

## On the long time behavior of homogeneous Ricci flows

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**Abstract.** In this paper we prove the following structure results for homogeneous Ricci flow solutions: Any homogeneous Ricci flow solution with finite extinction time develops a Type I singularity. Any homogeneous Ricci flow solution on a compact homogeneous space, not diffeomorphic to a torus, has finite extinction time. Any immortal homogeneous Ricci flow solution develops a Type III singularity and the natural blow downs subconverge to an immortal locally homogeneous Ricci flow solution.

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### 1. Introduction

A family  $(g(t))_{t \in [0, T)}$  of smooth, complete, Riemannian metrics on a smooth manifold  $M^n$  is called a solution to Hamilton's Ricci flow [20], if it satisfies the geometric evolution equation

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{ric}(g(t)) \quad \text{and} \quad g(0) = g_0.$$

We call a Ricci flow solution a homogeneous Ricci flow, if the initial metric  $g_0$  is homogeneous. In this case the evolved metrics are homogeneous as well, in fact the isometry groups do not change [24]. The Ricci flow on homogeneous spaces has been investigated by many authors, in particular in low dimensions and on Lie groups (see e.g., [1, 3, 11, 13, 14, 21, 22, 28–30, 35]). Still, in general the long time behavior of homogeneous Ricci flows is completely understood only in very special cases.

If a solution to the Ricci flow cannot be extended smoothly past time  $T$ , then we call  $T \in (0, \infty]$  a singular time. If the singular time  $T$  is finite, the Ricci flow solution is said to have finite extinction time. A Ricci flow solution with finite extinction time is said to develop a Type I singularity, if there exists a constant  $C_{g_0} > 0$ , such that

$$\sup_{M^n} \|R(g(t))\|_{g(t)} \cdot (T - t) \leq C_{g_0}$$

for all  $t \in [0, T)$ . Here  $R(g(t))$  denotes the curvature tensor of the metric  $g(t)$ .

By a recent result of Lafuente [26] a homogeneous Ricci flow has finite extinction time if and only if the scalar curvature of the evolved metrics becomes positive close to extinction time.

Our first main result is

**Theorem 1.** *A homogeneous Ricci flow with finite extinction time develops a Type I singularity.*

We will also show that for such homogeneous Ricci flows the norm of the curvature tensor can be controlled by the scalar curvature as soon as the scalar curvature is positive (Remark 2.2). By [23], [16], [19] the homogeneity assumption in Theorem 1 cannot be dropped, since on the Euclidean plane and on spheres there exist rotationally invariant metrics, which lead to Type II singularities.

Our second main result is

**Theorem 2.** *Let  $M^n$  be a compact homogeneous space not diffeomorphic to the torus  $T^n$ . Then any homogeneous Ricci flow solution has finite extinction time.*

A compact homogeneous space admits in general homogeneous metrics with negative scalar curvature; the spaces which do not have been classified by Wang and Ziller [44] (see also [8]). Notice that any homogeneous metric on a torus is flat.

By general results of Naber [33] and Enders, Müller, Topping [17] on Type I singularities of the Ricci flow, it follows that along any sequence of times converging to the finite extinction time  $T$ , parabolic rescalings will subconverge to a nonflat homogeneous gradient shrinking soliton. By work of Petersen and Wylie [37] such a shrinking soliton is in our situation a finite quotient of a nonflat product metric of a homogeneous Einstein metric with positive scalar curvature and a flat metric on Euclidean space. Notice that the flat factor might be absent.

We turn to the question whether the compact homogeneous Einstein space  $E_\infty$  appearing in the limit soliton can be related to the homogeneous space considered. Recall that a homogeneous space is diffeomorphic to a coset space  $G/H$ , where  $G$  is a Lie group acting isometrically and transitively on  $M^n$  and  $H$  is the compact isotropy subgroup of a point.

**Theorem 3.** *Let  $M^n = G/H$  be a compact homogeneous space not diffeomorphic to the torus  $T^n$ . Suppose that the isotropy representation decomposes into pairwise inequivalent summands. Then for any homogeneous Ricci flow on  $G/H$  there exists a compact intermediate subgroup  $K$ , such that  $E_\infty = K/H$ .*

The intermediate subgroup  $K$  corresponds to the most shrinking direction of the metrics  $g(t)$  (see Section 5) and depends only on the initial metric  $g_0$ . Notice though, that for different initial metrics the group  $K$  may vary as can be seen easily from considering homogeneous product metrics on  $S^2 \times S^2$ .

Since there exist homogeneous spaces  $K/H$  not admitting any  $K$ -invariant Einstein metrics (see [44], [9]), in general not all intermediate subgroups can occur.

For instance, let  $G = \text{SO}(2p+q)$ ,  $L = \text{SO}(2p)\text{SO}(q)$  and  $H = \text{SO}(p)\text{U}(1)\text{SO}(q)$ , where  $\text{SO}(p)\text{U}(1) \subset \text{U}(p) \subset \text{SO}(2p)$ . Then, if  $p \geq 5$  and  $q = 3$ , the spaces  $G/H$  and  $L/H$  do not admit homogeneous Einstein metrics by [34], [44]. As a consequence the only possible intermediate subgroup is  $K = \text{U}(p)\text{SO}(q)$ . If  $p = 3$  and  $q \geq 4$ , the space  $G/H$  does admit homogeneous Einstein metrics, that is one can also have  $K = G$  for appropriate initial metrics.

We will show in Theorem 5.14, that for any sequence of times converging to  $T$ , the restriction of appropriately rescaled metrics  $g(t)$  to  $K/H$  subconverges to an Einstein metric of positive scalar curvature. The limit Einstein metric depends only on the initial metric and not on any subsequences chosen, if on  $K/H$  there exist only finitely many solutions to the homogeneous Einstein equation of fixed volume (cp. finiteness conjecture of [10]). Since in the first of the above two examples the space  $K/H$  is isotropy irreducible, we get for any initial metric the same limit soliton. Let us mention, that if the isotropy representation has two inequivalent summands these results were obtained in [13].

We expect Theorem 3 to be true for arbitrary homogeneous spaces. In general the most shrinking direction of the evolved metrics corresponds to a distribution, which becomes integrable only in the limit. As a consequence, the Ricci flow on such homogeneous spaces is much more difficult to deal with.

We turn to homogeneous Ricci flows on noncompact homogeneous spaces. Bérard-Bergery [5] has shown that a homogeneous space admits a homogeneous metric of positive scalar curvature if and only if the universal covering space is not diffeomorphic to Euclidean space. By [26] it follows that on homogeneous spaces with Euclidean universal covering space any homogeneous Ricci flow solution will be immortal, that is  $T = \infty$ . Recall that a homogeneous Ricci flat metric is flat by [2]. As a consequence, for an immortal homogeneous Ricci flow solution, which is not flat, the scalar curvature is negative and must converge to zero.

An immortal solution to the Ricci flow is said to develop a Type III singularity, if there exists a constant  $C_{g_0} > 0$ , such that for all  $t \in [0, \infty)$

$$\sup_{M^n} \|\mathbf{R}(g(t))\|_{g(t)} \cdot t \leq C_{g_0}.$$

Our third main result is

**Theorem 4.** *An immortal homogeneous Ricci flow develops a Type III singularity.*

As an immediate consequence of the above results we obtain

**Corollary 5.** *For homogeneous spaces with compact or Euclidean universal covering space the following holds: The homogeneous Ricci flows on these spaces develop either a Type I or a Type III singularity, irrespectively of the chosen initial metric.*

It is an open problem, whether this dichotomy holds for a arbitrary homogeneous space. If true, this would imply the long standing conjecture of Alekseevskii on noncompact homogeneous Einstein spaces (see [6], 7.57).

We turn to the question to which extend there should be counterparts of the above mentioned results of Naber and Enders, Müller, Topping for Type III singularities of homogeneous Ricci flows on noncompact homogeneous spaces.

We consider for  $s > 0$  the immortal solution  $g_s(t) := \frac{1}{s} \cdot g(st)$ . It follows from Hamilton's compactness theorem that if the injectivity radius of  $(M^n, g(t))$  is bounded from the below by  $C_{g_0} \sqrt{t}$ , then for any sequence  $\{s_i\}_{i=1}^\infty$  converging to infinity the sequence  $(M^n, g_{s_i}(t))$  of blow downs subconverges to a homogeneous immortal limit Ricci flow  $(M_\infty^n, g_\infty(t))$  on a possibly different homogeneous space  $M_\infty^n$ . In general, by work of Glickenstein [18] and Lott ([30], Corollary 5.14), one obtains subconvergence to a limit flow on an  $n$ -dimensional, étale groupoid. In our situation such a groupoid is nothing but a locally homogeneous space, which in general will be incomplete (see Section 6).

Our fourth main result is

**Theorem 6.** *For any immortal homogeneous Ricci flow solution the above defined blow downs subconverge to an immortal locally homogeneous Ricci flow solution.*

By [41] a locally homogeneous space with nonpositive Ricci curvature can be extended to a (complete) homogeneous space. In general this is not true anymore. There exist even Einstein metrics of positive scalar curvature on locally homogeneous spaces, which do not extend to a complete Einstein metric [39], [25].

Lott proved in [30], that if the sequence  $(M^n, g_s(t))$  of blow downs has a limit for  $s \rightarrow \infty$ , then this limit Ricci flow is an expanding Ricci soliton. In special cases such as in dimension three and four [30] and for homogeneous metrics on nilpotent or certain solvable Lie groups this is known to be true [28], [3].

**Problem.** *Show that for any immortal homogeneous Ricci flow solution any blow down subconverges to an expanding Ricci soliton on a locally homogeneous space.*

Notice first, that such an expanding limit soliton might be flat even if the scalar curvature of the approximating Ricci flow solution is negative for all times. For instance on the isometry group  $E(2)$  of the Euclidean plane there exists a homogeneous immortal solution to the Ricci flow such that  $\|\mathbf{R}(g(t))\|_{g(t)} \approx \exp(-ct)$  and  $\text{scal}(g(t)) \approx -\exp(-2ct)$  for  $c > 0$  (see [21]). It follows, that the curvature is so rapidly decreasing that any geometric limit solution must be flat. Notice also that the norm of the curvature tensor is not controlled by the absolute value of the scalar curvature in this example in contrast to Type I singularities.

Let us now turn to immortal homogeneous solutions, for which there exists a constant  $c_{g_0} > 0$ , such that  $c_{g_0} \leq \sup_{M^n} \|\mathbf{R}(g(t))\| \cdot t$  for all  $t \in [0, \infty)$ . At first hand, one might hope that the bracket flow introduced by Lauret in [29] might be helpful establishing the existence of a nonflat expanding limit soliton. Recall that the bracket flow is a geometric flow on the set of Lie brackets, which is equivalent to the Ricci flow. In example 4.4 in [27] Lafuente and Lauret provide an example of a Type III solution (in fact an expanding soliton), such that the norm of the corresponding bracket flow solution tends to infinity. It follows of course that the

above blow downs will converge to the nonflat expanding soliton given. This limit soliton can however never be obtained by considering any normalized bracket flow.

The paper is organized as follows: In Section 2 we prove Theorem 1, in Section 3 Theorem 2, and in Section 4 Theorem 4. Theorem 3 is proved in Section 5, and in Section 6 we present a proof a Spiro's result.

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## 2. Finite time singularities of homogeneous Ricci flows

In this section we will provide the proof of Theorem 1. Furthermore we will show that the norm of the curvature tensor is controlled by the scalar curvature.

Recall that by [26] we may assume that for a homogeneous Ricci flow  $g(t)_{t \in [0, T)}$  with finite extinction time  $T < \infty$  we have  $\text{scal}(g(0)) = 1$ .

**Theorem 2.1.** *A homogeneous Ricci flow  $(M^n, g(t))_{t \in [0, T)}$  with finite extinction time  $T$  develops a Type I singularity.*

*Proof.* From the evolution equation for the scalar curvature along a solution to the Ricci flow we know  $s'(t) \geq \frac{2}{n} \cdot s^2(t)$ , where we have set  $s(t) := \text{scal}(g(t))$ . Let  $t_0 \in [0, T)$ . As is well known, if  $s(t_0) \neq 0$ , this implies

$$s(t) \geq \frac{1}{\frac{1}{s(t_0)} - \frac{2}{n} \cdot (t - t_0)}. \quad (2.1)$$

Since by assumption  $s(0) = 1$ , we conclude  $T \leq \frac{n}{2}$ . Furthermore, we get

$$s(t) \leq \frac{n}{2(T - t)} \quad (2.2)$$

for all  $t \in [0, T)$ . If not, then there exists  $t_0 \in [0, T)$  such that  $s(t_0) = \frac{(1+\varepsilon)n}{2(T-t_0)}$  for some  $\varepsilon > 0$ . From (2.1) we deduce for all  $t \in [t_0, T)$  that

$$t < \frac{T - t_0}{1 + \varepsilon} + t_0 = \frac{T + \varepsilon \cdot t_0}{1 + \varepsilon} = T - \frac{\varepsilon(T - t_0)}{1 + \varepsilon}.$$

Contradiction.

Let now  $K(t) := \|\mathbf{R}(g(t))\|_{g(t)}$  denote the norm of the curvature tensor at time  $t \in [0, T)$ . We will show below that there exists a constant  $C > 0$ , such that  $\frac{K(t)}{s(t)} \leq C$  for all  $t \in [0, T)$ . This implies by (2.2)

$$(T - t) \cdot K(t) \leq (T - t) \cdot s(t) \cdot C \leq \frac{C \cdot n}{2},$$

which shows that we have a Type I singularity.

