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On Welschinger invariants of symplectic 4-manifolds

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Abstract. We prove the vanishing of many Welschinger invariants of real symplectic 4-manifolds. In some particular instances, we also determine their sign and show that they are divisible by a large power of 2. Those results are a consequence of several relations among Welschinger invariants obtained by a real version of symplectic sum formula. In particular, this note contains proofs of results announced in [4].

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1. Introduction

A *real symplectic manifold* (X, ω, τ) is a symplectic manifold (X, ω) equipped with an antisymplectic involution τ. The *real part* of (X, ω, τ), denoted by $\mathbb{R}X$, is by definition the fixed point set of τ. We say that an almost complex structure J tamed by ω is τ-compatible if τ is J-antiholomorphic, i.e. $J \circ d\tau = -d\tau \circ J$.

Let $X_{\mathbb{R}} = (X, \omega, \tau)$ be a real symplectic manifold of dimension 4. Let C be an immersed real rational J-holomorphic curve in X for some τ-compatible almost complex structure J, and denote by L the connected component of $\mathbb{R}X$ containing the 1-dimensional part $\widetilde{\mathbb{R}C}$ of $\mathbb{R}C$. Fix also a τ-invariant class F in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. Any half of $C \setminus \widetilde{\mathbb{R}C}$ defines a class $\underline{C}$ in $H_2(X, L; \mathbb{Z}/2\mathbb{Z})$ whose intersection number modulo 2 with F, denoted by $\underline{C} \cdot F$, is well defined and does not depend on the chosen half. We further denote by $m(C)$ the number of nodes of C in L with two τ-conjugated branches, and we define the F-mass of C as

$$m_{L,F}(C) = m(C) + \underline{C} \cdot F.$$

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Choose a connected component L of $\mathbb{R}X$, a class $d \in H_2(X;\mathbb{Z})$, and $r, s \in \mathbb{Z}_{\geq 0}$ such that

$$c_1(X)\cdot d - 1 = r + 2s.$$

Choose a configuration $\underline{x}$ made of r points in L and s pairs of τ-conjugated points in $X \setminus \mathbb{R}X$. Given a τ-compatible almost complex structure J, we denote by $\mathcal{C}(d, \underline{x}, J)$ the set of real rational J-holomorphic curves in X realizing the class d, passing through $\underline{x}$, and such that L contains $\widetilde{\mathbb{R}C}$. For a generic choice of J, the set $\mathcal{C}(d, \underline{x}, J)$ is finite, and the integer

$$W_{X_{\mathbb{R}},L,F}(d,s) = \sum_{C\in\mathcal{C}(d,\underline{x},J)} (-1)^{m_{L,F}(C)}$$

depends neither on $\underline{x}$, J, nor on the deformation class of $X_{\mathbb{R}}$ (see [14, 25])[1]. We call these numbers the *Welschinger invariants of* $X_{\mathbb{R}}$. When $F = [\mathbb{R}X \setminus L]$, we simply denote $W_{X_{\mathbb{R}},L}(d,s)$ instead of $W_{X_{\mathbb{R}},L,[\mathbb{R}X\setminus L]}(d,s)$. Note that Welschinger invariants are non-trivial to compute only in the case of rational manifolds.

A *real Lagrangian sphere* of $X_{\mathbb{R}}$ is a Lagrangian sphere globally invariant under τ. Two disjoint surfaces S and S' in X are said to be *connected by a chain of real Lagrangian spheres* if there exists real Lagrangian spheres $S_1, \dots, S_k$ in X such that $S_i \cap S_j = \emptyset$ if $|i-j| \geq 2$, and S_i and S_{i+1} intersect transversely in a single point, as well as S and S_0, and S' and S_k.

The next two theorems are the main results of this note.

Theorem 1.1. *Let $X_{\mathbb{R}}$ be a real symplectic 4-manifold, and suppose that F has a τ-invariant representative connected to L by a chain of real Lagrangian spheres.*

(1) *If $r \geq 2$, then*

$$W_{X_{\mathbb{R}},L,F}(d,s) = 0.$$

(2) *If $r = 1$ and $c_1(X)\cdot d \geq 2$, then*

$$2^{\frac{c_1(X)\cdot d - 4}{2}} \mid W_{X_{\mathbb{R}},L,F}(d,s).$$

If in addition $F = [\mathbb{R}X \setminus L]$, then

$$(-1)^{\frac{d^2 - c_1(X)\cdot d + 2}{2}} W_{X_{\mathbb{R}},L}(d,s) \geq 0.$$

[1] Welschinger originally considered in [25] only the case when $F = [\mathbb{R}X \setminus L]$. In this case $m_{L,F}(C)$ is the number of solitary nodes of $\mathbb{R}C$. Later, Itenberg, Kharlamov, and Shustin observed in [14] that Welschinger's proof extends literally to arbitrary τ-invariant classes in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. See also [9] for a related discussion.

Note that our convention differs slightly from [14], where the sign of a curve in $\mathcal{C}(d, \underline{x}, J)$ depends on the parity of $m(C) + \underline{C}\cdot(F + [\mathbb{R}X \setminus L])$ instead of $m(C) + \underline{C}\cdot(F)$.

Theorem 1.1 is an immediate consequence of Theorem 2.3 and Corollary 2.6 respectively given in Sections 2.3 and 2.4. The invariant $W_{X_{\mathbb{R}},L,0}(d,s)$ does not seem to satisfy a vanishing statement analogous to Theorem 1.1(1) (see [5, 11, 14]), implying that the set $\mathcal{C}(d,\underline{x},J)$ is usually non-empty. Theorem 1.1(2) partially generalizes [24, Theorems 1.1, 2.1, 2.2, and 2.3] and [5, Proposition 8.2].

Theorem 1.1 can be specialized to real algebraic rational surfaces, whose classification is well known (see [17, 21] for example). A real algebraic rational surface is always implicitly assumed to be equipped with some Kähler form.

Let $\mathcal{G}$ be the subgroup of the τ-invariant classes in $H_2(X\setminus L;\mathbb{Z}/2\mathbb{Z})$ generated by the kernel of the natural map $H_2(X\setminus L;\mathbb{Z}/2\mathbb{Z})\to H_2(X;\mathbb{Z}/2\mathbb{Z})$, and by the classes realized by smooth real symplectic curves with either positive genus or self-intersection at least -1. We show in Propositions 4.2 and 4.3 that $W_{X_{\mathbb{R}},L,F}$ and $W_{X_{\mathbb{R}},L,F'}$ are equal in absolute value if $F-F'\in\mathcal{G}$. We denote by $\mathcal{H}(X_{\mathbb{R}},L)$ the group of τ-invariant classes in $H_2(X\setminus L;\mathbb{Z}/2\mathbb{Z})$ quotiented by $\mathcal{G}$. All groups $\mathcal{H}(X_{\mathbb{R}},L)$ are computed in the case of real algebraic rational surfaces in Section 4. In particular, we prove in Proposition 4.8 that they only depend on a minimal model of $X_{\mathbb{R}}$ and on the choice of L.

Theorem 1.2. *Let $X_{\mathbb{R}}$ be a real symplectic 4-manifold equal, up to deformation and equivariant symplectomorphism, to a real algebraic rational surface, and suppose that F is non-zero in $\mathcal{H}(X_{\mathbb{R}},L)$. Then the conclusions of Theorem 1.1 hold in the following cases:*

- *$X_{\mathbb{R}}$ is obtained from a minimal model by blowing up pairs of complex conjugated points and real points on at most two connected components of $\mathbb{R}X$, one of them being L;*
- *$X_{\mathbb{R}}$ is a Del Pezzo surface;*
- *$F=[\mathbb{R}X\setminus L]$.*

Remark. In a burst of enthusiasm, we forgot in [4, Proposition 3.3] the assumption that $X_{\mathbb{R}}$ has to be symplectomorphic/deformation equivalent to a real algebraic rational surface.

Theorem 1.2 follows from the classification of real algebraic rational surfaces and Theorem 1.1, which in its turn is a direct consequence of Theorem 2.3 and Corollary 2.6 below. Our strategy to prove these latters is to degenerate $X_{\mathbb{R}}$ into a reducible real symplectic manifold $X_{\sharp,\mathbb{R}}$, and to relate enumeration of curves in $X_{\sharp,\mathbb{R}}$ and in $X_{\mathbb{R}}$. This degeneration can be thought as a degeneration of $X_{\mathbb{R}}$ to a real nodal symplectic manifold, and can be described by the contraction of a real Lagrangian sphere S_V by stretching the neck of a τ-compatible almost complex structure in a neighborhood of S_V (see [8, 24]). In this note we use an equivalent description in terms of symplectic sum ([10, 15]), see section 2.2 for more details.

In particular, Corollary 2.6 follows from Theorem 2.5, which can be seen as a real version of the Abramovich–Bertram–Vakil formula [1, Theorem 3.1.1], [23, Theorem 4.5]. Another but related treatment of contraction of Lagrangian spheres contained in $\mathbb{R}X$ has previously been proposed by Welschinger in [24].

The paper is organized as follows. We state Theorems 2.3 and 2.5 in Section 2, and give their proof in Section 3 using a real version of the symplectic sum formula. We end this paper by explicit computations in the case of real algebraic rational surfaces in Section 4.

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2. Auxiliary results

2.1. Preliminaries. In the whole text, we denote by $X_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$, by ω_{FS} the Fubini–Study form on $\mathbb{C}P^n$, and by l_1 and l_2 respectively the homology classes $[\mathbb{C}P^1 \times \{0\}]$ and $[\{0\} \times \mathbb{C}P^1]$ in $H_2(X_0;\mathbb{Z})$. Recall that $H_2(X_0;\mathbb{Z})$ is the free abelian group generated by l_1 and l_2. Up to conjugation by an automorphism, there exist four different real structures on $(\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$, and the class $l_1 + l_2$ is invariant for exactly three of them, see for example [17, 21]. These latter are given in coordinate by:

- $\tau_{hy}(z,w) = (\overline{z}, \overline{w})$, $\mathbb{R}X_{hy} = S^1 \times S^1$;
- $\tau_{el}(z,w) = (\overline{w}, \overline{z})$, $\mathbb{R}X_{el} = S^2$;
- $\tau_{em}(z,w) = (-\frac{1}{\overline{z}}, -\frac{1}{\overline{w}})$, $\mathbb{R}X_{em} = \emptyset$.

Note that τ_{hy} and τ_{em} act trivially on $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$, while τ_{el} exchanges the classes l_1 and l_2. Note also, with the convention that $\chi(\emptyset) = 0$, that

$$\chi(\mathbb{R}X_{hy}) = \chi(\mathbb{R}X_{em}) = 0, \quad \text{and} \quad \chi(\mathbb{R}X_{el}) = 2.$$

Lemma 2.1. *Let E be a smooth symplectic curve in $(X_0, \omega_{FS} \oplus \omega_{FS})$ realizing the class $l_1 + l_2$ in $H_2(X_0;\mathbb{Z})$. The group $H_2(X_0 \setminus E;\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and is generated by any representative disjoint from E of the class $l_1 + l_2$ in $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$.*

Proof. The first Chern class of $(X_0, \omega_{FS} \oplus \omega_{FS})$ is dual to $l_1 + l_2$. Hence it follows from the adjunction formula [19, Chapter 2] that E is an embedded sphere. The lemma can be proved exactly as Lemma 4.1, nevertheless we provide an alternate proof.

Let J be an almost complex structure on X_0 tamed by $\omega_{FS} \oplus \omega_{FS}$ such that E is J-holomorphic. Since both classes l_1 and l_2 have the same symplectic area, a class al_1+bl_2 has positive symplectic area if and only if $a+b > 0$. As a consequence, any J-holomorphic curve realizing the class l_1 is an embedded sphere. The Gromov–Witten invariant of $(X_0, \omega_{FS} \oplus \omega_{FS})$ for the class l_1 is equal to 1, and $l_1^2 = 0$, so there exists a unique J-holomorphic sphere realizing the class l_1 and passing through any given point of X_0. Recall that any intersection of two distinct J-holomorphic curves is positive. Since $[E]\cdot l_1 = 1$, we deduce a S^2-fibration $X_0 \to E$ whose fiber over a point $p \in E$ is the J-holomorphic sphere realizing the class l_1 and passing through p. In its turn, this induces a $\mathbb{R}^2$-fibration $X_0 \setminus E \to E$, and so $X_0 \setminus E$ has the same homotopy type than E. This proves that $H_2(X_0 \setminus E; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Since $[E]^2 = 2$ in X_0, there exists a representative F of the class $l_1 + l_2$ in $H_2(X_0; \mathbb{Z}/2\mathbb{Z})$ disjoint from E. The class $[F]$ is obviously non-zero in $H_2(X_0 \setminus E; \mathbb{Z}/2\mathbb{Z})$, and so generates the group. □

Lemma 2.2. *Suppose that E is a smooth real symplectic curve in $(X_0, \omega_{FS} \oplus \omega_{FS}, \tau_{el})$ realizing the class $l_1 + l_2$ in $H_2(X_0; \mathbb{Z})$, and that $\mathcal{D}$ is an embedded τ_{el}-invariant disk with $\partial\mathcal{D} \subset E$. Then the group $H_2(X_0, E; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by $\mathcal{D}$.*

Proof. Recall that E is an embedded sphere. The long exact sequence of pairs gives the exact sequence

$$H_2(E; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} H_2(X_0; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{j} H_2(X_0, E; \mathbb{Z}/2\mathbb{Z}) \to 0.$$

The map i is clearly injective, so $H_2(X_0, E; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and generated by $j(l_1) = j(l_2)$.

