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Livšic theorem for low-dimensional diffeomorphism cocycles

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Abstract. We prove a Livšic type theorem for cocycles taking values in groups of diffeomorphisms of low-dimensional manifolds. The results hold without any localization assumption and in very low regularity. We also obtain a general result (in any dimension) which gives necessary and sufficient conditions to be a coboundary.

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1. Introduction

In the study of hyperbolic dynamical systems there is a general (and vague) idea that can be summarized with the following sentence:

> *Most of the dynamical interesting information on a hyperbolic system is concentrated in its periodic orbits.*

An archetypal example of a result supporting this idea is the celebrated Livšic's theorem [16, 17] claiming that given a hyperbolic homeomorphism $f: M \circlearrowleft$, a Hölder function $\Phi: M \to \mathbb{R}$ is a *coboundary*, i.e. there exists a continuous function $u: M \to \mathbb{R}$ satisfying

$$u \circ f - u = \Phi,$$

if and only if

$$\sum_{j=0}^{n-1} \Phi\big(f^j(p)\big) = 0,$$

for every periodic point $p \in M$, with $f^n(p) = p$.

Due to the interest this result has received since its appearance and the large amount of consequences that follow from it, several generalizations have been studied. Some of them consider more general dynamics on the base. For instance, in the

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works [13, 23] the cohomology of real cocycles over partially hyperbolic systems is analyzed.

In this paper, we consider a different kind of generalization: given a complete metric group G, a *G-cocycle* is just a continuous map $\Phi\colon M \to G$ and one wants to determine whether the condition

$$\Phi\big(f^{n-1}(p)\big)\Phi\big(f^{n-2}(p)\big)\cdots\Phi(p) = e_G, \quad \forall p \in \mathrm{Fix}(f^n),\ \forall n \geqslant 1,$$

where e_G is the identity element of G, is not just necessary but also sufficient to guarantee the existence of a "transfer function" $u\colon M \to G$ satisfying

$$\Phi(x) = u(f(x))u(x)^{-1}, \quad \forall x \in M.$$

Livšic himself gave in [16] an affirmative answer to this question for cocycles taking values on a topological group admitting a complete bi-invariant distance (e.g. Abelian or compact groups). However, the general situation is considerably more complicated.

So far, the main technique to handle this problem when the group G does not admit a bi-invariant metric has consisted in considering a left-invariant metric on G and to try to control the distortion produced by right translations to be able to apply the very same scheme of proof used in the Abelian case.

In order to get such a control of distortion of the distance, some *localization hypotheses* have been considered in the literature. For instance, in [17] Livšic gave a positive answer to above question for linear cocycles (i.e. where $G = \mathrm{GL}_d(\mathbb{R})$) which are not too far away from the identity constant cocycle. Improvements of this result using weaker localization hypotheses have been obtained for cocycles taking values in arbitrary finite-dimensional Lie groups (see [5, 14] and references therein) until the recent complete solution of the global Livšic problem for linear cocycles [10] (see also the recent preprint [7]).

In the infinite dimensional case, particularly when G is a group of diffeomorphisms of a compact manifold, the study began with the seminal paper of Niţică and Török [20]. Diffeomorphism groups seem to be the most interesting infinite dimensional groups for applications to rigidity theory (see [14] and references therein).

In contrast with the finite dimensional case, all the results for groups of diffeomorphisms obtained so far (see [5] for a survey with references) involve non-sharp localization hypotheses (in the sense that not every coboundary satisfies them) and require higher regularity for the diffeomorphism group (i.e. $G = \mathrm{Diff}^r(N)$, with $r \geqslant 4$). Moreover, the control of distortion techniques used in [5, 20] yield a loss of regularity in the solution of the cohomological equation. A recent result for infinite dimensional groups which does not fit in the previous description is due to Navas and Ponce [19]. They prove a *Livšic theorem* for cocycles taking values in the group of analytic germs at the origin of $\mathbb{C}$.

In this paper we use completely different techniques, with a more geometric flavor, which allow us to deal with the low regularity case (cocycles can take values in the group of C^1-diffeomorphisms). The main novelty of our approach is Theorem 3.1 which can be of independent interest. Regarding this result within the context of localization arguments, we could say that it is proven that our *non-uniform localization hypothesis* (i.e. vanishing of fibered Lyapunov exponents) is equivalent to be a coboundary.

In Section 4 we show that in the low-dimensional case the periodic orbit condition implies the vanishing of fibered Lyapunov exponents, proving in such cases the general (or global) *Livšic theorem* for groups of diffeomorphisms. We conjecture that such a result holds in any dimension.

Since our result was announced, several new ones concerning the cohomology of cocycles having equal periodic data have appeared. We refer the reader to [3] and references therein for more information on this important problem.

1.1. Main results.

The main results of this article are the following Livšic type theorems which, to the best of our knowledge, are the first general (i.e. global) results for cocycles taking values in groups of diffeomorphisms of compact manifolds:

Theorem A. *Let* $f\colon M \circlearrowleft$ *be a hyperbolic homeomorphism and* $\Phi\colon M \to \mathrm{Diff}^1(\mathbb{R}/\mathbb{Z})$ *be an* α*-Hölder cocycle such that the so called* periodic orbit obstructions *vanish; i.e.*

$$\Phi(f^{n-1}(p)) \circ \Phi(f^{n-2}(p)) \circ \cdots \circ \Phi(p) = id_{\mathbb{R}/\mathbb{Z}}, \quad \forall p \in \mathrm{Fix}(f^n),\ \forall n \in \mathbb{N}.$$

Then, there exists an α*-Hölder map* $u\colon M \to \mathrm{Diff}^1(\mathbb{R}/\mathbb{Z})$ *satisfying*

$$\Phi(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

Remark 1.1. The very same argument we use to prove Theorem A works for cocycles taking values in $\mathrm{Diff}^1([0,1])$

For higher regularity, invoking the main result of [6], one easily gets the following consequence of Theorem A:

Corollary 1.2. *Let* $f\colon M \circlearrowleft$ *be a hyperbolic homeomorphism and* $\Phi\colon M \to \mathrm{Diff}^r(\mathbb{R}/\mathbb{Z})$*, with* $r \geqslant 1$*, be an* α*-Hölder cocycle for which the* periodic orbit obstruction *vanishes.*

Then, there exists an α*-Hölder map* $u\colon M \to \mathrm{Diff}^r(\mathbb{R}/\mathbb{Z})$ *satisfying*

$$\Phi(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

Assuming higher regularity on the dynamics of the base and the cocycle, applying the results of [9, 21] one can improve the regularity of the solution of the cohomological equation. Since this kind of results is beyond the scope of this article, we suggest the interested reader to consult [14, 23] for further information.

In dimension 2, we can obtain a similar result for the group of area-preserving diffeomorphisms:

Theorem B. *Let M be a smooth closed manifold and $f: M \circlearrowleft$ be a $C^{1+\theta}$ transitive Anosov diffeomorphism. Let S denote a compact surface, let vol be a Lebesgue probability measure on S and let $\mathrm{Diff}^r_{vol}(S)$ be the group of C^r-diffeomorphisms of S that leave vol invariant.*

Let $\Phi: M \to \mathrm{Diff}^{1+\alpha}_{vol}(S)$ be a $C^{1+\alpha}$-cocycle[1] *for which the periodic orbit obstruction vanishes.*

Then, there exists an α-Hölder map $u: M \to \mathrm{Diff}^1_{vol}(S)$ such that

$$\Phi(x) = u(f(x)) \circ u(x)^{-1}, \quad \forall x \in M.$$

It should be noted that Theorem B, unlike Theorem A, is stated for Anosov diffeomorphisms on the base (and in paricular, the base space M should be a manifold) instead of a hyperbolic homeomorphisms. This restriction appears as a technical simplification which will be explained in the proof, but the theorem should remain true assuming weaker hypotheses.

Proofs of Theorems A and B consist in two steps. The first one concerns the vanishing of Lyapunov exponents for cocycles satisfying the periodic orbit condition and relies heavily in the low-dimensionality of the fibers. The second one holds in any dimension and can be of independent interest (see Theorem 3.1).

To end the introduction, let us mention that as far as we know, the following question is still open:

Question 1.3. *Does there exist a complete metric group*[2] *G such that a Livšic like theorem does not hold for G-cocyles? More precisely, does there exist a hyperbolic homeomorphism $f: M \circlearrowleft$ and a C^α-cocycle $\Phi: M \to G$ with vanishing periodic orbit obstructions but which is not a G-coboundary?*

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[1]This means that the induced map $M \times S \ni (x, y) \mapsto \Phi(x)(y) \in S$ is $C^{1+\alpha}$.

[2]A topological group which admits a complete distance compatible with the topology of the group.

2. Preliminaries and notations

2.1. Hölder continuity. All along this paper, (M, d) will denote a compact metric space. If (M', d') denotes another arbitrary metric space and $0 < \alpha \leqslant 1$, a map $\psi \colon M \to M'$ is said to be *α-Hölder* whenever

$$|\psi|_\alpha := \sup_{x \neq y} \frac{d'\big(\psi(x), \psi(y)\big)}{d(x,y)^\alpha} < \infty.$$

Most of the functions and maps we shall deal with in this paper will be at least Hölder because, as it was already observed in [15], in general C^0-regularity is not appropriate for dynamical cohomology.

When $\alpha < 1$, the space of α-Hölder maps from M to M' will be denoted by $C^\alpha(M, M')$. As usual, we use the term *Lipschitz* as a synonym of 1-Hölder, and to avoid confusions with the differentiable case, we write $C^{\mathrm{Lip}}(M, M')$ for the space of Lipschitz functions.

For such a real constant α, we can define a new distance on M by

$$d_\alpha(x, y) := d(x, y)^\alpha, \quad \forall x, y \in M. \tag{2.1}$$

Observe that the topologies induced by d and d_α coincide and a map $\psi \colon (M, d) \to (M', d')$ is α-Hölder if and only if $\psi \colon (M, d_\alpha) \to (M', d')$ is Lipschitz.