It remains to show that  $\frac{K(t)}{s(t)}$  is bounded for  $t \in [0, T)$ . Suppose the contrary: Then there exist times  $t_i \in [0, T)$  with  $t_i \rightarrow T$  and  $\frac{K(t_i)}{s(t_i)} = i$ . Moreover, we can assume that

$$i = \max \left\{ \frac{K(t)}{s(t)} \mid t \in [0, t_i] \right\} .$$

Since the scalar curvature, i.e.  $s(t)$ , is not decreasing,  $K(t_i) \geq K(t)$  for all  $t \in [0, t_i]$ . We set  $Q_i := K(t_i)$  and rescale parabolically at  $t = t_i$  by setting

$$g_i(t) := Q_i \cdot g \left( t_i + \frac{t}{Q_i} \right) .$$

The solutions  $g_i$  live on  $(-Q_i \cdot t_i, Q_i \cdot (T - t_i))$ . Moreover the norm of the curvature tensor of  $g_i(0)$  equals to 1 for all  $i$ .

Recall that the doubling time estimate for the Ricci flow roughly says, that the maximum of the norm of the curvature operator cannot grow too fast too soon as time increases. More precisely, if  $(M^n, g(t))$  is a complete solution to the Ricci flow with bounded curvature and if  $\|R(g(0))\|_{g(0)} \leq K$  for a constant  $K > 0$ , then for all  $t \in [0, \frac{1}{16K}]$  one has  $\|R(g(t))\|_{g(t)} \leq 2K$  (cf. [15], p. 213). In particular the Ricci flow exists on the entire interval  $[0, \frac{1}{16K}]$ .

In the above situation this implies that any of the solutions  $g_i(t)$  exist as long as  $t \in [0, \frac{1}{16}]$ . It follows that the above intervals converge (along a subsequence possibly) to  $(-\infty, T_\infty)$  with  $T_\infty > 0$ .

By the choice of the  $t_i$  we have  $\|R(g_i(t))\|_{g_i(t)} \leq 1$  for  $t \in (-t_i \cdot Q_i, 0]$ . In general, we do have of course no injectivity radius bound from the below. But due to Theorem 5.12 in [30] there exists a convergent subsequence converging to a solution on an étale groupoid which is Riemannian.

For the convenience of the reader let us recall how a complete Riemannian manifold  $(M^n, g)$  with section curvatures between  $-K^2$  and  $K^2$  for some  $K > 0$  can be considered a Riemannian groupoid  $G$  (see Example 5.7 in [30]). Given  $r \in (0, \frac{\pi}{K})$ , for any  $p \in M^n$  the exponential map  $\exp_p : T_p M^n \rightarrow M^n$  restricts to a local diffeomorphism from the  $r$ -ball  $B_r(0_p)$  in  $T_p M^n$  to  $B_r(p) \subset M^n$ . We endow  $B_r(0_p)$  with the pull back metric  $(\exp_p)^*(g)$ . Let now  $\{p_i\}_{i \in I}$  be points in  $M^n$ , such that  $\bigcup_{i \in I} B_r(p_i) = M^n$ . One defines a Riemannian groupoid  $G$  with

$$G^{(1)} = \bigsqcup_{i, j \in I} \left\{ (v_i, v_j) \in B_r(0_{p_i}) \times B_r(0_{p_j}) : \exp_{p_i}(v_i) = \exp_{p_j}(v_j) \right\}$$

and

$$G^{(0)} = \bigsqcup_{i \in I} B_r(0_{p_i})$$

by  $r(v_i, v_j) = v_i$ ,  $s(v_i, v_j) = v_j$  and  $(v_i, v_j) \cdot (v_j, v_k) = (v_i, v_k)$ . Then  $G$  is isometrically equivalent to  $(M^n, g)$ .

For  $v \in G^{(0)}$ , that is  $v \in B_r(0_p)$ , the orbit  $O_v$  consists of precisely those points  $v_j \in B_r(0_{p_j})$  with  $\exp_{p_j}(v_j) = \exp_p(v)$ . A pointed groupoid  $(G, O_v)$  is a groupoid  $G$  equipped with a preferred orbit  $O_p$ . Now the smooth convergence of a sequence of pointed  $n$ -dimensional Riemannian groupoids with uniform sectional curvature bounds as above implies in particular, that on each of the balls  $B_r(0_{p_i})$  of fixed radius  $r > 0$  the above defined pull back metrics converge to a limit metric on  $B_r(0_{p_\infty})$  in  $C^\infty$ -topology (see Definition 5.8 in [30]).

Since in our situation all the metrics considered are homogeneous the pull back metrics are locally homogeneous. Now by [36] a Riemannian manifold  $(M^n, g)$  is locally homogeneous if and only if any function on  $M^n$  that can be expressed as a polynomial in the covariant derivatives of the curvature tensor  $\nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_r} R_{jklm}$  and the inverse metric tensor  $g^{ij}$ , by contracting indices, is actually constant on  $M^n$ . This clearly shows that the limit metric on the limit ball  $B_r(0_{p_\infty})$  is locally homogeneous as well.

We conclude that this groupoid is a locally homogeneous, ancient solution  $g_\infty(t)$  to the Ricci flow with nonnegative scalar curvature. At time  $t = 0$  we have  $\|R(g_\infty(0))\|_{g_\infty(0)} = 1$ , but  $\text{scal}(g_\infty(0)) = 0$ , since the function  $\frac{K(t)}{s(t)}$  is scale invariant and by assumption we had  $i = \frac{K(t_i)}{s(t_i)} = \frac{1}{\text{scal}(g_i(0))}$ . Hence the limit solution is locally homogeneous and Ricci flat, hence by Theorem 6.2 flat. This is a contradiction to the fact that the norm of limit curvature tensor is 1 at  $t = 0$ .  $\square$

*Remark 2.2.* We have shown above that for any homogeneous Ricci flow solution  $(g(t))_{t \in [0, T]}$  with  $\text{scal}(g(0)) > 0$  there exists a constant  $C_{g_0} > 0$  such that for all  $t \in [0, T]$  we have

$$\|R(g(t))\|_{g(t)} \leq C_{g_0} \cdot \text{scal}(g(t)).$$

### 3. An algebraic proof of Bochner’s theorem

As is well known by a theorem of Bochner [7] a compact homogeneous manifold cannot admit a Riemannian metric of nonpositive Ricci curvature unless it is flat. Moreover, the only compact homogeneous manifold admitting flat homogeneous metrics is the torus ([6], 7.61). Hence Theorem 2 follows from Theorem 3.2.

In this section we will provide an algebraic proof of the above result for compact, locally homogeneous spaces (see Section 6). A locally homogeneous space  $G/H$  is called compact, if the Lie algebra  $\mathfrak{g}$  of  $G$  is the Lie algebra of a compact Lie group  $\bar{G}$ . Recall that  $\bar{G} = (G_1 \times \cdots \times G_s \times T^r)/\Gamma$ , where  $G_1, \dots, G_s$  are compact, simply connected, simple Lie groups and  $\Gamma$  is a finite subgroup of the center of  $\bar{G}$ . As a consequence  $G = G_1 \times \cdots \times G_s \times \mathbb{R}^n$ .

Since the isotropy group  $H$  is not assumed to be a closed subgroup of  $G$ , in general the locally homogeneous space  $G/H$  cannot be extended to a globally homogeneous space (see [25]). In particular, Stoke’s theorem is not applicable.

**Theorem 3.1.** *Let  $G/H$  be a connected, compact, locally homogeneous space. Let  $g$  be a homogeneous metric on  $G/H$ , which is not flat. Then the Ricci curvature of  $g$  is not nonpositive.*

*Proof.* Since  $G/H$  is a compact, locally homogeneous space, there exists an  $\text{Ad}(G)$ -invariant scalar product  $Q$  of  $\mathfrak{g}$ . Let  $\mathfrak{p}$  denote the orthogonal complement of the Lie algebra  $\mathfrak{h}$  of  $H$  in  $\mathfrak{g}$ . Then  $\mathfrak{p}$  is  $\text{Ad}(H)$ -invariant. Let  $B$  denote the Killing form of  $G$  and let  $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{a}$  be the decomposition of  $\mathfrak{g}$  into its semisimple part  $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$  and its center  $\mathfrak{a} = \mathfrak{z}(\mathfrak{g})$ . Notice that  $\mathfrak{a}$  is the kernel of  $B$ , whereas on  $\mathfrak{g}_s$  the Killing form  $B$  is negative definite. Let  $\mathfrak{p}_\alpha := \mathfrak{p} \cap \mathfrak{a}$  and let  $\mathfrak{p}_s$  denote the  $Q$ -orthogonal complement of  $\mathfrak{p}_\alpha$  in  $\mathfrak{p}$ . Then, since  $\mathfrak{p}_\alpha$  is  $\text{Ad}(H)$ -invariant, so is  $\mathfrak{p}_s$ . Notice that on  $\mathfrak{p}_s$  the Killing form  $B$  is negative definite.

Any  $G$ -invariant metric on  $G/H$  corresponds to an  $\text{Ad}(H)$ -invariant scalar product  $g$  on  $\mathfrak{p}$ . Using  $\bar{Q} := Q|_{\mathfrak{p}}$  we may write

$$g(v, w) = \bar{Q}(P \cdot v, w),$$

where  $P$  is an  $\text{Ad}(H)$ -equivariant endomorphisms of  $\mathfrak{p}$ , which is positive definite. Using the decomposition  $\mathfrak{p} = \mathfrak{p}_s \oplus \mathfrak{p}_\alpha$  we write

$$P = \begin{pmatrix} P_{ss} & P_{sa} \\ P_{sa}^T & P_{aa} \end{pmatrix}. \tag{3.1}$$

The endomorphism  $P_{ss}$  of  $\mathfrak{p}_s$  is positive definite. Next, let  $(\hat{e}_1, \dots, \hat{e}_n)$  denote an  $Q$ -orthonormal basis of Eigenvectors of  $P_{ss}$  corresponding to eigenvalues  $p_1, \dots, p_n > 0$ . We set  $e_i := \hat{e}_i / \sqrt{p_i}$  for  $1 \leq i \leq n$ . Then  $(e_1, \dots, e_n)$  is a  $g$ -orthonormal basis of  $\mathfrak{p}_s$ . We extend  $(e_1, \dots, e_n)$  to an  $g$ -orthonormal basis  $(e_1, \dots, e_n, e_{n+1}, \dots, e_{n+l})$  of  $\mathfrak{p}$ . Then by [6], (7.38) for  $x \in \mathfrak{p}$  we have

$$\text{ric}(g)(x, x) = -\frac{1}{2}B(x, x) - \frac{1}{2} \sum_{i=1}^{n+l} \|[x, e_i]_{\mathfrak{p}}\|_g^2 + \frac{1}{4} \sum_{i,j=1}^{n+l} \bar{Q}([e_i, e_j]_{\mathfrak{p}}, P(x))^2.$$

Using  $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{p}} = [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{p}_s}$  we arrive at

$$\begin{aligned} \text{ric}(g)(x, x) &\geq -\frac{1}{2}B(x, x) - \frac{1}{2} \sum_{i=1}^n \|[x, e_i]_{\mathfrak{p}_s}\|_g^2 + \frac{1}{4} \sum_{i,j=1}^n \bar{Q}([e_i, e_j]_{\mathfrak{p}_s}, P(x))^2 \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=n+1}^{n+l} \bar{Q}([e_i, e_j]_{\mathfrak{p}_s}, P(x))^2 - \frac{1}{2} \sum_{j=n+1}^l \|[x, e_j]_{\mathfrak{p}_s}\|_g^2. \end{aligned}$$

Next, we show that the sum of the fourth and fifth term is nonnegative for an eigenvector  $x \in \mathfrak{p}_s$  of  $P_{ss}$  corresponding to the largest eigenvalue  $p$ . We have for  $i \in \{1, \dots, n\}$  and  $j \in \{n + 1, \dots, n + l\}$

$$\begin{aligned} \bar{Q}([e_i, e_j]_{\mathfrak{p}_s}, P(x))^2 - \bar{Q}([x, e_j]_{\mathfrak{p}_s}, P(e_i))^2 \\ = p^2 \cdot Q([e_i, e_j], x)^2 - p_i^2 \cdot Q([x, e_j], e_i)^2 \\ = (p^2 - p_i^2) \cdot Q([e_i, e_j], x)^2 \geq 0, \end{aligned}$$

since  $Q$  is  $\text{Ad}(G)$ -invariant and consequently  $\text{ad}(v)$  skew symmetric for any  $v \in \mathfrak{g}$ .

It remains to show that in the above estimate for  $\text{ric}(g)(x, x)$ , for  $x \in \mathfrak{p}_s$  as above, the sum of the first three terms is positive. Firstly, the second and the third term depends only of  $P_{ss}$  and not on the other components of  $P$ . Secondly, we denote by  $\mathfrak{g}'$  the  $Q$ -orthogonal complement of  $\mathfrak{p}_\alpha$  in  $\mathfrak{g}$ , that is  $\mathfrak{g}' = \mathfrak{h} \oplus \mathfrak{p}_s$ . Then  $\mathfrak{g}'$  is a subalgebra of  $\mathfrak{g}$ . Moreover, the Killing forms of  $\mathfrak{g}$  and  $\mathfrak{g}'$  restricted to their semisimple part  $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}', \mathfrak{g}']$  agree. It follows that we may assume that  $B$  is negative definite on  $\mathfrak{p}$  and that  $P = P_{ss}$ .

