits a normal projective completion $S \hookrightarrow V$ whose boundary is a chain of proper smooth rational curves supported on the regular locus of V.*

Proof. It is well known that every $\mathbb{A}^1$-fibration $q : S \to C$ over a smooth affine curve C arises as the algebraic quotient morphism $q : S \to S/\!\!/\mathbb{G}_a = \mathrm{Spec}(\Gamma(S, \mathcal{O}_S)^{\mathbb{G}_a})$ of a non trivial $\mathbb{G}_a$-action on S. In particular, the general fibers of such fibrations coincide with the general orbits of a $\mathbb{G}_a$-action on S. Since a flexible surface admits at least two $\mathbb{G}_a$-actions with distinct general orbits, this provides two $\mathbb{A}^1$-fibrations on S with distinct general fibers and whose respective base curves are isomorphic to $\mathbb{A}^1$ due to the fact that they are dominated by a general fiber of the other fibration. Conversely, let $q_i : S \to \mathbb{A}^1$, $i = 1, 2$, be $\mathbb{A}^1$-fibrations on S

associated with a pair of $\mathbb{G}_a$-actions σ_1 and σ_2 on S with distinct general orbits. Since the morphism $q_1 \times q_2 : S \to \mathbb{A}^2$ is quasi-finite [8, Lemma 2.21], it follows on the one hand that general orbits of σ_1 and σ_2 intersect each other transversally and on the other hand that the intersection S_0 of the fixed point loci of σ_1 and σ_2 is finite. This implies in turn that every point in $S \setminus S_0$ can be mapped by an element of the sub-group of $\mathrm{Aut}(S)$ generated by σ_1 and σ_2 to a point $p \in S$ at which a general orbit of σ_1 intersects a general orbit of σ_2 transversally. Such a point p is certainly flexible. Therefore every point outside the finite closed subset S_0 is a flexible point of S which proves the equivalence between a) and b). For the equivalence b) $\Leftrightarrow$ c) we refer the reader to [8] (in which the statement of Theorem 2.4 should actually be corrected to read: A normal affine surface with no non constant invertible functions is completable by a zigzag if and only if it admits two $\mathbb{A}^1$-fibrations whose general fibers do not coincide). □

2.2. Stable rigidity/stable flexibility.

Rigidity property for line bundles.

2.2.1. The total space of a line bundle $\mathrm{p} : L \to X$ over an affine variety $X = \mathrm{Spec}(A)$ always admits $\mathbb{G}_a$-actions by generic translations along the fibers of p, associated with locally nilpotent A-derivations of $\Gamma(L, \mathcal{O}_L)$. More precisely, these derivations corresponds to $\mathbb{G}_{a,X}$-actions on L, i.e. $\mathbb{G}_a$-actions on L by X-automorphisms, and are in one-to-one correspondence with global sections $s \in H^0(X, L)$ of L. Indeed, letting $\mathrm{p} : L = \mathrm{Spec}(\mathrm{Sym}(M^\vee)) \to X$, where $M \simeq H^0(X, L)$ is a locally free A-module of rank 1, one has $\Omega^1_{\mathrm{Sym}(M^\vee)/A} \simeq \mathrm{Sym}(M^\vee) \otimes_A M^\vee$ and the isomorphism

$$\mathrm{Der}_A(\mathrm{Sym}(M^\vee)) \simeq \mathrm{Hom}_{\mathrm{Sym}(M^\vee)}(\Omega^1_{\mathrm{Sym}(M^\vee)/A}, \mathrm{Sym}(M^\vee)) \simeq \mathrm{Sym}(M^\vee) \otimes_A M$$

identifies A-derivations of $\Gamma(L, \mathcal{O}_L) \simeq \mathrm{Sym}(M^\vee)$ with global sections of the pull-back p^*L of L to its total space. Since a $\mathbb{G}_{a,X}$-action on L corresponding to a locally nilpotent A-derivation ∂ of $\mathrm{Sym}(M^\vee)$ restricts on every fiber of $\mathrm{p} : L \to X$ to a $\mathbb{G}_a$-action which is either trivial or a translation, it follows that the corresponding section of p^*L is constant along the fibers of $\mathrm{p} : L \to X$ whence is the pull-back by p of a certain section $s_\partial \in H^0(X, L)$. Consersely, every global section $s \in H^0(X, L)$ gives rise to a $\mathbb{G}_{a,X}$-action on L defined by $\sigma_s(t, \ell) = \ell + ts(\mathrm{p}(\ell))$ where the fiberwise addition and multiplication are given by the vector space structure. More formally, viewing $\mathrm{p} : L \to X$ as a locally constant group scheme for the law $\mu : L \times_X L \to L$ induced by the addition of germs of sections, global sections $s \in H^0(X, L)$ give rise to homomorphisms $s : \mathbb{G}_{a,X} \to L$ of group schemes over X whence to $\mathbb{G}_{a,X}$-actions $\sigma_s = \mu \circ (s \times \mathrm{id}_L) : \mathbb{G}_{a,X} \times_X L \to L$ on L.

2.2.2. Even though they are no longer rigid, it is natural to wonder whether total spaces of line bundles $\mathrm{p} : L \to X$ over rigid varieties X stay "as rigid as possible" in the sense that they do not admit any $\mathbb{G}_a$-actions besides the $\mathbb{G}_{a,X}$-actions described above. For the trivial line bundle $\mathrm{pr}_X : X \times \mathbb{A}^1 \to X$, the question was settled affirmatively by Makar-Limanov [14] (see also [12, Proposition 9.23]). Let us briefly recall the argument for the convenience of the reader: viewing $\Gamma(X \times \mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}) = A[x] = \bigoplus_{i\geq 0} A\cdot x^i$ as a graded A-algebra, every nonzero locally nilpotent derivation ∂ of $A[x]$ associated with a non trivial $\mathbb{G}_a$-action on $X \times \mathbb{A}^1$ decomposes into a finite sum $\partial = \sum_{i\in\mathbb{Z}} \partial_i$ of nonzero homogeneous derivations $\partial_i : A[x] \to A[x]$ of degree $i \in \mathbb{Z}$, the top homogeneous component ∂_m being itself locally nilpotent. Note that $m \geq -1$ for a nonzero derivation and that derivations of the form $a\partial_x$ for a certain $a \in A \setminus \{0\}$ correspond to the case $m = -1$. On the other hand, if $m \geq 0$ then $\partial_m = x^m\tilde{\partial}_0$ for a certain derivation of degree 0 and since $\partial_m(x) \in x^{m+1}A \subset xA$, x must belong to the kernel of ∂_m. This implies that $\tilde{\partial}_0$ is a locally nilpotent derivation of degree 0 whose restriction to $A = A \cdot x^0 \subset A[x]$ is trivial as X is rigid. But since $x \in \mathrm{Ker}(\tilde{\partial}_0) = \mathrm{Ker}(\partial_m)$, $\tilde{\partial}_0$ whence ∂ would be the zero derivation, a contradiction.

