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Families of p-adic Galois representations and (φ, Γ)-modules

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Abstract. We investigate the relation between p-adic Galois representations and overconvergent (φ, Γ)-modules in families. Especially we construct a natural open subspace of a family of (φ, Γ)-modules, over which it is induced by a family of Galois-representations.

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1. Introduction

The theory of (φ, Γ)-modules describes $\mathbb{Q}_p$-valued continuous representations of the absolute Galois group of a local field in terms of semi-linear algebra objects. This theory was generalized by Dee [9] to the case of coefficients in a complete local noetherian $\mathbb{Z}_p$-algebra. Finally Berger and Colmez [3] generalize the theory of *overconvergent* (φ, Γ)-modules to families parametrized by p-adic Banach algebras. More preceisely their result gives a fully faithful functor from the category of vector bundles with continuous Galois action on a rigid analytic variety to the category of families of étale overconvergent (φ, Γ)-modules. This functor fails to be essentially surjective. However it was shown by Kedlaya and Liu in [17] that this functor can be inverted locally around rigid analytic points.

It was already pointed out in our previous paper [12] that the right category to handle these objects is the category of adic spaces (locally of finite type over $\mathbb{Q}_p$) as introduced by Huber, see [14]. Using the language of adic spaces, we show in this paper that given a family $\mathcal{N}$ of (φ, Γ)-modules over the relative Robba ring $\mathscr{B}^\dagger_{X,\mathrm{rig}}$ on an adic space X locally of finite type over $\mathbb{Q}_p$ (see below for the construction of the sheaf $\mathscr{B}^\dagger_{X,\mathrm{rig}}$), one can construct natural open subspaces X^{int} resp. X^{adm}, where the family $\mathcal{N}$ is étale resp. induced by a family of Galois representations. This generalizes our paper [12] to the set up of (φ, Γ)-modules. Moreover we show that the inclusion $X^{\mathrm{adm}} \subset X$ is partially proper, i.e. contains all its specializations inside X, and we further investigate the difference between the open subspaces X^{int}

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and X^{adm}: we show that X^{adm} contains the tube over a point in the special fiber of some formal model of X^{int}.

Our main results are as follows. Let K be a finite extension of $\mathbb{Q}_p$ and write G_K for its absolute Galois group. Further we fix a cyclotomic extension $K_\infty = \bigcup K(\mu_{p^n})$ of K and write $\Gamma = \mathrm{Gal}(K_\infty/K)$.

Theorem 1.1. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$, and let $\mathcal{N}$ be a family of (φ,Γ)-modules over the relative Robba ring $\mathscr{B}^\dagger_{X,\mathrm{rig}}$.*

(i) *There is a natural open subspace $X^{\mathrm{int}} \subset X$ such that the restriction of $\mathcal{N}$ to X^{int} is étale, i.e. locally on X^{int} there is a family of étale lattices $\mathfrak{N} \subset \mathcal{N}$.*

(ii) *The formation $(X,\mathcal{N}) \mapsto X^{\mathrm{int}}$ is compatible with base change in X, and $X = X^{\mathrm{int}}$ whenever the family $\mathcal{N}$ is étale.*

In the classical theory of overconvergent (φ,Γ)-modules, the slope filtration theorem of Kedlaya, [15, Theorem 1.7.1] asserts that a φ-module over the Robba ring admits an étale lattice if and only if it is pure of slope zero. The latter condition is a semi-stability condition which only involves the slopes of the Frobenius. The question whether there is a generalization of this result to p-adic families was first considered by R. Liu in [18], where he shows that an étale lattice exists locally around rigid analytic points.

The condition of being étale is a local condition and asks for the existence of a lattice. As these lattices are only unique up to p-isogeny one can not expect that they glue together to give an étale lattice globally. Hence it is not easy to define this kind of structure over a formal model of the given rigid analytic space. However, if we relax the condition of being locally free and consider *classical* (φ,Γ)-modules in the sense of Fontaine instead of the *overconvergent* (φ,Γ)-modules (i.e. consider modules over $\mathscr{B}_{X,K}$ instead of modules over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$ in the notations of the body of the paper) we have the following replacement. Again we refer to the body of the paper for the precise definitions.

Theorem 1.2. *Let X be a reduced adic space of finite type over $\mathbb{Q}_p$ and let $\mathcal{N}$ be an étale family of (φ,Γ)-modules over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$ with associated (φ,Γ)-module $\hat{\mathcal{N}}$ over $\mathscr{B}_{X,K}$. Then there exists a coherent $\mathscr{A}_{X,K}$-sub (φ,Γ)-module $\hat{N} \subset \mathcal{N}$ that is étale and generates $\hat{\mathcal{N}}$ over $\mathscr{B}_{X,K}$. Moreover there exists a formal model $\mathcal{X}$ of X such that $\hat{N}$ admits a model over $\mathcal{X}$.*

On the other hand, we construct an *admissible* subset $X^{\mathrm{adm}} \subset X$ for a family of (φ,Γ)-modules over X. This is the subset over which there exists a family of Galois representations. It will be obvious that we always have an inclusion $X^{\mathrm{adm}} \subset X^{\mathrm{int}}$.

Theorem 1.3. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and $\mathcal{N}$ be a family of (φ, Γ)-modules over the relative Robba ring $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$.*

(i) *There is a natural open and partially proper subspace $X^{\mathrm{adm}} \subset X$ and a family $\mathcal{V}$ of G_K-representations on X^{adm} such that $\mathcal{N}|_{X^{\mathrm{adm}}}$ is associated to $\mathcal{V}$ by the construction of Berger–Colmez.*

(ii) *The formation $(X, \mathcal{N}) \mapsto (X^{\mathrm{adm}}, \mathcal{V})$ is compatible with base change in X, and $X = X^{\mathrm{adm}}$ whenever the family $\mathcal{N}$ comes from a family of Galois representations.*

(iii) *Assume that X is quasi-compact and $\mathcal{N}$ is étale, i.e. $X = X^{\mathrm{int}}$. Let $\mathcal{X}$ be a formal model of X that admits a model for an étale submodule $\hat{N} \subset \hat{\mathcal{N}}$ in the sense of Theorem 1.2. Let $Y \subset X$ be the tube of a closed point in the special fiber of $\mathcal{X}$. Then $Y \subset X^{\mathrm{adm}}$.*

In a forthcoming paper we will apply the theory developed in this article to families of trianguline (φ, Γ)-modules and the construction of a (conjectural) local Galois-theoretic theoretic counterpart of eigenvarieties. This should give an alternative construction of Kisin's *finite slope space*.