For any  $\text{Ad}(H)$ -invariant scalar product  $g$  on  $\mathfrak{p}$ , there exists a decomposition  $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_r$  of  $\mathfrak{p}$  into  $\text{Ad}(H)$ -irreducible summands, such that  $P$  is diagonal with respect to  $\bar{Q}$ . That is, we have

$$g = p_1 \cdot \bar{Q}|_{\mathfrak{p}_1} \perp \dots \perp p_r \cdot \bar{Q}|_{\mathfrak{p}_r}$$

with  $p_1, \dots, p_r > 0$ . Likewise, we have  $P|_{\mathfrak{p}_i} = p_i \cdot \text{id}|_{\mathfrak{p}_i}$ . For each  $1 \leq i \leq r$ , we set  $-B|_{\mathfrak{p}_i} = b_i \cdot Q|_{\mathfrak{p}_i}$  and  $d_i = \dim \mathfrak{p}_i$ . Notice, that by assumption we have  $b_i > 0$  for all  $i$ . Moreover, we set

$$[ijk] = \sum Q([\hat{e}_\alpha, \hat{e}_\beta], \hat{e}_\gamma)^2,$$

where the sum is taken over  $\{\hat{e}_\alpha\}$ ,  $\{\hat{e}_\beta\}$ , and  $\{\hat{e}_\gamma\}$ ,  $Q$ -orthonormal bases for  $\mathfrak{p}_i$ ,  $\mathfrak{p}_j$ , and  $\mathfrak{p}_k$ , respectively. Notice, that  $[ijk]$  is invariant under permutations. By [40], (see also [44]), if  $x \in \mathfrak{p}_1$  is an eigenvector of  $P$  with  $\bar{Q}(x, x) = 1$  corresponding to the largest eigenvalue  $p = p_1$  of  $P$ , then we have by the above estimate

$$\text{ric}(g)(x, x) \geq \frac{1}{4d_1} \cdot \left( 2d_1 b_1 - \sum_{j,k=1}^r [ijk] \frac{2p_k^2 - p^2}{p_j p_k} \right).$$

Notice that it is not difficult to deduce this formula from the above formula for the Ricci tensor of  $g$ . By [44], the identity

$$d_i b_i = 2d_i c_i + \sum_{j,k=1}^r [ijk] \tag{3.2}$$

holds. For a  $Q$ -orthonormal basis  $\{z_i\}$  of  $\mathfrak{h}$ , here  $C_{Q|\mathfrak{h}} = -\sum_i \text{ad } z_i \circ \text{ad } z_i$  denotes the Casimir operator and  $(C_{Q|\mathfrak{h}})|_{\mathfrak{p}_i} = c_i \cdot \text{id}_{\mathfrak{p}_i}$ . Recall that  $c_i \geq 0$  with  $c_i = 0$  if and

only if  $[\mathfrak{h}, \mathfrak{p}_i] = 0$ . Notice that the proof of Wang and Ziller carries over to compact, locally homogeneous spaces. We obtain

$$d_1 b_1 - \sum_{j,k=1}^r [1jk] \frac{2p_k^2 - p^2}{p_j p_k} = 2d_1 c_1 + \sum_{j,k=1}^r [1jk] \cdot \frac{p^2 - p_j^2 - p_k^2 + p_j p_k}{p_j p_k}.$$

Since  $p$  is the largest eigenvalue of  $P$  we have  $p^2 - p_j^2 - p_k^2 + p_j p_k \geq 0$  for all  $1 \leq j, k \leq r$ . As a consequence  $\text{ric}(g)(x, x) \geq \frac{b_1}{4} > 0$ . This shows the claim under the assumption that  $\dim \mathfrak{p}_s > 0$ .

If  $\mathfrak{p}_s = 0$ , then  $\mathfrak{p} \subset \mathfrak{a}$ , that is  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}$ . Since  $\mathfrak{h}$  is an ideal in  $\mathfrak{g}$  we have  $\mathfrak{h} = 0$  (by our definition locally homogeneous spaces are almost effective, see Section 6). Consequently  $\mathfrak{g}$  is abelian, which implies that any locally homogeneous metric on  $G/H$  is flat (cf. [6], (7.30)).  $\square$

As we have seen in the proof of the above theorem, a homogeneous metric  $g$  on  $G/H$  can be considered an endomorphism  $P$  of  $\mathfrak{p}$  using a background metric  $\tilde{Q}$ . It follows from the above proof that if  $g$  is not flat, then there exists  $\bar{x} \in \mathfrak{p}_s$  with  $P_{ss}(\bar{x}) = p \cdot \bar{x}$ ,  $g(\bar{x}, \bar{x}) = 1$  and

$$\text{ric}(\bar{x}, \bar{x}) \geq \frac{b}{4p}.$$

Here,  $-b$  denotes the largest negative eigenvalue of the Killing form  $B$ , restricted to the semisimple part  $\mathfrak{g}_s$  of  $\mathfrak{g}$ . An immediate corollary of this is

**Theorem 3.2.** *A homogeneous Ricci flow on a compact locally homogeneous space  $M^n = G/H$  has finite extinction time, if  $G/H$  does not admit flat metrics.*

*Proof.* As in the proof of Theorem 3.1, given an  $\text{Ad}(G)$ -invariant scalar product  $Q$  on  $\mathfrak{g}$ , we consider the decomposition  $\mathfrak{p} = \mathfrak{p}_s \oplus \mathfrak{p}_\alpha$  of the  $Q$ -orthogonal complement  $\mathfrak{p}$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ , where  $\mathfrak{p}_\alpha = \mathfrak{p} \cap \mathfrak{z}(\mathfrak{g})$  and  $\mathfrak{p}_s$  denotes the  $Q$ -orthogonal complement of  $\mathfrak{p}_\alpha$  in  $\mathfrak{p}$ .

Along a locally homogeneous solution  $(g(t))_{t \in [0, T]}$  to the Ricci flow on  $G/H$  we consider the function

$$\varphi(t) := \max\{g(t)(y, y) \mid y \in \mathfrak{p}_s \text{ and } \|y\|_Q = 1\}.$$

Recall that  $\varphi(t)$  is nothing but the largest eigenvalue of  $P_{ss}(t)$  in (3.1). Then by [15, p. 531], for the Dini derivative

$$\frac{d^+ \varphi}{dt}(t) := \limsup_{s \rightarrow 0, s > 0} \frac{\varphi(t + s) - \varphi(t)}{s}$$

we have

$$\frac{d^+ \varphi}{dt}(t) = \max\{-2\text{ric}(g(t))(x, x) \mid g(t)(x, x) = \varphi(t) \text{ and } x \in \mathfrak{p}_s, \|x\|_Q = 1\}.$$

Using the estimate of Theorem 3.1 we conclude  $\text{ric}(g(t))(x, x) \geq \frac{b}{4} > 0$  for any such  $x \in \mathfrak{p}_s$ , where  $b$  denotes the smallest eigenvalue of  $(-B)|_{\mathfrak{g}_s \times \mathfrak{g}_s}$ . This shows that  $T < +\infty$ .  $\square$

If a compact, locally homogeneous space  $G/H$  admits a flat metric, by [41]  $G/H$  is globally homogeneous and flat. It follows from the proof of Theorem 3.1 that  $G$  does not have any compact factor, thus  $G$  is abelian. But then any homogeneous metric on  $G/H$  is flat.

#### 4. Immortal solutions of homogeneous Ricci flows

In this section we give the proof of Theorem 4. Before doing so let us note that on the isometry group  $E(2)$  of the Euclidean plane there exists a homogeneous immortal solution to the Ricci flow such that  $\|R(g(t))\|_{g(t)} \approx \exp(-ct)$  and  $\text{scal}(g(t)) \approx \exp(-2ct)$  for a positive constant  $c > 0$  (see [21]). Hence for immortal solutions the norm of the curvature tensor is not controlled by the absolute value of the scalar curvature, which was true for homogeneous solutions with a Type I singularity by Remark 2.2. Moreover, the above example provides parabolic rescalings  $g_i(t)$ , defined on  $[0, \infty)$  with  $\|R(g_i(0))\| = 1$ , which converge on  $(0, \infty)$  to the flat metric. Recall also that locally homogeneous metrics maybe incomplete.

**Theorem 4.1.** *An immortal locally homogeneous Ricci flow develops a Type III singularity.*

*Proof.* Let  $(g(t))_{[0, \infty)}$  denote an immortal locally homogeneous Ricci flow solution. If the initial metric is flat the claim follows. If the initial metric is not flat, then by rescaling we may assume that  $\text{scal}(g_0) = -1$ . As above we write  $s(t) = \text{scal}(g(t))$ . By (2.1) we get  $\lim_{t \rightarrow \infty} s(t) = 0$ .

In a first step we show that  $K(t) := \|R(g(t))\|_{g(t)}$  must be bounded. Suppose this is not the case. Then there exists a sequence  $\{t_i\}$  of times converging to infinity, such that  $i = K(t_i)$  and  $K(t) < i$  for all  $t \in [0, t_i)$ . We may assume that  $t_i > \frac{1}{16 \cdot K(0)}$ . Next, we choose  $\bar{t}_i \in (0, t_i)$ , such that

$$t_i - \bar{t}_i = \frac{1}{16 \cdot K(\bar{t}_i)}. \tag{4.1}$$

This is possible, since for  $\bar{t}_i = 0$  we have  $t_i > \frac{1}{16 \cdot K(0)}$ , whereas for  $\bar{t}_i = t_i$  we have  $0 < \frac{1}{16 \cdot K(\bar{t}_i)}$ . Setting  $Q_i := K(\bar{t}_i)$ , we consider the parabolic rescaling

$$g_i(t) := Q_i \cdot g \left( \bar{t}_i + \frac{t}{Q_i} \right). \tag{4.2}$$

Notice that at time  $t = 0$  the norm of the curvature tensor of the metric  $g_i(0)$  equals to 1 for all  $i$  and that by (4.1) for given  $t_i, \bar{t}_i$  we have  $t_i = \bar{t}_i + \frac{t}{Q_i}$  if and only

if  $t = \frac{1}{16}$ . As a consequence, by the doubling time property a locally homogeneous limit flow  $g_\infty(t)$  will exist on  $[0, \frac{1}{16}]$  with  $\|R(g_\infty(\frac{1}{16}))\| > 1$  since  $K(t_i) > K(\bar{t}_i)$ .

Of course the scalar curvature of the limit flow is nonpositive. On the other hand side the scalar curvature of the limit metric must vanish at  $t = \frac{1}{16}$  since  $i = K(t_i)$  and  $|\text{scal}(g(t_i))| \leq 1$ . As in Theorem 2.1 we obtain a contradiction.

In the second step we suppose that  $K(t)$  is bounded but does not converge to zero for  $t \rightarrow \infty$ . Hence there exists a sequence  $\{t_i\}$  of times converging to  $+\infty$  and a constant  $\varepsilon > 0$ , such that  $K(t_i) \geq \varepsilon$ . Since  $\lim_{t \rightarrow \infty} s(t) = 0$ , we can argue as above to exclude this case as well.

We conclude that  $K(t)$  must converge to zero for  $t \rightarrow \infty$ . We suppose that the solution  $g(t)$  does not form a Type III singularity. Then there exists a sequence  $\{t_i\}$  of times converging to infinity such that

$$K(t_i) = \frac{i}{t_i + 1}$$

and  $K(t) < \frac{i}{t+1}$  for all  $t \in [0, t_i)$ . We may assume  $t_i > \frac{1}{16 \cdot K(0)}$ . As above we choose  $\bar{t}_i \in (0, t_i)$ , such that the identity (4.1) holds. Again we consider the parabolic rescalings  $g_i(t)$  as in (4.2). By the choice of the sequence  $\{t_i\}$  we have

$$K(\bar{t}_i) = \frac{r(i)}{\bar{t}_i + 1} \tag{4.3}$$

with  $r(i) < i$ . On the other hand side the doubling time property yields

$$2 \geq \frac{K(t_i)}{K(\bar{t}_i)} = \frac{i}{K(\bar{t}_i) \cdot (t_i + 1)} = \frac{i}{K(\bar{t}_i) \cdot (\bar{t}_i + 1 + \frac{1}{16 \cdot K(\bar{t}_i)})} = \frac{i}{r(i) + \frac{1}{16}}.$$

We deduce

$$i > r(i) \geq \frac{i}{2} - \frac{1}{16}.$$

In the final step we show that  $\|R(g_i(\frac{1}{16}))\|$  is bounded from the below by a positive number. This follows from (4.1) and (4.3), since

$$\left\| R\left(g_i\left(\frac{1}{16}\right)\right) \right\| = \frac{i}{r(i)} \cdot \frac{\bar{t}_i + 1}{t_i + 1} = \frac{i}{r(i)} \cdot \left(1 - \frac{1}{16 \cdot r(i) + 1}\right).$$

Let  $g_\infty(t)$  denote a locally homogeneous limit solution for the sequence  $(g_i(t))$ , defined on  $(0, \frac{1}{16}]$ . By the above computation we have  $\|R(g_i(\frac{1}{16}))\| \in [\frac{1}{2}, 3]$  for large  $i$ . Since  $s(0) = -1$  we deduce from (2.1) that  $|s(t)| \cdot (1 + \frac{2}{n} \cdot t) \leq 1$  for all  $t \geq 0$ . As a consequence, at time  $t = t_i$  we have

$$\frac{K(t_i)}{|s(t_i)|} \geq \frac{i}{t_i + 1} \cdot \left(1 + \frac{2}{n} \cdot t_i\right) \geq \frac{2 \cdot i}{n}. \tag{4.4}$$

It follows, that the scalar curvature of the limit solution  $g_\infty(t)$  at time  $t = \frac{1}{16}$  vanishes. Again, this is a contradiction. □

**5. Collapsing of homogeneous Ricci flows**

In this section we will prove Theorem 5.14, from which Theorem 3 and the results on the longtime behavior of the Ricci flow mentioned in the introduction below Theorem 3 follow.

Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on  $M^n$ , which develops a Type I singularity. Recall that by [26] we may assume that  $\text{scal}(g(0)) = 1$ . As is well known there exist constants  $c(n) > 0$  and  $C_{g_0} > 0$  such that

$$\frac{c(n)}{T - t} \leq \|R(g(t))\|_{g(t)} \leq \frac{C_{g_0}}{T - t}$$

for all  $t \in [0, T)$  (see e.g., [17]). Moreover, since for such homogeneous Ricci flows the norm of the curvature tensor is controlled by the scalar curvature (see Remark 2.2), there exist constants  $c_{g_0} > 0$  and  $\bar{C}_{g_0} > 0$  such that

$$\frac{c_{g_0}}{T - t} \leq \text{scal}(g(t)) \leq \frac{\bar{C}_{g_0}}{T - t} \tag{5.1}$$

for all  $t \in [0, T)$ . Let now  $(t_i)_{i \in \mathbb{N}}$  be any sequence in  $[0, T)$  with  $\lim_{i \rightarrow \infty} t_i = T$ . Let  $Q_i := \text{scal}(g(t_i))$  and

$$g_i(t) := Q_i \cdot g\left(t_i + \frac{t}{Q_i}\right).$$

The homogeneous Ricci flow  $(g_i(t))$  is defined for  $t \in [-Q_i t_i, (T - t_i)Q_i)$  and we have  $\text{scal}(g_i(0)) = 1$  for all  $i \in \mathbb{N}$ .

By Theorem 1.4 in [17], when setting  $\lambda_i := \text{scal}(g(t_i))$  and

$$\tilde{g}_i(\tilde{t}) := \lambda_i \cdot g\left(T + \frac{\tilde{t}}{\lambda_i}\right),$$

then  $(\tilde{g}_i(\tilde{t}))_{i \in \mathbb{N}}$  subconverges to a nonflat shrinking soliton  $(\tilde{g}_\infty(\tilde{t}))_{\tilde{t} \in (-\infty, 0)}$ , which of course is homogeneous. By (5.1) we know that  $\text{scal}(g(t_i)) = \frac{c_i}{T - t_i}$  for  $c_i \in [c_{g_0}, \bar{C}_{g_0}]$ . Notice now that

$$T + \frac{\tilde{t}}{\text{scal}(g(t_i))} = t_i + T - t_i + \frac{\tilde{t}}{\text{scal}(g(t_i))} = t_i + \frac{c_i + \tilde{t}}{\text{scal}(g(t_i))}.$$

Hence  $g_i(t) := \tilde{g}_i(t - c_i)$ . As a consequence,  $(g_i(t))_{i \in \mathbb{N}}$  subconverges to a nonflat homogeneous shrinking soliton  $(g_\infty(t))_{t \in (-\infty, T_\infty)}$ , with  $T_\infty \geq c_{g_0} > 0$ .

The results of [37] imply that the homogeneous shrinking soliton  $g_\infty(0)$  is up to finite covering the Riemannian product of a compact homogeneous Einstein manifold  $(E_\infty^k, g_\infty^1)$  with positive scalar curvature and a flat space  $(\mathbb{R}^{n-k}, g_{\text{flat}}^2)$  endowed with a Gaussian shrinking soliton. In particular the Ricci curvature of any such limit soliton is nonnegative.

*Remark 5.1.* In [11] an example  $(g(t))_{t \in [0, T]}$  of a homogeneous Ricci flow solution on  $M^{12} = \text{Sp}(3)/\text{Sp}(1)^3$  has been discussed, where the Ricci curvature of  $g(t)$  is not nonnegative for all  $t \geq T_0$  and some  $T_0 \in (0, T)$ .

Since  $\text{scal}(g_\infty(0)) = 1$ , it follows that there exists a constant  $r_\infty \geq \frac{1}{n}$ , such that the eigenvalues of  $\text{Ric}(g_\infty(0))$  are either equal to  $r_\infty$  or to zero. Here, for a Riemannian metric  $g$  the Ricci-endomorphism  $\text{Ric}(g)$  is defined by

$$\text{ric}(g)(\cdot, \cdot) = g(\text{Ric}(g)\cdot, \cdot).$$

**Theorem 5.2.** *For a homogeneous Ricci flow with finite extinction time the dimension of the Einstein factor of any limit shrinking soliton does only depend on the initial metric.*

*Proof.* There exists  $\bar{T} < T$  such that for all  $t \geq \bar{T}$  the eigenvalues of the Ricci endomorphism of

$$\bar{g}(t) := \text{scal}(g(t)) \cdot g(t)$$

come in two blocks: The positive ones are bounded from the below by  $\frac{3}{4n}$  and the small ones are bounded from the above by  $\frac{1}{4n}$ . Otherwise, there exists a sequence  $(t_i)_{i \in \mathbb{N}}$  of times converging to  $T$ , such that at least one eigenvalue of  $\text{Ric}(g_i(0))$  is contained in  $[\frac{1}{4n}, \frac{3}{4n}]$ . By passing to a subsequence the same is true for the Ricci endomorphism of a limit soliton  $g_\infty(0)$ . Contradiction.  $\square$

Notice though, that for different initial metrics the dimension of the Einstein factor may vary: For the product Einstein metric on  $S^2 \times S^2$  the Einstein factor of the limit soliton is of course 4-dimensional, whereas for any other homogeneous initial metric the dimension of the Einstein factor is 2-dimensional.

We turn now to a class of homogeneous spaces, where we can describe how the limit Einstein factor  $E_\infty^k$  is related to the original homogeneous space  $M^n$ . Let  $M^n = G/H$  be a connected compact homogeneous space such that  $G$  and  $H$  are compact Lie groups not necessarily connected. Let  $Q$  denote an  $\text{Ad}(G)$ -invariant scalar product on  $\mathfrak{g}$  and let  $\mathfrak{p}$  denote the  $Q$ -orthogonal complement to  $\mathfrak{h}$  in  $\mathfrak{g}$ . Then for any  $G$ -invariant metric  $g$  on  $G/H$  there exists a  $Q$ -orthogonal decomposition  $\mathfrak{p} = \mathfrak{p}_1 \oplus \dots \oplus \mathfrak{p}_r$  into  $\text{Ad}(H)$ -irreducible summands, such that

$$g = x_1 \cdot Q|_{\mathfrak{p}_1} \perp \dots \perp x_r \cdot Q|_{\mathfrak{p}_r} \tag{5.2}$$

where  $x_1, \dots, x_r > 0$ . For any nonempty index set  $J \subset \{1, \dots, r\}$  we set

$$\mathfrak{p}_J := \bigoplus_{j \in J} \mathfrak{p}_j.$$

**Assumption 5.3.** We assume that the moduls  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  are pairwise inequivalent.

An example would be a homogeneous space where both  $G$  and  $H$  have the same rank. As is well known, under Assumption 5.3 the Ricci endomorphism also respects the above decomposition of  $\mathfrak{p}$ ; in general this is not true anymore.

*Remark 5.4.* There exist homogeneous spaces  $G/H$  which admit diagonal metrics as in (5.2) allowing some of the moduls  $\mathfrak{p}_i$  and  $\mathfrak{p}_j$  being equivalent, such that the Ricci endomorphism of all such diagonal metrics still respects this decomposition. All the results proved in this section can also be obtained in that more general case.

So let  $(g(t))_{[0,T)}$  be a homogeneous Ricci flow and assume that Assumption 5.3 holds. We write

$$g(t) = x_1(t) \cdot Q|_{\mathfrak{p}_1} \perp \dots \perp x_r(t) \cdot Q|_{\mathfrak{p}_r} \tag{5.3}$$

for all  $t \in [0, T)$  and call  $x_1(t), \dots, x_r(t)$  the eigenvalues of  $g(t)$ .

**Lemma 5.5.** Let  $(g(t))_{[0,T)}$  be a homogeneous Ricci flow with finite extinction time and  $\text{scal}(g(0)) = 1$ . Then, under the Assumption 5.3 there exists a nonempty subset  $I \subset \{1, \dots, r\}$  and a positive constant  $r_\infty \in [\frac{1}{n}, \frac{1}{2}]$ , such that the following holds true: For any  $\varepsilon > 0$  there exist  $T(\varepsilon) < T$ , such that for all  $t \geq T(\varepsilon)$

$$\| \text{Ric}(\bar{g}(t))|_{\mathfrak{p}_I} - r_\infty \cdot \text{id}_{\mathfrak{p}_I} \|, \quad \| \text{Ric}(\bar{g}(t))|_{\mathfrak{p}_{I^C}} \| < \varepsilon.$$

Moreover, for any  $m \in I, l \in I^C$  and  $t \geq T(\varepsilon)$  we have

$$x_m(t) \leq x_m(T(\varepsilon)) \cdot \left( \frac{T-t}{T-T(\varepsilon)} \right)^{\frac{c_{g_0}}{n}} \tag{5.4}$$

$$x_l(t) \geq x_l(T(\varepsilon)) \cdot \left( \frac{T-t}{T-T(\varepsilon)} \right)^{2\varepsilon \bar{C}_{g_0}}. \tag{5.5}$$

*Proof.* From Theorem 5.2 it follows that there exists an index set  $I \subset \{1, \dots, r\}$  and  $T_0 < T_\infty$ , such that for all  $t \geq T_0$  the eigenvalues of  $\text{Ric}(\bar{g}(t))$  corresponding to  $\mathfrak{p}_I$  are bounded from the below by  $\frac{3}{4n}$ , whereas all the eigenvalues of  $\text{Ric}(\bar{g}(t))$  corresponding to  $\mathfrak{p}_{I^C}$  are bounded from the above by  $\frac{1}{4n}$ . Here  $I^C$  denotes the complement of  $I$  in  $\{1, \dots, r\}$ . Since for any sequence of times  $(t_i)_{i \in \mathbb{N}}$  converging to  $T$  there is a limit shrinking soliton (along a subsequence), the eigenvalues of  $\text{Ric}(\bar{g}(t))$  must pinch more and more. This shows the first claim.

Let  $r_m(t)$  denote the eigenvalue of the Ricci endomorphism  $\text{Ric}(g(t))$  restricted to  $\mathfrak{p}_m, m = 1, \dots, r$ . Then, under Assumption 5.3 the Ricci flow equation is nothing but  $x'_m(t) = -2 \cdot x_m(t) \cdot r_m(t), m = 1, \dots, r$ . Since  $\text{Ric}(g(t)) = \text{scal}(g(t)) \cdot \text{Ric}(\bar{g}(t))$  we deduce from (5.1) and the first claim the estimates (5.4) and (5.5) by integrating the above differential equation. □

In the next steps, we describe the index set  $I$  in Lemma 5.5 more precisely.

**Definition 5.6.** Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on a compact homogeneous space  $G/H$  with finite extinction time. Suppose that Assumption 5.3 holds. Let  $\{t_a\}$  denote a sequence of times converging to  $T$ , such that the metrics

$$\bar{g}_a = \text{scal}(g(t_a)) \cdot g(t_a)$$

converge to a nonflat limit soliton metric  $g_\infty$ .

We assume that at such times  $t_a$  the eigenvalues  $x_1(t_a), \dots, x_r(t_a)$  of  $g(t_a)$  can be ordered in blocks in the following manner. There exist nonempty, pairwise disjoint subsets  $I_1, \dots, I_\ell$  of  $\{1, \dots, r\}$  with

$$\{1, \dots, r\} = I_1 \cup \dots \cup I_\ell$$

such that the following holds

- (1) For any  $1 \leq s \leq \ell$  there exists a constant  $D_s > 0$  such that for any  $i, j \in I_s$  we have  $\frac{x_i(t_a)}{x_j(t_a)} \leq D_s$  for all  $a \in \mathbb{N}$ .
- (2) For  $1 \leq s < \tilde{s} \leq \ell$  and any  $i \in I_s$  and any  $j \in I_{\tilde{s}}$  we have  $\frac{x_j(t_a)}{x_i(t_a)} \rightarrow \infty$  for  $a \rightarrow \infty$ .
- (3) For any  $1 \leq s \leq \ell$  there exists a permutation  $\pi_s$  of  $I_s = \{n_s^1, \dots, n_s^{|I_s|}\}$ ,  $n_s^1 < \dots < n_s^{|I_s|}$ , such that  $x_{\pi_s(n_s^1)}(t_a) \leq \dots \leq x_{\pi_s(n_s^{|I_s|})}(t_a)$  for all  $a \in \mathbb{N}$ .

Let  $\{t_i\}_{i \in \mathbb{N}}$  be any sequence of times converging to  $T$ . We claim, that we can always extract a subsequence  $\{t_a\}$  as in Definition 5.6. To this end, as noticed above we can assume that along a subsequence  $\{t_a\}$  of  $\{t_i\}_{i \in \mathbb{N}}$ , the metrics  $\bar{g}_a$  converge to a nontrivial homogeneous shrinking soliton. Furthermore, it is clear that the eigenvalues  $x_1(t_a), \dots, x_r(t_a)$  can be assumed to be ordered along a subsequence, that is there exists a permutation  $\pi$  of  $\{1, \dots, r\}$ , such that  $x_{\pi(1)}(t_a) \leq \dots \leq x_{\pi(r)}(t_a)$  for all  $a \in \mathbb{N}$ . Next, the ratio  $\frac{x_{\pi(2)}(t_a)}{x_{\pi(1)}(t_a)}$  is either bounded along a subsequence of  $\{t_a\}$ , or it diverges. This shows that the index sets  $I_1, \dots, I_\ell$  can be chosen as above. Notice that the index sets  $I_1, \dots, I_\ell$  may depend on the subsequence  $\{t_a\}$  chosen.

**Corollary 5.7.** Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on a compact homogeneous space  $G/H$  with finite extinction time  $T$ . Suppose that Assumption 5.3 holds. Let  $\{t_a\}$  denote a sequence of times converging to  $T$  as in Definition 5.6. Then the index set  $I$  from Lemma 5.5 satisfies  $I = I_1 \cup \dots \cup I_s$  for some  $1 \leq s \leq \ell$ .

*Proof.* This follows directly from the estimates (5.4) and (5.5) in Lemma 5.5 and the definition of the index sets  $I_1, \dots, I_\ell$  in Definition 5.6. □

This result says, that the "small" eigenvalues of  $g(t_a)$  are the eigenvalues  $x_i(t_a)$  with  $i \in I$  and the "large" eigenvalues are those  $x_j(t_a)$  with  $j \in I^C = \{1, \dots, r\} \setminus I$ .

