

The arithmeticity of a Kodaira fibration is determined by its universal cover

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Abstract. Let $S \rightarrow C$ be a Kodaira fibration. Here we show that whether or not the algebraic surface S is defined over a number field depends only on the biholomorphic class of its universal cover.

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1. Introduction and statement of results

Let $X \subset \mathbb{P}^n$ be a complex projective variety and k a subfield of the field of the complex numbers \mathbb{C} . We shall say that X is defined over k or that k is a *field of definition* for X if there exists a collection of homogenous polynomials f_0, \dots, f_m with coefficients in k so that the variety they define is isomorphic to X . We will say that X is *arithmetic* if it is defined over \mathbb{Q} or equivalently over a number field.

While it is classically known that there are only three simply connected Riemann surfaces, there is a huge amount of possibilities for the holomorphic universal cover of a complex surface S . It would be interesting to understand the extent to which the arithmeticity of a projective surface can be read off from its holomorphic universal cover. In this short note we study this question for a very important class of complex surfaces known in the literature as Kodaira fibrations.

A *Kodaira fibration* consists of a non-singular compact complex surface S , a compact Riemann surface C and a surjective holomorphic map $S \rightarrow C$ everywhere of maximal rank such that the fibers are connected and not mutually isomorphic Riemann surfaces. The genera g of the fibre and b of C are called the genus of the fibration and of the base respectively. It is known that such a surface S must be an algebraic surface of general type and that necessarily $g \geq 3$ and $b \geq 2$. We notice

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that an important theorem by Arakelov [1] implies that, up to isomorphism, there are only finitely many Kodaira fibrations over a given algebraic curve C .

In 1967, Kodaira [13] used fibrations of this kind to show that the signature of a differentiable fiber bundle need not be multiplicative. Soon after Kas [12] studied the deformation space of the surfaces constructed by Kodaira, and two years later Atiyah [2] and Hirzebruch [10] studied further properties concerning the signature of Kodaira fibrations in a volume dedicated to Kodaira himself.

Explicit constructions of Kodaira surfaces have been made by González-Diez and Harvey [8], Bryan and Donagi [4], Zaal [16] and Catanese and Rollenske [6].

We now state the main results of the paper.

Theorem 1.1. *Let k be an algebraically closed subfield of the complex numbers and $S_1 \rightarrow C_1$ and $S_2 \rightarrow C_2$ two Kodaira fibrations so that their respective holomorphic universal covers are biholomorphically equivalent. Then S_1 is defined over k if and only if S_2 is defined over k . In particular, S_1 is arithmetic if and only if S_2 is arithmetic.*

To prove this theorem we will have to show first the following result which is interesting in its own right.

Theorem 1.2. *Let k be an algebraically closed subfield of the complex numbers and $S \rightarrow C$ a Kodaira fibration. Then S is defined over k if and only if C is defined over k . In particular, S is an arithmetic surface if and only if C is an arithmetic curve.*

Theorem 1.1 states that the arithmeticity of a Kodaira fibration can be recognized in its holomorphic universal cover. We anticipate that the holomorphic universal cover of S is a contractible bounded domain $\mathcal{B} \subset \mathbb{C}^2$ (see Section 2). Clearly, Theorem 1.1 implies that the biholomorphism class of \mathcal{B} varies together with the variation of S in moduli space. We note that in general Kodaira surfaces are not rigid ([12], [6]).

2. Uniformization of Kodaira Surfaces

It is well-known that the universal cover of a Riemann surface is biholomorphically equivalent to the projective line \mathbb{P}^1 , the complex plane \mathbb{C} or the upper half-plane \mathbb{H} . Understanding universal covers of complex manifolds of higher dimension seems to be a very complicated task. However, thanks to the work of Bers [3] and Griffiths [9] on uniformization of algebraic varieties, it is possible to describe the universal cover of a Kodaira fibration $f : S \rightarrow C$ in a very explicit way.