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2. Sheaves of period rings

In this section we define relative versions of the classical period rings used in the theory of (φ, Γ)-modules and in p-adic Hodge-theory. Some of these sheaves were already defined in [12, 8].

Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. Fix an algebraic closure $\bar{K}$ of K and write $G_K = \mathrm{Gal}(\bar{K}/K)$ for the absolute Galois group of K. As usual we choose a compatible system $\epsilon_n \in \bar{K}$ of p^n-th root of unity and write $K_\infty = \bigcup K(\epsilon_n)$. Let $H_K \subset G_K$ denote the absolute Galois group of K_∞ and write $\Gamma = \mathrm{Gal}(K_\infty/K)$. Finally we denote by $W = W(k)$ the ring of Witt vectors with coefficients in k and by $K_0 = \mathrm{Frac}\, W$ the maximal unramified extension of $\mathbb{Q}_p$ inside K. Moreover we write $W' = \mathcal{O}_{K_0'}$ for the ring of integers of the maximal unramified extension K_0' of $\mathbb{Q}_p$ inside K_∞.

2.1. The classical period rings.

We briefly recall the definitions of the period rings, as defined in [1] for example, see also [17, 1]. Write

$$\tilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}.$$

This is a perfect ring of characteristic p which is complete for the valuation $\mathrm{val}_{\mathbf{E}}$ given by $\mathrm{val}_{\mathbf{E}}(x_0, x_1, \dots) = \mathrm{val}_p(x_0)$. Let

$$\tilde{\mathbf{E}} = \operatorname{Frac}\tilde{\mathbf{E}}^{+} = \tilde{\mathbf{E}}^{+}[\tfrac{1}{\underline{\epsilon}}],$$

where $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots) \in \tilde{\mathbf{E}}^{+}$. Further we define

$$\begin{aligned} \tilde{\mathbf{A}}^{+} &= W(\tilde{\mathbf{E}}^{+}), & \tilde{\mathbf{A}} &= W(\tilde{\mathbf{E}}),\\ \tilde{\mathbf{B}}^{+} &= \tilde{\mathbf{A}}^{+}[1/p], & \tilde{\mathbf{B}} &= \tilde{\mathbf{A}}[1/p]. \end{aligned}$$

On all these ring we have an action of the Frobenius morphism φ which is induced by the p-th power map on $\tilde{\mathbf{E}}$. Let $W'((T)) = W'[[T]][1/T]$ denote the ring of Laurent series with coefficient in W'. Further we consider the ring $\mathbf{A}_K$ which is the p-adic completion of $W'((T))$ and denote by $\mathbf{B}_K = \mathbf{A}_K[1/p]$ its rational analogue. We embed these rings into $\tilde{\mathbf{B}}$ by mapping T to the lift of a uniformizer of the field of norms of K. The morphism φ restricts to an endomorphism, again denoted by φ, on $\mathbf{A}_K$, resp. $\mathbf{B}_K$. Further G_K acts on $\mathbf{A}_K$ through the quotient $G_K \to \Gamma$.

For $r < s \in \mathbb{Z}$ we define

$$\begin{aligned} \mathbf{A}^{[r,s]} &= \left\{ \sum_{n\in\mathbb{Z}} a_n T^n \;\middle|\; a_n \in K_0',\ \begin{matrix} 0 \le \mathrm{val}_p(a_n p^{n/r}) \to \infty, & n \to -\infty \\ 0 \le \mathrm{val}_p(a_n p^{n/s}) \to \infty, & n \to \infty \end{matrix} \right\},\\ \tilde{\mathbf{A}}^{\dagger,r} &= \left\{ \sum_{n\ge 0} [x_n] p^n \;\middle|\; x_n \in \tilde{\mathbf{E}},\ 0 \le \mathrm{val}_{\mathbf{E}}(x_n) + \tfrac{prn}{p-1} \to \infty, \quad n \to \infty \right\},\\ \tilde{\mathbf{B}}^{\dagger,r} &= \left\{ \sum_{n\gg -\infty} [x_n] p^n \;\middle|\; x_n \in \tilde{\mathbf{E}},\ \mathrm{val}_{\mathbf{E}}(x_n) + \tfrac{prn}{p-1} \to \infty, \quad n \to \infty \right\}. \end{aligned}$$

The rings $\tilde{\mathbf{A}}^{\dagger,r}$ and $\tilde{\mathbf{B}}^{\dagger,r}$ are endowed with the valuation

$$w_r : \sum p^k [x_k] \longmapsto \inf_k \left\{ \mathrm{val}_{\mathbf{E}}(x_k) + \tfrac{prk}{p-1} \right\}.$$

Using these definitions the perfect period rings (on which the Frobenius φ is bijective) are defined as follows:

$$\begin{aligned} \tilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}} &= \text{Frechet completion of } \tilde{\mathbf{B}}^{\dagger,s} \text{ for the valuations } w_{s'},\ s' \ge s,\\ \tilde{\mathbf{B}}^{\dagger} &= \varinjlim_s \tilde{\mathbf{B}}^{\dagger,s},\\ \tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} &= \varinjlim_s \tilde{\mathbf{B}}^{\dagger,s}_{\mathrm{rig}}. \end{aligned} \tag{2.1}$$

Further we have the usual imperfect period rings (where the Frobenius is not bijective):

$$\begin{aligned}
\mathbf{B}^{[r,s]} &= \mathbf{A}^{[r,s]}[1/p],\\
\mathbf{B}^{\dagger,r} &= \mathbf{B}_K \cap \tilde{\mathbf{B}}^{\dagger,r}\\
\mathbf{A}^{\dagger,r} &= \mathbf{A}_K \cap \tilde{\mathbf{A}}^{\dagger,r}\\
\mathbf{B}^{\dagger,s}_{\mathrm{rig}} &= \text{Frechet completion of } \mathbf{B}^{\dagger,s} \text{ for the valuations } w_{s'},\, s' \geq s,\\
\mathbf{B}^{\dagger} &= \varinjlim_r \mathbf{B}^{\dagger,r},\\
\mathbf{B}^{\dagger}_{\mathrm{rig}} &= \varinjlim_r \mathbf{B}^{\dagger,r}_{\mathrm{rig}},\\
\mathbf{A}^{\dagger} &= \mathbf{A}_K \cap \mathbf{B}^{\dagger}.
\end{aligned} \tag{2.2}$$

Note that these definitions equip all rings with a canonical topology. There are canonical actions of G_K on all of these rings which are continuous for their canonical topologies. The H_K-invariants of $\tilde{\mathbf{R}}$ for any of the rings in (2.1) are given by the corresponding ring without a tilde $\mathbf{R}$ in (2.2), where $\mathbf{R}$ is identified with a subring of $\tilde{\mathbf{R}}$ by mapping T to a lift of a uniformizer of the field of norms of K. Hence there is a natural continuous Γ-action on all the rings in (2.2).

