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# Ergodic properties of equilibrium measures for smooth three dimensional flows

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**Abstract.** Let $\{T^t\}$ be a smooth flow with positive speed and positive topological entropy on a compact smooth three dimensional manifold, and let $\mu$ be an ergodic measure of maximal entropy. We show that either $\{T^t\}$ is Bernoulli, or $\{T^t\}$ is isomorphic to the product of a Bernoulli flow and a rotational flow. Applications are given to Reeb flows.

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## 1. Introduction and statement of main results

**Introduction.** In 1973, Ornstein and Weiss proved that the geodesic flow of a compact smooth surface with constant negative curvature is Bernoulli with respect to the Liouville measure [36]. Ratner extended this to variable negative curvature [43]. In the case of non-positive and non identically zero curvature, Pesin showed that some ergodic component of the Liouville measure is open, dense, and Bernoulli [39], [4, Thm 12.2.13]. It follows from his work that all other ergodic components (if they exist) have zero entropy. Katok and Burns extended Pesin's work to Reeb flows [20]. Burns and Gerber proved that geodesic flows on certain surfaces with some positive curvature ("Donnay's examples") are Bernoulli [11]. Hu, Pesin and Talitskaya constructed smooth volume-preserving Bernoulli flows on every compact manifold of dimension at least three [18].

Ratner's work extends to general Anosov flows equipped with ergodic equilibrium measures of Hölder continuous potentials [43]. In this case the flow is either Bernoulli, or isomorphic to a Bernoulli flow times a rotational flow (this happens in the non-mixing case). Pesin's work extends to all $C^{1+\varepsilon}$ flows preserving an ergodic hyperbolic measure whose conditional measures on the unstable manifolds are absolutely continuous with respect to the induced Riemannian measure [21,27,32,38], with the same modification in the non-mixing case.

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The measure of maximal entropy does not have absolutely continuous conditional measures, except in special cases [19]. The purpose of this paper is to determine the ergodic theoretic structure of this measure in the context of general smooth three dimensional flows with positive topological entropy. Our methods also apply to ergodic equilibrium measures of Hölder potentials with positive entropy.

**Basic definitions.** Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space.

MEASURABLE FLOW: A quadruple $\mathsf{T} = (X, \mathscr{B}, \mu, \{T^t\})$ such that $(t, x) \mapsto T^t(x)$ is measurable, and the *time–t map* $(X, \mathscr{B}, \mu, T^t)$ is probability preserving, $\forall t \in \mathbb{R}$.

EIGENFUNCTION: A non-constant measurable function $f$ is an *eigenfunction* of $\mathsf{T}$ (with eigenvalue $e^{i\alpha}$) if for a.e. $x \in X$, $f(T^t x) = e^{i\alpha t} f(x)$ for all $t \in \mathbb{R}$. $\mathsf{T}$ is called *ergodic* if 1 is not an eigenvalue, and *weak-mixing* if it has no eigenfunctions at all.

ENTROPY: The *entropy* of $\mathsf{T}$ is the entropy of the time–1 map $T^1$.

ROTATIONAL FLOW: Given $c > 0$, the *rotational flow* is $T^t(x) := x + t/c \pmod 1$ on $\mathbb{R}/\mathbb{Z}$ equipped with the Haar measure. $c$ is called the *period*, and it is an invariant of the flow since $c = \min\{t > 0 : T^t = \mathrm{Id}\}$.

BERNOULLI FLOW: $\mathsf{T}$ is called *Bernoulli* if $T^1$ is a Bernoulli automorphism. $\mathsf{T}$ is called *Bernoulli up to a period* if $\mathsf{T}$ is Bernoulli, or if $\mathsf{T}$ is isomorphic to the product of a Bernoulli flow and a rotational flow.

If $\mathsf{T}$ is a Bernoulli flow then $T^t$ is a Bernoulli automorphism, $\forall t \neq 0$ [33]. Entropy is a complete set of invariants for Bernoulli flows [34], and entropy and period (if it exists) are a complete set of invariants for Bernoulli up to a period flows since the Bernoulli term is determined by the entropy and the rotational term is the Pinsker factor, see [52, Prop. 4.4].

**Main results.** Let $M$ be a three dimensional compact $C^\infty$ Riemannian manifold without boundary, let $\mathscr{B}$ be its Borel $\sigma$–algebra, let $X : M \to TM$ be a $C^{1+\varepsilon}$ vector field on $M$ such that $X_p \neq 0$, $\forall p \in M$, let $\mathsf{T}$ be the flow on $M$ generated by $X$, and let $\mu$ be a $\mathsf{T}$–invariant probability measure.

EQUILIBRIUM MEASURE: $\mu$ is an *equilibrium measure* of a *potential* $F : M \to \mathbb{R}$ if $h_\mu(T^1) + \int_M F d\mu = \sup\{h_\nu(T^1) + \int_M F d\nu\}$, where sup ranges over all $\mathsf{T}$–invariant probability measures $\nu$. If $F = 0$, then $\mu$ is called a *measure of maximal entropy*.

Equilibrium measures always exist if $X$ is $C^\infty$ and $F$ is continuous [29].

**Theorem 1.1.** *Under the above assumptions on $M, X, \mathsf{T}$, every equilibrium measure of a Hölder continuous potential has at most countably many ergodic components with positive entropy. Each of them is Bernoulli up to a period.*

Periods can exist (e.g. for the constant suspension of an Anosov diffeomorphism), but sometimes they can be discounted. Let $\{T^t\}$ be a Reeb flow on a compact smooth three dimensional contact manifold $M$ (see Section 7 for definitions). For example, $\{T^t\}$ could be the geodesic flow of a surface, or the Hamiltonian flow of a system with two degrees of freedom on a regular energy surface [1]. Katok and Burns showed that every ergodic absolutely continuous invariant measure with positive entropy is Bernoulli [20]. The following result covers other measures of interest, such as the measures of maximal entropy.

**Theorem 1.2.** *If $T$ is a three dimensional Reeb flow, then every equilibrium measure of a Hölder continuous potential has at most countably many ergodic components with positive entropy. Each of them is Bernoulli.*

**Corollary 1.3.** *Let $S$ be a compact smooth orientable surface without boundary, with nonpositive and non-identically zero curvature. Then the geodesic flow of $S$ is Bernoulli with respect to its (unique) measure of maximal entropy.*

*Proof.* Let $m$ be the invariant Liouville measure. By the curvature assumptions, $m$ has positive metric entropy, see for example [40, Corollary 3]. Hence the geodesic flow has positive topological entropy. Also by the curvature assumptions, $S$ is a rank one manifold [3], therefore there is a unique measure of maximal entropy [24]. By uniqueness, it is ergodic. By Theorem 1.2, it is Bernoulli. □

The "geometric potential" $J(x) := -\frac{d}{ds}|_{s=0} \log \|dT^s|_{E^u(x)}\|$ and its scalar multiples (see [8] and Section 8) are not directly covered by Theorems 1.1 and 1.2, because they are not necessarily Hölder continuous or even globally defined on $M$. But our methods do apply to them and give the following:

**Theorem 1.4.** *Under the assumptions of Theorem 1.1, every equilibrium measure of $tJ$ ($t \in \mathbb{R}$) has at most countably many ergodic components with positive entropy. Each is Bernoulli up to a period. If $T$ is a Reeb flow, each is Bernoulli.*

**Corollary 1.5.** ([39, Thm 9.7]) *Let $S$ be a compact smooth orientable surface without boundary, with nonpositive and non-identically zero curvature. Then the geodesic flow of $S$ is Bernoulli with respect to every positive entropy ergodic component of the invariant Liouville measure. There are at most countably many such components.*

*Proof.* The invariant Liouville measure is an equilibrium measure for the geometric potential $J(x)$, by the Pesin Entropy Formula and the Ruelle Entropy Inequality. It has positive metric entropy, as shown in the proof of Corollary 1.3. □

**Methodology.** Our approach is similar to that of [43,44]: First we code the flow as a topological Markov flow (Hölder suspension of a topological Markov shift), and then we analyze equilibrium measures for the symbolic model. The first step was done in [28]. The second step is the subject of the present work.

The ergodic behavior of equilibrium measures on topological Markov flows depends on the height function $r$. If $r$ is cohomologous to a function taking values in a discrete subgroup, then one can choose a coding with constant height function, and deduce that the flow is isomorphic to the product of a Bernoulli flow and a rotational flow. If $r$ is not cohomologous to a function taking values in a discrete subgroup, then one can exhibit a generating sequence of very weak Bernoulli partitions as in [36,43], and conclude that the flow is Bernoulli. An important step in the proof of the very weak Bernoulli property is to prove the K property. This is done using the method of Gurevič [17].

In Ratner's case the flow is Anosov, and the symbolic flow is a suspension over a topological Markov shift with finite alphabet [42]. In our case the flow is a general $C^{1+\varepsilon}$ flow on a three dimensional manifold, and the topological Markov shift has countable alphabet [28]. The thermodynamic formalism for countable Markov shifts [12] provides us with the local product structure we need to implement the ideas of [17,36,43,44].

The paper is divided into two parts. The first contains the analysis of topological Markov flows. The second contains the application to smooth flows, and in particular to Reeb flows and geodesic flows.

# Part I. Topological Markov flows

## 2. Topological Markov flows

**Topological Markov shifts (TMS).** Let $\mathscr{G}$ be a directed graph with countable set of vertices $V$. We write $v \to w$ if there is an edge from $v$ to $w$. We assume throughout that for every $v$ there are $u, w$ such that $u \to v, v \to w$, and that $\mathscr{G}$ is not a cycle.

Topological Markov shift (TMS): The *topological Markov shift* (TMS) associated to $\mathscr{G}$ is the discrete-time topological dynamical system $\sigma : \Sigma \to \Sigma$ where

$$\Sigma = \Sigma(\mathscr{G}) := \{\text{paths on } \mathscr{G}\} = \{\{v_i\}_{i\in\mathbb{Z}} : v_i \to v_{i+1}, \forall i \in \mathbb{Z}\},$$

and $\sigma : \{v_i\}_{i\in\mathbb{Z}} \mapsto \{v_{i+1}\}_{i\in\mathbb{Z}}$ is the *left shift*.

Points in $\Sigma$ will be denoted by $x = \{x_i\}_{i\in\mathbb{Z}}$. The topology of $\Sigma$ is given by the metric $d(x, y) := \exp[-\min\{|n| : x_n \neq y_n\}]$. The Borel $\sigma$-algebra $\mathscr{B}(\Sigma)$ is generated by the *cylinders*

$$_m[a_0, \ldots, a_{n-1}] := \{x \in \Sigma : x_{i+m} = a_i \text{ for all } i = 0, \ldots, n-1\}.$$

The index $m$ denotes the left-most coordinate of the constraint. If it is zero, we will simply write $[\underline{a}] := {}_0[\underline{a}]$. The parameter $n$ is called the *length* of the cylinder, also denoted by $|\underline{a}|$. A cylinder is non-empty iff $a_0 \to \cdots \to a_{n-1}$ is a path on $\mathscr{G}$. In this case we call the word $\underline{a}$ *admissible*.

For $x \in \Sigma$ and $i < j$ in $\mathbb{Z}$, let $x_i^j := (x_i, \ldots, x_j)$, $x_i^\infty := (x_i, x_{i+1}, \ldots)$, and $x_{-\infty}^i := (\ldots, x_{i-1}, x_i)$.

A TMS is *topologically transitive* iff for every $u, v \in V$ there is a finite path on $\mathscr{G}$ from $u$ to $v$. It is *topologically mixing* iff for every $u, v \in V$ there is $N = N(u, v)$ such that for every $n \geq N(u, v)$ there is a path of length $n$ on $\mathscr{G}$ from $u$ to $v$.

Every ergodic $\sigma$–invariant probability measure on $\Sigma$ is carried by a topologically transitive TMS inside $\Sigma$. If the measure is mixing, then the TMS is topologically mixing.

Every topologically transitive TMS has a *spectral decomposition* $\Sigma = \biguplus_{i=0}^{p-1} \Sigma_i$ where each $\Sigma_i$ is the union of cylinders of length one at the zeroth position, $\sigma^p : \Sigma_i \to \Sigma_i$ is topologically conjugate to a topologically mixing TMS for every $i$, and $\sigma(\Sigma_i) = \Sigma_{i+1 (\mathrm{mod}\ p)}$ [23].

**Topological Markov flows (TMF).** Let $r : \Sigma \to \mathbb{R}^+$ be Hölder continuous, bounded away from zero and infinity, and let $\Sigma_r := \{(x, t) : x \in \Sigma, 0 \leq t < r(x)\}$.

Topological Markov flow (TMF): The *topological Markov flow* (TMF) with *roof function* $r$ and *basis* $\sigma : \Sigma \to \Sigma$ is the flow $\{\sigma_r^\tau\}$ on $\Sigma_r$ which increases the $t$ coordinate at unit speed subject to the identifications $(x, r(x)) \sim (\sigma(x), 0)$.

Formally, $\sigma_r^\tau$ is defined as $\sigma_r^\tau(x, t) := (\sigma^n(x), t + \tau - r_n(x))$ for the unique $n \in \mathbb{Z}$ such that $0 \leq t + \tau - r_n(x) < r(\sigma^n(x))$ where $r_n$ is the $n$–th *Birkhoff sum*. Recall that $r_n := r + r \circ \sigma + \cdots + r \circ \sigma^{n-1}$ for $n \geq 1$, and that there is a unique way to extend the definition to $n \leq 0$ so that the *cocycle identity* $r_{m+n} = r_n + r_m \circ \sigma^n$ holds for all $m, n \in \mathbb{Z}$. It is given by $r_0 := 0$ and $r_n := -r_{|n|} \circ \sigma^{-|n|}$ for $n < 0$. The cocycle identity guarantees that $\sigma_r^{\tau_1 + \tau_2} = \sigma_r^{\tau_1} \circ \sigma_r^{\tau_2}$ for all $\tau_1, \tau_2 \in \mathbb{R}$.

A TMF is topologically transitive iff its basis is a topologically transitive TMS, but the same is not true for topological mixing. For instance, if the roof function is constant then the TMF is never topologically mixing. By the spectral decomposition [23], every TMF whose basis is a topologically transitive TMS can be recoded as a TMF whose basis is a topologically mixing TMS. Just replace $\Sigma$ by $\Sigma_0$ and $r$ by $r_p$. Let $\mu$ be a $\sigma_r$–invariant probability measure on $\Sigma_r$.

Induced measure: The *induced measure* of $\mu$ is the unique $\sigma$–invariant probability measure $\nu$ on $\Sigma$ such that $\mu = \frac{1}{\int_\Sigma r d\nu} \int_\Sigma \int_0^{r(x)} \delta_{(x,t)} dt\, d\nu(x)$.

Above, $\delta$ denotes the Dirac measure. A $\sigma_r$–invariant measure is ergodic iff its induced measure is. Every ergodic $\sigma_r$–invariant measure on $\Sigma_r$ is carried by a TMF whose basis is a topologically transitive TMS.

**Bowen–Walters Metric [9].** This is a metric which makes $\sigma_r : \Sigma_r \to \Sigma_r$ continuous. Suppose first that $r \equiv 1$ (constant suspension).

Let $\psi : \Sigma_1 \to \Sigma_1$ be the suspension flow, and introduce the following terminology:

- *Horizontal segments*: Ordered pairs $[z, w]_h \in \Sigma_1 \times \Sigma_1$ where $z = (x, t)$ and $w = (y, t)$ have the same height $0 \le t < 1$. The *length* of a horizontal segment $[z, w]_h$ is defined as $\ell([z, w]_h) := (1 - t)d(x, y) + t\,d(\sigma(x), \sigma(y))$.
- *Vertical segments*: Ordered pairs $[z, w]_v \in \Sigma_1 \times \Sigma_1$ where $w = \psi^t(z)$ for some $t$. The *length* of a vertical segment $[z, w]_v$ is $\ell([z, w]_v) := \min\{|t| > 0 : w = \psi^t(z)\}$.
- *Basic paths* from $z$ to $w$: $\gamma := (z_0 = z \xrightarrow{t_0} z_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{n-2}} z_{n-1} \xrightarrow{t_{n-1}} z_n = w)$ with $t_i \in \{h, v\}$ such that $[z_{i-1}, z_i]_{t_{i-1}}$ is a horizontal segment if $t_{i-1} = h$, and a vertical segment if $t_{i-1} = v$. Define $\ell(\gamma) := \sum_{i=0}^{n-1} \ell([z_i, z_{i+1}]_{t_i})$.

Bowen–Walters Metric on $\Sigma_1$: $d_1(z, w) := \inf\{\ell(\gamma)\}$ where $\gamma$ ranges over all basic paths from $z$ to $w$.

Next we consider the general case $r \not\equiv 1$. The idea is to use a canonical bijection from $\Sigma_r$ to $\Sigma_1$ and declare it to be an isometry.

Bowen–Walters metric on $\Sigma_r$: $d_r(z, w) := d_1(\vartheta_r(z), \vartheta_r(w))$, where $\vartheta_r : \Sigma_r \to \Sigma_1$ is given by $\vartheta_r(x, t) := (x, t/r(x))$.

**Lemma 2.1** ([9, 28]). *$d_r$ is a metric, and $\sigma_r^t : \Sigma_r \to \Sigma_r$ is continuous with respect to $d_r$. Moreover, $(t, x) \mapsto \sigma_r^t(x)$ is Hölder continuous on $[-1, 1] \times \Sigma$.*

**Roof functions independent of the past or future.** We say that $r : \Sigma \to \mathbb{R}$ is *independent of the past* if $r(x) = f(x_0, x_1, \ldots)$ for some function $f$, and it is *independent of the future* if $r(x) = g(\ldots, x_{-1}, x_0)$ for some function $g$ (note that we allow dependence on the zeroth coordinate). The next lemma is an adaptation of [43, Lemma 2]. Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a TMF and $\mu$ be an ergodic $\sigma_r$–invariant probability measure.

**Lemma 2.2.** *$(\Sigma_r, \sigma_r, \mu)$ is isomorphic to a TMF with roof function independent of the past, and to a TMF with roof function independent of the future.*

*Proof.* Let us prove the first statement (the second is proved similarly). If $\mu$ is supported on a periodic orbit, then every function is independent of the past on the support of $\mu$. Henceforth we assume that $\mu$ does not sit on a periodic orbit.

It is well known that there is a bounded continuous function $h^s : \Sigma \to \mathbb{R}$ such that $r^s := r - h^s + h^s \circ \sigma$ is bounded, Hölder continuous and independent of the past. Proofs for $\Sigma = \Sigma(\mathscr{G})$ with $\mathscr{G}$ finite can be found in [6, 51]. As noted in [14], these proofs extend without much difficulty to the case where $\mathscr{G}$ is countable. Since the $r^s$ produced by the proofs may take negative values, we now explain how to change $r$ and $h^s$ to have $r^s > 0$.