Let $\pi : \mathbb{H} \rightarrow C$ be the universal covering map of C and Γ the covering group so that $C \cong \mathbb{H}/\Gamma$. By considering the pull-back $h : \pi^*S \rightarrow \mathbb{H}$ of f by π , we obtain a new fibration in which, for each $t \in \mathbb{H}$, the fiber $h^{-1}(t)$ agrees

with the Riemann surface $f^{-1}(\pi(t))$. Teichmüller theory enables us to choose uniformizations $h^{-1}(t) = D_t/K_t$ possessing the following properties:

- (a) K_t is a Kleinian group acting on a bounded domain D_t of \mathbb{C} which is biholomorphically equivalent to a disk.
- (b) The union of all these disks $\mathcal{B} := \cup_{t \in \mathbb{H}} D_t$ is a contractible bounded domain of \mathbb{C}^2 which is biholomorphic to the universal cover of S , that is, $S \cong \mathcal{B}/\mathbb{G}$, where $\mathbb{G} < \text{Aut}(\mathcal{B})$ is the covering group.
- (c) The group \mathbb{G} is endowed with a surjective homomorphism of groups $\Theta : \mathbb{G} \rightarrow \Gamma$ which induces an exact sequence of groups

$$1 \longrightarrow \mathbb{K} \longrightarrow \mathbb{G} \xrightarrow{\Theta} \Gamma \longrightarrow 1$$

where, for each $t \in \mathbb{H}$, the subgroup \mathbb{K} preserves D_t and acts on it as the Kleinian group K_t .

We note that \mathcal{B} carries itself a fibration structure $\mathcal{B} \rightarrow \mathbb{H}$ whose fiber over $t \in \mathbb{H}$ is D_t (i.e. \mathcal{B} is a *Bergman domain* in Bers' terminology).

In [14] and [15] Shabat studied the automorphism groups of universal covers of families of Riemann surfaces and proved a deep result which in the case of Kodaira fibrations amounts to the following theorem.

Theorem (Shabat). *Let $f : S \rightarrow C$ be a Kodaira fibration and \mathcal{B} the holomorphic universal cover of S . Then:*

- (a) *the covering group \mathbb{G} of S has finite index in $\text{Aut}(\mathcal{B})$.*
- (b) *$\text{Aut}(\mathcal{B})$ is a discrete group.*

3. Proof of Theorems 1.1 and 1.2

We denote by $\text{Gal}(\mathbb{C})$ the group of field automorphisms of \mathbb{C} . The natural action of $\text{Gal}(\mathbb{C})$ on the ring of polynomials $\mathbb{C}[x_0, \dots, x_n]$ induces a well-defined action $(\sigma, X) \rightarrow X^\sigma$ on the set of isomorphism classes of algebraic varieties. Throughout this section k will denote an algebraically closed subfield of \mathbb{C} and $\text{Gal}(\mathbb{C}/k)$ the subgroup of $\text{Gal}(\mathbb{C})$ consisting of all automorphisms σ fixing the elements of k .

3.1. Proof of Theorem 1.2. Let $f : S \rightarrow C$ be a Kodaira fibration. Let us assume that the curve C is defined over k . Then $C^\sigma = C$ for all $\sigma \in \text{Gal}(\mathbb{C}/k)$, and so, by Arakelov's finiteness Theorem, there are only finitely many pairwise non-isomorphic Kodaira fibrations $f^\sigma : S^\sigma \rightarrow C^\sigma$ with $\sigma \in \text{Gal}(\mathbb{C}/k)$. This implies that S is defined over k [7, Crit. 2.1].

In order to prove the converse, we begin by recalling that a complex manifold X is named *hyperbolic* if every holomorphic map $\mathbb{C} \rightarrow X$ is a constant map. We claim

that Kodaira fibrations are hyperbolic. In fact, let $f : S \rightarrow C$ be a Kodaira fibration and $\varphi : \mathbb{C} \rightarrow S$ a non-constant holomorphic map. As C has genus greater than one, the map $f \circ \varphi : \mathbb{C} \rightarrow C$ must be constant and therefore the image of φ has to be contained in a fiber $f^{-1}(x)$ for some $x \in C$. Since the fibers are also hyperbolic, φ must be constant too.