Remark 2.1. Let us point out that some of the above rings have a geometric interpretation. We write $\mathbb{B}$ for the closed unit disc over K_0' and $\mathbb{U} \subset \mathbb{B}$ for the open unit disc. Then

$$\begin{aligned}
\mathbf{A}^{[r,s]} &= \Gamma(\mathbb{B}_{[p^{-1/r},p^{-1/s}]}, \mathcal{O}^+_{\mathbb{B}}),\\
\mathbf{B}^{[r,s]} &= \Gamma(\mathbb{B}_{[p^{-1/r},p^{-1/s}]}, \mathcal{O}_{\mathbb{B}}),
\end{aligned}$$

where $\mathbb{B}_{[a,b]} \subset \mathbb{B}$ is the subspace of inner radius a and outer radius b and $\mathbb{U}_{\geq a} \subset \mathbb{U}$ is the subspace of inner radius a.

The ring $\mathbf{B}^{\dagger,r}_{\mathrm{rig}}$ is known to be identified with the ring of rigid analytic functions in the variable T that converge on the annulus $0 < v_p(T) \leq 1/r$, i.e. we have the identification

$$\mathbf{B}^{\dagger,r}_{\mathrm{rig}} = \varprojlim_s \mathbf{B}^{[r,s]} = \Gamma(\mathbb{U}_{\geq p^{-1/r}}, \mathcal{O}_{\mathbb{U}}),$$

and $\mathbf{B}^{\dagger,r}$ is identified with its subring of functions that are bounded in that annulus. We write $\mathbf{A}^{[r,\infty)} \subset \mathbf{B}^{\dagger,r}$ for the subring[1] of power series with coefficients in $W' = \mathcal{O}_{K_0'}$. Note that

$$\mathbf{A}^{\dagger} = \varprojlim_s \mathbf{A}^{[r,\infty)}$$

but that $\mathbf{A}^{[r,\infty)}$ is strictly larger than $\varprojlim_s \mathbf{A}^{[r,s]}$, as for example $T^{-1} \in \mathbf{A}^{[r,\infty)}$, but T^{-1} is not bounded by 1 on any annulus $0 < v_p(T) \leq 1/r$.

[1]This ring is denoted $\mathcal{R}^{\mathrm{int},r}$ in [17].

The Frobenius endomorhpism φ of $\tilde{\mathbf{B}}$ induces a ring homomorphisms

$$\mathbf{A}^{[r,s]} \longrightarrow \mathbf{A}^{[pr,ps]}$$
$$\mathbf{A}^{[r,\infty)} \longrightarrow \mathbf{A}^{[pr,\infty)}$$
$$\tilde{\mathbf{A}}^{\dagger,s} \longrightarrow \tilde{\mathbf{A}}^{\dagger,ps},$$

for $r, s \gg 0$ and in the limit endomorphisms of the rings

$$\mathbf{A}^{\dagger}, \mathbf{B}^{\dagger}, \mathbf{B}^{\dagger}_{\mathrm{rig}}, \tilde{\mathbf{B}}^{\dagger}, \tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}.$$

These homomorphisms will be denoted by φ and commute with the action of Γ, resp. G_K.

2.2. Sheafification. Let X be an adic space locally of finite type over $\mathbb{Q}_p$ in the sense of Huber [14]. Recall that X comes along with a sheaf $\mathcal{O}_X^+ \subset \mathcal{O}_X$ of open and integrally closed subrings.

Let A^+ be a reduced $\mathbb{Z}_p$-algebra topologically of finite type. Recall that for $i \geq 0$ the completed tensor products

$$A^+\widehat{\otimes}_{\mathbb{Z}_p} W_i(\tilde{\mathbf{E}}^+) \quad \text{and} \quad A^+\widehat{\otimes}_{\mathbb{Z}_p} W_i(\tilde{\mathbf{E}})$$

are the completions of the ordinary tensor product for the topology that is given by the discrete topology on $A^+/p^i A^+$ and by the natural topology on $W_i(\tilde{\mathbf{E}}^+)$ resp. $W_i(\tilde{\mathbf{E}})$, see [12, 8.1].

Let X be a *reduced* adic space locally of finite type over $\mathbb{Q}_p$. As in [12, 8.1] we can define sheaves $\tilde{\mathscr{E}}_X^+$, $\tilde{\mathscr{E}}_X$, $\tilde{\mathscr{A}}_X^+$ and $\tilde{\mathscr{A}}_X$ by demanding

$$\begin{aligned}
\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{E}}_X^+) &= A^+\widehat{\otimes}_{\mathbb{Z}_p}\tilde{\mathbf{E}}^+,\\
\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{E}}_X) &= A^+\widehat{\otimes}_{\mathbb{Z}_p}\tilde{\mathbf{E}},\\
\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{A}}_X^+) &= \varprojlim_i A^+\widehat{\otimes}_{\mathbb{Z}_p} W_i(\tilde{\mathbf{E}}^+),\\
\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{A}}_X) &= \varprojlim_i A^+\widehat{\otimes}_{\mathbb{Z}_p} W_i(\tilde{\mathbf{E}}),
\end{aligned}$$

for an affinoid open subset $\mathrm{Spa}(A, A^+) \subset X$. It follows from [12, Lemma 8.1] that these are well defined sheaves.