**Claim.** *It is possible to change $r, h^s$ such that $0 < h^s < \frac{1}{2}r$. In particular $r^s > 0$.*

*Proof.* Since $h^s$ is bounded, we can add a large constant to get a new $h^s$ that is positive. The other inequality is more complicated. Let $c = \sup(h^s) < \infty$, and take $n_0 \in \mathbb{N}$ with $c < \frac{1}{2} n_0 \inf(r)$. Let $\nu$ be the induced measure of $\mu$. Since $\mu$ is ergodic and does not sit on a periodic orbit, $\nu$ is non-atomic, hence there is a cylinder $[\underline{b}]$ such that $0 < \nu[\underline{b}] < \frac{1}{n_0}$. Let $\varphi_{\underline{b}}(x) = \inf\{n \geq 1 : \sigma^n(x) \in [\underline{b}]\}$. By the Kac formula, $\frac{1}{\nu[\underline{b}]} \int_{[\underline{b}]} \varphi_{\underline{b}} d\nu > n_0$. Thus there exists an admissible word $\underline{a} = \underline{b}\,\underline{\xi}\,\underline{b}$ such that $\nu[\underline{a}] > 0$ and $\varphi_{\underline{b}} \restriction_{[\underline{a}]} > n_0$.

Recode the flow using the Poincaré section $[\underline{a}] \times \{0\}$ to obtain a suspension flow with basis $\sigma^{\varphi_{\underline{a}}} : [\underline{a}] \to [\underline{a}]$ and roof function $R = r_{\varphi_{\underline{a}}}$, where $\varphi_{\underline{a}}(x) = \inf\{n \geq 1 : \sigma^n(x) \in [\underline{a}]\}$. The map $\sigma^{\varphi_{\underline{a}}} : [\underline{a}] \to [\underline{a}]$ admits a countable Markov partition

$$S := \{[\underline{a}, \underline{\xi}, \underline{a}] : \varphi_{\underline{a}} \restriction_{[\underline{a},\underline{\xi},\underline{a}]} = |\underline{a}| + |\underline{\xi}|\} \setminus \{\varnothing\}.$$

Coding with $S$, $\sigma^{\varphi_{\underline{a}}} : [\underline{a}] \to [\underline{a}]$ becomes a TMS, therefore the suspension flow is a TMF. Under this new coding, $R^s := R - h^s + h^s \circ \sigma^{\varphi_{\underline{a}}}$ is independent of the past and Hölder continuous. Note that $\varphi_{\underline{a}} \geq \varphi_{\underline{b}} > n_0 \Rightarrow \inf R > n_0 \inf(r) > 2c \Rightarrow h^s < \frac{1}{2} R$.

Henceforth we assume, without loss of generality, that $0 < h^s < \frac{1}{2} r$ for the original flow. Then $r^s$ is bounded, positive and *uniformly bounded away from zero*. This allows us form the TMF $\sigma_{r^s} : \Sigma_{r^s} \to \Sigma_{r^s}$. This TMF is isomorphic to $\sigma_r : \Sigma_r \to \Sigma_r$ via the conjugacy

$$\vartheta_s(x, \xi) = \begin{cases} (x, \xi - h^s(x)), & \text{if } \xi \geq h^s(x) \\ (\sigma^{-1}(x), \xi + r(\sigma^{-1}(x)) - h^s(\sigma^{-1}(x))), & \text{if } 0 \leq \xi < h^s(x), \end{cases}$$

which recodes $\Sigma_r$ using the Poincaré section $\{(x, h^s(x)) : x \in \Sigma\}$. □

**Strong manifolds and the Bowen-Marcus Cocycles [7].** The *strong stable* and *strong unstable manifolds of* $(x, t)$ are:

- $W^{ss}(x,t) := \{(y,s) : d_r(\sigma_r^\tau(x,t), \sigma_r^\tau(y,s)) \xrightarrow[\tau\to\infty]{} 0\}$.
- $W^{su}(x,t) := \{(y,s) : d_r(\sigma_r^{-\tau}(x,t), \sigma_r^{-\tau}(y,s)) \xrightarrow[\tau\to\infty]{} 0\}$.

These are not manifolds, but they play the same role as their smooth analogues in hyperbolic dynamics.

To calculate $W^{ss}, W^{su}$ we make the following definitions. Assume $x$ is not pre-periodic (i.e. there are no $m, n$ such that $x_m^\infty$ or $x_{-\infty}^n$ is a periodic sequence). Let

$$W^{ws}(x) := \{y \in \Sigma : \exists m, n \text{ such that } y_m^\infty = x_n^\infty\}$$

and define $P^s(x, \cdot) : W^{ws}(x) \to \mathbb{R}$ by

$$P^s(x, y) := \lim_{k\to\infty} [r_{m+k}(y) - r_{n+k}(x)]$$

for some (every) $m, n$ such that $y_m^\infty = x_n^\infty$.

Similarly, let

$$W^{wu}(x) := \{y \in \Sigma : \exists m, n \text{ such that } y_{-\infty}^{m} = x_{-\infty}^{n}\},$$

and set $P^u(x,\cdot): W^{wu}(x) \to \mathbb{R}$ by

$$P^u(x,y) := \lim_{k\to-\infty} [r_{m+k}(y) - r_{n+k}(x)]$$

for some (every) $m, n$ such that $y_{-\infty}^{m} = x_{-\infty}^{n}$.

These definitions are independent of the choice of $m, n$, because in the non-pre-periodic case any two possible pairs $(m,n), (m',n')$ satisfy $m' = m+k_0, n' = n+k_0$ for some $k_0 \in \mathbb{Z}$. The limits which define $P^\tau(\cdot,\cdot)$ exist because they are the limits of the partial sums of the series

$$r_m(y) - r_n(x) + \sum_{k=0}^{\infty} [r(\sigma^{m+k}(y)) - r(\sigma^{n+k}(x))] \qquad (\tau = s)$$

or

$$r_m(y) - r_n(x) - \sum_{k=1}^{\infty} [r(\sigma^{m-k}(y)) - r(\sigma^{n-k}(x))] \qquad (\tau = u).$$

Since $r$ is Hölder continuous, the summands decay exponentially fast, and these series converge. Define $W^s\mathrm{loc}(x) = \{y \in \Sigma : y0^\infty = x0^\infty\}$ and $W^u\mathrm{loc}(x) = \{y \in \Sigma : y{-}\infty^0 = x{-}\infty^0\}$.

**Lemma 2.3** ([7]). *Suppose $x$ is not pre-periodic, then for $\tau = s, u$ it holds:*

(1) Bowen-Marcus condition: *$(y,s) \in W^{s\tau}(x,t)$ iff $y \in W^{w\tau}(x)$ and $s - t = P^\tau(x,y)$.*

(2) Shift identity: *$P^\tau(\sigma x, \sigma y) - P^\tau(x,y) = r(x) - r(y)$ wherever defined.*

(3) Cocycle equation: *For all $y, z \in W^{w\tau}(x)$, $P^\tau(x,y) + P^\tau(y,z) = P^\tau(x,z)$. In particular, $P^\tau(x,x) = 0$ and $P^\tau(x,y) = -P^\tau(y,x)$.*

(4) Hölder property: *There are $C > 0$, $0 < \alpha < 1$ such that $|P^\tau(x,y)| \le C d(x,y)^\alpha$ for all $y \in W^\tau_{\mathrm{loc}}(x)$.*

$P^s(\cdot,\cdot)$, $P^u(\cdot,\cdot)$ are called the *Bowen-Marcus cocycles*.

## 3. Equilibrium measures for topological Markov flows

**Equilibrium measures.** Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a TMF, and let $\Phi : \Sigma_r \to \mathbb{R}$ be bounded and continuous. The *(variational) topological pressure* of $\Phi$ is

$$P_{\mathrm{top}}(\Phi) := \sup \left\{ h_\mu(\sigma_r^1) + \int \Phi d\mu : \mu \text{ is } \sigma_r\text{–invariant Borel probability measure} \right\}.$$

Equilibrium measure: $\mu$ is called an *equilibrium measure* (for the potential $\Phi$ and the flow $\{\sigma_r\}$) if $h_\mu(\sigma_r^1) + \int \Phi d\mu = P_{\mathrm{top}}(\Phi)$.

In this section, we will describe the equilibrium measures when $\Sigma$ is topologically mixing, $\Phi$ is bounded and Hölder continuous, and $P_{\text{top}}(\Phi) < \infty$. Instead of describing them directly, we describe the one-sided version of their induced measures. Let $\mu$ be a $\sigma_r$–invariant probability measure, and let $\nu$ be its induced measure, a $\sigma$–invariant. $\nu$ is a $\sigma$–invariant probability measure on $\Sigma$.

One-sided TMS: Let $\pi_s : x \in \Sigma \mapsto (x_0, x_1, \ldots)$. The *one-sided TMS* is the discrete-time topological dynamical system $\sigma_s : \Sigma^s \to \Sigma^s$ where

$$\Sigma^s = \{\pi_s(x) : x \in \Sigma\}$$

and $\sigma_s : \{x_i\}_{i\geq 0} \mapsto \{x_{i+1}\}_{i\geq 0}$ is the *one-sided left shift*.

One-sided version of $\nu$: The *one-sided version* of $\nu$ is the probability measure $\nu^s := \nu \circ \pi_s^{-1}$. It is a $\sigma_s$–invariant probability measure on $\Sigma^s$.

The probability measure $\nu^s$ determines $\nu$ since $\nu \circ \sigma^{-1} = \nu$, and $\nu$ determines $\mu$. Here is the description of $\nu^s$.

**Theorem 3.1.** *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF and $\Phi : \Sigma_r \to \mathbb{R}$ be bounded and Hölder continuous with $P_{\text{top}}(\Phi) < \infty$. Let $\mu$ be an equilibrium measure for $\Phi$, and $\nu$ its induced measure. Then the one-sided version of $\nu$ has the form $\nu^s = h^s\xi^s$, where:*

(1) *$h^s$ is a positive function on $\Sigma^s$, and $\xi^s$ is a positive measure on $\Sigma^s$.*

(2) *There is $\phi^s : \Sigma^s \to \mathbb{R}$ bounded Hölder continuous with $P_{\text{top}}(\phi^s) < \infty$ such that $Lh^s = \lambda h^s$ and $L^*\xi^s = \lambda\xi^s$, where $\lambda = \exp[P_{\text{top}}(\phi^s)]$ and $L$ is the Ruelle operator of $\phi^s$, $(Lf)(x_0^\infty) = \sum_{\sigma_s(y_0^\infty)=x_0^\infty} \exp[\phi^s(y_0^\infty)]f(y_0^\infty)$ for all $f : \Sigma^s \to \mathbb{R}$.*

(3) *$h^s(x) = \lim_{n\to\infty} \frac{1}{\xi^s[\underline{a}]}\mathfrak{l}^{-n}(L^n 1_{[\underline{a}]})(x)$ for every cylinder $[\underline{a}]$ and $x \in \Sigma^s$.*

(4) *$\log h^s$ is uniformly Hölder continuous on cylinders of length one at the zeroth position.*

(5) *$\nu^s$ is ergodic.*

*Proof.* Bowen and Ruelle proved the theorem in [8] for TMF built from finite graphs, using Ruelle's Perron–Frobenius Theorem [6,25,48]. Since Ruelle's Perron–Frobenius Theorem is false for general infinite graphs, we sketch the modifications needed to treat our case.

**Claim 1.** *$\nu$ is an equilibrium measure for $\phi(x) := \int_0^{r(x)} \Phi(x,t)dt - P_{\text{top}}(\Phi)r(x)$. The function $\phi : \Sigma \to \mathbb{R}$ is bounded Hölder continuous with $P_{\text{top}}(\phi) = 0$.*

*Proof of Claim 1.* This is proved exactly as in [8]. The function $\phi$ is clearly bounded and Hölder continuous. By the Abramov entropy formula [2], $h_\mu(\sigma_r) = \frac{1}{\int r d\nu} h_\nu(\sigma)$. Hence

$$\frac{h_\nu(\sigma) + \int_\Sigma \int_0^{r(x)} \Phi(x,t)dt\,d\nu(x)}{\int r d\nu} \leq P_{\text{top}}(\Phi),$$

with equality iff $\mu$ is an equilibrium measure for $\Phi$. This can be rewritten as $h_\nu(\sigma) + \int_\Sigma \phi(x)d\nu(x) \leq 0$, with equality iff $\nu$ is an equilibrium measure for $\phi$. Therefore $P_{\text{top}}(\phi) = 0$, and $\mu$ is an equilibrium measure for $\Phi$ iff its induced measure $\nu$ is an equilibrium measure for $\phi$.

**Claim 2.** *The measure $\nu$ is an equilibrium measure for a bounded Hölder continuous potential that is independent of the past and has zero pressure.*

*Proof of Claim 2.* By [6, 14, 51] there is a bounded Hölder continuous function $v : \Sigma \to \mathbb{R}$ such that $\phi + v - v \circ \sigma$ is independent of the past. Since $\int (v - v \circ \sigma) dm = 0$ for every $\sigma$–invariant probability measure $m$, $P_{\text{top}}(\phi + v - v \circ \sigma) = P_{\text{top}}(\phi) = 0$.

Now we proceed to the proof of Theorem 3.1. By Claims 1–2, there is $\phi^s : \Sigma^s \to \mathbb{R}$ bounded Hölder continuous such that $\phi^s \circ \pi_s = \phi + v - v \circ \sigma$, $\nu$ is an equilibrium measure for $\phi^s \circ \pi_s$, and $P_{\text{top}}(\phi^s \circ \pi_s) = 0$. We want to conclude that $\nu^s$ is an equilibrium measure for $\phi^s$, and that $P_{\text{top}}(\phi^s) = 0$.

If $\nu$ is a $\sigma$–invariant probability measure then $(\Sigma, \nu, \sigma)$ is the natural extension of $(\Sigma^s, \nu^s, \sigma_s)$. Conversely, if $\nu^s$ is a $\sigma_s$–invariant probability measure then it is the one-sided version of some $\sigma$–invariant probability measure $\nu$ (its natural extension). Since natural extensions preserve entropy, $P_{\text{top}}(\phi^s) = P_{\text{top}}(\phi^s \circ \pi_s) = 0$, and $\nu$ is an equilibrium measure for $\phi^s \circ \pi_s$ iff $\nu^s$ is an equilibrium measure for $\phi^s$.

The structure of equilibrium measures for Hölder continuous potentials on one-sided TMS was determined in [12]. There it is shown that if $\Sigma^s$ is topologically mixing (a consequence of the topological mixing of $\Sigma$), then $\phi^s$ is positive recurrent in the sense of [49], and parts (1)–(3) of the theorem hold. Also, if the equilibrium measure exists then it is unique [12, Thm 1.1], and this gives part (5). Part (4) follows from part (3) and the boundedness and Hölder continuity of $\phi^s$. □

**Corollary 3.2.** *Suppose $\Sigma_r$ is a topologically transitive TMF, and $\Phi$ is a bounded Hölder continuous potential with finite pressure. Then $\Phi$ has at most one equilibrium measure and if this measure exists then it is ergodic.*

*Proof.* By Theorem 3.1, $\nu^s$ is ergodic. Therefore its natural extension $\nu$ is ergodic. If the induced measure is ergodic, then the original measure is ergodic. It follows that every equilibrium measure is ergodic. This implies that the equilibrium measure is unique: if there were two equilibrium measures, then their average would have been a non-ergodic equilibrium measure. □

**Conditional measures of the induced measure.** Theorem 3.1 can be used to construct the conditional measures $\nu(\cdot|x_0^\infty)$ for all, rather than almost all, $x \in \Sigma^s$. The basic tool is the *g–function* of $\nu$. This is the function $g : \Sigma^s \to \mathbb{R}$ given by

$$g := \frac{e^{\phi^s} h^s}{\lambda h^s \circ \sigma} = \frac{d\nu^s}{d(\nu^s \circ \sigma_s)}.$$

The reader can check that $g > 0$ and $\sum_{\sigma_s(y_0^\infty)=x_0^\infty} g(y_0^\infty) = 1$, whence $0 < g \leq 1$. Thus $g$ is a $g$–function in the sense of [22]. The function $\log g$ is bounded and uniformly Hölder continuous on cylinders of length two, since $\phi^s, \log h^s$ are bounded and uniformly Hölder continuous on cylinders of length one at the zeroth position.

**Theorem 3.3** ([26]). *Let $\nu$, $\nu^s$, $L$ as in Theorem 3.1.*

(1) *If $f \in L^1(\nu^s)$ then $\mathbb{E}_{\nu^s}(f|x_1^\infty) = \sum_{\sigma_s(y_0^\infty)=x_1^\infty} g(y_0^\infty) f(y_0^\infty)$ $\nu^s$–a.e.*

(2) $\nu^s(x_0|x_1, x_2, \ldots) := \lim_{k\to\infty} \nu([x_0]|_1[x_1, \ldots, x_k]) = g(x_0^\infty)$ *$\nu^s$–a.e.*

(3) $\lim_{k\to\infty} \nu({}_{-n}[x_{-n}, \ldots, x_{-1}]|_0[x_0, \ldots, x_k])$ *is equal $\nu$–a.e. to*

$$\nu(x_{-n}, \ldots, x_{-1}|x_0^\infty) := g_n(x_{-n}^\infty) := g(x_{-n}^\infty) g(x_{-n+1}^\infty) \cdots g(x_{-1}^\infty). \quad (3.1)$$

*Proof.* Part (1) follows from the equations $\nu^s = h^s \xi^s$, $Lh^s = \lambda h^s$, $L^* \xi^s = \lambda \xi^s$ as in [26]. Part (2) follows from part (1) and the martingale convergence theorem. Part (3) follows from part (2) and the invariance of $\nu$. □

One should view (3.1) as a consistent set of equations which determine the conditional probability measure $\nu(\cdot|x_0^\infty)$ on $W^s_{\text{loc}}(x)$, by specifying the weights these measures give to cylinders. Consistency follows from $\sum_{\sigma_s(y_0^\infty)=x_0^\infty} g(y_0^\infty) = 1$. Henceforth, we define $\nu(\cdot|x_0^\infty)$ as follows.

MEASURE $\nu(\cdot|x_0^\infty)$: $\nu(\cdot|x_0^\infty)$ is the unique probability measure on $W^s_{\text{loc}}(x)$ such that $\nu(\underline{a}|x_0^\infty) := g_n(\underline{a}x_0^\infty)$ for all admissible words $\underline{a}$ of length $n$.

**Lemma 3.4.** *Let $\nu$ be as in Theorem 3.1. If $\Sigma_r$ is topologically transitive and $\Sigma_r$ is not a union of cycles, then $\nu(\cdot|x_0^\infty)$ is non-atomic for $\nu$–a.e. $x \in \Sigma$.*

*Proof.* Since $\Sigma_r$ is topologically transitive and $\Sigma_r$ is not a union of cycles, the same is true for $\Sigma$. In particular there is a state $b$ with in-degree at least two. Fix one such edge $a \to b$. Since $\sum_{\sigma_s(z_0^\infty)=y_0^\infty} g(z_0^\infty) = 1$, we have $g(z) < 1$ for every $z \in \Sigma^s$ such that $(z_0, z_1) = (a, b)$. By the Hölder continuity of $\log g$, we can find a word $\underline{w} := (a, b, b_2, \ldots, b_n)$ such that $g \restriction_{[\underline{w}]} < 1$. By (3.1), $\nu(\{z\}|x_0^\infty) = 0$ whenever $z \in Z := \{z \in \Sigma : z_n^{n+|\underline{w}|-1} = \underline{w}$ for infinitely many $n < 0\}$. The conclusion is that $\nu(\cdot|x_0^\infty)$ is non-atomic for every $x_0^\infty$ such that $\nu(Z|x_0^\infty) = 1$.

