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Nonnegatively curved Euclidean submanifolds in codimension two

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Abstract. We provide a classification of compact Euclidean submanifolds $M^n \subset \mathbb{R}^{n+2}$ with nonnegative sectional curvature, for $n \geq 3$. The classification is in terms of the induced metric (including the diffeomorphism classification of the manifold), and we study the structure of the immersions as well. In particular, we provide the first known example of a nonorientable quotient $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\mathbb{Z}_2 \subset \mathbb{R}^{n+2}$ with nonnegative curvature. For the 3-dimensional case, we show that either the universal cover is isometric to $\mathbb{S}^2 \times \mathbb{R}$, or M^3 is diffeomorphic to a lens space, and the complement of the (nonempty) set of flat points is isometric to a twisted cylinder $(N^2 \times \mathbb{R})/\mathbb{Z}$. As a consequence we conclude that, if the set of flat points is not too big, there exists a unique flat totally geodesic surface in M^3 whose complement is the union of one or two twisted cylinders over disks.

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1. Introduction

It is well known that a compact immersed positively curved hypersurface in Euclidean space is diffeomorphic to a sphere (and is in fact the boundary of a convex body), the diffeomorphism simply given by the Gauss map. A deeper result, that such a submanifold still has to be homotopy equivalent to a sphere in codimension 2, is due to A. Weinstein [20] and D. Moore [16]. However, the situation where the sectional curvature of the submanifold is only nonnegative is more delicate. Although it is still not known whether there exists an isometric immersion of $\mathbb{RP}^2$ into $\mathbb{R}^4$ with nonnegative curvature, the higher dimensional problem was studied in [2, 3]. Using recent results about the Ricci flow, we strengthen these by showing the following.

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Theorem 1. *Let $f : M^n \to \mathbb{R}^{n+2}$, $n \geq 3$, be an isometric immersion of a compact Riemannian manifold with nonnegative sectional curvature. Then one of the following holds:*

(a) *M^n is diffeomorphic to $\mathbb{S}^n$;*

(b) *M^n is isometric to a product metric on $\mathbb{S}^k \times \mathbb{S}^{n-k}$ for some $2 \leq k \leq n-2$, and f is the product embedding of two convex Euclidean hypersurfaces;*

(c) *M^n is isometric to $(\mathbb{S}^{n-1} \times \mathbb{R})/\Gamma$ with a product metric on $\mathbb{S}^{n-1} \times \mathbb{R}$ and $\Gamma \simeq \mathbb{Z}$ acting isometrically. As a manifold, M^n is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ if orientable, or diffeomorphic to the nonorientable quotient $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$ otherwise, where $\Delta\mathbb{Z}_2$ denotes some diagonal $\mathbb{Z}_2$ action;*

(d) *M^n is diffeomorphic to a 3 dimensional lens space $\mathbb{S}^3/\mathbb{Z}_k$.*

Not only is this statement stronger than previously known results, but thanks to the use of the Ricci flow its proof becomes substantially simpler as well. More importantly, in the process we will also provide new strong restrictions on the structure of these submanifolds. In fact, the particular nature of the submanifolds in case (c), and even if a nonorientable immersion exists, was unsettled in the literature. Furthermore, if case (d) is possible at all was never discussed. The main purpose of our paper is to address both issues.

Regarding case (a), there are plenty of immersions of $\mathbb{S}^n$ with nonnegative curvature in codimension two. For example, take any compact convex hypersurface and then a composition with a flat (not necessarily complete) hypersurface. On the other hand, the submanifolds in case (b) are isometrically rigid.

It was the study of the structure of the immersions that led us to the following non-orientable example in case (c).

Example 1: The nonorientable quotient $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$ embedded in $\mathbb{R}^{n+2}$. Consider a flat strip isometrically immersed in $\mathbb{R}^3$

$$\beta : \mathbb{R} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

such that $\beta(x_0 + 1, -x_1) = \beta(x_0, x_1)$. The image is a flat Moebius band immersed or embedded in $\mathbb{R}^3$. Large families of analytic Moebius bands of this type, together with some classification results, have been given in [10, 19, 22].

The product immersion of β with the identity gives a flat hypersurface

$$h = \beta \times \mathrm{Id}_{\mathbb{R}^{n-1}} : \mathbb{R} \times (-\epsilon, \epsilon) \times \mathbb{R}^{n-1} \to \mathbb{R}^3 \times \mathbb{R}^{n-1} = \mathbb{R}^{n+2}.$$

Using a convex hypersurface $g\colon N^{n-1} \cong \mathbb{S}^{n-1} \to (-\epsilon, \epsilon) \times \mathbb{R}^{n-1}$ invariant under the reflection in the first coordinate of $\mathbb{R}^{n-1}$, we define the cylinder

$$\mathrm{Id}_{\mathbb{R}} \times g\colon \mathbb{R} \times N^{n-1} \to \mathbb{R} \times (-\epsilon, \epsilon) \times \mathbb{R}^{n-1}.$$

The composition

$$f = h \circ (\mathrm{Id}_{\mathbb{R}} \times g)\colon \mathbb{R} \times N^{n-1} \to \mathbb{R}^{n+2}$$

then satisfies $f \circ \tau = f$, where τ is the map

$$\tau(x_0, x_1, x_2, \dots, x_n) = (x_0 + 1, -x_1, x_2, \dots, x_n).$$

Hence the image of f is isometric to the *twisted cylinder* $(\mathbb{R} \times N^{n-1})/\mathbb{Z}$, where $\mathbb{Z}$ is generated by the orientation reversing isometry of N^{n-1} induced by τ. Thus $h \circ f$ descends to the desired immersion

$$f' : (\mathbb{S}^1 \times N^{n-1})/\Delta\mathbb{Z}_2 \simeq (\mathbb{R} \times N^{n-1})/\mathbb{Z} \to \mathbb{R}^{n+2}.$$

Observe that we can also choose β as a cylinder and obtain immersions of

$$(N^{n-1} \times \mathbb{R})/\mathbb{Z} \simeq \mathbb{S}^{n-1} \times \mathbb{S}^1$$

which are not products of immersions as in part (b). Actually, they are not even locally product of immersions, as is the case in Example 2 below.

We say that an isometric immersion f is a *composition (of j)* when $f = h \circ j$, where $j : M^n \to N_0^{n+1}$ is an isometric immersion into a (not necessarily complete) flat Euclidean hypersurface $h\colon N_0^{n+1} \to \mathbb{R}^{n+2}$. We will see that all submanifolds in Theorem 1 (c) are almost everywhere compositions of a cylinder over a strictly convex Euclidean hypersurface when the Ricci curvature is 2-positive. For complete simply connected nowhere flat nonnegatively curved manifolds which split off a line, this structure of a composition was shown in [4] under certain regularity assumptions. It is not clear though what additional restrictions hold, or whether regularity is necessary, if the immersion f descends to a compact quotient, as required in Theorem 1 (c).

The following example, where the induced intrinsic metric is well known in the theory of graph manifolds, illustrates the structure necessary for case (d) to occur.

Example 2: The switched $\mathbb{S}^3$ in $\mathbb{R}^5$. Consider a closed strictly convex hemisphere inside a closed halfspace,

$$\mathbb{S}^2_+ \subset \mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}_+,$$

such that the boundary is a closed geodesic in $\mathbb{S}^2_+$ along which the Gauss curvature vanishes to infinite order, and with image contained in $\mathbb{R}^2 \times \{0\}$. Its product with another $\mathbb{S}^1 \subset \mathbb{R}^2$ gives the nonnegatively curved three manifold

$$N^3_+ = \mathbb{S}^2_+ \times \mathbb{S}^1 \subset \mathbb{R}^5_+ = \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \times \mathbb{R}^2,$$

whose boundary is the totally geodesic flat torus

$$T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \{0\} \times \mathbb{R}^2 = \mathbb{R}^4 \subset \mathbb{R}^5.$$

Now, reversing the role of the $\mathbb{R}^2$ factors we similarly construct another manifold

$$N_-^3 = \mathbb{S}^1 \times \mathbb{S}_-^2 \subset \mathbb{R}_-^5 = \mathbb{R}^2 \times \mathbb{R}_{\leq 0} \times \mathbb{R}^2$$

with the same boundary $T^2 \subset \mathbb{R}^4$. Thus

$$M^3 = N_+^3 \cup N_-^3 \subset \mathbb{R}^5$$

gives an embedded Euclidean submanifold with nonnegative sectional curvature:

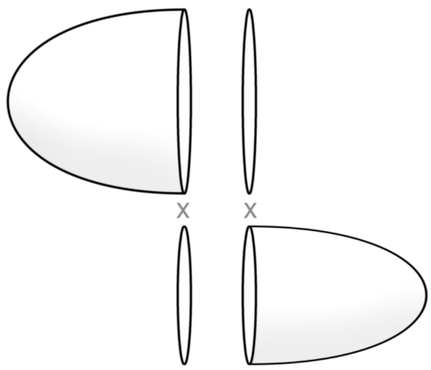

Figure 1. A switched $\mathbb{S}^3$ isometrically embedded in $\mathbb{R}^5$.