Denote by D_1 and D_2 the two halves of $E \setminus \partial\mathcal{D}$. Since E is a real symplectic curve, the involution τ_{el} exchanges D_1 and D_2. The surface $D_i \cup \mathcal{D}$ realizes a class in $H_2(X_0; \mathbb{Z}/2\mathbb{Z})$, and we have

$$l_1 + l_2 = [D_1 \cup D_2] = [D_1 \cup \mathcal{D}] + [D_2 \cup \mathcal{D}] \quad \text{in } H_2(X_0; \mathbb{Z}/2\mathbb{Z}).$$

Since τ_{el} exchanges the classes $[D_1 \cup \mathcal{D}]$ and $[D_2 \cup \mathcal{D}]$, both of them are non-null, i.e. $[D_1 \cup \mathcal{D}] = l_i$ and $[D_2 \cup \mathcal{D}] = l_{3-i}$. Hence by the long exact sequence of pairs, the class realized by $D_1 \cup \mathcal{D}$ in $H_2(X_0, E; \mathbb{Z}/2\mathbb{Z})$, which equals the class realized by $\mathcal{D}$, generates the group. □

Example. In the case when $X_0 \setminus E$ is the affine quadric with equation $x^2 + y^2 + z^2 = 1$ in $\mathbb{C}^3$, the sphere $X_0 \cap \mathbb{R}^3$ is an example of generator of $H_2(X_0 \setminus E; \mathbb{Z}/2\mathbb{Z})$, and the disk $X_0 \cap (i\mathbb{R} \times i\mathbb{R} \times \mathbb{R}_{>0})$ is an example of generator of $H_2(X_0, E; \mathbb{Z}/2\mathbb{Z})$, see Figure 2c.

2.2. Vanishing Lagrangian spheres. Let $X_{\mathbb{R}} = (X, \omega, \tau)$ be a real symplectic manifold of dimension 4. A class V in $H_2(X; \mathbb{Z}/2\mathbb{Z})$ is called a *real vanishing cycle* if it can be represented by a real Lagrangian sphere S_V. By stretching the neck of a τ-compatible almost complex structure in a neighborhood of S_V, one decomposes X into the union of $X \setminus S_V$ and T^*S_V. This operation can be thought as a degeneration of $X_{\mathbb{R}}$ to a real nodal symplectic manifold for which V is precisely the vanishing cycle. Equivalently, the class V is a real vanishing cycle if and only if, up to deformation, $X_{\mathbb{R}}$ can be represented as the real symplectic sum of two real symplectic manifolds (X_1, ω_1, τ_1) and $(X_0, \omega_{FS} \oplus \omega_{FS}, \tau_0)$ along an embedded symplectic sphere E of self-intersection -2 in X_1 (hence of self-intersection 2 in X_0) where:

- E is real and realizes the class $l_1 + l_2$ in $H_2(X_0; \mathbb{Z})$;
- V is represented by the deformation in X of a representative of the non-trivial class in $H_2(X_0 \setminus E; \mathbb{Z}/2\mathbb{Z})$.

By abuse, we still denote by V the non-trivial class in $H_2(X_0 \setminus E; \mathbb{Z}/2\mathbb{Z})$. We refer to Section 3.2 for more details about the symplectic sum operation. We denote by $X_\sharp$ the union of (X_1, ω_1, τ_1) and $(X_0, \omega_{FS} \oplus \omega_{FS}, \tau_0)$ along E, by $L_\sharp$ the degeneration of L as $X_{\mathbb{R}}$ degenerates to $X_\sharp$, and by L_i the intersection $L_\sharp \cap X_i$. Note that by construction we have $\partial L_i \subset \mathbb{R}E$.

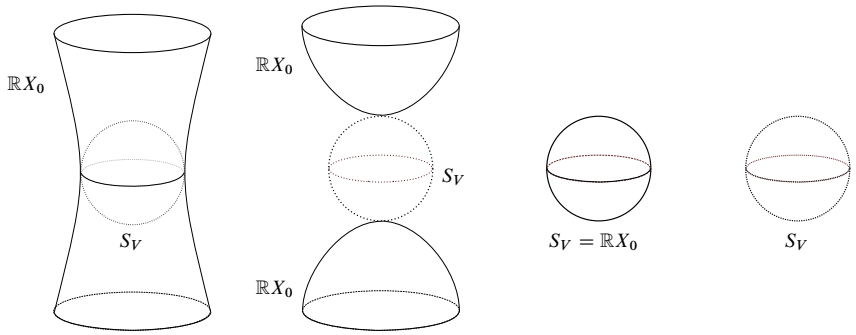


Figure 1. Possibilities for (X_0, τ_0) and S_V

Recall that T^*S^2 is equivariantly symplectomorphic to the complement of a smooth real hyperplane section E of a smooth real quadric in $\mathbb{C}P^3$. This real quadric is precisely the summand $(X_0, \omega_{FS} \oplus \omega_{FS}, \tau_0)$ of $X_\sharp$. We depicted in Figure 1 all possibilities for $\mathbb{R}X_0$ and S_V. Choose a diffeomorphism Ψ between T^*S^2 and the line bundle $\mathcal{O}_{\mathbb{C}P^1}(-2)$ of degree -2 over $E = \mathbb{C}P^1$, which restricts to a symplectomorphism between the complements of the zero sections (note that Ψ does not preserve the fibration). The summand (X_1, ω_1, τ_1) of $X_\sharp$ is obtained by

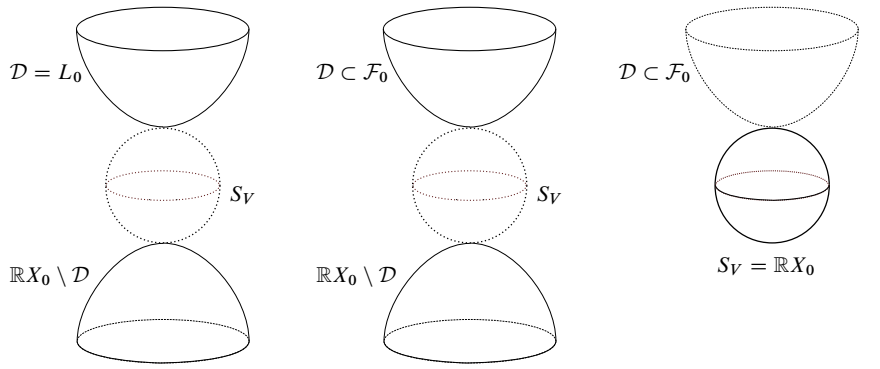
removing from X a small tubular neighborhood of S_V, and by gluing back via Ψ a small neighborhood of the zero section of $\mathcal{O}_{\mathbb{C}P^1}(-2)$. The homology groups $H_2(X_1;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$ are canonically identified, the class $[E]$ being identified with the class $\Psi_*^{-1}([E])$. We implicitly use this identification throughout the text.

Let F be a τ-invariant class in $H_2(X_\sharp \setminus L_\sharp;\mathbb{Z}/2\mathbb{Z})$ having a τ-invariant representative $\mathcal{F}$, and define $\mathcal{F}_i = \mathcal{F} \cap X_i$. Note that by construction we have $\partial\mathcal{F}_i \subset E$. Throughout the text, we always assume that $\mathcal{F}$ satisfies the following conditions:

- either $\mathcal{F} \cap \mathbb{R}E = \emptyset$, or there exists a neighborhood U of $\mathbb{R}E$ in $X_\sharp$ such that $\mathcal{F} \cap U \subset \mathbb{R}X_\sharp$ (i.e. $\mathcal{F}$ is either disjoint from $\mathbb{R}E$, or is locally contained in $\mathbb{R}X_\sharp$ around $\mathbb{R}E$);
- one of the two following assumptions hold:

  (H_1) $\mathcal{F}_0 \cup L_0$ is a cycle representing a multiple of V in $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$;

  (H_2) $\tau_0 = \tau_{el}$, and $\mathcal{F}_0 \cup L_0 = \mathcal{D}$ or $\mathcal{F}_0 \cup L_0 = \mathcal{D} \cup S_V$, where $\mathcal{D}$ is a τ-invariant embedded disk with $\partial\mathcal{D} \subset E$ (all possibilities are depicted in Figure 2).



a)$\mathbb{R}E \neq \emptyset$ and $\mathcal{F}_0 \subset S_V$ b)$\mathbb{R}E \neq \emptyset$ and $L_0 = \emptyset$ c) $\mathbb{R}E = \emptyset$ and $L_0 = \emptyset$

Figure 2. Possibilities for $\mathcal{F}_0 \cup L_0$ under the assumption (H_2)

2.3. Vanishing Welschinger invariants. Next theorem is a key ingredient in the proof of Theorem 1.1, and will be proved in Section 3.4.

Theorem 2.3. *Suppose that $\mathcal{F}_0 \cup L_0$ satisfies assumption (H_1) and contains $\mathbb{R}X_0$, and that L_0 is a disk.*

(1) *If $r \geq 2$, then*

$$W_{X_\mathbb{R},L,F}(d,s) = 0.$$

(2) *If* $r = 1$ *and* $c_1(X) \cdot d - 1 \geq 2$, *then*

$$2^{\frac{c_1(X)\cdot d-4}{2}} \mid W_{X_{\mathbb{R}},L,F}(d,s) \quad \textit{and} \quad (-1)^{\frac{d^2-c_1(X)\cdot d+2}{2}} W_{X_{\mathbb{R}},L}(d,s) \geq 0.$$

Note that the assumptions of Theorem 2.3 imply that $\tau_0 = \tau_{el}$ and $\mathbb{R}E \neq \emptyset$. In the Lagrangian sphere contraction presentation, the condition that L_0 is a disk translates to the condition that $L \cap S_V$ is reduced to a single intersection point.

2.4. From X_1 to X. Here we reduce the computation of Welschinger invariants of $X_{\mathbb{R}}$ to enumeration of real J-holomorphic curves in (X_1, ω_1, τ_1) for a τ_1-compatible almost complex structure J for which E is J-holomorphic.

Definition 2.4. Let J be a τ_1-compatible almost complex structure on (X_1, ω_1, τ_1) for which the curve E is J-holomorphic, and let C_1 be an immersed real rational J-holomorphic curve intersecting E transversely. We denote by a the number of points in $\mathbb{R}C_1 \cap \mathbb{R}E$, by b the number of pairs of τ_1-conjugated points in $C_1 \cap E$, and by $m_{L_1,\mathcal{F}_1}(C_1)$ the number of intersection points of a half of $C_1 \setminus \mathbb{R}C_1$ with $L_1 \cup \mathcal{F}_1$. Finally, let $k \geq 0$ be an integer.

(1) If $\mathcal{F}_0$ satisfies assumption (H_1), then we define

$$\mu^0_{L_\sharp,\mathcal{F}_0,k}(C_1) = (-1)^{m_{L_1,\mathcal{F}_1}(C_1)+\gamma(a+b)} \sum_{k=a_k+2b_k} \binom{a}{a_k}\binom{b}{b_k}$$

and

$$\mu^2_{L_\sharp,\mathcal{F}_0,k}(C_1) = \begin{cases} (-1)^{m_{L_1,\mathcal{F}_1}(C_1)+\gamma b}\, 2^b & \text{if } a = 0 \text{ and } k = b; \\ 0 & \text{otherwise.} \end{cases},$$

where $\gamma = 0, 1$ is such that $[\mathcal{F}_0 \cup L_0] = \gamma V$ in $H_2(X_0; \mathbb{Z}/2\mathbb{Z})$.

(2) If $\mathcal{F}_0$ satisfies assumption (H_2), then we define

$$\mu_{L_\sharp,\mathcal{F}_0,k}(C_1) = \begin{cases} (-1)^{m_{L_1,\mathcal{F}_1}(C_1)} & \text{if } k = a = b = 0; \\ 0 & \text{otherwise} \end{cases}$$

As above let $d \in H_2(X;\mathbb{Z})$ and $r, s \in \mathbb{Z}_{\geq 0}$ such that

$$c_1(X) \cdot d - 1 = r + 2s.$$

Choose a configuration $\underline{x}$ made of r points in L_1 and s pairs of τ-conjugated points in $X_1 \setminus \mathbb{R}X_1$. Let J be a τ_1-compatible almost complex structure for which E is J-holomorphic.

For each integer $k \geq 0$, we denote by $\mathcal{C}_{1,k}(d, \underline{x}, J)$ the set of all irreducible rational real J-holomorphic curves in (X_1, ω_1, τ_1) passing through all points in $\underline{x}$,

realizing the class $d - k[E]$, and such that L_1 contains the 1-dimensional part of $\mathbb{R}C_1$. For a generic choice of J satisfying the above conditions, it follows from Lemma 3.1 and Proposition 3.3 that the set $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is finite, and that any curve in $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is nodal and intersects E transversely. Moreover $\mathcal{C}_{1,k}(d, \underline{x}, J)$ is non-empty only for finitely many values of k.

We prove next theorem in Section 3.3. Recall that notations have been introduced in Section 2.2.

Theorem 2.5. *Suppose that $L_1 \neq \emptyset$ if $r > 0$. Then for a generic choice of J, the two following claims hold.*

(1) *If $\mathcal{F}_0$ satisfies assumption (H_1), then, with the convention that $\chi(\emptyset) = 0$, one has*

$$W_{X_{\mathbb{R}},L,F}(d,s) = \sum_{k\geq 0} \sum_{C_1 \in \mathcal{C}_{1,k}(d,\underline{x},J)} \mu_{L_\sharp,\mathcal{F}_0,k}^{\chi(\mathbb{R}X_0)}(C_1).$$

(2) *If $\mathcal{F}_0$ satisfies assumption (H_2), then one has*

$$W_{X_{\mathbb{R}},L,F}(d,s) = \sum_{C_1 \in \mathcal{C}_{1,0}(d,\underline{x},J)} \mu_{L_\sharp,\mathcal{F}_0,0}(C_1).$$

Applying Theorem 2.5(1) with $F = [\mathbb{R}X \setminus L]$, one obtains [4, Theorem 2.2]. Some instances of Theorem 2.5(1) when $\mathbb{R}X_0 = S^1 \times S^1$ have been known for sometimes, e.g. [6, 7, 16, 20]. Since the publication of [4], an algebro-geometric proof of Theorem 2.5(1) appeared in [5] and in [11] in the particular cases when X is a Del Pezzo surface of degree two or more. Theorem 2.5(2) immediately implies the following corollary.

Corollary 2.6. *Suppose that $V \in H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ and that $\mathcal{F}_0$ satisfies assumption (H_2). Then*

$$W_{X_{\mathbb{R}},L,F}(d,s) = W_{X_{\mathbb{R}},L,F+V}(d,s).$$

2.5. Applications of Theorem 2.5(1). We do not explicitly use Theorem 2.5(1) in the proof of Theorem 1.1, nevertheless its proof is almost contained in the proof of Theorem 2.5(2). Theorem 2.5(1) has many interesting applications, in particular in explicit computations of Welschinger invariants, see [5, 11]. We present two other consequences.