Let us show that this last condition is true $\nu$–a.e. By Theorem 3.1, $\nu$ is ergodic and positive on cylinders, hence $\nu(Z) = 1$, i.e. $\int \nu(Z|x_0^\infty) d\nu(x) = \nu(Z) = 1$, so $\nu(Z|x_0^\infty) = 1$ for $\nu$–a.e. $x \in \Sigma$. □

**Local product structure of the induced measure.** Let $\sigma : \Sigma \to \Sigma$ be a TMS. The following definitions are motivated by smooth ergodic theory, see e.g. [4]:

- $W^s(x) := \{y \in \Sigma : d(\sigma^n(x), \sigma^n(y)) \xrightarrow[n\to\infty]{} 0\}$
  $= \{y \in \Sigma : \exists n \text{ such that } y_n^\infty = x_n^\infty\}.$
- $W^u(x) := \{y \in \Sigma : d(\sigma^n(x), \sigma^n(y)) \xrightarrow[n\to-\infty]{} 0\}$
  $= \{y \in \Sigma : \exists n \text{ such that } y_{-\infty}^n = x_{-\infty}^n\}.$
- $W^s_{\text{loc}}(x) := \{y \in \Sigma : y_0^\infty = x_0^\infty\}.$
- $W^u_{\text{loc}}(x) := \{y \in \Sigma : y_{-\infty}^0 = x_{-\infty}^0\}.$

Smale bracket of points: Let $x, y \in \Sigma$ with $x_0 = y_0$. The *Smale bracket* of $x, y$ is $[x, y] := z$ where $z_i = x_i$ for $i \le 0$ and $z_i = y_i$ for $i \ge 0$.

If $x_0 = y_0 = v$, then

$$[W^s_{\text{loc}}(x), W^u_{\text{loc}}(y)] = \{[x', y'] : x' \in W^s_{\text{loc}}(x), y' \in W^u_{\text{loc}}(y)\}$$
$$= [v] = \{z \in \Sigma : z_0 = v\}.$$

We can also consider the Smale products of measures. Let $\alpha_x^s$, $\beta_y^u$ be finite measures on $W^s_{\text{loc}}(x)$, $W^u_{\text{loc}}(y)$, respectively.

Smale bracket of measures: The *Smale bracket* of $\alpha_x^s, \beta_y^u$ is a finite measure on $[W^s_{\text{loc}}(x), W^u_{\text{loc}}(y)] = [v]$ defined by

$$(\alpha_x^s \star \beta_y^u)(E) := \int_{W^u_{\text{loc}}(y)} \int_{W^s_{\text{loc}}(x)} 1_E([x', y'])\, d\alpha_x^s(x')\, d\beta_y^u(y'), \quad (E \text{ Borel measurable}).$$

The Smale product produces measures on $\Sigma$ out of measures on $W^s_{\text{loc}}(x)$, $W^u_{\text{loc}}(y)$. We can also produce measures on $W^s_{\text{loc}}(x)$, $W^u_{\text{loc}}(y)$ from measures on $\Sigma$. Let:

- $p_x^s : [x_0] \to W^s_{\text{loc}}(x)$, $p_x^s(\cdot) = [\cdot, x]$.
- $p_x^u : [x_0] \to W^u_{\text{loc}}(x)$, $p_x^u(\cdot) = [x, \cdot]$.

Projection measures: The *projections* of $\nu$ on $W^s_{\text{loc}}(x)$, $W^u_{\text{loc}}(y)$ are

$$\begin{cases} \nu_x^s := \nu \circ (p_x^s)^{-1}, & \text{a measure on } W^s_{\text{loc}}(x), \\ \nu_y^u := \nu \circ (p_y^u)^{-1}, & \text{a measure on } W^u_{\text{loc}}(y). \end{cases} \tag{3.2}$$

Note that for $\tau = u, s$:

- $\nu_x^\tau = \nu_y^\tau$ iff $W^\tau_{\text{loc}}(x) = W^\tau_{\text{loc}}(y)$.
- $\nu_x^\tau = (\nu_y^\tau \circ p_y^\tau) \restriction_{W^\tau_{\text{loc}}(x)}$ whenever $x_0 = y_0$.

Local product structure: $\nu$ is said to have *local product structure* if for every $x, y \in \Sigma$ such that $x_0 = y_0 = v$ we have $\nu^s_x \star \nu^u_y \sim \nu \restriction_{[v]}$.

**Theorem 3.5.** *Let $\mu$ be an equilibrium measure of a bounded Hölder continuous potential with finite pressure on a topologically transitive TMF, and let $\nu$ be its induced measure. Then $\nu$ is globally supported, and $\nu$ has local product structure.*

*Proof.* Let $\sigma : \Sigma \to \Sigma$ be the associated TMS, and let $\mathscr{G}$ be a directed graph associated to $\Sigma$. Since the TMF is topologically transitive, $\sigma : \Sigma \to \Sigma$ is topologically transitive, hence any two vertices on $\mathscr{G}$ can be joined by a path.

**Claim.** *Every non-empty cylinder on $\Sigma$ has positive $\nu$–measure, and for every edge $v \to w$ there is a constant $C_{vw} > 1$ such that if $m < 0, n > 0$ and ${}_m[v_m, \dots, v_n] \neq \varnothing$, then*

$$C^{-1}_{v_0 v_1} \leq \frac{\nu({}_m[v_m, \dots, v_n])}{\nu({}_m[v_m, \dots, v_0])\nu({}_0[v_0, \dots, v_n])} \leq C_{v_0 v_1}.$$

*Proof of the claim.* Let $\nu^s$ be the one-sided version of $\nu$. Theorem 3.1 implies, as in [50, Corollary 3.2], the existence of constants $K_v, D_{vw} > 0$ such that

(a) $K^{-1}_{a_{n-1}} \leq \dfrac{\nu^s([a_0, \dots, a_{n-1}, b_0, \dots, b_{k-1}])}{\nu^s([a_0, \dots, a_{n-1}])\nu^s([b_0, \dots, b_{k-1}])} \leq K_{a_{n-1}}$ for all $\underline{a}, \underline{b}$ such that $[\underline{a}, \underline{b}] \neq \varnothing$,

(b) $D^{-1}_{a_{n-1} b_0} \leq \dfrac{\nu^s([b_0, \dots, b_{k-1}])}{\nu^s([a_{n-1}, b_0, \dots, b_{k-1}])} \leq D_{a_{n-1} b_0}$ whenever $[a_{n-1}, \underline{b}] \neq \varnothing$.

By (a)–(b), there are constants $C_{vw}$ such that for all $\underline{a}, \underline{b}$ with $[\underline{a}, \underline{b}] \neq \varnothing$ we have:

$$C^{-1}_{a_{n-1} b_0} \leq \frac{\nu^s([a_0, \dots, a_{n-1}, b_0, \dots, b_{k-1}])}{\nu^s([a_0, \dots, a_{n-1}])\nu^s([a_{n-1}, b_0, \dots, b_{k-1}])} \leq C_{a_{n-1} b_0}.$$

Substituting $\underline{a} = (v_m, \dots, v_0), \underline{b} = (v_1, \dots, v_n)$ gives the claim.

By the claim, if $E$ is a cylinder contained in $[v, w]$ and $x, y \in [v]$ then:

$$C^{-1}_{vw} \times (\nu^s_x \star \nu^u_y)(E) \leq \nu(E) \leq C_{vw} \times (\nu^s_x \star \nu^u_y)(E). \tag{3.3}$$

The collection of cylinders $E \subset [v, w]$ satisfying (3.3) is closed under increasing unions and decreasing intersections. By the monotone class theorem, (3.3) holds for every Borel set $E \subset [v, w]$, whence $\nu^s_x \star \nu^u_y \sim \nu \restriction_{[v]}$. □

**Corollary 3.6.** *Let $\nu$ be as in the previous theorem. If $E \subset \Sigma$ is Borel and $\nu(E) = 0$, then $\nu^s_x(E) = \nu^u_x(E) = 0$ for $\nu$–a.e. $x$.*

*Proof.* Let $\Omega_v := \{x \in \Sigma : x_0 = v \text{ and } \nu_x^s(E) > 0\}$, and assume by contradiction that $\nu(\Omega_v) > 0$ for some $v$. Since $\nu$ has local product structure, if $x, y \in [v]$ then:

$$\int_{W^u_{\text{loc}}(y)} \int_{W^s_{\text{loc}}(x)} 1_{\Omega_v}([x', y'])d\nu_x^s(x')d\nu_y^u(y') > 0.$$

Note that $[x', y'] \in \Omega_v \Leftrightarrow \nu^s_{[x',y']}(E) > 0 \overset{!}{\Leftrightarrow} \nu^s_{y'}(E) > 0 \Leftrightarrow y' \in \Omega_v$ ($\overset{!}{\Leftrightarrow}$ is because $\nu^s_{[x',y']} = \nu^s_{y'}$). Hence $1_{\Omega_v}([x', y']) = 1_{\Omega_v}(y')$. Calculating the double integral, we find that $\nu_y^u[\Omega_v]\nu_x^s[W^s_{\text{loc}}(x)] > 0 \Rightarrow \nu_y^u[\Omega_v] > 0$. We use this to get a contradiction.

Let $y' \in \Omega_v$. Using that $\nu^s_{y'} = (\nu_x^s \circ p_x^s) \restriction_{W^s_{\text{loc}}(y')}$, we have

$$\begin{aligned} 0 < \nu^s_{y'}(E) = (\nu_x^s \circ p_x^s)[E \cap W^s_{\text{loc}}(y')] &= \nu_x^s\{x' \in W^s_{\text{loc}}(x) : [x', y'] \in E\} \\ &= \int_{W^s_{\text{loc}}(x)} 1_E([x', y'])d\nu_x^s(x'). \end{aligned}$$

Since $\nu^u_{y'}[\Omega_v] > 0$, if we integrate this inequality we obtain

$$\int_{W^u_{\text{loc}}(y)} \left( \int_{W^s_{\text{loc}}(x)} 1_E([x', y'])d\nu_x^s(x') \right) d\nu_y^u(y') > 0,$$

thus $(\nu_x^s \star \nu_y^u)(E) > 0$. Since $\nu$ has local product structure, this gives that $\nu(E) > 0$, a contradiction. We have just proved that $\nu[\Omega_v] = 0$ for every vertex $v$, whence $\nu_x^s(E) = 0$ for $\nu$–a.e. $x$. By symmetry, $\nu_x^u(E) = 0$ for $\nu$–a.e. $x$. □

## 4. The Pinsker factor of a topological Markov flow

**Review of general theory.** Let $(X, \mathscr{B}, \mu, T)$ be an *automorphism*, i.e. $(X, \mathscr{B}, \mu)$ is a non-atomic Lebesgue probability space and $T$ is an invertible transformation preserving $\mu$. Given $E \in \mathscr{B}$, let $\alpha_E = \{E, X \setminus E\}$.

PINSKER FACTOR: $E \in \mathscr{B}$ is called a *Pinsker set* if $h_\mu(T, \alpha_E) = 0$. The *Pinsker σ–algebra* is $\mathscr{P}(T) := \{E \in \mathscr{B} : E \text{ is a Pinsker set}\}$. $(X, \mathscr{P}(T), \mu, T)$ is called the *Pinsker factor* of $(X, \mathscr{B}, \mu, T)$.

The $\sigma$–algebra $\mathscr{P}(T)$ is $T$–invariant [41], hence $(X, \mathscr{P}(T), \mu, T)$ is indeed a factor. $(X, \mathscr{P}(T), \mu, T)$ has zero entropy, and if $\mathscr{A} \subset \mathscr{B}$ such that $(X, \mathscr{A}, \mu, T)$ is a factor of zero entropy then $\mathscr{A} \subset \mathscr{P}(T)$ modulo $\mu$. Therefore $(X, \mathscr{P}(T), \mu, T)$ is the largest factor of $(X, \mathscr{B}, \mu, T)$ with zero entropy.

COMPLETELY POSITIVE ENTROPY: $(X, \mathscr{B}, \mu, T)$ is said to have *completely positive entropy* if it has a trivial Pinsker factor, i.e. if $\mathscr{P}(T) = \{\varnothing, X\}$ modulo $\mu$.

Note that $(X, \mathscr{B}, \mu, T)$ has completely positive entropy iff all of its non-trivial factors have positive entropy.

TAIL $\sigma$–ALGEBRA: Given a $\sigma$–algebra $\mathscr{A} \subset \mathscr{B}$ with $T^{-1}\mathscr{A} \subset \mathscr{A}$, the *tail $\sigma$–algebra* of $\mathscr{A}$ is $\text{Tail}(\mathscr{A}) := \bigcap_{n\geq 0} T^{-n}\mathscr{A}$.

K PROPERTY: $(X, \mathscr{B}, \mu, T)$ has the *K property* if there is a $\sigma$–algebra $\mathscr{A} \subset \mathscr{B}$ such that:

(a) $T^{-1}\mathscr{A} \subset \mathscr{A}$,

(b) $\bigvee_{i=0}^{\infty} T^i\mathscr{A} = \mathscr{B}$ modulo $\mu$,

(c) $\text{Tail}(\mathscr{A}) = \{\varnothing, X\}$ modulo $\mu$.

**Theorem 4.1** (Rokhlin & Sinai [46]). *$(X, \mathscr{B}, \mu, T)$ has the K property iff it has completely positive entropy.*

The K property is stronger than mixing. It implies continuous Lebesgue spectrum [45], and the mixing property below, called *K-mixing*, see [13, §10.8]. Write $\delta$–a.e. when a property holds for a set of atoms with total measure $\geq 1 - \delta$.

**Theorem 4.2.** *Let $(X, \mathscr{B}, \mu, T)$ be an automorphism with the K property, $B \in \mathscr{B}$, and $\beta$ a finite measurable partition of $X$. Then for every $\delta > 0$ there is $N_0 = N_0(B, \delta)$ such that for all $N' > N \geq N_0$ and $\delta$–a.e. $A \in \bigvee_{k=N}^{N'} T^k\beta$ it holds $|\mu(B|A) - \mu(B)| < \delta$.*

Now let $\mathsf{T} = (X, \mathscr{B}, \mu, \{T^t\})$ be a flow. It is known that $h_\mu(T^t) = |t|h_\mu(T^1)$ and $\mathscr{P}(T^t) = \mathscr{P}(T^1)$, $\forall t \neq 0$ [2, 17]. The *Pinsker $\sigma$–algebra* of $\mathsf{T}$ is defined as $\mathscr{P}(T^1)$. $\mathsf{T}$ is said to have *completely positive entropy* if its Pinsker factor is trivial iff $\exists t \neq 0$ such that $(X, \mathscr{B}, \mu, T^t)$ is an automorphism with completely positive entropy. $\mathsf{T}$ is said to have the *K property* if $(X, \mathscr{B}, \mu, T^1)$ is an automorphism with the K property iff $\exists t \neq 0$ such that $(X, \mathscr{B}, \mu, T^t)$ is an automorphism with the K property. $\mathsf{T}$ has the K property iff it has completely positive entropy, and is in this case K-mixing [13]. The next theorem is a tool for proving the K property. Given a $\sigma$–algebra $\mathscr{A}$ with $T^{-t}\mathscr{A} \subset \mathscr{A}$, $\forall t > 0$, let $\text{Tail}(\mathscr{A}) := \bigcap_{t>0} T^{-t}\mathscr{A}$ be the *tail $\sigma$–algebra* of $\mathscr{A}$.

**Theorem 4.3** (Rokhlin & Sinai [46]). *Let $\mathsf{T} = (X, \mathscr{B}, \mu, \{T^t\})$ be a flow, and let $\mathscr{A} \subset \mathscr{B}$ be a $\sigma$–algebra such that:*

(a) $T^{-t}\mathscr{A} \subset \mathscr{A}$, $\forall t > 0$,

(b) $\bigvee_{t>0} T^t\mathscr{A} = \mathscr{B}$ *modulo* $\mu$.

*Then $\mathscr{P}(T) \subset \text{Tail}(\mathscr{A})$ modulo $\mu$.*

**An upper bound for the Pinsker factor of a TMF .** We now construct $\sigma$–algebras as in Theorem 4.3 for a topologically transitive TMF. The construction follows [17,44].

Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF. By Lemma 2.2, $\sigma_r : \Sigma_r \to \Sigma_r$ is isomorphic to a TMF $\sigma_{r^s} : \Sigma_{r^s} \to \Sigma_{r^s}$ such that $r^s$ is independent of the past. Let $\vartheta_s : \Sigma_r \to \Sigma_{r^s}$ be the isomorphism, $\vartheta_s \circ \sigma_r^t = \sigma_{r^s}^t \circ \vartheta_s$, $\forall t \in \mathbb{R}$. Points in $\Sigma_{r^s}$ will be decorated by over bars as in $(\overline{x}, \overline{\xi})$.

Given $(x, \xi) \in \Sigma_r$, let $(\overline{x}, \overline{\xi}) := \vartheta_s(x, \xi)$ and define

$$W_{\text{loc}}^{ss}(x, \xi) := \vartheta_s^{-1}\{(\overline{y}, \overline{\xi}) \in \Sigma_{r^s} : \overline{y}_0^\infty = \overline{x}_0^\infty\}.$$

Any two such sets are either equal or disjoint, hence $\{W_{\text{loc}}^{ss}(x, \xi)\}$ is a partition of $\Sigma_r$. Let $\mathscr{W}_{\text{loc}}^{ss}$ be the $\sigma$–algebra generated by $\{W_{\text{loc}}^{ss}(x, \xi)\}$. $\mathscr{W}_{\text{loc}}^{ss}$ is generated by the countable collection of sets $\vartheta_s^{-1}\{(\overline{y}, \overline{\xi}) \in \Sigma_{r^s} : \overline{y}_0^{N-1} = \underline{a}, \overline{\xi} \in (\alpha, \beta)\}$ where $N \in \mathbb{N}$, $\underline{a}$ is an admissible word of length $N$, and $\alpha, \beta \in \mathbb{Q}$.

Using that $r^s$ is independent of the past and that $\vartheta_s \circ \sigma_r^t = \sigma_{r^s}^t \circ \vartheta_s$, one shows:

(a) $\sigma_r^{-t}[\mathscr{W}_{\text{loc}}^{ss}] \subset \mathscr{W}_{\text{loc}}^{ss}$, $\forall t > 0$.

(b) $\bigvee_{t>0} \sigma_r^t[\mathscr{W}_{\text{loc}}^{ss}] = \mathscr{B}$ modulo $\mu$.

Let $\mathscr{W}^{ss} := \text{Tail}(\mathscr{W}_{\text{loc}}^{ss})$. By Theorem 4.3, $\mathscr{P}(\sigma_r) \subset \mathscr{W}^{ss}$ modulo $\mu$.