As a smooth manifold M^3 is diffeomorphic to the 3-sphere since this is the usual description of $\mathbb{S}^3$ as the union of 2 solid tori. The set of flat points of M^3 contains the totally geodesic flat torus $T^2 \subset M^3$ which disconnects M^3. Moreover, each connected component of $M^3 \setminus T^2$ is obtained as a composition of the cylinder $\mathbb{R} \times \mathbb{S}_+^2 \subset \mathbb{R}^4$ with a local isometric immersion from $\mathbb{R}^4$ into $\mathbb{R}^5$. By replacing the torus T^2 with a small flat cylinder $C_\epsilon = \mathbb{S}^1 \times [0, \epsilon] \times \mathbb{S}^1 \subset \mathbb{R}^5$, we obtain

$$\mathbb{S}^3 \cong M_\epsilon^3 = N_+^3 \cup C_\epsilon \cup N_-^3 \subset \mathbb{R}^5.$$

The resulting embeddings $M_\epsilon^3 \subset \mathbb{R}^5$ for $\epsilon \geq 0$ have thus the following properties:

- M_ϵ^3 is compact with nonnegative sectional curvature;
- The set of flat points contains a totally geodesic flat torus that disconnects M_ϵ^3;
- M_ϵ^3 is locally an isometric product of two manifolds for $\epsilon > 0$ (almost everywhere for $\epsilon = 0$);
- The embedding $M_\epsilon^3 \subset \mathbb{R}^5$ is locally, yet not globally, a composition of a cylindrical hypersurface for $\epsilon > 0$ (almost everywhere for $\epsilon = 0$);
- M_ϵ^3 has no points with positive Ricci curvature;
- M_ϵ^3 has 2-positive Ricci curvature outside the set of flat points.

The submanifold M^3_ϵ can be changed further so that the boundary of the set of flat points is not regular, and arbitrarily complicated sets of flat points can be introduced in $\mathbb{S}^2_\pm$ as well. Observe that, by taking the two disks $\mathbb{S}^2_\pm$ with strictly convex boundaries as plane curves, all these properties can be achieved even with 1-regular full embeddings, i.e. with the dimension of the space spanned by the second fundamental form having constant dimension 2 everywhere.

The following result recovers some of this structure for any immersion of a lens space with nonnegative sectional curvature.

Theorem 2. *Let $f : M^3 = \mathbb{S}^3/\mathbb{Z}_k \to \mathbb{R}^5$ with $k > 1$ be an isometric immersion with nonnegative sectional curvature. Then the set M_0 of flat points of M^3 is nonempty, and $M^3 \setminus M_0$ is isometric to a twisted cylinder $(N^2 \times \mathbb{R})/\mathbb{Z}$, where N^2 is a (not necessarily connected) surface with positive Gaussian curvature. Moreover, f is a composition almost everywhere on $M^3 \setminus M_0$.*

The natural intrinsic problem raised by this result is to try to understand how twisted cylinders can be glued together with flat regions in order to build a compact manifold. One way is to glue the cylinders through compact totally geodesic hypersurfaces, as we did for M^3_ϵ in Example 2 above. In [13] we study this intrinsic structure under the additional assumption that the set of nonflat points $M \setminus M_0$ is dense and locally finite (without assuming that the curvature is necessarily nonnegative). The main result in [13] is that under this assumption there exist pairwise disjoint flat totally geodesic compact surfaces in M^3 whose complement is a union of twisted cylinders. We also give examples of metrics on T^3 which show that the assumption on the nonflat points is necessary. In the case of nonnegative curvature, Theorem B above, together with Theorem C in [13], implies that:

Corollary. *Let $f : M^3 = \mathbb{S}^3/\mathbb{Z}_k \to \mathbb{R}^5$, $k > 1$, be an isometric immersion with nonnegative sectional curvature, such that $M^3 \setminus M_0$ is dense and its number of connected components is locally finite. Then, either M_0 contains a unique flat totally geodesic torus T^2 that disconnects M^3 and $M^3 = V_1 \sqcup T^2 \sqcup V_2$, or M^3 is the lens space $L(4p, 2p+1)$ and M_0 contains a unique flat totally geodesic Klein bottle K^2 satisfying $M^3 = V_1 \sqcup K^2$. In both cases, $V_i = (D^2_i \times \mathbb{R})/\mathbb{Z}$ is a twisted cylinder, where D^2_i is a disc with nonnegative Gaussian curvature and boundary a closed geodesic.*

Yet, despite the rigid structure described above, we still do not know if a nonnegatively curved lens space in $\mathbb{R}^5$ actually exists.

For the proof of Theorem 1 we will need a fundamental property of our submanifolds that was proved in [7] and [16]. In order to state it, recall that we have the *type numbers* τ_k of the immersion: if $h_v = \langle v, \cdot \rangle : M^n \to \mathbb{R}$ is a height function, and $\mu_k(v)$ the number of critical points of h_v with index k, then τ_k is the average of $\mu_k(v)$, integrated over the unit sphere in $\mathbb{R}^{n+2}$. If M^n is compact with nonnegative sectional curvature immersed in $\mathbb{R}^{n+2}$, then the type numbers satisfy

$$\tau_0 + \tau_n \geq \tau_1 + \cdots + \tau_{n-1}.$$

Furthermore, if the inequality is strict, we will see in Section 3 that M^n is diffeomorphic to a sphere. We are thus mainly interested in the case of equality, and call such immersions *wide*. In the situation of Theorem 2, we will show that

$$\tau_k = \frac{1}{8\pi^2} \int_{N^2} K(x)\kappa_g(x)\,\mathrm{d}x$$

for $0 \le k \le 3$, where $\kappa_g(x)$ denotes the total curvature of the image of the $\mathbb{R}$ factor through $x \in N^2$, and K the Gaussian curvature of N^2.

For codimension two Euclidean submanifolds with nonnegative curvature, A. Weinstein [20] showed that at every point there exists a basis $\{\xi, \eta\}$ of the normal space such that its shape operators A and B are nonnegative, and hence the curvature operator is nonnegative. The crucial new property that we will prove is that for a wide immersion we have in addition that

$$\text{either } \ker A \cap \ker B \neq 0 \quad \text{or} \quad \ker A \oplus \ker B = T_pM \quad \text{with} \quad A, B \neq 0,$$

and call the (not necessarily complete) immersions which satisfy only this property *locally wide*. These immersions already have strong rigidity properties. We examine their behavior on the open subsets

$$U_k = \big\{p \in M^n : \operatorname{rank} A(p) = n-k,\ \operatorname{rank} B(p) = k \text{ and } \ker A(p) \cap \ker B(p) = 0\big\},$$

which fill out the complement of the points of positive relative nullity. We will show in particular that:

- If $2 \le k \le [n/2]$, then $f|_{U_k}$ is locally a product immersion of two strictly convex Euclidean hypersurfaces;
- If $\pi : \tilde{U}_1 \to U_1$ is the universal cover, then $(f \circ \pi)$ is a composition of a convex Euclidean hypersurface with constant index of relative nullity one (see Theorem 3);
- If M^n is complete and has no points of positive Ricci curvature, but has 2-positive Ricci curvature, then $\tilde{M}^n = \mathbb{S}^{n-1} \times \mathbb{R}$, where $g : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is a compact strictly convex hypersurface.