We define the sheaf $\mathscr{A}_{X,K}$ to be the p-adic completion of $(\mathcal{O}_X^+ \otimes_{\mathbb{Z}_p} W)((T))$, that is

$$\mathscr{A}_{X,K}(\mathrm{Spa}(A, A^+)) = \big(A^+ \otimes_{\mathbb{Z}_p} W)((T))\big)^{\wedge}$$

for some reduced affinoid Tate algebra (A, A^+). As p-adic completion is left exact it is clear that this rule again defines a sheaf, not just a pre-sheaf. Further we set

$$\mathscr{B}_{X,K} = \mathscr{A}_{X,K}[1/p].$$

Let A^+ be as above and $A = A^+[1/p]$. We define

$$A^+\widehat{\otimes}_{\mathbb{Z}_p}\mathbf{A}^{[r,s]} \quad \text{and} \quad A^+\widehat{\otimes}_{\mathbb{Z}_p}\tilde{\mathbf{A}}^{\dagger,s}$$

to be the completion of the ordinary tensor product for the p-adic topology on A^+ and the natural topology on $\mathbf{A}^{[r,s]}$ resp. $\tilde{\mathbf{A}}^{\dagger,s}$. These completed tensor products can be viewed as subrings of $\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{A}}_{\mathrm{Spa}(A,A^+)})$. For a reduced adic space X locally of finite type over $\mathbb{Q}_p$, we define the sheaves $\mathscr{A}_X^{[r,s]}$ and $\tilde{\mathscr{A}}_X^{\dagger,s}$ by demanding

$$\Gamma(\mathrm{Spa}(A, A^+), \mathscr{A}_X^{[r,s]}) = A^+\widehat{\otimes}_{\mathbb{Z}_p}\mathbf{A}^{[r,s]},$$
$$\Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{A}}_X^{\dagger,s}) = A^+\widehat{\otimes}_{\mathbb{Z}_p}\tilde{\mathbf{A}}^{\dagger,s},$$

for an open affinoid $\mathrm{Spa}(A, A^+) \subset X$. In order to show that these rules really define sheaves, we proceed as follows: The rings $\mathbf{A}^{[r,s]}$ is a lattice in a Banach-algebra over $\mathbb{Q}_p$ and so is $\tilde{\mathbf{A}}^{\dagger,s}$ (it is complete for the valuation w_s) and we may use [17, Definition 3.2, Lemma 3.3] in order to prove the sheaf axiom. The claim of loc. cit. is formulated for Banach algebras, but the proof works the same for lattices in Banach algebras.

Similarly we define the sheaf $\tilde{\mathscr{B}}_X^{\dagger,s}$. Finally, as in the case above, we can use these sheaves to define the sheafified versions of (2.1):

$$\begin{aligned} \tilde{\mathscr{B}}_X^{\dagger} &= \varinjlim_s \tilde{\mathscr{B}}_X^{\dagger,s}, \\ \tilde{\mathscr{B}}_{X,\mathrm{rig}}^{\dagger} &= \varinjlim_s \tilde{\mathscr{B}}_{X,\mathrm{rig}}^{\dagger,s}, \end{aligned} \tag{2.3}$$

where the direct limits are (by definition) direct limits in the category of sheaves (i.e. sheafification of the direct limit as pre-sheaves).

Moreover we define $\tilde{\mathscr{B}}_{X,\mathrm{rig}}^{\dagger,r}$ to be the sheaf associated to

$$\mathrm{Spa}(A, A^+) \mapsto A\hat{\otimes}_{\mathbb{Q}_p}\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$$

for $\mathrm{Spa}(A, A^+) \subset X$ affinoid open. Again it is easy to see that this indeed defines a sheaf: The ring $A\hat{\otimes}_{\mathbb{Q}_p}\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ is the Frechet completion of

$$A\hat{\otimes}_{\mathbb{Q}_p}\tilde{\mathbf{B}}^{\dagger,r} = \Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{B}}_X^{\dagger,s})$$

with respect to the family of norms $w_{s'}$, for $s' \geq s$. But as completion is left exact, for some open covering $\mathrm{Spa}(A, A^+) = \bigcup_i U_i$, the exact sequence

$$0 \longrightarrow \Gamma(\mathrm{Spa}(A, A^+), \tilde{\mathscr{B}}_X^{\dagger,r}) \longrightarrow \prod_i \Gamma(U_i, \tilde{\mathscr{B}}_X^{\dagger,r}) \longrightarrow \prod_{i,j} \Gamma(U_i \cap U_j, \tilde{\mathscr{B}}_X^{\dagger,r})$$

stays exact after completion. We deduce the sheaf property from $\tilde{\mathscr{B}}_{X,\mathrm{rig}}^{\dagger,r}$ from the sheaf property of $\tilde{\mathscr{B}}_X^{\dagger,r}$.

Moreover we have the sheafified versions of the rings (2.2) (by a direct limit we always mean the direct limit in the category of sheaves, i.e. the sheafification of the direct limit in the category of presheaves):

$$\begin{aligned} \mathscr{B}_X^{[r,s]} &= \mathscr{A}_X^{[r,s]}[1/p], \\ \mathscr{B}_X^{\dagger,r} &= \mathscr{B}_{X,K} \cap \tilde{\mathscr{B}}_X^{\dagger,r} \\ \mathscr{A}_X^{\dagger,r} &= \mathscr{A}_{X,K} \cap \tilde{\mathscr{A}}_X^{\dagger,r} \\ \mathscr{B}_X^{\dagger} &= \varinjlim_r \mathscr{B}_X^{\dagger,r}, \\ \mathscr{A}_X^{\dagger} &= \mathscr{A}_{X,K} \cap \mathscr{B}_X^{\dagger}. \end{aligned} \tag{2.4}$$

Note that all the rational period rings (i.e. those period rings in which p is inverted) can also be defined on a non-reduced space X by locally embedding the space into a reduced space Y and restricting the corresponding period sheaf from Y to X, compare [12, 8.1].

Remark 2.2. As in the absolute case there is a geometric interpretation of some of these sheaves of period rings:

$$\begin{aligned} \mathscr{A}_X^{[r,s]} &= \mathrm{pr}_{X,*}\big(\mathcal{O}^+_{X\times\mathbb{B}_{[p^{-1/r},p^{-1/s}]}}\big), \\ \mathscr{B}_X^{[r,s]} &= \mathrm{pr}_{X,*}\big(\mathcal{O}_{X\times\mathbb{B}_{[p^{-1/r},p^{-1/s}]}}\big). \end{aligned}$$

Here pr_X denotes the projection from the product to X.