Next we work with an isomorphism $\vartheta_u : \Sigma_r \to \Sigma_{r^u}$ where $r^u$ is independent of the future and $\vartheta_u \circ \sigma_r^t = \sigma_{r^u}^t \circ \vartheta_u$, $\forall t \in \mathbb{R}$. Denoting points in $\Sigma_{r^u}$ also as $(\overline{x}, \overline{\xi}) := \vartheta_u(x, \xi)$, we can define for each $(x, \xi) \in \Sigma_r$ the set

$$W_{\text{loc}}^{su}(x, \xi) := \vartheta_u^{-1}\{(\overline{y}, \overline{\xi}) : \overline{y}_{-\infty}^0 = \overline{x}_{-\infty}^0\}$$

and $\mathscr{W}_{\text{loc}}^{su}$ as the $\sigma$–algebra generated by the partition $\{W_{\text{loc}}^{su}(x, \xi)\}$. Similarly, $\sigma_r^t[\mathscr{W}_{\text{loc}}^{su}] \subset \mathscr{W}_{\text{loc}}^{su}$, $\forall t > 0$, and $\bigvee_{t>0} \sigma_r^{-t}[\mathscr{W}_{\text{loc}}^{su}] = \mathscr{B}$ modulo $\mu$. Let $\mathscr{W}^{su} := \text{Tail}(\mathscr{W}_{\text{loc}}^{su})$. Applying Theorem 4.3 to the inverse flow $\{\sigma_r^{-t}\}$ and using that it has the same Pinsker $\sigma$–algebra as $\{\sigma_r^t\}$, we find that $\mathscr{P}(\sigma_r) \subset \mathscr{W}^{su}$ modulo $\mu$. We just proved:

**Theorem 4.4** ([17, 44]). *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a TMF, and let $\mu$ be an ergodic $\sigma_r$–invariant probability measure, not supported on a single orbit. Then $\mathscr{P}(\sigma_r) \subset \mathscr{W}^{ss} \cap \mathscr{W}^{su}$ modulo $\mu$.*

**Corollary 4.5.** *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a TMF, and let $\mu$ be an ergodic $\sigma_r$–invariant probability measure, not supported on a single orbit. If $f : \Sigma_r \to \mathbb{R}$ is $\mathscr{P}(\sigma_r)$–measurable, then there is a set $X$ of full $\mu$–measure such that for every $(x, \xi), (y, \eta) \in X$:*

(1) *If $(y, \eta) \in W^{ss}(x, \xi)$ then $f(x, \xi) = f(y, \eta)$.*

(2) *If $(y, \eta) \in W^{su}(x, \xi)$ then $f(x, \xi) = f(y, \eta)$.*

*Proof.* Recall the definitions of $W^{ss}(x, \xi)$ and $W^{su}(x, \xi)$ on page 71. We prove (1), and leave (2) to the reader. It is enough to prove this for $f = 1_E$ where $E \in \mathscr{P}(\sigma_r)$.

Since $\mathscr{P}(\sigma_r) \subset \mathscr{W}^{ss} = \mathrm{Tail}(\mathscr{W}^{ss}_{\mathrm{loc}})$, there is a sequence of sets $E_i \in \sigma_r^{-i}(\mathscr{W}^{ss}_{\mathrm{loc}})$ such that $\mu(E \triangle E_i) = 0$. The set $X := \Sigma_r \setminus [(\cup_{i\geq 1} E \triangle E_i) \cup \{(x,\xi) : x \text{ is pre-periodic}\}]$ has full $\mu$–measure.

If $(x,\xi), (y,\eta) \in X$ with $(y,\eta) \in W^{ss}(x,\xi)$, then $\sigma_r^t(y,\eta) \in W^{ss}_{\mathrm{loc}}(\sigma_r^t(x,\xi))$ for $t$ large enough. In particular, this holds for some $t = i \in \mathbb{N}$. We want to show that $(x,\xi) \in E \Leftrightarrow (y,\eta) \in E$. By symmetry, it is enough that $(x,\xi) \in E \Rightarrow (y,\eta) \in E$.

Let $(x,\xi) \in E$. Then $(x,\xi) \notin E \triangle E_i \Rightarrow (x,\xi) \in E_i \Rightarrow \sigma_r^i(x,\xi) \in \sigma_r^i(E_i) \in \mathscr{W}^{ss}_{\mathrm{loc}}$. The atom of $\mathscr{W}^{ss}_{\mathrm{loc}}$ which contains $\sigma_r^i(x,\xi)$ is $W^{ss}_{\mathrm{loc}}((\sigma_r^i(x,\xi))$, so $\sigma_r^i(y,\eta) \in W^{ss}_{\mathrm{loc}}((\sigma_r^i(x,\xi)) \subset \sigma_r^i(E_i) \Rightarrow (y,\eta) \in E_i \overset{!}{\Rightarrow} (y,\eta) \in E$ ($\overset{!}{\Rightarrow}$ is because $(y,\eta) \in X$). □

**The Pinsker factor in the non-arithmetic case.** Let $\sigma : \Sigma \to \Sigma$ be a TMS. A Hölder continuous $r : \Sigma \to \mathbb{R}$ is called *arithmetic*, if there are $\theta \in \mathbb{R}$, $\theta \neq 0$, and $h : \Sigma \to S^1$ Hölder continuous such that $e^{i\theta r} = h/h \circ \sigma$ [16].

**Theorem 4.6.** *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF, and let $\mu$ be an equilibrium measure of a bounded Hölder continuous function with finite pressure. The following are equivalent:*

(1) *$r$ is not arithmetic.*

(2) *$\mu$ is weak mixing.*

(3) *$\mu$ is mixing.*

(4) *$\mu$ has the K property, whence a trivial Pinsker factor.*

*In particular, if one equilibrium measure of a bounded Hölder continuous function with finite pressure satisfies one of* (2)–(4)*, then all equilibrium measures of bounded Hölder continuous functions with finite pressure satisfy all of* (2)–(4).

If $\Sigma$ is a subshift of finite type, then the equivalences of (2)–(4) are due to Ratner [44] (a special case was done before by Gurevich [17]), and (1) $\Leftrightarrow$ (2) is due to Parry and Pollicott [37, Prop. 6.2].

*Proof.* (4) $\Rightarrow$ (3) by general theory, and (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1) because if $e^{i\theta r} = h/h \circ \sigma$ for some $\theta \neq 0$ and $h : \Sigma \to S^1$ continuous, then $F(x,\xi) := e^{-i\theta\xi} h(x)$ satisfies $F \circ \sigma_r^t = e^{-i\theta t} F$, $\forall t \in \mathbb{R}$. By the weak mixing assumption, $F$ is constant $\mu$–a.e., whence everywhere (equilibrium measures of Hölder potentials on a topologically transitive TMF are globally supported). Thus $\theta = 0$.

It remains to show that (1) $\Rightarrow$ (4). We prove that if the Pinsker $\sigma$–algebra is not trivial then $r$ is arithmetic. Assume that $\mathscr{P}(\sigma_r)$ is not trivial, and fix a bounded Pinsker-measurable function $F$ that is not constant $\mu$–a.e. Let $F_\delta := \frac{1}{\delta}\int_0^\delta F \circ \sigma_r^t dt$. Note that $F_\delta \xrightarrow[\delta\to 0^+]{L^1} F$, thus $F_\delta$ is not constant $\mu$–a.e for any $\delta$ small enough. Fix one such $\delta$ and let $H := F_\delta$. $H$ is a bounded Pinsker-measurable function that is not

constant $\mu$–a.e. for which the map $t \mapsto (H \circ \sigma_r^t)(x,\xi)$ is continuous, $\forall (x,\xi) \in \Sigma_r$. We will use $H$ to prove that $r$ is arithmetic. Let $\nu$ be the induced measure of $\mu$. Recall the definition of the cocycles $P^s$, $P^u$ (see Lemma 2.3) and the measures $\nu_x^s$ on $W^s_{\text{loc}}(x)$ defined in (3.2).

**Claim 1.** *There is a Borel set $E \subset \Sigma$ of full $\nu$–measure such that:*

(1) *$E$ is $\sigma$–invariant and contains no pre-periodic points.*

(2) *For every $(x,\xi),(y,\eta)$ such that $x, y \in E$:*

(2.1) *If $(y,\eta) \in W^{ss}(x,\xi)$ then $H(y,\eta) = H(x,\xi)$.*

(2.2) *If $(y,\eta) \in W^{su}(x,\xi)$ then $H(y,\eta) = H(x,\xi)$.*

(3) *For every $x \in E$, $\nu_x^s(E^c) = \nu_x^u(E^c) = 0$.*

*Proof of Claim* 1. Let $E_0 := \{x \in \Sigma : x \text{ is not pre-periodic}\}$. $E_0$ has full $\nu$–measure, since $\nu$ is ergodic and globally supported. By Corollary 4.5, there is $X \subset \Sigma_r$ of full $\mu$–measure such that (2) holds for all $(x,\xi),(y,\eta) \in X$. Since $\mu$ is equivalent to $\nu \times d\xi$,

$$E_1 := \{x \in E_0 : (x,\xi) \in X \text{ for Lebesgue a.e. } \xi \in [0, r(x))\}$$

has full $\nu$–measure. We claim that $E_1$ satisfies (2).

We prove (2.1) and leave (2.2) to the reader. Since $x, y \in E_1$, there is an open neighborhood $U \subset \mathbb{R}$ of 0 such that $(x,\xi+t),(y,\eta+t) \in X$ for Lebesgue a.e. $t \in U$. Find $t_k \xrightarrow[k\to\infty]{} 0$ such that $(x,\xi+t_k),(y,\eta+t_k) \in X$. By Lemma 2.3(1), $(y,\eta+t_k) \in W^{ss}(x,\xi+t_k)$, therefore by the definition of $X$ we have $H(x,\xi+t_k) = H(y,\eta+t_k)$. Passing to the limit, and using that $t \mapsto (H \circ \sigma_r^t)(x,\xi)$ and $t \mapsto (H \circ \sigma_r^t)(y,\eta)$ are continuous, we conclude that $H(x,\xi) = H(y,\eta)$.

Now consider $E_2 := \bigcap_{n\in\mathbb{Z}} \sigma^n(E_1)$. The set $E_2$ has full $\nu$–measure and satisfies (1)–(2) but not necessarily (3). We define $E_3, E_4, \ldots$ by induction as

$$E_n := \{x \in E_{n-1} : \nu^s_{\sigma^k(x)}(E^c_{n-1}) = \nu^u_{\sigma^k(x)}(E^c_{n-1}) = 0, \forall k \in \mathbb{Z}\}.$$

$\{E_n\}$ is a decreasing sequence of $\sigma$–invariant sets of full $\nu$–measure each, by Corollary 3.6, thus $E := \bigcap_{n=4}^{\infty} E_n$ is $\sigma$–invariant set of full $\nu$–measure. The set $E$ satisfies (1)–(2) of the claim, since $E \subset E_0 \cap E_1$. To see that it also satisfies (3), just note that if $x \in E$ and $\tau = s, u$ then $\nu_x^\tau(E^c) = \nu_x^\tau(\bigcup_{n\geq 3} E_n^c) = \lim \nu_x^\tau(E_n^c) = 0$.

Construction of the holonomy group: Recall the *weak stable* and *weak unstable manifolds* of $x \in \Sigma$:

- $W^{ws}(x) := \{y : \exists m, n \text{ such that } x_m^\infty = y_n^\infty\}$.
- $W^{wu}(x) := \{y : \exists m, n \text{ such that } x_{-\infty}^m = y_{-\infty}^n\}$.

The following constructions are motivated by [10]:

- *su–path*: A finite sequence of points $\gamma = \langle x^0, \dots, x^n \rangle$ in $E$ such that $x^i \in W^{w\tau_i}(x^{i-1})$ for some $\tau_i \in \{s, u\}$. If $x^0 = x^n = x$, then $\gamma$ is called an *su–loop at* $x$.
- *Lift of su–path:* Suppose $0 \le \theta < r(x^0)$. The *lift* of $\gamma = \langle x^0, \dots, x^n \rangle$ at $z_0 := (x^0, \theta)$ is $\langle z_0, \dots, z_n \rangle \subset \Sigma_r$ where $z_i = \sigma_r^{\theta + t_i}(x^i, 0)$, and $z_i \in W^{s\tau_i}(z_{i-1})$, $i = 1, \dots, n$. The parameters $t_i$ are uniquely determined by the Bowen-Marcus condition, see Lemma 2.3(1): $t_0 := 0$, $t_i = t_{i-1} + P^{\tau_i}(x^{i-1}, x^i)$.
- *Weight of su–loop:* $P(\gamma) := t_n = \sum_{i=1}^n P^{\tau_i}(x^{i-1}, x^i)$.

For $x \in E$, let $G'_x := \{P(\gamma) : \gamma \text{ is an } su\text{–loop at } x\}$. We will show that there is a closed subgroup $G \subset \mathbb{R}$ such that $G_x := \overline{G'_x} = G$, $\forall x \in E$.

Holonomy group: It is the closed subgroup $G \subset \mathbb{R}$ such that $G_x = G$ for some (all) $x \in E$.

We first show that $G_x = c\mathbb{Z}$ for some $c \neq 0$ independent of $x \in E$, and then use this to prove that $\exp[\frac{2\pi i}{c} r]$ is a multiplicative coboundary.

**Claim 2.** *There exists* $c \neq 0$ *such that* $G_x = c\mathbb{Z}$, $\forall x \in E$.

*Proof of Claim* 2. We divide the proof into few steps. Fix $x \in E$.

Step 1. $G'_x, G_x$ *are additive subgroups of* $\mathbb{R}$, *and* $G'_{\sigma(x)} = G'_x$, $G_{\sigma(x)} = G_x$.

*Proof.* It is enough to prove the claims for $G'_x$. $G'_x$ is an additive group:

- $G'_x + G'_x \subset G'_x$, because $P(\gamma_1) + P(\gamma_2) = P(\gamma_1 \vee \gamma_2)$ where $\gamma_1 \vee \gamma_2$ is the concatenation of $\gamma_1$ and $\gamma_2$.
- $G'_x \ni 0$, because $P(\langle x, x \rangle) = 0$.
- $G'_x = -G'_x$, because $P(\langle x^n, \dots, x^0 \rangle) = -P(\langle x^0, \dots, x^n \rangle)$.

Now we show that $G'_{\sigma(x)} = G'_x$. Let $\gamma = \langle x^0, \dots, x^n \rangle$ be an $su$–loop at $x$, and let $\sigma(\gamma) := \langle \sigma(x^0), \dots, \sigma(x^n) \rangle$. By Lemma 2.3(2),

$$P^{\tau_i}(\sigma(x^{i-1}), \sigma(x^i)) - P^{\tau_i}(x^{i-1}, x^i) = r(x^{i-1}) - r(x^i).$$

Summing this over $i$ gives $P(\sigma(\gamma)) - P(\gamma) = r(x^n) - r(x^0) = 0$.

Step 2. *There is a closed subgroup* $G \subset \mathbb{R}$ *such that* $G_x = G$, $\forall x \in E$.

*Proof.* We claim that $x \mapsto G_x$ is constant on $E \cap [v]$, for every state $v$. Take $x, y \in E \cap [v]$, and define $\pi_{xy} : W^s_{\mathrm{loc}}(x) \to W^s_{\mathrm{loc}}(y)$ by $\pi_{xy}(\cdot) = [\cdot, y]$. $\pi_{xy}$ is measure-preserving:

$$\nu^s_x \circ \pi^{-1}_{xy} = \nu \circ (p^s_x)^{-1} \circ \pi^{-1}_{xy} = \nu \circ (\pi_{xy} \circ p^s_x)^{-1} = \nu \circ (p^s_y)^{-1} = \nu^s_y.$$

$E$ has full $\nu^s_x$–measure in $W^s_{\mathrm{loc}}(x)$. Since $\pi_{xy}$ is measure-preserving, $\pi_{xy}[E \cap W^s_{\mathrm{loc}}(x)]$ has full $\nu^s_y$–measure in $W^s_{\mathrm{loc}}(y)$. Thus $\pi_{xy}[E \cap W^s_{\mathrm{loc}}(x)] \cap E \neq \varnothing$, therefore $\exists z \in E \cap W^s_{\mathrm{loc}}(x)$ such that $w := [z, y] \in E \cap W^s_{\mathrm{loc}}(y)$. By the definition of the Smale product, $W^u_{\mathrm{loc}}(z) = W^u_{\mathrm{loc}}(w)$. In summary, we found $z \in W^s_{\mathrm{loc}}(x) \cap E$, $w \in W^s_{\mathrm{loc}}(y) \cap E$ such that $W^u_{\mathrm{loc}}(z) = W^u_{\mathrm{loc}}(w)$.

Every element of $G'_x$ equals $P(\gamma)$ for some $su$–loop $\gamma$ at $x$. Consider the concatenation $\gamma' := \langle y, w, z, x\rangle \vee \gamma \vee \langle x, z, w, y\rangle$. This is an $su$–loop at $y$ with $P(\gamma') = P(\langle y, w, z, x, z, w, y\rangle) + P(\gamma) = P(\gamma)$. Since $\gamma$ is arbitrary, this gives the inclusion $G_x \subset G_y$. By symmetry, $G_x = G_y$.

We see that for every $v$, there is a group $G_v$ such that $G_x = G_v$, $\forall x \in E \cap [v]$. Fix some state $v_0$. Since $\sigma : \Sigma \to \Sigma$ is topologically transitive, for any state $v$ there is an admissible path $v_0 = a_0 \to \cdots \to a_n = v$. The measure $\nu$ is globally supported, thus we can take $z \in E \cap [\underline{a}]$. By Step 1, $G_{v_0} = G_z = G_{\sigma(z)} = \cdots = G_{\sigma^n(z)} = G_v$, whence $G_v = G_{v_0}$ for all vertices $v$. This proves Step 2.

Step 3. *$G$ equals $c\mathbb{Z}$ for some $c \in \mathbb{R}$.*

*Proof.* $G$ is a closed additive subgroup of $\mathbb{R}$, so either $G = \mathbb{R}$ or $G = c\mathbb{Z}$ for some $c \in \mathbb{R}$. We will show that if $G = \mathbb{R}$ then $H$ is constant $\mu$–a.e., a contradiction.

We implement the classical Hopf argument. The key observation is that $H$ is constant on the intersection of the strong (un)stable manifolds of $\sigma_r$ with $E$, thanks to Claim 1(2). Suppose $\gamma = \langle x^0, \ldots, x^n\rangle$ is an $su$–path, fix some $0 \le \theta < r(x^0)$, and let $\langle z_0, \ldots, z_n\rangle \subset \Sigma_r$ be the lift of $\gamma$ at $z_0 := (x^0, \theta)$. Since $x^i \in E$, we have $H(z_0) = H(z_1) = \cdots = H(z_n)$. In particular, if $x \in E$ and $\gamma$ is an $su$–loop at $x$, then $H(x, \theta) = (H \circ \sigma_r^{P(\gamma)})(x, \theta)$.