2. Preliminaries

Let $f : M^n \to \mathbb{R}^{n+2}$, $n \ge 3$, be an isometric immersion of a Riemannian manifold with nonnegative sectional curvature. A. Weinstein observed that, in this case, the curvature operator $\hat{R}$ is nonnegative. Indeed, if α is the second fundamental form, the Gauss equations imply that $\langle \alpha(X,X), \alpha(Y,Y)\rangle \ge 0$ at every $p \in M$. Hence, in the connected subset $\{\alpha(X,X) : X \in T_pM\} \subset T_p^{\perp}M$ intersected with the unit circle all points have distance at most $\pi/2$. This implies that this set lies in the first quadrant with respect to some orthonormal basis $\{\xi, \eta\}$, while it lies in its interior if it has positive sectional curvature. We thus have the shape operators

$$A = A_\xi \quad \text{and} \quad B = A_\eta, \quad \text{with} \quad A \geq 0 \quad \text{and} \quad B \geq 0. \tag{2.1}$$

since $\langle A_\tau(X), Y\rangle = \langle \alpha(X, Y), \tau\rangle$.

By the Gauss equations we have $\hat{R} = \Lambda^2 A + \Lambda^2 B$, and hence $\hat{R} \geq 0$ (and $\hat{R} > 0$ if the sectional curvature is positive). Furthermore, the cone of shape operators which are positive semidefinite contains the first quadrant, while the cone of shape operators which are negative semidefinite contains the third.

We will see that the interesting immersions are quite rigid as a consequence of the following purely algebraic key result which will be applied to the shape operators in (2.1).

Lemma 2.1. *Let A and B be nonnegative semidefinite self–adjoint operators on an Euclidean space V. Then, $|\det(A + tB)| \geq |\det(A - tB)|$, $\forall\, t > 0$. Moreover, equality holds for all $t > 0$ if and only if either* $\ker A \cap \ker B \neq 0$, *or* $\ker A \oplus \ker B = V$.

Proof. First, observe that the lemma follows easily if one of the operators is nonsingular. Indeed, if, say, B is invertible, $\det(A + tB) = \det(B)\phi(t)$, where ϕ is the characteristic polynomial of the positive semidefinite operator $B^{-1/2}AB^{-1/2}$. Since all the coefficients of ϕ are nonnegative the inequality holds, with equality if and only if $A = 0$. Therefore, we can assume that both A and B are singular.

Let $K = \ker B$, and decompose A as its block endomorphisms (C, D, D^t, E), according to the orthogonal decomposition $K \oplus K^\perp$,

$$A(u, v) = (Cu + Dv, D^t u + Ev), \quad \forall u \in K, v \in K^\perp.$$

Since $A \geq 0$, we have $C \geq 0$ and $E \geq 0$.

For $u \in \ker C$ we get $0 \leq \langle A(su, v), (su, v)\rangle = 2s\langle D^t u, v\rangle + \langle Ev, v\rangle$, for all $s \in \mathbb{R}$. Hence, $D^t u = 0$ and $u \in K \cap \ker A$. Therefore either $C > 0$ or the kernels intersect and equality holds. So let us then assume further that $C > 0$ and $K \cap \ker A = 0$.

If we call $\hat{B}$ the symmetric operator B restricted and projected to $K^\perp$, then $\hat{B} > 0$ and, for all $u \in K, v \in K^\perp$,

$$(A + tB)(u, v) = (Cu + Dv, D^t u + (E + t\hat{B})v).$$

Thus,

$$\det(A + tB) = \det(C) \det\left(E + t\hat{B} - D^t C^{-1} D\right) = \det(C) \det(\hat{A} + t\hat{B}),$$

where $\hat{A}$ is the symmetric operator on $K^\perp$ given by $\hat{A} = E - D^t C^{-1} D$. Now, observe that $\hat{A}$ is nonnegative since

$$\begin{aligned}\langle (E - D^t C^{-1} D)v, v\rangle &= \langle Ev, v\rangle - \langle C^{-1} Dv, Dv\rangle \\ &= \langle A(-C^{-1}Dv, v), (-C^{-1}Dv, v)\rangle \geq 0.\end{aligned}$$

This gives the inequality in the lemma as in the nonsingular case. Moreover, this computation also shows that $v \in \ker \hat{A}$ if and only if $(-C^{-1}Dv, v) \in \ker A$. Thus, $\det(A + tB)$ is again, up to a positive constant, the characteristic polynomial of the positive semidefinite operator $\hat{B}^{-1/2}\hat{A}\hat{B}^{-1/2}$, which is an odd or even function if and only if $\hat{A} = 0$. In this case, $\dim \ker A = \dim K^{\perp}$, as claimed.

The converse of the equality case is straightforward. □

Since $\hat{R} \geq 0$ we can use the following rigidity result, where hol denotes the Lie algebra of the holonomy group of M (see [21, Theorem 1.13]):

Proposition 2.2. *If M is a compact and simply connected Riemannian manifold with $\hat{R} \geq 0$, then one of the following holds:*

- $\mathrm{hol} = \mathfrak{so}(n)$ *and M is diffeomorphic to $\mathbb{S}^n$, with the Ricci flow converging to a metric with constant positive curvature;*
- $\mathrm{hol} = \mathfrak{u}(n)$ *and M is diffeomorphic to $\mathbb{C}\mathbb{P}^n$, with the Ricci flow converging to the Fubini Study metric;*
- *M is isometric to an irreducible symmetric space G/K with* $\mathrm{hol} = \mathfrak{k}$*;*
- *Arbitrary isometric products of the above cases.*

Notice in particular that if $\mathrm{hol} = \mathfrak{so}(n)$ in the above, then M is diffeomorphic to a sphere since, if the Lie algebra $\mathfrak{k}$ of K for a symmetric space G/K satisfies $\mathfrak{k} = \mathfrak{so}(n)$, then G/K is in fact isometric to a round n-sphere.

We will also need the next result due to Bishop [5] about the holonomy group of a general compact Euclidean submanifold in codimension two.

Proposition 2.3. *If M^n is a compact submanifold immersed in $\mathbb{R}^{n+2}$, then either $n = 4$ and $\mathrm{hol} = \mathfrak{u}(2)$, or $\mathrm{hol} = \mathfrak{so}(k) \times \mathfrak{so}(n-k)$ for some $0 \leq k \leq [n/2]$.*

Here $k = 0$ corresponds to $\mathrm{hol} = \mathfrak{so}(n)$ and $k = 1$ to $\mathrm{hol} = \mathfrak{so}(n-1)$, i.e. the manifold splits a flat factor locally. In particular, there can be at most a one dimensional flat factor.

3. Rigidity

In this section we provide the proof of Theorem 1. In the process, we reprove, simplify and strengthen known results about compact Euclidean submanifolds in codimension 2 with nonnegative sectional curvature by using Proposition 2.2. Along the way, we will see how the proof implies further rigidity of the immersion.

Let f and M^n be as in Theorem 1. Since M^n is compact with nonnegative sectional curvature, a well known consequence of the Cheeger–Gromoll splitting theorem, see [8, Theorem C], implies that its universal cover $\tilde{M}^n$ splits isometrically as

$$\tilde{M}^n = \mathbb{R}^{\ell} \times N^{n-\ell}, \tag{3.1}$$

where $N^{n-\ell}$ is compact and simply connected. Since the Lie algebra of the holonomy group of M and $\tilde{M}$ coincide, Proposition 2.3 implies that $\ell \leq 1$.

Now observe that in our context $n = 4$ and $\mathrm{hol} = u(2)$ is not possible in Proposition 2.3 since then the universal cover would be compact, and by Proposition 2.2 diffeomorphic to $\mathbb{C}\mathbb{P}^2$. But $\mathbb{C}\mathbb{P}^2$ does not admit an immersion in $\mathbb{R}^6$ as follows, e.g., from $p_1(\mathbb{C}\mathbb{P}^2) = 3$, the product formula for Pontrjagin classes, and $p_1(E) = e(E)^2$ for the normal 2 plane bundle E of the immersion.

Therefore, the existence of an isometric immersion implies that

$$\mathrm{hol} = \mathfrak{so}(k) \oplus \mathfrak{so}(n-k), \tag{3.2}$$

for some $0 \leq k \leq [n/2]$. We can now combine (3.1), (3.2) and Proposition 2.2. First, it follows that $\ell \leq 1$ and $\ell = 1$ if and only if $k = 1$.