2.2.3. In contrast, as mentioned in the introduction, it was discovered by Bandman and Makar-Limanov [3] that the above property can fail for non trivial line bundles. The fact that the rigid surfaces considered in Theorem 1.1 admit line bundles $\mathrm{p} : L \to S$ with 1-flexible total spaces (see §4.0.4 below) shows that such total spaces can be in general very far from being rigid.

Lifting flexibility in codimension one to split vector bundles.

2.2.4. The total space of the trivial line bundle $\mathrm{pr}_X : X \times \mathbb{A}^1 \to X$ over a 1-flexible (resp. flexible in the sense of [2]) affine variety $X = \mathrm{Spec}(A)$ is again 1-flexible (resp. flexible). Indeed, every locally nilpotent derivation ∂ of A canonically extends to a locally nilpotent derivation $\tilde{\partial}$ of $A[x]$ containing x in its kernel in such way that the projection $\mathrm{pr}_X : X \times \mathbb{A}^1 \to X$ is equivariant for the corresponding $\mathbb{G}_a$-actions on X and $X \times \mathbb{A}^1$ respectively. It follows that for every point $p \in X \times \mathbb{A}^1$ dominating a flexible point x of X, say for which $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)_x$ is generated by the images of locally nilpotent derivations $\partial_1, \ldots, \partial_r$ of A, the $\mathcal{O}_{X\times\mathbb{A}^1,p}$-module $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_{X\times\mathbb{A}^1})_p$ is generated by the images of $\tilde{\partial}_1, \ldots, \tilde{\partial}_r$ together with the image of the locally nilpotent A-derivation ∂_x of $A[x]$. This implies that $\mathrm{pr}_X^{-1}(X_{\mathrm{flex}}) \subset (X \times \mathbb{A}^1)_{\mathrm{flex}}$ and hence that the set of non flexible points in $X \times \mathbb{A}^1$ has codimension at least two. Furthermore, $(X \times \mathbb{A}^1)_{\mathrm{flex}}$ coincides with $(X \times \mathbb{A}^1)_{\mathrm{reg}}$ in the case where $X_{\mathrm{flex}} = X_{\mathrm{reg}}$.

2.2.5. Even though different results related with lifts of $\mathbb{G}_a$-actions on an affine variety X to $\mathbb{G}_a$-actions on total spaces of line bundles $\mathrm{p} : L \to X$ over it exist in the literature (in particular, [3, Lemma 9] and [1, Corollary 4.5]), it seems that the question whether 1-flexibility or flexibility of X lifts to total spaces of arbitrary line

bundles over it has not been clearly settled yet. This is fixed by the following cost free generalization:

Lemma 2.3. *Let X be a normal affine variety and let $\mathrm{p} : E \to X$ be a vector bundle which splits as a direct sum of line bundles. If X is 1-flexible (resp. flexible) then so is the total space of E.*

Proof. Since E is isomorphic to the fiber product $L_1 \times_X L_2 \cdots \times_X L_r$ of line bundles $\mathrm{p}_i : L_i \to X$, we are reduced by induction to the case of a line bundle $\mathrm{p} : L \to Y$ over a 1-flexible (resp. flexible) affine variety. Recall that for a connected algebraic group G acting on a normal variety Y, there exists an exact sequence of groups

$$0 \to H^1_{\mathrm{alg}}(G, \Gamma(Y, \mathcal{O}_Y^*)) \to \mathrm{Pic}^G(Y) \xrightarrow{\alpha} \mathrm{Pic}(Y) \to \mathrm{Pic}(G)$$

where $\mathrm{Pic}^G(Y)$ denotes the group of G-linearized line bundles on Y and where $H^1_{\mathrm{alg}}(G, \Gamma(Y, \mathcal{O}_Y^*))$ parametrizes isomorphy classes of G-linearizations of the trivial line bundle over Y (see e.g. [7, Chap. 7]). In the case where $G = \mathbb{G}_a$, this immediately implies that every line bundle $\mathrm{p} : L \to Y$ admits a $\mathbb{G}_a$-linearization (note furthermore that such a linearization is unique up to isomorphism provided that $\Gamma(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$).

It follows in particular that every $\mathbb{G}_a$-action on Y can be lifted to a $\mathbb{G}_a$-action on L preserving the zero section $Y_0 \subset L$ and for which the structure morphism $\mathrm{p} : L \to Y$ is $\mathbb{G}_a$-invariant. So the 1-flexibility (resp. the flexibility) of L follows from that of Y thanks to [1, Corollary 4.5]. But let us provide a self-contained argument: the above property translates algebraically to the fact that every locally nilpotent derivation ∂ of $\Gamma(Y, \mathcal{O}_Y)$ extends to a locally nilpotent derivation $\tilde{\partial}$ of $\Gamma(L, \mathcal{O}_L)$ mapping the ideal I_{Y_0} of Y_0 into itself and such that the induced derivation on $\Gamma(Y_0, \mathcal{O}_{Y_0}) = \Gamma(L, \mathcal{O}_L)/I_{Y_0}$ coincides with ∂ via the isomorphism $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\sim} \Gamma(Y_0, \mathcal{O}_{Y_0})$ induced by the restriction of p. Since Y is affine, given any point $\ell \in L$, we can find a global section $s \in H^0(Y, L)$ which does not vanish at $y = \mathrm{p}(\ell)$. Now if y is a flexible point of Y, say for which $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_Y)_y$ is generated by the images of locally nilpotent derivations $\partial_1, \ldots, \partial_r$ of $\Gamma(Y, \mathcal{O}_Y)$ then $\ell_0 = \mathrm{p}^{-1}(y) \cap Y_0$ is a flexible point of L at which $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_L)_{\ell_0}$ is generated by the lifts $\tilde{\partial}_1, \ldots, \tilde{\partial}_r$ of $\partial_1, \ldots, \partial_r$ together with the locally nilpotent derivation ∂_s of $\Gamma(L, \mathcal{O}_L)$ corresponding to the $\mathbb{G}_{a,Y}$-action $\sigma_s : \mathbb{G}_{a,Y} \times_Y L \to L$ associated with s (see §2.2.1 above). Furthermore, since s does vanish at y, the $\mathbb{G}_a$-action induced by σ_s on $\mathrm{p}^{-1}(y)$ is transitive, and so $\mathrm{p}^{-1}(y)$ consists of flexible points of L. This shows that $\mathrm{p}^{-1}(Y_{\mathrm{flex}}) \subset L_{\mathrm{flex}}$ and completes the proof. □

3. Construction of rigid and 1-flexible $\mathbb{A}^1$-fibered surfaces over over $\mathbb{P}^1$

In this section, we first consider affine surfaces S_R which arise as complements of well-chosen hyperplane sections of a smooth cubic surface in $\mathbb{P}^3$. We check that

they are rigid by computing their automorphism groups and we exhibit certain $\mathbb{A}^1$-fibrations $\pi_R : S_R \to \mathbb{P}^1$ on them. We then construct auxiliary 1-flexible $\mathbb{A}^1$-fibered surfaces $\pi_F : S_F \to \mathbb{P}^1$ which will be used later on in section 4 for the proof of Theorem 1.1.