If $G = \mathbb{R}$ then the set of weights $P(\gamma)$ is dense in $\mathbb{R}$. Since $t \mapsto (H \circ \sigma^t_r)(x, \theta)$ is continuous, $H(x, \theta) = (H \circ \sigma^t_r)(x, \theta)$ for all $t \in \mathbb{R}$. This proves that $H \circ \sigma^t_r = H$ on $\{(x, \theta) \in \Sigma_r : x \in E\}$. Using that $\mu$ is ergodic (Corollary 3.2), we conclude that $H$ is constant $\mu$–a.e., a contradiction. Thus $G = c\mathbb{Z}$ for some $c \in \mathbb{R}$.

Step 4. *$c \neq 0$.*

*Proof.* Suppose by contradiction that $G = \{0\}$. We will show that $r = U \circ \sigma - U$ for some $U : \Sigma \to \mathbb{R}$ continuous, and derive a contradiction. Recall the definitions of $W^{ws}(x), W^s_{\mathrm{loc}}(x)$ on page 71. Fix $x \in E$ and define $\widetilde{U}$ on $W^{ws}(x) \cap E$ by $\widetilde{U}(y) = P^s(y, x)$. By Lemma 2.3(3),

$$\widetilde{U}(\sigma(y)) - \widetilde{U}(y) = P^s(\sigma(y), x) + P^s(x, y) = P^s(\sigma(y), y) = r(y).$$

Our plan is to show that $W^{ws}(x) \cap E$ is dense in $\Sigma$, and $\widetilde{U}$ is uniformly continuous on $W^{ws}(x) \cap E$. Thus the unique continuous extension to $\Sigma$ satisfies $U \circ \sigma - U = r$.

*Proof that $W^{ws}(x) \cap E$ is dense in $\Sigma$.* Let $C := {}_{-n}[v_{-n}, \ldots, v_n]$ be a non-empty cylinder in $\Sigma$. Since $\sigma : \Sigma \to \Sigma$ is topologically transitive, there is an admissible path $v_n \to v_{n+1} \to \cdots \to v_{n+k} \to x_0$. Now proceed as follows:

- Pick some $w \in C$, and define $y$ by $y^n_{-\infty} = w^n_{-\infty}$, $y^{n+k}_{n+1} = (v_{n+1}, \ldots, v_{n+k})$, $y^{\infty}_{n+k+1} = x^{\infty}_0$. Then $y \in W^{ws}(x) \cap C$, and there are integers $\ell, m > n$ such that $\sigma^m(y) \in W^s_{\mathrm{loc}}(\sigma^\ell(x)) \cap \sigma^m(C)$, whence $\sigma^m(C) \cap [x_\ell] \neq \varnothing$.
- Necessarily $\nu^s_{\sigma^\ell(x)}(\sigma^m C) = \nu[(p^s_{\sigma^\ell(x)})^{-1}(\sigma^m C)] = \nu(\sigma^m(C) \cap [x_\ell])$. Since $\nu$ is globally supported, $\nu^s_{\sigma^\ell(x)}(\sigma^m C) > 0$.
- Since $E$ is $\sigma$–invariant and $x \in E$, $\sigma^\ell(x) \in E$ and $\nu^s_{\sigma^\ell(x)}(\sigma^m(C) \cap E) \neq 0$.
- $\nu^s_{\sigma^\ell(x)}$ is supported on $W^s_{\mathrm{loc}}(\sigma^\ell(x))$, thus $W^s_{\mathrm{loc}}(\sigma^\ell(x)) \cap \sigma^m(C) \cap E \neq \varnothing$.
- Therefore $W^{ws}(x) \cap E \cap C \supseteq \sigma^{-m}[W^s_{\mathrm{loc}}(\sigma^\ell(x)) \cap \sigma^m(C) \cap E] \neq \varnothing$.

We see that $W^{ws}(x) \cap E$ intersects every non-empty cylinder $C$ in $\Sigma$.

*Proof that $\widetilde{U}$ is uniformly continuous on $W^{ws}(x) \cap E$.* Fix $y, z \in W^{ws}(x) \cap E$ such that $y \neq z$ and $y_0 = z_0$. We construct $y^1 \in W^s_{\mathrm{loc}}(y) \cap E$ such that

(i) $z^1 := [y^1, z] \in W^{ws}(x) \cap E$,

(ii) $d(z, z^1) \leq d(y, z)$ and $d(z^1, y^1) \leq d(z, y)$,

(iii) $d(y, y^1) \leq 3d(y, z)$.

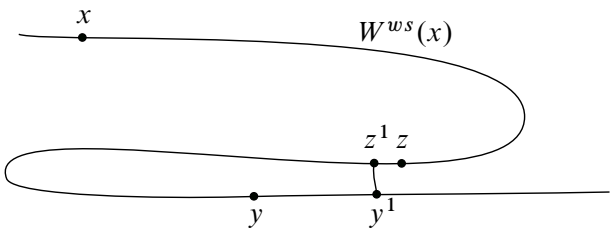


Figure 1.

Here is how to do this. First, find $z^1 \in W^s_{\mathrm{loc}}(z) \cap E$ arbitrarily close to $z$ such that $y^1 := [z^1, y] \in W^s_{\mathrm{loc}}(y) \cap E$. Such points exist because $\nu^s_z(E^c) = 0$, $\nu^s_y(E^c) = 0$, $\nu^s_z$ has full support in $W^s_{\mathrm{loc}}(z)$, and $\nu^s_z = \nu^s_y \circ \pi_{zy}$ for $\pi_{zy}(\cdot) = [\cdot, y]$. Automatically $z^1 = [y^1, z]$, and if $z^1$ is close enough to $z$, then $d(z^1, z) < d(z, y)$ and $d(z^1, y) = d(z, y)$ (the first place where $z^1, y$ disagree is the first place where $z, y$ disagree). Since $y^1 = [z^1, y]$, $d(z^1, y^1) \leq d(z^1, y) = d(z, y)$, proving (ii). Part (iii) follows from (ii) and the triangle inequality.

Let $\gamma = \langle y, z^1, y^1, y\rangle$. Using that $y \in E$ and $G = \{0\}$, we have

$$P^s(y, z^1) + P^u(z^1, y^1) + P^s(y^1, y) = 0. \tag{4.1}$$

By Lemma 2.3(3), $|\widetilde{U}(y) - \widetilde{U}(z^1)| = |P^s(y, z^1)| \le |P^u(z^1, y^1)| + |P^s(y^1, y)|$. Since $y^1 \in W^u_{\text{loc}}(z^1)$, $|P^u(z^1, y^1)| \le Cd(y, z)^\alpha$, where $C, \alpha$ are given by Lemma 2.3(4). Similarly, $|P^s(y^1, y)| \le 3^\alpha Cd(y, z)^\alpha$. Thus $|\widetilde{U}(y) - \widetilde{U}(z^1)| \le 4Cd(y, z)^\alpha$. Also by the cocycle equation, $|\widetilde{U}(z) - \widetilde{U}(z^1)| = |P^s(z, z^1)| \le Cd(y, z)^\alpha$. It follows that $|\widetilde{U}(y) - \widetilde{U}(z)| < 5Cd(y, z)^\alpha$, proving that $\widetilde{U}$ is uniformly continuous on $W^{ws}(x) \cap E$.

Therefore $\widetilde{U}$ extends continuously to a function $U : \Sigma \to \mathbb{R}$. Since $r = \widetilde{U} \circ \sigma - \widetilde{U}$ on $W^{ws}(x) \cap E$, $r = U \circ \sigma - U$ on $\Sigma$. This cannot happen as it implies, by the Poincaré recurrence theorem, that $\liminf r_n = \liminf[U \circ \sigma^n - U] < \infty$ a.e., whereas we know that $\inf r > 0$, so $\liminf r_n = \infty$. Thus $G \neq \{0\}$.

**Claim 3.** *There exists $h : \Sigma \to S^1$ Hölder continuous such that* $\exp[\frac{2\pi i}{c} r] = h / h \circ \sigma$.

Let $\theta := \frac{2\pi}{c}$, fix $x \in E$ and let $\widetilde{h} : W^{ws}(x) \cap E \to S^1$ by $\widetilde{h}(y) := \exp[-i\theta P^s(y, x)]$. By Lemma 2.3(3), $\widetilde{h}/\widetilde{h} \circ \sigma = \exp[i\theta r]$ on $W^{ws}(x) \cap E$. The idea is to show that $\widetilde{h}$ is Hölder continuous on $W^{ws}(x) \cap E$ and then deduce as in the previous proof that it extends Hölder continuously to a function $h : \Sigma \to S^1$. The proof is the same as in the last step of Claim 2, except that one needs to replace (4.1) by

$$\exp[i\theta(P^s(y, z^1) + P^u(z^1, y^1) + P^s(y^1, y))] = 1.$$

As before, this implies that

$$\frac{\widetilde{h}(y)}{\widetilde{h}(z^1)} = e^{i\varepsilon_1} \quad \text{with} \quad |\varepsilon_1| \le 4C|\theta| d(y, z)^\alpha,$$

and

$$\frac{\widetilde{h}(z^1)}{\widetilde{h}(z)} = e^{i\varepsilon_2} \quad \text{with} \quad |\varepsilon_2| \le C|\theta| d(y, z)^\alpha.$$

So

$$\frac{\widetilde{h}(y)}{\widetilde{h}(z)} = e^{i\varepsilon} \quad \text{with} \quad |\varepsilon| \le 5C|\theta| d(y, z)^\alpha,$$

whence the Hölder continuity of $\widetilde{h} : W^{ws}(x) \cap E \to S^1$.

Claim 3 completes the proof that if the Pinsker $\sigma$–algebra of $\sigma_r$ is not trivial then $r$ is arithmetic. Equivalently, (1) $\Rightarrow$ (4) in the statement of Theorem 4.6, and this completes the proof of the theorem. □

**The Pinsker factor in the arithmetic case.** In the last section we saw that if the roof function is arithmetic, then the Pinsker factor of every equilibrium measure of a bounded Hölder continuous potential with finite pressure is non-trivial. In this section we show that in this case the Pinsker factor is isomorphic to a rotational flow. In fact we will show more, that the flow is isomorphic to the direct product of a Bernoulli flow and a rotational flow.

**Theorem 4.7.** *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF such that $e^{i\theta r} = \frac{h}{h\circ\sigma}$ for some $\theta \neq 0$ and $h : \Sigma \to \mathbb{R}$ continuous. There exists $p \in \mathbb{N}$ such that for every equilibrium measure $\mu$ of a bounded Hölder continuous potential with finite pressure, the following hold:*

(1) *$(\Sigma_r, \sigma_r, \mu)$ is isomorphic to a topologically transitive TMF with constant roof function equal to $2\pi/\theta$.*

(2) *$(\Sigma_r, \sigma_r, \mu)$ is isomorphic to the product of a Bernoulli flow and a rotational flow with period $2\pi p/\theta$.*

(3) *The Pinsker factor of $(\Sigma_r, \sigma_r, \mu)$ is isomorphic to a rotation with period $2\pi p/\theta$.*

Before the proof of the theorem, let us prove that constant suspensions over Bernoulli automorphisms are the same as the product of a Bernoulli flow and a rotational flow.

**Lemma 4.8.** *Let $\mathsf{T} = (X, \mu, \{T^t\})$ be a measurable flow. The following are equivalent:*

(1) *$\mathsf{T}$ is isomorphic to a constant suspension over a Bernoulli automorphism.*

(2) *$\mathsf{T}$ is isomorphic to the product of a Bernoulli flow and a rotational flow.*

*Proof.* (1) $\Rightarrow$ (2). Assume that the roof function is $\equiv 1$. Then we can write $\mathsf{T} = (\Sigma_1, \mu, \{T^t\})$, $T^t(x,s) = (S^{\lfloor t+s\rfloor}(x), t+s-\lfloor t+s\rfloor)$, where:

- $(\Sigma, \nu, S)$ is a Bernoulli automorphism.
- $\Sigma_1$ is the suspension space over $\Sigma$ with roof function $\equiv 1$.
- $\mu = \int_\Sigma \int_0^1 \delta_t\, dt\, d\nu(x)$.

By Ornstein Theory, $(\Sigma, \nu, S)$ embeds into a Bernoulli flow $(\Sigma, \nu, \{S^t\})$, see [33]. Let $\{R^t\}$ be the rotational flow with period 1. We claim that $\mathsf{T}$ is isomorphic to $(\Sigma\times\mathbb{T}, \nu\times dt, \{S^t\times R^t\})$, the product of a Bernoulli flow and a rotational flow. The conjugacy is the bijection $\rho : \Sigma_1 \to \Sigma\times\mathbb{T}$, $\rho(x,s) = (S^s(x), s \pmod 1)$. First note that $\rho$ is well-defined since $\rho(x,1) = (S^1(x),0) = (Sx,0) = \rho(Sx,0)$. Also:

$$\begin{aligned}(\rho\circ T^t)(x,s) &= \rho(S^{\lfloor t+s\rfloor}(x), t+s-\lfloor t+s\rfloor) = (S^{t+s}(x), t+s \pmod 1)\\ &= (S^t\times R^t)(S^s(x), s \pmod 1) = [(S^t\times R^t)\circ\rho](x,s).\end{aligned}$$

For all measurable $A \subset \Sigma$ and interval $I \subset \mathbb{T}$ not containing zero, $(\mu \circ \rho^{-1})(A \times I) = \mu(A \times I) = \nu(A) \cdot |I|$, hence $\mu \circ \rho^{-1} = \nu \times dt$, which completes the proof that $\rho$ is a conjugacy between $\mathsf{T}$ and $\{S^t \times R^t\}$.

(2) $\Rightarrow$ (1). With the same notation as above, assume that $\mathsf{T} = (\Sigma \times \mathbb{T}, \nu \times dt, \{S^t \times R^t\})$. Then $\mathsf{T}$ is isomorphic to the suspension flow $(\Sigma_1, \mu, \{T^t\})$, where the basis dynamics is the Bernoulli automorphism $(\Sigma, \nu, S^1)$. The conjugacy is the same $\rho$ as above, and the proof is analogous to (1) $\Rightarrow$ (2). □

*Proof of Theorem 4.7.* Part (1) is the content of [28, Theorem 7.2]. Denote this TMF by $\sigma_{\widetilde{r}} : \widetilde{\Sigma}_{\widetilde{r}} \to \widetilde{\Sigma}_{\widetilde{r}}$, with $\widetilde{r} \equiv 2\pi/\theta$.

Let $p$ denote the period of $\widetilde{\Sigma}$. Recall from page 69 that, using the spectral decomposition of $\widetilde{\Sigma}$ [23], $\sigma_{\widetilde{r}} : \widetilde{\Sigma}_{\widetilde{r}} \to \widetilde{\Sigma}_{\widetilde{r}}$ is topologically conjugate to a TMF $\sigma_{\widehat{r}} : \widehat{\Sigma}_{\widehat{r}} \to \widehat{\Sigma}_{\widehat{r}}$ where $\sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ is topologically mixing, and $\widehat{r} = \widetilde{r}_p = 2\pi p/\theta =: \alpha$.

Let $\widehat{\mu}$ be the measure on $\widehat{\Sigma}_{\widehat{r}}$ corresponding to $\mu$, and let $\widehat{\nu}$ be the induced measure of $\widehat{\mu}$. $\widehat{\nu}$ is an equilibrium measure of a bounded Hölder continuous potential on $\widehat{\Sigma}$ with finite pressure. Since $\sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ is topologically mixing and $\widehat{\Sigma}$ is not a singleton, $\sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ is Bernoulli [5, 50]. By Lemma 4.8, $\sigma_{\widehat{r}} : \widehat{\Sigma}_{\widehat{r}} \to \widehat{\Sigma}_{\widehat{r}}$ is isomorphic to the product of a Bernoulli flow and a rotational flow with period $\alpha$.

Since the Pinsker factor of a direct product is isomorphic to the direct product of the Pinsker factors [52, Prop. 4.4], and since Bernoulli flows have trivial Pinsker factor, it follows that the Pinsker factor of $(\Sigma_r, \sigma_r, \mu)$ is isomorphic to $\mathscr{P}(R^t) = \mathscr{P}(R^1) = R^1$, a rotation with period $2\pi p/\theta$. □

## 5. The Bernoulli property

We have proved so far that if $\sigma_r : \Sigma_r \to \Sigma_r$ is a topologically transitive TMF and $\mu$ is an equilibrium measure of a bounded Hölder continuous potential with finite pressure, then $(\Sigma_r, \sigma_r, \mu)$ is isomorphic to a Bernoulli flow times a rotational flow when $r$ is arithmetic, and $(\Sigma_r, \sigma_r, \mu)$ is a K flow when $r$ is not arithmetic. The purpose of this section is to complete the picture and prove the following result.

**Theorem 5.1.** *Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF. If $r$ is not arithmetic, then for every equilibrium measure $\mu$ of a bounded Hölder continuous function with finite pressure $(\Sigma_r, \sigma_r, \mu)$ is a Bernoulli flow.*

The theorem above strengthens Theorem 4.6 by saying that for equilibrium measures of bounded Hölder potentials with finite pressure, weak mixing is equivalent to the Bernoulli property.

**Review of general theory.** Let $(X, \mathscr{B}, \mu)$ be a non-atomic Lebesgue probability space, and let $\alpha = \langle A_1, \ldots, A_N \rangle$ and $\beta = \langle B_1, \ldots, B_N \rangle$ be ordered partitions of $(X, \mathscr{B}, \mu)$. Given $x \in X$, define $\alpha(x) := i$ if $x \in A_i$.

PARTITION DISTANCE: $d(\alpha, \beta) := \sum_{i=1}^{N} \mu(A_i \triangle B_i) = 2 \int 1_{[\alpha(x) \neq \beta(x)]} d\mu(x)$.

Let $\{\alpha_i\}_1^n$ be a finite sequence of ordered partitions of $(X, \mathscr{B}, \mu)$, and let $\{\beta_i\}_1^n$ be a finite sequence of ordered partitions of another non-atomic Lebesgue probability space $(Y, \mathscr{C}, \nu)$. Suppose that each partition has $N$ elements, say $\alpha_i = \langle A_1^i, \ldots, A_N^i \rangle$ and $\beta_i = \langle B_1^i, \ldots, B_N^i \rangle$.

SAME DISTRIBUTION: We say that $\{\alpha_i\}_1^n, \{\beta_i\}_1^n$ *have the same distribution*, and write $\{\alpha_i\}_1^n \sim \{\beta_i\}_1^n$, if

$$\mu[A_{i_1}^1 \cap \cdots \cap A_{i_n}^n] = \nu[B_{i_1}^1 \cap \cdots \cap B_{i_n}^n], \quad \forall (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n.$$

This is equivalent to the existence of a measure preserving map

$$\theta : (X, \mathscr{B}, \mu) \to (Y, \mathscr{C}, \nu)$$

such that

$$\theta[A_{i_1}^1 \cap \cdots \cap A_{i_n}^n] = B_{i_1}^1 \cap \cdots \cap B_{i_n}^n$$

modulo $\nu$, $\forall (i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$. This notion can be weakened in the following way.