If $\ell = k = 0$, then $\tilde{M}^n$ is compact and hence diffeomorphic to $\mathbb{S}^n$, while if $\ell = 0$ and $k \geq 2$, then $\tilde{M}^n$ is diffeomorphic to $\mathbb{S}^k \times \mathbb{S}^{n-k}$. In the latter case, by [1] the immersion must be a product immersion of convex hypersurfaces and, in particular, M^n itself is simply connected and diffeomorphic to $\mathbb{S}^k \times \mathbb{S}^{n-k}$.

If $\ell = k = 1$, and hence $\mathrm{hol} = \mathfrak{so}(n-1)$, (3.1) implies that $\tilde{M}^n$ splits isometrically $\tilde{M}^n = N^{n-1} \times \mathbb{R}$, with N^{n-1} compact and simply connected. But then $\mathrm{hol}(N^{n-1}) = \mathfrak{so}(n-1)$, and again by Proposition 2.2 we get that N^{n-1} is diffeomorphic to $\mathbb{S}^{n-1}$. Thus $M^n = (\mathbb{S}^{n-1} \times \mathbb{R})/\Gamma$ isometrically, with Γ a discrete group acting by isometries. We will see in Corollary 3.5 below that $\Gamma = \mathbb{Z}$ and that M^n is diffeomorphic to either $\mathbb{S}^{n-1} \times \mathbb{S}^1$ if M^n is orientable, or to the nonorientable quotient $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$ otherwise.

To conclude the proof of Theorem 1, it remains to study the possibilities for the fundamental group when $\ell = k = 0$ and $\tilde{M}^n = \mathbb{S}^n$, or when $\ell = k = 1$ and $\tilde{M}^n = \mathbb{S}^{n-1} \times \mathbb{R}$ for some metrics of nonnegative curvature on the spheres. In order to do this, we first recall the relationship, due to Kuiper, between height functions and shape operators of compact Euclidean submanifolds.

Following [16], for each $v \in \mathbb{S}^{n+1}$ consider the height function

$$h_v = \langle v, \cdot \rangle : M^n \to \mathbb{R}$$

and define

$$\mu_k(v) = \text{number of critical points of } h_v \text{ with index } k,$$

and

$$\tau_k = \frac{1}{\mathrm{vol}(\mathbb{S}^{n+1})} \int_{\mathbb{S}^{n+1}} \mu_k(v)\, \mathrm{d}v.$$

Since on a set of full measure the height functions are nondegenerate, we can apply the usual Morse inequalities for Morse functions. After integrating them we get

$$\tau_k \geq b_k$$

and

$$\sum_{k=0}^{\ell}(-1)^{\ell-k}\,\tau_k \geq \sum_{k=0}^{\ell}(-1)^{\ell-k}\,b_k, \quad \forall\,\ell = 1,\dots n,$$

where $b_k = \dim H_k(M, F)$ for some field F.

Clearly, p is a critical point of h_v if and only if $v = \beta$ for some $(p,\beta) \in T_1^{\perp}M$, where $T_1^{\perp}M$ is the normal bundle of the immersion. Moreover, p is a nondegenerate critical point of h_β if and only if A_β is nonsingular, and the index of the critical point is equal to $\operatorname{ind}(A_\beta)$, the number of negative eigenvalues of A_β. We now use the Gauss map $G\colon T_1^{\perp}M \to \mathbb{S}^{n+1}$ with $G(p,\beta) = \beta$, which for compact submanifolds is surjective. Since $G_{*(p,\beta)} = \operatorname{diag}(A_\beta, \mathrm{Id})$, we have that p is a nondegenerate critical point of h_β if and only if (p,β) is a regular point of G. Thus if $C \subset \mathbb{S}^{n+1}$ is the set of regular values of G (an open and dense set of full measure), then h_v with $v \in C$ is a Morse function and hence μ_k is constant on every connected component of C. Furthermore, for $v \in C$, the set $G^{-1}(v)$ contains $\mu_k(v)$ many points of index k. Thus by change of variables,

$$\tau_k = \int_C \mu_k(v)\,\mathrm{d}v = \frac{1}{\operatorname{vol}(\mathbb{S}^{n+1})}\int_{\operatorname{ind}(A_\beta)=k} \left|\det A_\beta\right|\,\mathrm{dvol}_{T_1^{\perp}M}. \tag{3.3}$$

Since $\mu_k(v) = \mu_{n-k}(-v)$, we also have $\tau_{n-k} = \tau_k$ for all k.

We proceed by applying Lemma 2.1 to the shape operators A, B in (2.1) in order to estimate $\left|\det A_\beta\right|$ for $(p,\beta) \in T_1^{\perp}M$, by first integrating over a circle in the normal space $T_p^{\perp}M$, and then over $p \in M$. For a fixed normal space $T_p^{\perp}M$, Lemma 2.1 implies that

$$|\det(\cos\theta A + \sin\theta B)| \geq |\det(\cos\theta A - \sin\theta B)| \quad \text{for all } \ 0 \leq \theta \leq \frac{\pi}{2}. \tag{3.4}$$

Furthermore, if a regular value $\beta = \cos(\theta)\xi + \sin(\theta)\eta$ lies in the first quadrant, then $\operatorname{ind}(A_\beta) = 0$, and if it lies in the third quadrant then $\operatorname{ind}(A_\beta) = n$, whereas all saddle points (and possibly some local maxima and minima as well) lie in the second and fourth quadrants. Thus (3.4), together with $\tau_{n-k} = \tau_k$, implies that

$$\tau_0 + \tau_n \geq \tau_1 + \cdots + \tau_{n-1}. \tag{3.5}$$

As we will see, in the case of strict inequality M is diffeomorphic to a sphere. We are thus interested from now on in the equality case, where one expects a certain amount of rigidity. This motivates our next key definition.

Definition. Given a compact Riemannian manifold M^n with nonnegative sectional curvature, an isometric immersion $f : M^n \to \mathbb{R}^{n+2}$ is said to be *wide* if

$$\tau_0 + \tau_n = \tau_1 + \cdots + \tau_{n-1}.$$

We can now use Lemma 2.1 to express the rigidity in terms of the shape operators A and B in (2.1).

Proposition 3.1. *If $f\colon M^n \to \mathbb{R}^{n+2}$ is wide, then at each point $p \in M^n$ we have that either* $\ker A \oplus \ker B = T_pM$ *with* $A \neq 0$ *and* $B \neq 0$, *or* $\ker A \cap \ker B \neq 0$.

Proof. Equality in (3.5) holds if and only if we have equality in (3.4) at every point and for every angle θ. Thus Lemma 2.1 implies that either $\ker A \oplus \ker B = T_pM$ or $\ker A \cap \ker B \neq 0$. Furthermore, if $A = 0$ at some point, then B is nonsingular, and hence in a neighborhood of the point as well. But then this neighborhood will not have any saddle points, and we would have a strict inequality in (3.5). □

Notice that, along the (open) set where the first case holds, the orthonormal basis $\{\xi, \eta\}$ in (2.1), and hence A and B, are unique and thus smooth. On the other hand, no such information can be obtained in the second case.

Altogether, we have the following improvement of a result in [16], where we obtain additional information on the structure of the immersion.

Proposition 3.2. *If M^n is compact, nonnegatively curved and immersed in Euclidean space in codimension* 2, *then we have for any coefficient field F over which M^n is orientable that*

$$b_1 + b_2 + \cdots + b_{n-1} \leq 2,$$

with equality if and only if

$$\tau_k = b_k \;\; \forall\, 2 \leq k \leq n-2, \;\; \tau_1 - \tau_0 = b_1 - 1 \text{ and } \tau_0 + \tau_n = \tau_1 + \cdots + \tau_{n-1}.$$

In particular, equality implies that the immersion is wide.