2.2. Borel probability measures. Given an arbitrary locally compact metric space X, we write $\mathfrak{M}(X)$ for the space of Borel probability measures on X and we will always consider it endowed with (the restriction of) the weak-$\star$ topology. If Y denotes another compact metric space, any continuous map $h \colon X \to Y$ naturally induces a linear map $h_\star \colon \mathfrak{M}(X) \to \mathfrak{M}(Y)$ characterized by the following property:

$$\int_Y \phi \,\mathrm{d}(h_\star\mu) := \int_X \phi \circ h \,\mathrm{d}\mu, \quad \forall \phi \in C^0_c(Y), \ \forall \mu \in \mathfrak{M}(X).$$

In this way, if $f \colon X \circlearrowleft$ is a continuous map, one defines the space of f-invariant measures by

$$\mathfrak{M}(f) := \{\mu \in \mathfrak{M}(M) : f_\star\mu = \mu\}.$$

2.3. Hyperbolic homeomorphisms. Let (M, d) be a compact metric space and $f \colon M \circlearrowleft$ be a homeomorphism. Given any $x \in M$ and $\epsilon > 0$, one defines the *local stable* and *unstable sets* by

$$W^s_\varepsilon(x, f) := \{y \in M : d(f^n(x), f^n(y)) \leqslant \varepsilon, \ \forall n \geqslant 0\},$$
$$W^u_\varepsilon(x, f) := \{y \in M : d(f^n(x), f^n(y)) \leqslant \varepsilon, \ \forall n \leqslant 0\},$$

respectively. Where there is no risk of ambiguity, we just write $W^s_\epsilon(x)$ instead of $W^s_\epsilon(x, f)$, and the same holds for local unstable sets.

Following [2], we introduce the following definition.

Definition 2.1. A homeomorphism $f: M \circlearrowleft$ is said to be *hyperbolic with local product structure* whenever there exist constants $\varepsilon_0, \delta_0, K_0, \lambda > 0$ and functions $\nu_s, \nu_u: M \to (0, \infty)$ such that the following conditions are satisfied:

(h1) $d(f(y_1), f(y_2)) \leqslant \nu_s(x) d(y_1, y_2), \forall x \in M, \forall y_1, y_2 \in W^s_{\varepsilon_0}(x)$;

(h2) $d(f(y_1), f(y_2)) \geqslant \nu_u(x) d(y_1, y_2), \forall x \in M, \forall y_1, y_2 \in W^u_{\varepsilon_0}(x)$;

(h3) $\nu_s^{(n)}(x) := \nu_s(f^{n-1}(x)) \dots \nu_s(x) < K_0 e^{-\lambda n}, \forall x \in M, \forall n \geqslant 1$;

(h4) $\nu_u^{(n)}(x) := \nu_u(f^{n-1}(x)) \dots \nu_u(x) > K_0 e^{\lambda n}, \forall x \in M, \forall n \geqslant 1$;

(h5) If $d(x, y) \leqslant \delta_0$, then $W^u_{\varepsilon_0}(x)$ and $W^s_{\varepsilon_0}(y)$ intersect in a unique point which is denoted by $[x, y]$, and it depends continuously on x and y.

Remark 2.2. For the sake of simplicity of the exposition and to avoid unnecessary repetitions, from now on we shall assume that all hyperbolic homeomorphisms are transitive and exhibit local product structure.

For such homeomorphisms, one can define the *stable* and *unstable sets* by

$$W^s(x, f) := \bigcup_{n \geqslant 0} f^{-n}\big(W^s_\epsilon(f^n(x))\big) \quad \text{and} \quad W^u(x, f) := \bigcup_{n \geqslant 0} f^n\big(W^u_\epsilon(f^{-n}(x))\big),$$

respectively.

Notice that shifts of finite type and basic pieces of Axiom A diffeomorphisms are particular examples of hyperbolic homeomorphisms with local product structure (see for instance [18, Chapter IV, §9] for details).

Remark 2.3. For our purposes, it is important to notice that the notion of hyperbolicity for homeomorphisms is invariant under Hölder changes of metric. More precisely, a homeomorphism $f: (M, d) \circlearrowleft$ is hyperbolic if and only $f: (M, d_\alpha) \circlearrowleft$ is hyperbolic, for any $\alpha \in (0, 1)$, where the distance d_α is defined by (2.1).

The following result is proven for locally maximal hyperbolic sets of smooth diffeomorphisms in [12, Chapter 6]. However, by inspection on the proof (see [12, Corollary 6.4.17 and Proposition 6.4.16]) it can be easily check that the very same proof works for hyperbolic homeomorphisms with local product structure (see also [10, p. 1026]):

Theorem 2.4 (Anosov closing lemma). *Let $f: (M, d) \circlearrowleft$ be a hyperbolic homeomorphism. Then, there exist constants $c, \delta_1 > 0$ such that for every $x \in M$ and any $n > 0$ satisfying $d(x, f^n(x)) < \delta_1$, there exist unique points $p \in \mathrm{Fix}(f^n)$ and $y \in M$ such that:*

(1) $d(f^i(x), f^i(p)) \leqslant c\, d(x, f^n(x)) e^{-\lambda \min\{i, n-i\}}$;

(2) $d(f^i(p), f^i(y)) \leqslant c\, d(x, f^n(x)) e^{-\lambda i}$;

(3) $d(f^i(x), f^i(y)) \leqslant c\, d(x, f^n(x)) e^{-\lambda(n-i)}$;

for every $i \in \{0, \dots, n-1\}$, where $\lambda > 0$ is the constant given in Definition 2.1.

Remark 2.5. Notice that by uniqueness, we have that $y = [x, p]$, where the brackets $[\cdot, \cdot]$ are given by Definition 2.1.

2.4. Cocycles and coboundaries. Let G denote a topological group whose topology is induced by a complete distance function d_G, and let $f\colon (M, d) \circlearrowleft$ be a homeomorphism.

In this work all cocycles we consider will be at least continuous. In fact, a G-*cocycle* (over f) is just a continuous map $\Phi\colon M \to G$. As usual, we use the following notation

$$\Phi^{(n)}(x) := \begin{cases} e_G, & \text{if } n = 0; \\ \Phi(f^{n-1}(x))\Phi^{(n-1)}(x), & \text{if } n > 0; \\ \left(\Phi^{(-n)}(f^n(x))\right)^{-1}, & \text{if } n < 0; \end{cases}$$

where $e_G \in G$ denotes the identity element of G. We say that Φ is a Hölder cocycle when $\Phi\colon (M, d) \to (G, d_G)$ is an α-Hölder map, for some $\alpha \in (0, 1]$.

A G-cocycle Φ is said to be a G-*coboundary* when there exists a continuous map $u\colon M \to G$ such that

$$\Phi(x) = u(f(x)) \cdot (u(x))^{-1}, \quad \forall x \in M.$$

Notice this implies that $\Phi^{(n)}(x) = u(f^n(x)) \cdot (u(x))^{-1}$, for any $n \in \mathbb{Z}$ and any $x \in M$.

The first family of natural obstructions one encounters for a G-cocycle to be a G-coboundary is that over periodic orbits, the cocycle must vanish.

$$\Phi^{(n)}(p) = e_G, \quad \forall n \geqslant 1, \ \forall p \in \mathrm{Fix}(f^n). \tag{POO}$$

Equation (POO) implies the *vanishing of the periodic orbit obstructions*.

In this work we mainly concentrate in the case where $G = \mathrm{Diff}^1(N)$. To deal with such objects, we need a slight generalization of the concept of cocycle that we introduce in the following paragraph.

2.5. Fiber bundle maps and cocycles. Let N denote a compact differentiable manifold and (M, d) be compact metric space as above. Given any $\alpha \in (0, 1]$ and $r \geqslant 0$, a $C^{\alpha,r}$-*fiber bundle* over M with fiber N is an object $N \to \mathcal{E} \xrightarrow{\pi} M$, where $\mathcal{E}$ is a topological space and π is a surjective α-Hölder map such that there exists a finite open cover $\{U_j\}_1^n$ of M with the following properties:

- For each $j \in \{1, \dots, n\}$, there exists a homeomorphism $\varphi_j\colon \pi^{-1}(U_j) \to U_j \times N$;
- If $U_i \cap U_j \neq \varnothing$, there is an α-Hölder map $g_{ij}\colon U_i \cap U_j \to \mathrm{Diff}^r(N)$ such that

$$\varphi_i \circ \varphi_j^{-1}(x, y) = (x, (g_{ij}(x))(y)), \quad \forall (x, y) \in (U_i \cap U_j) \times N.$$

As usual, one defines *the fiber over x* by $\mathcal{E}_x := \pi^{-1}(x) \subset \mathcal{E}$. Due to the nature of maps (g_{ij}) involved in change of coordinates, each fiber can naturally endowed with a C^r-differentiable structure turning it into a C^r-manifold C^r-diffeomorphic to N.

As usual, when $N = \mathbb{R}^d$ and the maps g_{ij} are α-HÃűlder and have their image contained in $\mathrm{GL}_d(\mathbb{R})$, we say that $\mathbb{R}^d \to \mathcal{E} \xrightarrow{\pi} M$ is a C^α-vector bundle.

The total space of any $C^{\alpha,r}$-fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$ can be endowed with a distance function $d_{\mathcal{E}}$ constructed as follows:

Let d_N be a distance function compatible with the smooth structure of the fiber manifold N, $\{U_j\}_1^n$ and $\{\varphi_j\}_1^n$ be a local trivialization atlas as above, $L > 0$ be the Lebesgue number of the open covering $\{U_j\}_1^n$ and define

$$d_{\mathcal{E}}(\zeta, \eta) := \min \left\{ d\big(\pi(\zeta), \pi(\eta)\big) + \inf_{1 \leqslant j \leqslant n} \left\{ d_N \Big(\mathrm{pr}_2\big(\varphi_j(\zeta)\big), \mathrm{pr}_2\big(\varphi_j(\eta)\big) \Big) \right\} ; L \right\}. \tag{2.2}$$

where $\mathrm{pr}_2 \colon M \times N \to N$ denotes the projection on the second coordinate, and by convention, we declare that $d_N\Big(\mathrm{pr}_2\big(\varphi_j(\zeta)\big), \mathrm{pr}_2\big(\varphi_j(\eta)\big)\Big) = \mathrm{diam}_{d_N} N$, whenever either ζ or η does not belong to $\pi^{-1}(U_j)$.