$d$–BAR DISTANCE: The *$d$–bar distance* between $\{\alpha_i\}_1^n, \{\beta_i\}_1^n$ is

$$\overline{d}(\{\alpha_i\}_1^n, \{\beta_i\}_1^n) := \inf \left\{ \frac{1}{n} \sum_{i=1}^{n} d(\overline{\alpha}_i, \beta_i) : \begin{array}{l} \{\overline{\alpha}_i\}_1^n \text{ are ordered partitions of} \\ (Y, \mathscr{C}, \nu) \text{ such that } \{\overline{\alpha}_i\}_1^n \sim \{\alpha_i\}_1^n \end{array} \right\}.$$

To understand how the $d$–bar distance weakens the notion of same distribution, we first weaken the notion of measure preserving maps.

$\varepsilon$–MEASURE PRESERVING MAP: An invertible measurable map $\theta : (X, \mathscr{B}, \mu) \to (Y, \mathscr{C}, \nu)$ is called *$\varepsilon$–measure preserving* if $\exists E \in \mathscr{B}$, $\mu(E) < \varepsilon$, such that $\left| \frac{\nu(\theta(A))}{\mu(A)} - 1 \right| \leq \varepsilon$ for all $A \subset X \setminus E$ measurable.

**Lemma 5.2** ([36]). *If $\theta : (X, \mathscr{B}, \mu) \to (Y, \mathscr{C}, \nu)$ is $\varepsilon$–measure preserving such that*

$$\frac{1}{n} \sum_{i=1}^{n} 1_{[\alpha_i(x) \neq \beta_i(\theta(x))]} \leq \varepsilon$$

*on a set of measure $\geq 1 - \varepsilon$, then $\overline{d}(\{\alpha_i\}_1^n, \{\beta_i\}_1^n) \leq 16\varepsilon$.*

In other words, $\{\alpha_i\}_1^n, \{\beta_i\}_1^n$ are close in $d$–bar distance if there exists an $\varepsilon$–measure preserving map $\theta$ that matches $\alpha_i(x)$ and $\beta_i(\theta(x))$ on the average, for most points. That is why the $d$–bar distance weakens the notion of same distribution.

We now explain the property we will use to prove an automorphism is Bernoulli. Let $(X, \mathscr{B}, \mu, T)$ be an automorphism. Given $A \in \mathscr{B}$ with $\mu(A) > 0$, let $(A, \mathscr{B}_A, \mu_A)$ be the induced non-atomic Lebesgue probability space, i.e. $\mathscr{B}_A := \{B \cap A : B \in \mathscr{B}\}$ and $\mu_A(\cdot) = \mu(\cdot|A)$. Every partition $\alpha$ of $(X, \mathscr{B}, \mu)$ defines a *conditional partition* $\alpha|A = \{C \cap A : C \in \alpha\}$ of $(A, \mathscr{B}_A, \mu_A)$. Write *"$\varepsilon$–a.e. $A \in \alpha$"* when refering to a property that holds for a collection of atoms of $\alpha$ whose union has measure $\geq 1 - \varepsilon$.

Very weak Bernoulli property[1]: $\alpha$ is called *very weak Bernoulli (VWB)* if for every $\varepsilon > 0$ there is $N_0 = N_0(\varepsilon)$ such that for all $n \geq 0$ and $N' \geq N \geq N_0$ it holds

$$\overline{d}\left(\{T^{-i}\alpha\}_1^n, \{T^{-i}\alpha|A\}_1^n\right) < \varepsilon \text{ for } \varepsilon\text{–a.e. } A \in \bigvee_{k=N}^{N'} T^k\alpha.$$

$\vee$ denotes the joining of partitions. Taking $A \in \bigvee_{k=N}^{N'} T^k\alpha$ means that we are fixing the far past of $T$.

**Theorem 5.3** ( [30, 33, 36]). *Let $\mathsf{T} = (X, \mathscr{B}, \mu, \{T^t\})$ be a probability preserving measurable flow. If for some $t$, $(X, \mathscr{B}, \mu, T^t)$ has an increasing sequence of VWB partitions which generates $\mathscr{B}$, then $\mathsf{T}$ is a Bernoulli flow.*

**Construction of VWB partitions for equilibrium measures [36, 43].** Let $\sigma_r : \Sigma_r \to \Sigma_r$ be a topologically transitive TMF. Throughout this section we assume that $r$ is *not* arithmetic, and independent of the *past* (which we can assume because of Lemma 2.2). Fix an equilibrium measure $\mu$ of a bounded Hölder continuous potential with finite pressure, and let $\nu$ be the induced measure of $\mu$, i.e.

$$\mu = \frac{1}{\int_\Sigma r d\nu} \int_\Sigma \int_0^{r(x)} \delta_{(x,t)} dt\, d\nu(x).$$

Let $\pi_1, \pi_2 : \Sigma_r \to \Sigma$ be the projections on the first and second coordinates, respectively. We now define three $\sigma$–algebras:

- $\alpha$ = partition of $\Sigma$ into cylinders of length one at the zeroth position. $\bigvee_{i=0}^{\infty} \sigma^{-i}\alpha$ is the $\sigma$–algebra with information on the coordinates $x_0^\infty$ of $x \in \Sigma$.
- $\mathscr{F}_{-n} := \pi_1^{-1}(\bigvee_{i=-n}^{\infty} \sigma^{-i}\alpha)$, the $\sigma$–algebra with information on $x_{-n}^\infty$ of $(x, t) \in \Sigma_r$.
- $\mathscr{H} := \pi_2^{-1}[\mathscr{B}(\mathbb{R})]$, where $\mathscr{B}(\mathbb{R})$ is the Borel $\sigma$–algebra of $\mathbb{R}$. $\mathscr{H}$ is the $\sigma$–algebra with information on $t$ of $(x, t) \in \Sigma_r$.

[1] This is the formulation in [36] and it implies the definition in [35]. The two definitions are equivalent for Bernoulli automorphisms, since in this case every partition is VWB.

We will abuse notation and write $\mathbb{E}_\mu(\cdot|x_{-n}^\infty, t)$ instead of $\mathbb{E}_\mu(\cdot|\mathscr{F}_{-n} \vee \mathscr{H})(x,t)$ and $\mu(E|x_{-n}^\infty, t)$ instead of $\mathbb{E}_\mu(1_E|\mathscr{F}_{-n} \vee \mathscr{H})(x,t)$. Since $r$ is independent of past coordinates, it can be easily checked that for all $n \geq 0$:

$$\mu(\cdot|x_{-n}^\infty, t) = 1_{[r(x_0^\infty)>t]}(x,t) \cdot [\nu(\cdot|x_{-n}^\infty) \times \delta_t] \quad \text{for } \mu\text{–a.e. } (x,t). \tag{5.1}$$

Actually, there is a way to make sense of the right-hand-side for *every* $(x,t)$: use (3.1) to define $\nu(\cdot|x_0^\infty)$ for all $x$, and the identity $\nu \circ \sigma^{-n} = \nu$ to extend to other $n$:

$$\nu(E|x_{-n}^\infty) := \nu(\sigma^{-n}(E)|\sigma^{-n}(x)_0^\infty). \tag{5.2}$$

Given an admissible word $\underline{a}$, let $\rho(\underline{a}) := \inf\{r(x) : x_{-n}^n = \underline{a}\}$. Let $0 < \delta < 1$, $n \geq 0$. Consider the following definitions.

$(n,\delta)$–CUBE: A set $C = \{(x,t) : x_{-n}^n = \underline{a}, t \in [\tau, \tau+\delta)\}$, where $\underline{a}$ is an admissible word of length $2n+1$ and $\tau \geq 0$ such that $[\tau, \tau+\delta) \subset [0, \rho(\underline{a}))$.

CANONICAL PARTITION INTO $(n,\delta)$–CUBES: A finite or countable partition whose atoms are $(n,\delta)$–cubes, with the exception of an atom of the form $\{(x,t) : \rho(x_{-n}^n) \leq t < r(x)\}$ with measure $\leq \delta$.

PSEUDO-CANONICAL PARTITION INTO $(n,\delta)$–CUBES: A finite partition that can be refined to a canonical partition into $(n,\delta)$–cubes.

**Lemma 5.4** ([43]). *If $n_0 \geq 0$ and $0 < t_0 < \inf(r)$, then every pseudo-canonical partition into $(n_0,\delta_0)$–cubes is very weak Bernoulli for $(\Sigma_r, \sigma_r^{t_0}, \mu)$.*

*Proof.* This was proved (with different terminology and notation) in [36] for geodesic flows, and in [43] for TMF built on subshifts of finite type. What follows is a detailed exposition of the argument in [43], with some missing details added, and one (minor) point clarified.

Let $\gamma$ be a pseudo-canonical partition into $(n_0,\delta_0)$–cubes, and take $N' \geq N > \frac{n_0}{t_0}\sup(r)$. Every $A \in \bigvee_{k=N}^{N'} \sigma_r^{t_0 k}\gamma$ is a countable union of sets of the form

$$\{(x,t) : x \in D_i, a_i(x) \leq t < b_i(x)\},$$

where $D_i$ are cylinders in $\bigvee_{j=n_1(i)}^{n_2(i)} \sigma^i\alpha$ with

$$\left\lfloor \frac{t_0 N}{\sup(r)} \right\rfloor - n_0 - 1 \leq n_1(i) \leq n_2(i) \leq \left\lceil \frac{t_0 N'}{\inf(r)} \right\rceil + n_0 + 1, \tag{5.3}$$

and $a_i, b_i$ are independent of the past coordinates.

Fix $\varepsilon > 0$, and let $n \geq 0, \delta \in (0,1)$ to be determined later. Partition $\Sigma_r$ into finitely many $(n,\delta)$–cubes $\mathcal{C}_{n,\delta} := \{C_1, \ldots, C_m\}$ plus an additional "error set" with measure $\leq \delta$.

STEP 1. *$\exists N_0 = N_0(n,\delta) > 0$ such that for all $C \in \mathcal{C}_{n,\delta}$, for all $N' \geq N \geq N_0$, and for $\delta$–a.e. $A \in \bigvee_{n=N}^{N'} \sigma_r^{t_0 n}\gamma$, it holds $\left|\frac{\mu(A\cap C)}{\mu(A)\mu(C)} - 1\right| < \delta$.*

*Proof.* Since $r$ is not arithmetic, $(\Sigma_r, \sigma_r^{t_0}, \mu)$ is a K automorphism (Theorem 4.6). Now use Theorem 4.2 and the finiteness of $\mathcal{C}_{n,\delta}$.

STEP 2. *For all $A, C$ as in step 1, there is $(z,s) \in A \cap C$ such that $\mu(A \cap C\,|\,z_{-n}^{\infty}, s) > 0$ and $\mu(\cdot|z_{-n}^{\infty}, s)$ is non-atomic. We choose one such pair for each $A, C$ and write $(z,s) := (z(A,C), s(A,C))$.*

*Proof.* By Lemma 3.4 and (5.2), $\nu(\cdot|z_{-n}^{\infty})$ is non-atomic for $\nu$–a.e. $z$, so $\mu(\cdot|z_{-n}^{\infty}, s) = 1_{[r(x_0^{\infty})>s]}[\nu(\cdot|z_{-n}^{\infty}) \times \delta_s]$ is non-atomic for $\mu$–a.e. $(z,s) \in A \cap C$. Also $\left|\frac{\mu(A\cap C)}{\mu(A)\mu(C)} - 1\right| < \delta < 1 \Rightarrow \mu(A \cap C) > 0 \Rightarrow \mu(A \cap C\,|\,z_{-n}^{\infty}, s) > 0$ for a subset $(z,s) \in A \cap C$ of positive $\mu$–measure. Therefore there is $(z,s) \in A \cap C$ satisfying step 2.

Given $\delta > 0$, let us write $a = e^{\pm\delta}$ whenever $e^{-\delta} \le a \le e^{\delta}$.

STEP 3. *Given $\delta > 0$, the following holds for all $n$ large enough. If $(x,t), (z,s) \in C \in \mathcal{C}_{n,\delta}$, then the map $\Theta_{x,t}^{z,s} : (C, \mu(\cdot|x_{-n}^{\infty}, t)) \to (C, \mu(\cdot|z_{-n}^{\infty}, s))$, $\Theta_{x,t}^{z,s}(y,t) = (\vartheta(y), s)$, where $\vartheta(y) = (y_{-\infty}^{-n-1}, z_{-n}^{\infty})$, has Radon-Nikodym derivative equal to $e^{\pm\delta}$.*

*Proof.* Write $C = B \times I$, where $B =_{-n} [b_{-n}, \ldots, b_n]$ contains $x, z$ and $I$ is an interval of length $\delta$ containing $t, s$. The Radon-Nikodym derivative of $\Theta_{x,t}^{z,s}$ equals the Radon-Nikodym derivative of $\vartheta : (B, \nu(\cdot|x_{-n}^{\infty})) \to (B, \nu(\cdot|z_{-n}^{\infty}))$. To estimate this latter derivative, let $B' :=_{-(n+m)} [b_{-(n+m)}, \ldots, b_n] \subset B$ be a cylinder, and let $\varepsilon_n := \sum_{k \ge n} \mathrm{var}_k(\log g)$. By (3.1) and (5.2),

$$\frac{\nu(B'|x_{-n}^{\infty})}{\nu(\vartheta(B')|z_{-n}^{\infty})} = \frac{g_m(b_{-(n+m)}^{n}, x_{n+1}^{\infty})}{g_m(b_{-(n+m)}^{n}, z_{n+1}^{\infty})} = e^{\pm\varepsilon_n},$$

thus $\nu(B'|x_{-n}^{\infty}) = e^{\pm\varepsilon_n}\nu(\vartheta(B')|z_{-n}^{\infty})$ for every cylinder $B' \subset B$. Since the cylinders generate the $\sigma$–algebra of Borel sets of $B$, $\nu(E|x_{-n}^{\infty}) = e^{\pm\varepsilon_n}\nu(\vartheta(E)|z_{-n}^{\infty})$ for all Borel sets $E \subset B$, hence the Radon-Nikodym derivative of $\vartheta$ equals $e^{\pm\varepsilon_n}$. Since $\log g$ is Hölder continuous, $\varepsilon_n \xrightarrow[n\to\infty]{} 0$, thus $\varepsilon_n < \delta$ for all $n$ large enough.

STEP 4. *For all $A, C, (z,s)$ as in steps 1–2, there is an invertible bi-measurable map $\Psi : (C, \mu(\cdot|z_{-n}^{\infty}, s)) \to (A \cap C, \mu(\cdot|z_{-n}^{\infty}, s))$ with constant Radon-Nikodym derivative. Call the constant $D(A,C)$.*

*Proof.* Any two non-atomic Lebesgue probability spaces are measure theoretically isomorphic. $(C, \mu(\cdot|z_{-n}^{\infty}, s))$ and $(A \cap C, \mu(\cdot|z_{-n}^{\infty}, s))$ are non-atomic Lebesgue measure spaces, so instead of an isomorphism there is an invertible bi-measurable map $\Psi : (C, \mu(\cdot|z_{-n}^{\infty}, s)) \to (A \cap C, \mu(\cdot|z_{-n}^{\infty}, s))$ with constant Radon-Nikodym derivative equal to $D(A,C) := \frac{\mu(A\cap C|z_{-n}^{\infty}, s)}{\mu(C|z_{-n}^{\infty}, s)}$.

Let $\Omega := \bigcup_{i=1}^{m} C_i$, $\mu(\Omega) > 1 - \delta$.

STEP 5. *If $\delta$ is sufficiently small, $n$ is sufficiently large, and $N_0 = N_0(n, \delta)$ as in Step 1, then for all $N' \geq N \geq N_0$, for $\delta$–a.e. $A \in \bigvee_{k=N}^{N'} \sigma_r^{t_0 k} \gamma$, there is a map $\Xi : (\Sigma_r, \mu) \to (A, \mu(\cdot|A))$ such that:*

(1) $\Xi(x,t) = (y,t)$ *with* $y_{-n}^{\infty} = x_{-n}^{\infty}$ *for* $(x,t) \in \Omega$,

(2) $\Xi$ *is invertible and bi-measurable,*

(3) $\Xi$ *is $5\delta$–measure preserving.*

*Proof.* For each $A, C, (z,s)$ as in steps 1–2, define $\Xi \restriction_C : C \to A \cap C$ by

$$\Xi(x,t) = (\Theta_{z,s}^{x,t} \circ \Psi \circ \Theta_{x,t}^{z,s})(x,t).$$

Now define $\Xi$ on $\Sigma_r \setminus \Omega$ to take values on $A \setminus \Omega$ via a bijective measure preserving map. Thus (1) holds[2].

To prove (2), first note that $C \in \mathscr{F}_{-n} \vee \mathscr{H}$, hence we can write

$$\mu \restriction_C = \text{const} \int_C \mu(\cdot | x_{-n}^{\infty}, t) d\mu(x,t).$$

By Steps 3–4, $\Xi \restriction_C : (C, \mu(\cdot|x_{-n}^{\infty}, t)) \to (A \cap C, \mu(\cdot|x_{-n}^{\infty}, t))$ is an absolutely continuous bijection, thus $\Xi \restriction_C : (C, \mu) \to (A \cap C, \mu)$ is bijective a.e., which gives (2).

Let us now prove (3). By steps 3–4, $\Xi \restriction_C : (C, \mu(\cdot|x_{-n}^{\infty}, t)) \to (A \cap C, \mu(\cdot|x_{-n}^{\infty}, t))$ has Radon-Nikodym derivative $e^{\pm 2\delta} D(A, C)$, thus if $E \subset C$ is measurable then

$$\begin{aligned} \mu(\Xi(E)) &= \text{const} \int_C \mu(\Xi(E)|x_{-n}^{\infty}, t) d\mu(x,t) \\ &= \text{const}\, e^{\pm 2\delta} D(A,C) \int_C \mu(E|x_{-n}^{\infty}, t) d\mu(x,t) \\ &= \text{const}\, e^{\pm 2\delta} D(A,C) \mu(E). \end{aligned}$$

Therefore $\Xi : C \to A \cap C$ is absolutely continuous with Radon-Nikodym derivative equal to $e^{\pm 2\delta} K$ for some constant $K = K(A, C)$. Since $\Xi$ is a bijection a.e., $K = e^{\pm 2\delta} \frac{\mu(A \cap C)}{\mu(C)}$. If $\delta$ is so small that $1 - \delta > e^{-2\delta}$, Step 1 gives that $K = e^{\pm 4\delta} \mu(A)$. Since $C \in \mathcal{C}_{n,\delta}$ is arbitrary, $\Xi \restriction_{\Omega} : (\Omega, \mu) \to (A \cap \Omega, \mu)$ has Radon-Nikodym derivative equal to $e^{\pm 4\delta} \mu(A)$. After normalizing the measure of $A$, we find that the Radon-Nikodym derivative of $\Xi : (\Sigma_r, \mu) \to \big(A, \mu(\cdot|A)\big)$ equals $e^{\pm 4\delta}$ on $\Omega$. If $\delta$ is so small that $e^{4\delta} < 1 + 5\delta$, we conclude that $\Xi$ is $5\delta$–measure preserving.

[2]Our construction of $\Xi$ differs from [43], since it is not clear to us that her construction leads to a measurable map. Instead, we follow the construction used in [36].