We may further set

$$\begin{aligned} \mathscr{B}_{X,\mathrm{rig}}^{\dagger,r} &= \varprojlim_s \mathscr{B}_X^{[r,s]} = \mathrm{pr}_{X,*}\big(\mathcal{O}_{X\times\mathbb{U}_{\geq p^{-1/r}}}\big), \\ \mathscr{B}_{X,\mathrm{rig}}^{\dagger} &= \varinjlim_r \mathscr{B}_{\mathrm{rig}}^{\dagger,r}. \end{aligned}$$

By construction all the "perfect" sheaves $\tilde{\mathscr{R}}_X$ (i.e. those of the period sheaves with a tilde) are endowed with a continuous $\mathcal{O}_X$-linear G_K-action and an endomorphism φ commuting with the Galois action. The "imperfect" sheaves $\mathscr{R}_X$ (i.e. those period rings without a tilde) are endowed with a continuous Γ-action and an endomorphism φ commuting with the action of Γ.

Notation. In the following we will use the notation $X(\bar{\mathbb{Q}}_p)$ for the set of rigid analytic points of an adic space X locally of finite type over $\mathbb{Q}_p$, i.e.

$$X(\bar{\mathbb{Q}}_p) = \{x \in X \mid k(x)/\mathbb{Q}_p \text{ finite}\}.$$

Proposition 2.3. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $\boldsymbol{R}$ be any of the integral period rings* (*i.e. a period ring in which p is not inverted*) *defined above. Let $\mathscr{R}_X$ be the corresponding sheaf of period rings on X.*

(i) *The canonical map*

$$\Gamma(X, \mathscr{R}_X) \longrightarrow \prod_{x \in X(\bar{\mathbb{Q}}_p)} k(x)^+ \otimes_{\mathbb{Z}_p} \boldsymbol{R}$$

is an injection.

(ii) *Let $\boldsymbol{R}' \subset \boldsymbol{R}$ be another integral period ring with corresponding sheaf of period rings $\mathscr{R}'_X \subset \mathscr{R}_X$ and let $f \in \Gamma(X, \mathscr{R}_X)$. Then $f \in \Gamma(X, \mathscr{R}'_X)$ if and only if*

$$f(x) \in k(x)^+ \otimes_{\mathbb{Z}_p} \boldsymbol{R}' \subset k(x)^+ \otimes_{\mathbb{Z}_p} \boldsymbol{R}$$

for all rigid analytic points $x \in X$.

Proof. This is proven along the same lines as [12, Lemma 8.2] and [12, Lemma 8.6]. $\square$

Corollary 2.4. *Let X be an adic space locally of finite type over $\mathbb{Q}_p$, then*

$$\big(\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}}\big)^{\varphi=\mathrm{id}} = \mathcal{O}_X, \qquad \big(\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}}\big)^{H_K} = \mathscr{B}^{\dagger}_{X,\mathrm{rig}},$$
$$\big(\tilde{\mathscr{B}}^{\dagger}_{X}\big)^{\varphi=\mathrm{id}} = \mathcal{O}_X, \qquad \big(\tilde{\mathscr{B}}^{\dagger}_{X}\big)^{H_K} = \mathscr{B}^{\dagger}_{X}.$$

Proof. If the space is reduced this follows from the above by chasing through the definitions. Otherwise we can locally on X choose a finite morphism to a reduced space Y (namely a polydisc) and study the φ- resp. H_K-invariants in the fibers over the rigid analytic points of Y, compare [12, Corollary 8.4, Corollary 8.8] $\square$

Notation. Let X be an adic space locally of finite type and $\mathscr{R}$ be any of the sheaves of topological rings defined above. If $x \in X$ is a point then we will sometimes write $\mathscr{R}_x$ for the completion of the fiber $\mathscr{R} \otimes k(x)$ of $\mathscr{R}$ at x with respect to the canonical induced topology.

3. Coherent $\mathcal{O}_X^+$-modules and lattices

As the notion of being étale is defined by using lattices we make precise what we mean by (families of) lattices.

Let X be an adic space locally of finite type over $\mathbb{Q}_p$. The space X is endowed with a structure sheaf $\mathcal{O}_X$ and a sheaf of open and integrally closed subrings $\mathcal{O}_X^+ \subset \mathcal{O}_X$ consisting of the power bounded sections of $\mathcal{O}_X$. Recall that for any ringed space, there is the notion of a coherent module, see [10, 5.3].

Definition 3.1. Let X be an adic space (locally of finite type over $\mathbb{Q}_p$) and let E be a sheaf of $\mathcal{O}_X^+$-modules on X.

(i) The $\mathcal{O}_X^+$ module E is called *of finite type* or *finitely generated*, if there exist an open covering $X = \bigcup_{i\in I} U_i$ and for all $i \in I$ exact sequences

$$(\mathcal{O}_{U_i}^+)^{d_i} \longrightarrow E|_{U_i} \longrightarrow 0.$$

(ii) The module is called *coherent*, if it is of finite type and for any open subspace $U \subset X$ the kernel of any morphism $(\mathcal{O}_U^+)^d \to E|_U$ is of finite type.

(iii) The sheaf E is called *quasi-coherent* if there is an open covering $X = \bigcup U_i$ and there exist exact sequences

$$(\mathcal{O}_{U_i}^+)^{\oplus J_{1,i}} \longrightarrow (\mathcal{O}_{U_i}^+)^{\oplus J_{2,i}} \longrightarrow E|_{U_i} \longrightarrow 0.$$

for some index sets $J_{1,i}, J_{2,i}$.

Let $X = \mathrm{Spa}(A, A^+)$ be an affinoid adic space. Then any finitely generated A^+-module M defines a coherent sheaf of $\mathcal{O}_X^+$-modules E by the usual procedure

$$\Gamma(\mathrm{Spa}(B, B^+), E) = M \otimes_{A^+} B^+$$

for an affinoid open subspace $\mathrm{Spa}(B, B^+) \subset X$.