A *Riemannian structure* on a $C^{\alpha,r}$-fiber bundle $\mathcal{E}$ consists of choosing a Riemannian metric on each fiber $\mathcal{E}_x$ which varies Hölder-continuously with $x \in M$.

From now on, we will assume every fiber bundle is endowed with a fixed Riemannian structure and a distance function constructed as above. All the concepts we will consider about fiber bundles are completely independent of these chosen structures.

In this setting, given another $C^{\alpha,r}$-fiber bundle $N \to \tilde{\mathcal{E}} \xrightarrow{\tilde{\pi}} M$ and a C^α-homeomorphism $f \colon M \circlearrowleft$, a $C^{\alpha,r}$*-bundle map* over f is a homeomorphism $F \colon \mathcal{E} \to \tilde{\mathcal{E}}$ satisfying $\tilde{\pi} \circ F = f \circ \pi$ and such that the map $F_x := F|_{\mathcal{E}_x} \colon \tilde{\mathcal{E}}_x \to \mathcal{E}_{f(x)}$ is a C^r-diffeomorphism, for every $x \in M$.

As usual, the fiber bundle $N \to \mathcal{E} = M \times N \xrightarrow{\mathrm{pr}_1} M$, where $\mathrm{pr}_1 \colon M \times N \to M$ denotes the projection on the first coordinate, is called the *trivial fiber bundle.*

Observe any C^α cocycle $\Phi \colon M \to \mathrm{Diff}^r(N)$ naturally induces a $C^{\alpha,r}$-bundle map on the trivial fiber bundle $N \to M \times N \to M$ and over any $f \in \mathrm{Homeo}(M)$ just defining $F = F_{\Phi,f} \colon M \times N \circlearrowleft$ by

$$F(x, y) := \big(f(x), \Phi(x)(y)\big), \quad \forall (x, y) \in M \times N. \tag{2.3}$$

Inn such a case, $F_x = \Phi(x)$, for every $x \in M$.

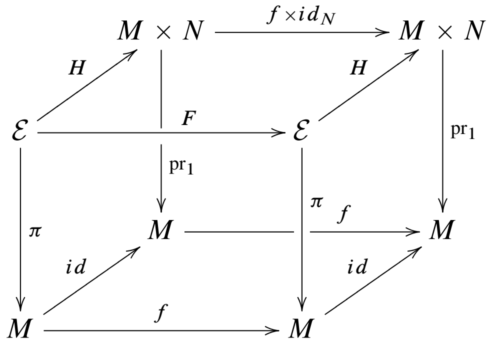
Such a particular bundle map is usually called the *skew-product* induced by Φ and f. So, bundle maps can be considered as generalizations of cocycles taking values in groups of diffeomorphisms.

2.6. POO and coboundaries for bundle maps. We can easily extend the notion of *vanishing of the periodic orbit obstructions* to fiber bundle maps defined on non-trivial fiber bundles.

Given a bundle map $F:\mathcal{E} \circlearrowleft$ over $f:M \circlearrowleft$, we say that the *periodic orbit obstruction* vanishes whenever, for every $n \geqslant 1$, it holds

$$F_p^n(\zeta) = \zeta, \quad \forall p \in \mathrm{Fix}(f^n),\ \forall \zeta \in \mathcal{E}_p. \tag{POO}$$

Now we finish this paragraph extending the notion of *coboundary* for (*à priori* more general) fiber bundle maps: if $N \to \mathcal{E} \xrightarrow{\pi} M$ denotes a $C^{\alpha,r}$-fiber bundle, a $C^{\alpha,r}$-bundle map $F:\mathcal{E} \circlearrowleft$ is said to be a $C^{\alpha,s}$*-coboundary*, with $s \leqslant r$, when there exists a $C^{\alpha,s}$-bundle map $H:\mathcal{E} \to M \times N$ over the identity map $id: M \circlearrowleft$ such that the following diagram commutes:



(2.4)

Observe that with our definition, coboundaries just exist on tivial fiber bundles. This definition is mainly motivated by Theorem 3.1.

2.7. Lyapunov exponents for bundle maps. Let $\pi:\mathcal{E} \to M$ be a $C^{\alpha,r}$-fiber bundle (endowed with a Riemannian structure) and consider a $C^{\alpha,r}$-bundle map $F:\mathcal{E} \circlearrowleft$ over $f:M \circlearrowleft$. Given any point $\zeta \in \mathcal{E}$ and $n \in \mathbb{Z}$, we have the linear map

$$D\left(F^n|_{\mathcal{E}_{\pi(\zeta)}}\right)_\zeta : T_\zeta \mathcal{E}_{\pi(\zeta)} \to T_{F^n(\zeta)} \mathcal{E}_{f^n(\pi(\zeta))} \tag{2.5}$$

between normed vector spaces, and hence it makes sense to talk about its norm. For the sake of simplicity, such linear operator will be just denoted by $\partial_{\mathrm{fib}} F^n(\zeta)$.

Then, one defines the *extremal Lyapunov exponents* of F along the fibers at $\zeta \in \mathcal{E}$ by

$$\lambda^+(F,\zeta) = \lim_{n\to+\infty} \frac{1}{n} \log \left\|\partial_{\mathrm{fib}} F^n(\zeta)\right\|;$$
$$\lambda^-(F,\zeta) = \lim_{n\to+\infty} \frac{1}{n} \log \left\|\left(\partial_{\mathrm{fib}} F^n(\zeta)\right)^{-1}\right\|^{-1};$$

whenever these limits exist. As a consequence of the sub-additive ergodic theorem, it is well known that these limits exist almost everywhere with respect to any F-invariant probability measure.

Given any $\hat{\mu} \in \mathfrak{M}(F)$, one defines the *extremal Lyapunov exponents of F with respect to $\hat{\mu}$* by

$$\lambda^{\pm}(F, \hat{\mu}) := \int_{\mathcal{E}} \lambda^{\pm}(F, \zeta)\, \mathrm{d}\hat{\mu}(\zeta).$$

The topological version of the invariance principle of Avila and Viana [2] gives strong consequences on certain invariant measures with vanishing extremal Lyapunov exponents.

2.8. Dominated bundle maps. As above, let us suppose $F\colon \mathcal{E} \circlearrowleft$ is a $C^{\alpha,r}$-bundle map over a transitive hyperbolic homeomorphism $f\colon M \circlearrowleft$. Given $\beta > 0$, we say that F is (u,β)-*dominated* whenever there exists $\ell \geqslant 1$ such that

$$\left\| \partial_{\mathrm{fib}} F^{\ell}(\zeta) \right\| \leqslant \frac{\left(\nu_u^{(\ell)}(\pi(\zeta))\right)^{\beta}}{2}, \quad \forall \zeta \in \mathcal{E}, \tag{2.6}$$

where ν_u is the (multiplicative) cocycle over f given in Definition 2.1.

Analogously, one says that F is (s,β)-*dominated* when there exists $\ell \geqslant 1$ such that

$$\left\| (\partial_{\mathrm{fib}} F^{\ell}(\zeta))^{-1} \right\|^{-1} \geqslant 2\left(\nu_s^{(\ell)}(\pi(\zeta))\right)^{\beta}, \quad \forall \zeta \in \mathcal{E}. \tag{2.7}$$

And F is just said to be β-*dominated* if it is simultaneously both (s,β)- and (u,β)-dominated. The definition of domination implicitly assumes that the dynamics on the base space is given by a hyperbolic homeomorphism $f\colon M \circlearrowleft$.

The following result is a consequence of classical graph transform arguments used in [8] (see [2, Proposition 5.1] for an indication of the proof in this exact context).

Proposition 2.6. *If F is an $(s,1)$-dominated $C^{\mathrm{Lip},1}$-fiber bundle over a hyperbolic homeomorphism $f\colon M \circlearrowleft$, then there exists a unique partition of $\mathcal{E}$*

$$\mathcal{W}^s = \{\mathcal{W}^s(\zeta) \subset \mathcal{E} : \zeta \in \mathcal{E}\},$$

exhibiting the following properties:

(i) for every $\zeta \in \mathcal{E}$, $\mathcal{W}^s(\zeta)$ is the image of a Lipschitz section $W^s(\pi(\zeta), f) \to \mathcal{E}$ whose Lipschitz constant is uniform, i.e. it can be chosen independently of ζ;

(ii) it is F-invariant, i.e.

$$F\big(\mathcal{W}^s(\zeta)\big) = \mathcal{W}^s\big(F(\zeta)\big), \quad \forall \zeta \in \mathcal{E}.$$

Of course, if F is $(u,1)$-dominated, a completely analogous result holds and in such a case the "unstable" partition is denoted by $\mathcal{W}^u$.

The following result is a combination of Proposition 5.1 and Theorem D of [2]:

Theorem 2.7. *Let $F\colon \mathcal{E} \circlearrowleft$ be a 1-dominated $C^{\mathrm{Lip},1}$-fiber bundle map over a hyperbolic homeomorphism $f\colon M \circlearrowleft$ and let $\hat{\mu}$ be an F-invariant probability measure whose projection $\mu := \pi_{\star}(\hat{\mu})$ to M has local product structure and full support. Then, if $\lambda^{\pm}(F,\hat{\mu}) = 0$, the support of $\hat{\mu}$ is saturated by $\mathcal{W}^s$ and $\mathcal{W}^u$.*

2.9. Solving the cohomological equation in Lie groups. We state in this section two results due to Kalinin [10] which will play an important role in our proof of Theorem 3.1. However, it is interesting to remark that our Theorem A is completely independent of those results of Kalinin and, in fact, the classical Livšic theorems [16, 17] are enough to deal with the case where the fibers are one-dimensional.