STEP 6. *If $\delta$ is sufficiently small and $n$ is sufficiently large, then for all $m \geq 0$, for all $N' \geq N \geq N_0(n,\delta)$, and for $\delta$–a.e. $A \in \bigvee_{i=N}^{N'} \sigma_r^{it_0}\gamma$,*

$$\frac{1}{m}\#\{1 \leq i \leq m : \sigma_r^{it_0}(x,t), \sigma_r^{it_0}(\Xi(x,t)) \text{ are in different } \gamma\text{–atoms}\} < \varepsilon$$

*holds for a set $(x,t) \in \Sigma_r$ of measure $\geq 1-\delta$.*

*Proof.* This follows, as in [36,43], from the fact that $\Xi(x,t) = (y,t)$ with $y_{-n}^{\infty} = x_{-n}^{\infty}$ for $(x,t) \in \Omega$. Let us recall the argument.

Let $\widehat{\gamma}$ denote the (countable) canonical partition into $(n_0,\delta_0)$–cubes which refines $\gamma$, and assume that $n > n_0$. If $\sigma_r^{it_0}(x,t), \sigma_r^{it_0}(y,s)$ belong to different $\gamma$–atoms, then they belong to different $\widehat{\gamma}$–atoms. At least one of these atoms is an $(n_0,\delta_0)$–cube of the form $C := B \times [a, a+\delta_0)$ with $B \in \bigvee_{j=-n_0}^{n_0} \sigma^j \alpha$. Using that $n > n_0$, that $r$ is independent of the past, and that $x_{-n}^{\infty} = y_{-n}^{\infty}$, we get that $\sigma_r^{it_0}(x,t)$ belongs to $\partial_\delta(C) := \bigcup_{|\theta|<\delta} \sigma_r^{\theta}(B \times \{a, a+\delta_0\})$. Let $\partial_\delta(\widehat{\gamma})$ be the union of all $\partial_\delta(C)$, $C$ as above.

Defining $Z_m(x,t) := \frac{1}{m}\#\{1 \leq i \leq m : \sigma_r^{it_0}(x,t), \sigma_r^{it_0}(\Xi(x,t))$ are in different $\gamma$–atoms$\}$ and $Y_m(x,t) := \sum_{i=1}^{m} 1_{\partial_\delta(\widehat{\gamma})}(\sigma_r^{it_0}(x,t))$, the previous paragraph and the Markov inequality imply that

$$\mu[Z_m \geq \varepsilon] \leq \mu[Y_m \geq \varepsilon m] \leq \frac{1}{\varepsilon m}\int Y_m d\mu \leq \varepsilon^{-1}\mu[\partial_\delta(\widehat{\gamma})].$$

If we choose $\delta$ so small that $\mu[\partial_\delta(\widehat{\gamma})] < \varepsilon^2$, then $\mu[Z_m \geq \varepsilon] < \varepsilon$ as required.

*Completion of the proof of Lemma 5.4.* Given $\varepsilon > 0$, let $\delta$ be sufficiently small and $n$ sufficiently large such that steps 1–6 hold, and $100\delta < \varepsilon$. By Lemma 5.2, for all $m \geq 0$, for all $N' \geq N \geq N_0(n,\delta)$, and for $\delta$–a.e. $A \in \bigvee_{k=N}^{N'} \sigma_r^{kt_0}\gamma$ it holds $\overline{d}(\{\sigma_r^{-it_0}\gamma\}_1^m, \{\sigma_r^{-it_0}\gamma|A\}_1^m) < 16 \times 5\delta < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\gamma$ is VWB. □

**Proof of Theorem 5.1.** Fix $t_0 \neq 0$, and construct an increasing sequence of pseudo-canonical partitions into $(n_k,\delta_k)$–cubes, with $n_k \to \infty$ and $\delta_k \to 0$. This sequence of partitions generates the full $\sigma$–algebra of $\Sigma_r$. Since each of these pseudo-canonical partitions is VWB for $\sigma_r^{t_0}$ (Lemma 5.4), it follows from Ornstein Theory [30,33,36] that $(\Sigma_r, \sigma_r, \mu)$ is a Bernoulli flow. □

# Part II. Smooth flows in three dimensions

## 6. Proof of Theorem 1.1

Let $M$ be a three dimensional compact $C^\infty$ Riemannian manifold, let $X : M \to TM$ be a non-vanishing $C^{1+\varepsilon}$ vector field, and let $\{T^t\}$ be the flow on $M$ generated

by $X$. Let $F : M \to \mathbb{R}$ be a bounded Hölder continuous function, and let $\nu$ be an equilibrium measure of $F$. Our task is to show that $\nu$ has at most countably many ergodic components $\nu_i$ with positive entropy, and that $\{T^t\}$ is Bernoulli up to a possible period with respect to each $\nu_i$.

That $\nu$ has at most countably many ergodic components with positive entropy was proved in [28] in the special case $F \equiv 0$. The same proof works for general bounded Hölder continuous $F$ almost verbatim. Let us recap the idea. For a fixed $\chi > 0$, we prove that $F$ has at most countably many $\chi$–hyperbolic[3] ergodic equilibrium measures. This happens because every ergodic equilibrium measure on a TMS is carried by a topologically transitive TMS. If there were uncountably many $\chi$–hyperbolic equilibrium measures for $F$, then some convex combination would generate a $\chi$–hyperbolic equilibrium measure on a TMS with uncountably many ergodic components. Taking the union over $\chi_n = 1/n$ gives countability.

It remains to show that if $\nu$ is ergodic with positive entropy, then $\nu$ is Bernoulli up to a possible period. Given a TMF $\sigma_r : \Sigma_r \to \Sigma_r$, let

$$\Sigma_r^{\#} := \{(x,t) \in \Sigma_r : \{x_i\}_{i>0}, \{x_i\}_{i<0} \text{ have constant subsequences}\}.$$

By the Poincaré recurrence theorem, $\Sigma_r^{\#}$ has full measure for every $\sigma_r$–invariant probability measure.

Apply [28, Theorem 1.2] to the flow $(M, \nu, \{T^t\})$ to get a TMF $\sigma_r : \Sigma_r \to \Sigma_r$ and a Hölder continuous map $\pi_r : \Sigma_r \to M$ such that:

(1) $\pi_r \circ \sigma_r^t = T^t \circ \pi_r$, $\forall t \in \mathbb{R}$.

(2) $\pi_r[\Sigma_r^{\#}]$ has full $\nu$–measure.

(3) $\pi_r : \Sigma_r^{\#} \to M$ is finite-to-one.

Notice that $\Phi := F \circ \pi_r$ is a bounded Hölder continuous function. Arguing as in [28, Theorem 6.2], one can prove that $\Phi$ has an ergodic equilibrium measure $\mu$ such that $\mu \circ \pi_r^{-1} = \nu$. By ergodicity, $\mu$ is carried by a topologically transitive TMF of $\Sigma_r$. By Theorems 4.7 and 5.1, $\mu$ is Bernoulli up to a period. Therefore $(M, \nu, \{T^t\})$ is a finite-to-one factor of a flow which is Bernoulli up to a period, so it is enough to prove the lemma below.

**Lemma 6.1.** *If a measurable flow is Bernoulli up to a period, then so are its finite-to-one factors.*

*Proof.* Suppose $\pi : X \to Y$ is a finite-to-one factor map between $\mathsf{T} = (X, \mu, \{T^t\})$ and $\mathsf{S} = (Y, \eta, \{S^t\})$. Suppose $\mathsf{T}$ is Bernoulli up to a period.

If $\mathsf{T}$ is Bernoulli, then $T^1$ is Bernoulli. Since factors of Bernoulli automorphisms are Bernoulli automorphisms [31], $S^1$ is a Bernoulli automorphism. By [33], $\mathsf{S}$ is a Bernoulli flow.

Assume that $\mathsf{T}$ is isomorphic to a Bernoulli flow times a rotational flow. By Lemma 4.8, it is enough to prove the claim below.

[3] $\nu$ is $\chi$–hyperbolic if $\nu$–a.e. point has one Lyapunov exponent $> \chi$ and another $< -\chi$.

**Claim.** *If* $\mathsf{T}$ *is a constant suspension over a Bernoulli automorphism, then* $\mathsf{S}$ *is a constant suspension over a Bernoulli automorphism.*

*Proof.* Assume without loss of generality that the roof function of $\mathsf{T}$ is $\equiv 1$, i.e. $\mathsf{T} = (\Sigma_1, \mu, \{T^t\})$ where $T^t(x,s) = (\tau^{\lfloor t+s \rfloor}(x), t+s-\lfloor t+s \rfloor)$, $\mu = \int_\Sigma \int_0^1 \delta_{(x,t)} dt\, d\mu_0(x)$, and $(\Sigma, \mu_0, \tau)$ is a Bernoulli automorphism.

Let $Y_0 := \pi(\Sigma \times \{0\})$. We claim that $Y_0$ is a Poincaré section for $\mathsf{S}$. For each $y \in \pi(\Sigma_1)$, let $I_y := \{t > 0 : S^t(y) \in Y_0\}$.

- $I_y \neq \varnothing$: $y = \pi(x,s) \Rightarrow 1 - s \in I_y$.
- $I_y \cap (0,1)$ is finite: if $S^{t_n}(y) = \pi(x_n, 0)$ for infinitely many $t_n, x_n$, then $y$ has infinitely many pre-images $(\tau^{-1}(x_n), 1 - t_n)$.
- $I_y$ is infinite: $y = \pi(x,t) \Rightarrow S^{n-t}(y) = \pi(\tau^n(x), 0) \Rightarrow n - t \in I_y$, $\forall n > 0$.

By symmetry, $\{t < 0 : S^t(y) \in Y_0\}$ is non-empty, infinite, and has no accumulation points. Therefore $Y_0$ is a Poincaré section for $\mathsf{S}$.

$r(y) := \min\{t > 0 : S^t(y) \in Y_0\}$ is well-defined and positive $\eta$–a.e. Using that $\pi$ commutes $\mathsf{T}$ and $\mathsf{S}$, we have $r \circ S^1 = r$, thus $r$ is constant $\eta$–a.e. Let $U : Y_0 \to Y_0$, $U(y) = S^{r(y)}(y)$, and let $\eta_0 := (\mu_0 \times \delta_0) \circ \pi^{-1}$. $\mathsf{S}$ is a constant suspension over $(Y_0, \eta_0, U)$. But $(Y_0, \eta_0, U)$ is a factor of $(\Sigma, \mu_0, \tau)$, hence it is a Bernoulli automorphism. □

## 7. Reeb flows

Let $M$ be a compact three dimensional smooth Riemannian manifold without boundary, equipped with the following objects [15]:

A Contact form: A smooth 1–form $\alpha$ on $M$ such that $\omega := \alpha \wedge d\alpha$ is a volume form. In this case, $\ker(d\alpha)_x := \{v \in T_x M : d\alpha(v, \cdot) \equiv 0\}$ is one-dimensional for all $x \in M$.

The Reeb vector field (of $\alpha$): The unique vector field $X$ such that $X_x \in \ker(d\alpha)_x$ and $\alpha(X_x) = 1$ for all $x \in M$. Necessarily $i_X \omega = d\alpha$.

The Reeb flow (of $\alpha$): The flow $\{T^t\}$ generated by the Reeb vector field of $\alpha$. This is a smooth flow with positive speed. $\{T^t\}$ preserves $\alpha$, i.e. $\alpha(dT^t v) = \alpha(v)$ for all $v$, since $\frac{d}{dt}(T^t)^*\alpha = (T^t)^* L_X \alpha = (T^t)^*[d\, i_X \alpha + i_X(d\alpha)] = (T^t)^*[0+0] = 0$. This setup covers geodesic flows of surfaces, and Hamiltonian flows of a system with two degrees of freedom restricted to regular energy surfaces [1].

We now add the assumption that $\{T^t\}$ has positive topological entropy. Let $\mu$ be an ergodic equilibrium measure of a Hölder continuous potential with positive metric entropy. By Theorem 1.1, $\mathsf{T} = (M, \mu, \{T^t\})$ is Bernoulli up to a period. We will

show that $\mathsf{T}$ is Bernoulli. A similar result for absolutely continuous measures is due to Katok [20, Theorem 3.6].

In dimension three, every ergodic invariant probability measure with positive metric entropy is non-uniformly hyperbolic [47], hence there is a $\mathsf{T}$–invariant set $M_0 \subset M$ of full $\mu$–measure such that for all $x \in M_0$ we have $T_x M = E^u(x) \oplus E^s(x) \oplus \mathrm{span}(X(x))$ where $E^u(x), E^s(x)$ are one-dimensional linear subspaces satisfying:

- $\lim_{t\to\pm\infty} \frac{1}{t} \log \|dT_x^t v\| < 0$ for all non-zero $v \in E^s(x)$,
- $\lim_{t\to\pm\infty} \frac{1}{t} \log \|dT_x^{-t} v\| < 0$ for all non-zero $v \in E^u(x)$,
- $dT_x^t E^s(x) = E^s(T^t(x))$ and $dT_x^t E^u(x) = E^u(T^t(x))$, $\forall t \in \mathbb{R}$,
- There is an immersed smooth curve $W^s(x) \ni x$ such that $T_y W^s(x) = E^s(y)$ and $d(T^t(x), T^t(y)) \xrightarrow[t\to\infty]{} 0$, $\forall y \in W^s(x)$. An analogous result holds for $W^u(x)$.

See [4, §8.2].

Quadrilateral: A *quadrilateral* is a closed embedded curve $\gamma : [0,1] \to M$ such that there are four distinct points $x_0, x_1, x_2, x_3 \in M_0$ with:

- $x_{i+1} \in W^{\tau_i}(x_i)$ for some $\tau_i \in \{s, u\}$ (here $x_4 = x_0$),
- If $\gamma(t_i) = x_i$, then $\gamma \restriction_{(t_i, t_{i+1})}$ is smooth with $\gamma'(t) \in E^{\tau_i}(\gamma(t))$, $\forall t \in (t_i, t_{i+1})$.

Quadrilaterals are the four-legged geometrical version of $su$–loops considered in page 83. Call $x_0, \ldots, x_3$ the *vertices* of the quadrilateral. The next lemma is standard.

**Lemma 7.1.** *Let $\mathsf{T} = (M, \mu, \{T^t\})$ be as above. Then $E^s(x) \oplus E^u(x) = \ker(\alpha_x)$, $\forall x \in M_0$. In particular, if $\gamma$ is a quadrilateral then $\int_\gamma \alpha = 0$.*

*Proof.* Let $v \in E^s(x)$. By the $\mathsf{T}$–invariance of $\alpha$, $\alpha(v) = \lim_{t\to+\infty} \alpha(dT^t v) = 0$, hence $E^s(x) \subset \ker(\alpha_x)$. Since contact forms are non-degenerate, $\dim \ker(\alpha_x) = 2$ whence $E^s(x) \oplus E^u(x) = \ker(\alpha_x)$. If $\gamma$ is a quadrilateral then $\gamma'(t) \in E^s(\gamma(t)) \oplus E^u(\gamma(t))$ except at the vertices, therefore $\int_\gamma \alpha = \int_0^1 \alpha(\gamma'(t))dt = 0$. □

*Proof of Theorem 1.2.* Using the same notation of Section 6, there is a TMF $\sigma_r : \Sigma_r \to \Sigma_r$ and a Hölder continuous map $\pi_r : \Sigma_r \to M$ such that:

(1) $\pi_r \circ \sigma_r^t = T^t \circ \pi_r$, $\forall t \in \mathbb{R}$,

(2) $\pi_r[\Sigma_r^\#]$ has full $\mu$–measure,

(3) $\pi_r : \Sigma_r^\# \to M$ is finite-to-one,

(4) $(\Sigma_r, \mu \circ \pi_r^{-1}, \sigma_r)$ is Bernoulli up to a period, and it has a period iff $r$ is arithmetic, iff the holonomy group equals $c\mathbb{Z}$ for some $c > 0$ (see Theorem 4.6).

Assume by way of contradiction that there is a period.

Let $\nu$ be the induced measure of $\mu \circ \pi_r^{-1}$, then $\nu$ is globally supported on $\Sigma$ and has local product structure (Theorem 3.5). Let $\nu_x^s, \nu_x^u$ be the projection measures of $\nu$, as in (3.2). These are globally supported measures on $W^s_{\text{loc}}(x), W^u_{\text{loc}}(x)$.

Let $E$ be the set constructed on page 82, then the holonomy group equals the closure of the set of weights of $su$–loops with vertices in $E$. The assumption that $\mathsf{T}$ has a period translates to the holonomy group being equal to $c\mathbb{Z}$ with $c > 0$.

The Bowen-Marcus cocycle $P^\tau(\cdot,\cdot)$ is Hölder continuous (Lemma 2.3(4)), therefore $\exists \delta > 0$ such that $d(x,y) < \delta \Rightarrow P^\tau(x,y) < c/5$ wherever defined. We claim there exist four distinct points $w_0, \dots, w_3 \in E$ such that $d(w_i, w_j) < \delta$ for all $i, j$ and $\gamma_0 = \langle w_0, w_1, w_2, w_3, w_0\rangle$ is a $su$–loop with $P(\gamma_0) = 0$. This can be done as follows:

- Fix $x, y \in E$ such that $d(x,y) < \delta$ and $y \notin W^s_{\text{loc}}(x)$.
- By Claim 1 of Theorem 4.6, $\nu_x^s(E^c) = \nu_y^s(E^c) = 0$, hence $\{w \in E \cap W^s_{\text{loc}}(x) : [w,y] \in E\}$ has full $\nu_x^s$–measure.
- $\nu_x^s$ is globally supported on $W^s_{\text{loc}}(x)$, thus there exist $w_0, w_1 \in \{w \in E \cap W^s_{\text{loc}}(x) : [w,y] \in E\}$ with $d(w_0, w_1) < \delta$. Take $w_2 = [w_1, y]$, $w_3 = [w_0, y]$.
- $\gamma_0 = \langle w_0, w_1, w_2, w_3, w_0\rangle$ is a $su$–loop with $|P(\gamma_0)| < \frac{4c}{5} < c \Rightarrow P(\gamma_0) = 0$.

Let $\widehat{\gamma}_0$ be the lifted $su$–path of $\gamma_0$, and $\gamma := \pi_r(\widehat{\gamma}_0)$. Since $\pi_r : \Sigma_r^\# \to M$ is finite-to-one and $\nu_x^s, \nu_x^u$ have global support, we can choose $w_0, w_1$ so that $\operatorname{diam}(\gamma) < \delta$.

We claim that if $\delta, \varepsilon$ are small enough, then for every $|t| < \varepsilon$, the quadrilateral $T^t\gamma$ is the boundary of a piecewise smooth *immersed* surface $T^tU$ such that:

- $T^tU$ is the union of compact smooth *embedded* surfaces $T^tU_i$, $i = 0,1,2,3$.
- $T^tU_i$ are uniformly transverse to the Reeb vector field.
- $T^tU_i$ have piecewise smooth boundaries and $\int_{T^t\gamma} = \int_{T^t(\partial U)} = \sum_{i=0}^3 \int_{T^t(\partial U_i)}$.

Had $\gamma$ been a Euclidean rectangle, we could take $U$ to be its interior, and $U_i$ the four triangles described by the principal diagonals. The general case is similar. It is enough to treat $t = 0$, since the case of small $t$ follows from uniform transversality.

