Comment. Math. Helv. 90 (2015), 75–120
DOI 10.4171/CMH/347

Commentarii Mathematici Helvetici

The Yang–Mills α-flow in vector bundles over four manifolds and its applications

Min-Chun Hong, Gang Tian and Hao Yin

Abstract. In this paper we introduce an α-flow for the Yang–Mills functional in vector bundles over four dimensional Riemannian manifolds, and establish global existence of a unique smooth solution to the α-flow with smooth initial value. We prove that the limit of the solutions of the α-flow as $\alpha \to 1$ is a weak solution to the Yang–Mills flow. By an application of the α-flow, we then follow the idea of Sacks and Uhlenbeck [22] to prove some existence results for Yang–Mills connections and improve the minimizing result of the Yang–Mills functional of Sedlacek [26].

Mathematics Subject Classification (2010). 58E15.

Keywords. Yang–Mills flow, Sacks–Uhlenbeck functional.

1. Introduction

Suppose that M is a connected compact four dimensional Riemannian manifold and E is a vector bundle over M. For each connection D_A, the Yang–Mills functional is defined by

$$\mathrm{YM}(A; M) = \int_M |F_A|^2 \, dv,$$

where F_A is the curvature of D_A. In a local trivialization, we can express D_A as $d + A$, where $A \in \Gamma(\mathrm{End}E \otimes T^*M)$ is the connection matrix.

We say that a connection D_A is a *Yang–Mills connection* if it is a critical point of the Yang–Mills functional; i.e. D_A satisfies the Yang–Mills equation

$$D_A^* F_A = 0\,. \tag{1.1}$$

Yang–Mills equations originated from the theory of classical fields in particle physics. It turns out that Yang–Mills theory has substantial applications in pure mathematics, especially in dimension 4. In [3], Atiyah, Hitchin, Drinfel'd and Manin established the fundamental existence result of instantons on S^4. Uhlenbeck [33, 34] established important analytic theorems for Yang–Mills connections on

4-manifolds. Donaldson [7] successfully applied the Yang–Mills theory to four dimensional geometric topology.

The Yang–Mills equation is a typical example of partial differential equations involving gauge invariant of a group action. Besides its applications to geometry and topology, the study of the existence of Yang–Mills connections is very interesting in itself. Motivated by the seminal work of Eells–Sampson [10] on harmonic maps, Atiyah and Bott [2] suggested to use the method of the Yang–Mills flow to establish the existence of Yang–Mills connections. The Yang–Mills flow equation is

$$\frac{\partial D_A}{\partial t} = -D_A^* F_A, \tag{1.2}$$

with initial condition $D_A(0) = D_0$, where D_0 is a given smooth connection on E. In [8], Donaldson used the Yang–Mills flow to establish the important result that an irreducible holomorphic vector bundle E over a compact Kähler surface X admits a unique Hermitian-Einstein connection if and only if it is stable. Without the holomorphic structure of the bundle E, it is still open whether the Yang–Mills flow in four dimensional manifolds develop a singularity in finite time. Struwe [30] proved the existence of the weak solution to the Yang–Mills flow in vector bundles on four manifolds, where the weak solution is regular away from finitely many singularities in $M \times (0, \infty)$. Schlatter [24] gave the details for the blow-up analysis at each singular point and the longtime behaviour of the Yang–Mills flow in dimension four. If the Yang–Mills flow blows up at a finite time $T > 0$, the weak solution constructed by Struwe [30] after the time T lies on the new vector bundle $\tilde{E}$, which might have different second Chern number from the original bundle E.

The Yang–Mills functional in dimension four is conformally invariant, which is similar to the conformal invariance of the Dirichlet energy of maps in dimension two, so there are general expectations that those results, which hold for harmonic maps from surfaces, should remain true in some sense for Yang–Mills connections in dimension four, if the gauge invariance problem is treated properly. In their celebrated paper [22], Sacks and Uhlenbeck proposed to study the perturbed energy of a map u from M to N

$$E_\alpha(u) = \int_M (1 + |du|^2)^\alpha \, dv.$$

For $\alpha > 1$, the functional $E_\alpha(u)$ satisfies the Palais–Smale condition and therefore it is not difficult to find critical points of E_α. They then analyzed the limit of the critical points when α goes to 1. In spite of the possible blow-up phenomena, several interesting applications concerning the existence of harmonic maps were made. One of the major goals of this paper is to develop a parallel theory for the Yang–Mills functional in dimension four. Namely, we introduce the Yang–Mills α-functional

$$YM_\alpha(A) = \int_M (1 + |F_A|^2)^\alpha \, dv.$$

The Euler–Lagrange equation for the functional YM_α is

$$D_A^*\left((1+|F_A|^2)^{\alpha-1}F_A\right)=0. \tag{1.3}$$

A solution to the Yang–Mills α-equation (1.3) is called a Yang–Mills α-connection. In order to show the existence of smooth α-connections, one maybe check the Palais–Smale condition for YM_α and then prove the regularity of the weak solution of (1.3). Instead, in this paper we introduce the Yang–Mills α-flow

$$\frac{\partial A}{\partial t}=-D_A^*F_A+(\alpha-1)\frac{*(d\,|F_A|^2\wedge *F_A)}{1+|F_A|^2} \tag{1.4}$$

with initial condition $A(0)=A_0$. Then we apply the Yang–Mills α-flow to deform any given connection to a smooth Yang–Mills α-connection. More precisely, we prove

Theorem 1.1. For a given smooth connection A_0, there exists a unique global smooth solution $A_\alpha(x,t)$ to the evolution problem (1.4) in $M\times[0,\infty)$ for $\alpha-1$ sufficiently small. Moreover, for any $t_i\to\infty$, by passing to a subsequence, $A_\alpha(\cdot,t_i)$ converges up to transformations to a limiting connection A_α^∞ in $C^k(M)$ for any $k\geq 1$, and the connection A_α^∞ is a smooth solution of (1.3).

Remark. *Recently, L. Schabrun* [23] *proved that the solution of the Yang–Mills α-flow converges to a unique limit A_α^∞ as $t\to\infty$.*

To prove the global existence of the smooth solution of the Yang–Mills α-flow is not easy since the Yang–Mills α-flow is not parabolic. For the local existence of the flow, we modify an idea of Donaldson [8] to study a equivalent flow. The main difficulty in proving the global existence is to establish the local solution of the flow for a fixed time t_0 depending on initial values (see Theorem 2.4). Due to the energy inequality, the Yang–Mills energy of the solution to the α-flow does not concentrate at any time $T>0$ for each fixed $\alpha>1$. However, we cannot follow the same proof of Struwe in [29] to control the norm H^2 of the curvature F since the extra terms $\int_M|\nabla F|^4\,dv$ and $\int_M|F|^4\,dv$ come out due to the complexity of the α-flow. Instead, we work on the gauge-equivalent flow and prove that for any $t>0$, the Yang–Mills α-flow has a smooth solution in $M\times[t,t+t_0]$ for a fixed $t_0>0$, which depends on $YM_\alpha(A_\alpha)$, so that we can extend the smooth solution to $M\times[0,\infty)$ (see Theorem 2.3).

Following an idea from [17], we apply the Yang–Mills α-flow to obtain a new proof of the existence of a weak solution of the Yang–Mills flow, which might be a different global weak solution from the one obtained by Struwe in [30], as in the following.

Theorem 1.2. Let A_α be the smooth solution of the Yang–Mills α-flow with the same initial condition A_0 for each $\alpha>1$. Then, there is a closed singularity set

$\Sigma \subset M \times (0, \infty)$ with finite 2-dimensional parabolic Hausdorff measure such that $\Sigma_t = \Sigma \cap (M \times \{t\})$ is at most a finite set for any t. There is a smooth bundle $\tilde{E}$ over $M \times [0, \infty) \setminus \Sigma$ with $\tilde{E}|_{M \times \{0\}}$ isomorphic to E and a smooth connection $A_\infty(t)$ on $\tilde{E}|_{M \times \{t\} \setminus \Sigma_t}$ such that (1) $A_\infty(t)$ is a solution of the Yang–Mills flow; (2) for each compact set $K \subset M \times [0, \infty) \setminus \Sigma$, there are gauge transformations ϕ_α over K with $\phi_\alpha^* A_\alpha$ converging smoothly to A_∞ over K as $\alpha \to 1$.

To prove Theorem 1.2, we establish a Bochner type estimate uniformly in α and a local parabolic monotonicity formula for the Yang–Mills α-flow, which is similar to one in [29] and [16]. Then we follow an idea of Schoen [25] (also see [29]) to obtain a uniform estimate on $|F_{A_\alpha}|$ in α. However, there is a technical difficulty that we do not have Bochner formulas for higher order derivatives of F_{A_α}, so we cannot apply the Moser estimate to obtain the unform estimates of higher order derivatives of F_{A_α}. To overcome this difficulty, we obtain the uniform Sobolev norms of $\nabla_{A_\alpha}^k F_{A_\alpha}$ for all integers $k \geq 1$ by using the equation of F_{A_α} (see Lemma 3.6).

With the analytic tools developed in the proof of the previous two theorems, we investigate further applications of the α-flow. It is not hard to establish an ε-regularity result for studying the blow-up of a sequence of Yang–Mills α-connections. When a blow-up phenomenon happens, we will study the change of the topology of the bundle. More precisely, the original bundle E, on which the blow-up sequence lies, is the connected sum of the weak limit bundle over M and the bubbling bundles over S^4. Following the idea of Sacks and Uhlenbeck's paper [22], we apply the existence of smooth Yang–Mills α-connections of Theorem 1.1 to show

Theorem 1.3. If $\pi_3(G)$ is a free abelian group of rank r, then there exist at least r different Yang–Mills G-connections over S^4.

Remark. *It is well known that any simple compact Lie group G has $\pi_3(G) = \mathbb{Z}$. So the result is useful only for semi-simple compact Lie groups, for example $SO(4)$.*

Furthermore, we can apply the Yang–Mills α-flow to improve the minimizing theory of the Yang–Mills functional on E. In [26], Sedlacek studied the direct minimizing method for the Yang–Mills functional in E. More precisely, let D_i be a minimizing sequence in the given bundle E over M. Using the weak compactness result of Uhlenbeck [34], Sedlacek proved that D_i weakly converges in $W^{1,2}(M \backslash \{x_1, ... x_l\})$ to a limiting connection D_∞ which can be extended to a Yang–Mills connection in a (possibly) new bundle E' over M with the same topological invariant $\eta(E') = \eta(E)$, which is an element of $H^2(M, \pi_1(G))$. Because there is only $W^{2,2}$ control of the transition functions, one can not use the gluing argument of Uhlenbeck in [34] to obtain a bundle map. Therefore, the relation between the original bundle and the limit bundle E' (which may be different) is not quite clear. It is known that the topology of a vector bundle over a 4-manifold is determined by some η invariant, and the vector Pontryagin number (see the appendix in [26]). By using the α-flow, we modify the minimizing sequence to obtain a better control and new minimizing sequence, which converges to the same limit in the smooth topology up to gauge transformation away from finite singular points. Moreover, for the modified minimizing sequence, a blow-up analysis is discussed and an energy identity is proved.

Theorem 1.4. Let E be a vector bundle over M with structure group G. Assume that D_i is a minimizing sequence of the Yang–Mills functional YM among smooth connections on E, which converges weakly to some limit connection D_∞ by Sedlacek's result. There is a modified minimizing sequence D_i', a finite set $S \subset M$ and a sequence of gauge transformations ϕ_i defined on $M \setminus S$, such that for any compact $K \subset M \setminus S$, $\phi_i^* D'_i$ converges to D'_∞ smoothly in K, where D'_∞ is gauge equivalent to the connection D_∞. Moreover, there are a finite number of bubble bundles $E_1, \cdots, E_l$ over S^4 and Yang–Mills connections $\tilde{D}_1, \cdots, \tilde{D}_l$ such that

$$\lim_{i\to\infty} YM(D_i) = YM(D_\infty) + \sum_{j=1}^{l} YM(\tilde{D}_j).$$

This improves Theorem 5.5 of [26] because the convergence of $\phi_i^* D'_i$ is smooth. (See [18] for a similar discussion using Sobolev bundles and the weak convergence.)

Finally, we would like to discuss some potential application of the Yang–Mills α-flow to the Morse theory of the Yang–Mills functional. It is well known that the Yang–Mills functional in dimension four does not satisfy the Palais–Smale condition. Many efforts have been made in this direction (see [32] and the references therein). Following an idea in [22], one expects to study the limiting solutions of the α-equations (1.3) as α goes to 1. It seems that the Yang–Mills α-flow provides a new analytic tool to prove the existence of Yang–Mills connections. In Subsection 4.4, we use it as the analytic tool to provide a new proof of the existence of the nonminimal Yang–Mills connection on S^4, which is due to Sibner, Sibner and Uhlenbeck [27].

The rest of the paper is organized as follows: In Section 2, we prove Theorem 1.1 and some other analytic results needed for the applications. In Section 3, we study the limit of the α-flow as α goes to 1 and prove Theorem 1.2. In the final section, we study serval applications of the α-flow.

2. Existence of the α-flow and its equivalent flow

2.1. Local existence of the α-flow.

It is well known that (1.4) is not a parabolic system and that this difficulty can be overcome by using a kind of Deturk trick. Throughout this paper, let D_{ref} be a fixed smooth background connection.

Let $D_0 = D_{ref} + A_0$ be a given smooth connection in E.

Following [29], we consider an equivalent flow

$$\frac{\partial \bar{D}}{\partial t} = -\bar{D}^* F_{\bar{D}} + (\alpha - 1)\frac{*(d|F_{\bar{D}}|^2 \wedge *F_{\bar{D}})}{1 + \left|F_{\bar{D}}\right|^2} - \bar{D}(\bar{D}^* a), \tag{2.1}$$

with $\bar{D}(t) = D_{ref} + a(t)$ and $a(0) = A_0$. Then the equivalent flow is a nonlinear parabolic system. By the well-known theory of partial differential equations, there is a unique smooth solution of (2.1) defined on $M \times [0, T]$ for some $T > 0$. By the theory of ordinary differential equations, there is a unique solution to the following initial problem:

$$\frac{d}{dt} S = -S \circ (\bar{D}^* a), \tag{2.2}$$

$M \times [0, T]$, with initial value $S(0) = I$. Here $S(t)$ is a global gauge transformation and I is the trivial one.

Setting

$$D = (S^{-1})^* \bar{D},$$

we have (e.g. see [29], [14])

$$F_{\bar{D}} = S^{-1} F S, \quad \bar{D}(\bar{D}^* a) = \bar{D} \circ (\bar{D}^* a) - \bar{D}^* a \circ \bar{D}.$$

Combining (2.1), (2.2) with the above facts yields

$$\begin{aligned}
\frac{d}{dt} D &= \frac{dS}{dt} \circ \bar{D} \circ S^{-1} + S \circ \frac{d\bar{D}}{dt} \circ S^{-1} + S \circ \bar{D} \circ \frac{dS^{-1}}{dt} \\
&= S \left(-\bar{D}^* F_{\bar{D}} + (\alpha - 1)\frac{*(d|F_{\bar{D}}|^2 \wedge *F_{\bar{D}})}{1 + \left|F_{\bar{D}}\right|^2} \right) S^{-1} \\
&= -D_A^* F_A + (\alpha - 1)\frac{*(d|F_A|^2 \wedge *F_A)}{1 + |F_A|^2}.
\end{aligned}$$

This shows that $D = (S^{-1})^* \bar{D}$ satisfies the Yang–Mills α-flow with $D(0) = D_0$ in $M \times [0, T]$ for some $T > 0$.

Next, we remark that the smooth solution of the Yang–Mills α-flow is unique. In fact, let $D_i = D_{ref} + A_i (i = 1, 2)$ be two smooth solutions to the Yang–Mills α-flow with $A_i(0) = A_0$. By the theory of parabolic equations, there is a unique local smooth solution of the parabolic system of second order:

$$\frac{d}{dt} S_i = -(D_{ref} + A_i)^* [A_i S_i + D_{ref} S_i] \tag{2.3}$$

with $S(0) = I$. By computation, we can check that the connections $\bar{D}_i = S^*(D_i)$ are two solutions to the modified flow (2.1) with the same initial value. Hence, $\bar{D}_1$ and $\bar{D}_2$ are the same. Moreover, (2.3) is nothing but the ODE (2.2). By the uniqueness of ODEs, we know S_i and hence D_i are the same.