3.1. A family of rigid affine cubic surfaces . Most of the material of this subsection is borrowed from [10] to which we refer the reader for the details.

3.1.1. Given a smooth cubic surface $V \subset \mathbb{P}^3$ and a line L on it, the restriction to V of the linear pencil $\mathcal{H}_L = |\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}_L|$ on $\mathbb{P}^3$ generated by planes containing L can be decomposed as $\mathcal{H}_L\,|_V = \mathcal{L} + L$ where $\mathcal{L}$ is a base point free pencil defining a conic bundle $\Phi_{\mathcal{L}} : V \to \mathbb{P}^1$ with five degenerate fibers each consisting of the union of two lines. The restriction $\Phi_{\mathcal{L}}\,|_L : L \to \mathbb{P}^1$ is a double cover and for every branch point $x \in \mathbb{P}^1$ of $\Phi_{\mathcal{L}}\,|_L$, the intersection of V with the corresponding hyperplane $H_x \in \mathcal{H}$ consists either of a smooth conic tangent to L or two distinct lines intersecting L in a same point, which is then an Eckardt point of V. The second case does not occur if V is chosen general. We fix from now on a cubic surface V, a line L on it and a hyperplane section $D = H \cap V$ for which $D = L + C$ where C is a smooth conic tangent to L at a point $p \in L$.

3.1.2. Given a pair (V, D) where $D = L + C$ as above, the surface $S_R = V \setminus D$ is affine as D is a hyperplane section of V. It comes equipped with an $\mathbb{A}^1$-fibration $\pi_R : S_R \to \mathbb{P}^1$ which is obtained as follows: we let $\mu : V \to \mathbb{P}^2$ be the birational morphism obtained by contracting a 6-tuple of disjoint lines $L, F_1, \ldots, F_5 \subset V$ with the property that each F_i, $i = 1, \ldots, 5$, intersects C transversally. Since L is tangent to C, the image $\mu_*(C)$ of C in $\mathbb{P}^2$ is a cuspidal cubic. The rational pencil on $\mathbb{P}^2$ generated by $\mu_*(C)$ and three times its tangent T at its unique singular point $\mu(p)$ lifts to a rational pencil $\overline{q} : V \dashrightarrow \mathbb{P}^1$ having the divisors $C + \sum_{i=1}^5 F_i$ and $3T + L$ as singular members. Letting $\tau : \hat{V} \to V$ be a minimal resolution of $\overline{q}$, the induced morphism $\overline{q} \circ \tau : \hat{V} \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration whose restriction to $S_R = V \setminus D \simeq \hat{V} \setminus \tau^{-1}D$ is an $\mathbb{A}^1$-fibration $\pi_R : S_R \to \mathbb{P}^1$ with two degenerate fibers: one is irreducible of multiplicity three consisting of the intersection of the proper transform of T with S_R and the other is reduced, consisting of the disjoint union of the curves $F_i \cap S_R \simeq \mathbb{A}^1$, $i = 1, \ldots, 5$ (see Figure 3.1).

Remark 3.1. Choosing an alternative 6-tuple of disjoint lines $F_{0,1}F_{0,2}, F_{\infty,1}, \ldots, F_{\infty,4}$ such that $F_{\infty,1}, \ldots, F_{\infty,4}$ intersect C transversally while $F_{0,1}$ and $F_{0,2}$ intersects L but not C, we obtain another contraction morphism $\tilde{\mu} : V \to \mathbb{P}^2$ for which the proper transforms of C and L are respectively a conic and its tangent line at the point $\tilde{\mu}(p)$. One checks that the lift to V of the rational pencil on $\mathbb{P}^2$ generated by $\tilde{\mu}_*(C)$ and $2\mu_*(L)$ restricts on S_R to an $\mathbb{A}^1$-fibration $\pi'_R : S_R \to \mathbb{P}^1$ with two reducible degenerate fibers: one consisting of the disjoint union of the curves $F_{0,i} \cap S \simeq \mathbb{A}^1$, $i = 1, 2$, both occuring with multiplicity two and the other one consisting of the disjoint union of the reduced curves $F_{\infty,i} \cap S_R \simeq \mathbb{A}^1$, $i = 1, \ldots, 4$.

The description of the degenerate fibers shows that this second $\mathbb{A}^1$-fibration is not isomorphic to the one $\pi_R : S_R \to \mathbb{P}^1$, so that S_R carries at least two distinct types of $\mathbb{A}^1$-fibrations over $\mathbb{P}^1$.

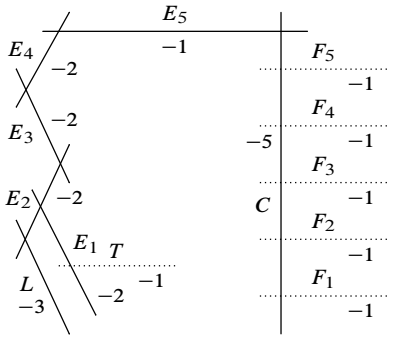


Figure 3.1. The total transform of $\mu_*(C) \cup T \subset \mathbb{P}^2$ in the minimal resolution $\tau : \hat{V} \to V$ of $\overline{q}$. The plain curves correspond to the irreducible components of $\tau^{-1}(D)$ and the exceptional divisors E_i of τ are numbered according to the order they are extracted.