We first relate some tropical Welschinger invariants to genuine Welschinger invariants of the quadric ellipsoid. We refer to [13] for the definition of tropical Welschinger invariants. The only homology classes of $(X_0, \omega_{FS} \oplus \omega_{FS}, \tau_{el})$ realized by real curves are of the form $d(l_1 + l_2)$ with $d \in \mathbb{Z}_{>0}$. We say that a tropical curve in $\mathbb{R}^2$ is of class (a, b) in the tropical second Hirzebruch surface $\mathbb{TF}_2$ if its Newton polygon has vertices $(0, 0)$, $(0, a)$, (b, a), and $(2a + b, 0)$. We denote by $W_{\mathbb{TF}_2}(d)$ the irreducible tropical Welschinger invariant of $\mathbb{TF}_2$ for curves of class $(d, 0)$.

Proposition 2.7. *For any $d \in \mathbb{Z}_{>0}$, we have*

$$W_{X_{0,el},S^2}(dl_1 + dl_2, 0) = W_{\mathbb{TF}_2}(d).$$

Proof. We consider the second Hirzebruch surface $\mathbb{F}_2$ equipped with its real structure induced by the blow up at the origin of the real quadratic cone with equation $x^2 + y^2 - z^2 = 0$. We denote respectively by h and f the class in $H_2(\mathbb{F}_2;\mathbb{Z})$ of a hyperplane section and of a fiber. According to [18], if $\underline{x}^{\mathbb{T}}$ is a tropically generic configuration of $4d-1$ points in $\mathbb{R}^2$, then any rational tropical curve in $\mathbb{TF}_2$ of class $(d-k, 2k)$ and containing $\underline{x}^{\mathbb{T}}$ has $4d$ unbounded edges of weight 1. Still by [18], this implies the existence of a generic configuration $\underline{x}$ of $4d-1$ points in $\mathbb{RF}_2$ such that any real algebraic rational curve in $\mathbb{F}_2$ of class $(d-k)h + 2kf$ and containing $\underline{x}$ intersect the (-2)-curve only in real points. Now the corollary follows from Theorem 2.5(1) applied with $\mathbb{R}X_0 = S^2$. □

It is proved in [12] that given a real toric Del Pezzo surface X equipped with its tautological real toric structure and a class $d \in H_2(X;\mathbb{Z})$, we have

$$W_{X_{\mathbb{R}},\mathbb{R}X}(d,0) \geq W_{X_{\mathbb{R}},\mathbb{R}X}(d,1).$$

The same idea we used in the proof of Proposition 2.7 combined with Theorem 2.5 and [5, Theorem 3.12] provide a natural generalization of this formula in the particular cases when X is a Del Pezzo surface of degree at least three.

Proposition 2.8. *Let (X,ω) be a symplectic 4-manifold symplectomorphic/deformation equivalent to a Del Pezzo surface of degree at least three. If $X_{\mathbb{R}} = (X,\omega,\tau_1)$ and $X'_{\mathbb{R}} = (X,\omega,\tau_2)$ are two real structures on (X,ω), then for any $d \in H_2(X;\mathbb{Z})$ one has*

$$W_{X_{\mathbb{R}},L_1}(d,0) \geq W_{X'_{\mathbb{R}},L_2}(d,0) \geq 0 \quad \text{if} \quad \chi(\mathbb{R}X) \leq \chi(\mathbb{R}X').$$

Proof. We first prove the proposition in the case when (X,ω) is deformation equivalent to $\mathbb{C}P^2$ blown up at six points. We consider $\mathbb{C}P^2$ and its blown up equipped with the standard complex structure J_{st}. Let us denote by $\mathbb{C}P^2_6(\kappa)$ the blow up of $\mathbb{C}P^2$ in $6-2\kappa$ real points and κ pairs of conjugated points, such that these 6 blown-up points do not lye on a conic, and no 3 of them lye on the same line. We further denote by $\widetilde{\mathbb{C}P^2_6}(\kappa)$ the blow up of $\mathbb{C}P^2$ in $6-2\kappa$ real points and κ pairs of conjugated points, such that this 6 blown-up points lye on a smooth real conic with a non-empty real part, but no 3 of them lye on the same line. We denote by E the strict transform of this conic in $\widetilde{\mathbb{C}P^2_6}(\kappa)$. We also denote by $\mathbb{C}P^2_6(4)$ the real structure on the blow up $\mathbb{C}P^2$ in 6 points with a disconnected real part (see [17,21]). We have

$$\chi(\mathbb{R}P^2_6(\kappa)) = -5 + 2\kappa \quad \text{for all } \kappa \in \{0,1,2,3,4\}.$$

Note that $\mathbb{C}P^2_6(\kappa)$ contains no complex algebraic curves C with $C^2 \leq -2$, and that the curve E and its multiples are the only algebraic curves in $\widetilde{\mathbb{C}P^2_6}(\kappa)$ with self-intersection strictly less that -1. Hence it follows from [19, Lemma 3.3.1] that J_{st}

is generic enough for our purposes, as long as we consider generic configuration of points in $\widetilde{\mathbb{C}P}^2_6(\kappa)$. Theorem 2.5(1) applied to E in $X_1 = \widetilde{\mathbb{C}P}^2_6(\kappa)$ allows one to compute $W_{\mathbb{C}P^2_6(\kappa),L_1}(d,s)$ (when $\mathbb{R}X_0 = S^1 \times S^1$) and $W_{\mathbb{C}P^2_6(\kappa+1),L_2}(d,s)$ (when $\mathbb{R}X_0 = S^2$) out of the sets $\mathcal{C}_{1,k}(d, \underline{x}, J_{st})$. When $s = 0$, it follows from [5, Theorem 3.12] that there exists a configuration of real points $\underline{x}$ in $\widetilde{\mathbb{C}P}^2_6(\kappa)$ such that for any $k \geq 0$, any curve in $\mathcal{C}_{1,k}(d, \underline{x}, J_{st})$ intersects E only in real points (i.e. $b = 0$), and

$$\sum_{C_1 \in \mathcal{C}_{1,k}(d,\underline{x},J_{st})} (-1)^{m_{L_1,\mathbb{R}X_1\setminus L_1}(C_1)} \geq 0.$$

Hence by Theorem 2.5(1) we obtain

$$W_{\mathbb{C}P^2_6(\kappa),L_1}(d,0) - W_{\mathbb{C}P^2_6(\kappa+1),L_2}(d,0) = \sum_{k\geq 1} \binom{d\cdot[E]}{k} \sum_{\substack{C_1 \in \\ \mathcal{C}_{1,k}(d,\underline{x},J_{st})}} (-1)^{m_{L_1,\mathbb{R}X_1\setminus L_1}(C_1)}$$
$$\geq 0.$$

The proof in the case of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is analogous using floor diagrams from [3]. □

Note that Proposition 2.8 does not generalize immediately to any symplectic 4-manifold. Indeed, according to [2, Section 7.3] one has

$$W_{\mathbb{C}P^2,\mathbb{R}P^2}(9,12) < W_{\mathbb{C}P^2,\mathbb{R}P^2}(9,13),$$

i.e. Proposition 2.8 does not hold in the case of $\mathbb{C}P^2$ blown up in 26 points.

3. Real symplectic sums and enumeration of real curves

This section is devoted to the proof of Theorems 2.3 and 2.5. We start by performing some preliminary computations in Section 3.1. We recall the symplectic sum construction in Section 3.2, as well as a basic application to complex enumerative problems. We prove Theorems 2.3 and 2.5 in Sections 3.3 and 3.4, by adapting results from Section 3.2 to the real setting.

An isomorphism between two J-holomorphic maps $f_1 : C_1 \to X$ and $f_2 : C_2 \to X$ is a biholomorphism $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$. Maps are always considered up to isomorphisms.

Given $\alpha = (\alpha_i)_{i\geq 1} \in \mathbb{Z}^\infty_{\geq 0}$, we use the following notation:

$$|\alpha| = \sum_{i=1}^{+\infty} \alpha_i, \quad \text{and} \quad I\alpha = \sum_{i=1}^{+\infty} i\alpha_i.$$

The vector in $\mathbb{Z}^\infty_{\geq 0}$ whose all coordinates are equal to 0, except the ith one which is equal to 1, is denoted by e_i.

3.1. Curves with tangency conditions. Let (X,ω) be a compact and connected 4-dimensional symplectic manifold, and let $E \subset X$ be an embedded symplectic curve in X. Let $d \in H_2(X;\mathbb{Z})$ and $\alpha,\beta \in \mathbb{Z}_{\geq 0}^{\infty}$ such that

$$I\alpha + I\beta = d \cdot [E].$$

Choose a configuration $\underline{x} = \underline{x}^{\circ} \sqcup \underline{x}_E$ of points in X, with $\underline{x}^{\circ}$ a configuration of $c_1(X)\cdot d - 1 - d\cdot[E] + |\beta|$ points in $X \setminus E$, and $\underline{x}_E = \{p_{i,j}\}_{0<j\leq\alpha_i, i\geq 1}$ a configuration of $|\alpha|$ points in E. Given J an almost complex structure on X tamed by ω and for which E is J-holomorphic, we denote by $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ the set of rational J-holomorphic maps $f : \mathbb{C}P^1 \to X$ such that

- $f_*[\mathbb{C}P^1] = d$;
- $\underline{x} \subset f(\mathbb{C}P^1)$;
- E does not contain $f(\mathbb{C}P^1)$;
- $f(\mathbb{C}P^1)$ has order of contact i with E at each points $p_{i,j}$;
- $f(\mathbb{C}P^1)$ has order of contact i with E at exactly β_i distinct points on $E \setminus \underline{x}_E$.

For a generic choice of J, the set of simple maps in $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ is 0-dimensional. However $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ might contain components of positive dimension corresponding to non-simple maps.

Lemma 3.1. *Suppose that $\beta = (d\cdot[E])$ and $\alpha = 0$, or $\beta = (d\cdot[E]-1)$ and $\alpha = (1)$. Then for a generic choice of J, the set $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ only contains simple maps.*

Proof. Suppose on the contrary that $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ contains a non-simple map which factors through a non-trivial ramified covering of degree δ of a simple map $f_0 : \mathbb{C}P^1 \to X$. Let d_0 denotes the homology class $(f_0)_*[\mathbb{C}P^1]$. Since $f_0(\mathbb{C}P^1)$ passes through $\delta c_1(X)\cdot d_0 - 1$ points, we have

$$c_1(X)\cdot d_0 - 1 \geq \delta c_1(X)\cdot d_0 - 1 \geq 0,$$

which is impossible. □

Next proposition shows that the set of images of non-simple maps in $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ is 0-dimensional.

Proposition 3.2. *Suppose that $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ contains a non-simple map f which factors through a non-trivial ramified covering of a simple map $f_0 : \mathbb{C}P^1 \to X$. Denote by d_0 the homology class $(f_0)_*[\mathbb{C}P^1]$, and let $\alpha',\beta' \in \mathbb{Z}_{\geq 0}^{\infty}$ such that $f_0 \in \mathcal{C}^{\alpha',\beta'}(d_0,\underline{x},J)$. Then for a generic choice of J, we have*

$$c_1(X)\cdot d_0 - 1 - d_0\cdot[E] + |\beta'| = |\underline{x}^{\circ}| = k_1 \quad \text{and} \quad |\alpha'| = k_2$$

with $(k_1,k_2) = (1,0),(0,1)$, or $(0,0)$. Moreover in the first two cases, the set of such ramified coverings f is finite, and $|\alpha'| + |\beta'| \geq 2$.

Proof. Let $\delta \geq 2$ be the degree of the covering map through which f factors. In particular we have $d = \delta d_0$. By Riemann–Hurwitz Formula, we have

$$\delta(|\alpha'| + |\beta'|) - |\alpha| - |\beta| \leq 2\delta - 2.$$

Combining the latter identity with $|\alpha'| = |\alpha|$, we get

$$|\beta| - |\beta'| \geq (\delta - 1)(|\alpha'| + |\beta'| - 2). \tag{3.1}$$

Since $f_0(\mathbb{C}P^1)$ contains all points in $\underline{x}^\circ$, we have

$$c_1(X) \cdot d_0 - 1 - d_0 \cdot [E] + |\beta'| \geq |\underline{x}^\circ| = \delta c_1(X) \cdot d_0 - 1 - \delta d_0 \cdot [E] + |\beta|,$$

and so

$$(\delta - 1)(d_0 \cdot [E] - c_1(X) \cdot d_0) \geq |\beta| - |\beta'|.$$

Combining this identity with (3.1), we obtain

$$0 \geq (\delta - 1)(c_1(X) \cdot d_0 - d_0 \cdot [E] + |\alpha'| + |\beta'| - 2).$$

Since we have

$$\delta \geq 2, \quad c_1(X) \cdot d_0 - d_0 \cdot [E] + |\beta'| - 1 \geq 0, \quad \text{and} \quad |\alpha'| \geq 0,$$

we deduce that

$$c_1(X) \cdot d_0 - 1 - d_0 \cdot [E] + |\beta'| = k_1 \quad \text{and} \quad |\alpha'| = k_2$$

with $(k_1, k_2) = (1, 0), (0, 1)$, or $(0, 0)$. Moreover in the first two cases, all inequalities above are in fact equalities. In particular there exists finitely many coverings $\pi : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree δ such that $f_0 \circ \pi \in \mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$. Since $|\beta| - |\beta'| \geq 0$, we also deduce from (3.1) that $|\alpha'| + |\beta'| \geq 2$. □

Remark. The three cases from Proposition 3.2 show up, even in simple situations. Let us consider for example X to be $\mathbb{C}P^2$ blown up at a point q. Denote by l the homology class of a line, by l_{exc} the class of the exceptional divisor, and by E the pull back of a conic not passing through q. Then for any choice of J, the sets $\mathcal{C}^{0,2e_\delta}(\delta(l - l_{exc}), \{p\}, J)$, $\mathcal{C}^{e_\delta,e_\delta}(\delta(l - l_{exc}), \emptyset, J)$, and $\mathcal{C}^{0,e_{2\delta}}(\delta(l - l_{exc}), \emptyset, J)$ with $\delta \geq 2$ contain a non-trivial ramified covering of a line, the third set being of dimension $\delta - 1$.