Proof. By the Morse inequalities $\tau_k \geq b_k$ for $k = 2, \ldots, n-2$ and $\tau_1 - \tau_0 \geq b_1 - b_0$, i.e. $b_1 \leq \tau_1 - \tau_0 + 1$. If M^n is compact and orientable mod F, then $b_k = b_{n-k}$ and, in general, $\tau_k = \tau_{n-k}$. Hence,

$$b_1 + (b_2 + \cdots + b_{n-2}) + b_{n-1} \leq \tau_1 - \tau_0 + 1 + (\tau_2 + \cdots + \tau_{n-2}) + \tau_{n-1} - \tau_n + 1,$$

which together with (3.5) yields

$$b_1 + b_2 + \cdots + b_{n-1} \leq \tau_1 \cdots + \tau_{n-1} - (\tau_0 + \tau_n) + 2 \leq 2,$$

with equality as claimed. □

So far we have shown that there are three situations: either $\tilde{M}^n \cong \mathbb{S}^n$ diffeomorphically, or $\tilde{M}^n$ splits isometrically as either $\tilde{M}^n \equiv \mathbb{S}^{n-1} \times \mathbb{R}$ or $M^n \equiv \mathbb{S}^k \times \mathbb{S}^{n-k}$, for certain metrics of nonnegative curvature on the spheres. As a corollary of Proposition 3.2, we conclude that in the first case M itself is a sphere when $n \geq 4$:

Corollary 3.3. *If $n \geq 4$ and $\pi_1(M^n)$ is finite, then M^n is simply connected. In particular, either M^n is diffeomorphic to $\mathbb{S}^n$, or isometric to a Riemannian product $\mathbb{S}^k \times \mathbb{S}^{n-k}$, $2 \leq k \leq n-2$, for certain metrics of nonnegative sectional curvature on the spheres. In the latter case, the immersion is a product of two convex hypersurfaces, and thus it is wide.*

Proof. Assume M^n is not simply connected, and let $g \in \pi_1(M^n)$ be an element of prime order q, so $\langle g\rangle \cong \mathbb{Z}_q \subseteq \pi_1(M)$. Then, the lens space $\mathbb{S}^n/\mathbb{Z}_q$ is a finite cover of M^n and with the covering metric it has an isometric immersion into $\mathbb{R}^{n+2}$. But $b_k(\mathbb{S}^n/\mathbb{Z}_q, \mathbb{Z}_q) = 1$ for all k which contradicts Proposition 3.2 if $n \geq 4$. Notice that if $q > 2$ the lens space is orientable, and if $q = 2$ it is orientable mod 2.

In the case where M^n is isometric to a Riemannian product $\mathbb{S}^k \times \mathbb{S}^{n-k}$, [1] implies that the immersion is a product of two convex hypersurfaces. One easily sees that in this case the immersion is wide. □

For the three dimensional case we have the following.

Corollary 3.4. *If $n = 3$ and $\pi_1(M^3) \neq 0$ is finite, then M^3 is diffeomorphic to a lens space $\mathbb{S}^3/\mathbb{Z}_k$, and the immersion is wide with $\tau_0 = \tau_1 = \tau_2 = \tau_3$. Furthermore, the height functions satisfy $\mu_0 = \mu_1$ as well as $\mu_2 = \mu_3$.*

Proof. M^3 is again covered by a lens space with $b_1 = b_2 = 1$ and we have equality in Proposition 3.2, which implies that $\tau_0 = \tau_1 = \tau_2 = \tau_3$ (they do not have to be 1, though). The Morse inequality tells us that $\mu_1(v) - \mu_0(v) \geq b_1 - b_0 = 0$ for a.e. $v \in \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, and since $\tau_1 = \tau_0$, it follows that $\mu_1(v) = \mu_0(v)$. From $\mu_k(v) = \mu_{n-k}(-v)$ we get $\mu_3 = \mu_2$ for a.e. v. Therefore for almost all v the Morse function h_v has the same number of critical points of index 0 and 1 (first for the lens space, and then for M^3 as well). But if this is the case, the cancellation result in [17] says that there exists another (abstract) Morse function on M^3 with only one critical point of index 0 and 1. In particular, this means that M^3 is a CW complex whose 1 skeleton is a circle, which implies by transversality that the map $\pi_1(S^1) \to \pi_1(M^3)$ induced by the inclusion is onto and hence $\pi_1(M^3)$ is finite cyclic. □

For infinite fundamental group, we have:

Corollary 3.5. *If $\pi_1(M^n)$ is infinite, then $\pi_1(M^n) \cong \mathbb{Z}$ is also cyclic. Moreover, $\tilde{M}^n = \mathbb{S}^{n-1} \times \mathbb{R}$ splits isometrically for some metric of nonnegative curvature in $\mathbb{S}^{n-1}$, and M^n is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ if M^n is orientable, or to the nonorientable quotient $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$ otherwise. In addition, $\tau_0 = \tau_1 = \tau_{n-1} = \tau_n$, and $\tau_2 = \cdots = \tau_{n-2} = 0$, and hence the immersion is wide. Furthermore, the height functions satisfy $\mu_0 = \mu_1$ as well as $\mu_{n-1} = \mu_n$, and $\mu_2 = \cdots = \mu_{n-2} = 0$.*

Proof. We already saw that under this assumption, M^n is diffeomorphic to $(\mathbb{S}^{n-1} \times \mathbb{R})/\Gamma$ for some group Γ acting properly discontinuously. Since the quotient is compact, there exists a subgroup $\Gamma' \simeq \mathbb{Z}$ which acts via translations on the $\mathbb{R}$

factor. Choose a covering $f: M^* \to M$ such that the image of the fundamental group under f_* is equal to Γ'. Thus $b_1(M^*, \mathbb{Z}_2) = b_{n-1}(M^*, \mathbb{Z}_2) = 1$, and we can apply Proposition 3.2 to the induced immersion of M^*. As in the proof of Corollary 3.4, it follows that for the height functions on M^* we have $\mu_1(v) = \mu_0(v)$ and $\mu_{n-1}(v) = \mu_n(v)$ for a.e. v. The same thus holds for the height functions of the immersion of M^n, and again as in the proof of Corollary 3.4, we see that $\pi_1(M)$ is cyclic, hence Γ is isomorphic to $\mathbb{Z}$. Finally, since the functions μ_i are nonnegative, and $\tau_2 = \cdots = \tau_{n-2} = 0$, it follows that $\mu_2 = \cdots = \mu_{n-2} = 0$ as well.

Now, projection onto the first factor gives rise to a fiber bundle $M^n \to \mathbb{S}^1$ with fiber $\mathbb{S}^{n-1}$ which is hence isomorphic to $\mathbb{S}^{n-1} \times [0,1]/(p,0) \sim (\sigma(p),1)$ for some diffeomorphism σ. If σ is orientation preserving, σ is homotopic to the identity and hence the bundle is trivial, in which case M^n is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. If on the other hand σ is orientation reversing, M^n is nonorientable and the orientable double cover is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and hence M^n is diffeomorphic to $(\mathbb{S}^{n-1} \times \mathbb{S}^1)/\Delta\mathbb{Z}_2$. Notice that, in both cases, the product structure is not necessarily isometric. □

The presence of points with positive curvatures imposes further restrictions.

Corollary 3.6. *If there exists a point with positive Ricci curvature, then either M^n is diffeomorphic to $\mathbb{S}^n$, or M^n is isometric to a product $\mathbb{S}^k \times \mathbb{S}^{n-k}$ and f is a product of two convex hypersurfaces, for some $2 \le k \le n-2$. In particular, if there is a point with positive sectional curvature, then M^n is diffeomorphic to $\mathbb{S}^n$.*

Proof. The fundamental group must be finite since otherwise Corollary 3.5 implies that $\tilde{M}^n$ splits off a real line. If $n = 3$, then Proposition 3.1 implies that at a point with Ric > 0, one of A, B, say A, has rank 2, and B has rank one. But then $\hat{R} = \Lambda^2 A + \Lambda^2 B = \Lambda^2 A$ and hence all 2-planes containing ker A have curvature 0, thus contradicting Ric > 0. In the remaining case Corollary 3.3 proves our claim. □

In particular, Corollaries 3.3, 3.4 and 3.5 imply the following.

Corollary 3.7. *If M^n is not diffeomorphic to a sphere, then the immersion is wide.*

4. Wide and locally wide immersions

In this section we obtain further information about the way the manifold M^n is immersed, that is, about the isometric immersion f. As we saw in Corollary 3.7, unless M^n is diffeomorphic to the sphere, for which the space of nonnegatively curved immersions is quite rich, the isometric immersion f must be wide. Hence the purpose of this section is to understand these immersions. We have chosen this approach since there are interesting wide immersions of spheres, as shown in Example 2.