The first result of Kalinin is the following one:

Theorem 2.8 (Theorem 1.4 in [10]). *Let $A\colon M \to \mathrm{GL}_d(\mathbb{R})$ be a Hölder cocycle over a transitive hyperbolic homeomorphism f. Assume that (POO) holds for A. Then the cocycle A has zero Lyapunov exponents with respect to any f-invariant ergodic measure on M.*

We would like to remark that we can get another proof of Theorem 2.8 as a consequence of our Theorem 4.1 below.

In order to show that the solution of cohomological equations are sufficiently regular, we must show that the holonomy maps of certain foliations are smooth. We will show this proving that certain linear cocycles are indeed coboundaries. As it was already mentioned above, that can be done invoking "classical" Livšic theorem when fibers are one-dimensional. In higher dimensions, we need the following Livšic type theorem for linear cocycles due to Kalinin:

Theorem 2.9 (Theorem 1.1 in [10]). *Let $A\colon M \to \mathrm{GL}_d(\mathbb{R})$ be a Hölder cocycle over a transitive hyperbolic homeomorphism $f\colon M \circlearrowleft$. Assume that (POO) holds for A. Then, for every $x_0 \in M$, there exists a unique Hölder map $U\colon M \to \mathrm{GL}_d(\mathbb{R})$ such that*

$$A(x) = U(f(x))U(x)^{-1}, \quad \forall x \in M,$$

and $U(x_0) = Id_{\mathbb{R}^d}$.

Moreover, there exists a constant $C > 0$ depending only on f such that

$$|U|_\alpha \leqslant C\, |A|_\alpha\,.$$

The following consequence of the previous result will be needed later:

Proposition 2.10. *Let $f\colon M \circlearrowleft$ be a hyperbolic homeomorphism and $(A_t\colon M \to \mathrm{GL}_d(\mathbb{R}))_{t\in N}$ be a continuous family (parametrized on a topological manifold N) of α-Hölder cocycles such that (POO) holds for every $t \in N$. Then, there exists a continuous family of α-Hölder transfer functions $(U_t\colon M \to \mathrm{GL}_d(\mathbb{R}))_{t\in N}$ satisfying*

$$A_t(x) = U_t(f(x))U_t(x)^{-1}, \quad \forall t \in N,\ \forall x \in M.$$

Proof. Consider a point $x_0 \in M$ whose forward orbit by f is dense in M and let $U_t : M \to \mathrm{GL}_d(\mathbb{R})$ be the unique solution of the cohomological equation $A_t(x) = U_t(f(x))U_t(x)^{-1}$ such that $U_t(x_0) = Id$, given by Theorem 2.9. Notice that $|U_t|_\alpha$ is uniformly bounded on $t \in N$.

Now we have to show that U_t depends continuously on $t \in N$. Let us fix $t_0 \in N$ and $\varepsilon > 0$. Let n be sufficiently large so that the segment of orbit $x_0, \dots, f^n(x_0)$ is δ-dense, where

$$\delta := \varepsilon \left(6C \max_{t \in N} |A_t|_\alpha \right)^{-\frac{1}{\alpha}},$$

and C is the positive constant given by Theorem 2.9.

By continuity of the family A_t with respect to t, and observing that $U_t(f^k(x_0)) = A_t^{(k)}(x_0)$ for every $k \geqslant 0$, there exists a neighborhood V of t_0 such that it holds

$$\left\| U_t\big(f^k(x_0)\big) - U_{t_0}\big(f^k(x_0)\big) \right\| \leqslant \frac{\varepsilon}{2}, \quad \text{for } 0 \leqslant k \leqslant n,\ \forall t \in V.$$

Invoking the uniform Hölder estimates, we deduce that the C^0-distance between the functions U_t and U_{t_0} is smaller than ε, for every $t \in V$. □

To end this section, let us explain the main difference between our approach and Kalinin's. In [10], the author's strategy consists in estimating the distortion of a left $\mathrm{GL}_d(\mathbb{R})$-invariant metric via the control of the Lyapunov exponents of the cocycle, and then he can apply Livšic's classical argument (see for instance [12, Theorem 19.2.1]). On the other hand, our approach is quite closer in spirit to the one used by Wilkinson in [23], where one lifts the dynamics to a certain skew-product and, using partial hyperbolicity theory, one is able to lift invariant foliations that are used to construct the solution of the cohomological equation.

3. Domination, zero Lyapunov exponents and coboundaries

In this section we study the relation between domination, nullity of Lyapunov exponents and cohomology of bundle maps. The main result we present here is the following:

Theorem 3.1. *Let $N \to \mathcal{E} \xrightarrow{\pi} M$ be a $C^{\alpha,1}$-fiber bundle and $F\colon \mathcal{E} \circlearrowleft$ be a $C^{\alpha,1}$-bundle map over an α-Hölder hyperbolic homeomorphism $f\colon M \circlearrowleft$. Let us assume that (POO) holds for F. Then, the following statements are equivalent:*

(i) $\lambda^{\pm}(F, \hat{\mu}) = 0$, for all $\hat{\mu} \in \mathfrak{M}(F)$;

(ii) F is α-dominated;

(iii) F is a $C^{\alpha,1}$-coboundary and, according to Section 2.6, the fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$ admits a $C^{\alpha,1}$-trivialization.

The most relevant implication in this result is *(ii)* ⇒ *(iii)*, while the other two are rather classical. Moreover, condition *(iii)* automatically implies condition (POO), while *(i)* implies *(ii)* regardless of this condition.

Remark 3.2. We shall use a rather classical trick (see for example [22]) which allows us to reduce the general α-Hölder case to the Lipschitz one: if $N \to \mathcal{E} \xrightarrow{\pi} (M, d)$ is a $C^{\alpha,1}$-fiber bundle, $f: (M, d) \circlearrowleft$ is a hyperbolic homeomorphism and $F: \mathcal{E} \circlearrowleft$ is a $C^{\alpha,1}$-fiber bundle map which is α-dominated, then changing the metric d by d_α (see (2.1)) on M, we obtain a $C^{\mathrm{Lip},1}$-fiber bundle, f continues to be hyperbolic (see Remark 2.3) and F turns to be a $C^{\mathrm{Lip},1}$-bundle map which is 1-dominated. Moreover, F is a $C^{\mathrm{Lip},1}$-coboundary when M is endowed with the d_α metric if and only if it is a $C^{\alpha,1}$-coboundary when M is equipped with d.

In view of Theorem 3.1, it is natural to ask:

Question 3.3. *Let $N \to \mathcal{E} \xrightarrow{\pi} M$ be a $C^{\alpha,1}$-fiber bundle and $F: \mathcal{E} \circlearrowleft$ be a $C^{\alpha,1}$-bundle map over a hyperbolic C^α-homeomorphism $f: M \circlearrowleft$. Suppose that* (POO) *holds for F. Then, is it true that F is α-dominated?*

By Remark 3.2, in order to simplify the notation from now on and until the end of this section, we shall assume that $\alpha = 1 = \mathrm{Lip}$.

To start with the proof of Theorem 3.1, we first show that *(i)* implies *(ii)*. This result maybe belongs to the folklore, but since our context is slightly different from usual ones, we decided to include an outline of the proof:

Proposition 3.4. *Let us assume that*

$$\lambda^{\pm}(F, \hat{\mu}) = 0, \quad \forall \hat{\mu} \in \mathfrak{M}(F),$$

Then F is 1*-dominated.*

Proof. Let us show that F is $(u, 1)$-dominated. The $(s, 1)$-domination follows from completely analogous arguments. To prove that, we shall only use the hypothesis $\lambda^+(F, \hat{\mu}) = 0$, for every $\hat{\mu} \in \mathfrak{M}(F)$.

Then, let us consider the fiber bundle $\pi_{\mathbb{P}}: \mathbb{P} \to \mathcal{E}$, where the fiber over an arbitrary $\zeta \in \mathcal{E}$ is given by the projectivized tangent space of the submanifold $\mathcal{E}_{\pi(\zeta)} \subset \mathcal{E}$.

Now, the derivative-along-fiber operator $\partial_{\mathrm{fib}} F$ defined by (2.5) naturally induces a bundle map $[\partial_{\mathrm{fib}} F]: \mathbb{P} \circlearrowleft$ over $F: \mathcal{E} \circlearrowleft$.

Then we consider the continuous real cocycle $\psi: \mathbb{P} \to \mathbb{R}$ over $[\partial_{\mathrm{fib}} F]$ given by

$$\psi([v]) := \log \frac{\|\partial_{\mathrm{fib}} F \cdot v\|_{F(\zeta)}}{\|v\|_\zeta}, \quad \forall \zeta \in \mathcal{E},\ \forall v \in T\mathcal{E}_{\pi(\zeta)} \backslash \{0\},$$

where $[v]$ denotes the element of $\mathbb{P}$ induced by v.