Proposition 3.3. *Suppose that E is an embedded symplectic sphere with $[E]^2 \geq -2$, and that $|\beta| \geq [E] \cdot d - 1$. Then for a generic choice of J, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ contains finitely many simple maps. As a consequence, the set*

$$\mathcal{C}^{\alpha,\beta}_*(d, \underline{x}, J) = \left\{ f(\mathbb{C}P^1) \mid (f : \mathbb{C}P^1 \to X) \in \mathcal{C}^{\alpha,\beta}(d, \underline{x}, J) \right\}$$

is also finite.

Proof. Suppose that $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ contains infinitely many simple maps. By Gromov compactness Theorem, there exists a sequence $(f_n)_{n\geq 0}$ of simple maps in $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ which converges to some J-holomorphic map $\overline{f}:\overline{C}\to X$. By genericity of J, the set of simple maps in $\mathcal{C}^{\alpha,\beta}(d,\underline{x},J)$ is discrete. Hence either $\overline{C}$ is reducible, or $\overline{f}$ is non-simple. Let $\overline{C}_1,\dots,\overline{C}_m,\overline{C}'_1,\dots,\overline{C}'_{m'}$ be the irreducible components of $\overline{C}$, labeled in such a way that

- $\overline{f}(\overline{C}_i)\not\subset E$;
- $\overline{f}(\overline{C}'_i)\subset E$, and $\overline{f}_*[\overline{C}'_i]=k_i[E]$.

Define $k=\sum_{i=1}^{m'}k_i$. The restriction of $\overline{f}$ to $\bigcup_{i=1}^{m}\overline{C}_i$ is subject to

$$c_1(X)\cdot d-1-d\cdot[E]+|\beta|$$

points conditions, so we have

$$c_1(X)\cdot(d-k[E])-m\geq c_1(X)\cdot d-1-d\cdot[E]+|\beta|.$$

Since E is an embedded sphere, the adjunction formula implies that $c_1(X)\cdot[E]=[E]^2+2$. Hence we get

$$c_1(X)\cdot d-2k-k[E]^2-m\geq c_1(X)\cdot d-1-d\cdot[E]+|\beta|,$$

that is

$$0\geq -d\cdot[E]+|\beta|+m-1+2k+k[E]^2.$$

Since $d\cdot[E]\geq|\beta|$, we are in one of the following situations:

(1) $d\cdot[E]=|\beta|$ (in particular $\alpha=0$):
   (a) $k=0$, and $m=1$;
   (b) $[E]^2=-2$, $k>0$, and $m=1$;

(2) $d\cdot[E]=|\beta|+1$ (in particular either $\beta=(d\cdot[E]-1)$ and $\alpha=(1)$, or $\beta=(d\cdot[E]-2,1)$):
   (a) $k=0$, and $m=1$;
   (b) $k=0$, and $m=2$;
   (c) $[E]^2=1$, $k=1$, and $m=1$;
   (d) $[E]^2=-2$, $k>0$, and $m=1$;
   (e) $[E]^2=-2$, $k>0$, and $m=2$.

We end the proof of the proposition by ruling out all these cases one by one.

(1)(a) $d \cdot [E] = |\beta|$, $k = 0$, and $m = 1$:

As explained above, the map $\overline{f}$ has to factorize through a non-trivial ramified covering of a simple map $f_0 : \mathbb{C}P^1 \to X$. But then f_0 is subject to more point constraints that the dimension of its space of deformation, which provides a contradiction.

(1)(b) $d \cdot [E] = |\beta|$, $[E]^2 = -2$, $k > 0$, and $m = 1$:

By genericity, the curve $\overline{f}(\overline{C}_1)$ is fixed by the $c_1(X) \cdot d - 1$ point constraints and intersect E transversely. Any intersection point of $\overline{f}(\overline{C}_1 \backslash (\overline{C}'_1 \cup \ldots \cup \overline{C}'_{m'}))$ and E deforms to an intersection point of $f_n(\mathbb{C}P^1)$ and E for $n >> 1$. Since $(d - k[E]) \cdot [E] = d \cdot [E] + 2k$, at least $d \cdot [E] + k$ intersection points of $\overline{f}(\overline{C}_1)$ and E deform to an intersection point of $f_n(\mathbb{C}P^1)$ and E for $n >> 1$. But this contradicts the fact that two J-holomorphic curves intersect positively.

(2)(a) $d \cdot [E] = |\beta| + 1$, $k = 0$, and $m = 1$:

Since $\overline{f}$ is a non-simple map, it factorizes through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{C}P^1 \to X$. If $d_0 = (f_0)_*[\mathbb{C}P^1]$, the adjunction formula implies that the image of f_0 has

$$\frac{d_0^2 - c_1(X) \cdot d_0 + 2}{2}$$

nodes. Each of this node deforms to 2δ intersection point of $f_n(\mathbb{C}P^1)$ and $f_0(\mathbb{C}P^1)$ for $n >> 1$. Since $\underline{x} \subset f_n(\mathbb{C}P^1) \cap f_0(\mathbb{C}P^1)$, we get

$$d \cdot d_0 \geq 2\delta \, \frac{d_0^2 - c_1(X) \cdot d_0 + 2}{2} + c_1(X) \cdot d - 2 = d \cdot d_0 + 2(\delta - 1) > d \cdot d_0$$

which is a contradiction.

(2)(b) $d \cdot [E] = |\beta| + 1$, $k = 0$, and $m = 2$:

By genericity, the curve $\overline{f}(\overline{C}_1 \cup \overline{C}_2)$ is fixed by the $c_1(X) \cdot d - 2$ point constraints, and intersect E transversely at non-prescribed points. This contradicts the fact that either $\alpha \neq 0$ or $\beta_2 \neq 0$.

(2)(c) $d \cdot [E] = |\beta| + 1$, $[E]^2 = 1$, $k = 1$, and $m = 1$:

By genericity, the curve $\overline{f}(\overline{C}_1)$ is fixed by the $c_1(X) \cdot d - 2$ point constraints, and intersect E transversely in $d \cdot [E] + 1$ non-prescribed points. Any such intersection point distinct from $\overline{f}(\overline{C}_1 \cap \overline{C}'_1)$ deforms to an intersection point of $f_n(\mathbb{C}P^1)$ and E for $n >> 1$. Moreover since all intersection points of $\overline{f}(\overline{C}_1)$ and E are transverse and non-prescribed, the component $\overline{C}'_1$ contains the limit of the point corresponding to the extra constraint α_1 or β_2. Therefore $f_n(\mathbb{C}P^1)$ and E for $n >> 1$ must have at least $d \cdot [E] + 1$ intersection points for $n >> 1$, which is a contradiction.

(2)(d) $d \cdot [E] = |\beta| + 1$, $[E]^2 = -2$, $k > 0$, and $m = 1$:

Suppose first that $\overline{f}_{|\overline{C}_1}$ factorizes through a non-trivial ramified covering of degree $\delta \geq 2$ of a simple map $f_0 : \mathbb{C}P^1 \to X$. Since $f_0(\mathbb{C}P^1)$ satisfies $c_1(X) \cdot d - 2$ point conditions, we have that

$$c_1(X) \cdot d_0 - 1 \geq c_1(X) \cdot d - 2 = \delta c_1(X) \cdot d_0 - 2 \geq 0,$$

where $d_0 = (f_0)_*[\mathbb{C}P^1]$. Hence we obtain that $c_1(X) \cdot d_0 = 1$ and $\delta = 2$. In particular the curve $f_0(\mathbb{C}P^1)$ is rigid, and intersect E at smooth points. Now the same arguments used in the case (2)(a) provide a contradiction.

Hence $\overline{f}_{|\overline{C}_1}$ is a simple map. Since $\overline{f}_{|\overline{C}_1}$ satisfies $c_1(X) \cdot d - 2$ point constraints, it has at most one tangency point with E. The same argument used in the case (1)(b) implies that $k = 1$ and $\overline{f}(\overline{C}_1)$ is tangent to E at $\overline{f}(\overline{C}_1 \cap \overline{C}_1')$. Hence $\overline{f}_{|\overline{C}_1}$ is fixed by this tangency condition and the $c_1(X) \cdot d - 2$ other point conditions, and the component $\overline{C}_1'$ contains the limit of the point corresponding to the extra constraint α_1 or β_2. Thus we obtain again a contradiction with the positivity of intersection points of E and $f_n(\mathbb{C}P^1)$ for $n >> 1$.

(2)(e) $d \cdot [E] = |\beta| + 1$, $[E]^2 = -2$, $k > 0$, and $m = 2$:

By genericity, the curve $\overline{f}(\overline{C}_1 \cup \overline{C}_2)$ is fixed by the $c_1(X) \cdot d - 2$ point constraints, and intersect E transversely at non-prescribed points. Hence the same argument used in the case (1)(b) implies that $k = 1$. Thus the component $\overline{C}_1'$ contains the limit of the point corresponding to the extra constraint α_1 or β_2, which gives a contradiction as in the case (2)(d).

The finiteness of the set $\mathcal{C}_*^{\alpha,\beta}(d, \underline{x}, J)$ follows from Proposition 3.2 and the finiteness of simple maps in $\mathcal{C}^{\alpha',\beta'}(d_0, \underline{x}, J)$ for all possible α', β', and d_0 with $d = \delta d_0$. □

In the case when $X = \mathbb{C}P^1 \times \mathbb{C}P^1$, $[E] = l_1 + l_2$, and $|\underline{x}^\circ| \leq 1$, the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ is always finite and made of simple maps.

Proposition 3.4. *Suppose that $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $[E] = l_1 + l_2$. Then the set $\mathcal{C}^{\alpha,\beta}(d, \underline{x}, J)$ with $|\underline{x}^\circ| \leq 1$ is empty for a generic choice of J, except in the following situations where it contains a unique element:*

- $\mathcal{C}^{e_1,0}(l_i, \emptyset, J)$, $i = 1, 2$;
- $\mathcal{C}^{0,e_1}(l_i, \{p\}, J)$, $i = 1, 2$;
- $\mathcal{C}^{2e_1,0}(l_1 + l_2, \{p\}, J)$;
- $\mathcal{C}^{e_2,0}(l_1 + l_2, \{p\}, J)$.

Moreover, this unique element is an embedding.

Proof. The first Chern class of X is dual to $2(l_1 + l_2)$, so

$$c_1(X) \cdot (al_1 + bl_2) - 1 - (al_1 + bl_2) \cdot [E] + |\beta| = a + b - 1 + |\beta|.$$

Suppose that $a + b - 1 + |\beta| = 0$ and that $\mathcal{C}^{\alpha,\beta}(al_1 + bl_2, \emptyset, J) \neq \emptyset$. Since $(al_1 + bl_2) \cdot [E] = a + b \geq |\beta|$, and since two J-holomorphic curves intersect positively, we obtain $a + b = 1$ and $|\beta| = 0$. By genericity of J, we have that $(al_1 + bl_2)^2 \geq -1$, i.e. $2ab \geq -1$. From $a + b = 1$, we deduce that $a = 0$ or 1.

In the case $a + b - 1 + |\beta| = 1$ and $\mathcal{C}^{\alpha,\beta}(al_1 + bl_2, \{p\}, J) \neq \emptyset$, we prove analogously that we are in one of the following situations:

- $(a, b) = (1, 0)$ or $(0, 1)$, and $|\beta| = 1$;
- $(a, b) = (1, 1), (2, 0)$, or $(0, 2)$, and $|\beta| = 0$.

If X is equipped with the symplectic form $\omega_{FS} \oplus \omega_{FS}$ and its standard complex structure J_{st}, it is easy to check that the sets $\mathcal{C}^{e_1,0}(l_i, \emptyset, J_{st})$, $\mathcal{C}^{0,e_1}(l_i, \{p\}, J_{st})$, $\mathcal{C}^{2e_1,0}(l_1 + l_2, \{p\}, J_{st})$, and $\mathcal{C}^{e_2,0}(l_1 + l_2, \{p\}, J_{st})$ consists of a unique element. This implies that when we vary both ω and J, the corresponding sets still contain at least one element. Moreover they cannot contain more than one element, since the imposed constraints imply that two distinct curves would have an intersection number strictly bigger than the one imposed by their homology class. Finally, all J-holomorphic maps under consideration are embeddings thanks to the adjunction formula.

Suppose now that $\mathcal{C}^{\alpha,0}(2l_i, \{p\}, J)$ contains an element $f : \mathbb{C}P^1 \to X$. We proved in the previous paragraph that there exists a map $f_0 : \mathbb{C}P^1 \to X$ in $\mathcal{C}^{0,e_1}(l_i, \{p\}, J)$. Since we have $f_*[\mathbb{C}P^1] \cdot (f_0)_*[\mathbb{C}P^1] = 2l_i^2 = 0$, we deduce that f factors through f_0 and a degree 2 ramified covering of $\mathbb{C}P^1$. This contradicts Proposition 3.2. □

3.2. Symplectic sums. Here we describe a very particular case of the symplectic sum formula from [15]. Recall that (X_1, ω_1) is a compact and connected symplectic manifold of dimension 4, containing an embedded symplectic sphere E with $[E]^2 = -2$. We furthermore assume the existence of a symplectomorphism ϕ from E to a symplectic curve realizing the class $l_1 + l_2$ in $(X_0, \omega_0) = (\mathbb{C}P^1 \times \mathbb{C}P^1, \omega_{FS} \oplus \omega_{FS})$. By abuse, we still denote by E the image $\phi(E)$ in X_0. Since the self-intersection of E in X_0 and X_1 are opposite, there exists a symplectic bundle isomorphism ψ between the normal bundle of E in X_0 and the dual of the normal bundle of E in X_1. Out of these data, one produces a family of symplectic 4-manifolds (Y_t, ω_t) parametrized by a small complex number t in $\mathbb{C}^*$, see [10]. All those manifolds are deformation equivalent, and are called symplectic sums of (X_0, ω_0) and (X_1, ω_1) along E. Next theorem says that this family can be seen as a symplectic deformation of the singular symplectic manifold $X_\sharp = X_0 \cup_E X_1$ obtained by gluing (X_0, ω_0) and (X_1, ω_1) along E.