A similar method to prove uniqueness was used for the Ricci flow and also for the Seiberg–Witten flow [15]. Therefore, we have shown that the α-flow has a unique solution in $M \times [0, T)$ for some $T > 0$.

2.2. Energy inequality of the α-flow.

Lemma 2.1. Let $A(t)$ be a solution to the Yang–Mills α-flow in $M \times [0, T)$ with initial value $A(0) = A_0$. For each $0 < t < T$, we have

$$\int_M (1+|F|^2)^\alpha \, dv + 2\alpha \int_0^t \int_M (1+|F|^2)^{\alpha-1} \left|\frac{\partial A}{\partial s}\right|^2 dv\, ds = \int_M (1+|F_{A_0}|^2)^\alpha \, dv. \tag{2.4}$$

Proof. Note $\frac{\partial F}{\partial t} = D\frac{\partial A}{\partial t}$. Then, multiplying (1.4) by $(1+|F|^2)^{\alpha-1}\partial_t A$ and integrating by parts, we have

$$\begin{aligned}
\frac{d}{dt}\int_M (1+|F|^2)^\alpha dv &= 2\alpha \int_M \left\langle (1+|F|^2)^{\alpha-1} F, \frac{\partial F}{\partial t}\right\rangle dv \\
&= 2\alpha \int_M \left\langle D^*((1+|F|^2)^{\alpha-1} F), \frac{\partial A}{\partial t}\right\rangle dv \\
&= -2\alpha \int_M (1+|F|^2)^{\alpha-1} \left|\frac{\partial A}{\partial t}\right|^2 dv.
\end{aligned}$$

Then (2.4) follows from integrating over $[0, t]$. □

Lemma 2.2. Let $A(t)$ be a solution to the Yang–Mills α-flow in $M \times [0, T)$. For each $0 < t_1 < t_2 < T$, we have

$$\int_{B_R(x)} (1+|F|^2)^\alpha (t_2) dv \le \int_{B_{2R}(x)} (1+|F|^2)^\alpha (t_1) dv + C\frac{t_2 - t_1}{R^2} YM_\alpha(A(0)). \tag{2.5}$$

Proof. Let φ be a cut-off function supported in $B_{2R}(x)$ and $\varphi \equiv 1$ on $B_R(x)$.

$$\begin{aligned}
\frac{d}{dt}\int_M \varphi^2(1+|F|^2)^\alpha dv &= 2\alpha \int_M \varphi^2 \left\langle D^*((1+|F|^2)^{\alpha-1} F), \frac{\partial A}{\partial t}\right\rangle \\
&\qquad + \varphi(1+|F|^2)^{\alpha-1} F \# \nabla\varphi \# \frac{\partial A}{\partial t}\, dv \\
&\le -\int_M \varphi^2 (1+|F|^2)^{\alpha-1} \left|\frac{\partial A}{\partial t}\right|^2 \\
&\qquad + (1+|F|^2)^{\alpha-1} |\nabla\varphi|^2 |F|^2 \, dv.
\end{aligned}$$

The lemma follows from integration over $[t_1, t_2]$. □

We also need a similar result in the other direction.

Lemma 2.3. Let $A(t)$ be a solution to the Yang–Mills α-flow in $M \times [0,T)$. For each $0 < t_1 < t_2 < T$, we have

$$\int_{B_R(x)} (1+|F|^2)^\alpha(t_1)dv \le \int_{B_{2R}(x)} (1+|F|^2)^\alpha(t_2)dv + C\frac{t_2-t_1}{R^2}YM_\alpha(A(0)) + \int_{t_1}^{t_2}\int_M (1+|F|^2)^{\alpha-1}\left|\frac{dA}{dt}\right|^2 dvdt.$$

Proof. The claim follows from the above proof in Lemma 2.2. □

2.3. Global existence of the α-flow. In this section, we will show that the solution of the Yang–Mills α-flow (for small $\alpha - 1$) exists in $M \times [0,T)$ for all $T > 0$.

Theorem 2.4. Let $D_0 = D_{ref} + A_0$ be a smooth connection in E. Then there is a smooth solution A to the α-flow (1.4) with initial value A_0 in $M \times [0,t_0)$ for a constant $t_0 > 0$ depending only on $YM_\alpha(D_0)$.

We note that together with Lemma 2.1 and the uniqueness of smooth solution to (1.4), Theorem 2.4 implies the global existence part of Theorem 1.1.

The proof involves higher order estimates for parabolic systems. For that purpose, we resort to the modified flow (2.1) again. To start the proof, we need the following lemma.

Lemma 2.5. Let D be a smooth connection on E with $YM_\alpha(D)$ bounded, and let D_{ref} be some fixed reference connection on E. Then there exists a global smooth gauge transformation s such that

$$\|s^*D - D_{ref}\|_{W^{1,2\alpha}(M)} \le C.$$

Here C is some constant depending only on D_{ref} and $YM_\alpha(D)$.

Proof. Although not explicitly stated, the proof is essentially contained in the paper [34] of Uhlenbeck. We briefly indicate how it follows from [34].

If the lemma is not true, then there exists a sequence of D_i with $YM_\alpha(D_i)$ uniformly bounded such that for any smooth gauge transformation s_i, we have

$$\|s_i^*D_i - D_{ref}\|_{W^{1,2\alpha}(M)} \ge i. \tag{2.6}$$

It is shown in [34] that by passing to some subsequence, there exists s_i such that $s_i^*D_i$ converges weakly in $W^{1,p}$ to some D_∞ for $p = 2\alpha$.

In the proof, Uhlenbeck chose some j sufficiently large and wrote $s_i^*D_i$ in local trivialization $\sigma_\alpha(j)$ as

$$d + \rho_\alpha^{-1}(i)d\rho_\alpha(i) + \rho_\alpha^{-1}(i)A(\alpha,i)\rho_\alpha(i).$$

Here we refer the reader to [34] to see the definitions of $\sigma_\alpha(i)$, $\rho_\alpha(i)$ and $A(\alpha, i)$. Moreover, Uhlenbeck proved that

$$\rho_\alpha^{-1}(i)d\rho_\alpha(i) + \rho_\alpha^{-1}(i)A(\alpha, i)\rho_\alpha(i)$$

is bounded in $W^{1,p}$ uniformly in i. Although the local expression of D_{ref} in the trivialization $\sigma_\alpha(j)$ has no explicit bound, it is independent of i. Hence $s_i^* D_i - D_{ref}$ is bounded in $W^{1,p}$ uniformly in i locally in the trivialization $\sigma_\alpha(j)$. Since $s_i^* D_i - D_{ref}$ is a tensor and we may show the same bound in $\sigma_\beta(j)$ for $\beta \neq \alpha$. We get a contradiction with (2.6) and the lemma is proved. □

With this lemma, we may assume without loss of generality that A_0 in Theorem 2.4 has bounded $W^{1,2\alpha}$ norm.

Proof of Theorem 2.4. Instead of (1.4), we shall discuss (2.1). By our discussion in Subsection 2.1, we know this is sufficient.

For some $\varepsilon > 0$ to be determined later, the Hölder inequality and Lemma 2.5 imply that there exist $r_0 > 0$ and $C_1 > 0$ such that for all $x \in M$,

$$\int_{B_{r_0}(x)} |A_0|^2 + \left|\nabla_{ref} A_0\right|^2 dx \le \varepsilon/2 \tag{2.7}$$

and

$$\int_M |A_0|^2 + \left|\nabla_{ref} A_0\right|^2 dx \le C_1. \tag{2.8}$$

Let $\{x_i \in M | i = 1, \cdots, L\}$ be a finite number of points in M such that $\{B_{r_0}(x_i)\}$ covers M and for each i there are at most k different j's ball $B_{r_0}(x_j)$ with $B_{2r_0}(x_i) \cap B_{r_0}(x_j) \neq \emptyset$. Although L depends on ε, it is important to note that k is a universal constant depending only on the dimension.

Let $D(t) = D_{ref} + a(t)$ be the local solution to (2.1) defined on $[0, T)$. Since $a(t)$ is smooth, there exists a $t_1 > 0$ which is the maximal time in $[0, T]$ such that for all $i = 1, \cdots, L$,

$$\sup_{0 \le t < t_1} \int_{B_{r_0}(x_i)} |a(t)|^2 + \left|\nabla_{ref} a(t)\right|^2 dx \le \varepsilon \tag{2.9}$$

and

$$\sup_{0 \le t < t_1} \int_M |a(t)|^2 + \left|\nabla_{ref} a(t)\right|^2 dx + \int_0^{t_1} \int_M \left|\nabla_{ref}^2 a\right|^2 dx dt \le 2C_1. \tag{2.10}$$

We shall find t_0 depending on $YM_\alpha(D_0)$ and α alone (the exact value of t_0 is determined in the process of proof) and prove that $T \ge t_0$, which concludes the proof of the theorem. If not, then either $t_1 < T < t_0$ or $t_1 = T < t_0$. It suffices to show that neither case is possible.

Before we give the details of the proof, we outline the idea of the proof. By Lemma 2.5, we have (2.7) and (2.8) for the initial value $a(0)$. Step 1 below shows that as long as the solution exists, (2.9) and (2.10) must remain true for $t \in [0, t_0]$ for some $t_0 > 0$ depending only on $YM_\alpha(D_0)$. The condition (2.9) is a 'smallness' condition, which will enable us to prove higher derivative estimates for the nonlinear parabolic system (2.11) of second order. This is done in Step 2 below.

Step 1: $t_1 < T < t_0$ is not possible.

To study the evolution of $a(t)$, we rewrite the flow equation (2.1) as

$$\begin{aligned}\frac{\partial a}{\partial t} \quad = \quad & \triangle_{ref} a + (\nabla_{ref} a \# a + a \# a \# a) - D^*_{ref} F_{ref} \\ & + (\alpha - 1)\psi(F_D)\#(\nabla^2_{ref} a + a \# \nabla_{ref} a + a \# a \# a + \nabla_{ref} F_{ref}),\end{aligned} \tag{2.11}$$

with the initial value $a(0) = A_0$, where $\psi(F_D)$ is a bounded function depending on F_D. For any i, let ϕ_i be a cut-off function supported in $B_{2r_0}(x_i)$ with $\phi_i \equiv 1$ on $B_{r_0}(x_i)$. For simplicity, we write ϕ when it applies to all ϕ_i.

Multiplying (2.11) by a and using Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int_M |a|^2\, dv + \int_M |\nabla_{ref} a|^2\, dv \\ &\leq \frac{1}{2}\int_M |\nabla_{ref} a|^2\, dv + C(\alpha - 1)\int_M |\nabla^2_{ref} a|^2\, dv + C\int_M |a|^4\, dv + C.\end{aligned} \tag{2.12}$$

By our choice of t_1, (2.10) and using the Sobolev embedding from $W^{1,2}$ to L^4, we have for $t < t_1$,

$$\frac{d}{dt}\int_M |a|^2\, dv + \frac{1}{2}\int_M \left|\nabla_{ref} a\right|^2 dv \leq C(\alpha - 1)\int_M \left|\nabla^2_{ref} a\right|^2 dv + C.$$

Multiplying (2.11) by $\triangle_{ref} a$, we have

$$\begin{aligned}\frac{d}{dt}&\int_M |\nabla_{ref} a|^2\, dv + \int_M |\triangle_{ref} a|^2\, dv \\ &\leq \frac{1}{2}\int_M |\triangle_{ref} a|^2\, dv + C(\alpha - 1)\int_M |\nabla^2_{ref} a|^2\, dv \\ &\qquad + \int_M (|\nabla_{ref} a|^2 |a|^2 + |a|^6)\, dv + C.\end{aligned} \tag{2.13}$$

By Hölder's inequality and the Sobolev inequality, we obtain

$$\begin{aligned}
\int_M |a|^6\,dv &\le \sum_i \int_{B_{r_0}(x_i)} |a|^6\,dv \\
&\le \sum_i \left(\int_{B_{r_0}(x_i)} |a|^4\right)^{1/2} \left(\int_{B_{r_0}(x_i)} |a|^8\,dv\right)^{1/2} \\
&\le C\varepsilon \sum_i \int_{B_{r_0}(x_i)} \left|\nabla_{ref} a\right|^2 |a|^2 + |a|^4\,dv \\
&\le C\varepsilon \int_M \left|\nabla_{ref} a\right|^2 |a|^2 + |a|^4\,dv \\
&\le C\varepsilon \int_M \left|\nabla_{ref} a\right|^2 |a|^2\,dv + C.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_M \left|\nabla_{ref} a\right|^2 |a|^2\,dv &\le \sum_i \int_{B_{r_0}(x_i)} \left|\nabla_{ref} a\right|^2 |a|^2\,dv \\
&\le \sum_i \left(\int_{B_{r_0}(x_i)} |a|^4\right)^{1/2} \left(\int_{B_{r_0}(x_i)} \left|\nabla_{ref} a\right|^4\,dv\right)^{1/2} \\
&\le C\varepsilon \sum_i \int_{B_{r_0}(x_i)} \left|\nabla^2_{ref} a\right|^2 + \left|\nabla_{ref} a\right|^2\,dv \\
&\le C\varepsilon \int_M \left|\nabla^2_{ref} a\right|^2 + \left|\nabla_{ref} a\right|^2\,dv
\end{aligned}$$

Using integration by parts, we have

$$\int_M \left|\nabla^2_{ref} a\right|^2 dv \le \int_M \left|\triangle_{ref} a\right|^2 dv + C\int_M \left|\nabla_{ref} a\right|^2 dv,$$

which implies

$$\frac{3}{4}\int_M \left|\nabla^2_{ref} a\right|^2 dv \le \int_M \left|\triangle_{ref} a\right|^2 dv + C.$$

In summary, by choosing $\alpha - 1$ and ε small, we have

$$\frac{d}{dt}\int_M |a|^2 + \left|\nabla_{ref} a\right|^2 dv + \frac{1}{4}\int_M \left|\nabla_{ref} a\right|^2 + \left|\nabla^2_{ref} a\right|^2 dv \le C$$

for $t \in [0, t_1]$. Integrating the above inequality yields that there exists $t_0 > 0$ such that (2.10) remains true for $t_1 \le t_0$.

For (2.9), we need a local version of the above computation. Multiplying (2.11) by $\phi_i^2 a$ and using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_M |a|^2\phi_i^2\,dv + \frac{1}{2}\int_M |\nabla_{ref}a|^2\phi_i^2\,dv \\ \le C(\alpha-1)\int_M |\nabla^2_{ref}a|^2\phi_i^2\,dv + C.\end{aligned} \tag{2.14}$$

Here we have used the bound on $|\nabla\phi_i|$ and $\int_M |a|^4\,dv$ for $t\le t_1$. Multiplying (2.11) by $\phi_i^2\triangle_{ref}a$, we have

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla_{ref}a|^2\phi_i^2\,dv + \frac{1}{2}\int_M |\triangle_{ref}a|^2\phi_i^2\,dv \\ &\le C(\alpha-1)\int_M |\nabla^2_{ref}a|^2\phi_i^2\,dv + \int_M (|\nabla_{ref}a|^2|a|^2\phi_i^2 + |a|^6\phi_i^2)\,dv + C \\ &\quad + C\int_M \left|\nabla_{ref}a\right|^2|\nabla\phi_i|^2\,dv.\end{aligned} \tag{2.15}$$

By integration by parts, we have

$$\begin{aligned}\frac{3}{4}\int_M \left|\nabla^2_{ref}a\right|^2\phi_i^2 dv &\le \int_M \left|\triangle_{ref}a\right|^2\phi_i^2 dv + C\int_M \left|\nabla_{ref}a\right|^2(\phi_i^2 + |\nabla\phi_i|^2)dv \\ &\le \int_M \left|\triangle_{ref}a\right|^2\phi_i^2 dv + C,\end{aligned}$$

where we have used (2.10) for $t<t_1$.