Remark 3.2. Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$. Then locally on X the sections $\Gamma(X, \mathcal{O}_X)$ as well as $\Gamma(X, \mathcal{O}_X^+)$ are noetherian rings: Indeed, this comes down to the following claim: Let $\mathrm{Spa}(A, A^+)$ be an affinoid adic space of finite type over $\mathbb{Q}_p$ and assume that A is reduced. We claim that A^+ is noetherian. But A^+ is identified with the ring of power bounded elements of A (by definition of being of finite type). By Noether normalization there exists a morphism

$$B = \mathbb{Q}_p\langle T_1, \dots, T_r\rangle \longrightarrow A$$

which makes A into a finite B-module. As the valuation on $\mathbb{Q}_p$ is discrete it follows from [4, 6.4.1, Corollary 6] that A^+ is finite over the ring of power bounded elements $B^+ = \mathbb{Z}_p\langle T_1, \dots, T_r\rangle$ of $B = \mathbb{Q}_p\langle T_1, \dots, T_r\rangle$. As B^+ is noetherian, so is A^+.

It follows form this remark that an $\mathcal{O}_X^+$-module which is locally associated with a module of finite type is coherent.

Remark 3.3. The same definition of course also applies to the sheaves of period rings that we defined above. However, as in this case the sections over open affinoids are (in general) not noetherian, the analogue of Remark 3.2 does not apply.

On the other hand it is not true that all coherent $\mathcal{O}_X^+$-modules on an affinoid space arise in that way, as shown by the following example. The reason is that the cohomology $H^1(X, E)$ of a coherent $\mathcal{O}_X^+$-sheaf E does not necessarily vanish on affinoid spaces.

Example 3.4. Let $X = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$ be the closed unit disc. Let

$$U_1 = \{x \in X \mid |x| \le |p|\}$$
$$U_2 = \{x \in X \mid |p| \le |x| \le 1\}.$$

Define the $\mathcal{O}_X^+$-sheaf $E_1 \subset \mathcal{O}_X$ by glueing $\mathcal{O}_{U_1}^+$ and $p^{-1}T\mathcal{O}_{U_2}^+$ over $U_1 \cap U_2$ and $E_2 \subset \mathcal{O}_X$ by glueing $\mathcal{O}_{U_1}^+$ and $pT^{-1}\mathcal{O}_{U_2}^+$. Then E_1 and E_2 are coherent $\mathcal{O}_X^+$-modules. We have

$$\Gamma(X, E_1) = (1, p^{-1}T)\Gamma(X, \mathcal{O}_X^+),$$
$$\Gamma(X, E_2) = p\Gamma(X, \mathcal{O}_X^+).$$

Especially E_2 is not generated by global sections. If $\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2 \cong \widehat{\mathbb{A}}^1_{\mathbb{Z}_p} \cup \widehat{\mathbb{P}}^1_{\mathbb{Z}_p}$ is the canonical formal model of $X = U_1 \cup U_2$, then E_2 is defined by the coherent $\mathcal{O}_{\mathcal{X}}$-sheaf which is trivial on the formal affine line and which is the twisting sheaf $\mathcal{O}(1)$ on the formal projective line, while E_1 is defined by its dual $\mathcal{O}(-1)$ on the formal projective line.

Let X be an adic space of finite type over $\mathbb{Q}_p$ (especially X is quasi-compact) and E be a coherent $\mathcal{O}_X^+$-module on X. As E is not necessarily associated to an A^+-module on an affinoid open $\mathrm{Spa}(A, A^+) \subset X$, the sheaf E does not necessarily have a model $\mathcal{E}$ over any formal model $\mathcal{X}$ of X: The sheaf $\mathcal{U} \mapsto \Gamma(\mathcal{U}^{\mathrm{ad}}, E)$ does not define E in the generic fiber in general. However there is a covering $X = \bigcup U_i$ of X by finitely many open affinoids such that $E|_{U_i}$ is the sheaf defined by the finitely generated $\Gamma(U_i, \mathcal{O}_X^+)$-module $\Gamma(U_i, E)$. Hence there is a formal model $\mathcal{X}$ of X such that E is defined by a coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{E}$. Namely $\mathcal{X}$ is a formal model on which one can realize the covering $X = \bigcup U_i$ as a covering by open formal subschemes.

Remark 3.5. If E is a coherent $\mathcal{O}_X^+$-module on an adic space X of finite type over $\mathbb{Q}_p$ and if $U = \mathrm{Spa}(A, A^+) \subset X$ is an affinoid open, then $\Gamma(U, E)$ is a finitely generated A^+-module. In fact there is a formal model $\tilde{\mathcal{U}}$ of U that is an admissible blow up of $\mathrm{Spf}\, A^+$ and such that there is a model $\tilde{\mathcal{E}}$ of $E|_U$ over $\tilde{\mathcal{U}}$. Then the claim follows from standard finiteness results for coherent sheaves and projective morphisms.

Let E be a coherent $\mathcal{O}_X^+$-module on an adic space X and let $x \in X$. Let $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ denote the maximal ideal of function vanishing at x and write $\mathfrak{m}_x^+ = \mathfrak{m}_x \cap \mathcal{O}_{X,x}^+$, i.e. $\mathcal{O}_{X,x}^+/\mathfrak{m}_x^+ = k(x)^+$ is the integral subring of $k(x)$. We write $E \otimes k(x)^+$ for the fiber of E at x, that is for the quotient of the $\mathcal{O}_{X,x}^+$-module

$$E_x = \varinjlim_{U \ni x} \Gamma(U, E)$$

by the ideal $\mathfrak{m}_x^+$.

Let $\mathcal{X}$ be a formal model of X and $\mathcal{E}$ be a coherent $\mathcal{O}_{\mathcal{X}}$-module defining E in the generic fiber. Further let $\operatorname{Spf} k(x)^+ \hookrightarrow \mathcal{X}$ denote the morphism defining x in the generic fiber. Then $\mathcal{E} \otimes k(x)^+ = E \otimes k(x)^+$. If we write $\bar{\mathcal{X}}$ for the special fiber of $\mathcal{X}$ and $\bar{\mathcal{E}}$ for the restriction of $\mathcal{E}$ to $\bar{\mathcal{X}}$ and if $x_0 \in \bar{\mathcal{X}}$ denotes the specialization of x, then it follows that

$$\bar{\mathcal{E}} \otimes k(x_0) = (\mathcal{E} \otimes k(x)^+) \otimes_{k(x)^+} k(x_0) = (E \otimes k(x)^+) \otimes_{k(x)^+} k(x_0).$$