Now, let K_0, λ be the constants and ν_u be the multiplicative cocycle associated to f given by Definition 2.1, and suppose F is not $(u, 1)$-dominated. Then, there exists a sequence of points $(\zeta_n)_{n \geqslant 1}$ in $\mathcal{E}$ and a strictly increasing sequence of natural numbers $(\ell_n)_{n \geqslant 1}$ such that

$$\left\| \partial_{\mathrm{fib}} F^{\ell_n}(\zeta_n) \right\| \geqslant \frac{\nu_u^{(\ell_n)}\big(\pi(\zeta_n)\big)}{2}, \quad \forall n \geqslant 1. \tag{3.1}$$

This implies that for each $n \in \mathbb{N}$ we can find $[v_n] \in \mathbb{P}_{\zeta_n}$ such that

$$\psi^{(\ell_n)}\Big([v_n]\Big) = \sum_{j=0}^{\ell_n - 1} \psi\Big([\partial_{\mathrm{fib}} F^j][v_n]\Big) = \log \Big\| \partial_{\mathrm{fib}} F^{\ell_n}(\zeta_n) \Big\|. \tag{3.2}$$

Then, by Banach–Alaoglu theorem, there is no lost of generality assuming that there exists $\tilde{\eta} \in \mathfrak{M}(\mathbb{P})$ such that

$$\frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} [\partial_{\mathrm{fib}} F^j]_\star(\delta_{[v_n]}) \to \tilde{\eta}, \quad \text{as } n \to \infty, \tag{3.3}$$

where the convergence is in the weak-$\star$ topology.

Putting together (3.1), (3.2) and (3.3), we can easily show that

$$\int_{\mathbb{P}} \psi \, \mathrm{d}\tilde{\eta} = \lim_{n\to\infty} \frac{\psi^{(\ell_n)}([v_n])}{\ell_n} \geqslant \lim_{n\to\infty} \frac{1}{\ell_n}\Big(\log K_0 + \lambda \ell_n - 2 \Big) = \lambda. \tag{3.4}$$

Finally, defining $\eta := \pi_{\mathbb{P}\star}(\tilde{\eta})$, we get $\eta \in \mathfrak{M}(F)$ and from (3.4) it easily follows

$$\lambda^+(F, \eta) \geqslant \lambda > 0,$$

contradicting our hypothesis. □

Next we show that *(ii)* implies *(iii)* in Theorem 3.1.

Since we are assuming F is 1-dominated, by Proposition 2.6 we know we can lift the stable and unstable sets of f to $\mathcal{E}$. We shall need the following regularity result about these lifts.

Lemma 3.5. *If $\mathcal{W}^\sigma$ denotes the lift of W^σ (with $\sigma \in \{s, u\}$), then there exists a constant $K \geqslant 1$ such that*

$$d_{\mathcal{E}}(\zeta, \eta) \leqslant K d\big(\pi(\zeta), \pi(\eta)\big),$$

for every $\zeta \in \mathcal{E}$, $\eta \in \mathcal{W}^\sigma(\zeta)$ and such that $\pi(\eta) \in W^\sigma_{\delta_0}(\pi(\zeta), f)$, where δ_0 is the constant associated to f by Definition 2.1.

Proof. This is a straightforward consequence of Proposition 2.6, i.e. the fact that the elements of $\mathcal{W}^\sigma$ are graphs of Lipschitz functions with uniformly bounded constant over the stable and unstable sets of f. □

Then we need the following result that plays a key role in the construction of solutions for the cohomological equation. Given a homeomorphism $g\colon X \to X$ and a point $x \in X$, we shall write $\mathcal{O}_g(x) = \{g^n(x)\}_{n\in\mathbb{Z}}$ for the g-orbit of x.

Proposition 3.6. *If F is 1-dominated and (POO) holds, then the closure of every F-orbit is the image of a Lipschitz section. More precisely, for every $\zeta \in \mathcal{E}$, there exists a Lipschitz section $V_\zeta \colon \overline{\mathcal{O}_f\big(\pi(\zeta)\big)} \subset M \to \mathcal{E}$ such that*

$$\overline{\mathcal{O}_F(\zeta)} = \Big\{V_\zeta(y) \in \mathcal{E} : y \in \overline{\mathcal{O}_f\big(\pi(\zeta)\big)}\Big\}.$$

Proof. In order to show that the closure of any F-orbit coincides with the image of a continuous section of the fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$, it is enough to show the following

Claim 1. For every $\zeta \in \mathcal{E}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_{\mathcal{E}}\big(\zeta, F^n(\zeta)\big) < \varepsilon,$$

whenever $d\big(\pi(\zeta), f^n(\pi(\zeta))\big) < \delta$.

Indeed, if the Claim is verified, it follows that the map from $\mathcal{O}_f(\pi(\zeta))$ to $\mathcal{E}$ which maps $f^k(\pi(\zeta)) \mapsto F^k(\zeta)$ is uniformly continuous and therefore extends continuously to its closure.

To prove Claim 1, let $\zeta \in \mathcal{E}$ and $\varepsilon > 0$ be arbitrary. Then, let us choose $\delta := \min\big(\delta_0, \delta_1, \varepsilon(4cK)^{-1}\big)$, where constants δ_1 and c are given by Theorem 2.4 and K is given by Lemma 3.5.

Then, suppose $n \in \mathbb{N}$ is given such that $d\big(\pi(\zeta), f^n(\pi(\zeta))\big) < \delta$. Since f is a hyperbolic homeomorphism and $\delta \leqslant \delta_1$, we can apply Theorem 2.4 to guarantee the existence of $p \in \mathrm{Per}(f)$ and $y := \big[\pi(\zeta), p\big] \in M$ satisfying (1), (2) and (3) in Theorem 2.4. Thus, taking into account that the fiber bundle projection π is one-to-one on $\mathcal{W}^u(\zeta) \subset \mathcal{E}$ and $y \in W^u(\pi(\zeta), f) = \pi\big(\mathcal{W}^u(\zeta)\big)$, there exists a unique point $\zeta_y \in \mathcal{E}_y \cap \mathcal{W}^u(\zeta)$. Analogously, π is one-to-one from $\mathcal{W}^s(\zeta_y)$ onto $W^s(y)$ and hence, there exists a unique point $\zeta_p \in \mathcal{E}_p \cap \mathcal{W}^s(\zeta_y)$.

Now, observing that $\zeta_p \in \mathcal{W}^s(\zeta_y)$ and $\zeta_y \in \mathcal{W}^u(\zeta)$, we can combine Theorem 2.4 and Lemma 3.5 to guarantee that

$$\begin{aligned} d_{\mathcal{E}}(\zeta, \zeta_p) &\leqslant d_{\mathcal{E}}(\zeta, \zeta_y) + d_{\mathcal{E}}(\zeta_y, \zeta_p) \\ &\leqslant K\Big[d\big(\pi(\zeta), y\big) + d(y, p)\Big] < 2Kc\delta \leqslant \frac{\varepsilon}{2}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d_{\mathcal{E}}\big(F^n(\zeta), F^n(\zeta_p)\big) &\leqslant d_{\mathcal{E}}\big(F^n(\zeta), F^n(\zeta_y)\big) + d_{\mathcal{E}}\big(F^n(\zeta_y), F^n(\zeta_p)\big) \\ &\leqslant K\Big[d\big(f^n(\pi(\zeta), f^n(y)\big) + d\big(f^n(y), f^n(p)\big)\Big] \\ &\leqslant 2Kcd\big(f^n(\pi(\zeta)), \pi(\zeta)\big) < 2Kc\delta \leqslant \frac{\varepsilon}{2}, \end{aligned} \tag{3.6}$$

Finally observe that, since $f^n(p) = p$ and (POO) holds, it follows that $F^n(\zeta_p) = \zeta_p$. Then, putting together (3.5) and (3.6), we get $d_{\mathcal{E}}(\zeta, F^n(\zeta)) < \varepsilon$, and our claim is proven.

To finish, notice that in the proof of Claim 1, the constant

$$\delta = \min\{\delta_0, \delta_1, \varepsilon(4cK)^{-1}\}$$

depends linearly on ε (when ε is small enough) and therefore, the map is indeed Lipschitz. □

It is interesting to notice that the exponential shadowing given by Anosov closing lemma (Theorem 2.4) was not used in the proof of Proposition 3.6. In fact, the classical Shadowing Lemma is enough because the Hölder regularity (in this case Lipschitz) was already used in Lemma 3.5.

Now, let us consider a point $x_0 \in M$ such that its forward and backward f-orbits are dense. Then, by Proposition 3.6, assuming F is 1-dominated, for every $\zeta \in \mathcal{E}_{x_0}$ there exists a continuous section $V_\zeta\colon M \to \mathcal{E}$ such that $\overline{\mathcal{O}_F(\zeta)}$ coincides with the image of V_ζ. To simplify the notation, the image of section V_ζ will be denoted by $\mathcal{I}_\zeta$, i.e. we define $\mathcal{I}_\zeta := \{V_\zeta(x) \in \mathcal{E} : x \in M\}$, for every $\zeta \in \mathcal{E}_{x_0}$.

Then we will show that the family $\{\mathcal{I}_\zeta\}_{\zeta \in \mathcal{E}_{x_0}}$ determines a continuous lamination in $\mathcal{E}$. To do this, we first prove the following

Proposition 3.7. *If F is 1-dominated, then for each $\zeta \in \mathcal{E}_{x_0}$ the image of the section V_ζ defined above is saturated by leaves of the lamination $\mathcal{W}^s$* ($\mathcal{W}^u$, respectively.) *More precisely, for every $\zeta \in \mathcal{E}_{x_0}$ and any $\eta \in \mathcal{I}_\zeta$,*

$$\mathcal{W}^\sigma(\eta) \subset \mathcal{I}_\zeta, \quad \textit{for } \sigma \in \{s, u\}.$$

Proof. We will prove the proposition for $\sigma = s$. One can recover the proof for $\sigma = u$ by considering F^{-1} instead of F.

Let us suppose the assertion is not true. Then, there exists some $\zeta \in \mathcal{E}_{x_0}$ and $\eta' \in \mathcal{I}_\zeta$ such that $\mathcal{W}^s(\eta') \not\subset \mathcal{I}_\zeta$. By continuity of the section V_ζ and the stable lamination, we can choose a point $\eta \in \mathcal{I}_\zeta$ such that the forward f-orbit of $\pi(\eta)$ is dense in M and $\mathcal{W}^s(\eta) \not\subset \mathcal{I}_\zeta$. Then, we take a point $\xi \in \mathcal{W}^s(\eta) \setminus \mathcal{I}_\zeta$.