Recall that on a homogeneous space  $G/H$  an intermediate  $\text{Ad}(H)$ -invariant subalgebra  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_{\mathfrak{k}}$ ,  $\mathfrak{p}_{\mathfrak{k}} \subset \mathfrak{p}$ , is called toral (nontoral), if  $\mathfrak{p}_{\mathfrak{k}}$  is (not) an abelian subalgebra of  $\mathfrak{g}$ .

**Lemma 5.8.** *Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on a compact homogeneous space  $G/H$  with finite extinction time. Suppose that Assumption 5.3 holds and let  $\{t_a\}$  be a sequence of times as in Definition 5.6. Then there exists  $s_0 \in \{1, \dots, \ell\}$ , such that  $\mathfrak{k}_{s_0-1} := \mathfrak{h} \oplus \mathfrak{p}_{I_1} \oplus \dots \oplus \mathfrak{p}_{I_{s_0-1}}$  is a toral and  $\mathfrak{k}_{s_0} := \mathfrak{k}_{s_0-1} \oplus \mathfrak{p}_{I_{s_0}}$  a nontoral subalgebra of  $\mathfrak{g}$ , respectively. Moreover there exists a constant  $C(G, H) > 0$ , only depending on  $G/H$ , such that*

$$\text{scal}(g(t_a)) \leq \frac{C(G, H)}{x_{\pi_{s_0}(n_{s_0}^1)}(t_a)}.$$

*Proof.* By [44], for a homogeneous metric  $g$  as in (5.2) we have

$$\text{scal}(g) = \frac{1}{2} \sum_{i=1}^r \frac{d_i b_i}{x_i} - \frac{1}{4} \sum_{i,j,k=1}^r [ijk] \cdot \frac{x_i}{x_j x_k}. \tag{5.6}$$

Since  $\lim_{a \rightarrow \infty} \text{scal}(g(t_a)) = +\infty$ , we deduce that the terms  $[ijk] \cdot \frac{x_i(t_a)}{x_k(t_a)}$  must be bounded for  $i \in I_1, j \in I_s$  and  $k \in I_{s'}$  with  $s \neq s'$ . Consequently

$$[I_1 I_s I_{s'}] = 0 \tag{5.7}$$

for  $s \neq s'$ . This shows that  $\mathfrak{k}_1 := \mathfrak{h} \oplus \mathfrak{p}_{I_1}$  is a subalgebra. Moreover, it follows from (5.6) that  $\text{scal}(g(t_a)) \leq \frac{C(G, H)}{x_{\pi_1(n_1^1)}(t_a)}$  for a constant  $C(G, H) > 0$  depending only on the pair  $H \subset G$ . If  $\mathfrak{k}_1$  is a nontoral subalgebra, the claim follows.

So suppose that  $\mathfrak{k}_1$  is a toral subalgebra. By  $[I_1 I_1 I_1] = 0$  we deduce

$$\begin{aligned} \text{scal}(g(t_a)) &= \frac{1}{2} \sum_{i \in I_1} \frac{d_i b_i}{x_i(t_a)} - \frac{1}{4} \sum_{i \in I_1, j, k \in I_1^C} [ijk] \cdot \left( \frac{x_i(t_a)}{x_j(t_a)x_k(t_a)} + 2 \frac{x_j(t_a)}{x_i(t_a)x_k(t_a)} \right) \\ &\quad + \frac{1}{2} \sum_{i \in I_1^C} \frac{d_i b_i}{x_i(t_a)} - \frac{1}{4} \sum_{i, j, k \in I_1^C} [ijk] \cdot \frac{x_i(t_a)}{x_j(t_a)x_k(t_a)}. \end{aligned}$$

Using the identity  $d_i b_i = 2c_i d_i + \sum_{j,k=1}^n [ijk]$  (cf. (3.2)) and the vanishing of the Casimir operator restricted to  $\mathfrak{p}_{I_1}$ , we see that the terms of the first line of the right hand side can be simplified to the following nonpositive term:

$$\frac{1}{2} \cdot \sum_{i \in I_1} \frac{1}{x_i(t_a)} \sum_{j, k \in I_1^C} [ijk] \cdot \left( 1 - \frac{x_j(t_a)}{x_k(t_a)} - \frac{x_i^2(t_a)}{2x_j(t_a)x_k(t_a)} \right) \leq 0.$$

Here we have used  $[ijk] = [ikj]$  and the estimate  $2 - x - \frac{1}{x} \leq 0$  for  $x > 0$ . Since  $\text{scal}(g(t_a)) \rightarrow +\infty$  for  $a \rightarrow \infty$  we obtain by induction the claim.  $\square$

For an intermediate  $\text{Ad}(H)$ -invariant subalgebra  $\mathfrak{k}$  of  $\mathfrak{g}$  we denote by  $H(\mathfrak{k})$  the subgroup of  $G$  generated by  $H$  and the connected Lie subgroup  $K_0$  of  $G$  with Lie algebra  $\mathfrak{k}$ . Notice that the Lie algebra of  $H(\mathfrak{k})$  equals to  $\mathfrak{k}$ , but that in general  $H(\mathfrak{k})$  might be disconnected.

For an  $\text{Ad}(H)$ -invariant intermediate subalgebra  $\mathfrak{k}$  and for a  $G$ -invariant metric  $g$  on  $G/H$  as in (5.2) we denote by

$$g_{K/H} = g|_{\mathfrak{p}_{\mathfrak{k}}}$$

the induced metric on  $K/H$ , where  $K = H(\mathfrak{k})$ .

**Lemma 5.9.** *Let  $g$  be a homogeneous metric on a compact homogeneous space  $G/H$  as in (5.2). Suppose that there is an index set  $I_{\mathfrak{k}} \subset \{1, \dots, r\}$ , such that  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_{I_{\mathfrak{k}}}$  is a subalgebra of  $\mathfrak{g}$ . Then for  $m \in I_{\mathfrak{k}}$  we have*

$$\text{Ric}(g)|_{\mathfrak{p}_m} = \text{Ric}(g_{K/H})|_{\mathfrak{p}_m} + \left( \sum_{j,k \in I_{\mathfrak{k}}^C} \frac{[mjk]}{4d_m} \cdot \left( 2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_m^2}{x_j x_k} \right) \right) \cdot \frac{\text{id}_{\mathfrak{p}_m}}{x_m}$$

*Proof.* For  $m \in \{1, \dots, r\}$  the following identity for the Ricci tensor of a homogeneous metric  $g$  as in (5.2) holds true (cf. Section 3):

$$\begin{aligned} \text{Ric}(g)|_{\mathfrak{p}_m} &= \left( \frac{b_m}{2x_m} - \frac{1}{2d_m} \sum_{j,k=1}^r [jkm] \cdot \frac{x_k}{x_m x_j} + \frac{1}{4d_m} \sum_{j,k=1}^r [jkm] \cdot \frac{x_m}{x_j x_k} \right) \cdot \text{id}_{\mathfrak{p}_m} \quad (5.8) \\ &= \left( 2d_m c_m + \frac{1}{2} \sum_{j=1}^r [mjj] \frac{x_m^2}{x_j^2} + \sum_{j < k} [mjk] \cdot \left( 2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_m^2}{x_j x_k} \right) \right) \cdot \frac{\text{id}_{\mathfrak{p}_m}}{2d_m x_m}, \end{aligned}$$

where we have used (3.2) again. By the above, the Ricci tensor of  $(K/H, g_{K/H})$  is given by

$$\text{Ric}(g_{K/H})|_{\mathfrak{p}_m} = \left( d_m c_m + \sum_{j,k \in I_{\mathfrak{k}}} \frac{[mjk]}{4} \cdot \left( 2 - \frac{x_k}{x_j} - \frac{x_j}{x_k} + \frac{x_m^2}{x_j x_k} \right) \right) \cdot \frac{\text{id}_{\mathfrak{p}_m}}{d_m x_m}.$$

Since  $\mathfrak{k}$  is a subalgebra of  $\mathfrak{g}$ , we deduce from the identity  $[I_{\mathfrak{k}} I_{\mathfrak{k}} I_{\mathfrak{k}}^C] = 0$  for  $m \in I_{\mathfrak{k}}$  the claim.  $\square$

The next lemma shows, that even when a nontoral subalgebra  $\mathfrak{k}$  is not the Lie algebra of a compact subgroup of  $G$  we may chose a subalgebra  $\tilde{\mathfrak{k}}$  of  $\mathfrak{k}$ , which is the Lie algebra of a compact subgroup. Under the Assumption 5.3 this is trivial if both groups  $G$  and  $H$  are connected, since then the isotropy representation contains at most one trivial summand, which must be one-dimensional.

**Lemma 5.10.** *Let  $G/H$  be a compact homogeneous space and let  $\mathfrak{k}$  be a proper, intermediate, nontoral,  $\text{Ad}(H)$ -invariant subalgebra of  $\mathfrak{g}$ . Then there exists a nontoral  $\text{Ad}(H)$ -invariant subalgebra  $\tilde{\mathfrak{k}}$  of  $\mathfrak{k}$  containing  $\mathfrak{h}$  properly, which is the Lie algebra of a compact subgroup of  $G$ .*

*Proof.* By [12], Chapter 0, Theorem 5.1 there exists  $N \in \mathbb{N}$  with  $G \subset \text{O}(2N)$ . The negative of the Killing form on  $\text{O}(2N)$ , denoted by  $Q_N$ , induces an  $\text{Ad}(G)$ -invariant scalar product  $\tilde{Q}$  on  $\mathfrak{g}$ , which we will use in the proof of this lemma.

Let  $\mathfrak{a}$  be any abelian subalgebra of  $\mathfrak{g}$ , corresponding to a compact subgroup  $A$  of  $G$ . Let  $\mathfrak{t}$  denote any maximal torus of  $\mathfrak{g}$  with  $\mathfrak{a} \subset \mathfrak{t}$ . Our first claim is, that the  $\tilde{Q}$ -orthogonal complement  $\mathfrak{a}^\perp$  of  $\mathfrak{a}$  in  $\mathfrak{t}$  also corresponds to a compact subalgebra of  $G$ . This is seen as follows: Conjugating the group  $G$  in  $\text{O}(2N)$  we may assume that  $\mathfrak{t}$  is contained in the standard maximal torus  $T_N = (S^1)^N$  of  $\text{O}(2N)$ . We denote by  $s_1, \dots, s_N$  the standard  $Q_N$ -orthonormal basis of  $T_e T_N$  and identify  $T_e T_N$  with  $\mathbb{R}^N$ . Since  $A$  is compact by assumption, the subalgebra  $\mathfrak{a}$ , considered as a subset of  $\mathbb{R}^N$ , has a rational basis  $a_1, \dots, a_m \in \mathbb{Q}^N$ . As a consequence the  $Q_N$ -orthogonal complement  $\mathfrak{a}_N^\perp$  of  $\mathfrak{a}$  in  $\mathbb{R}^N$  also has a rational basis, hence corresponds to a compact subgroup of  $T_N$ . Now the  $\tilde{Q}$ -orthogonal complement  $\mathfrak{a}^\perp$  of  $\mathfrak{a}$  in  $\mathfrak{t}$  equals to  $\mathfrak{t} \cap \mathfrak{a}_N^\perp$ . Since both  $\mathfrak{t}$  and  $\mathfrak{a}_N^\perp$  are Lie algebras of compact subgroups the above claim follows.

Next, we claim, that if  $\mathfrak{b} \subset \mathfrak{t}$  is the Lie algebra of a compact subgroup and if  $\pi : \mathfrak{t} \rightarrow \mathfrak{a}^\perp$  denotes the  $Q$ -orthogonal projection onto  $\mathfrak{a}^\perp$ ,  $\mathfrak{a}$  as above, then  $\pi(\mathfrak{b})$  is the Lie algebra of a compact subgroup as well. When considering  $\mathfrak{t}$  a subspace of  $\mathbb{R}^N$  as above, both  $\mathfrak{a}$  and  $\mathfrak{a}^\perp$ , respectively, have a basis  $v_1, \dots, v_m \in \mathbb{Z}^N$  and a basis  $w_1, \dots, w_{p-m} \in \mathbb{Z}^N$ ,  $p = \dim \mathfrak{t}$ . Now any vector in  $\mathfrak{t} \cap \mathbb{Z}^N$  can be written in this basis with rational coefficients. The second claim follows.

We decompose  $\mathfrak{k} = \mathfrak{k}_s \oplus \mathfrak{z}(\mathfrak{k})$  and  $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{z}(\mathfrak{h})$  into its semisimple part and its center. Since  $\mathfrak{k}_s = [\mathfrak{k}_s, \mathfrak{k}_s]$ , it follows from

$$\tilde{Q}(\mathfrak{z}(\mathfrak{k}), \mathfrak{k}_s) = \tilde{Q}(\mathfrak{z}(\mathfrak{k}), [\mathfrak{k}_s, \mathfrak{k}_s]) = \tilde{Q}([\mathfrak{z}(\mathfrak{k}), \mathfrak{k}_s], \mathfrak{k}_s) = 0$$

that both these decompositions are  $\tilde{Q}$ -orthogonal. Using  $\mathfrak{h}_s \subset \mathfrak{k}_s$ , we can assume that  $\mathfrak{k}_s \neq \mathfrak{h}_s$ , since otherwise  $\mathfrak{k}$  is a toral subalgebra.