Let $w_0, \dots, w_3 \in {}_{-N}[a_{-N}, \dots, a_N]$, where $N$ is large to be chosen later. Let $u_0, \dots, u_3$ be the vertices of $\gamma$. If $\delta$ is small enough, then $\gamma$ is covered by a chart of $M$ and we can think of $\vec{u}_i := u_i$, $\vec{\gamma}(t) := \gamma(t)$ as vectors in $\mathbb{R}^3$. If $N$ is sufficiently large, then $\vec{\gamma}'(t)$ is nearly parallel to $E^u(u_0)$ or $E^s(u_0)$ at all points. Therefore $\vec{\gamma}$ is made of four curves which are $C^1$ close to the sides of a parallelogram such that $\vec{u}_1 - \vec{u}_0, \vec{u}_2 - \vec{u}_3$ are nearly parallel to $E^s(u_0)$ and $\vec{u}_2 - \vec{u}_1, \vec{u}_3 - \vec{u}_0$ are nearly parallel to $E^u(u_0)$. There is no loss of generality in assuming that these vectors have norm in $(\frac{1}{2}, 2)$. Let $\varepsilon_0 := C^1$ distance between $\gamma$ and a parallelogram with sides $\vec{u}_1 - \vec{u}_0$ and $\vec{u}_3 - \vec{u}_0$. Then $\varepsilon_0 \to 0$ as $N \to \infty$.

Let $\vec{z} := \frac{1}{4}(\vec{u}_0 + \cdots + \vec{u}_3)$, then $\vec{z} - \frac{1}{2}(\vec{u}_i + \vec{u}_{i+1}) = \frac{1}{2}(\vec{u}_{i-1} - \vec{u}_i) + O(\varepsilon_0)$, where $\vec{u}_i := \vec{u}_{i\,(\text{mod } 4)}$ (the approximation is an identity for real parallelograms). We

define $U_i$ to be the cone with vertex $z$ and base $\vec{\gamma}_i$, where $\vec{\gamma}_i : [0,1] \to \mathbb{R}^3$ is the "leg" of $\vec{\gamma}$ from $\vec{u}_i$ to $\vec{u}_{i+1}$:

$$U_i := \{\vec{x}_i(s,t) := s\vec{\gamma}_i(t) + (1-s)\vec{z} : s,t \in [0,1]\}, \quad i = 0,\dots,3.$$

$U_i$ are embedded, and $\int_\gamma = \int_{\partial U} = \sum_{i=0}^3 \int_{\partial U_i}$. At $\vec{x}_i(s,t)$, $U_i$ is perpendicular to

$$\vec{n} = (\vec{\gamma}_i(t) - \vec{z}) \times \gamma_i'(t) = \left(\vec{\gamma}_i(t) - \frac{\vec{u}_i + \vec{u}_{i+1}}{2}\right) \times \vec{\gamma}_i'(t) + \left(\frac{\vec{u}_i + \vec{u}_{i+1}}{2} - \vec{z}\right) \times \vec{\gamma}_i'(t).$$

The first summand is $O(\varepsilon_0 |\vec{\gamma}_i'(t)|)$, being the product of vectors at angle $O(\varepsilon_0)$. The second summand is of size $\sim |\vec{\gamma}_i'(t)|$ and $\varepsilon_0$–parallel to $\vec{e}^u(u_0) \times \vec{e}^s(u_0)$. By Lemma 7.1, $U_i$ is almost parallel to $\ker(\alpha)$, whence uniformly transverse to the Reeb flow.

Fix $t_0 > 0$ so small that $D_i := \bigcup_{t\in[0,t_0]} T^t U_i$ is a flow box. So

$$\begin{aligned} 0 \neq \sum_{i=0}^3 \int_{D_i} \omega = \sum_{i=0}^3 \int_0^{t_0} \left( \int_{T^t U_i} i_X \omega \right) dt &= \sum_{i=0}^3 \int_0^{t_0} \left( \int_{T^t U_i} d\alpha \right) dt \\ &= \int_0^{t_0} \left( \int_{T^t U} d\alpha \right) dt. \end{aligned}$$

But by the Stokes Theorem, this equals $\int_0^{t_0} \left(\int_{T^t \gamma} \alpha\right) dt = 0$, since the inner integral is zero by Lemma 7.1. We obtain a contradiction. □

## 8. Equilibrium states for the geometric potential

Let $M$ be a three dimensional compact $C^\infty$ Riemannian manifold, let $X : M \to TM$ be a non-vanishing $C^{1+\varepsilon}$ vector field, and let $\mathsf{T}$ be the flow on $M$ generated by $X$. Throughout this section we assume $\mathsf{T}$ has positive topological entropy.

The subset $M_{\text{hyp}} \subset M$: $p \in M_{\text{hyp}}$ if there are unit vectors $\underline{e}^s_p, \underline{e}^u_p \in T_p M$ such that

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log \|d T^t_p \underline{e}^s_p\| < 0 \quad \text{and} \quad \lim_{t\to\pm\infty} \frac{1}{|t|} \log \|d T^t_p \underline{e}^u_p\| > 0.$$

If $\underline{e}^s_p, \underline{e}^u_p$ exist, then they are unique up to a sign, hence $M_{\text{hyp}}$ is $\mathsf{T}$–invariant. By the Oseledets theorem and the Ruelle entropy inequality, any $\mathsf{T}$–invariant and ergodic measure with positive metric entropy is carried by $M_{\text{hyp}}$.

The geometric potential of $\mathsf{T}$ [8]: $J : M_{\text{hyp}} \to \mathbb{R}$ given by

$$J(p) = -\frac{d}{dt}\Big|_{t=0} \log \|d T^t_p \underline{e}^u_p\| = -\lim_{t\to 0} \tfrac{1}{t} \log \|d T^t_p \underline{e}^u_p\|.$$

$J$ is bounded, since $\{T^t\}$ is $C^{1+\varepsilon}$.

**Some facts from [28].** We recall some facts from [28, §2]. There exists a Poincaré section $\Lambda \subset M$ with return map $f : \Lambda \to \Lambda$ and roof function $R : \Lambda \to \mathbb{R}$ such that:

(1) $\Lambda$ is the union of disjoint discs transverse to $X$.

(2) $\inf R > 0$ and $\sup R < \infty$.

(3) Let $\mathfrak{S} \subset \Lambda$ denote the *singular set* of $f : \Lambda \to \Lambda$, consisting of points $p$ which do not have a (relative) neighborhood $V \subset \Lambda \setminus \partial\Lambda$ which is diffeomorphic to a disc, such that $f|_V$, $f^{-1}|_V$ are diffeomorphisms onto their images. There is a constant $\mathfrak{C}$ such that $R$, $f$, $f^{-1}$ are differentiable on $\Lambda' := \Lambda \setminus \mathfrak{S}$ with $\sup_{p\in\Lambda'} \|dR_p\| < \mathfrak{C}$, $\sup_{p\in\Lambda'} \|df_p\| < \mathfrak{C}$, $\sup_{p\in\Lambda'} \|(df_p)^{-1}\| < \mathfrak{C}$, and $\|f|_U\|_{C^{1+\varepsilon}} < \mathfrak{C}$, $\|f^{-1}|_U\|_{C^{1+\varepsilon}} < \mathfrak{C}$ for all open and connected $U \subset \Lambda'$. See [28, Lemma 2.5].

(4) For all $p \in \Lambda_{\text{hyp}} := (\Lambda \setminus \bigcup_{n\in\mathbb{Z}} f^n(\mathfrak{S})) \cap M_{\text{hyp}}$ there are $\vec{v}^s_p, \vec{v}^u_p \in T_p\Lambda$ unitary such that

$$\lim_{n\to\pm\infty} \tfrac{1}{|n|} \log \|df^n_p \vec{v}^s_p\| < 0 \quad \text{and} \quad \lim_{n\to\pm\infty} \tfrac{1}{|n|} \log \|df^n_p \vec{v}^u_p\| > 0.$$

See [28, Lemma 2.6] and its proof.

Suppose $\mu$ is a hyperbolic $\mathsf{T}$–invariant probability measure on $M$, and $\mu_\Lambda$, the "induced measure", is the measure on $\Lambda$ such that

$$\mu = \frac{1}{\int_\Lambda R d\mu_\Lambda} \int_\Lambda \left[ \int_0^{R(p)} \delta_{T^t p} dt \right] d\mu_\Lambda(p).$$

Then $\Lambda$ can be chosen with the additional properties below.

(5) The induced measure $\mu_\Lambda$ on $\Lambda$ satisfies:

(5.1) $\mu_\Lambda(\mathfrak{S}) = 0$.

(5.2) $\lim_{n\to\infty} \frac{1}{n} \mathrm{dist}_\Lambda(f^n(p), \mathfrak{S}) = 0$ $\mu_\Lambda$–a.e.

See [28, Thm 2.8].

(6) There are a TMF $\sigma_r : \Sigma_r \to \Sigma_r$ and Hölder continuous maps $\pi : \Sigma \to \Lambda$ and $\pi_r : \Sigma_r \to M$ such that:

(6.1) $\pi \circ \sigma = f \circ \pi$, $\pi[\Sigma^\#]$ has full $\mu_\Lambda$–measure, and every $x \in \pi[\Sigma^\#]$ has finitely many pre-images in $\Sigma^\#$.

(6.2) $\pi_r(x,t) = T^t\pi(x)$, $\pi_r \circ \sigma_r = T \circ \pi_r$, $\pi_r[\Sigma^\#_r]$ has full $\mu$–measure, and every $p \in \pi_r[\Sigma_r]$ has finitely many pre-images in $\Sigma^\#_r$.

See [28, Thm 5.6]. Here $\Sigma^\#$, $\Sigma^\#_r$ denote the regular parts of $\Sigma$, $\Sigma_r$, see [28, §1].

Finally, if $\mu$ is ergodic with positive entropy then $h_{\mu_\Lambda}(f) > 0$ and $\mu_\Lambda(\Lambda_{\text{hyp}}) = 1$.

GEOMETRIC POTENTIAL OF $f : \Lambda \to \Lambda$: $J^f : \Lambda_{\text{hyp}} \to \mathbb{R}$, $J^f(p) = -\log \|df_p \vec{v}^u_p\|$.

$J^f$ is bounded, since $\sup_{p\in\Lambda'} \|df_p\| < \mathfrak{C}$, $\sup_{p\in\Lambda'} \|(df_p)^{-1}\| < \mathfrak{C}$. Even though $J, J^f$ are not globally defined, we can define their equilibrium measures.

EQUILIBRIUM MEASURES OF $J$ AND $J^f$: $\mu$ is called an *equilibrium measure of* $J$ if $h_\mu(T^1) + \int J d\mu = P_{\text{top}}(J)$, where

$$P_{\text{top}}(J) := \sup \left\{ h_\nu(T^1) + \int_M J d\nu : \begin{array}{c} \nu \text{ is T–invariant Borel probability} \\ \text{measure with } \nu(M_{\text{hyp}}) = 1 \end{array} \right\}.$$

$P_{\text{top}}(J)$ is called the *topological pressure* of $J$. Similar definitions hold for $J^f$ with $T^1, M_{\text{hyp}}$ replaced by $f, \Lambda_{\text{hyp}}$.

$P_{\text{top}}(J), P_{\text{top}}(J^f) < \infty$, since $J, J^f$ are bounded. Similar definitions can also be given for functions of the form $aJ, bf, a, b \in \mathbb{R}$.

**Lemma 8.1.** *Assume that $\Lambda$, $f$, $R$ and $\mu$ satisfy conditions* (1)–(6) *above. Then $\mu$ is an equilibrium measure of $J$ iff $\mu_\Lambda$ is an equilibrium measure of $J^f - P_{\text{top}}(J)R$.*

*Proof.* Let $\overline{J} : \Lambda_{\text{hyp}} \to \mathbb{R}$, $\overline{J}(p) = \int_0^{R(p)} J(T^s p) ds$. As in claim 1 of the proof of Theorem 3.1, $\mu$ is an equilibrium measure of $J$ iff $\mu_\Lambda$ is an equilibrium measure of $\overline{J} - P_{\text{top}}(J)R$. We will show that $\int_\Lambda \overline{J} d\nu = \int_\Lambda J^f d\nu$ for every $f$–invariant $\nu$ with $\nu(\Lambda_{\text{hyp}}) = 1$, and deduce that $\mu$ is an equilibrium measure of $J$ iff $\mu_\Lambda$ is an equilibrium measure of $J^f - P_{\text{top}}(J)R$.

A simple calculation[4] shows that

$$\overline{J}(p) = -\int_0^{R(p)} \frac{d}{dt}\Big|_{t=0} \log \|dT^t \underline{e}^u_{T^s p}\| ds = -\log \|dT^{R(p)} \underline{e}^u_p\|.$$

Since $f(p) = T^{R(p)}(p)$, we have $df_p v = dT_p^{R(p)} v + \langle \nabla R(p), v\rangle X_{f(p)}$, $\forall v \in T_p\Lambda$. Write $\vec{v}^u_p = \alpha(p)\underline{e}^u_p + \beta(p)X_p$ (necessarily $\alpha(p) \neq 0$). Then

$$\begin{aligned} df\vec{v}^u_p &= dT^{R(p)}\vec{v}^u_p + \langle \nabla R(p), \vec{v}^u_p\rangle X_{f(p)} \\ &= \alpha(p) dT^{R(p)} \underline{e}^u_p + \beta(p) dT^{R(p)} X_p + \langle \nabla R(p), \vec{v}^u_p\rangle X_{f(p)} \\ &= \pm\alpha(p)\|dT^{R(p)}\underline{e}^u_p\| \underline{e}^u_{f(p)} + [\beta(p) + \langle \nabla R(p), \vec{v}^u_p\rangle] X_{f(p)}. \end{aligned}$$

Similarly

$$df\vec{v}^u_p = \pm\|df\vec{v}^u_p\|\vec{v}^u_{f(p)} = \pm(\alpha(f(p))\|df\vec{v}^u_p\|\underline{e}^u_{f(p)} + \beta(f(p))\|df\vec{v}^u_p\| X_{f(p)}).$$

[4] Let $h(t) := -\log \|dT^t \underline{e}^u_p\|$, then $h(0) = 0$ and $-\log \|dT^t \underline{e}^u_{T^s p}\| = h(t+s) - h(s)$, therefore $-\frac{d}{dt}\Big|_{t=0} \log \|dT^t \underline{e}^u_{T^s p}\| = \frac{d}{dt}\Big|_{t=0} [h(t+s) - h(s)] = h'(s)$. By the fundamental theorem of calculus, $\overline{J}(p) = \int_0^{R(p)} h'(s) ds = h(R(p)) = -\log \|dT^{R(p)} \underline{e}^u_p\|$.

Comparing the $\underline{e}^u_{f(p)}$ components, we get $|\alpha(p)|\|dT^{R(p)}\underline{e}^u_p\| = |\alpha(f(p))|\|df\vec{v}^u_p\|$. Hence $U : \Lambda_{\text{hyp}} \to \mathbb{R}$, $U(p) := -\log|\alpha(p)|$ is a measurable function with

$$\overline{J} = J^f + U \circ f - U.$$

We use this to show that $\mu$ is an equilibrium measure for $J$ iff $\mu_\Lambda$ is an equilibrium measure for $J^f - P_{\text{top}}(J)R$. By (4) and (5), $\mu(M_{\text{hyp}}) = \mu_\Lambda(\Lambda_{\text{hyp}})$. Let $\nu$ be an ergodic $f$–invariant probability measure with $\nu(\Lambda_{\text{hyp}}) = 1$. By the Birkhoff ergodic theorem, $\lim_{n\to\infty}\frac{1}{n}\overline{J}_n = \int_\Lambda \overline{J}d\nu$ and $\lim_{n\to\infty}\frac{1}{n}J^f_n = \int_\Lambda J^f d\nu$ $\nu$–a.e. By the Poincaré recurrence theorem, $\liminf_{n\to\infty}|U(f^n(p)) - U(p)| < \infty$ $\nu$–a.e., hence for $\nu$–a.e. $p \in \Lambda$ we have $\int_\Lambda \overline{J}d\nu = \liminf_{n\to\infty}\frac{1}{n}\overline{J}(p) = \liminf_{n\to\infty}\frac{1}{n}J^f(p) = \int_\Lambda J^f d\nu$. By the ergodic decomposition, $\int_\Lambda \overline{J}d\nu = \int_\Lambda J^f d\nu$ for every $f$–invariant $\nu$ such that $\nu(\Lambda_{\text{hyp}}) = 1$. The lemma follows from the discussion at the beginning of the proof. □

**Lemma 8.2.** *$[J^f - P_{\text{top}}(J)f]\circ\pi$ is a Hölder continuous potential on $\Sigma$ with respect to the symbolic metric.*

*Proof.* $R \circ \pi : \Sigma \to \mathbb{R}$ is Hölder by construction: $R \circ \pi = r$ and roof functions of TMF are Hölder. $J^f \circ \pi$ is Hölder, because $df$ is uniformly Hölder on $\Lambda'$ and $x \in \Sigma \to \vec{v}^u_{\pi(x)}$ is Hölder by [28, Lemma 5.7]. □

*Proof of Theorem 1.4.* Fix $\chi > 0$, and let $\mu$ be a $\chi$–hyperbolic[5] equilibrium measure of $J$ with $h_\mu(T^1) > 0$. Take $\Lambda, f, R$ satisfying (1)–(6) above. Since $\mu$ is carried by $M_{\text{hyp}}$, Lemma 8.1 implies that $\mu_\Lambda$ is an equilibrium measure of $J^f - P_{\text{top}}(J)R$. Arguing as in [28, Theorem 6.2], the function $[J^f - P_{\text{top}}(J)R]\circ\pi : \Sigma \to \mathbb{R}$ has an equilibrium measure $\widehat{\mu_\Lambda}$ such that $\widehat{\mu_\Lambda}\circ\pi^{-1} = \mu_\Lambda$. The potential $[J^f - P_{\text{top}}(J)R]\circ\pi$ is Hölder continuous. Since ergodic equilibrium measures of Hölder potentials on a TMS are carried by topologically transitive TMS, $\widehat{\mu_\Lambda}$ has at most countably many ergodic components. This shows that $J$ has at most countably many $\chi$–hyperbolic ergodic equilibrium measures: if there were uncountably many, then some convex combination would generate a $\chi$–hyperbolic equilibrium measure with uncountably many ergodic components.

Assume now that $\mu$ is also ergodic. We can choose $\widehat{\mu_\Lambda}$ to be ergodic. The measure $\widehat{\mu_\Lambda}$ is the induced measure of some $\widehat{\mu}$ on $\Sigma_r$, hence $\widehat{\mu}\circ\pi_r^{-1} = \mu$. By Theorems 4.7 and 5.1 $(\Sigma_r, \widehat{\mu}, \sigma_r)$ is Bernoulli up to a period. Since $\widehat{\mu}$ projects to $\mu$, $(M, \mu, \{T^t\})$ is also Bernoulli up to a period.

If additionally $\mathsf{T}$ is a Reeb flow, then it is Bernoulli and so is $\mathsf{T}$. □

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[5] $\mu$ is $\chi$–hyperbolic if $\mu$–a.e. point has one Lyapunov exponent $> \chi$ and another $< -\chi$.

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