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Bounded cohomology with coefficients in uniformly convex Banach spaces

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Abstract. We show that for acylindrically hyperbolic groups Γ (with no nontrivial finite normal subgroups) and arbitrary unitary representation ρ of Γ in a (nonzero) uniformly convex Banach space the vector space $H_b^2(\Gamma;\rho)$ is infinite dimensional. The result was known for the regular representations on $\ell^p(\Gamma)$ with $1 < p < \infty$ by a different argument. But our result is new even for a non-abelian free group in this great generality for representations, and also the case for acylindrically hyperbolic groups follows as an application.

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1. Introduction

1.1. Quasi-cocycle and quasi-action. Let G be a group and E a normed vector space (usually complete, either over $\mathbb{R}$ or over $\mathbb{C}$). The linear or rotational part of an isometric G-action on E determines a representation $\rho : G \to O(E)$ where $O(E)$ is the group of norm-preserving linear isomorphisms $E \to E$. We will refer to ρ as a *unitary representation*. We will usually write $\rho(g)x$ as $g(x)$ or gx.

The translational part of the G-action is a *cocycle* (with respect to ρ). Namely the translational part is a function $F : G \to E$ that satisfies

$$F(gg') = F(g) + gF(g') \tag{1.1}$$

for all $g, g' \in G$. Going in the other direction, if ρ is a unitary representation and F a cocycle then the map $g \mapsto (x \mapsto \rho(g)x + F(g))$ determines an (affine) isometric G-action on E. Note that $F(g^{-1}) = -g^{-1}F(g)$. $\rho(g)$ is sometimes called the linear part of the action.

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For an isometric quasi-action of G on E the linear part will still be a unitary representation. However, the translational part F will become a *quasi-cocycle* and will only satisfy (1.1) up to a uniformly bounded error so that

$$\Delta(F) := \sup_{g,g' \in G} |F(gg') - F(g) - gF(g')| < \infty. \tag{1.2}$$

The quantity $\Delta(F)$ is the *defect* of the quasi-cocycle.

A basic question is if there are quasi-actions that are not boundedly close to an actual action. Such a quasi-action is *essential*. Since quasi-actions determine unitary representations a more refined question is if there are essential quasi-actions for a given unitary representation.

The above discussion is perhaps more familiar in its algebraic form where it can be rephrased in terms of bounded cohomology. A quasi-cocycle F can be viewed as 1-cochain in the group cohomology twisted by the representation ρ. Condition (1.2), is equivalent to the coboundary δF being a bounded 2-cocycle and will therefore determine a cohomology class in $H^2_b(G;\rho)$, the second bounded cohomology group. Now this cocycle will clearly be trivial in the regular second cohomology group $H^2(G;\rho)$ as it is the coboundary of a 1-cochain. If the cochain F is a bounded distance from a cocycle then δF will also be trivial in $H^2_b(G;\rho)$ so we are interested in the kernel of the map

$$H^2_b(G;\rho) \to H^2(G;\rho)$$

from bounded cohomology to regular cohomology. In particular this kernel is the vector space $QC(G;\rho)$ of all quasi-cocycles modulo the subspace generated by bounded functions and cocycles. We denote this quotient space $\widetilde{QC}(G;\rho)$. This is the vector space of *essential* quasi-cocycles and it is the main object of study of this paper.

For the trivial representation on $\mathbb{R}$ a cocycle is just a homomorphism to $\mathbb{R}$ and a quasi-cocycle is usually called a quasi-morphism. When $G = F_2$, the free group on two generators, Brooks [7] gave a combinatorial construction of an infinite dimensional family of essential quasi-morphisms.

1.2. Uniformly convex Banach space and main result. Following the work of Brooks, there is a long history of generalizations of this construction to other groups. Initially, the work focused on the trivial representation. See [4, 5, 12]. This was followed by generalizations to the same groups G but with coefficients in the regular representation $\ell^p(G)$, $1 \le p < \infty$. See [14, 16].

In this paper we will extend this work to unitary representations in *uniformly convex* Banach spaces. Note that this essentially includes the previous cases since $\ell^p(G)$ is uniformly convex when $1 < p < \infty$.

If one is a bit more careful about how the counting is done then Brooks construction of quasi-morphisms can also be used to produce quasi-cocycles. In

Brooks' original work (i.e., for trivial representations) it is easy to see that the quasi-morphisms are essential. Here we will have to work harder to get the following result.

Theorem 1.1 (Theorem 3.9). *Let ρ be a unitary representation of F_2 on a uniformly convex Banach space $E \neq 0$. Then* $\dim \widetilde{QC}(F_2; \rho) = \infty$.

To show $\widetilde{QC}(F_2; \rho)$ is non-trivial is already hard. We will argue that for a certain Brooks' quasi-cocycle H into a Banach space E, there exists a sequence of elements in F_2 on which H is unbounded. For that we use that E is uniformly convex in an essential way (Lemma 3.4). We also show those quasi-cocycles are not at bounded distance from any cocycle using that E is reflexive (using Lemma 3.6). Those two steps are the novel part of the paper. It seems that the uniform convexity is nearly a necessary assumption for the conclusion. See the examples at the end of this section.