We denote now by  $\pi_s$ , respectively  $\pi_{\mathfrak{z}}$ , the  $\tilde{Q}$ -orthogonal projections from  $\mathfrak{k}$  onto  $\mathfrak{k}_s$  and  $\mathfrak{z}(\mathfrak{k})$ , respectively. Both projections are  $\text{Ad}(H)$ -invariant. Furthermore, we denote by  $\mathfrak{a}_s := \pi_s(\mathfrak{z}(\mathfrak{h}))$  and by  $\mathfrak{a}_{\mathfrak{z}} := \pi_{\mathfrak{z}}(\mathfrak{z}(\mathfrak{h}))$ . Let  $\mathfrak{t}$  be a maximal torus of  $\mathfrak{g}$  with  $\mathfrak{a}_s \oplus \mathfrak{z}(\mathfrak{k}) \subset \mathfrak{t}$ . Notice that  $\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{t}$ . Next, we set  $\mathfrak{t}_s := \mathfrak{t} \cap \mathfrak{k}_s$ . Then  $\mathfrak{t}_s$  is a maximal torus of  $\mathfrak{k}_s$  with  $\mathfrak{a}_s \subset \mathfrak{t}_s$ . Let  $\mathfrak{t}_s^\perp$  denote the  $\tilde{Q}$ -orthogonal complement of  $\mathfrak{t}_s$  in  $\mathfrak{t}$ . Then, since  $\mathfrak{z}(\mathfrak{k})$  is orthogonal to  $\mathfrak{k}_s$ , it is orthogonal to  $\mathfrak{t}_s$ , hence contained in  $\mathfrak{t}_s^\perp$ . As a consequence  $\mathfrak{a}_{\mathfrak{z}} \subset \mathfrak{t}_s^\perp$ . Since  $\mathfrak{t}_s$  is the Lie algebra of a compact subgroup of  $\mathfrak{g}$ , by the above so is  $\mathfrak{t}_s^\perp$ . We denote by  $\pi : \mathfrak{t} \rightarrow \mathfrak{t}_s^\perp$  the  $\tilde{Q}$ -orthogonal projection. By the above  $\pi(\mathfrak{z}(\mathfrak{h})) = \mathfrak{a}_{\mathfrak{z}}$ , since  $\mathfrak{z}(\mathfrak{k}) \subset \mathfrak{t}_s^\perp$  and  $\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{t}_s \oplus \mathfrak{z}(\mathfrak{k})$ . As a consequence,  $\mathfrak{a}_{\mathfrak{z}}$

is the Lie algebra of a compact subgroup of  $G$ . It is also clear that  $\mathfrak{a}_3$  is  $\text{Ad}(H)$ -invariant. We set now  $\tilde{\mathfrak{k}} = \mathfrak{k}_s \oplus \mathfrak{a}_3$ . Clearly  $\tilde{\mathfrak{k}}$  is a proper  $\text{Ad}(H)$ -invariant subalgebra of  $\mathfrak{h}$  and  $\mathfrak{k}$ , which is compact.  $\square$

Next, we show that under Assumption 5.3,  $\text{Ad}(H)$ -invariant intermediate subalgebras always correspond to totally geodesic immersed submanifolds of  $G/H$ .

**Lemma 5.11.** *Let  $G/H$  be a compact homogeneous space and suppose that Assumption 5.3 holds true. Let  $K$  be a proper intermediate subgroup, that is  $H < K < G$ . Then  $K/H$  is a totally geodesic submanifold of  $G/H$ , possibly immersed though.*

*Proof.* Since  $H < K$ , the Lie algebra  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_{\mathfrak{k}}$  is  $\text{Ad}(H)$ -invariant, hence  $\mathfrak{p}_{\mathfrak{k}}$  as well. It follows from Assumption 5.3 that  $\mathfrak{p}_{\mathfrak{k}} = \mathfrak{p}_{i_1} \oplus \cdots \oplus \mathfrak{p}_{i_s}$  for  $1 \leq i_1 < \cdots < i_s \leq r$ . Due to formula (7.27) in [6], for Killing vector fields  $X, Y, N$  on  $G/H$  one has

$$2g(\nabla_X Y, N) = g([X, Y], N) - g([N, X], Y) + g([Y, N], X).$$

By assumption, every  $G$ -invariant metric on  $G/H$  has the special form described in (5.2). We deduce that for  $X, Y \in \mathfrak{p}_{\mathfrak{k}}$  and  $N \perp \mathfrak{k}$  we have  $g(\nabla_X Y, N) = 0$  since  $Q([X, Y], N) = Q([X, N], Y) = 0$ .  $\square$

If the intermediate subgroup  $K$  is compact, then  $K/H$  is compact and totally geodesic.

After these algebraic and geometric preliminaries we come to our first important result on homogeneous Ricci flow solutions on  $G/H$ , satisfying Assumption 5.3. For a homogeneous Ricci flow solution  $(g(t))_{t \in [0, T)}$  we denote by  $x_{\min}(t)$  the minimum of the eigenvalues  $x_1(t), \dots, x_r(t)$  from (5.2). Notice that for a sequence  $\{t_a\}$  as in Definition 5.6 we have  $x_{\pi_1(n_1)}(t_a) = x_{\min}(t_a)$ .

**Lemma 5.12.** *Let  $(g(t))_{t \in [0, T)}$  be a homogeneous Ricci flow on  $G/H$  with finite extinction time. Suppose that Assumption 5.3 holds and let  $I \subset \{1, \dots, r\}$  be the subset from Lemma 5.5. Suppose that  $\text{scal}(g(0)) = 1$ . Then there exists an  $\varepsilon > 0$  such that for all  $t \in [0, T)$  one has*

$$\text{scal}(g(t)) \geq \varepsilon \cdot \frac{1}{x_{\min}(t)}.$$

*Proof.* If such an  $\varepsilon > 0$  does not exist, then there exists a function  $c : \mathbb{N} \rightarrow \mathbb{N}$  with  $\lim_{a \rightarrow \infty} c(a) = +\infty$  and a sequence  $\{t_a\}$  as in Definition 5.6 converging to  $T$ , such that for all  $a \in \mathbb{N}$  we have  $c(a) \cdot x_{\pi_1(n_1)}(t_a) \leq \bar{x}(a)$ , where

$$\bar{x}(a) := \frac{1}{\text{scal}(g(t_a))}.$$

To simplify notation we permute the moduls  $\mathfrak{p}_1, \dots, \mathfrak{p}_r$  and assume that

$$x_1(t_a) \leq x_2(t_a) \leq \cdots \leq x_r(t_a)$$

for all  $a \in \mathbb{N}$ . That is we have

$$I_1 = \{1, 2, \dots, i_1\}, I_2 = \{i_1 + 1, \dots, i_2\}, \dots, I_\ell = \{i_{\ell-1} + 1, \dots, i_\ell = r\}.$$

In this notation we have  $x_1(t_a) = x_{\min}(t_a)$  and the above estimate reads

$$c(a) \cdot x_1(t_a) \leq \bar{x}(a). \tag{5.9}$$

Notice that we can and will pass to a subsequence of  $\{t_a\}$  whenever this is convenient.

By Lemma 5.8 we know that  $\mathfrak{k}_1 = \mathfrak{h} \oplus \mathfrak{p}_{I_1}$  is an  $\text{Ad}(H)$ -invariant subalgebra of  $\mathfrak{g}$ . Suppose that  $\mathfrak{k}_1$  is nontoral. By Lemma 5.10 there exists an  $\text{Ad}(H)$ -invariant subalgebra  $\tilde{\mathfrak{k}}_1$  of  $\mathfrak{k}_1$ , which is the Lie algebra of a compact subgroup  $\tilde{K}_1$  of  $G$  containing  $H$ . By Lemma 5.11  $\tilde{K}_1/H$  is a compact, totally geodesic submanifold of  $(G/H, g(t_a))$ . We rescale the metric  $g(t_a)$  by  $\text{scal}(g(t_a)) = \frac{1}{\bar{x}_a}$  to obtain the metrics  $\bar{g}_a$ , which converge to a nonflat limit soliton. By the very definition of  $\mathfrak{k}_1$ , the induced metric  $g(t_a)_{\tilde{K}_1/H}$  is given by eigenvalues of order  $x_1(t_a)$ . When rescaled by  $\frac{1}{\bar{x}_a}$  this means by (5.9) that the eigenvalues of  $(\bar{g}_a)_{\tilde{K}_1/H}$  converge to zero for  $a \rightarrow \infty$ . It follows that the injectivity radius of  $(\tilde{K}_1/H, (\bar{g}_a)_{\tilde{K}_1/H})$  converges to zero for  $a \rightarrow \infty$  and consequently also the injectivity radius of  $(G/H, \bar{g}_a)$ . But this is impossible, since the limit soliton has the same dimension as  $G/H$ .

This shows that in the sequel we can assume that  $\mathfrak{k}_1$  is a toral subalgebra of  $\mathfrak{g}$ . By Lemma 5.8 we conclude that  $c \cdot x_{i_1+1}(t_a) \leq \bar{x}_a$  for all  $a \in \mathbb{N}$  and a constant  $c > 0$ . Next, we assume that there exists  $C > 0$  such that  $\bar{x}_a \leq C \cdot x_{i_1+1}(t_a)$  for all  $a \in \mathbb{N}$ . By Lemma 5.9, using that  $\text{Ric}(g_{K_1/H}) = 0$ , we see that the largest positive term in  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_{I_1}}$  is of order  $\frac{\bar{x}_a \cdot x_1(t_a)}{\bar{x}_{i_1+1}(t_a) x_{i_1+1}(t_a)}$ . Since this term converges to zero for  $a \rightarrow \infty$  due to  $\bar{x}_a \leq C \cdot x_{i_1+1}(t_a)$  we deduce  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_{I_1}} \rightarrow 0$  for  $a \rightarrow \infty$ . But this contradicts Corollary 5.7.

This shows that in the sequel we can assume that  $c(a) \cdot x_{i_1+1}(t_a) \leq \bar{x}(a)$ . Since  $\mathfrak{k}_1$  is a toral subalgebra  $\mathfrak{k}_2 = \mathfrak{k}_1 \oplus \mathfrak{p}_{I_2}$  is an  $\text{Ad}(H)$ -invariant subalgebra of  $\mathfrak{g}$  by Lemma 5.8. If  $\mathfrak{k}_2$  is nontoral, then we see as above, that the injectivity radius of  $(G/H, \bar{g}_a)$  converges to zero for  $a \rightarrow \infty$ . Contradiction.

As a consequence we can assume in the sequel that  $\mathfrak{k}_2$  is toral. By Lemma 5.8 we conclude that  $c \cdot x_{i_2+1}(t_a) \leq \bar{x}_a$  for all  $a \in \mathbb{N}$  and a constant  $c > 0$ . Next, we assume that there exists  $C > 0$  such that  $\bar{x}_a \leq C \cdot x_{i_2+1}(t_a)$  for all  $a \in \mathbb{N}$ . By Lemma 5.9, applied to the toral subalgebra  $\mathfrak{k}_2$  we see that the largest positive term in  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_{\mathfrak{k}_2}}$  is of order  $\frac{\bar{x}_a \cdot \bar{x}_{i_1+1}(t_a)}{x_{i_2+1}(t_a) x_{i_2+1}(t_a)}$ . Since this term converges to zero for  $a \rightarrow \infty$  due to  $\bar{x}_a \leq C \cdot x_{i_2+1}(t_a)$  we deduce  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_{\mathfrak{k}_2}} \rightarrow 0$  for  $a \rightarrow \infty$ . Again we obtain a contradiction. By induction the claim follows, since the limit soliton is not flat.  $\square$

Knowing that the scalar curvature of  $\text{scal}(g(t))$  is as large as possible we can deduce the following

**Lemma 5.13.** *Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on  $G/H$  with finite extinction time. Suppose that Assumption 5.3 holds and let  $I \subset \{1, \dots, r\}$  be the subset from Lemma 5.5. Then we have*

- (1) *The space  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_I$  is the Lie algebra of a compact subgroup  $K$  of  $G$  with  $\dim K/H = \dim E_\infty^k$ .*
- (2) *There exists  $C > 0$ , such that for all  $t \geq 0$  and all  $m, \bar{m} \in I$  we have  $|\frac{x_m(t)}{x_{\bar{m}}(t)}| \leq C$ .*
- (3) *The Ricci curvature of  $\bar{g}(t)$  restricted to  $\mathfrak{p}_I$  is up to lower order terms given by the Ricci curvature of  $\text{scal}(g(t)) \cdot g(t)_{K/H}$ .*

*Proof.* Let  $(t_a)_{a \in \mathbb{N}}$  be any sequence as in Definition 5.6. We suppose again, to simplify notation, that  $x_1(t_a) \leq \dots \leq x_r(t_a)$  for all  $a \in \mathbb{N}$  (see Lemma 5.12).

Recall that  $\bar{g}_a = \frac{1}{\bar{x}(a)} \cdot g(t_a)$  and that  $\text{scal}(\bar{g}_a) = 1$  for all  $a \in \mathbb{N}$ . Furthermore, by Lemma 5.8 and Lemma 5.12 there exists constants  $\varepsilon(G/H) > 0$  and  $C > 0$ , such that

$$\varepsilon(G/H) \cdot x_1(t_a) \leq \bar{x}(a) \leq C \cdot x_1(t_a) \tag{5.10}$$

for all  $a \in \mathbb{N}$ . From Lemma 5.8 it follows that  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_{I_1}$  is a nontoral subalgebra of  $\mathfrak{g}$  and that  $[I_1 I_s I_{s'}] = 0$  for  $s \neq s', 1 \leq s, s' \leq \ell$ .

We will show that for  $m \in I_s$  with  $s \geq 2$  we have  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_m} \rightarrow 0$  for  $a \rightarrow \infty$ , using (5.10) and the fact that the limit manifold  $(M_\infty^n, g_\infty)$  has nonnegative Ricci curvature. Notice that by Corollary 5.7 it is sufficient to show this for  $m = i_1 + 1$ . We deduce this from (5.8): We examine for  $m = i_1 + 1$  terms for  $g = g(t_a)$ , which when multiplied by  $x_1(t_a) \approx \bar{x}_a$  can possibly have a positive limit for  $a \rightarrow \infty$ . Since  $[I_1 I_1 I_2] = 0$  no such terms occur in the first sum of (5.8). In the second sum such terms could occur for  $j \in I_1$  and  $k \in I_2$ . However since  $k \geq m = i_1 + 1$  we have, using  $x_m(t_a) \leq x_k(t_a)$ , that  $-\frac{x_k(t_a)}{x_j(t_a)} + \frac{x_m^2(t_a)}{x_j(t_a)x_k(t_a)} \leq 0$ . This means that also such terms cannot force  $\text{Ric}(\bar{g}_a)|_{\mathfrak{p}_m}$  to become positive when  $a \rightarrow \infty$ . We deduce  $I = I_1$  from Corollary 5.7.