Definition 3.6. Let E be a vector bundle of rank d on an adic space X, locally of finite type over $\mathbb{Q}_p$. A *lattice* in E is a coherent $\mathcal{O}_X^+$-submodule $E^+ \subset E$ which is locally on X free of rank d over $\mathcal{O}_X^+$ and which generates E, i.e. the inclusion induces an isomorphism

$$E^+ \otimes_{\mathcal{O}_X^+} \mathcal{O}_X \cong E.$$

Let us assume for simplicity that the space X is reduced and let $\mathcal{X}$ be a formal model of X. In a similar way as above we can define coherent sheaves of $\mathscr{A}_{X,K}$ or $\mathscr{A}_X^{[r,\infty)}$-modules. Moreover we can define the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras $\mathscr{A}_{\mathcal{X},K}$ on $\mathcal{X}$ by

$$\Gamma(\mathcal{U}, \mathscr{A}_{\mathcal{X},K}) = \Gamma(\mathcal{U}^{\mathrm{ad}}, \mathscr{A}_{X,K}).$$

If $\mathcal{X} = \operatorname{Spf} A^+$ is affine and if $\mathfrak{N}$ is a finitely generate $A^+\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K = \Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X},K})$-module, then we can associate to $\mathfrak{N}$ a coherent $\mathscr{A}_{\mathcal{X},K}$-module by

$$\operatorname{Spf} B^+ \longmapsto \mathfrak{N}\hat{\otimes}_{A^+} B^+ = \mathfrak{N} \otimes_{A^+\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K} (B^+\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K),$$

for $\operatorname{Spf} B^+ \subset \operatorname{Spf} A^+$ open affine.

Similarly we associate to $\mathfrak{N}$ a coherent $\mathscr{A}_{X,K}$-module by

$$\operatorname{Spa}(B, B^+) \longmapsto \mathfrak{N}\hat{\otimes}_{A^+} B^+ = \mathfrak{N} \otimes_{A^+\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K} (B^+\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K),$$

for $\operatorname{Spa}(B, B^+) \subset \operatorname{Spa}(A, A^+)$ open affinoid.

Given again an arbitrary adic space of finite type over $\mathbb{Q}_p$ and a formal model $\mathcal{X}$ of X. Let N be a coherent $\mathscr{A}_{\mathcal{X},K}$-module on $\mathcal{X}$. As a coherent $\mathscr{A}_{\mathcal{X},K}$-module is of finite type it follows that there is an affine cover $\mathcal{X} = \bigcup_{i\in U} \operatorname{Spf} A_i^+$ such that $\mathfrak{N}|_{\operatorname{Spf} A_i^+}$ is associated to a module $\mathfrak{N}_i$ as above[2]. Then we can associate to N a coherent $\mathscr{A}_{X,K}$-module N^{ad} on X by defining $N^{\mathrm{ad}}|_{(\operatorname{Spf} A_i^+)^{\mathrm{ad}}}$ to be the sheaf associated to $\mathfrak{N}_i$. If a coherent $\mathscr{A}_{X,K}$-module is of the form N^{ad} for some coherent $\mathscr{A}_{\mathcal{X},K}$-module N, then we say that this module admits a model over $\mathcal{X}$, or that N is a model for N^{ad}.

Finally let $\mathcal{N}$ be a locally free $\mathscr{B}_{X,K}$-module. We say that a coherent $\mathscr{A}_{X,K}$-submodule $N \subset \mathcal{N}$ is a lattice in $\mathcal{N}$ if N is locally on X free as an $\mathscr{A}_{X,K}$-module and if $N \otimes_{\mathscr{A}_{X,K}} \mathscr{B}_{X,K} = N[\frac{1}{p}] = \mathcal{N}$.

Similar remarks and constructions apply to $\mathscr{A}_X^{[r,\infty)}$ as well.

[2] Note that we do not claim that if $\mathcal{X}$ is affine then every coherent $\mathscr{A}_{\mathcal{X},K}$-module is associated to a module over $\Gamma(\mathcal{X}, \mathscr{A}_{\mathcal{X},K})$.

4. (φ, Γ)-modules over the relative Robba ring

In this section we define certain families of φ-modules that will appear in the context of families of Galois representations later on. Some results of this section are already contained in [12, 6].

Definition 4.1. Let X be an adic space and $\mathscr{R} \in \{\mathscr{A}_{X,K}, \mathscr{A}_X^\dagger\}$.

An étale φ-module over $\mathscr{R}$ is a coherent $\mathscr{R}$-module N together with an isomorphism

$$\Phi : \varphi^* N \longrightarrow N\,.$$

Definition 4.2. Let $X \in \mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ and

$$\mathscr{R} \in \{\mathscr{B}_{X,K}, \mathscr{B}_X^\dagger, \mathscr{B}_{X,\mathrm{rig}}^\dagger\}.$$

Write $\mathscr{R}^+ \subset \mathscr{R}$ for the corresponding integral subring[3].

(i) A φ-module over $\mathscr{R}$ is an $\mathscr{R}$-module N which is locally on X free over $\mathscr{R}$ together with an isomorphism $\Phi : \varphi^* N \to N$.

(ii) A φ-module over $\mathscr{R}$ is called *étale* if it is locally on X induced from an étale φ-module that is free over $\mathscr{R}^+$.

Remark 4.3. Although our main interest is in objects that are (locally on some X) free, we need more flexibility in the case of Definition 4.1. Especially, given an étale φ-module on some affinoid space, we want to be able to treat its global sections as an étale φ-module.

Recall that K_∞ is a fixed cyclotomic extension of K and $\Gamma = \mathrm{Gal}(K_\infty/K)$ denotes the Galois group of K_∞ over K.

Definition 4.4. Let $X \in \mathrm{Ad}^{\mathrm{lft}}_{\mathbb{Q}_p}$ and $\mathscr{R}$ be any of the sheaves of rings defined above.

(i) A (φ, Γ)-module over $\mathscr{R}$ is a φ-module over $\mathscr{R}$ together with a continuous semi-linear action of Γ commuting with the semi-linear endomorphism Φ.

(ii) A (φ, Γ)-module over $\mathscr{R}$ is called étale if its underlying φ-module is étale.