We assume from now on that M^n is a Riemannian manifold with nonnegative sectional curvature, $n \geq 3$, and $f : M^n \to \mathbb{R}^{n+2}$ is an isometric immersion. To provide a deeper understanding of the local phenomena, we do not require M^n to be complete or compact unless otherwise stated. Instead, we only assume that f satisfies the local consequence in Proposition 3.1 of being wide. In other words, again following the notations in (2.1), we say that f is *locally wide* if, at every point $p \in M$, either

$$\ker A \cap \ker B \neq 0, \quad \text{or} \quad \ker A \oplus \ker B = T_pM \quad \text{with} \quad A, B \neq 0. \tag{4.1}$$

From now on we will also assume in the second case that rank $B \leq$ rank A.

Recall that the *index of nullity* μ and the *nullity* Γ of M are defined as

$$\Gamma(p) = \{u \in T_pM : R(a,b)u = 0 \;\; \forall\, a, b \in T_pM\}, \quad \text{and} \quad \mu(p) = \dim \Gamma(p).$$

Furthermore, we have the *index of relative nullity* $\nu(p)$ of f at p defined as the dimension of the *relative nullity* $\Delta(p)$ of f at p,

$$\Delta(p) = \{u \in T_pM : \alpha(u,v) = 0 \;\; \forall\, v \in T_pM\} = \ker A(p) \cap \ker B(p).$$

Notice that μ is an intrinsic invariant, while ν is extrinsic. By the Gauss equation, $\Delta \subseteq \Gamma$ and hence $\nu \leq \mu$. We will also use the well know fact (see e.g. [9, 14]) that the nullity distribution, as well as the relative nullity distribution, is integrable on any open set where it has constant dimension, and its leaves are totally geodesic in M^n. In the case of the relative nullity, the images of the leaves under f are open subsets of affine subspaces of the Euclidean space as well. In addition, if M^n is complete, the leaves of both distributions are complete on the open set of minimal nullity or minimal relative nullity.

Choosing an orthonormal normal frame $\{\xi, \eta\}$ as in (2.1), by (4.1) M^n can be written as the disjoint union

$$M^n = K \cup U_1 \cup \cdots \cup U_{[n/2]},$$

where K is the subset of positive index of relative nullity,

$$K = \{p \in M^n : \nu(p) > 0\},$$

and, for $1 \leq k \leq [n/2]$, U_k is given by

$$U_k = \{p \in M^n : \text{rank } A(p) = n-k, \text{ rank } B(p) = k \text{ and } \ker A(p) \cap \ker B(p) = 0\}.$$

From the continuity of the eigenvalues and $\hat{R} = \Lambda^2 A + \Lambda^2 B$ it follows that:

- K is closed, and the sets U_k are open with $\partial U_k \subset K$;
- $U_2 \cup \cdots \cup U_{[n/2]}$ is the set of all points with positive Ricci curvature, Ric > 0;
- On U_1 we have $\hat{R} = \Lambda^2 A$ and hence $\Gamma = \ker A$ and $\mu = 1$, $\nu = 0$.

Remark 4.1. The above discussion and Corollary 3.6 imply that, if M^n is compact and f is wide, then f is isometric to a product of convex hypersurfaces unless $U_k = \emptyset$ for all $k \geq 2$. Therefore, for the immersions of types (c) and (d) in Theorem 1 we have that $M^n = K \cup U_1$, and, in particular, $\mu \geq 1$ everywhere.

In what follows, on $M^n \setminus K$ where we have seen that the frame $\{\xi, \eta\}$ is unique and smooth, w will denote the normal connection form of f given by

$$w(X) := \langle \nabla^{\perp}_X \xi, \eta \rangle$$

for $X \in TM^n$, and thus $\nabla^{\perp}_X \xi = w(X)\eta$ and $\nabla^{\perp}_X \eta = -w(X)\xi$. Hence the Codazzi equations $(\nabla_X A)(Y, \beta) = (\nabla_Y A)(X, \beta)$ become

$$\begin{aligned} \nabla_X AY - A\nabla_X Y - w(X)BY &= \nabla_Y AX - A\nabla_Y X - w(Y)BX, \\ \nabla_X BY - B\nabla_X Y + w(X)AY &= \nabla_Y BX - B\nabla_Y X + w(Y)AX, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \nabla_X AY - \nabla_Y AX &= A([X, Y]) + B(w(X)Y - w(Y)X) \\ \nabla_X BY - \nabla_Y BX &= B([X, Y]) - A(w(X)Y - w(Y)X), \end{aligned}$$

while the Ricci equation is

$$\langle R^{\perp}(X, Y)\xi, \eta \rangle = dw(X, Y) = \langle [A, B]X, Y \rangle.$$

For the sake of completeness, let us first analyze the local behaviour of f on U_k for $k \geq 2$. We will see that it is already a product of strictly convex hypersurfaces, hence showing that case (b) in Theorem 1 has its roots in a local phenomenon.

Proposition 4.2. *Assume that f is locally wide and $2 \leq k \leq [n/2]$. Then the immersion $f|_{U_k}$ is locally a product immersion of two strictly convex Euclidean hypersurfaces whose respective normal vectors are ξ and η.*

Proof. Since their dimensions are constant, the distributions $\ker A$ and $\ker B$ are smooth, have rank bigger or equal than two, and satisfy $\ker A \oplus \ker B = TU_k$.

For $X, Y \in \ker A$, the Codazzi equations imply that

$$A[X, Y] = B(w(X)Y - w(Y)X).$$

Since $\operatorname{Im} A \cap \operatorname{Im} B = 0$, both sides have to be 0 and hence $\ker A$ is integrable. Furthermore, $\ker A \subset \ker w$ since $\ker A \cap \ker B = 0$ and $\dim \ker A \geq 2$. Indeed, If $AX = 0$, choose a linearly independent $Y \in \ker A \cap \ker \omega$, which implies $w(X)BY = w(Y)BX$ and hence $\omega(X) = 0$. Analogously, $\ker B$ is integrable and $\ker B \subset \ker w$. Therefore, $w|_{U_k} = 0$ which implies that $f|_{U_k}$ has flat normal bundle. From the Ricci equation it now follows that A and B commute, and hence $\ker A \perp \ker B$.

For $X, Y \in \ker A = \operatorname{Im} B$ and $U \in \ker B = \operatorname{Im} A$ the Codazzi equations imply that $A[X, U] = \nabla_X AU$. Since U is arbitrary, $\nabla_X U \in \operatorname{Im} A$ as well. Hence

$$\langle \nabla_X Y, U \rangle = -\langle Y, \nabla_X U \rangle = 0$$

which implies that the distribution $\ker A$ is totally geodesic, and similarly so is $\ker B$. We conclude that both $\ker A$ and $\ker B$ are mutually orthogonal transversal totally geodesic distributions, and hence both are parallel, and M^n is locally a product. Furthermore,

$$\alpha_f(X, U) = \langle AX, U \rangle \xi + \langle BX, U \rangle \eta = 0$$

for all $X \in \ker A$ and $U \in \ker B$. The proposition then follows from the Main Lemma in [15]. Observe that each factor is strictly convex since A and B are positive definite on the corresponding factors. $\square$

Before continuing we point out the following consequence.

Corollary 4.3. *If M^n is complete, $\operatorname{Ric}_M > 0$, and f is locally wide, then f is a global product of two strictly convex embedded hypersurfaces. In particular, f is rigid.*

Proof. Consider $\pi : \tilde{M}^n \to M^n$ the universal cover of M^n with the covering metric, and $\tilde{f} = f \circ \pi$. By hypothesis, K and U_1 are empty. Since $\partial U_k \subset K$ for all k, there is $k_0 \geq 2$ such that $U_{k_0} = \tilde{M}^n$. But then Proposition 4.2 and the deRham decomposition theorem imply that $\tilde{M}^n$ is globally a product. From [1] it follows that $\tilde{f}$ is a product immersion of two strictly convex Euclidean hypersurfaces. By [18], a complete strictly convex Euclidean hypersurface is the boundary of a strictly convex body and hence embedded. Thus $\tilde{f}$ is injective and so $\tilde{f} = f$ and $\tilde{M}^n = M^n$. $\square$

The description of f on the set U_1 is considerably more delicate. This case is of main interest to us since, as pointed out in Remark 4.1, for the immersion in case (c) and (d) of Theorem 1, we have $M = K \cup U_1$. Furthermore, U_1 is nonempty since otherwise the open subset of minimal relative nullity in K would be foliated by complete straight lines in Euclidean space, contradicting compactness.

We need the following definition from [11].