3.1.3. To determine the automorphism group of $S_R = V \setminus D$ we first notice that the subgroup $\mathrm{Aut}(V, D)$ of $\mathrm{Aut}(V)$ consisting of automorphisms of V which leave D globally invariant can be identified in a natural way with a subgroup of $\mathrm{Aut}(S_R)$. The latter is always finite, and even trivial if the cubic surface V is chosen general. On the other hand, S_R admits at least another natural automorphism which is obtained as follows: the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from the point $p = L \cap C$ induces a rational map $V \dashrightarrow \mathbb{P}^2$ with p as a unique proper base point and whose lift to the blow-up $\alpha : W \to V$ of V at p coincides with the morphism $\theta : W \to \mathbb{P}^2$ defined by the anticanonical linear system $|-K_W|$. The latter factors into a birational morphism $W \to Y$ contracting the proper transform of L followed by a Galois double cover $Y \to \mathbb{P}^2$ ramified over an irreducible quartic curve Δ with a unique double point located at the image of L. The non trivial involution of the double cover $Y \to \mathbb{P}^2$ induces an involution $G_W : W \to W$ fixing L and exchanging the proper transform of C with the exceptional divisor E of α. The former descends to a birational involution $G_{V,p} : V \dashrightarrow V$ which restricts further to a biregular involution $j_{G_{V,p}}$ of $S_R = V \setminus D$.

The following description of the automorphism group of S_R shows in particular that these surfaces are rigid:

Lemma 3.2. *For a surface $S_R = V \setminus D$ as above, there exists a split exact sequence*

$$0 \to \mathrm{Aut}(V, D) \to \mathrm{Aut}(S_R) \to \{\mathrm{id}_{S_R}, j_{G_{V,p}}\} \simeq \mathbb{Z}_2 \to 0.$$

Proof. We interpret every automorphism of S_R as a birational self-map $f : V \dashrightarrow V$ of V restricting to an isomorphism from $S_R = V \setminus D$ to itself. Since $f \in \mathrm{Aut}(V, D)$ in case it is biregular, it is enough to show that either f or $G_{V,p} \circ f$ is biregular. To establish this alternative, it suffices to check that the lift $f_W = \alpha^{-1} f \alpha : W \dashrightarrow W$ of f to W is a biregular morphism, hence an automorphism of the pair $(W, \alpha^{-1}(D)_{\mathrm{red}})$. Indeed if so, then f_W preserves the union of E and the proper transform of C as these are the only (-1)-curves contained in the support of $\alpha^{-1}(D)_{\mathrm{red}}$. Since by construction G_W exchanges E and the proper transform of C, it follows that either f_W or $G_W \circ f_W$ leaves E, the proper transform of C and the proper transform of L invariant. This implies in turn that either $f = \alpha f_W \alpha^{-1}$ or $\alpha G_W \circ f_W \alpha^{-1} = (\alpha G_W \alpha^{-1}) \circ (\alpha f_W \alpha^{-1}) = G_{V,p} \circ f$ is a biregular automorphism of V.

To show that f_W is a biregular automorphism of W, we consider the lift $\tilde{f} = \sigma^{-1} \circ f \circ \sigma : \tilde{V} \dashrightarrow \tilde{V}$ of f to the variety $\tilde{\alpha} : \tilde{V} \to W$ obtained from W by blowing-up further the intersection point of E and of the proper transform of C, say with exceptional divisor $\tilde{E}$. We identify S_R with the complement in $\tilde{V}$ of the SNC divisor $\tilde{D} = L \cup C \cup E \cup \tilde{E}$. Now suppose by contradiction that $\tilde{f}$ is strictly birational and consider its minimal resolution $\tilde{V} \xleftarrow{\beta} X \xrightarrow{\beta'} \tilde{V}'$. Recall that the minimality of the resolution implies in particular that there is no (-1)-curve in X which is exceptional for β and β' simultaneously. Furthermore, since $\tilde{V}$ is smooth and $\tilde{D}$ is an SNC divisor, β' decomposes into a finite sequence of blow-downs of successive (-1)-curves supported on the boundary $B = \beta^{-1}(\tilde{D})_{\mathrm{red}} = (\beta')^{-1}(\tilde{D})_{\mathrm{red}}$ with the property that at each step, the proper transform of B is again an SNC divisor. The structure of $\tilde{D}$ implies that the only possible (-1)-curve in B which is not exceptional for β is the proper transform of $\tilde{E}$, but after its contraction, the proper transform of B would no longer be an SNC divisor, a contradiction. So $\tilde{f} : \tilde{V} \to \tilde{V}$ is a morphism and the same argument shows that it does not contract any curve in the boundary $\tilde{D}$. Thus $\tilde{f}$ is a biregular automorphism of $\tilde{V}$, in fact, an element of $\mathrm{Aut}(\tilde{V}, \tilde{D})$. Since $\tilde{E}$ is the unique (-1)-curve contained in the support of $\tilde{D}$ it must be invariant by $\tilde{f}$ which implies in turn that $f_W = \tilde{\alpha} \tilde{f} \tilde{\alpha}^{-1}$ is a biregular automorphism of the pair $(W, \alpha^{-1}(D)_{\mathrm{red}})$, as desired. □

3.2. Flexible mates. In this subsection, we construct 1-flexible affine surfaces S_F admitting $\mathbb{A}^1$-fibrations $\pi_F : S_F \to \mathbb{P}^1$ whose degenerate fibers resemble the ones of the fibrations $\pi_R : S_R \to \mathbb{P}^1$ described in §3.1.2 above. A more precise interpretation of this resemblance, going beyond the bare fact that the number of their irreducible components and their respective multiplicities are the same, will be given in section 4 below.

3.2.1. For the construction, we start with a Hirzebruch surface $\pi_n : \mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \to \mathbb{P}^1$, $n \geq 0$, in which we fix an ample section $C \simeq \mathbb{P}^1$ of π_n and two distinct fibers, say $F_0 = \pi_n^{-1}(p_0)$ and $F_\infty = \pi_n^{-1}(p_\infty)$, where $p_0, p_\infty \in \mathbb{P}^1$. We let

$\sigma : X \to \mathbb{F}_n$ be the birational map obtained by the following sequence of blow-ups:

Step 1 consists of the blow-up of five distinct points $p_{\infty,1}, \ldots, p_{\infty,5}$ on $F_\infty \setminus C$ with respective exceptional divisors $G_{\infty,1}, \ldots, G_{\infty,5}$.

Step 2 consists of the blow-up of a point $p_{0,1} \in F_0 \setminus C$ with exceptional divisor E_1, followed by the blow-up of the intersection point $p_{0,2}$ of the proper transform of F_0 with E_1, with exceptional divisor E_2, then followed by the blow-up of the intersection point $p_{0,3}$ of the proper transform of F_0 with E_2, with exceptional divisor E_3. Finally, we blow-up a point $p_{0,4} \in E_3$ distinct from the intersection points of E_3 with the proper transforms of F_0 and E_2 respectively. We denote the last exceptional divisor produced by $G_{0,1}$.