Recently Osin [20] (see also [11]) has identified the class of *acylindrically hyperbolic groups* and this seems to be the most general context where the Brooks' construction can be applied. Osin has shown that acylindrically hyperbolic groups contain *hyperbolically embedded* copies of F_2 and then applying work of Hull–Osin [17] we have the the following corollary to Theorem 3.9. See Section 4 for the proof.

Corollary 1.2. *Let ρ be a unitary representation of an acylindrically hyperbolic group G on a uniformly convex Banach space $E \neq 0$ and assume that the maximal finite normal subgroup has a non-zero fixed vector. Then* $\dim \widetilde{QC}(G; \rho) = \infty$.

A wide variety of groups are acylindrically hyperbolic. In particular our results apply to the following examples. To apply our result, in all examples assume G has no nontrivial finite normal subgroups, or more generally that for the maximal finite normal subgroup N (see [11]) we have that $\rho(N)$ fixes a nonzero vector in E.

Examples 1.3 (Acylindrically hyperbolic groups).

- G is non-elementary word hyperbolic,
- G admits a non-elementary isometric action on a connected δ-hyperbolic space such that at least one element is hyperbolic and WPD,
- $G = Mod(S)$, the mapping class group of a compact surface which is not virtually abelian,
- $G = Out(F_n)$ for $n \geq 2$,
- G admits a non-elementary isometric action on a $CAT(0)$ space and at least one element is WPD and acts as a rank 1 isometry.

Remark 1.4. Recall that a Banach space is *superreflexive* if it admits an equivalent uniformly convex norm. It is observed in [1, Proposition 2.3] that if $\rho : G \to E$ is a unitary representation with E superreflexive, then there is an equivalent uniformly convex norm with respect to which ρ is still unitary. Thus in Corollary 1.2 we may replace "uniformly convex" with "superreflexive".

Remark 1.5. There is also a more direct approach to going from Theorem 3.9 to our the main theorem. The key point is that any group G covered in the the main theorem acts on a quasi-tree such that there is a free group $F \subset G$ that acts properly and co-compactly on a tree isometrically embedded in the quasi-tree. This is done using the *projection complex* of [2]. Using this one can apply the Brooks' construction to produce quasi-cocycles that when restricted to the free group are exactly the quasi-cocycles of Theorem 3.9. We carry this out in a separate paper [3].

1.3. Known examples with certain Banach spaces. Here are some known vanishing/non-vanishing examples in the literature.

- $E = \mathbb{R}$ and ρ is trivial. In this case $H_b^2(G;\rho)$ is the usual bounded cohomology and quasi-cocycles are quasi-morphisms. As we said this case was known for various kinds of groups.
- $E = \ell^p(G)$ and ρ is the regular representation, see [13, 15]. When $1 < p < \infty$, $\ell^p(G)$ is uniformly convex and our theorem applies. When $p = 1$ or $p = \infty$ then $\ell^p(G)$ is not uniformly, or even strictly, convex. However, for $p = 1$ summation determines a ρ-invariant functional and one can produce a family of quasi-cocycles that when composed with the invariant functional are an infinite dimensional family of non-trivial quasi-morphisms in $\widetilde{QH}(G)$ implying that $\dim \widetilde{QC}(G;\ell^1(G)) = \infty$.

On the other hand,

- When $p = \infty$ given any quasi-cocycle one can explicitly find a cocycle a bounded distance away so $\widetilde{QC}(G;\ell^\infty(G)) = 0$ for any group G.
- If G is countable and exact (e.g., F_2), then $H_b^2(G;\ell_0^\infty(G)) = 0$. In particular, $\widetilde{QC}(G;\ell_0^\infty(G)) = 0$ (Example 3.10). Here $\ell_0^\infty(G)$ is the subspace of $\ell^\infty(G)$ consisting of sequences which are asymptotically 0.

There are also examples where G is not acylindrically hyperbolic but where $\widetilde{QC}(G;\rho)$ is known to be non-zero for certain actions of G on ℓ^p spaces.

- If G has a non-elementary action on a $CAT(0)$ cube complex then $\widetilde{QC}(G;\rho) \neq 0$ where ρ is the representation of G on the space of ℓ^p-functions $(1 \le p < \infty)$ on a certain space where G naturally acts [8]. Note that this class of groups is closed under products so it contains groups that aren't acylindrically hyperbolic.

There are other examples where essentially nothing is known.

- $E = \ell_0^1(G) \subset \ell^1(G)$ is the space of ℓ^1-functions on G that sum to zero and ρ is the regular representation. Unlike with $\ell^1(G)$, $\ell_0^1(G)$ has no ρ-invariant functionals.
- $E = \mathcal{B}(\ell^2(G))$ the space of bounded linear maps of $\ell^2(G)$ to itself. This example was suggested to us by N. Monod as the non-commutative analogue to $\ell^\infty(G)$.

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2. Quasi-cocycles from trees

Fix $F_2 = \langle a, b \rangle$ and choose a word $w \in F_2$. For simplicity we will assume that w is cyclically reduced. Let E be a normed vector space and $\rho : G \to O(E)$ a linear representation. Also choose a nonzero $e \in E$. We now set up some notation that will be convenient for what we will do later.