Proposition 3.5 ([15, Theorem 2.1]). *There exists a symplectic 6-manifold (Y, ω_Y) and a symplectic fibration $\pi : Y \to D$ over a disk $D \subset \mathbb{C}$ such that the central fiber $\pi^{-1}(0)$ is the singular symplectic manifold $X_\sharp$, and $\pi^{-1}(t) = (Y_t, \omega_t)$ for $t \neq 0$.*

Topologically, Y_t is obtained by removing a tubular neighborhood of E in X_1, and gluing back $X_0 \setminus E$ via ψ. Note that the symplectomorphism ϕ induces a diffeomorphism Ψ from the normal bundle of E in X_1 and $X_0 \setminus E$. Hence the homology groups $H_2(X_1;\mathbb{Z})$ and $H_2(Y_t;\mathbb{Z})$ are identified, the class $[E]$ being identified with the class $\Psi_*[E]$. Without loss of generality we may assume that $\Psi_*[E] = l_1 - l_2$ in $H_2(X_0;\mathbb{Z})$.

Let $d \in H_2(Y_t;\mathbb{Z})$, and choose $\underline{x}(t)$ a set of $c_1(X)\cdot d - 1$ symplectic sections $D \to Y$ such that $\underline{x}(0) \cap E = \emptyset$. Choose an almost complex structure J on Y tamed by ω_Y, which restrict to an almost complex structure J_t tamed by ω_t on each fiber Y_t, and generic with respect to all choices we made.

Define $\mathcal{C}(d, \underline{x}(0), J_0)$ to be the set $\left\{\overline{f} : \overline{C} \to X_\sharp\right\}$ of limits, as stable maps, of maps in $\mathcal{C}(d, \underline{x}(t), J_t)$ as t goes to 0. Recall (see [15, Section 3]) that $\overline{C}$ is a connected nodal rational curve such that:

- $\underline{x}(0) \subset \overline{f}(\overline{C})$;
- any point $p \in \overline{f}^{-1}(E)$ is a node of $\overline{C}$ which is the intersection of two irreducible components $\overline{C}'$ and $\overline{C}''$ of $\overline{C}$, with $\overline{f}(\overline{C}') \subset X_0$ and $\overline{f}(\overline{C}'') \subset X_1$;
- if in addition neither $\overline{f}(\overline{C}')$ nor $\overline{f}(\overline{C}'')$ is entirely mapped to E, then the multiplicity of intersection of both $\overline{f}(\overline{C}')$ and $\overline{f}(\overline{C}'')$ with E are equal.

Given an element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, we denote by C_i the union of the irreducible components of $\overline{C}$ mapped to X_i.

Lemma 3.6. *Given an element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}(d, \underline{x}(0), J_0)$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that*

$$\overline{f}_*[C_1] = d - k[E] \quad \text{and} \quad \overline{f}_*[C_0] = kl_1 + (d\cdot[E] + k)l_2.$$

Moreover $c_1(X_1)\cdot \overline{f}_*[C_1] = c_1(Y_t)\cdot d$.

Proof. Let k such that $f_*[C_1] = d - k[E]$, and let $f_*[C_0] = al_1 + bl_2$. Using the above identification of $H_2(X_1;\mathbb{Z})$ and $H_2(Y_t;\mathbb{Z})$, we have

$$d\cdot[E] = \overline{f}_*[C_0]\cdot(l_1 - l_2) = b - a.$$

On the other hand, by considering a representative of E in X_1 and another in X_0, we obtain

$$a + b = \overline{f}_*[C_0]\cdot(l_1 + l_2) = (d - k[E])\cdot[E] = d\cdot[E] + 2k,$$

which gives $a = k$ and $b = d\cdot[E] + k$.

By [15, Lemma 2.2], we have

$$c_1(Y_t)\cdot d = c_1(X_1)\cdot \overline{f}_*[C_1] + c_1(X_0)\cdot \overline{f}_*[C_0] - 2\overline{f}_*[C_0]\cdot[E].$$

Since $c_1(X_0)$ is dual to $2(l_1+l_2)$, we deduce that $c_1(X_1)\cdot \overline{f}_*[C_1] = c_1(Y_t)\cdot d$. □

Proposition 3.7. *Assume that the set $\underline{x}(0)\cap X_0$ contains at most one point, and that $\underline{x}(0)\cap X_1 \neq \emptyset$ if $\underline{x}(0)\cap X_0 \neq \emptyset$. Then for a generic J_0, the set $\mathcal{C}(d,\underline{x}(0),J_0)$ is finite, and only depends on $\underline{x}(0)$ and J_0. Given $\overline{f}:\overline{C}\to X_\sharp$ an element of $\mathcal{C}(d,\underline{x}(0),J_0)$, the restriction of $\overline{f}$ to any component of $\overline{C}$ is a simple map, and no irreducible component of $\overline{C}$ is entirely mapped to E. Moreover the following are true.*

(1) *If $\underline{x}(0)\cap X_0 = \emptyset$, then the curve C_1 is irreducible, and the image of any irreducible component of C_0 realizes a class l_i. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(t),J_t)$ as t goes to 0.*

(2) *If $\underline{x}(0)\cap X_0 = \{p_0\}$, then the image of the irreducible component $\overline{C}'$ of C_0 whose image contains p_0 realizes either a class l_i or the class l_1+l_2, while any other irreducible component of C_0 realizes a class l_i.*

   (a) *If $\overline{f}(\overline{C}')$ realizes the class l_i, then the curve C_1 is irreducible and $\overline{f}_{|C_1}$ is an element of $\mathcal{C}^{e_1,(d\cdot[E]+2k-1)e_1}(d-k[E],\underline{x}(0)\cup\underline{x}_E,J_0)$, where $\underline{x}_E = \overline{f}(\overline{C}')\cap E$. The map $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(t),J_t)$ as t goes to 0.*

   (b) *If $\overline{f}(\overline{C}')$ realizes the class l_1+l_2, then $\overline{f}_{|\overline{C}'}$ is an element of $\mathcal{C}^{\alpha,0}(l_1+l_2,\{p_0\}\cup\underline{x}_E,J_0)$, where $\underline{x}_E\subset\overline{f}(C_1)\cap E$, and $\alpha = 2e_1$ or $\alpha = e_2$. In the former case, the curve C_1 has two irreducible components, and $\overline{f}$ is the limit of a unique element of $\mathcal{C}(d,\underline{x}(t),J_t)$ as t goes to 0; in the latter case, the curve C_1 is irreducible, and $\overline{f}$ is the limit of exactly two elements of $\mathcal{C}(d,\underline{x}(t),J_t)$ as t goes to 0.*

Proof. The fact that no component of $\overline{C}$ is entirely mapped to E follows from [15, Example 11.4 and Lemma 14.6]. By assumption we have $[E]^2 = -2$ in X_1, so the adjunction formula implies that $c_1(X_1)\cdot[E] = 0$. Since the curve $\overline{f}(C_1)$ passes through all the points in $\underline{x}(0)\cap X_1$ and realizes the class $d-k[E]$ in $H_2(X_1;\mathbb{Z})$, the following hold.

(1) If $\underline{x}(0)\cap X_0 = \emptyset$, then the map $\overline{f}_{|C_1}$ is constrained by $c_1(X)\cdot d - 1 = c_1(X_1)\cdot(d-k[E]) - 1$ points in X_1. Hence the curve C_1 is irreducible, the map $\overline{f}_{|C_1}$ is simple, and $\overline{f}(C_1)$ intersects E transversely in $d\cdot[E]+2k$ distinct points. The curve $\overline{C}$ is rational, and any J_0-holomorphic curve in X_0 intersects E, so we deduce that the curve C_0 has exactly $d\cdot[E]+2k$ irreducible components. Furthermore the image of any of them realizes a class l_i.

(2) If $\underline{x}(0) \cap X_0 = \{p_0\}$, then the map $\overline{f}_{|C_1}$ is constrained by $c_1(X_1) \cdot (d - k[E]) - 2$ points in X_1. Hence we are in one of the following situations.

(a) The curve C_1 is irreducible, and $\overline{f}^{-1}(E)$ consists in $d \cdot [E] + 2k$ distinct points. As above, the curve C_0 must have exactly $d \cdot [E] + 2k$ irreducible components, and the image of any of them realizes a class l_i. Since $\overline{f}(\overline{C}')$ contains p_0, the map $\overline{f}_{|C_1}$ is also constrained by the point $\overline{f}(\overline{C}') \cap E$. Hence $\overline{f}_{|C_1}$ is a simple map by Lemma 3.1.

(b) The curve C_1 has two connected components, the map $\overline{f}_{|C_1}$ is simple and fixed by $\underline{x}(0) \cap X_1$, and $\overline{f}(C_1)$ intersects E transversely in $d \cdot [E] + 2k$ distinct points. Hence the curve C_0 must have exactly $d \cdot [E] + 2k - 1$ irreducible components, one of them, say $\overline{C}''$, intersecting the two components of C_1. The curve $\overline{f}(\overline{C}'')$ has to realize the class $l_1 + l_2$, and the image of any other irreducible component of C_0 realizes a class l_i. Since all these latter components are constrained by $\overline{f}(C_1) \cap E$, we deduce that $\overline{C}' = \overline{C}''$.

(c) The curve C_1 is irreducible, and $\overline{f}^{-1}(E)$ consists in $d \cdot [E] + 2k - 1$ distinct points. Again, the curve C_0 must have exactly $d \cdot [E] + 2k - 1$ irreducible components, the image of one of them being tangent to E. As in the case (2)(b), we deduce that this component must be $\overline{C}'$, that its image must realize the class $l_1 + l_2$, and that the image of any other irreducible component of C_0 realizes a class l_i.

Suppose that $\overline{f}$ restricts to a non-simple map on C_1, and let $\rho : \mathbb{C}P^1 \to \mathbb{C}P^1$ be the ramified covering through which $\overline{f}_{|C_1}$ factors. Since $\underline{x}(0) \cap X_1 \neq \emptyset$, Proposition 3.2 implies that at least two ramification points of ρ should be mapped to E. Hence there should exist an irreducible component $\overline{C}''$ of C_0 distinct from $\overline{C}'$ and intersecting E non-transversely. This contradicts the fact that $\overline{f}_*[\overline{C}''] = l_i$.

The statement about the number of elements of $\mathcal{C}(d, \underline{x}(t), J_t)$ converging to $\overline{f}$ as t goes to 0 follows from [15]. Let us recall briefly the behavior, close to a smoothing of an intersection point p of C_1 and C_0, of an elements $f_t : C_t \to Y_t$ of $\mathcal{C}(d, \underline{x}(t), J_t)$ converging to $\overline{f}$. In local coordinates (t, x, y) at $\overline{f}(p)$, the manifold Y is given by the equation $xy = t$, the manifold X_0 (resp. X_1) being locally given by $\{t = 0 \text{ and } y = 0\}$ (resp. $\{t = 0 \text{ and } x = 0\}$). If the order of intersection of $\overline{f}_{C_0}$ and E at $f(p)$ is equal to s, then the maps $\overline{f}_{C_0}$ and $\overline{f}_{C_1}$ have expansions

$$x(z) = az^s + o(z^s) \quad \text{and} \quad y(w) = bw^s + o(w^s),$$

where z and w are local coordinates at p of C_0 and C_1 respectively.

For $0 < |t| << 1$, there exists a solution $\mu(t) \in \mathbb{C}^*$ of

$$\mu(t)^s = \frac{t}{ab},$$

such that the smoothing of $\overline{C}$ at p is locally given by $zw = \mu(t)$, and the map f_t is approximated by the map

$$\{zw = \mu(t)\} \subset \mathbb{C}^2 \mapsto (t, az^s, bw^s)$$

close to the smoothing of p (see [15, Section 6], and also [22, Section 6.2] for details). Furthermore, such maps $f_t \in \mathcal{C}(d, \underline{x}(t), J_t)$ converging to $\overline{f}$ are in one to one correspondence with a choice of such $\mu(t)$ for each point of $C_0 \cap C_1$. □

Next Corollary generalizes Abramovich–Bertram–Vakil formula.

Corollary 3.8. *Suppose that $\underline{x}(0) \cap X_0 = \emptyset$, and let $\overline{f} : \overline{C} \to X_\sharp$ be an element of $\mathcal{C}(d, \underline{x}(0), J_0)$. Define $\mathcal{C}_{\overline{f}}$ to be the set of elements $\overline{f}' : \overline{C}' \to X_\sharp$ in $\mathcal{C}(d, \underline{x}(0), J_0)$ such that $\overline{f}_{|C_1} = \overline{f}'_{|C_1'}$. If $\overline{f}_*[C_1] = d - k[E]$, then $\mathcal{C}_{\overline{f}}$ has exactly $\binom{d\cdot[E]+2k}{k}$ elements.*

Proof. It follows from Proposition 3.7 and Lemma 3.6 that $\overline{f}_*[C_1] = d - k[E]$ if and only if the image of exactly k irreducible components of C_0 realize the class l_1. Since $(d - k[E]) \cdot [E] = d \cdot [E] + 2k$, the result follows. □

3.3. Proof of Theorem 2.5. Theorems 2.3 and 2.5 are obtained by considering a real version of the symplectic sum described in Section 3.2. We first provide the proof of Theorem 2.5 since it is a immediate adaptation of Proposition 3.7 to the real setting. We equip the disc D from Proposition 3.5 with the standard complex conjugation, the symplectic manifold (X_i, ω_i) with a real structure τ_i, and (Y, ω_Y) with a real structure τ_Y such that the map $\pi : Y \to D$ is real. Furthermore we choose the set of sections $\underline{x} : D \to Y$ to be real. Note that each fiber Y_t comes naturally equipped with a real structure τ_t when $t \in \mathbb{R}$. If $\mathcal{F} \cap \mathbb{R}E = \emptyset$, then by perturbing $\mathcal{F}$ if necessary, we may assume that $\overline{f}(\overline{C}) \cap \mathcal{F} \cap E = \emptyset$ for all $\overline{f} \in \mathcal{C}(d, \underline{x}(0), J_0)$.

Theorem 2.5 is obtained by choosing $\underline{x}$ such that $\underline{x}(0) \cap X_0 = \emptyset$.