The structure morphism $\pi_n : \mathbb{F}_n \to \mathbb{P}^1$ lifts to a $\mathbb{P}^1$-fibration $\overline{p} = \pi_n \circ \sigma : X \to \mathbb{P}^1$ with two degenerate fibers $\overline{p}^{-1}(p_0) = F_0 + E_1 + 2E_2 + 3E_3 + 3G_{0,1}$ and $\overline{p}^{-1}(p_\infty) = F_\infty + \sum_{i=1}^5 G_{\infty,i}$. The inverse image by σ of the divisor $F_0 \cup C \cup F_\infty$ is pictured in Figure 3.2. Letting S_F be the open complement in X of the divisor $B = F_\infty \cup C \cup F_0 \cup E_3 \cup E_2 \cup E_1$, the restriction of $\overline{p}$ to S_F is an $\mathbb{A}^1$-fibration $\pi_F : S_F \to \mathbb{P}^1$ with two degenerate fibers: the one $\pi_F^{-1}(p_0)$ is irreducible of multiplicity three consisting of the intersection of $G_{0,1}$ with S_F and the other one $\pi_F^{-1}(p_\infty)$ is reduced, consisting of the disjoint union of the curves $G_{\infty,i} \cap S_F \simeq \mathbb{A}^1$, $i = 1, \ldots, 5$.

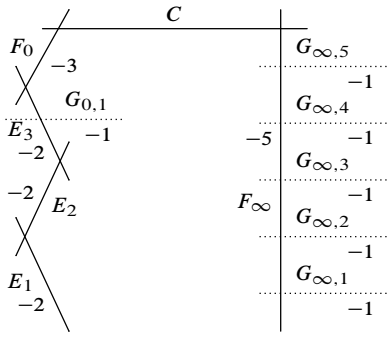


Figure 3.2. The total transform of $F_0 \cup C \cup F_\infty \subset \mathbb{F}_n$ in X. The plain curves correspond to irreducible components of the boundary divisor B.

Lemma 3.3. *A surface $S_F = X \setminus B$ as above is affine and 1-flexible.*

Proof. By construction, B is chain of smooth complete rational curves. So the 1-flexibility of S_F will follows from Theorem 2.2 provided that S_F is indeed affine and has no non constant invertible functions. Since $\pi_F : S_F \to \mathbb{P}^1$ is an $\mathbb{A}^1$-fibration, an invertible function on S_F is constant in restriction to every non degenerate fiber of π_F and hence has the form $f \circ \pi_F$ for a certain global invertible function f on $\mathbb{P}^1$. So such a function is certainly constant. To establish the affineness

of S_F, we first observe that S_F does not contain a complete curve. Indeed, otherwise since the points blown-up by $\sigma : X \to \mathbb{F}_n$ are contained in $\mathbb{F}_n \setminus C$, the image by σ of such curve would be a complete curve in $\mathbb{F}_n$ which does not intersect C, in contradiction with the ampleness of C in $\mathbb{F}_n$. On the other hand, since C has positive self-intersection in $\mathbb{F}_n$, whence in X, one checks by direct computation that for $a_\infty, a, a_0, a_1, a_2, a_3 \in \mathbb{Z}_{>0}$ such that $a_0 \gg a_3 \gg a_2 \gg a_1$ and $a \gg \max(a_0, a_\infty)$, the effective divisor $\tilde{B} = a_\infty F_\infty + aC + a_0 F_0 + a_1 E_1 + a_2 E_2 + a_3 E_3$ has positive self-intersection and positive intersection which each of its irreducible components. It then follows from the Nakai–Moishezon criterion that $\tilde{B}$ is an ample effective divisor supported on B, and hence that $S_F = X \setminus B$ is affine. □

Remark 3.4. In the construction of §3.2.1, one can replace Step 1 and 2 by the following alternative sequence of blow-ups $\sigma' : X' \to \mathbb{F}_n$:

Step 1′ consists of the blow-up of four distinct points $p'_{\infty,1}, \ldots, p'_{\infty,4}$ on $F_\infty \setminus C$ with respective exceptional divisors $G'_{\infty,1}, \ldots, G'_{\infty,4}$.

Step 2′ consists of the blow-up of a point $p'_{0,1} \in F_0 \setminus C$ with exceptional divisor E'_1, followed by the blow-up of the intersection point $p'_{0,2}$ of the proper transform of F_0 with E'_1, with exceptional divisor E'_2, then followed by the blow-up of a pair of distinct points $p'_{0,3}$ and $p''_{0,3}$ on E'_2 distinct from the intersection points of E'_2 with the proper transforms of F_0 and E'_1, with respective exceptional divisors $G'_{0,1}$ and $G'_{0,2}$.

The morphism $\overline{p}' = \pi_n \circ \sigma' : X' \to \mathbb{P}^1$ is then a $\mathbb{P}^1$-fibration with two degenerate fibers $\overline{p}'^{-1}(p_0) = F_0 + E'_1 + 2E'_2 + 2G'_{0,1} + 2G'_{0,2}$ and $\overline{p}'^{-1}(p_\infty) = F_\infty + \sum_{i=1}^4 G'_{\infty,i}$. The same argument as in the proof of Lemma 3.3 above shows that the complement in X' of the chain of smooth complete rational curves $B' = F_\infty \cup C \cup F_0 \cup E'_1 \cup E'_2$ is a 1-flexible affine surface, on which $\overline{p}'$ restricts to an $\mathbb{A}^1$-fibration $\pi'_F : S'_F \to \mathbb{P}^1$ with two degenerate fibers consisting respectively of the disjoint union of $G'_{0,i} \cap S'_F \simeq \mathbb{A}^1$, $i = 1, 2$ both occurring with multiplicity 2 and of the disjoint union of the reduced curves $G'_{\infty,i} \cap S'_F \simeq \mathbb{A}^1$, $i = 1, \ldots, 4$. So $\pi'_F : S'_F \to \mathbb{P}^1$ resembles the alternative $\mathbb{A}^1$-fibration $\pi'_R : S_R \to \mathbb{P}^1$ described in Remark 3.1 above.

4. Rigidity lost

The last ingredient needed to derive Theorem 1.1 is the following result:

Proposition 4.1. *Let $\pi_R : S_R \to \mathbb{P}^1$ and $\pi_F : S_F \to \mathbb{P}^1$ be a pair of $\mathbb{A}^1$-fibered surfaces as constructed in §3.1.2 and 3.2.1 above. Then there exists an algebraic space $\delta : \mathfrak{C} \to \mathbb{P}^1$ such that π_R and π_F factor respectively through étale locally trivial $\mathbb{A}^1$-bundles $\rho_R : S_R \to \mathfrak{C}$ and $\rho_F : S_F \to \mathfrak{C}$.*

Let us first explain how to derive the 1-flexibility of the cylinder $S_F \times \mathbb{A}^2$ from this Proposition.