Let $[g, h]$ be an oriented segment in the Cayley graph for F_2 with generators a and b. Then we write $[g, h] \overset{\circ}{\subset} [g', h']$ if $[g, h]$ is a subsegment of $[g', h']$ and the orientations of the two segments agree. We then define

$$w_+(g) = \{h \in G | [h, hw] \overset{\circ}{\subset} [1, g]\}$$

and

$$w_-(g) = \{h \in H | [h, hw] \overset{\circ}{\subset} [g, 1]\}.$$

Now define a function $H = H_{w,e} : F_2 \to E$ by

$$H(g) = \sum_{h \in w_+(g)} h(e) - \sum_{h \in w_-(g)} h(e)$$

In other words, to a translate $h \cdot w$ we assign $h(e)$ when traversed in the positive direction, and $-h(e)$ when traversed in negative direction. Note that it follows that $H(g^{-1}) = -g^{-1} H(g)$.

Proposition 2.1. *The function H constructed above is a quasi-cocycle.*

Proof. This is the standard Brooks argument. Consider the tripod spanned by 1, g, gf. Call the central point p. We will see that contributions of copies of w in the tripod that do not cross p cancel out leaving only a bounded number of terms.

If $h \cdot w \overset{\circ}{\subset} [1, p]$ then $h(e)$ enters with positive sign in $H(g)$ and in $H(gf)$, so it cancels in the expression $H(gf) - H(g)$. Likewise, if $h \cdot w \overset{\circ}{\subset} [p, 1]$ then $-h(e)$ enters both $H(g)$ and $H(gf)$, so it again cancels.

If $h \cdot w \overset{\circ}{\subset} [p, g]$ then $h(e)$ is a summand in $H(g)$. Since $h \cdot w \overset{\circ}{\subset} [gf, g]$ we also have $g^{-1} h \cdot w \overset{\circ}{\subset} [f, 1]$, so $-g^{-1} h(e)$ is a summand in $H(f)$, and thus we have cancellation in $-H(g) - gH(f)$. There is similar cancellation if $h \cdot w \overset{\circ}{\subset} [g, p]$.

If $h \cdot w \overset{\circ}{\subset} [p, gf]$ or $[gf, p]$ then similarly to the previous paragraph there is cancellation in $H(gf) - gH(f)$.

After the above cancellations in the expression $H(gf) - H(g) - gH(f)$ the only terms left are of the form $\pm h(e)$ where $h(w)$ is contained in the tripod and contains p in its interior. The number of such terms is clearly (generously) bounded by $6|w|$ so we deduce that $\Delta(H) \leq 6|w| \|e\|$. □

Remark 2.2. Note that if $h \cdot w$ does not overlap w for any $1 \neq h \in F_2$, then $\Delta(H) \leq 6\|e\|$. More generally, for a given w, write $w = u^n v$ as a word such that $|v| < |u|$ and $n > 0$ is maximal. Then, $\Delta(H) \leq 6(n+1)\|e\|$.

Example 2.3. Suppose $w = ab$. Then $H(a^n) = H(b^n) = 0$, while $H((ab)^n) = (1 + ab + (ab)^2 + \cdots + (ab)^{n-1})e \in E$. If the operator $1 - ab : E \to E$ has a continuous inverse (i.e. if $1 \in \mathbb{C}$ is not in the spectrum of ab) then H is uniformly bounded on the powers of ab since $(1-ab)H((ab)^n) = e - (ab)^n(e)$ has bounded norm. For example, this happens even for $E = \mathbb{R}^2$ when $\rho(ab)$ is a (proper) rotation.

On the other hand, for the representation $\ell^p(F_2)$ with $1 \leq p < \infty$ and with $e \in \ell^p(F_2)$ defined by $e(1) = 1$, $e(g) = 0$ for $g \neq 1$, the quasi-cocycle H is unbounded on the powers of ab.

3. Nontriviality of quasi-cocycles

In Brooks' original construction of quasi-morphisms $F_2 = \langle a, b \rangle \to \mathbb{R}$ it is easy to see that the quasi-morphisms are nontrivial. Choosing w to be a reduced word not of the form a^m or b^m it is clear that $H(w^n)$ will be unbounded while $H(a^n)$ and $H(b^n)$ will be zero. By this last fact if G is a homomorphism that is boundedly close to H then G must be bounded on powers of a and b and therefore $G(a) = G(b) = 0$. Since any homomorphism is determined by its behavior on the generators we have $G \equiv 0$ and the nontriviality of H follows.

When the Brooks construction is extended to quasi-cocycles it is no longer clear that the quasi-cocycle is nontrivial. In particular if $H = H_{w,e}$ it may be that $H(w^n)$ is bounded. See Examples 2.3 and 3.5. In fact if 1 is not in the spectrum of $\rho(w)$ then $H(w^n)$ will be bounded for all choices of vectors e. Even if 1 is in the spectrum, when e is chosen arbitrarily $H(w^n)$ may be bounded. To show that the Brooks quasi-cocycles are unbounded we will need to restrict to the class of *uniformly convex* Banach spaces and to look at a wider class of words than powers of w.