Proof of Theorem 2.5(1). Assume that $\underline{x}(0) \cap X_0 = \emptyset$ and $\underline{x}(0) \cap \mathbb{R}X_1 \subset L_1$, and let us choose a real element $\overline{f} : \overline{C} \to X_\sharp$ of $\mathcal{C}(d, \underline{x}(0), J_0)$. Denote by a (resp. b) the number of real (resp. pairs of τ_0-conjugated) points of intersections of E and $\overline{f}(\overline{C})$. The map $\overline{f} : C \to X$ is real, and by Corollary 3.8 the set $\mathcal{C}_{\overline{f}}$ has exactly

- $\sum_{k=a_i+2b_i} \binom{a}{a_i}\binom{b}{b_i}$ real elements if τ_0 acts trivially on $H_2(X_0; \mathbb{Z})$;
- 2^k if $a = 0$ and $b = k$, and 0 otherwise, real elements if τ_0 exchanges l_1 and l_2.

By assumption (H_1), we have that $\mathcal{F}_0 \cup L_0$ represents a cycle γV in $H_2(X_0;\mathbb{Z}/2\mathbb{Z})$ with $\gamma = 0, 1$, so in both cases above we have $m_{L_\sharp,\mathcal{F}_0}(\overline{f}(\overline{C})) = m_{L_1,\mathcal{F}_\sharp}(\overline{f}_{|C_1}) + \gamma(a+b)$. By Proposition 3.7, any element $\overline{f}$ of $\mathcal{C}(d, \underline{x}(0), J_0)$ is the limit of a unique element of $\mathcal{C}(d, \underline{x}(t), J_t)$, so this latter has to be real when $\overline{f}$ is real and $t \in \mathbb{R}^*$. Hence to end the proof of Theorem 2.5(1), it remains to show that no node appears in a neighborhood of $E \cap \overline{f}(\overline{C})$ when deforming $\overline{f}$. This follows from the description provided at the end of the proof of Proposition 3.7 of the local deformation of $\overline{f}$ (since $s = 1$ in the present case). An alternative proof is to observe that $\overline{f}(\overline{C})$ has as many nodes as any of its deformation:

$$\frac{(d-k[E])^2 - c_1(X_1)\cdot(d-k[E]) + 2}{2} + k(d\cdot[E]+k) \quad = \frac{d^2 - c_1(Y_t)\cdot d + 2}{2}$$

since $[E]^2 = -2$ and $c_1(X_1)\cdot[E] = 0$. □

Proof of Theorem 2.5(2). Assume again that $\underline{x}(0) \cap X_0 = \emptyset$ and $\underline{x}(0) \cap \mathbb{R}X_1 \subset L_1$. Recall that by assumption (H_2) we have $\tau_0 = \tau_{el}$, which implies in particular that $a = 0$. Suppose that $b \neq 0$, and choose a pair $\{p, \tau_0(p)\}$ of τ_0-conjugated intersection points of E with $\overline{f}(\overline{C})$. Let $\overline{f}' : \overline{C}' \to X_\sharp$ be an element of $\mathcal{C}_{\overline{f}}$, and denote by $\overline{C}_p$ (resp. $\overline{C}_{\tau_0(p)}$) the irreducible component of C_0 whose image contains p (resp. $\tau_0(p)$). Define the map $\overline{f}'' : \overline{C}' \to X_\sharp$ as follows: $\overline{f}''(x) = \overline{f}'(x)$ if $x \notin \overline{C}_p \cup \overline{C}_{\tau_0(p)}$, and $\overline{f}''(x) = \tau_0 \circ \overline{f}'(x)$ if $x \in \overline{C}_p \cup \overline{C}_{\tau_0(p)}$. The map $\overline{f}''$ is also an element of $\mathcal{C}_{\overline{f}}$, and it follows from Lemma 3.9 below that $m_{L_\sharp,\mathcal{F}_\sharp}(\overline{f}'(\overline{C}')) = -m_{L_\sharp,\mathcal{F}_\sharp}(\overline{f}''(\overline{C}'))$. Hence

$$\sum_{\overline{f}' \in \mathcal{C}_{\overline{f}}} m_{L,\mathcal{F}_\sharp}(\overline{f}'(\overline{C}')) = 0,$$

and Theorem 2.5(2) is proved. □

Given a point $p \in E$, we denote by C_p the (unique) $J_{0|X_0}$-holomorphic curve in the class l_1 passing through p.

Lemma 3.9. *Let $\mathcal{D} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ be an embedded τ_{el}-invariant disk with $\partial\mathcal{D} \subset E$. Then if $p \in E \setminus \partial\mathcal{D}$, the parity of the number of intersection points of $\mathcal{D}$ with C_p and $C_{\tau_{el}(p)}$ are different.*

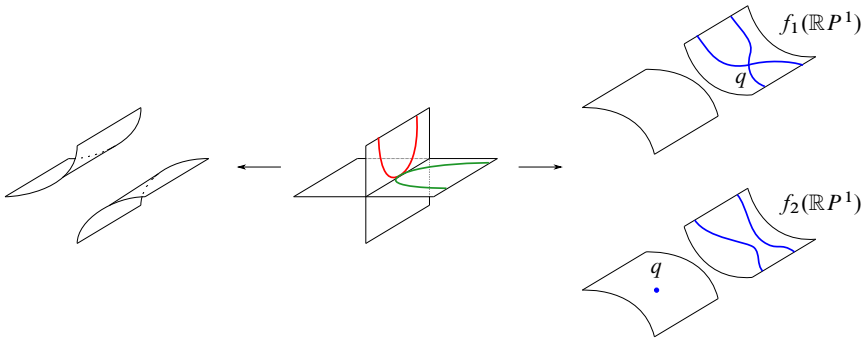
Proof. Denote by D_1 and D_2 the two halves of $E \setminus \partial\mathcal{D}$. According to the proof of Lemma 2.2, up to exchanging D_1 and D_2 we have $[D_1 \cup \mathcal{D}] = l_1$ and $[D_2 \cup \mathcal{D}] = l_2$ in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z}/2\mathbb{Z})$. Any $J_{0|X_0}$-holomorphic curve in the class l_i intersects E in exactly one point, so the result follows from the fact that both C_p and $C_{\tau_{el}(p)}$ intersect $D_1 \cup \mathcal{D}$ in an even number of points. □

3.4. Proof of Theorem 2.3. We prove Theorem 2.3 by choosing the set of sections $\underline{x}$ so that $\underline{x}(0) \cap X_0$ is reduced to a single point. In this case, it follows from Proposition 3.7 that an element $\overline{f}$ of $\mathcal{C}(d, \underline{x}(0), J_0)$ might be the limit of two distinct elements of $\mathcal{C}(d, \underline{x}(t), J_t)$. Next proposition is a real version of Proposition 3.7 in this case.

Proposition 3.10. *Suppose that $\underline{x}(0) \cap X_0 = \{p_0\}$ and $\underline{x}(0) \cap \mathbb{R}X_1 \neq \emptyset$. Let $\overline{f} : \overline{C} \to X_\sharp$ be a real element of $\mathcal{C}(d, \underline{x}(0), J_0)$, with a point $p \in C_1$ such that $\overline{f}(C_1)$ has a tangency with E at $\overline{f}(p)$. Given $t \neq 0$, let $f_1 : \mathbb{C}P^1 \to Y_t$ and $f_2 : \mathbb{C}P^1 \to Y_t$ be the two deformations of $\overline{f}$ in $\mathcal{C}(d, \underline{x}(t), J_t)$ (see Proposition 3.7).*

Then p is a real point of $\overline{C}$, and both $f_1(\mathbb{C}P^1)$ and $f_2(\mathbb{C}P^1)$ have a unique node q arising from the smoothing of $\overline{C}$ at p. Moreover, there exists $\varepsilon = \pm 1$ such that neither f_1 nor f_2 are real when $\varepsilon t < 0$, and both f_1 and f_2 are real when $\varepsilon t > 0$. In this latter case, up to exchanging f_1 and f_2, we have (see Figure 3):

- *$f_1^{-1}(q) \in \mathbb{R}P^1$;*
- *$f_2^{-1}(q) \notin \mathbb{R}P^1$ and $f_2(\mathbb{R}P^1) \cap U = \emptyset$, where U is the connected component that contains q of the intersection of Y_t with a small neighborhood in $\mathbb{R}Y$ of $\overline{f}(p)$.*



$\varepsilon t < 0$, no real deformation

$\varepsilon t > 0$, two real deformations

Figure 3. Real deformations of a real map $\overline{f} : \overline{C} \to X_\sharp$ which is the limit of two maps

Proof. It follows from Proposition 3.7 that the point p is unique, and hence real. Since $\overline{f}(\overline{C})$ has one node less that any of its deformation, we deduce that both $f_1(\mathbb{C}P^1)$ and $f_2(\mathbb{C}P^1)$ have a unique node q arising from the smoothing of $\overline{C}$ at p. Since q is unique, it has to be real if the deformation is real.

Recall from the end of the proof of Proposition 3.7 how looks like a deformation of $\overline{f}$ in a neighborhood of the smoothing of p. The manifold Y is given in local coordinates (t, x, y) at $\overline{f}(p)$ by the equation $xy = t$, the manifold X_0 (resp. X_1)

being locally given by $\{t = 0 \text{ and } y = 0\}$ (resp. $\{t = 0 \text{ and } x = 0\}$). Furthermore the maps $\overline{f}_{C_0}$ and $\overline{f}_{C_1}$ have expansions

$$x(z) = az^2 + o(z^2) \quad \text{and} \quad y(w) = bw^2 + o(w^2),$$

where $a, b \in \mathbb{R}^*$, and z and w are local coordinates at p of C_0 and C_1 respectively. For $0 < |t| << 1$, the two maps f_1 and f_2 correspond to the two solutions $\mu(t) \in \mathbb{C}^*$ of

$$\mu(t)^2 = \frac{t}{ab}.$$

For each solution, the smoothing of $\overline{C}$ at p is locally given by $zw = \mu(t)$, and the corresponding deformation is approximated by the map

$$g_t : \{zw = \mu(t)\} \subset \mathbb{C}^2 \mapsto (t, az^2, bw^2)$$

close to the smoothing of p.

If $tab < 0$, then the two solutions of $\mu(t)^2 = \frac{t}{ab}$ are complex conjugated, and neither f_1 nor f_2 are real. On the opposite, if $tab > 0$, then the two solutions of $\mu(t)^2 = \frac{t}{ab}$ are real, and both f_1 and f_2 are real. Moreover the arcs of $\mathbb{R}C_0 \setminus \{p\}$ and $\mathbb{R}C_1 \setminus \{p\}$ are glued in a different way for f_1 and f_2 (see Figure 4). In particular one of them, say f_1, satisfies $f_1^{-1}(q) \in \mathbb{R}P^1$, while f_2 satisfies $f_2^{-1}(q) \notin \mathbb{R}P^1$.

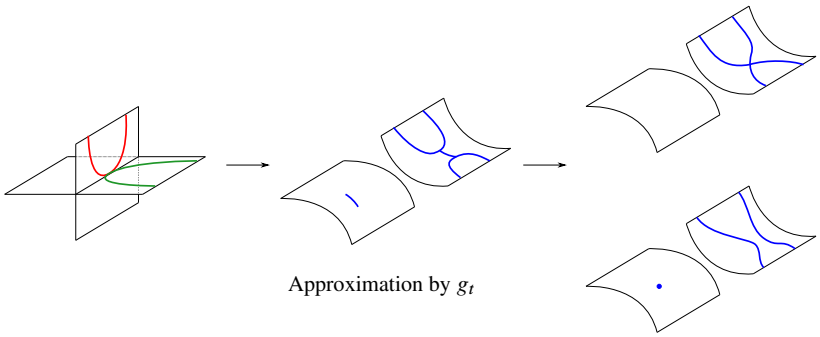


Figure 4. Real deformations of $\overline{f} : \overline{C} \to X_\sharp$, intermediate step

Let U be the connected component that contains q of the intersection of Y_t with a small neighborhood in $\mathbb{R}Y$ of $\overline{f}(p)$. We have to prove that $f_2(\mathbb{R}P^1) \cap U = \emptyset$. Suppose that this is not the case, and let $S \subset Y_t$ be a topological surface passing through q, and locally a cylinder in the variable t at q. Then the set $f_2^{-1}(S)$ would contain four points in a neighborhood of a smoothing of p. However the set $g_t^{-1}(S)$ has only two points in $\{zw = \mu(t)\}$, which contradicts the fact that f_2 is approximated by g_t close to the smoothing of p. □

Now we are ready to prove Theorem 2.3. Recall that by assumption L_0 is a disk, which in particular implies that $\tau_0 = \tau_{el}$.

Proof of Theorem 2.3(1). Suppose that $\underline{x}(0) \cap X_0 = \{p_0\}$, and $\underline{x}(0) \cap \mathbb{R}X_1$ is non-empty and contained in L_1. Without loss of generality, we may assume that $\underline{x}(t) \cap \mathbb{R}Y_t \subset L$ when $t > 0$ (and so $\underline{x}(t) \cap \mathbb{R}Y_t \not\subset L$ when $t < 0$, since L' contains the deformation of p_0). A schematic picture of the degeneration of $\mathbb{R}Y_t$ to $\mathbb{R}X_\sharp$ is provided in Figure 5. Denote by $\overline{L}$ the connected component of $\mathbb{R}X_1$ containing L_1. Since L_0 is a disk, we necessarily have $L' \neq L$, and $\overline{L} \setminus \mathbb{R}E$ is disconnected.

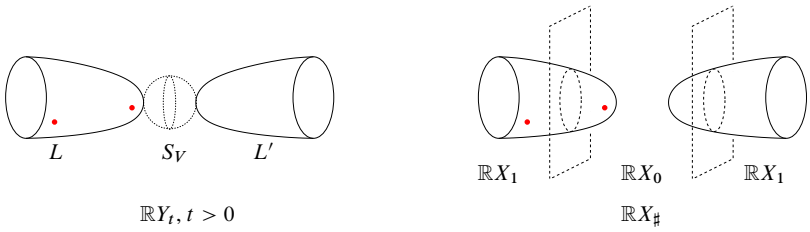


Figure 5. Degeneration of $\mathbb{R}Y_t$ to $\mathbb{R}X_\sharp$

Let $\overline{f} : \overline{C} \to X_\sharp$ be a real element of $\mathcal{C}(d, \underline{x}(0), J_0)$. Recall that $\overline{C}'$ denotes the irreducible component of C_0 whose image passes through the point p_0.