Observe that $\mathcal{O}_f^+(\pi(\xi))$ is dense in M. Hence, the section V_ξ given by Proposition 3.6 is defined on the whole space M. But, since $\xi \in \mathcal{W}^s(\eta)$, the set $\overline{\mathcal{O}_F(\xi)}$ intersects the fiber $\mathcal{E}_{\pi(\xi)}$ at two different points: at ξ and at $V_\zeta(\pi(\xi))$, contradicting Proposition 3.6. □

Remark 3.8. A less elementary proof of Proposition 3.7 can be easily gotten by invoking the topological version of the Invariance Principle of Avila and Viana (see Theorem 2.7). In fact, assuming domination and condition (POO), using Theorem 2.8 it can be shown that condition *(i)* of Theorem 3.1 holds; and then the Invariance Principle can be applied.

As a consequence of Proposition 3.7 we know the family $\{\mathcal{I}_\zeta\}_{\zeta \in \mathcal{E}_{x_0}}$ is a partition of the total space $\mathcal{E}$, and moreover, a continuous lamination whose leaves are (topologically) transverse to the fibers of the fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$. Thus, we can define the *holonomy maps* of this lamination as follows: given arbitrary points $x, y \in M$,the *holonomy map from x to y* is defined by

$$\mathcal{H}_{x,y} : \mathcal{E}_x \ni \zeta \mapsto V_{\hat{\zeta}}(y) \in \mathcal{E}_y, \tag{3.7}$$

where $\hat{\zeta}$ is the unique point in $\mathcal{E}_{x_0}$ such that $\zeta \in \mathcal{I}_{\hat{\zeta}}$. Observe that, by Proposition 3.7, holonomy maps are (at least) homeomorphisms. After some additional results, we shall show they are indeed C^1-diffeomorphisms.

Then we get the following

Proposition 3.9. *The fiber bundle map $F: \mathcal{E} \circlearrowleft$ is a $C^{\mathrm{Lip},0}$-coboundary. More precisely, the fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$ admits a continuous trivialization $H: \mathcal{E} \to M \times N$ that makes the diagram* (2.4) *commute.*

Proof. To show that the fiber bundle is trivial, let us consider the map $H: \mathcal{E} \to M \times N$ given by

$$H(\zeta) := \Big(\pi(\zeta), \mathrm{pr}_2\big(\phi_j(\mathcal{H}_{\pi(\zeta),x_0}(\zeta))\big)\Big), \quad \forall \zeta \in \mathcal{E}, \tag{3.8}$$

where $\phi_j: U_j \to M \times N$ is a fixed trivializing chart of the fiber bundle $N \to \mathcal{E} \xrightarrow{\pi} M$ such that $x_0 \in U_j$. Then, since holonomy maps are homeomorphisms, it is clear that H itself is a homeomorphism, and since $F(\mathcal{I}_{\hat{\zeta}}) = \mathcal{I}_{\hat{\zeta}}$, for every $\hat{\zeta} \in \mathcal{E}_{x_0}$, we conclude that

$$H\big(F(\zeta)\big) = (f \times id_N)\big(H(\zeta)\big), \quad \forall \zeta \in \mathcal{E},$$

as desired. □

At this point, it is worth mentioning that we took the effort of working on general fiber bundles because we understood this more general setting might actually arise in some cases (see for instance the vector bundle Ξ constructed after Lemma 3.10). However, as Proposition 3.9 shows, an *a priori* arbitrary fiber bundle supporting a fiber bundle map with vanishing of the periodic orbit obstruction is *a posteriori* trivial.

Finally, in order to show that F is a $C^{\mathrm{Lip},1}$-coboundary it remains to prove that the map $H: \mathcal{E} \to M \times N$ constructed in the proof of Proposition 3.9 is indeed a $C^{\mathrm{Lip},1}$-bundle map.

To do this, it is necessary to show that the holonomy maps defined in (3.7) are differentiable and this will be gotten by invoking Proposition 2.10. To use this result, we first need the following

Lemma 3.10. *For every $\zeta \in \mathcal{E}_{x_0}$, the section $V_\zeta: M \to \mathcal{E}$ (whose image is $\mathcal{I}_\zeta$) is Lipschitz.*

Proof. This is a straightforward consequence of the fact that the graph of V_ζ is saturated by $\mathcal{W}^s$ and $\mathcal{W}^u$, which are Lischitz themselves and have local product structure (see (h5) in Definition 2.1). □

Now, for every $\zeta \in \mathcal{E}_{x_0}$ consider the set

$$\Xi^\zeta := \bigsqcup_{x \in M} T_{V_\zeta(x)} \mathcal{E}_x,$$

where $\bigsqcup$ denotes the disjoint-union operator, and the "natural projection" map $\pi^\zeta \colon \Xi^\zeta \to M$ given by $(\pi^\zeta)^{-1}(x) = T_{V_\zeta(x)}\mathcal{E}_x$, for every $x \in M$. By Lemma 3.10, the set Ξ^ζ can naturally be endowed with an appropriate vector bundle structure turning $\mathbb{R}^d \to \Xi^\zeta \xrightarrow{\pi^\zeta} M$ into a C^{Lip}-vector bundle, where $d = \dim N$.

On the other hand, since every leaf $\mathcal{I}_\zeta$ is F-invariant and $F|_{\mathcal{E}_x} \colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a C^1-diffeomorphism, our fiber bundle map F naturally induces a C^{Lip}-vector bundle map $DF^\zeta \colon \Xi^\zeta \circlearrowleft$ over $f \colon M \circlearrowleft$ given by

$$DF^\zeta(v_x) = \partial_{\mathrm{fib}} F\big(V_\zeta(x)\big)(v_x), \quad \forall x \in M,\ \forall v_x \in \Xi^\zeta_x = T_{V_\zeta(x)}\mathcal{E}_x, \tag{3.9}$$

where $\partial_{\mathrm{fib}} F$ denotes the (partial) derivative along the fibers defined in Section 2.7.

Then we get the following

Proposition 3.11. *For every $\zeta \in \mathcal{E}_{x_0}$, the vector bundle $\mathbb{R}^d \to \Xi^\zeta \xrightarrow{\pi^\zeta} M$ is trivial and the vector bundle map DF^ζ is a C^{Lip}-coboundary, i.e. there exists a C^{Lip}-vector bundle map $U^\zeta \colon \Xi^\zeta \to M \times \mathbb{R}^d$ satisfying*

$$U^\zeta \circ DF^\zeta = (f \times Id_{\mathbb{R}^d}) \circ U^\zeta.$$

Moreover, the family $(U^\zeta)_{\zeta \in \mathcal{E}_{x_0}}$ can be chosen to vary continuously on ζ.

Proof. Since (POO) holds for F, one know that it vanishes for DF^ζ, too. Hence, by Theorem 2.8, DF^ζ has zero Lyapunov exponents with respect to any f-invariant probability measure. In particular, invoking Proposition 3.9 we conclude that the vector bundle

$$\mathbb{R}^d \to \Xi^\zeta \xrightarrow{\pi^\zeta} M$$

is trivial and we can apply Proposition 2.10 to obtain a continuous family U^ζ of solutions, as desired. □

Then we get the following

Corollary 3.12. *If F is 1-dominated and (POO) holds, then there exists $C > 0$ such that*

$$\|\partial_{\mathrm{fib}} F^n(v)\| \leqslant C, \quad \forall n \in \mathbb{Z},\ \forall \zeta \in \mathcal{E},\ \forall v \in T_\zeta \mathcal{E}_{\approx(\asymp)}.$$

Proof. This is a straightforward consequence of Propositions 3.11 and 2.10. □

Then we finally get

Proposition 3.13. *The holonomy maps given by (3.7) are differentiable and consequently, the map $H \colon \mathcal{E} \to M \times N$ defined by (3.8) is a $C^{\mathrm{Lip},1}$-bundle map.*

Proof. Given arbitrary points $x, y \in M$, we need to show that the holonomy map $\mathcal{H}_{x,y}\colon \mathcal{E}_x \to \mathcal{E}_y$ associated to the lamination $\{\mathcal{I}\}_{\zeta\in\mathcal{E}_{x_0}}$, which is clearly a homeomorphism, is indeed a C^1-diffeomorphism.

To do this, first observe that since each leaf of the lamination $\{\mathcal{I}_\zeta\}_{\zeta\in\mathcal{E}_{x_0}}$ is F-invariant, it holds

$$\mathcal{H}_{f^m(x),f^n(x)} = F^{n-m}|_{\mathcal{E}_{f^m(x)}}, \quad \forall x \in \mathcal{E},\ \forall m,n \in \mathbb{Z}. \tag{3.10}$$

Consequently, holonomy maps between any two points of the same f-orbit are indeed C^1-diffeomorphisms.

To deal with the general case, consider arbitrary points $x, y \in M$ and let $\varphi_i\colon \pi^{-1}(U_i) \to U_i \times N$, with $i = 1, 2$, be two trivializing charts such that $x \in U_1$ and $y \in U_2$. Recalling we have chosen $x_0 \in M$ so that its forward f-orbit is dense in M, we can find two sequences of natural numbers (m_i) and (n_i) such that $U_1 \ni f^{m_i}(x_0) \to x$ and $U_2 \ni f^{n_i}(x_0) \to y$, as $i \to \infty$.

Then, for each $i \geqslant 1$, let us define $\mathcal{H}_i \in \mathrm{Diff}^1(N)$ by

$$\mathcal{H}_i(p) := \mathrm{pr}_2 \circ \varphi_2 \circ \mathcal{H}_{f^{m_i}(x_0),f^{n_i}(x_0)} \circ \varphi_1^{-1}\big(f^{m_i}(x_0), p\big),$$

and $\mathcal{H} \in \mathrm{Homeo}(N)$ by

$$\mathcal{H}(p) = \mathrm{pr}_2 \circ \varphi_2 \circ \mathcal{H}_{x,y} \circ \varphi_1^{-1}\big(x, p\big),$$

for every $p \in N$. We want to show $\mathcal{H} \in \mathrm{Diff}^1(N)$, too.