We will also have to work harder to show that a cocycle G that is bounded on powers of the generators is bounded everywhere. In fact we cannot do this in general but instead will show that in a reflexive Banach space (which includes uniformly convex Banach spaces) either the cocycle is bounded or the original representation, when restricted to a non-abelian subgroup, has an eigenvector. In this latter case it is easy to construct many nontrivial quasi-cocycles.

3.1. Uniformly convex and reflexive Banach spaces.

We will use basic facts about Banach spaces. General references are [6,18]. The following concept was introduced by Clarkson [10].

Definition 3.1. A Banach space E is *uniformly convex if for every* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $x, y \in E$, $|x| \leq 1$, $|y| \leq 1$, $|x - y| \geq \epsilon$ *implies* $|\frac{x+y}{2}| \leq 1 - \delta$.

The original definition in [10] replaces $|x|, |y| \leq 1$ above with equalities, but it is not hard to see that the two are equivalent.

Proposition 3.2. *(i) ℓ^p spaces are uniformly convex for $1 < p < \infty$ [10]. ℓ^1 and ℓ^∞ spaces are not uniformly convex and not reflexive.*

(ii) A uniformly convex Banach space is reflexive (the Milman–Pettis theorem).

(iii) If E is uniformly convex, then for any $R > 0$ there are $\epsilon > 0$ and $\mu > 0$ so that the following holds. If $|v| \leq R$ and $f : E \to \mathbb{R}$ is a functional of norm 1 with $f(v) = |v|$ and if e is a vector of norm $\geq 1/2$ with $f(e) \geq -\mu$ then $|v + e| \geq |v| + \epsilon$.

Proof. We only prove (iii). Choose $\delta \in (0, 1)$ so that $|x|, |y| \leq 1$, $|x - y| \geq \frac{1}{2(R+1)}$ implies $|\frac{x+y}{2}| \leq 1 - \delta$. Then choose $\epsilon, \mu > 0$ so that $\epsilon < \frac{1}{8}$ and $\frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon} > 1 - \delta$. Suppose f, v, e satisfy the assumptions but $|v + e| < R + \epsilon$. If $|v| \leq 1/8$ then $|v + e| \geq |e| - |v| \geq 1/4 \geq |v| + 1/8$ and we are done. So assume that $|v| > 1/8$. Then for $x = \frac{v}{|v|+\epsilon}$, $y = \frac{v+e}{|v|+\epsilon}$ we have $|x|, |y| \leq 1$ and $|x - y| \geq \frac{1}{2(|v|+1)} \geq \frac{1}{2(R+1)}$, so we must have $|\frac{x+y}{2}| \leq 1 - \delta$. Thus

$$1 - \delta \geq \left|\frac{x + y}{2}\right| = \left|\frac{v + e/2}{|v| + \epsilon}\right| \geq \frac{|v| - \frac{\mu}{2}}{|v| + \epsilon} \geq \frac{\frac{1}{8} - \frac{\mu}{2}}{\frac{1}{8} + \epsilon}$$

since $f(v + e/2) = |v| + f(e)/2 \geq |v| - \frac{\mu}{2}$ and $|f| = 1$. This contradicts the choice of μ, ϵ. □

Lemma 3.3. *Let ρ be a unitary representation of a group F on a reflexive Banach space E. If there is a linear functional f and a vector $e \in E$ such that the F-orbit of e lies in the half space $\{f \geq \mu\}$ with $\mu > 0$ then there is an F-invariant vector $e' \neq 0 \in E$ and an F-invariant functional ϕ with $\phi(e') \geq \mu$. If e is F-invariant, then we can take $e' = e$.*

Proof. Let Λ be the convex hull of the F-orbit of e in the weak topology on E. Since E is reflexive, Λ is weakly compact. The convex hull Λ is also F-invariant so by the Ryll-Nardzewski fixed point theorem it will contain an F-invariant vector e'. Since $e' \in \Lambda$, $f(e') \geq \mu$ and therefore $e' \neq 0$.

Since e' is a functional on the reflexive Banach space E^* and the F-orbit of f will be contained in the half space $\{e' \geq \mu\}$ we similarly get a F-invariant vector $\phi \in E^*$ with $e'(\phi) = \phi(e') \geq \mu$. □

Note that if E contains a nonzero vector that is F-invariant, then the Hahn–Banach theorem supplies a functional that satisfies the conditions of the lemma and so there is also a nonzero F-invariant functional.

3.2. Detecting unboundedness.

Lemma 3.4. *Let ρ be any unitary representation of $F_2 = \langle a, b\rangle$ into a uniformly convex Banach space E. Then one of the following holds:*

(i) for every $e \neq 0 \in E$ and any $1 \neq w \in F_2$ not of the form $a^m b^n$ nor $b^m a^n$ the quasi-cocycle $H = H_{w,e}$ is unbounded on F_2, or

(ii) there is a free subgroup $F \subset F_2$ with $F \cong F_2$, a linear functional g, a vector e and a $\mu > 0$ such that the F-orbit of e is contained in the half-space $\{g \leq -\mu\}$. In particular, there is an F-invariant vector $e' \neq 0$ in the half space.