We can deal with the main nonlinear terms as before.

$$\begin{aligned}\int_M |a|^6\phi_i^2 dv &\le C\varepsilon\int_M \left|\nabla_{ref}(\varphi a^2)\right|^2 + \varphi^2|a|^4\,dv \\ &\le C\varepsilon\int_M (|\nabla\phi_i|^2 + \phi_i^2)|a|^4 + \phi_i^2|a|^2\left|\nabla_{ref}a\right|^2 dv \\ &\le C\varepsilon\int_M \phi_i^2|a|^2 + \left|\nabla_{ref}a\right|^2 dv + C\end{aligned}$$

and

$$\begin{aligned}\int_M \phi_i^2|a|^2\left|\nabla_{ref}a\right|^2 dv &\le C\varepsilon\int_M \left|\nabla_{ref}(\phi_i\nabla_{ref}a)\right|^2 + \phi_i^2\left|\nabla_{ref}a\right|^2 dv \\ &\le C\varepsilon\int_M \phi_i^2\left|\nabla^2_{ref}a\right|^2 dv + C.\end{aligned}$$

In summary, for $t<t_1$, we have

$$\frac{d}{dt}\int_M \phi_i^2(|a|^2 + \left|\nabla_{ref}a\right|^2)dv \le C.$$

Therefore, by choosing t_0 sufficiently small, we see that both (2.9) and (2.10) remain true for $t_1 \le t_0$. By our definition of t_1, this shows $t_1 < T < t_0$ is not possible.

Step 2: $t_1 = T < t_0$ is not possible.

As pointed out before in Step 1, we now show thtat (2.9) and (2.10) together with (2.11) imply higher order estimates up to T, so that the solution can be extended beyond T.

For that purpose, we consider the evolution equation of a. Let φ be a cut-off function in time. Precisely, $\varphi(t) \equiv 0$ for $t < t_1/4$ and $\varphi(t) \equiv 1$ for $t \in [t_1/4, t_1]$. Multiplying (2.11) with φ^3 and applying the L^p estimate (see Theorem 9.1 of [19]; pages 341–342), we obtain for $p = 4$,

$$\begin{aligned}\left\|\varphi^3 a\right\|_{W_p^{2,1}(M\times[0,t_1])} \le C(\alpha-1)\left\|\varphi^3\nabla^2_{ref}a\right\|_{L^p(M\times[0,t_1])} \\ + C\left\|\varphi^3\nabla_{ref}a\#a\right\|_{L^p(M\times[0,t_1])} \\ + C\left\|\varphi^3 a\#a\#a\right\|_{L^p(M\times[0,t_1])} + C.\end{aligned}$$

We denote $W_p^{2,1}$ by the space of functions whose space derivatives up to second order and first order time derivative belong to L^p. The L^p norm of $\varphi^2\partial_t\varphi a$ is bounded by (2.10), which is why we assume $p = 4$.

By choosing $\alpha - 1$ sufficiently small and using Young's inequality, we have

$$\left\|\varphi^3 a\right\|_{W_p^{2,1}(M\times[0,t_1])} \le C\left\|\varphi a\right\|^3_{L^{3p}(M\times[0,t_1])} + C\left\|\varphi^2\nabla_{ref}a\right\|^{3/2}_{L^{3p/2}(M\times[0,t_1])} + C.$$

Recall that M is covered by $B_{r_0}(x_i)$ and $\int_{B_{r_0}(x_i)}|a|^4\,dv \le C\varepsilon^2$. For simplicity, we write B_i for $B_{r_0}(x_i)$. An interpolation theorem of Nirenberg (Theorem 1 in [20]) implies that

$$\|\varphi a\|_{L^{3p}(B_i)} \le C\left\|\varphi^3\nabla^2_{ref}a\right\|^{1/3}_{L^p(B_i)}\|a\|^{2/3}_{L^4(B_i)} + C\|a\|_{L^4(B_i)}.$$

This implies that

$$\int_{B_i}|\varphi a|^{3p}\,dv \le C\varepsilon^p\int_{B_i}\left|\varphi^3\nabla^2_{ref}a\right|^p dv + C.$$

Hence,

$$\begin{aligned}\int_o^{t_1}\int_M|\varphi a|^{3p}\,dv &\le \int_0^{t_1}\sum_i\int_{B_{r_0}(x_i)}|\varphi a|^{3p}\,dv \\ &\le C\varepsilon^p\int_0^{t_1}\int_M\left|\varphi^3\nabla^2_{ref}a\right|^p dv + C.\end{aligned}$$

That is

$$\|\varphi a\|^3_{L^{3p}(M\times[0,t_1])} \leq C\varepsilon \left\|\nabla^2_{ref}(\varphi^3 a)\right\|_{L^p(M\times[0,t_1])} + C.$$

Similarly,

$$\left\|\varphi^2\nabla_{ref} a\right\|^{3/2}_{L^{3p/2}(M\times[0,t_1])} \leq C\varepsilon \left\|\nabla^2_{ref}(\varphi^3 a)\right\|_{L^p(M\times[0,t_1])} + C.$$

The proof is the same, except that we use another interpolation inequality

$$\left\|\varphi^2\nabla_{ref} a\right\|_{L^{3p/2}(B_i)} \leq C \left\|\varphi^3\nabla^2_{ref} a\right\|^{2/3}_{L^p(B_i)} \|a\|^{1/3}_{L^4(B_i)} + C\,\|a\|_{L^4(B_i)}\,.$$

By choosing ε small, we obtain an $W^{2,1}_p$ bound on a for $p = 4$, which allows us to apply the estimates for linear parabolic system for higher order estimates. In fact, the parabolic Sobolev embedding theorem (Lemma 3.3 of [19]; page 80) implies that $\varphi^2\partial_t\varphi a$ is in $L^p(M\times[0,t_1])$ for any $p > 1$. We then repeat the above argument and use the parabolic Sobolev embedding again to see that $\nabla_{ref} a$ is Hölder continuous. The higher order estimates now follow from Schauder estimates and (2.11).

□

2.4. Convergence for $t_i \to \infty$. We now complete the proof of Theorem 1.1 by considering $t_i \to \infty$. We first claim that we have some gauge transformations σ_i such that the $\sigma_i^*(A(t_i))$ are uniformly bounded in any C^k norm. To see this, let t_0 be as in Theorem 2.4 and set $s_i = t_i - t_0/2$. Consider the solution $\tilde{A}(t)$ to the modified flow (2.1) with initial value $\tilde{A}(s_i) = A(s_i)$. The proof in Step 2 of Theorem 2.4 in fact established a C^k estimate for $\tilde{A}(t_i)$, which is gauge equivalent to $A(t_i)$ by the discussion in Subsection 2.1. Therefore, there is a subsequence which converges smoothly up to gauge transformations. By similar argument above, we have uniform a bound on $\nabla^k F(x,t)$ for any k. Due to (1.4), we have a uniform bound for $\frac{\partial^k A}{\partial t^k}$ as well. Hence, there is $C > 0$ independent of t such that

$$\frac{\partial}{\partial s}\int_M (1+|F|^2)^{\alpha-1}\left|\frac{\partial A}{\partial s}\right|^2 dv \leq C.$$

Lemma 2.1 then implies that

$$\lim_{t\to\infty}\int_M \left|\frac{\partial A}{\partial t}\right|^2 dv = 0.$$

Hence, the limit obtained above is a Yang–Mills α-connection. This completes the proof of Theorem 1.1.

2.5. Stability of the modified flow. The results in this subsection are prepared for later applications. Since we shall use the Yang–Mills α-flow as a deformation in the space of connections, we need to show that this flow depends at least continuously on its initial value in some chosen topology.

Theorem 2.6. If $D_i = D_{ref} + A_i (i = 1, 2)$ are two initial connections satisfying

$$\|A_i\|_{C^{k,\beta}(M)} \leq K,$$

then by Theorem 2.4, there exists $t_0 > 0$, which now depends on K and the solution $A_i(t)$ to the modified flow (2.1), which is defined on $[0, t_0]$ and satisfies $A_i(0) = A_i$ and

$$\|A_i\|_{C^{k,\beta}(M\times[0,t_0])} \leq C(K),$$

Moreover, for any $\varepsilon > 0$, there exists $\delta(K) > 0$ such that if

$$\|A_1 - A_2\|_{C^{k,\beta}(M)} \leq \delta,$$

then

$$\|A_1(t) - A_2(t)\|_{C^{k,\beta}(M)} \leq \varepsilon,$$

for $t \in [0, t_0]$.

Proof. The proof of the first part is essentially contained in the proof of Theorem 2.4. At that time, we didn't have good control over the initial value, hence a cut-off function in time was used to produce higher order estimates on $M \times [t_0/2, t_0]$. For our purposes here, it suffices to remove the cut-off function φ in Step 2 of the proof there.

The proof of the second part follows from theory of linear partial differential equations and is perhaps well known. Both A_1 and A_2 satisfy the modified Yang–Mills flow, which for our purposes here is written as

$$\frac{\partial A_i}{\partial t} = \triangle A_i + (\alpha - 1)P(A_i, \nabla A_i)\#\nabla^2 A_i + Q(A_i, \nabla A_i).$$

The exact form of P and Q is not important for us. It suffices to know that P and Q are smooth functions of A_i and ∇A_i. Subtracting the two equations, we have

$$\begin{aligned}\frac{\partial A_1 - A_2}{\partial t} = \triangle(A_1 - A_2) + (\alpha - 1)P(A_1, \nabla A_1)\#\nabla^2(A_1 - A_2)\\ + (P(A_1, \nabla A_1) - P(A_2, \nabla A_2))\#\nabla^2 A_2\\ + Q(A_1, \nabla A_1) - Q(A_2, \nabla A_2).\end{aligned}$$

There are smooth functions R and S of A_i and ∇A_i such that

$$\begin{aligned}\frac{\partial A_1 - A_2}{\partial t} &= \triangle(A_1 - A_2) + (\alpha - 1)P(A_1, \nabla A_1)\#\nabla^2(A_1 - A_2)\\ &+ R(A_i, \nabla A_i, \nabla^2 A_2)(A_1 - A_2) + S(A_i, \nabla A_i, \nabla^2 A_2)(\nabla A_1 - \nabla A_2).\end{aligned}$$

If we take the above as a linear parabolic system for $A_1 - A_2$, then (i) the system is strictly parabolic in the sense of Petrovskiy (see page 4 of [11] by noting that P is always bounded and hence the principle part is a small perturbation of the Laplacian) and (ii) the coefficients are bounded in the $C^{k-2,\alpha}$ norm.

For the strictly parabolic linear systems in the sense of Petrovskiy, Eidel'man ([11], pages 243–244) constructed the heat kernel $\mathcal{Z}$ explicitly. Moreover, the solution to the linear system is expressed as the convolution

$$(A_1 - A_2)(x,t) = \int_M (A_1 - A_2)(y,0)\mathcal{Z}(x,t;y,0)dv.$$

Therefore

$$\|A_1 - A_2\|_{C^0(M\times[0,t_0])} \le C(K)\,\|A_1(\cdot,0) - A_2(\cdot,0)\|_{C^0(M)}\,.$$

We can now apply the Schauder estimate to see

$$\begin{aligned} \|A_1(\cdot,t) - A_2(\cdot,t)\|_{C^{k,\beta}(M)} &\le \|A_1 - A_2\|_{C^{k,\beta}(M\times[0,t_0])} \\ &\le C(K)\,\|A_1(\cdot,0) - A_2(\cdot,0)\|_{C^{k,\beta}(M)}\,. \end{aligned}$$

This proves our claim. □

3. Convergence of α-flow solutions

In this section, we study the convergence of the α-flow solutions as α goes to 1. We follow the same idea as in [17]. The key ingredients in the proof are a Bochner formula and a monotonicity formula, which are well known techniques but should still be computed for our new equation.

We start with the Bochner formula.

3.1. Bochner formula and the uniform bound of F. Let $A(t)$ be a solution of the Yang–Mills alpha flow; i.e.

$$\frac{\partial A}{\partial t} = -D^*F + 2(\alpha-1)\frac{*(\langle\nabla F, F\rangle \wedge *F)}{1+|F|^2}, \tag{3.1}$$

where $D = D_{ref} + A$. We recall that the curvature F of D satisfies

$$\frac{\partial F}{\partial t} = -DD^*F + 2(\alpha-1)D\frac{*(\langle\nabla F, F\rangle \wedge *F)}{1+|F|^2}. \tag{3.2}$$

For each point $p \in M$, let e^i be a normal frame of TM and ω^i the corresponding orthonormal basis of the cotangent bundle T^*M. Then at $p \in M$,

$$F = \sum_{i<j} F_{ij}\omega^i \wedge \omega^j.$$

At $p \in M$, we can assume that $\nabla e^i = 0$ and $\nabla\omega^i = 0$.

In order to derive a Bochner type formula, we need

Lemma 3.1. Let

$$\varphi := \langle \nabla F, F \rangle = \varphi_k \omega^k.$$

Then at $p \in M$, we have

$$*(\varphi \wedge *F) = \sum_{i=1}^{4} \sum_{j=1}^{4} \varphi_j F_{ij} \omega^i.$$

Proof. At $p \in M$, we have

$$\begin{aligned} F &= F_{12}\omega^1 \wedge \omega^2 + F_{13}\omega^1 \wedge \omega^3 + F_{14}\omega^1 \wedge \omega^4 \\ &+ F_{23}\omega^2 \wedge \omega^3 + F_{24}\omega^2 \wedge \omega^4 + F_{34}\omega^3 \wedge \omega^4. \end{aligned}$$

Applying the Hodge star operator $*$, we have

$$\begin{aligned} *F &= F_{12}\omega^3 \wedge \omega^4 - F_{13}\omega^2 \wedge \omega^4 + F_{14}\omega^2 \wedge \omega^3 \\ &+ F_{23}\omega^1 \wedge \omega^4 - F_{24}\omega^1 \wedge \omega^3 + F_{34}\omega^1 \wedge \omega^2. \end{aligned}$$

Hence

$$\begin{aligned} \varphi \wedge *F = &+\varphi_1 F_{12}\omega^1 \wedge \omega^3 \wedge \omega^4 - \varphi_1 F_{13}\omega^1 \wedge \omega^2 \wedge \omega^4 + \varphi_1 F_{14}\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &+\varphi_2 F_{12}\omega^2 \wedge \omega^3 \wedge \omega^4 - \varphi_2 F_{23}\omega^1 \wedge \omega^2 \wedge \omega^4 + \varphi_2 F_{24}\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &+\varphi_3 F_{13}\omega^2 \wedge \omega^3 \wedge \omega^4 - \varphi_3 F_{23}\omega^1 \wedge \omega^3 \wedge \omega^4 + \varphi_3 F_{34}\omega^1 \wedge \omega^2 \wedge \omega^3 \\ &+\varphi_4 F_{14}\omega^2 \wedge \omega^3 \wedge \omega^4 - \varphi_4 F_{24}\omega^1 \wedge \omega^3 \wedge \omega^4 + \varphi_4 F_{34}\omega^1 \wedge \omega^2 \wedge \omega^4. \end{aligned}$$

Applying the Hodge star operator again, we have

$$\begin{aligned} *(\varphi \wedge *F) &= (\varphi_2 F_{12} + \varphi_3 F_{13} + \varphi_4 F_{14})\omega^1 \\ &+ (-\varphi_1 F_{12} + \varphi_3 F_{23} + \varphi_4 F_{24})\omega^2 \\ &+ (-\varphi_1 F_{13} - \varphi_2 F_{23} + \varphi_4 F_{34})\omega^3 \\ &+ (-\varphi_1 F_{14} - \varphi_2 F_{24} - \varphi_3 F_{34})\omega^4 \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} \varphi_j F_{ij} \omega^i. \end{aligned}$$

This proves our claim. □

Lemma 3.2. (Bochner type formula 1) When $\alpha - 1$ is sufficiently small, there is a constant C such that

$$\frac{\partial}{\partial t} |F|^2 - \nabla_{e_i} \left((\delta_{ij} + 2(\alpha - 1)\frac{\langle F_{lj}, F_{li} \rangle}{1 + |F|^2})\nabla_{e_j} |F|^2 \right) + |\nabla F|^2 \le C |F|^2 (1 + |F|). \tag{3.3}$$

Proof. Recall that we use a local normal orthonomal frame $\{e_i\}$ and its dual $\{\omega_i\}$ at p. Noticing the fact that $\nabla_{e^j} e^j = 0$ at $p \in M$, we have

$$\nabla^*\nabla|F|^2 = -\sum_j \nabla^2_{e^j,e^j}|F|^2 = -\sum_j \nabla_{e^j}\nabla_{e^j}|F|^2$$

and

$$\sum_i \nabla^2_{e^i;e^i}\langle F, F\rangle = 2\sum_i \left\langle \nabla_{e_i} F, \nabla_{e_i} F\right\rangle + 2\sum_i \left\langle F, \nabla_{e_i;e_i} F\right\rangle.$$

Let $\triangle = DD^* + D^*D$ denote the Hodge Laplacian with respect to the connection D. The well-known Weizenböck formula is

$$\triangle F = \nabla^*\nabla F + F \circ (Ric \wedge g + 2R) + F\#F.$$

Here Ric is the Ricci curvature of M, R is the curvature operator, and $(Ric \wedge g + 2R)$ is a linear mapping from 2 forms to 2 forms. We refer to Theorem (3.10) of [4] for the exact statement and the proof. Since we are not interested in the exact form of the last term and it is quadratic in F, we denote it by $F\#F$.