By continuity of the lamination $\{\mathcal{I}_\zeta\}_{\zeta\in\mathcal{E}_{x_0}}$, when $i \to \infty$, $\mathcal{H}_i \to \mathcal{H}$ pointwise. By Corollary 3.12 and Arzelà–Ascoli theorem, we conclude the convergence $\mathcal{H}_i \to \mathcal{H}$ is C^0-uniform.

Now, Proposition 3.11 implies that the fiber bundle is trivial and the derivatives of the cocycle provide a continuous family of linear cocycles for which (POO) holds. Therefore, the hypotheses of Proposition 2.10 are verified and we get a continuous family of solutions for the corresponding family of cohomological equations. These solutions are indeed the derivatives of $\mathcal{H}$ so we deduce that the sequence of derivatives $(D\mathcal{H}_i(p))_{i\geqslant 1}$ is also convergent, for each $p \in N$. Consequently, $\mathcal{H}$ is C^1 and then, $\mathcal{H}_{x,y}\colon \mathcal{E}_x \to \mathcal{E}_y$ is a diffeomorphism, as desired. □

Finally, it remains to show that *(iii)* implies *(i)* in Theorem 3.1. But this is obvious, because a $C^{\mathrm{Lip},1}$-coboundary is, by the very same definition, conjugate to the map $(f \times id_N)\colon M \times N \circlearrowleft$ via a $C^{\mathrm{Lip},1}$-fiber bundle conjugacy, and therefore, every Lyapunov exponent must vanish.

4. Domination as a consequence of vanishing of the periodic orbit obstruction

In this section we shall review some contexts where condition (POO) alone implies that the cocycle is dominated, and as a consequence of Theorem 3.1, it is a coboundary.

We start proving Theorem B which follows from Theorem 3.1 and Katok closing lemma [11].

Proof of Theorem B. Let $S \to \mathcal{E} = M \times S \xrightarrow{\pi} M$ denote the trivial fiber bundle and $F: M \times S \circlearrowleft$ be the $C^{1+\alpha}$ skew-product over f induced by Φ as in (2.3). Oberserve that for F the (POO) holds.

Let us suppose there exists an F-invariant ergodic probability measure $\hat{\mu}$ such that $\lambda^+(F, \hat{\mu}) \neq 0$.

Since Φ takes values in the group of area-preserving diffeomorphisms of S, by Oseledets theorem we know that

$$\lambda^-(F, \hat{\mu}) + \lambda^+(F, \hat{\mu}) = 0.$$

So, we have

$$\lambda^-(F, \hat{\mu}) < 0 < \lambda^+(F, \hat{\mu}),$$

and since $f: M \circlearrowleft$ is an Anosov diffeomorphism, this implies $\hat{\mu}$ is a hyperbolic measure for F (i.e. all its Lyapunov exponents given by Oseledets theorem are different from zero).

So, applying Katok closing lemma [11, Corollary 4.3], we conclude that F exhibits a hyperbolic periodic point. But, invoking condition (POO), if $\zeta_0 \in \mathcal{E}$ is periodic with $F^n(\zeta_0) = \zeta_0$, then $F^n(\zeta) = \zeta$, for every $\zeta \in \mathcal{E}_{\pi\zeta_0}$. So ζ_0 is not an isolated point of $\mathrm{Fix}(F^n)$, and hence, it is not hyperbolic, getting a contradiction. □

The amount of regularity required in the fiber direction is essential in our argument and it is the usual one in Pesin's theory which allows to obtain a subexponential neighborhoods of a regular orbit with good estimates on the bundles of the Oseledet's splitting (see [12, Supplement]). The recent examples of [4] show that improving this regularity requires new ideas, and we do not see how low-dimensionality nor volume preservation would help.

On the other hand, the requirement on Theorem B that the base dynamics is smooth might not be essential. Most likely, similar arguments to those we will perform to prove Theorem A might allow to get the same result just requiring continuity for the base dynamics, but the details would become significantly more involved. Since in this case our result is partial, we have chosen to present it in this simplified context.

In order to prove Theorem A, we need the following result that should be considered as the main one of this section:

Theorem 4.1. *Let $N \to \mathcal{E} \xrightarrow{\pi} M$ be a $C^{\alpha,1}$-fiber bundle and $F: \mathcal{E} \circlearrowleft$ be a $C^{\alpha,1}$-bundle map over an α-Hölder hyperbolic homeomorphism $f: M \circlearrowleft$. If there exists an ergodic measure $\hat{\mu} \in \mathfrak{M}(F)$ with $\lambda^+(F, \hat{\mu}) < 0$, then there exists $\zeta_0 \in \mathrm{Per}(F)$ which is uniformly contracting along the fiber, i.e. if $n > 0$ denotes the period of ζ_0, then all the eigenvalues of the linear map $\partial_{\mathrm{fib}} F^n(\zeta_0): T_{\zeta_0}\mathcal{E}_{\pi(\zeta_0)} \circlearrowleft$ have modulus strictly smaller than* 1.

It is interesting to remark that applying Theorem 4.1 to the natural action induced by a linear cocycle on a suitable Grasmannian fiber bundle (corresponding to the dimension of the subspace with largest Lyapunov exponent), one can reprove part of Kalinin's result on approximation[3] of Lyapunov exponents [10, Theorem 1.4].

Now, we can prove Theorem A as a combination of Theorems 3.1 and 4.1:

Proof of Theorem A. Let $\mathbb{R}/\mathbb{Z} \to \mathcal{E} = M \times \mathbb{R}/\mathbb{Z} \xrightarrow{\pi} M$ denote the trivial fiber bundle and $F\colon M \times \mathbb{R}/\mathbb{Z} \circlearrowleft$ be the skew-product over f induced by Φ as in (2.3). Since (POO) holds for Φ, then it does for F, too.

Hence, for every $\zeta \in \mathrm{Per}(F)$ such that $F^n(\zeta) = \zeta$, it clearly holds $\partial_{\mathrm{fib}} F^n = D\Phi^{(n)} \equiv id$ and consequently, all the eigenvalues are equal to 1. So, applying Theorem 4.1 to F and F^{-1} we get

$$-\lambda^+(F^{-1}, \hat{\mu}) = \lambda^-(F, \mu) \leqslant 0 \leqslant \lambda^+(F, \hat{\mu}).$$

But since the fibers are one-dimensional, we can apply Birkhoff ergodic theorem to conclude that $\lambda^-(F, \hat{\mu}) = \lambda^+(F, \hat{\mu})$. Therefore, $\lambda^-(F, \hat{\mu}) = \lambda^+(F, \hat{\mu}) = 0$ and by Theorem 3.1, F is a $C^{\alpha,1}$-coboundary, as desired. □

4.1. Proof of Theorem 4.1. From the uniform continuity of $\partial_{\mathrm{fib}} F$ and f, it easily follows.

Lemma 4.2. *For every $\delta > 0$, there exists $\chi > 0$ such that for every $\eta, \xi \in \mathcal{E}$ satisfying*

$$d_{\mathcal{E}}\big(F^i(\eta), F^i(\xi)\big) \leqslant \chi, \quad \textit{for every } i \in \{0, \dots k\},$$

it holds

$$\prod_{i=0}^{k-1} \left\|\partial_{\mathrm{fib}} F\big(F^i(\eta)\big)\right\| \leqslant e^{k\delta} \prod_{i=0}^{k-1} \left\|\partial_{\mathrm{fib}} F\big(F^i(\xi)\big)\right\|.$$

Along the proof we shall assume that $\lambda^+ := \lambda^+(F, \hat{\mu}) < 0$.

It is a classical fact that one can choose measurable adapted metrics which see the contraction at each iterate (see for example Proposition 8.2 of [1]):

Lemma 4.3. *For every $\varepsilon > 0$ there exists an integer $N > 0$ and a measurable function $A : \mathcal{E} \to [1, +\infty)$ such that:*

- *The sequence $\big(A(F^n(\zeta)\big)_{n\in\mathbb{Z}}$ varies sub-exponentially (i.e. one has that for $\hat{\mu}$-almost every $\zeta \in \mathcal{E}$ the sequence $\frac{1}{|n|} \log |A(F^n(\zeta))|$ converges to 0 as $|n| \to \infty$).*

[3] The statement above implies, in particular, that if $\Phi\colon M \to \mathrm{GL}_d(\mathbb{R})$ is α-Hölder and the (POO) holds, then every measure has zero Lyapunov exponents. To obtain that all Lyapunov exponents can be approximated one can apply the arguments in this section to approximate the top Lyapunov exponent, and then uses the same trick of exterior power as in Kalinin's paper to obtain the other estimates.

- *If we define the metric* $\|\cdot\|'_\zeta$ *in* $T_\zeta \mathcal{E}_{\pi(\zeta)}$ *by*

$$\|v\|'_\zeta = \sum_{0 \leqslant k \leqslant N} e^{-k(\lambda^+ + \varepsilon)} A(F^k(\zeta)) \left\| \partial_{\text{fib}} F^k(\zeta) \cdot v \right\|_{F^k(\zeta)},$$

then, for $\hat{\mu}$ *almost every* $\zeta \in \mathcal{E}$ *and every* $v \in T_\zeta \mathcal{E}_{\pi(\zeta)}$ *one has*

$$\|\partial_{\text{fib}} F(\zeta) \cdot v\|'_{F(\zeta)} \leqslant e^{(\lambda^+ + \varepsilon)} \|v\|'_\zeta .$$

We shall fix $\varepsilon < \min\{-\frac{\lambda^+}{5}, \frac{\alpha\lambda}{2}\}$, where α is the Hölder exponent of F and λ is the hyperbolicity constant appearing in Theorem 2.4. Consider the function A and the metric $\|\cdot\|'$ given by the previous lemma and let us fix them from now on.