Suppose first that $\overline{f}_*[\overline{C}'] = l_i$. Since τ_{el} exchanges l_1 and l_2, there exists an irreducible component $\overline{C}''$ of C_0 such that $\tau_{el} \circ \overline{f}(\overline{C}'') = \overline{f}(\overline{C}')$. However $\overline{f}(\overline{C}'') \cap \overline{f}(\overline{C}') = \{p_0\}$, which contradicts that J_0 is generic.

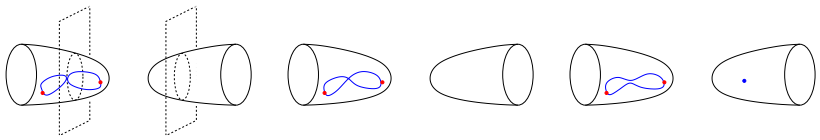


Figure 6. $\overline{f}(\overline{C})$ and its two real deformations

Hence $\overline{f}_*[\overline{C}'] = l_1 + l_2$ for any choice of $\overline{f}$. Since $\tau_0 = \tau_{el}$ and any other component of C_0 realizes a class l_i, the curve $\overline{C}'$ is the only real component of C_0, and $\overline{f}^{-1}(\mathbb{R}E)$ consists of at most 2 points. Suppose that $\overline{f}_{|\overline{C}'} \in \mathcal{C}^{2e_1,0}(l_1 + l_2, \{p_0\} \cup \underline{x}_E, J_0)$. In particular, C_1 has two irreducible components $\overline{C}_1$ and $\overline{C}_2$. Since $\underline{x}(0) \cap \mathbb{R}X_1 \neq \emptyset$, both $\overline{C}_1$ and $\overline{C}_2$ must be real with a non-empty real part. Since $\overline{C}$ has arithmetic genus 0, we deduce that $\overline{f}^{-1}(\mathbb{R}E)$ consists of precisely 2 points, which are the intersection points of $\overline{C}'$ with C_1. Hence both $\overline{f}(\mathbb{R}\overline{C}_1)$ and $\overline{f}(\mathbb{R}\overline{C}_2)$ intersect $\mathbb{R}E$ in exactly one point, where this intersection is transverse. But this contradicts the fact that $\overline{L} \setminus \mathbb{R}E$ is disconnected.

Hence $\overline{f}_{|\overline{C}'} \in \mathcal{C}^{e_2,0}(l_1 + l_2, \{p_0\} \cup \underline{x}_E, J_0)$ for any choice of $\overline{f}$, and Theorem 2.3(1) is now a consequence of Proposition 3.10 (see Figure 6). □

Proof of Theorem 2.3(2). Assume now that that $\underline{x}(0) \cap X_0 = \{p_0\}$ and $\underline{x}(0) \cap \mathbb{R}X_1 = \emptyset$. According to the proof of Theorem 2.3(1), the only maps $\overline{f} \in \mathcal{C}(d, \underline{x}(0), J_0)$ with a non-trivial contribution to $W_{X_\mathbb{R},L,F}(d,s)$ satisfy $\overline{f}_{|\overline{C}'} \in \mathcal{C}^{2e_1,0}(l_1+l_2, \{p_0\} \cup \underline{x}_E, J_0)$. In particular, the curve C_1 has two irreducible components, which are exchanged by the real structure on $\overline{C}$. There are $2^{\frac{c_1(X)\cdot d-4}{2}}$ ways of distributing the points in $\underline{x}(0) \cap X_1$ among these two components, which proves the result about divisibility of $W_{X_\mathbb{R},L,F}(d,s)$.

Moreover $\overline{C}'$ is the only real irreducible component of $\overline{C}$ and the map $\overline{f}_{|\overline{C}'}$ is an embedding. The adjunction formula implies that the number of real solitary nodes of $\overline{f}(\overline{C})$ has the same parity than $\frac{d^2-c_1(X)\cdot d+2}{2}$. □

4. Real algebraic rational surfaces

Here we deduce Theorem 1.2 from Theorem 1.1 and the classification of real rational algebraic surfaces (see for example [17, 21]). The proof goes by explicit computations of homology groups and direct application of Theorem 1.1. Recall that any real algebraic minimal rational surface with a non empty real part corresponds to exactly one of the following cases:

- $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the real structure τ_{el};
- $\mathbb{C}P^2$ equipped with the complex conjugation;
- minimal conic bundles;
- covering of degree 2 of $\mathbb{C}P^2$ ramified along a maximal real quartic;
- covering of degree 2 of the quadratic cone in $\mathbb{C}P^3$ ramified along a maximal real cubic section.

We treat all these cases in Sections 4.2, 4.3, 4.4, and 4.5, and prove Theorem 1.2 in Section 4.6.

4.1. Generalities. In this section, we fix once for all a real rational symplectic 4-manifold $X_\mathbb{R} = (X, \omega, \tau)$ and L a connected component of $\mathbb{R}X$. Since (X, ω) is diffeomorphic to either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or to $\mathbb{C}P^2$ blown-up at finitely many points, all homology groups of X are known, and the intersection form on $H_2(X; \mathbb{Z}/2\mathbb{Z})$ is non-degenerate.

Lemma 4.1. *The following hold:*

- $b_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) = b_2(X; \mathbb{Z}/2\mathbb{Z}) + b_1(L) + b_1(X \setminus L) - 1$;
- *the group $H_1(L; \mathbb{Z}/2\mathbb{Z})$ is naturally isomorphic to the kernel of the natural map $\iota : H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \to H_2(X; \mathbb{Z}/2\mathbb{Z})$;*
- *$b_1(X \setminus L) = 0$ if $[L] \neq 0$ in $H_2(X; \mathbb{Z}/2\mathbb{Z})$, and $b_1(X \setminus L) = 1$ otherwise.*

Proof. Let U be a tubular neighborhood of L in X (in particular U retracts to L). Since X is simply connected, the Mayer–Vietoris sequence applied to $X = (X \setminus L) \cup U$ gives the exact sequence

$$\begin{aligned} 0 \to H_2(U \setminus L; \mathbb{Z}/2\mathbb{Z}) &\xrightarrow{(i_2, j_2)} H_2(L; \mathbb{Z}/2\mathbb{Z}) \oplus H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \to H_2(X; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial} \\ \xrightarrow{\partial} H_1(U \setminus L; \mathbb{Z}/2\mathbb{Z}) &\xrightarrow{(i_1, j_1)} H_1(L; \mathbb{Z}/2\mathbb{Z}) \oplus H_1(X \setminus L; \mathbb{Z}/2\mathbb{Z}) \to 0. \end{aligned} \tag{4.1}$$

The space $U \setminus L$ retracts to an S^1-bundle $\psi : M \to L$ over L, hence it follows from Poincaré duality that $b_2(U \setminus L; \mathbb{Z}/2\mathbb{Z}) = b_1(U \setminus L; \mathbb{Z}/2\mathbb{Z})$. Together with the exact sequence (4.1), this implies that

$$b_2(X \setminus L; \mathbb{Z}/2\mathbb{Z}) = b_2(X; \mathbb{Z}/2\mathbb{Z}) + b_1(L) + b_1(X \setminus L) - 1.$$

Each loop γ in L produces a surface $\psi^{-1}(\gamma)$ in M. By the Gysin sequence, this induces an injective map $\kappa : H_1(L; \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H_2(M; \mathbb{Z}/2\mathbb{Z})$, and we have (intersection numbers are in $\mathbb{Z}/2\mathbb{Z}$)

$$b_2(M; \mathbb{Z}/2\mathbb{Z}) = b_1(L; \mathbb{Z}/2\mathbb{Z}) + 1 - L^2.$$

The map $\psi_* : H_2(M; \mathbb{Z}/2\mathbb{Z}) \to H_2(L; \mathbb{Z}/2\mathbb{Z})$ admits a section if and only if $L^2 = 0$. In this case the extra generator of $H_2(M; \mathbb{Z}/2\mathbb{Z})$ is precisely given by the image of such a section. By definition of the Mayer–Vietoris sequence (4.1), we obtain that

$$\operatorname{Ker} \iota \simeq \operatorname{Ker} i_2 = \operatorname{Im} \kappa \simeq H_1(L; \mathbb{Z}/2\mathbb{Z}).$$

Analogously, the natural map $\psi_* : H_1(M; \mathbb{Z}/2\mathbb{Z}) \to H_1(L; \mathbb{Z}/2\mathbb{Z})$ is surjective with kernel generated by the class ν realized by a fiber of ψ, and $\nu = 0$ if and only if $L^2 = 1$. By definition of the Mayer–Vietoris sequence (4.1), the same holds for the map i_1. We deduce that $b_1(X \setminus L) = 1 - \operatorname{rank} \partial$. If S is a closed surface in X intersecting L transversely in finitely many points $p_1, \ldots, p_k$, we have

$$\partial([S]) = [\psi^{-1}(p_1)] + \ldots + [\psi^{-1}(p_k)] = ([S] \cdot [L])\nu.$$

Hence the map ∂ is null if and only if $[L]$ is in the kernel of the intersection form on $H_2(X; \mathbb{Z}/2\mathbb{Z})$. This intersection form in non-degenerate, hence the map ∂ is null if and only if $[L] = 0$. □

The consideration of the group $\mathcal{H}(X_{\mathbb{R}}, L)$ is justified by the next two propositions.

Proposition 4.2. *Let δ be a τ-invariant class in $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$ realized by a smooth real symplectic curve E. Assume in addition that $\delta^2 \geq -1$ if E is a sphere. Then for any $d \in H_2(X; \mathbb{Z})$, we have*

$$W_{X_{\mathbb{R}}, L, F}(d, s) = (-1)^{\frac{d \cdot \delta}{2}} W_{X_{\mathbb{R}}, L, F + \delta}(d, s).$$

Proof. Choose a configuration $\underline{x}$ made of $c_1(X)\cdot d - 1 - 2s$ points in L and s pairs of τ-conjugated points in $X \setminus \mathbb{R}X$. Let J_0 be a generic τ-compatible almost complex structure on X such that E is J_0-holomorphic. By the same arguments used in the proof of Proposition 3.3, if $f : C \to X$ is a J_0-holomorphic map from a nodal curve of arithmetic genus 0, and such that $f_*[C] = d$ and $\underline{x} \subset f(C)$, then C is actually smooth and irreducible. Furthermore all intersection points of $f(C)$ and E are positive, so the intersection $E \cap f(C)$ is made of $d\cdot\delta$ distinct points if $f(C) \not\subset E$. If f is in addition real and such that L contains $f(\mathbb{R}C)$, since both curves E and $f(C)$ are real with disjoint real parts, we have that $\frac{f(C)}{2}\cdot\delta = \frac{d\cdot\delta}{2}$, and this equality is preserved modulo 2 under deformation of both f and J_0. □

Assume now that $X_{\mathbb{R}}$ is a real algebraic rational surface. The real part of a real symplectic curve C in X defines a class l_C in $H_1(X;\mathbb{Z}/2\mathbb{Z})$. It follows from the classification of real rational algebraic surfaces that any class in $H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$ is realizable by a real deformation of a real algebraic curve. If two real cycles in X intersect in finitely many points, the parity of this number only depends on the classes realized by these cycles in $H_2(X;\mathbb{Z}/2\mathbb{Z})$. Moreover the intersection form modulo 2 is non-degenerated on $H_1(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$. Hence the class l_C only depends on $[C] \in H_2(X;\mathbb{Z}/2\mathbb{Z})$, and we denote it by $l_{[C]}$.

Proposition 4.3. *Let δ an element of $Ker\ \iota \simeq H_1(L;\mathbb{Z}/2\mathbb{Z})$. Then for any $d \in H_2(X;\mathbb{Z})$, we have*

$$W_{X_{\mathbb{R}},L,F}(d,s) = (-1)^{\delta\cdot l_d}\ W_{X_{\mathbb{R}},L,F+\delta}(d,s).$$

Proof. Let C be a real symplectic curve in X. Recall that δ can be represented by the restriction over a loop γ of the boundary of a tubular neighborhood of L in X. We denote by γ' this representative of δ. Without loss of generality, we may further assume that γ intersect $\mathbb{R}C$ transversely and in finitely many points. Note that the tubular neighborhood of L in X can be chosen as small as needed. In particular, all intersection points of γ' and C are located in a neighborhood of $\mathbb{R}C \cap \gamma$, and each such point corresponds to a pair of τ-conjugated points of $\gamma' \cap C$. □

4.2. Surfaces with $\mathcal{H}(X_{\mathbb{R}}, L) = 0$. We start by giving the list of real algebraic minimal rational surfaces whose group $\mathcal{H}(X_{\mathbb{R}}, L)$ vanishes. There are exactly four of them.

Lemma 4.4. *If $X_{\mathbb{R}}$ is either $(\mathbb{C}P^1\times\mathbb{C}P^1, \tau_{el})$, or $(\mathbb{C}P^2, conj)$, or a minimal conic bundle with a connected real part, then $\mathcal{H}(X_{\mathbb{R}}, L) = 0$.*

Proof. One computes easily that $H_2((\mathbb{C}P^1\times\mathbb{C}P^1)\setminus S^2;\mathbb{Z}/2\mathbb{Z})$ (resp. $H_2(\mathbb{C}P^2 \setminus \mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})$) is generated by the class realized by a real algebraic curve in the class $l_1 + l_2$ (resp. a real conic) with an empty real part. There exist two minimal conic bundles with a connected real part, namely $X_{\mathbb{R}} = (\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_{hy})$, and a minimal conic bundle with $\mathbb{R}X = S^2$. This latter case is covered by

Section 4.3, so assume that $X_{\mathbb{R}} = (\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_{hy})$. By Lemma 4.1, we have that $b_2(X \setminus \mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) = b_2(X; \mathbb{Z}/2\mathbb{Z}) + 2$, and that the kernel of the natural map $\iota : H_2(X \setminus \mathbb{R}X; \mathbb{Z}/2\mathbb{Z}) \to H_2(X; \mathbb{Z}/2\mathbb{Z})$ is of dimension 2. Hence $H_2(X \setminus \mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathrm{Ker}\ \iota \times H_2(X; \mathbb{Z}/2\mathbb{Z})$. Any class in $H_2(X; \mathbb{Z}/2\mathbb{Z})$ is realized by a non-singular real rational algebraic curve with a non-empty real part, and has intersection number 1 with some other class in $H_2(X; \mathbb{Z}/2\mathbb{Z})$. This implies that $\mathrm{Ker}\ \iota$ is the set of τ_{hyp}-invariant classes in $H_2(X \setminus \mathbb{R}X; \mathbb{Z}/2\mathbb{Z})$, and the lemma is proved. □

4.3. Minimal conic bundles. Let (X, τ) be a minimal conic bundle whose real part is made of $n \geq 2$ spheres. Up to real deformation, we may assume that X has the following affine equation in $\mathbb{C}^3$:

$$y^2 + z^2 = \prod_{i=1}^{2n} (x - a_i)$$

where $a_1 < a_2 \ldots < a_{2n}$ are distinct real numbers, and τ is the restriction of the complex conjugation on $\mathbb{C}^3$. Forgetting the (y, z)-coordinates provides a real projection $\rho : X \to \mathbb{C}P^1$. Given $i = 1, \ldots n$, we denote by S_{2i-1} (resp. S_{2i}) the Lagrangian sphere $\rho^{-1}([a_{2i-2}; a_{2i-1}]) \cap \mathbb{R}^3$ (resp. $\rho^{-1}([a_{2i-1}; a_{2i}]) \cap \mathbb{R} \times (i\mathbb{R})^2$), with the obvious convention that $a_{2n+1} = a_0$, see Figure 7. We also denote by F a generic fiber, by E_2 an irreducible component of the singular fiber $\rho^{-1}(a_2)$, and by B a (non-real) section of ρ which does not intersect the curve E_2. The real Picard group of X is the free abelian group generated by F and $c_1(X)$ (see [17, 21]).