Proof. We first make some observations about words in F_2. Given a word w as in (i) we can find buffer words B and B' of the form $a^\ell b^\ell$ or $b^\ell a^\ell$ and a subgroup $F = \langle a^m, b^m\rangle$ with $m \gg \ell, |w|$ such that if $w' = BwB'$ and $y_1, y_2, \ldots, y_n \in F$ then in the reduced word for the element $x = y_1 w' y_2 w' \cdots y_n w'$ there is exactly one copy of w for each w' and no other copies of either w or w^{-1}. Note that the word $y_1 w' y_2 w' \cdots y_n w'$ may not be reduced and in its reduced version there may be cancellations in the w'. However, the buffer words will prevent these cancellations from reaching w. The restrictions on w ensure that w does not appear as a subword of some y_i. In particular, $|H(w')| = |e|$ and $H(xyw') = H(x) + xH(yw') = H(x) + xyH(w')$ for any $y \in F$.

For simplicity, normalize so that $|e| = 1$, so $|H(w')| = 1$. Assume that (ii) doesn't hold, and that H is bounded on F_2. Let F_w be the set of words of the form

$$y_1 w' y_2 w' \cdots y_n w', (y_i \in F)$$

and let $R = \sup_{x \in F_w} |H(x)| < \infty$. Let $\epsilon, \mu > 0$ be as in Proposition 3.2(iii). Choose an $x \in F_w$ such that $|H(x)| > R - \epsilon$. We will find a $y \in F$ such that $|H(xyw')| > R$ to obtain a contradiction since $xyw' \in F_w$.

Let ϕ be a linear functional of norm 1 such that $\phi(H(x)) = |H(x)|$. Let $\psi = \phi \circ x$. Since (ii) doesn't hold, there exists a $y \in F$ with $\psi(yH(w')) > -\mu$. (We are applying the negation of (ii) not to e but to $H(w')$, which is in the F_2-orbit of e, but it is easy to see that this follows from the corresponding fact for e by replacing F with a conjugate.) So, $\phi(xyH(w')) > -\mu$. Then by Proposition 3.2(iii), $|H(xyw')| = |H(x) + xyH(w')| \geq |H(x)| + \epsilon > R$, contradiction.

For an F-invariant vector in (ii), see the proof of Lemma 3.3. □

We give an application of Lemma 3.4.

Example 3.5. Choose an embedding $\rho : F_2 \subset U(2)$ so that every nontrivial element is conjugate to a matrix of the form

$$\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix}$$

with $t, s, \frac{t}{s}$ all irrational.

(Such representations can be constructed by noting that they form the complement of countably many proper subvarieties in $Hom(F_2, U(2))$.) Put $E = \mathbb{C}^2$.

Then any $H = H_{w,e}$ with $0 \neq e \in E$, $1 \neq w \in F_2$ is bounded on any cyclic subgroup, but many are globally unbounded. The second statement follows by noting that the orbit of any unit vector under a nontrivial cyclic subgroup is dense in a torus $S^1 \times S^1 \subset \mathbb{C}^2$, so (ii) of Lemma 3.4 fails, and (i) must hold. For the first statement, observe that for a fixed $g \in F$ the values $H(g^n)$ can be computed, up to a bounded error, by adding sums of the form

$$U_n = u(e) + gu(e) + \cdots + g^{n-1}u(e)$$

one for every g-orbit of occurrences of w or w^{-1} along the axis of g. Applying g we have

$$g(U_n) = gu(e) + \cdots + g^n u(e)$$

and so $|g(U_n) - U_n| \leq 2|e|$, which implies that $|U_n|$ is bounded, since $g : \mathbb{C}^2 \to \mathbb{C}^2$ moves every unit vector a definite amount. It follows $H(g^n)$ is bounded on n. This gives an isometric quasi-action of F_2 on $\mathbb{C}^2$ or $\mathbb{R}^4$ with unbounded orbits, but with every cyclic subgroup having bounded orbits.

In fact, since $H^1(F_2; \rho) \neq 0$, it follows that there are *isometric* actions of F_2 on $\mathbb{R}^4$ with unbounded orbits and with every element fixing a point.

The following is our basic method of detecting bounded cocycles. In the presence of reflexivity of the Banach space, bounded isometric actions have fixed points. Thus a cocycle $G : F_2 \to E$ is bounded if and only if for some $v \in E$ (a fixed point of the action) we have $G(g) = v - \rho(g)v$ for every $g \in F_2$.

Lemma 3.6. *Let ρ be a unitary representation of F_2 on a reflexive Banach space E and G a cocycle that is bounded on $\langle a^2, b\rangle$ and $\langle a^3, b\rangle$. Then one of the following holds.*

(i) G is bounded on F_2, or

(ii) There is a free subgroup $F \subset F_2$ with $F \cong F_2$ such that $\rho|_F$ fixes a nonzero vector in E.