Since the above is true for any such sequence  $\{t_a\}$  we deduce that (2) holds. To this end notice, that if (2) does not hold, say along a sequence  $\{t_i\}$  converging to  $T$ , then we may pass to a subsequence as in Definition 5.6. However, since (2) does not hold, we cannot have  $I_1 = I$  for such a subsequence. Contradiction.

We show now the claim (3): To this end we will show below that for any structure constants with  $[ijk] > 0$  and  $i \in I$  and  $j, k \in I^C$  one has  $\frac{x_j(t)}{x_k(t)} \rightarrow 1$  for  $t \rightarrow T$ . It follows from Lemma 5.9 that  $\text{Ric}(\bar{g}(t))|_{\mathfrak{p}_I}$  equals to  $\text{Ric}((\bar{g}(t))_{K/H})$  up to terms of lower order. This shows (3).

Moreover, since we know that  $\text{Ric}(g(t))|_{\mathfrak{p}_I}$  has a uniform positive lower bound for large  $t$  we deduce that the Ricci curvature of the  $g(t)_{K/H}$  on  $K/H$  has a positive lower bound, too. It follows that  $K/H$  is compact, hence  $K$  must be compact as well. This shows (1).

We suppose now that there exist indices with  $[im\tilde{m}] > 0, i \in I$  and  $m \neq \tilde{m} \in I^C$  such that for a subsequence  $\{t_a\}$  converging to  $T$  we have  $x_{\tilde{m}}(t_a) \geq (1 + \varepsilon_0)x_m(t_a)$  for all  $a \in \mathbb{N}$  and some  $\varepsilon_0 > 0$ . As above we assume  $x_1(t_a) \leq \dots \leq x_r(t_a)$ . By the above we have  $I = I_1$ . Moreover,  $\tilde{m} > m$  and  $m, \tilde{m} \in I_{s_0}$  for some  $2 \leq s_0 \leq \ell$  since  $[I_1 I_s I_{s'}] = 0$  for  $s \neq s'$ . We choose  $m$  and hence  $s_0$  minimal.

From (5.8) we deduce that the term  $-[im\tilde{m}] \cdot \frac{x_{\tilde{m}}(t_a)}{x_i(t_a)x_m(t_a)}$  in the formula for  $\text{Ric}(g(t_a))|_{\mathfrak{p}_m}$  is of order  $\frac{1}{x_1(t_a)}$ . Notice that for this term the corresponding positive term  $[im\tilde{m}] \cdot \frac{x_m(t_a)}{x_i(t_a)x_{\tilde{m}}(t_a)}$  in (5.8) is strictly smaller than the first term. This holds true for any such  $i \in I_1$  and any such  $\tilde{m} \in I_{s_0}$ .

Since the Ricci curvature of the limit space is nonnegative there must be further positive terms of order  $\frac{1}{x_1(t_a)}$ . Thus, there are indices  $i, j \in I_1 \cup \dots \cup I_{s_0-1}$  such that  $[ijm] > 0$  and  $\frac{x_m(t_a)}{x_i(t_a)x_j(t_a)}$  is of order  $\frac{1}{x_1(t_a)}$  or larger; indices in  $I_s, s > s_0$ , can obviously not occur and indices in  $I_{s_0}$  are already covered by the above. Assuming  $i \leq j$  we see that we cannot have  $i \in I_1$ , since then  $j \in I_{s_0}$  would follow from  $[I_1 I_s I_{\tilde{s}}] = 0$  for  $s \neq \tilde{s}$ .

We pick the minimal index  $i \in I_{s_1}, 2 \leq s_1 \leq s_0 - 1$ , with that property. From (5.8) we deduce that the term  $-[ijm] \cdot \frac{x_m(t_a)}{x_i(t_a)x_j(t_a)}$  in the formula for  $\text{Ric}(g(t_a))|_{\mathfrak{p}_i}$  is of order  $\frac{1}{x_1(t_a)}$  or larger. As above we conclude that there must also be large positive terms  $[ijk] \cdot \frac{x_i(t_a)}{x_j(t_a)x_k(t_a)}$ . By the minimality of the above  $m \in I_{s_0}$  if these indices are of type  $[I_1 I_{s_1} I_{s_1}]$ , that is  $j \in I_1$  and  $k \in I_{s_1}$ , then we must have  $\frac{x_i(t_a)}{x_k(t_a)} \rightarrow 1$  for  $a \rightarrow \infty$ . From (5.8) we deduce that such terms do not contribute to terms of order  $\frac{1}{x_1(t_a)}$ . It follows as above that  $j, k \in I_2 \cup \dots \cup I_{s_1-1}$ .

Assuming  $j \leq k$  we pick the minimal index  $j \in I_{s_2}, 2 \leq s_2 \leq s_1 - 1$ , with that property. Inductively we may assume  $j \in I_2$ . But the only positive terms which could make up for such a negative term are of type  $[ijk] \cdot \frac{x_j(t_a)}{x_i(t_a)x_k(t_a)}$  where  $i \in I_1$  and  $k \in I_2$ . Hence we have shown that in our above assumption we must have had  $m \in I_2$ . Since  $[I_1 I_1 I_2] = 0$  we obtain a contradiction.  $\square$

Recall also that for the intermediate subgroup  $K$  from Lemma 5.13 we denoted by  $g_{K/H}$  the metric on  $K/H$  induced by a homogeneous metric  $g$  on  $G/H$ .

**Theorem 5.14.** *Let  $(g(t))_{t \in [0, T]}$  be a homogeneous Ricci flow on  $G/H$  with finite extinction time. Suppose that Assumption 5.3 holds. Then there exists a compact, nontoral intermediate subgroup  $K$  such that the metrics*

$$\bar{g}_{K/H}(t) = \text{scal}(g(t)) \cdot g_{K/H}(t)$$

subconverge to an Einstein metric on  $K/H$  for any sequence  $\{t_i\}_{i \in \mathbb{N}}$  converging to  $T$ . Moreover, the limit Einstein factor  $E_\infty^k$  is diffeomorphic to  $K/H$ .

*Proof.* By Lemma 5.13, the eigenvalues of  $\bar{g}(t) = \text{scal}(g(t)) \cdot g(t)$  restricted to  $\mathfrak{p}_\mathfrak{k}$ , where  $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}_I$ , have a uniform positive lower and an upper bound. Moreover,  $\text{Ric}(\bar{g}(t))|_{\mathfrak{p}_\mathfrak{k}}$  equals to  $\text{Ric}((\bar{g}(t))_{K/H})$  up to terms of lower order.

We consider now a normalized Ricci flow, which keeps the volume of  $g_{K/H}(t)$  constant. To this end, let  $\text{scal}_{K/H}(g) := \text{tr}(\text{Ric}(g)|_{\mathfrak{p}_I})$  and  $k := \dim K - \dim H$ . Then the following  $K/H$ -volume normalized Ricci flow

$$\tilde{g}'(t) = -2(\text{ric}(\tilde{g}(t)) - \frac{1}{k} \cdot \text{scal}_{K/H}(\tilde{g}(t)) \cdot \tilde{g}(t)) \tag{5.11}$$

is equivalent to the Ricci flow on  $G/H$  and leaves the volume of  $g_{K/H}(t)$  constant. This follows the same way proving that unnormalized Ricci flow and Ricci flow are equivalent. The solution  $\tilde{g}(t)$  can be obtained (up to parametrization) from a solution  $g(t)$  to the Ricci flow by rescaling to keep the volume  $\tilde{g}_{K/H}(t)$  constant. It follows from Lemma 5.13, (2) that for a solution  $\tilde{g}(t)$  the eigenvalues of  $\tilde{g}(t)$  restricted to  $\mathfrak{p}_\mathfrak{k}$  are bounded uniformly. As a consequence, for any sequence of times  $\{t_i\}$  converging to  $T$  there exists a subsequence  $\{t_a\}$  such that  $g_{K/H}(t_a)$  converges to a limit metric  $\tilde{g}_{K/H}^\infty$  on  $K/H$ . Using Lemma 5.13 and Lemma 5.5 again, we conclude that  $\tilde{g}_{K/H}^\infty$  must be an Einstein metric with positive scalar curvature. Since by Lemma 5.12 the scalar curvature  $\text{scal}(g(t))$  is of order  $\frac{1}{x_{\min}(t)}$ , the metrics  $\bar{g}_{K/H}(t_a)$  on  $K/H$  converge to a limit Einstein metric, too. This shows the first claim.

Finally, we show that the homogeneous spaces  $(K/H, \bar{g}_{K/H}(t_a))$  converge to the Einstein manifold  $(E_\infty^k, g_\infty^1)$ . We use that  $(G/H, \bar{g}(t_a))$  converges to  $(M_\infty^n, g_\infty)$  in pointed  $C^\infty$ -topology. It follows that the tangent spaces of the totally geodesic submanifolds  $(K/H, \bar{g}_{K/H}(t_a))$  converge to the tangent space of the Einstein factor  $(E_\infty^k, g_\infty^1)$ , since these are precisely the directions where the Ricci curvature is positive. As a consequence  $(K/H, \bar{g}_{K/H}(t_a))$  converges to  $(E_\infty^k, g_\infty^1)$  in  $C^\infty$ -topology. By Theorem 1.1 in [10] the limit Einstein factor  $E_\infty^k$  is diffeomorphic to  $K/H$ .  $\square$

There are two easy consequence of the above convergence result:

**Corollary 5.15.** *Under the assumptions of Theorem 5.14 we have: If on  $K/H$  there exist only finitely many solutions to the homogeneous Einstein equation, then  $\bar{g}_{K/H}(t)$  converges for  $t \rightarrow T$  to an Einstein metric on  $K/H$ .*

*Proof.* If there are only finitely many solutions to the Einstein equation on  $K/H$ , then  $\bar{g}_{K/H}(t)$  must of course converge to one of those metrics by Theorem 5.14.  $\square$

**Corollary 5.16.** *Under the assumptions of Theorem 5.14 we have: It there exists a solution  $(g(t))_{t \in [0, T)}$  such that  $\bar{g}_{K/H}(t)$  does non converge to an Einstein metric on  $K/H$  for  $t \rightarrow T$ , then on  $K/H$  there exist infinitely many solutions to the homogeneous Einstein equation.*

## 6. Locally homogeneous Ricci flat metrics are flat

Recall the following result

**Theorem 6.1** (Alekseevskii, Kimel'fel'd [2]). *Let  $(M^n, g)$  be a complete, homogeneous Ricci flat manifold. Then  $(M^n, g)$  is flat.*

Notice that there is a very short proof using the Cheeger–Gromoll splitting theorem (cf. [6], p. 191). Later, Spiro observed in [41] that the above theorem is of local nature. For convenience, we provide a proof of Spiro's result following Spiro's original approach.

We recall that a Riemannian manifold  $(M^n, g)$ , not necessarily complete, is called locally homogeneous, if for any two points  $p, q \in M^n$  there exists a local isometry mapping  $p$  to  $q$ . Each locally homogeneous space is uniquely described by its so called infinitesimal model. The Nomizu construction associates to each infinitesimal model a Lie algebra  $\mathfrak{g}$ , a subalgebra  $\mathfrak{h}$  and a reductive decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  (see [42]). Notice that  $\mathfrak{g}$  is the Lie algebra of the full local isometry group of the Riemannian metric  $g$  on  $M^n$ .

Since there might be also smaller local Lie groups acting locally transitively on  $M^n$  the following algebraic definition of a locally homogeneous space has been introduced, for instance in [41] or in [29]:

Let  $G$  and  $H$  be connected Lie groups with  $G$  simply connected and  $H \subset G$ . Let  $\mathfrak{g}$  and  $\mathfrak{h}$  denote the Lie algebras of  $G$  and  $H$ , respectively. We call  $G/H$  a locally homogeneous space, if the following conditions hold:

- (h1) There exists an  $\text{Ad}(H)$ -invariant complement  $\mathfrak{p}$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ .
- (h2) There exists an  $\text{Ad}(H)$ -invariant scalar product  $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$  on  $\mathfrak{p}$ .
- (h3) The Lie algebra  $\mathfrak{h}$  does not contain any nontrivial ideal of  $\mathfrak{g}$ .

Notice that (h1) is equivalent to  $\text{ad}(x)(v) \in \mathfrak{p}$  for all  $x \in \mathfrak{h}$  and all  $v \in \mathfrak{p}$ , where  $\text{ad}(x)(v) = [x, v]$  just denotes the Lie bracket of  $\mathfrak{g}$ . Condition (h2) is equivalent to  $\langle \text{ad}(x)(v), w \rangle_{\mathfrak{p}} = -\langle v, \text{ad}(v)(w) \rangle_{\mathfrak{p}}$  for all  $x \in \mathfrak{h}$  and all  $v, w \in \mathfrak{p}$ . Notice that (h3) ensures that the locally homogeneous space  $G/H$  is almost effective.

Having such an algebraic locally homogeneous space one may determine a locally homogeneous metric  $g$  on a local quotient  $M^n$  as described in [41].

The space  $G/H$  is regular or globally homogeneous, if  $H$  is a closed subgroup of  $G$ . By [25] there exist locally homogeneous spaces which are not globally homogeneous.