4.0.2. Recall that since the automorphism group of $\mathbb{A}^1$ is the affine group $\mathrm{Aff}_1 = \mathbb{G}_m \ltimes \mathbb{G}_a$, every étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to \mathfrak{C}$ is in fact an affine-linear bundle. This means that there exists a line bundle $\mathrm{p} : L \to \mathfrak{C}$ such that $\rho : S \to \mathfrak{C}$ has the structure of an étale L-torsor, that is, an étale locally trivial principal homogeneous bundle under L, considered as a group space over $\mathfrak{C}$ for the group law induced by the addition of germs of sections. Isomorphy classes of such principal homogeneous L-bundles are then classified by the cohomology group $H^1_{\text{ét}}(\mathfrak{C}, L)$ (see e.g. [9, §1.2]).

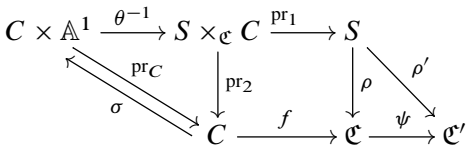
4.0.3. So Proposition 4.1 implies in particular that $\rho_R : S_R \to \mathfrak{C}$ and $\rho_F : S_F \to \mathfrak{C}$ can be equipped with the structure of principal homogeneous bundles under suitable line bundles $\mathrm{p}_R : L_R \to \mathfrak{C}$ and $\mathrm{p}_F : L_F \to \mathfrak{C}$ respectively. As a consequence, the fiber product $Z = S_R \times_{\mathfrak{C}} S_F$ is simultaneously equipped via the first and second projection with the structure of a principal homogeneous bundle under the line bundles $\rho_R^* L_F$ and $\rho_F^* L_R$ respectively. But since S_R and S_F are both affine, the vanishing of $H^1_{\text{ét}}(S_R, \rho_R^* L_F)$ and $H^1_{\text{ét}}(S_F, \rho_F^* L_R)$ implies that $\mathrm{pr}_1 : Z \to S_R$ and $\mathrm{pr}_2 : Z \to S_F$ are the trivial $\rho_R^* L_F$-torsor and $\rho_F^* L_R$-torsor respectively. In other word, Z carries simultaneously the structure of a line bundle over S_R and S_F.

4.0.4. Now since S_F is 1-flexible by virtue of Theorem 2.2, we deduce from Lemma 2.3 that Z is 1-flexible. Furthermore, the same Lemma implies that given any line bundle $\mathrm{p} : Z' \to S_R$, the total space of the rank 2 vector bundle $\mathrm{pr}_1 \times \mathrm{p} : E = Z' \times_X Z \to S_R$ over S_R is 1-flexible. On the other hand, it follows from [15, Theorem 3.2] that every rank 2 vector bundle $E \to S_R$ splits off a trivial factor, whence is isomorphic to the direct sum of its determinant $\det E$ and of the trivial line bundle. Choosing for Z' a line bundle representing the inverse of the class of $\mathrm{pr}_1 : Z \to S_R$ in the Picard group $\mathrm{Pic}(S_R)$ of S_R yields a vector bundle $E = Z' \times_X Z \to S_R$ with trivial determinant, whence isomorphic to the trivial bundle $S_R \times \mathbb{A}^2$, and with 1-flexible total space.

4.1. Proof of Proposition 4.1.

4.1.1. To prove Proposition 4.1, we first observe that if it exists, an algebraic space $\delta : \mathfrak{C} \to \mathbb{P}^1$ with the property that a given $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{P}^1$ on a smooth surface S factors as $\delta \circ \rho$, where $\rho : S \to \mathfrak{C}$ is an étale locally trivial $\mathbb{A}^1$-bundle, is unique up to isomorphism of spaces over $\mathbb{P}^1$. Indeed, suppose that $\delta' : \mathfrak{C} \to \mathbb{P}^1$ is another such space for which we have $\pi = \delta' \circ \rho'$ where $\rho' : S' \to \mathfrak{C}'$ is an étale locally trivial $\mathbb{A}^1$-bundle. The closed fibers of ρ and ρ' being both in one-to-one correspondence with irreducible components of closed fibers of π, it follows that for every closed point $c \in \mathfrak{C}$ there exists a unique closed point $c' \in \mathfrak{C}'$ such that $\delta(c) = \delta'(c')$ and $\rho^{-1}(c) = (\rho')^{-1}(c') \subset \pi^{-1}(\delta(c))$. So the correspondence $c \mapsto c'$ defines a bijection $\psi : \mathfrak{C} \to \mathfrak{C}'$ such that $\rho' = \psi \circ \rho$ and $\delta = \delta' \circ \psi$.

Letting $f : C \to \mathfrak{C}$ be an étale cover over which $\rho : S \to \mathfrak{C}$ becomes trivial, say with isomorphism $\theta : S \times_{\mathfrak{C}} C \xrightarrow{\sim} C \times \mathbb{A}^1$, and choosing a section $\sigma : C \to C \times \mathbb{A}^1$ of $\mathrm{pr}_C : C \times \mathbb{A}^1 \to C$, the composition $\psi \circ f = \psi \circ f \circ \mathrm{pr}_C \circ \sigma$ is equal to $\psi \circ \rho \circ \mathrm{pr}_1 \circ \theta^{-1} \circ \sigma$ and hence to $\rho' \circ \mathrm{pr}_1 \circ \theta^{-1} \circ \sigma$ by construction of ψ.



This implies that $\psi \circ f : C \to \mathfrak{C}'$ is a morphism whence that $\psi : \mathfrak{C} \to \mathfrak{C}'$ is a morphism since being a morphism is a local property with respect to the étale topology. The same argument on an étale cover $f' : C' \to \mathfrak{C}'$ over which $\rho' : S' \to \mathfrak{C}'$ becomes trivial implies that the set-theoretic inverse ψ^{-1} of ψ is also a morphism, and so $\psi : \mathfrak{C} \to \mathfrak{C}'$ is an isomorphism of spaces over $\mathbb{P}^1$.