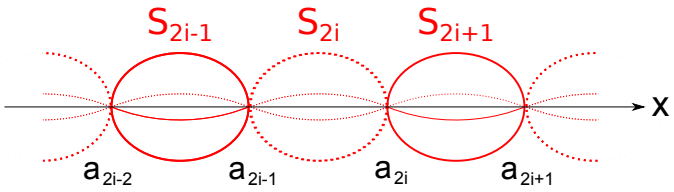


Figure 7. Real vanishing cycles of Conic bundles

Lemma 4.5. *A basis of* $\mathcal{H}(X_{\mathbb{R}}, S_1)$ *is given by* $([S_3], \ldots, [S_{2n-1}])$.

Proof. We have the following intersection products in $H_2(X; \mathbb{Z}/2\mathbb{Z})$:

$$[S_i] \cdot [S_j] = 0 \text{ if } |i - j| \neq 1, \qquad [S_i] \cdot [S_{i+1}] = 1,$$
$$[S_i] \cdot [E_2] = 0 \text{ if } i \neq 2, 3, \qquad [S_i] \cdot [E_2] = 1 \text{ if } i = 2, 3,$$
$$c_1(X) \cdot [E_2] = 1, \qquad c_1(X)^2 = [B] \cdot c_1(X) = [F] \cdot c_1(X) = [S_i] \cdot c_1(X) = 0,$$

and

$$[F]^2 = [F]\cdot[S_i] = [F]\cdot[E_2] = [B]\cdot[E_2] = [B]\cdot[S_i] = [B]^2 = 0,$$
$$[E_2]^2 = 1, \qquad [B]\cdot[F] = 1.$$

In particular $(c_1(X), [B], [F], [E_2], [S_2] \ldots, [S_{2n-1}])$ is a free family of $H_2(X;\mathbb{Z}/2\mathbb{Z})$, and hence is a basis since $b_2(X;\mathbb{Z}/2\mathbb{Z}) = 2n+2$.

From the intersection $[S_1]\cdot[S_2] = 1$, we deduce that $[S_1] \neq 0$ in $H_2(X;\mathbb{Z}/2\mathbb{Z})$, and Lemma 4.1 implies that $b_2(X \setminus L;\mathbb{Z}/2\mathbb{Z}) = b_2(X;\mathbb{Z}/2\mathbb{Z}) - 1$. A basis of $H_2(X \setminus L;\mathbb{Z}/2\mathbb{Z})$ is then given by

$$(c_1(X), [B], [F], [E_2], [S_3], \ldots, [S_{2n-1}]),$$

since its rank in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ is at most its rank in $H_2(X \setminus L;\mathbb{Z}/2\mathbb{Z})$. The classes $c_1(X)$, $[F]$, and $[S_i]$ are τ-invariant, and we have:

$$\tau_*[B] = [B] + n[F] + c_1(X), \quad \text{and} \quad \tau_*[E_2] = [E_2] + F.$$

It follows that $(c_1(X), [F], [S_3], \ldots, [S_{2n-1}])$ is a basis of the subspace of τ-invariant classes of $H_2(X \setminus L;\mathbb{Z}/2\mathbb{Z})$, and the lemma is proved. □

4.4. Minimal real Del Pezzo surface of degree 2. Let Q be the real quartic in $\mathbb{C}P^2$ whose real part together its position with respect to a bitangent H is depicted in Figure 8a. We denote by (X, τ) the real double covering $\rho : X \to \mathbb{C}P^2$ ramified along Q, whose real part consists of four spheres. The real Picard group of X is the free abelian group generated by $c_1(X)$ (see [17, 21]).

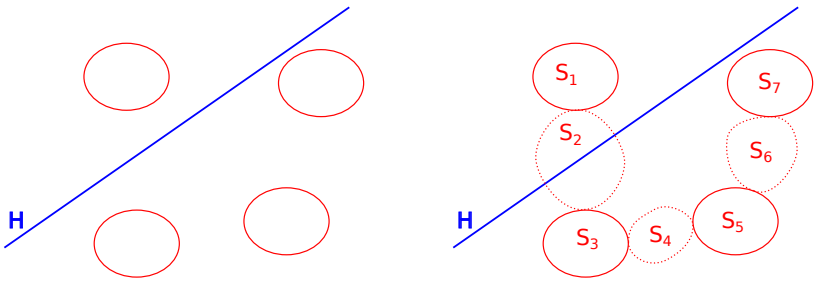


Figure 8. Real vanishing cycles of a minimal real Del Pezzo surface of degree 2

There exists a (-1)-curve E such that $\rho(E) = H$. Let S_1, S_3, S_5, and S_7 by the four spheres of $\mathbb{R}X$. By the rigid isotopy classification of real plane quartics, each pair of real spheres is connected by a τ-invariant vanishing Lagrangian sphere. Let S_2 (resp. S_4, S_6) such a sphere connecting S_1 and S_3 (resp. S_3 and S_5, S_5 and S_7) as depicted in Figure 8b.

Lemma 4.6. *A basis of* $\mathcal{H}(X_{\mathbb{R}}, S_1)$ *is given by* $([S_3], \ldots, [S_7])$.

Proof. We have the following intersection products in $H_2(X; \mathbb{Z}/2\mathbb{Z})$:

$$[S_i] \cdot [S_j] = 0 \text{ if } |i-j| \neq 1, \qquad [S_i] \cdot [S_{i+1}] = 1, \qquad [S_i] \cdot [E] = 0 \text{ if } i \neq 2,$$
$$[S_2] \cdot [E] = 1,$$
$$c_1(X) \cdot [E] = 1, \qquad c_1(X)^2 = [S_i] \cdot c_1(X) = 0, \quad \text{and} \quad [E]^2 = 1.$$

In particular $(c_1(X), [E], [S_2], \ldots, [S_7])$ is a basis of $H_2(X; \mathbb{Z}/2\mathbb{Z})$, and $(c_1(X), [E], [S_3], \ldots, [S_7])$ is a basis of $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$. The classes $c_1(X)$ and $[S_i]$ are τ-invariant, and $\tau_*[E] = c_1(X) + [E]$. Hence $(c_1(X), [S_3], \ldots, [S_{2n-1}])$ is a basis of the subspace of τ-invariant classes of $H_2(X \setminus L; \mathbb{Z}/2\mathbb{Z})$, and the lemma is proved. □

4.5. Minimal real Del Pezzo surface of degree 1. Let Q be the real cubic section of the quadratic cone Σ in $\mathbb{C}P^3$ whose real part together with its position with respect to a tritangent hyperplane section H is depicted in figure 9a. We denote by (X, τ) the real double covering $\rho : X \to \Sigma$ ramified along Q whose real part consists of four spheres and a real projective plane. The real Picard group of X is the free abelian group generated by $c_1(X)$ (see [17, 21]).

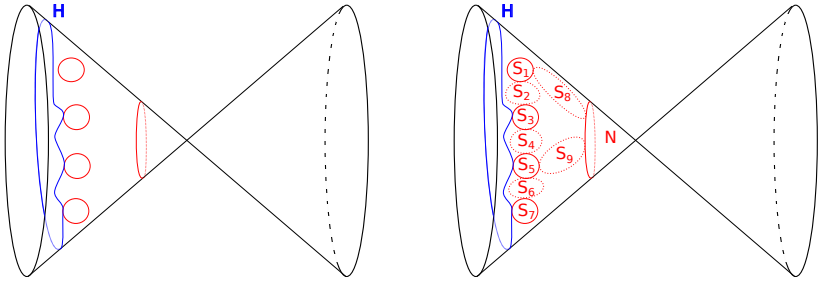


Figure 9. Real vanishing cycles of a minimal real Del Pezzo surface of degree 1

There exists a (-1)-curve E such that $\rho(E) = H$. Let S_1, S_3, S_5, S_7 and N be respectively the four spheres and the real projective plane of $\mathbb{R}X$. By the rigid isotopy classification of real cubic sections of Σ, there exist τ-invariant vanishing Lagrangian spheres S_2, S_4, S_6, S_8, S_9 as depicted in Figure 9b. Note that τ acts trivially on $H_2(X; \mathbb{Z}/2\mathbb{Z})$.

Lemma 4.7.

A basis of $\mathcal{H}(X_{\mathbb{R}}, S_1)$ *is given by* $([S_3], \ldots, [S_7], [S_9], [N])$.

A basis of $\mathcal{H}(X_{\mathbb{R}}, S_7)$ *is given by* $([S_1], \ldots, [S_5], [S_8], [N])$.

A basis of $\mathcal{H}(X_{\mathbb{R}}, N)$ *is given by* $([S_1], \ldots, [S_7])$.

Proof. All intersection products of $[S_i]$ with $[S_j]$, $[N]$, and $[E]$, can be read on Figure 9b. The other intersection products in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ are:

$$c_1(X)^2 = [E]^2 = [N]^2 = [E]\cdot c_1(X) = [N]\cdot c_1(X) = 1, \quad [N]\cdot[E] = 0.$$

Hence $(c_1(X), [S_1], \dots, [S_8])$ is a basis of $H_2(X;\mathbb{Z}/2\mathbb{Z})$, and we have

$$[N] = c_1(C) + [S_1] + [S_3] + [S_5] + [S_7] \quad \text{and} \quad [S_9] = [S_8] + [S_2] + [S_4].$$

The result about $\mathcal{H}(X_{\mathbb{R}}, S_1)$ and $\mathcal{H}(X_{\mathbb{R}}, S_7)$ follows immediately.

The generator of $Ker\ \iota$ can be represented by $B = \rho^{-1}(A)$, where A is a hyperplane section of Σ. Hence $([B], [E], [S_1], \dots, [S_7])$ is a basis of $H_2(X \setminus N;\mathbb{Z}/2\mathbb{Z})$. Since $[E] + \tau_*[E] = B$, we get that $([B], [S_1], \dots, [S_7])$ is a basis of the subspace of τ-invariant classes of $H_2(X \setminus N;\mathbb{Z}/2\mathbb{Z})$, and the lemma is proved. □

4.6. Proof of Theorem 1.2. Lemmas 4.4, 4.5, 4.6, and 4.7 and Theorem 1.1 provide a proof of Theorem 1.2 in the case of minimal real algebraic rational surfaces, and when $F = [\mathbb{R}X \setminus L]$. To end the proof, we start with the following remark: if $(\widetilde{X}, \widetilde{\tau})$ is a blow up of (X, τ) at a real point or at a pair of τ-conjugated points, and if $\widetilde{L}$ is the component of $\mathbb{R}\widetilde{X}$ corresponding to L, then there is a natural injective group homomorphism $\phi : \mathcal{H}(X_{\mathbb{R}}, L) \to \mathcal{H}(\widetilde{X}_{\mathbb{R}}, \widetilde{L})$. Theorem 1.2 now follows immediately from next proposition.

Proposition 4.8. *The map ϕ is an isomorphism.*

Proof. This is clearly true when $(\widetilde{X}, \widetilde{\tau})$ is a blow up of (X, τ) at a real point in $\mathbb{R}X \setminus L$ or at a pair of τ-conjugated points. Hence let us now assume that $(\widetilde{X}, \widetilde{\tau})$ is a blow up of (X, τ) at a point $p \in L$. Since $[\widetilde{L}] \neq 0$ in $H_2(X;\mathbb{Z}/2\mathbb{Z})$, by Lemma 4.1 we have

$$b_2(\widetilde{X} \setminus \widetilde{L};\mathbb{Z}/2\mathbb{Z}) = b_2(X \setminus L;\mathbb{Z}/2\mathbb{Z}) + 1$$

if $(X, \tau) = (\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_{hy})$, and

$$b_2(\widetilde{X} \setminus \widetilde{L};\mathbb{Z}/2\mathbb{Z}) = b_2(X \setminus L;\mathbb{Z}/2\mathbb{Z}) + 2$$

otherwise. In both cases, an extra generator of $H_2(\widetilde{X} \setminus \widetilde{L};\mathbb{Z}/2\mathbb{Z})$ is given by the extra generator of $H_1(\widetilde{L};\mathbb{Z}/2\mathbb{Z})$. In particular this proves the proposition in the case $(X, \tau) = (\mathbb{C}P^1 \times \mathbb{C}P^1, \tau_{hy})$, and implies

$$\dim \mathcal{H}(\widetilde{X}_{\mathbb{R}}, \widetilde{L}) \leq \dim \mathcal{H}(X_{\mathbb{R}}, L) + 1$$

otherwise. In this latter case, from the classification of minimal real algebraic surfaces up to deformation, we may assume that there exists an algebraic curve C in X such that $C \cap L = \{p\}$ and that this intersection is transverse. Hence the strict transform of C in $\widetilde{X}$ is a second extra generator of $H_2(\widetilde{X} \setminus \widetilde{L};\mathbb{Z}/2\mathbb{Z})$, which is either not τ-invariant or mapped to 0 in $\mathcal{H}(\widetilde{X}_{\mathbb{R}}, \widetilde{L})$. □

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