Using Bianchi's identity $DF = 0$, we have

$$-\langle DD^*F, F\rangle = \langle \nabla_{e^i}\nabla_{e^i} F + F\#F - F \circ (Ric \wedge g + 2R), F\rangle.$$

For simplicity, we set

$$b_{ij} = 2(\alpha - 1)\frac{\left\langle F_{lj}, F_{li}\right\rangle}{1 + |F|^2}.$$

Then we have

$$\begin{aligned}
\frac{\partial}{\partial t}|F|^2 &- \nabla_{e^i}\left((\delta_{ij} + b_{ij})\nabla_{e^j}|F|^2\right) \qquad (3.4)\\
&= 2\left\langle F, \frac{\partial}{\partial t} F\right\rangle - 2\nabla_{e^i}\langle \nabla_{e^i} F, F\rangle - \nabla_{e^i}\left(b_{ij}\nabla_{e^j}|F|^2\right)\\
&= 2\left\langle F, \frac{\partial F}{\partial t} + DD^*F\right\rangle + \langle F, F\#F - F \circ (Ric \wedge g + 2R)\rangle\\
&\qquad -2|\nabla F|^2 - \nabla_{e^i}\left(b_{ij}\nabla_{e^j}|F|^2\right).
\end{aligned}$$

By Lemma 3.1, we have

$$*(\varphi \wedge *F) = \sum \varphi_i F_{ij}\omega^j.$$

Let $f(|a|)$ denote a function , whose absolute value is smaller than a constant multiple of $|a|$; i.e. $|f(a)| \leq C|a|$ for a constant $C > 0$. Then at p, we have

$$\begin{aligned} D\frac{*(\varphi \wedge *F)}{1+|F|^2} &= \frac{D(*(\varphi \wedge *F))}{1+|F|^2} + d(1+|F|^2)^{-1} \wedge *(\varphi \wedge *F) \\ &= \frac{(\varphi_i F_{ij})_{;k}\omega^k \wedge \omega^j}{1+|F|^2} + f(|\nabla F|^2 \frac{1}{|F|}) \\ &= \frac{\varphi_{i;k} F_{ij}\omega^k \wedge \omega^j}{1+|F|^2} + f(|\nabla F|^2 \frac{1}{|F|}) \end{aligned}$$

which implies

$$\left\langle D\frac{*(\varphi \wedge *F)}{1+|F|^2}, F\right\rangle = \frac{\varphi_{i;k}\left\langle F_{ij}, F_{kj}\right\rangle}{1+|F|^2} + f(|\nabla F|^2).$$

On the other hand, we have at p

$$\begin{aligned} \nabla_{e^i}\left(b_{ij}\nabla_{e^j}|F|^2\right) &= \nabla_i\left(4(\alpha-1)\frac{\left\langle F_{lj}, F_{li}\right\rangle}{1+|F|^2}\varphi_j\right) \\ &= 4(\alpha-1)\frac{\left\langle F_{lj}, F_{li}\right\rangle}{1+|F|^2}\varphi_{j;i} + (\alpha-1)\frac{F\#F\#\nabla F\#\nabla F}{1+|F|^2} \\ &\qquad + (\alpha-1)\frac{F\#F\#\langle F, \nabla F\rangle^2}{(1+|F|^2)^2} \\ &\geq 4(\alpha-1)\left\langle D\frac{*(\varphi \wedge *F)}{1+|F|^2}, F\right\rangle - C(\alpha-1)|\nabla F|^2. \end{aligned} \tag{3.5}$$

Since p is an arbitrary point of M, we may combine (3.2), (3.4) and (3.5) to get (when $\alpha - 1$ small),

$$\begin{aligned} &\frac{\partial}{\partial t}|F|^2 - \nabla_{e_i}\left((\delta_{ij}+b_{ij})\nabla_{e_j}|F|^2\right) + |\nabla F|^2 \\ &\qquad \leq C|F|^3 - \langle F, F \circ (Ric \wedge g + 2R)\rangle. \end{aligned} \tag{3.6}$$

Since the manifold is compact and the curvatures are bounded, the lemma follows trivially from (3.6). We shall use this shaper estimate later to prove a gap theorem for Yang–Mills α-connections on S^4. □

As a consequence of Lemma 3.2 , we have

Lemma 3.3. (Bochner type formula 2) For each $\alpha > 1$, let A be the smooth solution of the Yang–Mills α-flow and $F := F_A$ the curvature of A. Then for $\alpha-1$ sufficiently small, we have

$$\frac{\partial}{\partial t}(1+|F|^2)^\alpha - \nabla_{e_i}\left((\delta_{ij} + 2(\alpha-1)\frac{\langle F_{lj}, F_{li}\rangle}{1+|F|^2})\nabla_{e_j}(1+|F|^2)^\alpha\right) \leq C(1+|F|^2)^\alpha(1+|F|) \quad (3.7)$$

for a constant $C > 0$.

Proof. In fact, one sees

$$\frac{\partial}{\partial t}(1+|F|^2)^\alpha = \alpha(1+|F|^2)^{\alpha-1}\frac{\partial |F|^2}{\partial t}$$

and

$$\nabla_{e_j}(1+|F|^2)^\alpha = \alpha(1+|F|^2)^{\alpha-1}\nabla_{e_j}|F|^2.$$

For simplicity, we set

$$a_{ij} = \delta_{ij} + 2(\alpha-1)\frac{\langle F_{lj}, F_{li}\rangle}{1+|F|^2}.$$

Then we have

$$\begin{aligned}
\nabla_{e_i}\left(a_{ij}\nabla_{e_j}(1+|F|^2)^\alpha\right) &= \alpha\nabla_{e_i}(a_{ij}(1+|F|^2)^{\alpha-1}\nabla_{e_j}|F|^2) \\
&= \alpha(1+|F_\alpha|^2)^{\alpha-1}\nabla_{e_i}(a_{ij}\nabla_{e_j}F|^2) \\
&\quad +\alpha(\alpha-1)(1+|F|^2)^{\alpha-2}a_{ij}\nabla_{e_i}|F|^2\nabla_{e_j}|F|^2.
\end{aligned}$$

By Lemma 3.2, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}(1+|F|^2)^\alpha &- \nabla_{e_i}\left(a_{ij}\nabla_{e_j}(1+|F|^2)^\alpha\right) \\
&= \alpha(1+|F|^2)^{\alpha-1}\left[\frac{\partial}{\partial t}|F|^2 - \nabla_{e_i}(a_{ij}\nabla_{e_j}|F|^2)\right] \\
&\qquad -\alpha(\alpha-1)(1+|F|^2)^{\alpha-2}a_{ij}\nabla_{e_i}|F|^2\nabla_{e_j}|F|^2. \\
&\leq C(1+|F|^2)^{\alpha-1}|F|^2(1+|F|).
\end{aligned}$$

This proves our claim. □

3.2. Monotonicity formula. The global parabolic monotonicity formula for harmonic maps was first established by Struwe in [28], and for the Yang–Mills flow in [5] and [13]. Next, we will derive a local parabolic type of monotonicity for the Yang–Mills α-flow as similar to one in [16].

Let $i(M)$ be the injectivity radius of M. For $z_0 = (x_0, t_0) \in M \times \mathbb{R}_+$, we write

$$T_R(z_0) = \left\{z = (x,t) : t_0 - 4R^2 < t < t_0 - R^2, x \in M\right\}.$$

When there is no ambiguity for z_0, we write T_R only.

If we take the normal coordinates $\{x^i\}$ in $B_{i(M)}(x_0)$, $dv = \sqrt{g(x)}dx$ and the Euclidean backward heat kernel to the (backward) heat equation with singularity at z_0 is

$$G_{z_0}(z) = \frac{1}{(4\pi(t_0 - t))^2}\exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \quad t < t_0.$$

As before, we write $G(x, t)$ when z_0 is obvious.

For a small $R_0 \le i(M)$ and some fixed $x_0 \in M$, let ϕ be a cut-off function supported in $B_{R_0}(x_0)$ with $\phi \equiv 1$ on $B_{R_0/2}(x_0)$. Assume that A is a solution of the α-flow (1.4) in $M \times \mathbb{R}_+$. For any $z_0 = (x_0, t_0) \in M \times [0, T]$, we set

$$\Phi_\alpha(R, A; z_0) = R^{4\alpha-2} \int_{T_R(z_0)} (1 + |F(z)|^2)^\alpha\, \phi^2(x - x_0)\, G_{z_0}(z)\, \sqrt{g}\, dx\, dt. \quad (3.8)$$

Lemma 3.4. (Local Monotonicity) Let A be a regular solution of the α-flow (1.4). Then, for $z_0 = (x_0, t_0) \in M \times (0, \infty)$ and for any two numbers R_1, R_2 with $0 < R_1 \le R_2 \le R_0$, we have

$$\Phi_\alpha(R_1, A; z_0) \le C \exp(C(R_2 - R_1))\Phi_\alpha(R_2, A; z_0) + C(R_2^2 - R_1^2)\mathrm{YM}_\alpha(A_0).$$

Proof. Although the main idea of the proof is similar to one for the Yang–Mills flow in [16], the proof becomes much more involved, so we have to give more details here.

Since the computation is local, we choose normal coordinates $\{x^i\}$ around x_0 and assume without loss of generality that $t_0 = 0$ and $x_0 = 0$.

In (3.8), we set $x = R\tilde{x}$ and $t = R^2\tilde{t}$ to obtain

$$\Phi_\alpha(R, A; z_0) = \int_{T_1} R^{4\alpha}(1 + |F|^2(x, t))^\alpha\, \phi^2(R\tilde{x})\, G(\tilde{z})\sqrt{g(R\tilde{x})}\, d\tilde{z}\,,$$

where $d\tilde{z} = d\tilde{x}\, d\tilde{t}$.

Then we compute

$$
\begin{aligned}
&\frac{d}{dR}\Phi_\alpha(R,A;z_0)\\
&= \int_{T_1}\frac{d}{dR}\Big[R^{4\alpha}[1+|F|^2(R\tilde{x},R^2\tilde{t})]^\alpha\,\phi^2(R\tilde{x})\,\sqrt{g(R\tilde{x})}\,\Big]\,G(\tilde{z})\,d\tilde{z}\\
&= 4\alpha R^{4\alpha-1}\int_{T_1}[1+|F|^2(R\tilde{x},R^2\tilde{t})]^\alpha\,\phi^2(R\tilde{x})\sqrt{g(R\tilde{x})}\,G(\tilde{z})\,d\tilde{z}\\
&+\alpha R^{4\alpha}\int_{T_1}\Big[[1+|F|^2(R\tilde{x},R^2\tilde{t})]^{\alpha-1}\tilde{x}^k\\
&\qquad\times\frac{\partial}{\partial x^k}|F|^2(R\tilde{x},R^2\tilde{t})\,\phi^2(R\tilde{x})\,\sqrt{g(R\tilde{x})}G(\tilde{z})\Big]\,d\tilde{z}\\
&+\alpha R^{4\alpha}\int_{T_1}\Big[[1+|F|^2(R\tilde{x},R^2\tilde{t})]^{\alpha-1}2R\tilde{t}\\
&\qquad\times\frac{\partial}{\partial t}|F|^2(R\tilde{x},R^2\tilde{t})\,\phi^2(R\tilde{x})\,\sqrt{g(R\tilde{x})}\,G(\tilde{z})\Big]\,d\tilde{z}\\
&+\int_{T_1}R^{4\alpha}[1+|F|^2(R\tilde{x},R^2\tilde{t})]^\alpha\,\tilde{x}^k\,\frac{\partial}{\partial x^k}\,(\phi^2\sqrt{g})(R\tilde{x})\,G(\tilde{z})\,d\tilde{z}\\
&:= I_1+I_2+I_3+I_4\,.
\end{aligned}
$$

In order to estimate I_1 and I_2, we note that in local coordinates, we have

$$F=\frac{1}{2}F_{ij}\,dx^i\wedge dx^j.$$

Let $\nabla_{A,x^k}F=\frac{1}{2}\nabla_{A,x^k}F_{ij}\,dx^i\wedge dx^j$ be the gauge-covariant derivative of F with respect to $\frac{\partial}{\partial x^k}$ satisfying $\nabla_{A,x^k}F_{ij}=\frac{\partial F_{ij}}{\partial x^k}+[A_k,F_{ij}]-\sum_s\Gamma^s_{ik}F_{sj}-\sum_s\Gamma^s_{jk}F_{is}$. Since A is compatible with the Riemannian structure, we have

$$\frac{\partial}{\partial x^k}|F|^2=\frac{1}{2}\Big\langle\nabla_{A,x^k}F_{ij}\,dx^i\wedge dx^j,\,F_{lm}dx^l\wedge dx^m\Big\rangle\,.$$

In local coordinates, the Bianchi identity $DF=0$ is equivalent to

$$\nabla_{A,x^k}F_{ij}=\nabla_{A,x^i}F_{kj}-\nabla_{A,x^j}F_{ki}\,.$$

Using the Bianchi identity, we have

$$
\begin{aligned}
x^k\frac{\partial}{\partial x^k}|F|^2 &= \frac{1}{2}x^k\Big\langle(\nabla_{A,x^i}F_{kj}-\nabla_{A,x^j}F_{ki})dx^i\wedge dx^j,\;F_{lm}dx^l\wedge dx^m\Big\rangle\\
&= \Big\langle\nabla_{A,x^i}(x^kF_{kj})dx^i\wedge dx^j,\;F_{lm}dx^l\wedge dx^m\Big\rangle-4|F|^2\\
&\qquad -\Big\langle x^kF_{sj}\Gamma^s_{ki}dx^i\wedge dx^j,\;F_{lm}dx^l\wedge dx^m\Big\rangle,
\end{aligned}
$$

where $\nabla_{A,x^i}(x^k F_{kj}) := \frac{\partial}{\partial x^i}(x^k F_{kj}) + [A_i, x^k F_{kj}] - x^k F_{ks}\Gamma^s_{ji}$ is the gauge-covariant derivative of $x^k F_{kj}$ with respect to $\frac{\partial}{\partial x^i}$. Changing back to (x, t), we have

$$\begin{aligned}
I_1 + I_2 &= \alpha R^{4\alpha-3}\int_{T_R}(1+|F|^2)^{\alpha-1}\Big[4(|F|^2+1) + x^k\frac{\partial |F|^2}{\partial x^k}\Big]\phi^2\, G\,\sqrt{g}\,dz \\
&= \alpha R^{4\alpha-3}\int_{T_R}\Big[(1+|F|^2)^{\alpha-1} \\
&\quad\times\Big[4+\Big\langle \nabla_{A,x^i}(x^k F_{kj})dx^i\wedge dx^j,\ F_{lm}dx^l\wedge dx^m\Big\rangle\Big]\phi^2\, G\,\sqrt{g}\Big]\,dz \\
&\quad-\alpha R^{4\alpha-3}\int_{T_R}\Big[(1+|F|^2)^{\alpha-1} \\
&\qquad\times\Big\langle x^k F_{sj}\Gamma^s_{ki}dx^i\wedge dx^j,\ F_{lm}dx^l\wedge dx^m\Big\rangle\,\phi^2\, G\,\sqrt{g}\Big]\,dz.
\end{aligned}$$