4.1.2. In what follows, given an $\mathbb{A}^1$-fibered surface $\pi : S \to \mathbb{P}^1$, we use the notation $S/\mathbb{A}^1$ to refer to an algebraic space $\delta : \mathfrak{C} \to \mathbb{P}^1$ with the property that π factors through an étale locally trivial $\mathbb{A}^1$-bundle $\rho : S \to \mathfrak{C}$. The previous observation implies that its existence is a local problem with respect to the Zariski topology on $\mathbb{P}^1$. More precisely, we may cover $\mathbb{P}^1$ by finitely many affine open subsets U_i, $i = 1, \ldots, r$ over which the restriction of $\pi : S \to \mathbb{P}^1$ is an $\mathbb{A}^1$-fibration with a most a degenerate fiber, say $\pi^{-1}(p_i)$ for some $p_i \in U_i$. Since the restriction of π over $U_{i,*} = U_i \setminus \{p_i\}$ is then a Zariski locally trivial $\mathbb{A}^1$-bundle, we see that if $\delta_i : \mathfrak{C}_i = \pi^{-1}(U_i)/\mathbb{A}^1 \to U_i$ exists then the restriction of δ_i over $U_i \setminus \{p_i\}$ is an isomorphism of schemes over U_i. This implies that the isomorphisms $\delta_j^{-1} \circ \delta_i \mid_{\delta^{-1}(U_{i,*} \cap U_{j,*})} : \delta_i^{-1}(U_{i,*} \cap U_{j,*}) \to \delta_j^{-1}(U_{i,*} \cap U_{j,*})$, $i, j = 1, \ldots, r$, satisfy the usual cocyle condition on triple intersections whence that the algebraic space $\delta : \mathfrak{C} = S/\mathbb{A}^1 \to \mathbb{P}^1$ with the desired property is obtained by gluing the local ones $\delta_i : \mathfrak{C}_i \to U_i$, $i = 1, \ldots, r$ along their respective open sub-schemes $\delta_i^{-1}(U_{i,*} \cap U_{j,*}) \subset \mathfrak{C}_i$, $i, j = 1, \ldots, r$ via these isomorphisms.

4.1.3. Now we turn more specifically to the case of the $\mathbb{A}^1$-fibrations $\pi_R : S_R \to \mathbb{P}^1$ and $\pi_F : S_F \to \mathbb{P}^1$ constructed in §3.1.2 and §3.2.1 respectively. Both have exactly two degenerate fibers, one irreducible of multiplicity three and the other one consisting of the disjoint union of five reduced curves. So up to an automorphism of $\mathbb{P}^1$ we may choose a pair of distinct point $p_0, p_\infty \in \mathbb{P}^1$ such that $\pi_R^{-1}(p_0) = 3T \cap S_R$, $\pi_F^{-1}(p_0) = 3G_{0,1} \cap S_F$, $\pi_R^{-1}(p_\infty) = \bigsqcup_{i=1}^5 F_i \cap S_R$ and $\pi_F^{-1}(p_\infty) = \bigsqcup_{i=1}^5 G_{\infty,i} \cap S_F$. Letting $U_0 = \mathbb{P}^1 \setminus \{p_\infty\}$ and $U_\infty = \mathbb{P}^1 \setminus \{p_0\}$, the existence and isomorphy of the algebraic spaces $\pi_R^{-1}(U_0)/\mathbb{A}^1$ and $\pi_F^{-1}(U_0)/\mathbb{A}^1$ (resp. $\pi_R^{-1}(U_\infty)/\mathbb{A}^1$ and $\pi_F^{-1}(U_\infty)/\mathbb{A}^1$) hence of those $S_R/\mathbb{A}^1$ and $S_F/\mathbb{A}^1$, follows from a reinterpretation of a description due to Fieseler [11]:

– Since the unique degenerate fiber of the restriction of π_R (resp. π_F) over U_∞ is reduced, consisting of five irreducible components, $\pi_R^{-1}(U_\infty)/\mathbb{A}^1$ and

$\pi_F^{-1}(U_\infty)/\mathbb{A}^1$ are isomorphic to the scheme $\delta_\infty : \mathfrak{C}_\infty \to U_\infty$ obtained from U_∞ by replacing the point p_∞ by five copies $p_{\infty,1}, \ldots, p_{\infty,5}$ of itself, one for each irreducible component of $\pi_R^{-1}(p_\infty)$ (resp. $\pi_F^{-1}(p_\infty)$). More explicitly, π_R restricts on $S_{R,\infty,i} = \pi_R^{-1}(U_\infty) \setminus \bigsqcup_{j\neq i}(F_j \cap S_R)$, $i = 1, \ldots, 5$, to an $\mathbb{A}^1$-fibration $\pi_{R,\infty,i} : S_{R,\infty,i} \to U_\infty$ with no degenerate fiber over the factorial base $U_\infty \simeq \mathbb{A}^1$ and hence is a trivial $\mathbb{A}^1$-bundle. So $S_{R,\infty,i}/\mathbb{A}^1 \simeq U_\infty$ and $\pi_R^{-1}(U_\infty)/\mathbb{A}^1$ is thus isomorphic to the U_∞-scheme obtained by gluing five copies $\delta_{\infty,i} : U_{\infty,i} \xrightarrow{\sim} U_\infty$, $i = 1, \ldots, 5$, of U_∞ by the identity along the open subsets $U_{\infty,i} \setminus \{p_{\infty,i}\}$, where $p_{\infty,i} = \delta_{\infty,i}^{-1}(p_\infty)$. The same description holds for $\pi_F^{-1}(U_\infty)/\mathbb{A}^1$.

– The situation for the open subsets $S_{R,0} = \pi_R^{-1}(U_0)$ and $S_{F,0} = \pi_F^{-1}(U_0)$ is a little more complicated. Letting $g : \tilde{U}_0 \to U_0$ be a Galois cover of order three ramified over p_0 and étale everywhere else, the inverse image of $\pi_F^{-1}(p_0)_{\mathrm{red}}$ in the normalization $\tilde{S}_{R,0}$ of the reduced fiber product $(S_R \times_{U_0} \tilde{U}_0)_{\mathrm{red}}$ is the disjoint union of three curves $\ell_{0,1}$, $\ell_{0,\varepsilon}$, and ℓ_{0,ε^2} (where $\varepsilon \in \mathbb{C}^*$ is a primitive cubic root of unity) which are permuted by the action of the Galois group μ_3 of cubic roots of unity. The $\mathbb{A}^1$-fibration $\mathrm{pr}_1 : S_{R,0} \times_{U_0} \tilde{U}_0 \to U_0'$ lifts to one $\tilde{\pi}_{R,0} : \tilde{S}_{R,0} \to \tilde{U}_0$ with a unique, reduced, degenerate fiber $(\pi')_{R,0}^{-1}(\tilde{p}_0)$, where $\tilde{p}_0 = g^{-1}(p_0)$, which consists of the union of the $\ell_{0,\alpha}$, $\alpha = 1, \varepsilon, \varepsilon^2$. The same argument as in the previous case implies then that $\tilde{\mathfrak{C}}_0 = \tilde{S}_{R,0}/\mathbb{A}^1$ is isomorphic to the $\tilde{U}_0$-scheme $\tilde{\delta}_0 : \tilde{\mathfrak{C}}_0 \to \tilde{U}_0$ obtained by gluing three copies $\tilde{\delta}_{0,\alpha} : \tilde{U}_{0,\alpha} \xrightarrow{\sim} \tilde{U}_0$, $\alpha = 1, \varepsilon, \varepsilon^2$, of $\tilde{U}_0$ by the identity outside the points $\tilde{p}_{0,\alpha} = (\tilde{\delta}_{0,\alpha})^{-1}(\tilde{p}_0)$. Furthermore, the action of the Galois group μ_3 on $\tilde{S}_{R,0}$ descends to a fixed point free action on $\tilde{\mathfrak{C}}_0$ defined locally by $\tilde{U}_{0,\alpha} \ni \tilde{p} \mapsto \varepsilon \cdot \tilde{p} \in \tilde{U}_{0,\varepsilon\alpha}$. A geometric quotient for this action on $\tilde{\mathfrak{C}}_0$ exists in the category of algebraic spaces in the form of an étale μ_3-torsor $\tilde{\mathfrak{C}}_0 \to \tilde{\mathfrak{C}}_0/\mu_3$ over a certain algebraic space $\tilde{\mathfrak{C}}_0/\mu_3$ and we obtain a commutative diagram