Let us now assume that S is defined over k , so that $S^\sigma = S$ for any $\sigma \in \text{Gal}(\mathbb{C}/k)$. Now as S is a Kähler hyperbolic manifold, the canonical divisor K_S is ample [5] and this implies that only finitely many curves R of genus greater than one can arise as the image of a surjective morphism $S \rightarrow R$ [11]. In particular the family $\{C^\sigma : \sigma \in \text{Gal}(\mathbb{C}/k)\}$ itself contains only finitely many isomorphism classes of curves. It then follows that C is defined over k [7, Crit. 2.1], as required.

3.2. Proof of Theorem 1.1. Let $f_2 : S_2 \rightarrow C_2$ be a Kodaira fibration and S_1 an arbitrary non-singular complex surface. Let us denote by \mathcal{B}_i the universal cover of S_i and suppose that there exists an isomorphism $\alpha : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ between them. Let \mathbb{G}_i be the uniformizing group of S_i so that $\mathcal{B}_i/\mathbb{G}_i \cong S_i$. By Shabat’s Theorem \mathbb{G}_2 has finite index in $\text{Aut}(\mathcal{B}_2)$. We claim that \mathbb{G}_1 has finite index in $\text{Aut}(\mathcal{B}_1)$ too. In fact, as $\mathcal{B}_1/\text{Aut}(\mathcal{B}_1) \cong \mathcal{B}_2/\text{Aut}(\mathcal{B}_2)$ and as $\text{Aut}(\mathcal{B}_2)$ is a discrete group, the projection map $S_1 = \mathcal{B}_1/\mathbb{G}_1 \rightarrow \mathcal{B}_1/\text{Aut}(\mathcal{B}_1)$ is a holomorphic map between (normal) compact complex surfaces; from here the claim follows.

By replacing \mathbb{G}_1 by $\alpha\mathbb{G}_1\alpha^{-1}$ we can assume that $\mathcal{B}_1 = \mathcal{B}_2$, so we denote \mathcal{B}_i simply by \mathcal{B} . As both \mathbb{G}_1 and \mathbb{G}_2 have finite index in $\text{Aut}(\mathcal{B})$, so must do their intersection $\mathbb{G}_{12} = \mathbb{G}_1 \cap \mathbb{G}_2$. The complex surface $S_{12} := \mathcal{B}/\mathbb{G}_{12}$ is endowed with two finite degree covers $\pi'_i : S_{12} \rightarrow S_i$ with $i = 1, 2$; in particular, S_{12} is also a projective surface. Moreover, if we denote by Θ_{12} the restriction of the epimorphism $\Theta : \mathbb{G}_2 \rightarrow \Gamma_2$ introduced in the previous section to \mathbb{G}_{12} , then we obtain an exact sequence of groups

$$1 \longrightarrow \mathbb{K}_{12} \longrightarrow \mathbb{G}_{12} \xrightarrow{\Theta_{12}} \Gamma_{12} \longrightarrow 1$$

where $\Gamma_{12} = \Theta_{12}(\mathbb{G}_{12})$ and $\mathbb{K}_{12} = \ker(\Theta_{12}) = \mathbb{K} \cap \mathbb{G}_{12}$. As in Section 2, this sequence defines a Kodaira fibration $f_{12} : S_{12} \rightarrow C_{12} := \mathbb{H}/\Gamma_{12}$ whose fiber over $[t]_{\Gamma_{12}}$ is isomorphic to the Riemann surface D_t/K_t^{12} where K_t^{12} is the Kleinian group that realizes the action of \mathbb{K}_{12} on D_t . We have the following commutative diagram

$$\begin{array}{ccc}
 & \mathcal{B} & \\
 & \swarrow \quad \searrow & \\
 S_{12} & \xrightarrow{\pi'_2} & S_2 \\
 f_{12} \downarrow & & \downarrow f \\
 C_{12} & \xrightarrow{p} & C_2
 \end{array}$$

where p is the projection induced by the finite index inclusion $\Gamma_{12} < \Gamma_2$.

Let us now assume that S_2 is defined over k . Then Theorem 1.2 ensures that C_2 is also defined over k . Furthermore, being an unbranched cover of C_2 , the curve C_{12} must also be defined over k [7, Th. 4.1]. Again, by Theorem 1.2 we conclude that S_{12} is defined over k . Now, as S_1 is a surface of general type arising as the image (by π'_1) of a surface defined over k , it must be defined over k as well [7, Prop. 3.2]. This proves Theorem 1.1.

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