Note that

$$D^*[(1+|F|^2)^{\alpha-1}F] = -g^{il}\nabla_{A,x^i}[(1+|F|^2)^{\alpha-1}F_{lm}]dx^m\,.$$

Then using Stokes' formula, we have

$$\begin{aligned}
&\int_{T_R}(1+|F|^2)^{\alpha-1}\Big\langle\nabla_{x^i}(x^k F_{kj})dx^i\wedge dx^j,\ F_{lm}dx^l\wedge dx^m\Big\rangle\,\phi^2\, G\,\sqrt{g}\,dz \\
&\quad= 2\int_{T_R}(1+|F|^2)^{\alpha-1}\Big\langle\nabla_{x^i}(x^k F_{kj})dx^j,\ g^{il}F_{lm}dx^m\Big\rangle\,\phi^2\, G\,\sqrt{g}\,dz \\
&\quad= 2\int_{T_R}\Big\langle x^k F_{kj}dx^j,\ D^*[(1+|F|^2)^{\alpha-1}F]\Big\rangle\,\phi^2\, G\,\sqrt{g}\,dz \\
&\qquad-2\int_{T_R}(1+|F|^2)^{\alpha-1}\Big\langle x^k F_{kj}dx^j,\ g^{il}F_{lm}dx^m\Big\rangle\,\phi^2\,\frac{\partial G}{\partial x^i}\sqrt{g}\,dz \\
&\qquad-4\int_{T_R}(1+|F|^2)^{\alpha-1}\Big\langle x^k F_{kj}dx^j,\ g^{il}F_{lm}dx^m\Big\rangle\,\phi\,\frac{\partial\phi}{\partial x^i}\, G\,\sqrt{g}\,dz\,.
\end{aligned}$$

Using the fact that

$$|g_{ij}(x)-\delta_{ij}|\le C|x|^2,\quad \left|\frac{\partial g_{ij}}{\partial x^k}\right|\le C|x|,\quad \frac{\partial G}{\partial x^i} = \frac{x^i}{2t}G,$$

we have

$$\begin{aligned}
I_1+I_2 &\ge 2\alpha R^{4\alpha-3}\int_{T_R}\Big\langle x^k F_{kj}dx^j, D^*((1+|F|^2)^{\alpha-1}F)\Big\rangle\,\phi^2\, G\,\sqrt{g}\,dz \\
&\quad+\ \alpha R^{4\alpha-3}\int_{T_R}(1+|F|^2)^{\alpha-1}|x^i g^{ik}F_{kj}dx^j|^2\frac{1}{|t|}\,G\,\phi^2\,\sqrt{g}\,dz \\
&\quad- C\alpha R^{4\alpha-3}\int_{T_{R_1}}(1+|F|^2)^{\alpha}(|x|^2\phi^2 + |x||\nabla\phi| + \frac{|x|^4}{|t|}\phi^2)\ G\,\sqrt{g}\,dz\,.
\end{aligned}$$

To estimate I_3,we note that the α-flow (1.4) is equivalent to

$$(1+|F|^2)^{\alpha-1}\frac{\partial A}{\partial t}=-D^*\left((1+|F|^2)^{\alpha-1}F\right).$$

Then using Stokes' formula, we have

$$\begin{aligned}
I_3 &= 2\alpha R^{4\alpha-3}\int_{T_R}(1+|F|^2)^{\alpha-1}\,t\frac{\partial}{\partial t}|F|^2\,\phi^2\,G\,\sqrt{g}\,dz\\
&= 4\alpha R^{4\alpha-3}\int_{T_R}t\left\langle(1+|F|^2)^{\alpha-1}F,\ D(\frac{\partial A}{\partial t})\right\rangle\phi^2\,G\,\sqrt{g}dz\\
&= 4\alpha R^{4\alpha-3}\int_{T_R}t\left\langle D^*\left[(1+|F|^2)^{\alpha-1}F\right],\ \frac{\partial A}{\partial t}\right\rangle\phi^2\,G\,\sqrt{g}\,dz\\
&- 4\alpha R^{4\alpha-3}\int_{T_R}t(1+|F|^2)^{\alpha-1}\left\langle\frac{\partial A}{\partial t},\ g^{il}F_{lm}dx^m\right\rangle\left(\frac{\partial G}{\partial x^i}\phi^2+2\phi\frac{\partial\phi}{\partial x^i}G\right)\sqrt{g}\,dz\\
&= 4\alpha R^{4\alpha-3}\int_{T_R}|t|\,(1+|F|^2)^{\alpha-1}|\frac{\partial A}{\partial t}|^2\,\phi^2\,G\,\sqrt{g}\,dz\\
&- 2\alpha R^{4\alpha-3}\int_{T_R}(1+|F|^2)^{\alpha-1}\left\langle\frac{\partial A}{\partial t},\ x^i g^{il}F_{lm}dx^m\right\rangle\phi^2 G\,\sqrt{g}\,dz\\
&- 4\alpha R^{4\alpha-3}\int_{T_R}t(1+|F|^2)^{\alpha-1}\left\langle\frac{\partial A}{\partial t},\ g^{il}F_{lm}dx^m\right\rangle 2\phi\frac{\partial\phi}{\partial x^i}G\,\sqrt{g}\,dz.
\end{aligned}$$

Using above estimates and also Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dR}\Phi(R;A) &= I_1+I_2+I_3+I_4\\
&\geq \frac{1}{2}\alpha R^{4\alpha-3}\int_{T_R}|t|(1+|F|^2)^{\alpha-1}\left|2\frac{\partial A}{\partial t}-\frac{x^i}{|t|}g^{il}F_{lm}dx^m\right|^2\phi^2\,G\,\sqrt{g}\,dz\\
&- CR^{4\alpha-3}\int_{T_R}(1+|F|^2)^{\alpha}(|x|^2\phi^2+|x||\nabla\phi|+\frac{|x|^4}{t}\phi^2+|t||\nabla\phi|^2)\,G\,\sqrt{g}\,dz,
\end{aligned}$$

where C is a constant depending on the geometry of M. We know that

$$R^{-1}|x|^2G\leq C(1+G),\quad R^{-1}|t|^{-1}|x|^4G\leq C(1+G)\quad\text{ on } T_R\,.$$

Moreover, since $\nabla\phi=0$ for $|x|<R_0/2$, we see that

$$(|x||\nabla\phi|+|t|\,|\nabla\phi|^2)G\leq C\quad\text{ on } T_R.$$

Combining these estimates with Lemma 2.1, we obtain

$$\frac{d}{dR}\Phi(R;A)\geq -C\,\Phi(R;u,A)-CR\,\mathrm{YM}_\alpha(A_0)\,.$$

The claim for Φ follows from integrating the above inequality in R. □

3.3. The $\varepsilon-$regularity and convergence.

Lemma 3.5. There exists a positive constant $\varepsilon_0 < i(M)$ such that for a solution A to (1.4), if for some R with $0 < R < \min\{\varepsilon_0, \frac{t_0^{1/2}}{2}\}$ the inequality

$$R^{4\alpha-6}\int_{P_R(x_0,t_0)}(1+|F|^2)^\alpha\, dv\, dt \leq \varepsilon_0$$

holds, we have

$$\sup_{P_{\frac{1}{4}R}(x_0,t_0)} |F|^2 \leq CR^{-4}\,,$$

where the constant C depends on M and the bound of $\mathrm{YM}_\alpha(A_0)$.

Proof. Without loss of generality, assume that $(x_0,t_0) = (0,0)$. For simplicity, we set $r_1 = \frac{1}{2}R$. As in [25], we choose $r_0 < r_1$ such that

$$(r_1-r_0)^{4\alpha}\sup_{P_{r_0}}(1+|F|^2)^\alpha = \max_{0\leq r\leq r_1}\left[(r_1-r)^{4\alpha}\sup_{P_r}(1+|F|^2)^\alpha\right],$$

and find $(x_1,t_1)\in P_{r_0}$ such that

$$e_0 := (1+|F|^2)^\alpha(x_1,t_1) = \sup_{P_{r_0}}(1+|F|^2)^\alpha\,.$$

We claim that

$$e_0 \leq 2^{4\alpha}(r_1-r_0)^{-4\alpha}\,. \tag{3.9}$$

Otherwise, we have

$$\rho_0 = e_0^{-\frac{1}{4\alpha}} \leq \frac{r_1-r_0}{2}\,.$$

Rescale

$$B(\tilde{x}) = \rho_0\, A(x_1+\rho_0\tilde{x}, t_1+\rho_0^2\tilde{t})\,.$$

and

$$e_{\rho_0} := (\rho_0^4+|F_B|^2)^\alpha = \rho_0^{4\alpha}\,(1+|F|^2)^\alpha.$$

Then we have

$$\begin{aligned}
1 = e_{\rho_0}(0,0) \leq \sup_{\tilde{P}_1} e_{\rho_0}(\tilde{x},\tilde{t}) &= \rho_0^{4\alpha}\sup_{P_{\rho_0}(x_1,t_1)}(1+|F(x,t)|^2)^\alpha\\
&\leq \rho_0^{4\alpha}\left(\frac{r_1-r_0}{2}\right)^{-4\alpha}\left(\frac{r_1-r_0}{2}\right)^{4\alpha}\sup_{P_{\frac{r_1+r_0}{2}}}(1+|F(x,t)|^2)^\alpha\\
&\leq \rho_0^{4\alpha}\left(\frac{r_1-r_0}{2}\right)^{-4\alpha}(r_1-r_0)^{4\alpha}\,e_0 = 2^{4\alpha},
\end{aligned}$$

with $\tilde{P}_1 := \{(\tilde{x},\tilde{t}) : (\tilde{x},\tilde{t})\in B_1(0)\times[-1,1]\}$.

This implies that

$$|F_B|^2 \le 16 \quad \text{on } \bar{P}_1.$$

Combining this with Lemma 3.3, we have

$$\begin{aligned}
&(\frac{\partial}{\partial \tilde{t}} e_{\rho_0} - \tilde{\nabla}_{e_i}\left((\delta_{ij} + 2(\alpha-1)\frac{\langle F_{Blj}, F_{Bli}\rangle}{\rho_0^4 + |F_B|^2})\tilde{\nabla}_{e_j} e_{\rho_0}\right) \\
&= \rho_0^{2+4\alpha}\left[\frac{\partial}{\partial t}(1+|F|^2)^\alpha - \nabla_{e_i}\left((\delta_{ij} + 2(\alpha-1)\frac{\langle F_{lj}, F_{li}\rangle}{1+|F|^2})\nabla_{e_j}(1+|F|^2)^\alpha\right)\right] \\
&\le C e_{\rho_0}, \text{ in } \tilde{P}_1 ,
\end{aligned}$$

where the constant C depends on $i(M)$ and $\sup_{x\in M}|R_m|$. Then Moser's parabolic Harnack inequality yields

$$1 = e_{\rho_0}(0,0) \le C\int_{\tilde{P}_1} e_{\rho_0}\, d\tilde{x}\, d\tilde{t} = C\rho_0^{4\alpha-6}\int_{P_{\rho_0}(x_1,t_1)}(1+|F|^2)^\alpha\, dv\, dt\, . \tag{3.10}$$

Taking $\sigma = 2\rho_0$ and noting that $z_1 = (x_1, t_1) \in P_{r_0}$ and $\sigma + r_0 \le \frac{R}{2}$, we apply Lemma 3.4 with $R_1 = \frac{\sigma}{2}$, $R_2 = R_0 = \frac{1}{2}R$ to obtain

$$\begin{aligned}
&\rho_0^{4\alpha-6}\int_{P_{\rho_0}(z_1)}(1+|F|^2)^\alpha\, dv\, dt \qquad (3.11)\\
&\quad\le C\int_{T_\sigma(x_1,t_1+2\sigma^2)}\sigma^{4\alpha-2}(1+|F|^2)^\alpha\, G_{(x_1,t_1+2\sigma^2)}\,\phi^2\, dv\, dt \\
&\quad\le C\int_{T_{\frac{1}{2}R}(x_1,t_1+2\sigma^2)} R^{4\alpha-2}(1+|F|^2)^\alpha\, G_{(x_1,t_1+2\sigma^2)}\,\phi^2\, dv\, dt \\
&\qquad\qquad + CR\mathrm{YM}_\alpha(A_0) \\
&\quad\le CR^{4\alpha-6}\int_{P_R}(1+|F|^2)^\alpha\, dv\, dt + CR\,\mathrm{YM}_\alpha(A_0) \le C\varepsilon_0,
\end{aligned}$$

where we used the fact that for $t_1+2\sigma^2-R^2 \le t \le t_1+2\sigma^2-\frac{R^2}{4}$ and $x \in B_R(x_1)$, there is a constant C such that

$$G_{x_1,t_1+2\sigma^2} = \frac{1}{(4\pi(t_1+2\sigma^2-t))^2}\exp\left(-\frac{(x-x_1)^2}{4(t_1+2\sigma^2-t)}\right) \le CR^{-4}.$$

Letting ε_0 be sufficiently small, (3.11) contradicts (3.10). Therefore, we have proved the claim (3.9), which implies

$$\sup_{P_{R/4}}(1+|F|^2)^\alpha \le (\frac{r_1}{2})^{-4\alpha}(r_1-r_0)^{4\alpha}e_0 \le 2^{4\alpha}R^{-4\alpha}\, .$$

This proves Lemma 3.5. □

With the curvature bound obtained by Lemma 3.5, we may obtain higher order derivative estimates of F.

Lemma 3.6. Suppose that A is a solution of the flow equation (3.1) on some parabolic ball $P_r(x_0, t_0)$ and that

$$\sup_{P_r(x_0,t_0)} |F| \leq C.$$

Then for each k, there is a constant C_k such that

$$\sup_{P_{r/2}(x_0,t_0)} \left|\nabla^k F\right| \leq C(k).$$

Proof. Assume that $r = 1$ and write P_r for $P_r(x_0, t_0)$. Recall that F satisfies

$$\frac{\partial F}{\partial t} = -DD^* F + 2(\alpha - 1) D \frac{*(\langle \nabla F, F \rangle \wedge *F)}{1 + |F|^2}.$$

By the Bianchi identity and Weizenböck formula, we have

$$\frac{\partial F}{\partial t} = \triangle F + 2(\alpha - 1) D \frac{*(\langle \nabla F, F \rangle \wedge *F)}{1 + |F|^2} + F\#F + \mathrm{Rm}\#F, \tag{3.12}$$

where $\triangle$ is the covariant Laplacian and Rm is the Riemannian curvature of M. The proof is by induction. Let φ be a cut-off function supported in B_1 with $\varphi \equiv 1$ on $B_{3/4}$. Multiplying both sides of (3.12) by $\varphi^2 F$ and integrating over B_1, we have

$$\frac{1}{2}\frac{d}{dt}\int_{B_1} \varphi^2 |F|^2 \, dv + \int_{B_1} \varphi^2 |\nabla F|^2 \, dv \leq C(\alpha - 1) \int_{B_1} \varphi^2 |\nabla F|^2 \, dv + \mathcal{L},$$

where $\mathcal{L}$ contains all 'lower order terms'.

In the above equation, it includes $\int_{B_1} \varphi^2 |F|^3 \, dv$ and $\int_{B_1} \varphi^2 |F|^2 \, dv$, which are bounded, and $\int_{B_1} |\nabla\varphi| \, \varphi \, |\nabla F| \, |F| \, dv$, which arises in the integration by parts. We shall see that

$$\mathcal{L} \leq \eta \int_{B_1} \varphi^2 |\nabla F|^2 \, dv + C. \tag{3.13}$$

In fact,

$$\int_{B_1} |\nabla\varphi| \, \varphi \, |\nabla F| \, |F| \, dv \leq C + \eta \int_{B_1} \varphi^2 |\nabla F|^2 \, dv.$$

By choosing $\alpha - 1$ and η small, we conclude that

$$\int_{P_{3/4}} |\nabla F|^2 \, dv dt \leq C.$$

We may choose a good time slice on which the space integration of $|\nabla F|^2$ is bounded. Instead of further shrinking the neighborhood, we assume

$\int_{P_1} |\nabla F|^2 \, dvdt \le C$ and $\int_{B_1} |\nabla F|^2 (\cdot, -1) dv \le C$, which is the starting point for the next step of induction.