$$\begin{array}{ccc}
\tilde{S}_{R,0} & \longrightarrow & \tilde{S}_{R,0}/\mu_3 \simeq S_{R,0} \\
{\scriptstyle \tilde{\rho}_{R,0}}\downarrow & & \downarrow{\scriptstyle \rho_{R,0}} \\
\tilde{\mathfrak{C}}_0 & \longrightarrow & \tilde{\mathfrak{C}}_0/\mu_3 \\
{\scriptstyle \tilde{\delta}_0}\downarrow & & \downarrow{\scriptstyle \delta_0} \\
\tilde{U}_0 & \longrightarrow & \tilde{U}_0/\mu_3 \simeq U_0
\end{array}$$

in which the top square is cartesian. It follows that the induced morphism $\rho_{R,0} : S_{R,0} \to \tilde{\mathfrak{C}}_0/\mu_3$ is an étale locally trivial $\mathbb{A}^1$-bundle which factors the restriction of π_R to $S_{R,0}$. So $\delta_0 : \tilde{\mathfrak{C}}_0/\mu_3 \to U_0$ is the desired algebraic space $S_{R,0}/\mathbb{A}^1$.

It is clear from the construction that the isomorphy type of $\tilde{\mathfrak{C}}_0/\mu_3$ as a space over U_0 depends only on the fact that $S_{R,0}$ is smooth and that $\pi_R\,|_{S_{R,0}}: S_{R,0} \to U_0$ is an $\mathbb{A}^1$-fibration with a unique degenerate fiber of multiplicity three over p_0, and not on the full isomorphy type of $S_{R,0}$ as a scheme over U_0. In other words, the same

construction applied $\pi_F\,|_{S_{F,0}}\colon S_{F,0} \to U_0$ yields an algebraic space $S_{F,0}/\mathbb{A}^1$ which is isomorphic to $\tilde{\mathfrak{C}}_0/\mu_3$ as spaces over U_0.

Finally, the desired algebraic space $\mathfrak{C} = S_R/\mathbb{A}_1 = S_F/\mathbb{A}_1$ is obtained by gluing $\mathfrak{C}_\infty$ and $\mathfrak{C}_0 = \tilde{\mathfrak{C}}_0/\mu_3$ by the identity along the open sub-schemes $\delta_\infty^{-1}(U_0 \cap U_\infty) \simeq U_0 \cap U_\infty \simeq \delta_0^{-1}(U_0 \cap U_\infty)$. This completes the proof of Proposition 4.1.

Remark 4.2. A similar construction applies to the $\mathbb{A}^1$-fibrations $\pi'_R : S_R \to \mathbb{P}^1$ and $\pi'_F : S_F \to \mathbb{P}^1$ considered in remarks 3.1 and 3.4 respectively. The desired algebraic space $\mathfrak{C}' = S_R/\mathbb{A}^1 = S_F/\mathbb{A}^1$ is again obtained as the gluing by the identity along $\delta'^{-1}_\infty(U_0 \cap U_\infty) \simeq U_0 \cap U_\infty \simeq \delta'^{-1}_0(U_0 \cap U_\infty)$ of two algebraic spaces $\delta'_\infty : \mathfrak{C}'_\infty \to U_\infty$ and $\delta'_0 : \mathfrak{C}'_0 \to U_0$ which are constructed as follows:

– The algebraic space $\mathfrak{C}'_\infty$ is obtained from U_∞ be replacing the point p_∞ by four copies of itself, one for each irreducible component in the reduced degenerate fiber $\pi'^{-1}_R(p_\infty)$ (resp. $\pi'^{-1}_F(p_\infty)$).

– Corresponding to the fact that the degenerate fiber $\pi'^{-1}_R(p_0)$ (resp. $\pi'^{-1}_F(p_0)$) has two irreducible components, both occurring with multiplicity two, the algebraic space $\mathfrak{C}'_0$ is now itself a compound object. First we let $g : \tilde{U}_0 \to U_0$ be Galois cover of degree two ramified at p_0 and étale elsewhere. Then we let $\tilde{\mathfrak{D}}'_0 \to \tilde{U}_0$ be the scheme obtained by gluing two copies $\tilde{U}_{0,\pm}$ of $\tilde{U}_0$ by the identity outside $\tilde{p}_0 = g^{-1}(p_0)$. The Galois group μ_2 acts freely on $\tilde{\mathfrak{D}}'_0$ by $\tilde{U}_{0,\pm} \ni \tilde{p} \mapsto -\tilde{p} \in \tilde{U}_{0,\mp}$ and we let $\gamma'_0 : \mathfrak{D}'_0 = \tilde{\mathfrak{D}}'_0/\mu_2 \to U_0 \simeq \tilde{U}_0/\mu_2$ be the geometric quotient taken in the category of algebraic spaces. Finally, $\delta'_0 : \mathfrak{C}'_0 \to U_0$ is obtained by gluing two copies $\gamma'_{0,i} : \mathfrak{D}'_{0,i} \to U_0$, $i = 1, 2$ of $\mathfrak{D}'_0$ by the identity along the open subschemes ${\gamma'_{0,1}}^{-1}(U_0 \setminus \{p_0\}) \simeq U_0 \setminus \{p_0\} \simeq {\gamma'_{0,2}}^{-1}(U_0 \setminus \{p_0\})$.

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