Definition 4.4. Given an isometric immersion $g\colon M^n \to \mathbb{R}^{n+1}$, we say that another isometric immersion $f\colon M^n \to \mathbb{R}^{n+2}$ is a *composition (of g)* when there is an isometric embedding $g'\colon M^n \hookrightarrow N_0^{n+1}$ into a flat manifold N_0^{n+1}, an isometric immersion $j\colon N_0^{n+1} \to \mathbb{R}^{n+1}$ (that is, a local isometry) satisfying $g = j \circ g'$, and an isometric immersion $h\colon N_0^{n+1} \to \mathbb{R}^{n+2}$ such that $f = h \circ g'$.

Observe that, for any open subset $U \subset M^n$ where $g_{|U}$ in the above is an embedding, we can assume that N_0^{n+1} is an open subset on $\mathbb{R}^{n+1}$ and j is the inclusion.

Compare the following local description of f on U_1 with the structure of Example 1 in the introduction.

Proposition 4.5. *If $\pi : \tilde{U}_1 \to U_1$ is the universal cover of U_1, then $f \circ \pi$ is a composition of a convex Euclidean hypersurface with constant index of relative nullity one.*

Proof. Since rank $B = 1$, and hence $\hat{R} = \Lambda^2 A$, the shape operator A alone satisfies the Gauss equation along $U_1 \subset M^n$. We claim that A also satisfies the Codazzi equation for hypersurfaces, that is, the skew symmetric tensor

$$S(X, Y) := \nabla_X AY - \nabla_Y AX - A[X, Y]$$

vanishes.

For any $X, Y \in TU_1$, the Codazzi equation for A tells us that

$$S(X, Y) = B(w(X)Y - w(Y)X).$$

For $X, Y \in \ker B$, the Codazzi equation for B says that

$$B[X, Y] = -A(w(X)Y - w(Y)X).$$

Since $\dim \ker B \geq 2$, it follows as in the proof of Proposition 4.2 that $\ker B \subset \ker w$. Thus either $\ker B = \ker w$ or $w = 0$. In the first case, since S is skew symmetric, we can assume that X and Y are linearly independent, and hence can assume that $X \in \ker \omega$. But then $S(X, Y) = 0$. Altogether, S vanishes.

By the Fundamental Theorem of Submanifolds, locally on U_1 (or globally on U_1 if it is simply connected), there exists a Euclidean hypersurface g whose second fundamental form is A. Since $\operatorname{rk} A = n - 1 \geq 2$, $\nu_g = \mu|_{U_1} \equiv 1$. Now, since $\ker B \subset \ker w$, by Proposition 8 in [11] we have that f is a composition of g (see also [12, Proposition 9]). ☐

Remark. Since $\nu_g = 1$, the nullity geodesics in U_1 are (locally) mapped by g into straight lines of $\mathbb{R}^{n+1}$. If $M = U_1$ is complete and simply connected, we will see in the proof of Theorem 3 that these straight lines are parallel and g is globally a cylinder.

We will also discuss properties of the metric and the immersion on the set of index of nullity 1, i.e., on $V := \mu^{-1}(1)$, and the open set U_1' where B does not vanish (and hence f is 1-regular, i.e., $\dim \operatorname{span} \operatorname{Im}(\alpha) = 2$),

$$U_1' = \{p \in V : \operatorname{rk} B(p) = 1\} \supseteq U_1.$$

Recall that by the Gauss equations on the set U_1 we have $\mu = 1$ and, by definition, $\nu = 0$ as well. Thus,

$$U_1 \subseteq U_1' \subseteq V \subseteq K \cup U_1 \subseteq M^n.$$

On the complement of U_1 in V we have $\nu = \mu = 1$. In any case, V is the set of minimal index of nullity of M^n, so its leaves of nullity are complete if M^n is complete.

The global version of Proposition 4.5 is the following. This in particular applies to any immersion in cases (c) and (d) of Theorem 1.

Theorem 3. *Assume that M^n is compact with nonnegative sectional curvature, and f is locally wide. Furthermore, assume that M^n has no points with positive Ricci curvature, i.e., $M = U_1 \cup K$. Let $\pi: \tilde{V} \to V$ the universal cover of V, $\tilde{f} = f \circ \pi$ the lift of $f|_V$, and set $\tilde{U}_1' = \pi^{-1}(U_1') \subset \tilde{V}$. Then we have:*

(i) *$\tilde{V}$ splits globally and isometrically as a product $\tilde{V} = N^{n-1} \times \mathbb{R}$, where $g: N^{n-1} \to \mathbb{R}^n$ is a strictly locally convex hypersurface. In particular, $\tilde{V}$ is itself an Euclidean hypersurface via the cylinder over g, $g \times \mathrm{Id}_{\mathbb{R}} : \tilde{V} \to \mathbb{R}^{n+1}$;*

(ii) *The restriction $\tilde{f}|_{\tilde{U}_1'}$ is a composition of the cylinder over $g|_{\tilde{U}_1'}$;*

(iii) *Along each connected component W of the interior of $\tilde{V} \setminus \tilde{U}_1'$, $\tilde{f}|_W$ is a composition of the cylinder over $g|_W$ with a linear inclusion $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$.*

Proof. Here we continue with the notations in the proof of Proposition 4.5. Define on V the *splitting tensor* of the (totally geodesic and complete) nullity distribution, $C : \Gamma^\perp \to \Gamma^\perp$ given by $CX = -\nabla_X T$, where T is a unit vector field tangent to Γ. Since Γ is totally geodesic, the distribution $\Gamma^\perp$ is totally geodesic if and only if C vanishes.

Observe that C satisfies the Riccati type differential equation $C' = C^2$ when restricted to a geodesic with $\gamma' = T \in \Gamma$. Indeed,

$$\begin{aligned} C'X = \nabla_T CX - C\nabla_T X &= -\nabla_T \nabla_X T + \nabla_{\nabla_T X} T \\ &= -\nabla_X \nabla_T T - \nabla_{[T,X]} T - R(T,X)T + \nabla_{\nabla_T X} T \\ &= \nabla_{\nabla_X T} T = C^2 X \end{aligned}$$

since $\nabla_T T = 0$ and $R(T,X)T = 0$. Since γ is complete, this ODE holds true over the entire real line, and then $C(\gamma(t)) = C_0(I - tC_0)^{-1}$, where $C_0 = C(\gamma(0))$. Therefore, along each nullity geodesic, all real eigenvalues of C vanish.

We claim that $C \equiv 0$. Observe first that every nullity line in V has to intersect U_1 since in $V \setminus U_1$ we have $\nu = 1$ and hence a complete nullity line is also a relative nullity line. But then its image under f is a straight line in $\mathbb{R}^{n+2}$, contradicting compactness of M^n. As we saw in the proof of Proposition 4.5, on U_1 the shape operator A satisfies the Codazzi equation of a hypersurface and hence

$$\nabla_T AX = \nabla_X AT + A[T,X] = A[T,X]$$

since on U_1 we also have $AT = 0$. Thus if $X \in TU_1$ and $X \in \Gamma^\perp$ we have

$$A'X = \nabla_T AX - A\nabla_T X = A[T,X] - A\nabla_T X = -A\nabla_X T = ACX. \tag{4.2}$$

Hence, from the symmetry of A', we have $AC = (AC)^t = C^t A$. Furthermore, since $A > 0$ on $\Gamma^\perp$, we have the inner product $\langle X, Y\rangle_1 = \langle AX, Y\rangle$, positive definite on $\Gamma^\perp$. But then C is self adjoint since

$$\langle X, CY\rangle_1 = \langle AX, CY\rangle = \langle C^t AX, Y\rangle = \langle ACX, Y\rangle = \langle CX, Y\rangle_1.$$

Thus all eigenvalues of C are real, and hence $C \equiv 0$ on V which proves the claim.

In particular, V is locally a Riemannian product of a line and a manifold with positive sectional curvature. Moreover, by (4.2), A is parallel along γ in U_1, which implies in particular that ξ and η are parallel along U_1 as well.

Now, in $V \setminus U_1$, T spans the relative nullity of f. For any parallel normal vector field σ along γ we obtain as in (4.2) that $A'_\sigma = A_\sigma C = 0$. In particular, by taking the parallel transport of ξ we see that, along the whole V, there is a unique smooth unit normal vector field ξ such that $A = A_\xi \geq 0$ has rank equal to $n-1$, and for a local unit normal vector field $\eta \perp \xi$, $B = A_\eta$ has rank at most 1. In addition, A is parallel along the complete lines of nullity in V, and $w(T) = 0$ along $V \setminus U_1$.