**Theorem 6.2** (Spiro [41]). *Let  $(X^n, g)$  be a locally homogeneous Ricci-flat manifold. Then  $(X^n, g)$  is flat.*

*Proof.* Let  $G, H$  be as above and let the homogeneous metric  $g$  be induced by an  $\text{Ad}(H)$ -invariant scalar product  $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$  on an  $\text{Ad}(H)$ -invariant complement  $\mathfrak{p}$  of  $\mathfrak{h}$  in  $\mathfrak{g}$ . We assume that  $H$  is not a closed subgroup of  $G$ .

We define a scalar product on  $\mathfrak{h}$  as follows:

$$\langle x, y \rangle_{\mathfrak{h}} := -\operatorname{tr}_{\mathfrak{p}}(\operatorname{ad}(x)|_{\mathfrak{p}} \circ \operatorname{ad}(y)|_{\mathfrak{p}}).$$

We claim, that  $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$  is  $\operatorname{Ad}(H)$ -invariant. This is seen as follows: By (h2) we have

$$\langle x, x \rangle_{\mathfrak{h}} = \sum \| [x, e_i] \|_{\mathfrak{p}}^2$$

for any orthonormal basis  $(e_1, \dots, e_n)$  of  $\mathfrak{p}$ . We need to show that for  $v \neq 0$  the right hand side is positive. If not, then a short computation shows that the kernel  $\mathfrak{k}$  of  $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$  is  $\operatorname{Ad}(H)$ -invariant and  $[\mathfrak{k}, \mathfrak{p}] = 0$ . Hence,  $\mathfrak{k}$  is a nontrivial ideal of  $\mathfrak{g}$ , which was excluded by (h3).

The  $\operatorname{Ad}(H)$ -invariant scalar product  $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$  and  $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$  on  $\mathfrak{h}$  and  $\mathfrak{p}$  respectively, induce an  $\operatorname{Ad}(H)$ -invariant scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$  by requiring  $\langle \mathfrak{h}, \mathfrak{p} \rangle = 0$ . Notice that for  $x \in \mathfrak{h}$   $\operatorname{ad}(x)$  acts skew symmetrically on  $\mathfrak{g}$  with respect to this scalar product.

Let  $\bar{H}$  denote the closure of  $H$  in  $G$  and let  $\bar{\mathfrak{h}}$  denote its Lie algebra. Furthermore let  $\mathfrak{h}^{\perp}$  denote the orthogonal complement of  $\mathfrak{h}$  in  $\bar{\mathfrak{h}}$ . Since  $\langle \cdot, \cdot \rangle$  is  $\operatorname{Ad}(H)$ -invariant it is  $\operatorname{Ad}(\bar{H})$ -invariant as well. Consequently,  $\operatorname{ad}(\bar{x})$  acts skew symmetrically on  $\mathfrak{g}$  for  $\bar{x} \in \mathfrak{h}^{\perp}$ .

We turn now to curvature computations. We denote by  $g_G$  the  $G$ -invariant metric on  $G$ , which corresponds to  $\langle \cdot, \cdot \rangle$  and by  $g_H$  the  $H$ -invariant metric on  $H$ , which corresponds to  $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ .

By Lemma 2.1 of [32], for  $\bar{x} \in \mathfrak{h}^{\perp}$  we have  $\operatorname{Ric}_{g_G}(\bar{x}, \bar{x}) \geq 0$  and  $\operatorname{Ric}_{g_G}(\bar{x}, \bar{x}) = 0$  if and only if  $\langle \bar{x}, [v, w] \rangle = 0$  for all  $v, w \in \mathfrak{g}$ . Using the O'Neill formula for the Riemannian submersion  $(H, g_H) \rightarrow (G, g_G) \rightarrow (G/H, g)$  with totally geodesic fibers (cf. [6], (9.36c), [6] 9.80 – this computation works also for locally homogeneous spaces), we conclude that  $\operatorname{Ric}_{g_G}(\bar{x}, \bar{x}) = 0$ , since  $\operatorname{Ric}_g \equiv 0$  by assumption.

It follows that  $\mathfrak{h}^{\perp} \perp [\mathfrak{g}, \mathfrak{g}]$ . Let  $\bar{\mathfrak{p}}$  denote the orthogonal complement of  $\bar{\mathfrak{h}}$  in  $\mathfrak{g}$ . Then  $[\mathfrak{g}, \mathfrak{h} \oplus \bar{\mathfrak{p}}] \perp \mathfrak{h}^{\perp}$ , that is  $\mathfrak{h} \oplus \bar{\mathfrak{p}}$  is an ideal in  $\mathfrak{g}$ . As is well known this ideal corresponds to a closed normal connected subgroup  $N$  of  $G$ , since  $G$  is simply connected (cf. [38], p.81). Since  $N$  is closed and  $H \subset N$  we would conclude  $\bar{H} \subset N$ . This is of course a contradiction, meaning that  $H$  must have been closed.  $\square$

It should be mentioned that Spiro's result shows that any locally homogeneous space with nonpositive Ricci curvature comes from a global homogeneous space. Notice also that by [43] the space of globally homogeneous spaces is dense in the space of locally homogeneous spaces.

**References**

- [1] S. Anastassiou and I. Chrysikos, The Ricci flow approach to homogeneous Einstein metrics on flag manifolds, *J. Geom. Phys.* **61** (2011), 1587–1600. [Zbl 1221.53095](#) [MR 2802495](#)
- [2] D. V. Alekseevskii and B. N. Kimel'fel'd, Structure of homogeneous Riemannian spaces with zero Ricci curvature, *Funkcional. Anal. i Priložen.* **9** (1975), no. 2, 5–11. [Zbl 0316.53041](#) [MR 402650](#)
- [3] R. Arroyo, The Ricci flow in a class of solvmanifolds, *Differential Geom. Appl.* **31** (2013), 472–485. [Zbl 1279.53061](#) [MR 3066027](#)
- [4] I. Bakas, S. Kong and L. Ni, Ancient solutions of Ricci flow on spheres and generalized Hopf fibrations, *J. Reine Angew. Math.* **663** (2012), 209–248. [Zbl 1269.53065](#) [MR 2889711](#)
- [5] L. Bérard-Bergery, Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes, *Ann. Sci. École Norm. Sup.* **11** (1978), 543–576. [Zbl 0426.53038](#) [MR 533067](#)
- [6] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1987. [Zbl 0613.53001](#) [MR 867684](#)
- [7] S. Bochner, Curvature and Betti numbers, *Ann. of Math.* **49** (1948), 379–390. [Zbl 0038.34401](#) [MR 25238](#)
- [8] C. Böhm, Homogeneous Einstein metrics and simplicial complexes, *J. Differential Geom.* **67** (2004), 79–165. [Zbl 1098.53039](#) [MR 2153482](#)
- [9] C. Böhm, Non-existence of homogeneous Einstein metrics, *Comment. Math. Helv.* **80** (2005), 123–146. [Zbl 1083.53046](#) [MR 2130570](#)
- [10] C. Böhm, M. Wang and W. Ziller, A variational approach for compact homogeneous Einstein manifolds, *Geom. Funct. Anal.* **4** (2004), 681–733. [Zbl 1068.53029](#) [MR 2084976](#)
- [11] C. Böhm, B. Wilking, Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature, *Geom. Funct. Anal.* **17** (2007), 665–681. [Zbl 1132.53035](#) [MR 2346271](#)
- [12] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, 46. Academic Press, New York-London, 1972. [Zbl 0246.57017](#) [MR 413144](#)
- [13] M. Buzano, Ricci flow on homogeneous spaces with two isotropy summands, *Ann. Global Anal. Geom.* **45** (2014), 25–45. [Zbl 1285.53051](#) [MR 3152086](#)
- [14] X. Cao and L. Saloff-Coste, Backward Ricci flow on locally homogeneous 3-manifolds, *Comm. Anal. Geom.* **17** (2009), 305–325. [Zbl 1184.53069](#) [MR 2520911](#)
- [15] B. Chow, P. Lu and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006. [Zbl 1118.53001](#) [MR 2274812](#)
- [16] P. Daskalopoulos and R. Hamilton, Geometric estimates for the logarithmic fast diffusion equation, *Comm. Anal. Geom.* **12** (2004), 143–164. [Zbl 1070.53041](#) [MR 2074874](#)
- [17] J. Enders, R. Müller and P. Topping, On type-I singularities in Ricci flow, *Comm. Anal. Geom.* **19** (2011), no. 5, 905–922. [Zbl 1244.53074](#) [MR 2886712](#)

- [18] D. Glickenstein, Precompactness of solutions to the Ricci flow in the absence of injectivity radius estimates, *Geom. Topol.* **7** (2003), 487–510. [Zbl 1044.53048](#) [MR 2026540](#)
- [19] H.-L. Gu and X.-P. Zhu, The existence of type II singularities for the Ricci flow on  $S^{n+1}$ , *Comm. Anal. Geom.* **18** (2008), 467–494. [Zbl 1152.53054](#) [MR 2429966](#)
- [20] R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.* **17** (1982), 255–306. [Zbl 0504.53034](#) [MR 664497](#)
- [21] J. Isenberg and M. Jackson, Ricci flow of locally homogeneous geometries on closed manifolds, *J. Differential Geom.* **35** (1992), no. 3, 723–741. [Zbl 0808.53044](#) [MR 1163457](#)
- [22] J. Isenberg, M. Jackson and P. Lu, Ricci flow on locally homogeneous closed 4-manifolds, *Comm. Anal. Geom.* **14** (2006), 345–386. [Zbl 1121.53045](#) [MR 2255014](#)
- [23] J. R. King, Self-similar behavior for the equation of fast nonlinear diffusion, *Phil. Trans. R. Soc. A*, **343** (1993), 337–375.
- [24] R. Kotschwar, Backwards uniqueness for the Ricci flow, *Int. Math. Res. Not. IMRN* **21** (2010), 4064–4097. [Zbl 1211.53086](#) [MR 2738351](#)
- [25] O. Kowalski, Counter-Example to the Second Singer’s Theorem, *Ann. Global Anal. Geom.* **8** (1990), no. 2, 211–214. [Zbl 0736.53047](#) [MR 1088512](#)
- [26] R. Lafuente, Scalar curvature behavior of homogeneous Ricci flows, to appear in *Journal of Geometric Analysis*, 2014.
- [27] R. Lafuente and J. Lauret, On homogeneous Ricci solitons, *Q. J. Math.* **65** (2014), 399–419. [Zbl 1301.53045](#) [MR 3230368](#)
- [28] J. Lauret, The Ricci flow for simply connected nilmanifolds, *Comm. Anal. Geom.* **19** (2011), 831–854. [Zbl 1244.53077](#) [MR 2886709](#)
- [29] J. Lauret, Ricci flow of homogeneous manifolds, *Math. Z.* **274** (2013), 373–403. [Zbl 1272.53055](#) [MR 3054335](#)
- [30] J. Lott, On the long time behaviour of type-III Ricci flow solutions, *Math. Annalen* (2007), no. 3, 627–666. [Zbl 1135.53046](#) [MR 2336062](#)
- [31] J. Lott, Dimensional reduction and the long-time behavior of Ricci flow, *Comment. Math. Helv.* **85** (2010), 485–534. [Zbl 1195.53094](#) [MR 2653690](#)
- [32] J. Milnor, Curvatures of left invariant metrics on Lie groups, *Advances in Math.* **21** (1976), no. 3, 293–329. [Zbl 0341.53030](#) [MR 425012](#)
- [33] A. Naber, Noncompact shrinking four solitons with nonnegative curvature, *J. Reine Angew. Math.* **645** (2010), 125–153. [Zbl 1196.53041](#) [MR 2673425](#)
- [34] J.-S. Park and Y. Sakane, Invariant Einstein metrics on certain homogeneous spaces, *Tokyo J. Math.* **20** (1997), 51–61. [Zbl 0884.53039](#) [MR 1451858](#)
- [35] T. L. Payne, The Ricci flow for nilmanifolds, *J. Mod. Dyn.* **4** (2010), 65–90. [Zbl 1206.53074](#) [MR 2643888](#)
- [36] F. Prüfer, F. Tricerri and L. Vanhecke, Curvature invariants, differential operators and local homogeneity, *Trans. Amer. Math. Soc.* **348** (1996), 4643–4652. [Zbl 0867.53032](#) [MR 1363946](#)
- [37] P. Petersen and W. Wylie, On gradient Ricci solitons with symmetry, *Proc. Amer. Math. Soc.* **137** (2009), no. 6, 2085–2092. [Zbl 1168.53021](#) [MR 2480290](#)

- [38] C. Procesi, *Lie groups. An approach through invariants and representations*, Universitext. Springer, New York, 2007. [Zbl 1154.22001](#) [MR 2265844](#)
- [39] E. D. Rodionov, Einstein metrics on a class of five-dimensional homogeneous spaces, *Comment. Math. Univ. Carolin.* **32** (1991), 389–393. [Zbl 0747.53037](#) [MR 1137801](#)
- [40] Y. Sakane, Homogeneous Einstein Metrics on flag manifolds, *Lobach. Jour. of Math.* **4** (1999), 71–87. [Zbl 0953.53029](#) [MR 1743146](#)
- [41] A. Spiro, A remark on locally homogeneous Riemannian spaces, *Results Math.* **24** (1993), no. 3–4, 318–325. [Zbl 0795.53016](#) [MR 1244285](#)
- [42] F. Tricerri, Locally homogeneous Riemannian manifolds, *Rend. Sem. Mat. Univ. Politec. Torino* **50** (1993), 411–426. [Zbl 0793.53056](#) [MR 1261452](#)
- [43] K. Tsukada, The moduli space of locally homogeneous spaces and locally homogeneous spaces which are not locally isometric to globally homogeneous spaces, *Differential geometry and applications (Brno, 1998)*, 149–159, Masaryk Univ., Brno, 1999. [Zbl 0951.53034](#) [MR 1707579](#)
- [44] M. Wang and W. Ziller, Existence and non-existence of homogeneous Einstein metrics, *Invent. Math.* **84** (1986), 177–194. [Zbl 0596.53040](#) [MR 830044](#)

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