Applying ∇ on (3.12), multiplying by $\varphi^4 \nabla F$ and integrating over B_1, we have

$$\frac{1}{2}\frac{d}{dt}\int_{B_1} \varphi^4 |\nabla F|^2 \, dv + \int_{B_1} \varphi^4 \left|\nabla^2 F\right|^2 dv \le C(\alpha - 1)\int_{B_1} \varphi^4 \left|\nabla^2 F\right|^2 + \varphi^4 |\nabla F|^4 \, dv + \mathcal{L}.$$

The lower order terms (still denoted by $\mathcal{L}$) which arise from switching the order of covariant derivatives, integration by parts and interchanging ∇ and $\frac{\partial}{\partial t}$ can be controlled by $\eta \int_{B_1} \varphi^4 |\nabla F|^4 + \varphi^4 \left|\nabla^2 F\right|^2 dv + C$ as before. For example,

$$\begin{aligned}
\int_{B_1} \left|\nabla^2 F\right| |\nabla F| \left|\nabla(\varphi^4)\right| dv \quad &\le \quad C\int_{B_1} \left|\varphi^2 \nabla^2 F\right| |\varphi \nabla F| \, |\nabla \varphi| \, dv \\
&\le \quad \eta \int_{B_1} \varphi^4 |\nabla F|^4 + \varphi^4 \left|\nabla^2 F\right|^2 dv + C.
\end{aligned}$$

Thanks to the boundedness of F, we have

$$\begin{aligned}
\int_{B_1} \varphi^4 |\nabla F|^4 \, dv \quad &= \quad \int_{B_1} \varphi^4 \langle \nabla F, \nabla F\rangle \, |\nabla F|^2 \, dv \\
&\le \quad C\int_{B_1} \varphi^4 \left|\nabla^2 F\right| |\nabla F|^2 \, dv + C\int_{B_1} |\nabla \varphi| \, \varphi^3 |\nabla F|^3 \, dv \\
&\le \quad \frac{1}{2}\int_{B_1} \varphi^4 |\nabla F|^4 + C + C\int_{B_1} \varphi^4 \left|\nabla^2 F\right|^2 dv.
\end{aligned} \tag{3.14}$$

By taking $\alpha - 1$ small, we have that $\int_{P_{3/4}} \left|\nabla^2 F\right| dvdt$ is bounded, due to the boundedness of $\int_{B_1} |\nabla F|^2 (\cdot, -1) dv$.

For $k > 2$, we give an indication of how the above process works. By a similar computation,

$$\frac{1}{2}\frac{d}{dt}\int_{B_1} \varphi \left|\nabla^k F\right|^2 dv + \int_{B_1} \varphi \left|\nabla^{k+1} F\right|^2 dv \le C(\alpha - 1)\int_{B_1} \varphi \cdot \left(\sum \prod_{i=1}^{l} |\nabla^{a_i} F|^{b_i}\right) + \mathcal{L}.$$

Here the summation $\sum$ is over all possible (a_i, b_i) satisfying (1) $a_i = 1, \cdots, k+1$, $b_i \in \mathbb{N}$ with $i = 1, \cdots, l$ for some $l \in \mathbb{N}$ and (2) $\sum_{i=1}^{l} a_i b_i = 2(k+1)$. The sum of those terms with $\sum_{i=1}^{l} a_i b_i < 2(k+1)$ are denoted by $\mathcal{L}$.

By Young's inequality, we have

$$\int_{B_1} \varphi \sum \prod_{i=1}^{l} |\nabla^{a_i} F|^{b_i} \, dv \le C \sum_{i=1}^{k+1} \int_{B_1} \varphi \left|\nabla^i F\right|^{\frac{2(k+1)}{i}} dv.$$

We now claim that for each $i = 1 \cdots k$, we have

$$\int_{B_1} \varphi \left|\nabla^i F\right|^{\frac{2(k+1)}{i}} dv \le C \int_{B_1} \varphi \left|\nabla^{i+1} F\right|^{\frac{2(k+1)}{i+1}} dv + C + \mathcal{L}.$$

The claim can be proved by induction from $i = 1$, which is essentially (3.14). For $i > 1$,

$$\begin{aligned}\int_{B_1} \varphi \left|\nabla^i F\right|^{\frac{2(k+1)}{i}} dv &\le C \int_{B_1} \varphi \left|\nabla^{i-1} F\right| \left|\nabla^{i+1} F\right| \left|\nabla^i F\right|^{\frac{2(k+1)}{i}-2} dv + \mathcal{L}. \\ &\le \eta \int_{B_1} \varphi \left|\nabla^i F\right|^{\frac{2(k+1)}{i}} dv + \eta \int_{B_1} \varphi \left|\nabla^{i-1} F\right|^{\frac{2(k+1)}{i-1}} dv \\ &\quad + C_\eta \int_{B_1} \varphi \left|\nabla^{i+1} F\right|^{\frac{2(k+1)}{i+1}} dv + C + \mathcal{L}.\end{aligned}$$

By the induction assumption and choosing η small, we see that the claim is true. □

Once we know that the C^k norm of the curvature is bounded in some parabolic neighborhood, it is natural to expect a good 'gauge' in which the connection form is bounded in C^{k+1}. This will be the parabolic analogue of Uhlenbeck's gauge fixing theorem. The precise statement and the proof of such a result will be interesting in its own right. For our purposes, since we have all C^k bounds and the connection is a solution of a parabolic equation, we can reduce the following result to its elliptic counterpart.

Lemma 3.7. Let $D(t)$ be a solution to the Yang–Mills α-flow defined on $B \times [t_1, t_2]$. Assume that

$$\sup_{B\times[t_1,t_2]} \left|\nabla^k F\right| \le C(k).$$

Then there is a trivialization (independent of t) in which $D(t) = d + A(t)$ and all derivatives (space and time) of $A(t)$ are bounded.

Proof. For $t = t_1$ fixed, we may apply Uhlenbeck's gauge fixing to find a trivialization such that at least all C^k norms of $A(t_1)$ are bounded (see Lemma 2.3.11 in [9]). We can now use (3.1) to see that $\frac{\partial A}{\partial t}$ is bounded for $B \times [t_1, t_2]$. The Newton–Leibnitz formula

$$A(t) = A(t_1) + \int_{t_1}^{t} \frac{\partial A}{\partial t} ds$$

then implies that $A(t)$ is uniformly bounded in $M \times [t_1, t_2]$. If we take derivatives of (3.1) both in space and time, by noticing that the right hand side involves only F, we know that $\frac{\partial^k A}{\partial t^k}$ are bounded on $B \times [t_1, t_2]$. By using the Newton–Leibnitz formula again, the space derivatives of A are uniformly bounded on $B \times [t_1, t_2]$. Since A is bounded, one can argue inductively that both covariant derivatives and the partial derivatives are bounded. $\square$

We now prove Theorem 1.2.

Proof. Let A_α be the smooth solution of the Yang–Mills α-flow in $M \times [0, \infty)$ with the same initial value A_0 for each $\alpha > 1$. The concentration set Σ is defined by

$$\Sigma = \bigcap_{0<R<R_M} \left\{ z \in M \times [0, \infty) : \liminf_{\alpha \to 1} R^{4\alpha-6} \int_{P_R(z)} (1 + |F_{A_\alpha}|^2)^\alpha \, dv \, dt \geq \varepsilon_0 \right\}$$

for some $\varepsilon_0 > 0$. It is standard to show that Σ is closed. The same argument as in [17] also yields that for any two positive t_1 and t_2, $\mathcal{P}^2(\Sigma \cap (M \times [t_1, t_2]))$ is finite, where $\mathcal{P}^2$ denotes the 2-dimensional parabolic Hausdorff measure. Moreover, for any $t \in (0, +\infty)$, $\Sigma_t = \Sigma \cap (M \times \{t\})$ consists of at most finitely many points.

For a point z_0 outside Σ, there is a constant $R > 0$ such that for sequence of $\alpha \to 1$, we have

$$R^{4\alpha-6} \int_{P_R(z_0)} (1 + |F_{A_\alpha}|^2)^\alpha \, dv \, dt \leq \varepsilon_0.$$

Then applying Lemma 3.5, we know that F_{A_α} is uniformly bounded in α inside $P_{R/2}(z_0)$.

Lemma 3.6 and Lemma 3.7 then imply that there is a trivialization on $P_{R/2}(z_0)$ such that $A_\alpha(t)$ is bounded in any C^k norm. We then choose a sequence of such neighborhoods $\{P_i\}$ covering $M \times [0, \infty) \setminus \Sigma$. Denote the transition functions by σ_{ij}^α. The C^k bound of σ_{ij}^α follows from those of A_i^α.

By taking a subsequence, we may assume that σ_{ij}^α converges to σ_{ij} and A_i^α to A_i smoothly as α goes to 1. The σ_{ij}'s define a bundle E_∞ over $M \times [0, \infty) \setminus \Sigma$ and the A_i's define a connection D_∞ of E_∞. Since the convergence is strong, we know from the evolution equation of A_i^α that $A_i(t)$ is a solution to the Yang–Mills flow. $\square$

Before we conclude this section, we would like to make some remarks. Both are related to the singular set Σ.

Remark 3.8. Let $T = \inf_{(x,t)\in\Sigma} t$ be the first concentration time in Theorem 1.2. We may follow from the argument of Theorem 1.3 in [17] to show that T is the same as the first singular time T' of the Yang–Mills flow.

As in [17], one may ask what more we can say about the singular set Σ. For the general case, not much is known. However, we do know something for a minimizing sequence.

Precisely, we have

Proposition 3.9. Let D_i be a minimizing sequence of $YM(\cdot)$ among all smooth connections of the bundle E. Then we choose a subsequence of $\alpha_i \to 1$ such that $YM_{\alpha_i}(D_i) < YM(D_i)+V(M)+\frac{1}{i}$, where $V(M)$ denotes the volume of M. Denote by $D_i(t)$ the α_i−flow solution with initial value D_i. If we consider $i \to \infty$, then the concentration set Σ as defined above satisfies

$$\Sigma = \bigcup_{j=1}^{l} \{p_j\} \times (0, \infty).$$

Proof. By the same proof of Theorem 1.2, the singular set Σ has the following form:

$$\Sigma = \bigcap_{0<R<R_M} \left\{ z \in M \times [0,\infty) : \liminf_{\alpha_i \to 1} R^{4\alpha-6} \int_{P_R(z)} (1 + |F_{D_i(t)}|^2)^{\alpha_i}\, dv\, dt \geq \varepsilon_0 \right\}.$$

For completeness, we give a proof of the finiteness of $\Sigma_t = \Sigma \cap (M \times \{t\})$. Let $\{x_j\}_{j=1}^{l}$ be any finite subset of Σ_t. By the definition of Σ, we know

$$\liminf_{\alpha_i \to 1} R^{4\alpha_i-6} \int_{P_R(x_j,t)} (1 + \left|F_{D_i(t)}\right|^2)^{\alpha_i} dvdt > \varepsilon_0$$

for any $R < R_M$. Let R be small positive number such that $B_R(x_i)$, $i = 1, \cdots, l$, are mutually disjoint. Hence, for α close to 1, we have

$$R^{-2} \int_{P_R(x_i,t)} (1 + \left|F_{D_i(t)}\right|^2)^{\alpha_i} dvdt > \varepsilon_0 R^{4-4\alpha_i} \geq C\varepsilon_0,$$

because R is small and $\alpha > 1$. Summing over i yields

$$lC\varepsilon_0 \leq R^{-2} \int_{t-R^2}^{t} \int_M (1 + \left|F_{D_i(t)}\right|^2)^{\alpha_i} dvdt.$$

By Lemma 2.1, l is bounded by a uniform bound of the total energy $YM(D_i)$ and ε_0, which implies the finiteness of Σ_t.

For any $t_4 > t_3 > 0$, since D_i is a minimizing sequence, by our suitable choice of $\alpha_i \to 1$ we have

$$\begin{aligned} V(M) + YM(D_i) + \frac{1}{i} &\geq YM_{\alpha_i}(D_i) \geq YM_{\alpha_i}(D_i(t_3)) \\ &\geq YM_{\alpha_i}(D_i(t_4)) \geq V(M) + YM(D_i), \end{aligned}$$

where we have used Lemma 2.1.

By Lemma 2.1 again, we have

$$\lim_{i\to\infty}\int_{t_3}^{t_4}\int_M (1+\left|F_{D_i(t)}\right|^2)^{\alpha_i-1}\left|\frac{dD_i(t)}{dt}\right|^2 dv dt = 0. \tag{3.15}$$

Moreover, the convergence is uniform with respect to t_3 and t_4. For any $t_2, t_1 > 0$, if $(x, t_1) \notin \Sigma$, we will show $(x, t_2) \notin \Sigma$ either. Since $(x, t_1) \notin \Sigma$, we have some $r_1 > 0$ such that for a subsequence (for simplicity, we still denote the subsequence by i),

$$\int_{B_{r_1}(x)} (1+\left|F_{D_i}(t_1)\right|^2)^{\alpha_i} dv \le \frac{\varepsilon_0}{4}.$$

Let φ be some cut-off function supported in $B_{r_1}(x)$. Then

$$\begin{aligned}
&\left|\frac{d}{dt}\int_M \varphi^2(1+\left|F_{D_i}\right|^2)^{\alpha_i} dv\right| \\
&\quad = \left|\int_M \alpha_i\varphi^2(1+\left|F_{D_i}\right|^2)^{\alpha_i-1}\langle F_{D_i}, \frac{\partial F_{D_i}}{\partial t}\rangle dv\right| \\
&\quad \le \int_M \left|\alpha_i\varphi^2\langle D_i^*\left((1+\left|F_{D_i}\right|^2)^{\alpha_i-1}F_{D_i}\right), \frac{\partial D_i}{\partial t}\rangle\right| \\
&\qquad\qquad + 2\alpha_i\varphi(1+\left|F_{D_i}\right|^2)^{\alpha_i-1}\left|\nabla\varphi\right|\left|F_{D_i}\right|\left|\frac{\partial D_i}{\partial t}\right| dv \\
&\quad \le \int_M \alpha_i\varphi^2(1+\left|F_{D_i}\right|^2)^{\alpha_i-1}\left|\frac{\partial D_i}{\partial t}\right|^2 dv \\
&\qquad\qquad + C\left(\int_M \alpha_i\varphi^2(1+\left|F_{D_i}\right|^2)^{\alpha_i-1}\left|\frac{\partial D_i}{\partial t}\right|^2 dv\right)^{1/2} \\
&\qquad\qquad\quad \cdot\left(\int_M \alpha_i\left|\nabla\varphi\right|^2(1+\left|F_{D_i}\right|^2)^{\alpha_i-1}\left|F_{D_i}\right|^2 dv\right)^{1/2}.
\end{aligned}$$

The term in the last line above is bounded by a constant depending on r_1 but not on i. Therefore, if we integrate from t_1 to t_3 and let $i \to \infty$, we have, thanks to (3.15),

$$\lim_{i\to\infty}\int_M \varphi^2(1+\left|F_{D_i}\right|^2)^{\alpha_i}(t_3)dv < \varepsilon_0/2.$$

Hence, by taking every $t_3 \in [t_2 - r_i^2, t_2 + r_i^2]$, we have (for some subsequence which we labeled by i)

$$\lim_{i\to\infty} r_1^{4\alpha_i-6}\int_{P_{r_i}(x,t_2)} (1+\left|F_{D_i}\right|^2)^{\alpha_i} dv dt \le \varepsilon_0.$$

Therefore (x, t_2) is not in Σ and the proof is done. □

4. Applications

In this section, we study the applications of the Yang Mills α-flow and the Yang Mills α-connection produced as the limit of the flow. The outline is as follows: in Subsection 4.1, we will prove the ε-regularity estimate for smooth Yang Mills α-connections. In Subsection 4.2, we will recall some facts about the topology of bundles and prove Theorem 1.3. In Subsection 4.3, we discuss a minimizing sequence of $YM(\cdot)$ and prove Theorem 1.4. Finally, we show how the Yang–Mills α-flow can be used to obtain a nonminimal Yang–Mills connections over S^4.