Proof. The cocycle G induces an action of F_2 on E by affine isometries and the image of G is the orbit of 0 under this action. If the restriction of this action to $\langle a^2, b\rangle$ is bounded (with respect to the norm topology) then the convex hull of the orbit (in the weak topology) will be $\langle a^2, b\rangle$-invariant and compact since E is reflexive so by the Ryll-Nardzewski fixed point theorem $\langle a^2, b\rangle$ will have a fixed point. Thus $Fix(a^2) \cap Fix(b) \neq \emptyset$. If this intersection is not a single point then (ii) holds since the representation ρ restricted to $F = \langle a^2, b\rangle$ fixes the difference of any two vectors in the intersection. (ρ is the derivative!) Similarly, (ii) holds if $Fix(a^3) \cap Fix(b) \neq \emptyset$ is not a single point. Now suppose each intersection is a single point. If the two intersections coincide then the intersection point is fixed by both $a = a^3(a^2)^{-1}$ and b, thus by the whole group F_2, which implies that G is bounded. If the intersections are distinct then $F = \langle a^6, b\rangle$ fixes two distinct points, so (ii) holds as before. □

3.3. Detecting essentiality and proof of Theorem 1.1. We now show that under suitable conditions our quasi-cocycles are essential. We consider two cases. If there is a free subgroup that fixes a nonzero vector $e \in E$, the argument essentially goes back to Brooks, since in this case we restrict to the trivial representation. This case is presented first.

Proposition 3.7. *Let ρ be a unitary representation of F_2 in a reflexive Banach space E and let F be a rank two free subgroup such that $\rho|_F$ has an invariant vector $e \neq 0$. Then quasi-cocycles of the form $H_{w,e}$ where w is a reduced word span an infinite dimensional subspace of $\widetilde{QC}(F_2;\rho)$.*

Proof. After possibly conjugating F we can assume that the minimal F-tree contains the identity in the Cayley graph for F_2 and allows us to find cyclically reduced words α and β in F such that the concatenation

$$w_k = \alpha^k\beta^k\alpha^k\beta^k$$

is cyclically reduced. Furthermore we can assume that α and β generate F. Let $H_k = H_{w_k,e}$. By Lemma 3.3 there exists an F-invariant (continuous) linear functional ϕ with $\phi(e) \geq \mu > 0$.

Then the restriction to F of the composition $\phi \circ G$ with any co-cycle G is a homomorphism, and similarly the restriction of the composition $\phi \circ H$ to F with any quasi-co-cycle H is a quasi-morphism.

We will show that the sequence $H_1, H_2, \cdots$ represents linearly independent elements in $\widetilde{QC}(F_2;\rho)$. Indeed, if $H = H_k - c_1H_1 - \cdots - c_{k-1}H_{k-1}$, with $1 < k$, for any constants c_i then the quasi-morphism $\phi \circ H$ on F is 0 on the powers of α and β, so if a co-cycle G is boundedly close H, then the homomorphism $\phi \circ G$ on F must be bounded, and therefore zero, on powers of α and β. Therefore $\phi \circ G$ is trivial when restricted to F. On the other hand a staightforward calculation shows that $\phi \circ H(w_k^n) \geq n\mu$ so $\phi \circ H$ is unbounded on F and H and G cannot be boundedly close. We showed that H is non-trivial in $\widetilde{QC}(F_2;\rho)$, so $H_1, H_2, \ldots, H_k$ are linearly independent. □

We now consider the opposite case when no reduction to the trivial representation is possible.

Proposition 3.8. *Let ρ be a unitary representation of $F_2 = \langle a, b\rangle$ on a uniformly convex Banach space and assume that no nonabelian subgroup of F_2 fixes a nonzero vector. Then for any fixed $e \neq 0$ the quasi-cocycles of the form $H_{w,e}$ span an infinite dimensional subspace of $\widetilde{QC}(F_2;\rho)$, where w ranges over cyclically reduced words.*

Proof. Let $w_m = a^{5m}b^{5m}a^{7m}b^{7m}$, $m \geq 1$, and $gcd(m,6) = 1$. By Lemma 3.4, $H_m = H_{w_m,e}$ is unbounded. Furthermore H_m is 0 on the subgroups $\langle a^2, b\rangle$ and $\langle a^3, b\rangle$ listed in Lemma 3.6.

We claim that those H_m's are linearly independent in $\widetilde{QC}(F_2;\rho)$. Fix m and let $H = H_m - \sum_{i<m} c_i H_i$ for constants c_i. Then H is also unbounded, since the H_i for $i < m$ are visibly 0 on all words in F_{w_m}, the set given in the proof of Lemma 3.4, but H_m is unbounded on F_{w_m}. H is bounded on $\langle a^2, b\rangle$ and $\langle a^3, b\rangle$.

Suppose H differs from a cocycle G by a bounded function. Then G is also bounded on the subgroups $\langle a^2, b\rangle$ and $\langle a^3, b\rangle$, therefore G is bounded on F_2 since (i) must hold in Lemma 3.6. So, H is bounded on F_2, contradiction. We showed that $H_i, i \leq m$ are linearly independent in $\widetilde{QC}(F_2;\rho)$. □

Theorem 1.1 now follows immediately.

Theorem 3.9. *Let ρ be a unitary representation of F_2 on a uniformly convex Banach space $E \neq 0$. Then* $\dim \widetilde{QC}(F_2;\rho) = \infty$.