Using this metric, by standard arguments one can show that it is possible to define sub-exponential neighborhoods (sometimes called *Pesin charts*) of typical points with respect to $\hat{\mu}$ such that the dynamics in those neighborhoods behaves similarly to the derivative (see for example [12, Supplement]).

For $\zeta \in \mathcal{E}$ we shall consider the exponential map $\exp : T_\zeta \mathcal{E}_{\pi(\zeta)} \to \mathcal{E}_{\pi(\zeta)}$ where the distances in $T_\zeta \mathcal{E}_{\pi(\zeta)}$ are measured with respect to the metric $\|\cdot\|'_\zeta$. We denote by $B'_\zeta(r)$ the ball of radius r centered at 0 in $T_\zeta \mathcal{E}_{\pi(\zeta)}$.

Lemma 4.4. *There exists a measurable function* $\rho : \mathcal{E} \to (0, +\infty)$ *such that if* $\varphi_\zeta = \exp|_{B'_\zeta(\rho(\zeta))}$, *then for* $\hat{\mu}$*-almost every point the map* $\varphi^{-1}_{F(\zeta)} \circ F \circ \varphi_\zeta : B'_\zeta(\rho(\zeta)) \to B'_{F(\zeta)}(\rho(F(\zeta)))$ *contracts vectors by a factor smaller than* $e^{(\lambda^+ + 2\varepsilon)}$. *Moreover, the sequence* $\rho(F^n(\zeta))$ *is sub-exponential and can be chosen so that* $e^{-\varepsilon}\rho(\zeta) < \rho(F(\zeta)) < e^{\varepsilon}\rho(\zeta)$.

It is relevant to remark here the fact that the sub-exponential growth of the function ρ is essential in Pesin's theory and this is the precise point where the Hölder regularity of the cocycle is usually invoked. Here, since we are working with measures whose Lyapunov exponents are all negative, C^1-regularity along the fibers is enough.

Remark 4.5. Notice there is a measurable function D which associates to each $\zeta \in \mathcal{E}$ an isometry $D_\zeta : \left(T_\zeta \mathcal{E}_{\pi(\zeta)}, \|\cdot\|_\zeta\right) \to \left(T_\zeta \mathcal{E}_{\pi(\zeta)}, \|\cdot\|'_\zeta\right)$. When this linear map is considered as a transformation from $\left(T_\zeta \mathcal{E}_{\pi(\zeta)}, \|\cdot\|_\zeta\right)$ to itself, the norm and co-norm of D_ζ are bounded by a number depending only on $A(\zeta)$.

Using Luisin's Theorem on approximation of measurable functions by continuous ones (see for example [12, Supplement]) one obtains a compact set $X \subset \mathcal{E}$ of positive $\hat{\mu}$-measure such that functions A and ρ are continuous on X and, thus, bounded (we define $A_X := \sup_{\zeta \in X} A(\zeta)$ and $\rho_X := \inf_{\zeta \in X} \rho(x)$).

Consider a point $\zeta_0 \in X$ which is recurrent inside X, i.e. there exists $n_j \to \infty$ such that $F^{n_j}(\zeta_0) \to \zeta_0$ and $F^{n_j}(\zeta_0) \in X$ for every $j > 0$.

Let us write $x_0 := \pi(\zeta_0)$ and, for each $j > 0$, let $p_j \in \operatorname{Fix}(f^{n_j})$ be the periodic point of f given by Anosov Closing Lemma (Theorem 2.4). One has that:

$$d(f^i(x_0), f^i(p_j)) \leqslant c e^{-\lambda \min\{i, n_j - i\}} d(f^{n_j}(x_0), x_0), \quad \text{for } i = 0, \dots, n_j.$$

where $c, \lambda > 0$ are the constants given in Theorem 2.4.

Notice that

$$d(f^{n_j}(x_0), x_0) \leqslant d_{\mathcal{E}}(F^{n_j}(\zeta_0), \zeta_0) \to 0.$$

Fix $\delta < \varepsilon$ and let χ be the constant given by Lemma 4.2 for such a δ.

The main step in the proof is the following

Lemma 4.6. *For n_j large enough, there exists a small open ball $B_j \subset \mathcal{E}_{p_j}$ such that*

$$F^{n_j}(\overline{B_j}) \subset B_j. \tag{4.1}$$

Moreover, $\operatorname{diam}(F^i(B_j)) \leqslant \chi/2$ *and* $d_{\mathcal{E}}(F^i(B_j), F^i(\zeta_0)) \leqslant \chi/2$*, for every* $0 \leqslant i \leqslant n_j$*.*

Let us now conclude the proof of Theorem 4.1 assuming this lemma:

By (4.1), we know there exists $\xi_j \in B_j$ such that $F^{n_j}(\xi_j) = \xi_j$, and it also holds $d_{\mathcal{E}}(F^i(\xi_j), F^i(\zeta_0)) \leqslant \chi$, for all $0 \leqslant i \leqslant n_j$. Hence, applying Lemma 4.2 and the fact that $\zeta_0, F^{n_j}(\zeta_0) \in X$, we prove that ξ_j is uniformly contracting along the fiber, as desired.

So, it only remains to prove Lemma 4.6.

Proof of Lemma 4.6. Since there exists a trivializing chart containing x_0 and p_j, there exists a point $\zeta_j \in \mathcal{E}_{p_j}$ such that $d_{\mathcal{E}}(\zeta_j, \zeta_0) = d(p_j, x)$.

From the choice of x_0 and p_j, if n_j is large enough, we can always assume that both $f^i(x_0)$ and $f^i(p_j)$ lie in the same trivializing chart for $0 \leqslant i \leqslant n_j$. So, fixing a trivializing chart containing $f^i(x_0)$ and $f^i(p_j)$, we have a projection $\mathrm{pr}_2 : \mathcal{E}_{f^i(p_j)} \to \mathcal{E}_{f^i(x_0)}$. Given two points $\xi \in \mathcal{E}_{f^i(x_0)}$ and $\eta \in \mathcal{E}_{f^i(p_j)}$ such that $\xi, \mathrm{pr}_2(\eta) \in B'_{F^i(\zeta_0)}(\rho(F^i(\zeta_0)))$, we can define $d'_{\mathcal{E}}(\xi, \eta) := d(\pi(\xi), \pi(\eta)) + d'(\xi, \mathrm{pr}_2(\eta))$, where d' is the distance in $B'_{F^i(\zeta_0)}(\rho(F^i(\zeta_0)))$ induced by the norm $\|\cdot\|'_{F^i(\zeta_0)}$.

For $1 \leqslant k \leqslant n_j$, and assuming that

$$d'_{\mathcal{E}}\big(F^{k-1}(\zeta_j), F^{k-1}(\zeta_0)\big) \leqslant \min\{\rho(F^{k-1}(\zeta_0)), \frac{\chi}{2}\},$$

we can invoke Lemma 4.2 to get

$$\begin{aligned} d'_{\mathcal{E}}\big(F^k(\zeta_j), F^k(\zeta_0)\big) &= d'_{\mathcal{E}}\Big(F\big(F^{k-1}(\zeta_j)\big), F\big(F^{k-1}(\zeta_0)\big)\Big) \\ &\leqslant \hat{c} e^{\varepsilon \min\{k, n_j - k\}} d_{C^1}\big(F_{f^{k-1}(p_j)}, F_{f^{k-1}(x_0)}\big) \\ &\quad + e^{\delta} \left\| \partial_{\mathrm{fib}} F(F^{k-1}(\zeta_0)) \right\|' d'_{\mathcal{E}}\big(F^{k-1}(\zeta_j), F^{k-1}(\zeta_0)\big), \end{aligned} \tag{4.2}$$

where the constant $\hat{c}$ only depends on A_X and ρ_X. The factor $\hat{c} e^{\varepsilon \min\{k, n_j - k\}}$ appears to take into account the distortion in the new metric, which is bounded by $\hat{c}$ at the points $\zeta_0, F^{n_j}(\zeta_0) \in X$ and the change of the distortion at each iterate is bounded by e^{ε}.

Now, let us define the sequences

$$a_k := \hat{c} e^{\varepsilon \min\{k, n_j - k\}} d_{C^1}\big(F_{f^{k-1}(p_j)}, F_{f^{k-1}(x_0)}\big)$$

and

$$b_k := e^{\delta} \left\| \partial_{\mathrm{fib}} F(F^{k-1}(\zeta_0)) \right\|'.$$

Observe that $b_k \leqslant e^{(\lambda^+ + 3\varepsilon)} < 1$, for every $k \geqslant 1$.

By induction and applying estimate (4.2), one gets

$$d'_\varepsilon\big(F^k(\zeta_j), F^k(\zeta_0)\big) \leqslant \sum_{i=1}^{k} a_i \Big(\prod_{j=i+1}^{k} b_j \Big) \leqslant \sum_{i=1}^{k} a_i e^{(k-i)(\lambda^+ + 3\varepsilon)}, \quad \forall k \geqslant 1. \tag{4.3}$$

One can estimate the size of a_i as follows (here is where Hölder continuity of F is essential):

$$a_i \leqslant e^{\varepsilon \min\{i, n_j - i\}} c' e^{-\alpha\lambda \min\{i, n_j - i\}} d(x_0, f^{n_j}(x_0))^\alpha, \quad \text{for } 0 \leqslant i \leqslant n_j, \tag{4.4}$$

where $c' > 0$ depends on the constant c appearing in Anosov closing lemma (Theorem 2.4), the Hölder norm of the $C^{\alpha,1}$-bundle map F and the constant $\hat{c}$ which was defined above.

Choosing n_j so that $d(x_0, f^{n_j}(x_0))$ is sufficiently small and recalling that $\varepsilon < \frac{1}{2}\alpha\lambda$, we can perform induction and thus ensure that the iterates $F^k(\zeta_j)$ of the point ζ_j remain always close enough to $F^k(\zeta_0)$.

Using the estimate of Lemma 4.2, one concludes there is a ball B_j with the desired properties. □

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