4.1. An ε-regularity lemma. This is an analogue of what Sacks and Uhlenbeck called 'main estimate'. It is necessary for the blow-up analysis. Please note that we use the α-flow to obtain a Yang–Mills α-connection as the limit as $t_i \to \infty$. It follows from Theorem 1.1 that the α-connection is smooth.

Lemma 4.1. There is $\varepsilon_1 > 0$ such that if D is a smooth α-Yang–Mills connection defined on B_1 with $\int_{B_1} |F|^2\, dv \le \varepsilon_1^2$, then in some trivialization with $D = d + A$,

$$\|A\|_{C^k(B_{1/2})} \le C(k)\, \|F\|_{L^2(B_1)}\,.$$

Although we can prove it directly, we show a parabolic version, from which Lemma 4.1 follows obviously.

Theorem 4.2. There is some $\varepsilon_1 > 0$ such that if $D(t)$ is a smooth solution to the α-Yang–Mills flow on $P_1 = B_1 \times [-1, 0]$ and

$$\sup_{t\in[-1,0]} \int_{B_1} |F|^2\, dv \le \varepsilon_1^2,$$

then

$$\sup_{t\in[-1/4,0]} \sup_{B_{1/2}} \left|\nabla^k F\right| \le C(k).$$

The proof is omitted because it is rather well known and follows the same method as in Lemma 3.5. It suffices to use the first Bochner formula (3.3). Moreover, the same method can be used to prove a stronger result by choosing a different blow-up factor. We need the following for the blow-up analysis

Theorem 4.3. There exists $\varepsilon_1 > 0$ such that if $D(t)$ is a smooth solution to the α-Yang–Mills flow satisfying

$$\sup_{[t_0-R^2,t_0]} \int_{B_R(x_0)} |F|^2\, dv \le \varepsilon_1^2,$$

then we have

$$\sup_{B_{R/2}(x_0)\times[t_0-R^2/4,t_0]} |F| \le \frac{C\varepsilon^{1/2}}{R^2},$$

where

$$\varepsilon := \sup_{t\in[t_0-R^2,t_0]} \int_{B_R(x_0)} |F|^2 \, dV.$$

Proof. By scaling and translation, we may assume that $R = 1$, $x_0 = 0$ and $t_0 = 0$. Set

$$P_r(x,t) = \{(x',t') | x' \in B_r(x) \text{ and } t - r^2 \le t' \le t\}.$$

It is $\sup_{P_{1/2}} |F|$ that we want to estimate. Find (x_1, t_1) in $P_{1/2}$ such that

$$|F| (x_1, t_1) \ge \frac{1}{2} \sup_{P_{1/2}} |F| .$$

It now suffices to bound $f_1 := |F| (x_1, t_1)$. If we are lucky, then we have

$$\sup_{P_{1/4}(x_1,t_1)} |F| \le 16 f_1. \tag{4.1}$$

If not, we can find (x_2, t_2) in $P_{1/4}(x_1, t_1)$ such that

$$|F| (x_2, t_2) = 16 f_1.$$

By induction, we claim that after finitely many times, we have $k \in \mathbb{N}$, such that

$$|F| (x_k, t_k) = 16^{k-1} f_1$$

and

$$\sup_{P_{1/4^k}(x_k,t_k)} |F| \le 16 |F| (x_k, t_k) = 16^k f_1.$$

In fact, if we write d_P for parabolic distance, then we have

$$d_P((x_k, t_k), (x_{k-1}, t_{k-1})) \le \frac{1}{4^{k-1}}.$$

Since (x_1, t_1) is in $P_{1/2}$, we know $(x_k, t_k) \in P_{5/6}$ for all k. However, F is smooth in P_1 and hence $\sup_{P_{5/6}} |F|$ is bounded.

We do a scaling and translation on $P_{1/4^k}(x_k, t_k)$ to get $\tilde{A}$ such that

$$\sup_{P_{\frac{1}{4} f_1^{1/2}}} \left|F_{\tilde{A}}\right| \le 16 \quad \text{and} \quad \left|F_{\tilde{A}}\right| (0,0) = 1 \tag{4.2}$$

and

$$\sup_{[-f_1/16,0]} \int_{B_{\frac{1}{4} f_1^{1/2}}} \left|F_{\tilde{A}}\right|^2 dV \le \varepsilon.$$

Applying (4.2) to Theorem 4.2 and noticing Lemma 3.2, we have

$$\frac{\partial}{\partial t}\left|F_{\tilde{A}}\right|^2 \leq \triangle\left|F_{\tilde{A}}\right|^2 + C\left|F_{\tilde{A}}\right|^2.$$

Consider $g(x,t) = e^{-Ct}\left|F_{\tilde{A}}\right|^2$ which is a subsolution to the heat equation. By Theorem 4.2, we know f_1 is bounded by a constant. Hence

$$\int_{-f_1/16}^{0}\int_{B_{\frac{1}{4}f_1^{1/2}}} g(x,t)dxdt \leq C\int_{-f_1/16}^{0}\int_{B_{\frac{1}{4}f_1^{1/2}}}\left|F_{\tilde{A}}\right|^2(x,t)dxdt.$$

By Mean Value inequality for linear heat equation,

$$1 = g(0,0) \leq Cf_1^{-2}\varepsilon,$$

which finishes the proof of this lemma. □

4.2. Connected sum of vector bundles. We recall some topological facts about vector bundles (principal bundles). Let G be a connected compact Lie group. There is a topological space BG, which is called the classifying space of G, and a G-bundle EG with BG as its base, which is called the universal bundle, such that for any $G-$bundle E over M, there is a map $f : M \to BG$ such that E is just the pull back bundle $f^*(EG)$. Moreover, the isomorphism classes of G-bundles are in one to one correspondence with the homotopy classes of maps from M to BG. Therefore, the classification of bundles is equivalent to the classification of continuous maps from M to BG.

The topology of BG is closely related to that of G. Since EG is contractible, the exact sequence of homotopy groups implies that

$$\pi_{i+1}(BG) = \pi_i(G).$$

Moreover, it is known that for all connected Lie groups G, $\pi_1(G)$ is a finitely generated abelian group, $\pi_2(G) = 0$ and $\pi_3(G)$ is a finitely generated free abelian group. An invariant of the classifying map f(hence of E) related to $\pi_1(G)$ is called an η invariant. It was defined via Čech cohomology in [26]. In particular, if $\pi_1(G) = 0$ or $M = S^4$, then η is always trivial. There is another invariant called the vector Pontryagin number related to $\pi_3(G)$. For our purposes, we shall restrict ourselves to the case $M = S^4$ below. Hence, it is nothing but an element in $\pi_4(BG) = \pi_3(G) = \mathbb{Z}^l$.

To define the connected sum of bundles, let us consider two bundles E_i over M_i for $i = 1, 2$. Pick any $p_i \in M_i$ and let B_i be a small ball around p_i such that $E_i|_{B_i}$ are trivial bundles. We obtain two manifolds with boundary $M_i \setminus B_i$ and two bundles $E_i|_{M_i\setminus B_i}$. We identify ∂B_i with orientation taken into account to obtain the connected sum $M = M_1\#M_2$. Such an identification is uniquely determined topologically. We still need an identification of $E_i|_{\partial B_i}$. Although they are trivial bundles over S^3, there are many different bundle isomorphisms between them. Among those isomorphisms, there is a natural one. $E_i|_{\partial B_i}$ admits a trivialization inherited from the trivialization of $E_i|_{B_i}$. By identifying the two trivializations, we obtain the natural isomorphism and a bundle E over M, which is called the connected sum of E_1 and E_2. Since we will always consider connected manifolds M_i, the definition is independent of the choice of p_i and the size of (small) B_i. We remark that $M\#S^4 = M$ for any closed 4−manifold M.

It is well known that when we consider the convergence of a sequence of Yang–Mills connections on bundle E with bounded energy, blow-up occurs. In fact, the same discussion works for α-Yang–Mills connections, or any other sequence of connections as long as we have the ε-regularity and a total energy bound. This results in a weak limit on some different bundle E' and finitely many bubble connections on E_i over S^4 for $i = 1 \cdots l$. The point is that $E = E'\#E_1\#\cdots\#E_l$. This follows from the removable singularity theorem of Uhlenbeck and some analysis on the neck region, which we briefly recall as follows.

Assume for simplicity that there is only one bubble. That is A_i, after gauge transformations, converges on $M\setminus B_\delta$ to the weak limit A', and after scaling, $A_i|_{B_{\lambda_i R}}$ converges on B_R to the bubble connection $\tilde{A}$. Since δ and R can be arbitrary, A' is defined on $M \setminus \{p\}$ and $\tilde{A}$ is defined on $\mathbb{R}^4$. The removable singularity theorem claims that in fact A' and $\tilde{A}$ are smooth connections of E' over M and $\tilde{E}$ over S^4. Topologically, there are different ways to extend a bundle over $M \setminus \{p\}$ to M. This amounts to the choice of a trivialization of $E|_{\partial B_\delta}$ (up to topological equivalence). There is one naturally dictated by the converging sequence A_i. By the ε−regularity, if we restrict A_i to $B_\delta \setminus B_{\delta/2}$ and scale to $B_2 \setminus B_1$, it is a connection with arbitrarily small curvature (in any norm). This decides a trivialization (see Lemma 2.4 in [33]). Similar analysis works for the bubble connection on $B_{2\lambda_i R} \setminus B_{\lambda_i R}$.

To see that E is the connected sum of E' and $\tilde{E}$, it suffices to show that the trivialization of E on $B_\delta \setminus B_{\delta/2}$ and $B_{2\lambda_i R} \setminus B_{\lambda_i R}$ agree with each other. This is related to how the bubble tree is constructed. If one follows the process of Ding and Tian [6], we know that the energy of the A_i restricted to $B_t \setminus B_{t/2}$ are smaller than any given ε_1 for $t \in [2\lambda_i R, \delta]$. For each t, the smallness of energy and ε−regularity implies a choice of trivialization. As t changes from $2\lambda_R$ to δ, we see that the two trivialization can be continuously deformed to each other. If one follows the construction of Parker [21], we have the total energy over the neck region $B_\delta \setminus B_{2\lambda_i R}$ is small (see (1.3) and (1.6) in [21]), say smaller than ε_1. Using the trivialization over $B_{2\lambda_i R} \setminus B_R$, we may extend the connection to B_R with a controlled amount of the energy. We can do the same at the infinity to obtain a smooth connection over S^4 whose energy is smaller than a multiple of ε_1. Hence, the bundle must be trivial and it implies that the two trivialization agree with each other.

The proof of Theorem 1.3 depends on the following lemma, which is well known.

Lemma 4.4. Let a_1, a_2 be two elements in $\pi_3(G)$ and E_1 and E_2 be the corresponding bundles over S^4. If $E = E_1 \# E_2$, then E corresponds to the element $a_1 + a_2$ in $\pi_3(G)$.

Proof. The key proof is to clarify the correspondence between homotopy class of maps from S^3 to G and the bundle over S^4. This can be done via the clutching functions.

Let S^4 be the unit sphere in $\mathbb{R}^5$ with coordinates $x_1, \cdots, x_5$. Let S^4_N be the north hemisphere given by $\{x_5 \geq 0\}$ and S^4_S be the south hemisphere. We also identify the equator $\{x_5 = 0\}$ by S^3. For any G bundle E over S^4, its restrictions to both hemispheres are trivial. Hence, we may choose the trivialization on both S^4_N and S^4_S. The topology of E is encoded in the gluing map $\theta : S^3 \to G$, which we call a clutching function. It is obvious that the isomorphism class of E corresponds to homotopic class of clutching functions θ.

Next, we study the connected sum of bundles in this setting. Let E_1 and E_2 be two bundles over S^4 as assumed. By abuse of notations, we may write a_1 and a_2 for the clutching functions of E_1 and E_2 respectively. In doing connected sum, we identify the trivialization on the south hemisphere part of E_1 with the trivialization on the north hemisphere part of E_2. Hence, the new bundle is glued from three pieces. The central one is a trivial bundle over $S^3 \times [0, 1]$. If we remove the central piece, we see the clutching function of the new bundle is $a_1 \cdot a_2$ (Lie group multiplication).

It remains to see that the homotopy class of $a_1 \cdot a_2$ is just the sum of a_1 and a_2. In fact, we may pick a map homotopic to a_1 (or a_2), still denoted by a_1 (or a_2), such that its restriction to a neighborhood of south (or north) hemisphere is the unit of G. Then, by the definition of group structure of $\pi_3(G)$ (as given on page 341 of Hatcher's book [12]), the homotopic class of $a_1 \cdot a_2$ is the sum of a_1 and a_2 in $\pi_3(G)$. □

We now prove Theorem 1.3.

Proof of Theorem 1.3. Recall that G-bundles over S^4 correspond to the homotopy classes of maps from S^4 to the classifying space BG of G, and that $\pi_4(BG) = \pi_3(G)$. Assume the theorem is not true. Then there are at most $r - 1$ G-bundles which admit Yang–Mills G−connections. Let $a_1, \cdots, a_{r-1}$ be elements in $\pi_4(BG)$ corresponding to these G-bundles. By our assumption, there is $a \in \pi_4(BG)$ which is not generated by $\{a_1, \cdots, a_{r-1}\}$.

Let E be the bundle corresponding to a. Pick any smooth connection on E. Consider the α-flow starting from it. Theorem 1.1 gives a Yang–Mills α-connection A_α for each $\alpha > 1$. Since E is not a trivial bundle and S^4 is simply connected, A_α cannot be flat. Take the limit as α to 1.

If the convergence is strong, then we find a Yang–Mills G-connection, which contradicts the choice of a. If not, the bundle E splits into a connected sum of $E_1,\dots,E_l$ over S^4, and each admits a Yang–Mills G-connection. By our assumption, $E_i (i = 1,\cdots,l)$ corresponds to one of $a_1,\cdots,a_{r-1}$. Moreover, by Lemma 4.4, the fact that E is a connected sum of $E_1,\cdots,E_l$ implies that a is a combination of $a_1,\cdots,a_{r-1}$ in $\pi_4(BG) = \pi_3(G)$. This is a contradiction to our choice of a. □

4.3. Minimizing sequences of $YM(\cdot)$. In this subsection, we prove Theorem 1.4. For a closed 4−manifold M and the G-bundle E, let $m(E)$ be the infimum of $YM(A)$ for all $G-$connections A of E.

First, let us show a general result which has nothing to do with the blow-up.

Proposition 4.5. If $E = E'\#E_1\#\cdots\#E_l$, where E' is a bundle over M and E_i are bundles over S^4, then

$$m(E) \le m(E') + \sum_{i=1}^{l} m(E_i).$$

Proof. For simplicity, consider $l = 1$. If suffices to show that for any $\varepsilon > 0$ and any two connections D_1 and D_2 of E' and E_1 respectively, we may construct a connection D of E such that

$$YM(D) \le YM(D_1) + YM(D_2) + \varepsilon.$$

(This is exactly Lemma 5.7 in [18]). For completeness, we also give a proof here.

Given any smooth connection D_i and a trivialization of the bundle over some ball B, by multiplying by a cut-off function, we may assume that D_i is flat in a smaller ball at the expense of any small change of the energy. More precisely, for any $\varepsilon > 0$, there is a $\delta > 0$ and we have another connection D_i' such that

(1) $D_i = D_i'$ outside B_δ;

(2) $D_i' = d$ on $B_{\delta/2}$;

(3) $\left|YM(D_i') - YM(D_i)\right| < \varepsilon$.

Indeed, if $D_i = d + A_i$ on B, due to the smoothness of A_i, there exists $\delta > 0$ such that if we scale B_δ to B_2, D_i becomes $d + \tilde{A}_i$ with $\|\tilde{A}_i\|_{C^k}$ as small as we need.

Let φ be a cut-off function: $\varphi \equiv 1$ on $B_2 \setminus B_{3/2}$ and $\varphi \equiv 0$ in B_1. Consider a new connection $d + (\varphi\tilde{A}_i)$. It agrees with $d + \tilde{A}_i$ outside $B_{3/2}$ and is d in B_1. We scale $d + (\varphi\tilde{A}_i)$ back to B_σ and denote the new connection by D_i'. It remains to see that the change in the energy is small. Due to the scaling invariance of energy, it suffices to check that any C^k norm of $F = d(\varphi\tilde{A}_i) + [\varphi\tilde{A}_i, \varphi\tilde{A}_i]$ is small on B_2.