We now show that $\tilde{V}$ is a cylinder over a strictly convex Euclidean hypersurface, thus proving part (i). First, the Ricci equation along $V \setminus U_1$ tells us that

$$\begin{aligned} 0 = R^\perp(X, T) &= T(w(X)) - X(w(T)) - w(\nabla_T X - \nabla_X T) \\ &= T(w(X)) - w(\nabla_T X), \end{aligned}$$

since $\Gamma = \Delta$ in $V \setminus U_1$. Therefore, $\ker w$ is parallel along γ. Moreover, since $B' = BC = 0$, $\ker B$ is also parallel along γ. But in the proof of Proposition 4.5 we showed that, on U_1, $\ker B \subset \ker w$. We conclude that $\ker B \subset \ker w$ on the whole V, and, again as in the proof of Proposition 4.5, A satisfies Gauss and Codazzi equations for V. So, the tensor $\tilde{A} = \pi^* A$ satisfies Gauss and Codazzi equations for $\tilde{V}$, and thus there exists an isometric immersion $g' : \tilde{V} \to \mathbb{R}^{n+1}$ with gauss map $\tilde{\xi}$ whose second fundamental form is $\tilde{A}$. By construction, Γ is now the relative nullity of g', and hence the complete geodesics of nullity are mapped into complete straight affine lines of $\mathbb{R}^{n+1}$. On the other hand, since

$$\tilde{\nabla}_X g'_* T = \langle \tilde{A}T, X\rangle \tilde{\xi} = 0, \quad \forall X \in TV,$$

for the standard connection $\tilde{\nabla}$ of $\mathbb{R}^{n+1}$, the complete leaves of Γ are mapped by g' into locally parallel lines of $\mathbb{R}^{n+1}$. Therefore, $\tilde{V}$ is globally a product $\tilde{V} = N^{n-1} \times \mathbb{R}$ and g' is a cylinder, i.e., $g' = g \circ \mathrm{Id}_{\mathbb{R}}$, where $g : N^{n-1} \to \mathbb{R}^n$ is a strictly convex hypersurface, as claimed.

Finally, since $\ker B \subset \ker w$, part (ii) follows from Proposition 8 in [11] as in Proposition 4.5, while part (iii) is immediate from $B = 0$ and $w = 0$ (i.e. η is locally constant) on $V \setminus U_1$. □

Remark 4.6. The map h in Definition 4.4 may fail to be an immersion along the image of j on the boundary points of $\tilde{U}'_1$.

The above easily implies the following corollary. Note that this applies in particular to the immersions in part (c) of Theorem 1 if the metric on $\mathbb{S}^{n-1}$ has positive sectional curvature.

Corollary 4.7. *Under the assumptions of Theorem 3, suppose further that M^n has 2-positive Ricci curvature. Then, for the universal cover $\pi : \tilde{M}^n \to M^n$ we have $\tilde{M}^n = \mathbb{S}^{n-1} \times \mathbb{R}$, where $g : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is a compact strictly convex embedded hypersurface. Moreover, if f is 1-regular, then $f \circ \pi$ is globally a composition of its cylinder: $f \circ \pi = h \circ (g \times \mathrm{Id}_{\mathbb{R}})$.*

Proof. The hypothesis on the Ricci curvature is equivalent to $\operatorname{rank} A = n - 1$ everywhere, and hence $V = M^n$. The 1-regularity of f implies that $U_1' = V = M^n$. The result then follows from Theorem 3. □

For $n = 3$, if M^3 is compact but not diffeomorphic to a sphere, we have the decomposition

$$M^3 = K \cup U_1 = F \cup V,$$

where F is the set of flat points of M^3. Indeed, in dimension 3, $\mu(p) > 1$ implies that $p \in F$.

Compare our next result with Examples 1 and 2 in the introduction.

Corollary 4.8. *Let $f : M^3 \to \mathbb{R}^5$ be an isometric immersion of a compact manifold with nonnegative sectional curvature. If M^3 has no complete geodesic of flat points, then M^3 is either diffeomorphic to $\mathbb{S}^3$, or its universal cover $\tilde{M}^3$ is isometric to $\mathbb{S}^2 \times \mathbb{R}$ for some metric of positive Gaussian curvature on the sphere $\mathbb{S}^2$.*

Proof. By Corollary 3.7, f is wide, hence locally wide. As we saw in the proof of Theorem 3, V is then foliated by complete geodesics of nullity, and therefore its boundary as well. But since $M^3 = F \cup V$, the boundary of V is made of flat points. Therefore, M^3 has no flat points, and $\mu = 1$ everywhere. By part (i) of Theorem 3, $\tilde{M}^3 = \tilde{V} = N^2 \times \mathbb{R}$ splits globally and isometrically. That N^2 is a sphere is a consequence of Theorem 1. □

Part (i) of Theorem 3 also immediately implies the next corollary, which has Theorem 2 as a consequence.

Corollary 4.9. *Let $f : M^3 = \mathbb{S}^3/\mathbb{Z}_k \to \mathbb{R}^5$ with $k > 1$ be an isometric immersion with nonnegative sectional curvature. Then M^3 has flat points, and the complement V of its flat points is isometric to a twisted cylinder $(N^2 \times \mathbb{R})/\mathbb{Z}$, where $N^2 \subset \mathbb{R}^3$ is a surface with positive Gaussian curvature.*

We can now use the above and Corollary 3.4 to compute the type numbers τ_k for case (d) in Theorem 1.

Corollary 4.10. *In the situation of Corollary 4.9 and hence Theorem 1 part (d), the type numbers are given by* $\tau_k = \frac{1}{8\pi^2}\int_{N^2} K(x)\kappa_g(x)dx$ *for all* $0 \le k \le 3$. *Here,* K *denotes the Gaussian curvature of* N^2 *and* $\kappa_g(x)$ *the total curvature of the leaf of* Γ *through* $x \in N^2$, *considered as a curve in* $\mathbb{R}^5$.

Proof. Notice first that the computation of τ_k in (3.3) only takes into account the normal bundle over the set with vanishing relative nullity, which in our case is $U_1 \subset V$. Writing $\beta = \cos(\theta)\xi + \sin(\theta)\eta$, a direct computation using rank $A \le 2$ and rank $B \le 1$ shows that $\det A_\beta = \cos(\theta)^2 \sin(\theta) \det(\hat{A})\langle BZ, Z\rangle$ on V, where $\hat{A}$ is the restriction of A to $\Gamma^\perp = TN^2$ and $Z = \gamma_x' \in \Gamma = \ker A$ for $\gamma_x(t) = [(x,t)]$. Along V the surface N^2 in Corollary 4.9 has $\det(\hat{A}) = K \ge 0$. Furthermore, $|\langle BZ, Z\rangle| = \|\tilde{\gamma}_x''\|$ is the geodesic curvature of $\tilde{\gamma}_x := f \circ \gamma_x$ in $\mathbb{R}^5$ since $AZ = 0$ and hence $\tilde{\gamma}_x'' = \alpha(\gamma', \gamma') = \langle BZ, Z\rangle\eta$. Thus (3.3) implies that

$$\tau_k = \frac{1}{4}\sum_{i=0}^{3}\tau_i = \frac{3}{32\pi^2}\int\limits_{T_1^\perp V} \left|\det A_\beta\right|$$
$$= \frac{1}{8\pi^2}\int\limits_{V} K(x)\|\tilde{\gamma}_x''(t)\| = \frac{1}{8\pi^2}\int\limits_{N^2} K(x)\kappa_g(x)\,\mathrm{d}x.$$

Here, $\kappa_g(x) = \int_0^a \|\tilde{\gamma}_x''(t)\|\,\mathrm{d}t$ is the total geodesic curvature of $\tilde{\gamma}_x \subset \mathbb{R}^5$ until the return time $a > 0$, i.e., the cyclic group $\mathbb{Z}$ in Corollary 4.9 is spanned by

$$(x,t) \mapsto (j(x), t+a) \in \mathrm{Iso}(N \times \mathbb{R}). \qquad \square$$

In particular, for the switched 3-sphere M_ϵ^3 in Example 2 we get $\tau_k = 1$, and hence the immersion is tight.

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