Proof. If there is a rank two free subgroup F in F_2 with an F-invariant vector $e \neq 0$, then use Proposition 3.7 to produce an infinite dimensional subspace. Otherwise, use Proposition 3.8. □

We remark that Pascal Rolli has a new construction, different from the Brooks construction, that he showed in [22] produces nontrivial quasi-cocycles on F_2 (and some other groups) when the Banach space E is an ℓ^p-space (or finite dimensional).

Example 3.10. To see the importance of uniform convexity we will look more closely at the examples $\ell^\infty(F_2)$ of bounded functions and $\ell^\infty_0(F_2)$ of bounded functions that vanish at infinity.

(1) For the regular representation on $\ell^\infty(F_2)$ (or any group G) the constant functions determine a one-dimensional invariant subspace. In particular, any quasi-morphism canonically determines a quasi-cocycle with image in this invariant subspace. If the original quasi-morphism is essential one may expect that the associated quasi-cocycle is also essential. However, for any quasi-cocycle H we can define the function $H_0 : F_2 \to \ell^\infty(F_2)$ by

$$H_0(g)(f) = H(f)(f) - \rho(g)H(g^{-1}f)(f) = H(f)(f) - H(g^{-1}f)(g^{-1}f)$$

and then we can check that H_0 is a cocycle (essentially it is the coboundary of the 0-cochain defined by the function $f \mapsto H(f)(f)$) and that $\|H - H_0\|_\infty \leq \Delta(H)$. In particular, $\widetilde{QC}(F_2;\ell^\infty(F_2)) = 0$ and $H^2_b(F_2;\ell^\infty(F_2)) = 0$.

(2) For the regular representation of F_2 on $\ell^\infty_0(F_2)$ neither F_2 nor any non-trivial subgroup fixes a non-trivial subspace so we cannot, as in the $\ell^\infty(F_2)$ case, use quasi-morphisms to construct unbounded quasi-cocycles. Furthermore for some choices

of the vector e, the quasi-cocyle $H_{w,e}$ will be bounded. For example if $e \in \ell_0^\infty(F_2)$ is defined by

$$e(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

then $\|H_{w,e}(x)\|_\infty = 0$ or 1 depending on whether x does or doesn't contain a copy of w. More generally if $e \in \ell^1(F_2) \subset \ell_0^\infty(F_2)$ we have that $\|H_{w,e}(x)\|_\infty \leq \|e\|_1$. On the other hand if we define $f \in \ell_0^\infty(F_2)$ by

$$f(x) = \begin{cases} 1/n & x = w^{-n}, n > 0 \\ 0 & \text{otherwise} \end{cases}$$

then $|H_{w,f}(w^n)(id)| = \sum_{i=1}^n 1/i$ so $\|H_{w,f}(w^n)\|_\infty$ is unbounded. We can still construct the cocycle H_0 as in the previous paragraph where $H = H_{w,f}$ but this cocycle will not lie in $\ell_0^\infty(F_2)$. This example emphasizes an inherent difficulty in extending our results to a wider class of Banach spaces.

Note that $H_b^n(G;\ell^\infty(G)) = 0 (n \geq 1)$ for any group G [19, Proposition 7.4.1] since $\ell^\infty(G)$ is a "relatively injective" Banach G-module [19, Chapter II], so some assumption on the Banach space is necessary.

(3) We also note that $H_b^2(G;\ell_0^\infty(G)) = 0$ for any countable, *exact* group (e.g. $G = F_2$, see [21]). This can be seen as follows. First, since $\ell^\infty(G)$ is a relatively injective Banach G-module, $H_b^n(G;\ell^\infty(G)) = 0$ for all $n > 0$. From the long exact sequence in bounded cohomology [19, Proposition 8.2.1] induced by the short exact sequence $0 \to \ell_0^\infty(G) \to \ell^\infty(G) \to \ell^\infty(G)/\ell_0^\infty(G) \to 0$, it suffices to show $H_b^1(G,\ell^\infty(G)/\ell_0^\infty(G)) = 0$. But this holds if G is countable and exact [9, Theorem 3]. We thank Narutaka Ozawa for pointing out his work to us.

To show $H_b^n(G;\ell_0^\infty(G)) = 0$ for all $n > 1$ it suffices to know

$$H_b^n(G,\ell^\infty(G)/\ell_0^\infty(G)) = 0$$

for all $n > 0$. Ozawa informs us that this is also true.

4. Hyperbolically embedded subgroups

Before proving our main theorem we need a couple of straightforward lemmas.

Lemma 4.1. *Let ρ be a unitary representation of a group G on E and K a finite normal subgroup. Let $E' \subset E$ be the closed subspace of K-invariant vectors and ρ' the unitary representation of G on E' obtained by restriction. Then every (quasi)-cocyle in $QC(G;\rho)$ is a bounded distance from a (quasi)-cocyle in $QC(G;\rho')$*

Proof. We first define a linear projection $\pi : E \to E'$ by

$$\pi(x) = \frac{1}{|K|}\sum_{k \in K}\rho(k)x.$$

If H is a (quasi)-cocycle in $QC(G;\rho)$ then $\tilde{H} = \pi \circ H$ is a (quasi)-cocycle in $QC(G;\rho')$. We need to show that $\tilde{H}$ is at bounded distance from H.