Fix $p \in M$ and $q \in S^4$. By the above construction, we may assume that in $B_\delta(p)$ and $B_\delta(q)$, there is a trivialization such that the connection is just d. Via the stereographic projection, D_1 is a connection over $\mathbb{R}^4$, which outside B_R is nothing but d in some trivialization. We further scale it down to assume that $R = \delta/2$. We can now obtain a new connection by gluing D' on $M \setminus B_{\delta/2}$ and D_1 on B_R. Since there is no energy at all in the overlap domain, the lemma is proved. □

We then consider a minimizing sequence. For a given bundle E, let D_i be a minimizing sequence with

$$\lim_{i\to\infty} YM(D_i) = m(E).$$

Since D_i is smooth, we can find α_i close to 1 such that

$$YM_{\alpha_i}(D_i) \le YM(D_i) + V(M) + \frac{1}{i}.$$

Let $D_i(t)$ be the solution of the α_i-Yang–Mills flow from D_i and set $D_i' = D_i(1)$. Then,

$$YM(D_i') + V(M) \le YM_{\alpha_i}(D_i') \le YM(D_i) + V(M) + \frac{1}{i}.$$

This implies that D_i' is another minimizing sequence.

In order to do the blow-up analysis for D_i', we need the following ε–regularity result,

Lemma 4.6. There exists $\varepsilon > 0$ such that if $B_r(x) \subset M$ satisfies

$$\limsup_{i\to\infty} \int_{B_r(x)} \left|F_{D_i'}\right|^2 dv \le \varepsilon,$$

then

$$\left\|\nabla^k_{D_i'} F_{D_i'}\right\|_{C^0(B_{r/4}(x))} \le Cr^{-k-2}.$$

Proof. The proof relies on Theorem 4.2 and ε will be determined by ε_1 and the energy bound for our minimizing sequence.

By our choice of α_i, we have

$$\lim_{i\to\infty} \int_M \left(1 + \left|F_{D_i'}\right|^2\right)^{\alpha_i} - \left(1 + \left|F_{D_i'}\right|^2\right) dv = 0.$$

Hence,

$$\limsup_{i\to\infty} \int_{B_r(x)} (1 + \left|F_{D_i'}\right|^2)^{\alpha_i} - 1\, dv \le 2\varepsilon.$$

The local energy inequality (Lemma 2.3) implies that there exists $\sigma > 0$ depending on the total energy and ε such that for i sufficiently large,

$$\sup_{t\in[1-\sigma r^2,1]} \int_{B_{r/2}(x)} (1 + \left|F_{D_i(t)}\right|^2)^{\alpha_i} - 1\, dv \le 3\varepsilon.$$

Therefore,

$$\sup_{t\in[1-\sigma r^2,1]} \int_{B_{r/2}(x)} \left|F_{D_i(t)}\right|^2 \, dv \leq 4\varepsilon.$$

Set $\varepsilon = \frac{1}{4}\varepsilon_1$ and the proof follows from Theorem 4.2. □

Now we can do the well-known blow-up analysis for D_i'. If there are nontrivial bubbles and $E = E'\#E_1\#\cdots\#E_l$, then

$$m(E) = \lim_{i\to\infty} YM(D_i') \geq m(E') + \sum_{i=1}^{l} m(E_i).$$

This together with Proposition 4.5 will imply the energy identity:

Proposition 4.7. Let D_i be a minimizing sequence of the Yang–Mills functional among all smooth connections of the bundle E over M. Then, there exist bundles E' over M and $E_1, \cdots, E_l$ over S^4 for some $l \geq 0$ and Yang–Mills connections D_∞' and $\tilde{D}_1, \cdots, \tilde{D}_l$ such that

$$\lim_{i\to\infty} YM(D_i) = YM(D_\infty') + \sum_{i=1}^{l} YM(\tilde{D}_i).$$

Next, it remains to study the relation between the limit connection D_∞' and the weak limit D_∞ of Sedlacek [26].

We try to prove that the two limit (two Yang–Mills connection on two smooth bundles) are globally the same up to gauge transformations. This is the best one could hope for.

Let S be the union of energy concentration sets, both for D_i in the Sedlacek limit and for D_i' above. Let $\{U^\beta\}$ be an open cover of $M \setminus S$. We shall consider three bundles.

(1) The original one where the minimizing sequences and their α-flow lies on is denoted by E.

(2) The weak limit bundle, E_1, where the weak limit of D_i lies. In the paper of Sedlacek, it is given by transition functions. However, it is convenient to think of it as an abstract bundle, with a set of trivialization.

(3) The strong limit bundle, E_2, where the weak limit of D_i' lies.

For each D_i, there is a trivialization e_i^β in which $D_i = d + A_i^\beta$, where $\left\|A_i^\beta\right\|_{W^{1,2}}$ is bounded. $g_i^{\beta\gamma}$ will denote the transition functions, by which we mean

$$e_i^\beta = g_i^{\beta\gamma} e_i^\gamma. \tag{4.3}$$

There is a trivialization e^β of E_1 when restricted to $M \setminus S$, in which the weak limit $D_\infty = d + A_\infty^\beta$. We denote the transition functions by $g^{\beta\gamma}$, which means that

$$e^\beta = g^{\beta\gamma} e^\gamma. \tag{4.4}$$

The convergence of the minimizing sequence D_i on E in [26] can be reformulated as follows. By part e) of Theorem 3.1 in [26], we have

$$\left\| A_i^\beta - A_\infty^\beta \right\|_{W^{1,2}} \to 0.$$

By part c) of the same theorem, $g_i^{\beta\gamma}$ converges to $g^{\beta\gamma}$ weakly in $W^{1,4}$.

Remark 4.8. This convergence was shown to be weakly $W^{1,2}$ in [26] and was shown to be strong by Isobe in [18].

There is a bundle map $\varphi_i^\beta : E|_{U^\beta} \to (E_1)|_{U^\beta}$ by identifying trivialization e_i^β and e^β. The above convergence can be written as

$$\left\| (\varphi_i^\beta)^* D_\infty - D_i \right\|_{W^{1,2}(U^\beta)} \to 0. \tag{4.5}$$

In [26], φ_i^β and φ_i^γ cannot be fitted together to get a larger bundle map. However, we have the following relation between them.

Let v be any vector of $E|_{U^\beta \cap U^\gamma}$. Suppose that

$$v = \tilde{v} e_i^\beta = g_i^{\beta\gamma} \tilde{v} e_i^\gamma.$$

By definition of v and (4.3), (4.4),

$$\begin{aligned} \varphi_i^\beta(v) &= \tilde{v} e^\beta \\ &= g^{\beta\gamma} \tilde{v} e^\gamma \\ &= g^{\beta\gamma} \tilde{v} (\varphi_i^\gamma e_i^\gamma) \\ &= \varphi_i^\gamma (g^{\beta\gamma} g_i^{\gamma\beta} \tilde{v} e_i^\beta) \\ &= g^{\beta\gamma} g_i^{\gamma\beta} \varphi_i^\gamma(v). \end{aligned} \tag{4.6}$$

The relation (4.6) will be important for us later.

Next, we describe the strong convergence of D_i' to D_∞'. We know there is a sequence of bundle maps σ_i from $E|_{M\setminus S}$ to $E_2|_{M\setminus S}$ such that

$$\left\| \sigma_i^* D_\infty' - D_i' \right\|_{C^k(K)} \to 0$$

for any compact K in $M \setminus S$. For any β, we have

$$\left\| \sigma_i^* D_\infty' - D_i' \right\|_{C^k(U^\beta)} \to 0. \tag{4.7}$$

By our construction, we know

$$\left\| D_i - D_i' \right\|_{L^2} \to 0.$$

Hence,

$$\left\| (\varphi_i^\beta)^* D_\infty - \sigma_i^* D_\infty' \right\|_{L^2(U^\beta)} \to 0.$$

That is

$$\left\| D_\infty - (\eta_i^\beta)^* D_\infty' \right\|_{L^2(U^\beta)} \to 0, \tag{4.8}$$

where $\eta_i^\beta = \sigma_i \circ (\varphi_i^\beta)^{-1}$ is a bundle map from $E_1|_{U^\beta}$ to $E_2|_{U^\beta}$.

We claim that η_i^β converges to η^β in weak $W^{1,2}$ topology and $D_\infty = (\eta^\beta)^* D_\infty'$ on U^β. To see this, consider the meaning of (4.8) in trivialization e^β and f^β. (Here f^β is a trivialization of E_2 on U^β.) Since $D_\infty = d + A_\infty$ and $D_\infty' = d + A_\infty'$, we have

$$\left\| A_\infty - (s_i^{-1} ds_i + s_i^{-1} A_\infty' s_i) \right\|_{L^2(U^\beta)} \le C.$$

Here s_i is the map η_i^β in a trivialization. Hence s_i is bounded in $W^{1,2}$ and our claim follows. Moreover, although the convergence is only weakly $W^{1,2}$, η^β is smooth since it maps smooth connections to smooth connections.

We next claim that η^β and η^γ agree over $U^\beta \cap U^\gamma$. Hence, this gives a global bundle map η from $E_1|_{M\setminus S}$ to $E_2|_{M\setminus S}$. To see this, it suffices to check that

$$\lim_{i\to\infty} \sigma_i \circ (\varphi_i^\beta)^{-1} = \lim_{i\to\infty} \sigma_i \circ (\varphi_i^\gamma)^{-1}.$$

Due to the smoothness of η^β and η^γ, it suffices to check the above for a dense set of $x \in U^\beta \cap U^\gamma$. Thanks to (4.6) and the $W^{1,4}$ weak convergence of $g_i^{\beta\gamma}$ to $g^{\beta\gamma}$ (Theorem 3.1 in [26]), we have a dense set W such that for $x \in W$ and any $v \in (E_1)_x$, we have

$$(\varphi_i^\beta)^{-1}(v) - (\varphi_i^\gamma)^{-1}(v) \to 0.$$

Because σ_i is a linear map and σ_i lies in $G \subset SO(r)$ (r is the rank of E), we have

$$\lim_{i\to\infty} \sigma_i \circ (\varphi_i^\beta)^{-1}(v) - \sigma_i \circ (\varphi_i^\gamma)^{-1}(v) = 0.$$

Now we have a bundle map η defined on $M \setminus S$ satisfying $\eta^* D_\infty' = D_\infty$. Finally, since D_∞ and D_∞' are smooth connections, η extends automatically to a global smooth gauge transformation with $\eta^* D_\infty' = D_\infty$. In fact, locally on $B \setminus \{0\}$,

$$A_\infty = \eta^{-1} d\eta + \eta^{-1} A_\infty' \eta,$$

which implies η and all its derivatives are bounded on $B \setminus \{0\}$ since A_∞ and A_∞' are smooth over B.

Hence, we finish the proof of Theorem 1.4.

4.4. Another approach for Min-Max of the Yang–Mills functional. It is well known that the Yang–Mills functional in dimension 4 does not satisfy the Palais–Smale condition, which caused great difficulty in applying Morse theory to show the existence of a nonminimal critical point. In 1989, Sibner, Sibner and Uhlenbeck [27] proved the existence of nonminimal Yang–Mills connections on the trivial $SU(2)$ bundle over S^4. They used the fundamental relationship between $m-$equivariant gauge fields on S^4 and monopoles on hyperbolic 3−space $\mathbb{H}^3$ as presented by Atiyah [1]. If we identify S^4 with $\mathbb{R}^4 \cup \{\infty\}$ by stereographic projection, we may introduce the following coordinates

$$(z, \theta, (x, y)) \mapsto (z \cos\theta, z \sin\theta, x, y) \in \mathbb{R}^4.$$

Hence, one can define a $U(1)$ action on S^4 by

$$q(\theta')(z, \theta, (x, y)) = (z, \theta + \theta'(\mathrm{mod} 2\pi), (x, y))$$

and leaving other points in S^4 not represented by this coordinate system fixed.

Let $\left\{\hat{i}, \hat{j}, \hat{k}\right\}$ be a standard basis for $\mathfrak{su}(2)$ and $s(\theta) = e^{\hat{i}m\theta}(m \geq 2)$ be a homeomorphism from $U(1)$ to $SU(2)$. A connection D is called *an m-equivariant connection* if

$$q(\theta)^* D = s(\theta)^{-1} \circ D \circ s(\theta)$$

for all $\theta \in U(1)$. Denote the set of all m-equivariant connections of the trivial $SU(2)$ bundle over S^4 by $\mathcal{M}$.

The authors of [27] followed a construction of Taubes [31] to find a non-contractible loop of connections $D^\gamma (\gamma \in S^1)$ of m-equivariant connections in $\mathcal{M}$, satisfying

$$YM(D^\gamma) < 8\pi m. \tag{4.9}$$

The connections in Lemma 2 of [27] are in $W^{1,\infty}$, but by approximation, we can assume that they are smooth and (4.9) remains true. Since they are smooth, we know

$$YM_\alpha(D^\gamma) < 8\pi m + \omega_4$$

for sufficiently small α. Here ω_4 is the volume of S^4.

We can now apply the Yang–Mills α-flow to the loop. The α-flow preserves symmetry, so that the flow stays in $\mathcal{M}$. By Theorem 2.6, we obtain a deformation of the circle in $\mathcal{M}$. We then claim that we obtain a nontrivial Yang–Mills α-connection D_α with $YM_\alpha(D_\alpha) < 8\pi m + \omega_4$. Otherwise, the flow will converge to the flat connection for any $\gamma \in S^1$, which will result in a contraction of the loop to a single point in $\mathcal{M}$. This is not possible.

The energy of these Yang–Mills α-connections D_α has a uniform lower bound. This is a generalized gap theorem similar to the result of Bourguignon and Lawson [4].

Lemma 4.9. There is $\kappa > 0$ depending only on G such that any nontrivial Yang–Mills α-connection D_α on S^4 satisfies

$$YM(D_\alpha) > \kappa.$$

Proof. Recall that we have proved a stronger Bochner formula (3.6) than stated in Lemma 3.2. For our purpose here, $\partial_t |F|^2$ vanishes and the $Ric \wedge g + 2R$ is just the 4 times of the identify map on 2−forms. Hence,

$$-\nabla_{e_i}\left((\delta_{ij} + b_{ij})\nabla_{e_j} |F|^2\right) \leq C\,|F|^3 - 3\,|F|^2,$$

when $\alpha - 1$ is small. Multiplying both sides by $|F|^2$ and integrating over S^4, we have

$$\int_{S^4} \left|\nabla |F|^2\right|^2 + |F|^4 \leq C \int_{S^4} |F|^5 .$$

By the Sobolev inequality and the Hölder inequality, we obtain

$$\left(\int_{S^4} |F|^8\right)^{1/2} \leq C \left(\int_{S^4} |F|^2\right)^{1/2} \left(\int_{S^4} |F|^8\right)^{1/2} .$$

This implies that F is identically zero if the energy is small. □

Now, we may pass to the limit $\alpha \to 1$. Note that $\kappa < YM(D_\alpha) < 8\pi m$. The rest of the proof goes just like Theorem 1 in [27]. If the convergence of D_α is strong, we obtain a nonminimal Yang–Mills connection on the trivial $SU(2)$ bundle over S^4. If not, the energy bound $8\pi m$ implies that either the weak limit or one of the bubbles is a nontrivial Yang–Mills connection on the trivial $SU(2)$ bundle (hence nonminimal), because the energy is not enough for two nontrivial bundles.

Acknowledgements. The research of the first author was supported by the Australian Research Council grant. A part of the work was done when Tian and Yin visited the University of Queensland in 2012. The authors would like to thank the referee for many useful comments and suggestions.

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Received March 05, 2013

M.-C. Hong, Department of Mathematics, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: hong@maths.uq.edu.au

G. Tian, Department of Mathematics, Princeton University, USA

E-mail: tian@princeton.edu

H. Yin, School of Mathematical Sciences, University of Science and Technology of China, Hefei, China

E-mail: haoyin@ustc.edu.cn