Recall that H is the translational part of an isometric G(-quasi)-action on E. By the normality of K if two points in E are in the same G-orbit then their K-orbits are (quasi)-isometric. Since $H(G)$ is the G-orbit of 0 under this (quasi)-action and $H(Kg)$ is at bounded distance from the K-orbit of $H(g)$ we have that the K-orbits of points in the image of H are uniformly bounded, and so π moves points in $Im(H)$ a uniformly bounded amount. □

Corollary 4.2. *The natural map* $\widetilde{QC}(G;\rho') \to \widetilde{QC}(G;\rho)$ *is an isomorphism.*

Lemma 4.3. *Let* ρ *be a unitary representation of* $G \times K$ *on* E *such that* K *is finite and* ρ *restricted to the* K*-factor is trivial. Then there is a natural isomorphism from* $\widetilde{QC}(G \times K;\rho) \to \widetilde{QC}(G;\rho)$.

Proof. Given $H \in QC(G \times K;\rho)$ define $\tilde{H} \in QC(G;\rho)$ by $\tilde{H}(g) = H(g, id)$. The linear map defined by $H \mapsto \tilde{H}$ descends to a linear map $\widetilde{QC}(G \times K;\rho) \to \widetilde{QC}(G;\rho)$. Any quasi-cocycle in $QC(G;\rho)$ determines a quasi-cocylce in $QC(G \times K;\rho)$ by extending it to be constant on the K-factor. This also descends to a map $\widetilde{QC}(G;\rho) \to \widetilde{QC}(G \times K;\rho)$, which is an inverse of our first map since $\|H(g,k) - H(g,id)\| \leq \Delta(H) + C$ where $C = \max\{\|H(id,k)\| | k \in K\}$. Hence we have the desired isomorphism. □

In [11], Dahmani, Guiradel and Osin defined the notion of a *hyperbolically embedded subgroup*. For convenience we recount the definition here. Let G be a group, H a subgroup and $X \subset G$ such that $X \cup H$ generates G. Let $\Gamma(G, X \sqcup H)$ be the Cayley graph with generating set $X \sqcup H$. Then H is hyperbolically embedded in G if

- $\Gamma(G, X \sqcup H)$ is hyperbolic;
- For all $n > 0$ and $h \in H$ there are at most finitely many $h' \in H$ that can be connected to h in $\Gamma(G, X \sqcup H)$ by a path of length $\leq n$ with no edges in H.

A quasi-cocycle is *anti-symmetric* if

$$H(g^{-1}) = -\rho(g^{-1})H(g).$$

A cocycle automatically satisfies this condition. Furthermore every quasi-cocycle is a bounded distance from an anti-symmetric quasi-cocycle. (Simply replace $H(g)$ with $\frac{1}{2}(H(g) - \rho(g)H(g^{-1}))$.) We have the following important theorem of Hull and Osin.

Theorem 4.4 ([17]). *Let* G *be a group and* F *a hyperbolically embedded subgroup. Then there exists a linear map*

$$\iota : QC_{as}(F;\rho) \to QC_{as}(G;\rho)$$

such that if $H \in QC_{as}(F;\rho)$ *then* $H = \iota(H)|_F$. *In particular,* $\dim \widetilde{QC}(F;\rho) \leq \dim \widetilde{QC}(G;\rho)$.

The action of a group G on a metric space X is *acylindrical* if for all $B > 0$ there exist D, N such that if $x, y \in X$ and with $d(x,y) > D$ then there are at most N elements $g \in G$ with $d(x, gx) < B$ and $d(y, gy) < B$. A group G is *acylindrically hyperbolic* if it has an acylindrical, non-elementary, action on a δ-hyperbolic space. To apply the previous theorem we need the following result of Dahmani–Guirardel–Osin and Osin:

Theorem 4.5 ([11, 20]). *Let G be an acylindrically hyperbolic group and K the maximal finite normal subgroup. Then G contains a hyperbolically embedded copy of $F_2 \times K$.*

Remark 4.6. Theorem 4.5 is a combination of two theorems. In [20, Theorem 1.2], Osin proves that an acylindrically hyperbolical group contains a non-degenerate hyperbolically embedded subgroup. In [11, Theorem 2.24], Dahmani–Guirardel–Osin show that if G contains a non-degenerate hyperbolically embedded subgroup then it contains a hyperbolically embedded copy of $F_2 \times K$. We note that this latter theorem relies on the projection complex defined in [2].

Proof of Corollary 1.2. Let $E' \subset E$ be the subspace fixed by K and ρ' the restriction of ρ to E'. By assumption $\dim E' > 0$. By Theorem 4.5 there is a copy of $F_2 \times K$ hyperbolically embedded in G. By Lemma 4.3 and Theorem 3.9 we have that $\dim \widetilde{QC}(F_2 \times K; \rho') = \dim \widetilde{QC}(F_2, \rho') = \infty$. Corollary 4.2 implies that $\dim \widetilde{QC}(F_2 \times K; \rho) = \infty$. The corollary then follows from Theorem 4.4. □

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