4.1. The étale locus.

If X is an adic space (locally of finite type over $\mathbb{Q}_p$) and $x \in X$ is any point, we will write $\iota_x : x \to X$ for the inclusion of x. If $\mathscr{R}$ is any of the sheaf of topological rings above and if $\mathcal{N}$ is a sheaf of $\mathscr{R}_X$-modules on X, we write

$$\iota_x^* \mathcal{N} = \iota_x^{-1} \mathcal{N} \otimes_{\mathscr{R}_X} \mathscr{R}_x$$

for the pullback of $\mathcal{N}$ to the point x. The following result is a generalization of [17, Theorem 7.4] to the category of adic spaces.

[3] The integral subring of $\mathscr{B}_{X,\mathrm{rig}}^\dagger$ is $\mathscr{A}_X^\dagger$.

Theorem 4.5. *Let X be an adic space locally of finite type over $\mathbb{Q}_p$ and $\mathcal{N}$ be a family of (φ,Γ)-modules over $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$.*

(i) *The set*
$$X^{\mathrm{int}} = \{x \in X \mid \iota_x^*\mathcal{N} \text{ is étale}\} \subset X$$
is open.

(ii) *There exists a covering $X^{\mathrm{int}} = \bigcup U_i$ and locally free étale $\mathscr{A}^{\dagger}_{U_i}$-modules $N_i \subset \mathcal{N}|_{U_i}$ which are stable under Φ such that*
$$N_i \otimes_{\mathscr{A}^{\dagger}_{U_i}} \mathscr{B}^{\dagger}_{U_i,\mathrm{rig}} = \mathcal{N}|_{U_i},$$
i.e. $\mathcal{N}|_{X^{\mathrm{int}}}$ is étale.

Proof. If X is reduced this is [12, Corollary 6.11]. In loc. cit. we use a different Frobenius φ. However the proof works verbatim in the case considered here. For non reduced spaces we follow the same proof using [11, Theorem 6.5] instead of [12, Theorem 6.9][4]. □

Remark 4.6. If we are interested in integral models it is in fact enough to work with locally free $\mathscr{B}^{\dagger,r}_X$-modules $\mathcal{N}$ instead of modules over $\mathscr{B}^{\dagger,r}_X$ that are locally over X free: A locally free $\mathscr{B}^{[r,s]}_X$-module (which is obtained by restricting a locally free $\mathscr{B}^{\dagger,r}_X$-module) is locally on X free over $\mathscr{B}^{[r,s]}_X$. Hence [12, Theorem 6.9] resp. [11, Theorem 6.5] still apply and the assumptions of [12, Proposition 6.5] are satisfied.

Theorem 4.7. *Let $f: X \to Y$ be a morphism of adic spaces locally of finite type over $\mathbb{Q}_p$. Let N_Y be a family of (φ,Γ)-modules over $\mathscr{B}^{\dagger}_{Y,\mathrm{rig}}$ and write N_X for its pullback over $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$. Then $f^{-1}(Y^{\mathrm{int}}) = X^{\mathrm{int}}$.*

Proof. This is [12, Proposition 6.14]. Again the same proof applies with the Frobenius considered here. □

4.2. Existence of étale submodules.

For later applications to Galois representations the existence of an étale lattice locally on X will not be sufficient. We cannot hope that the étale lattices glue together to a global étale lattice on the space X. However we have a replacement which will be sufficient for applications.

Convention. Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $(\mathcal{N},\Phi)$ be an étale φ-module over $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$ and $(\hat{\mathcal{N}},\hat{\Phi})$ be an (étale) φ-module

[4] There is a mistake in [12]. The proof of Theorem 6.9 only applies to reduced spaces. However, this is enough for the purposes of [12]. This mistake is fixed in [11].

over $\mathscr{B}_{X,K}$. We say that $(\hat{\mathcal{N}}, \hat{\Phi})$ *is induced from* $(\mathcal{N}, \Phi)$ if there exists a covering $X = \bigcup U_i$ and étale $\mathscr{A}^\dagger_{U_i}$ lattices $N_i \subset \mathcal{N}|_{U_i}$ such that

$$(\hat{\mathcal{N}}, \hat{\Phi})|_{U_i} = \big((N_i, \Phi)^\wedge\big)[\tfrac{1}{p}] = (N_i, \Phi) \otimes_{\mathscr{A}^\dagger_{U_i}} \mathscr{B}_{U_i,K}.$$

Note that $\mathscr{B}_{X,K}$ is not a sheaf of $\mathscr{B}^\dagger_{X,\mathrm{rig}}$-modules and hence we can only base change after passing to an étale lattice. Further note the every étale φ-module over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$ gives rise to a unique φ-module over $\mathscr{B}_{X,K}$, as an étale $\mathscr{A}^\dagger_X$-lattice is unique up to p-isogeny, compare [17, Prop. 6.5].

Proposition 4.8. *Let X be a reduced adic space of finite type and $(\hat{\mathcal{N}}, \hat{\Phi})$ be a φ-module over $\mathscr{B}_{X,K}$ which is induced from an étale φ-module $(\mathcal{N}, \Phi)$ over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$. Then there exists an étale φ-submodule $\hat{N} \subset \hat{\mathcal{N}}$ over $\mathscr{A}_{X,K}$ such that the inclusion induces an isomorphism after inverting p. Moreover there exists an formal model $\mathcal{X}$ of X such that $\hat{N}$ has a model over $\mathcal{X}$.*

Remark 4.9. Note that in this proposition we do not claim that $\hat{N}$ is locally free.

Proposition 4.10. *Let X be an reduced adic space of finite type over $\mathbb{Q}_p$. Let $\mathcal{N}$ be an étale φ-module over $\mathscr{B}^{\dagger,r}_{X,\mathrm{rig}}$, then there exists a quasi-coherent $\mathscr{A}^{[r,\infty)}_X$-submodule $N \subset \mathcal{N}$ which (locally on X) contains a basis of $\mathcal{N}$. Moreover, if $X = \bigcup_{i=1}^m U_i$ is a finite covering such that $\mathcal{N}|_{U_i}$ admits a free étale lattice N_i, then we can choose N such that $N|_{U_i} \subset N_i$.*

Proof. Let $X = \bigcup_{i=1}^m U_i$ be a finite covering such that $\mathcal{N}|_{U_i}$ is free and admits an étale lattice $N_i \subset \mathcal{N}|_{U_i}$. Write $V_i = \bigcup_{j=1}^i U_j$.

Let $M_1 = N_1$ on U_1. We claim that we can inductively extend M_i on V_i to M_{i+1} on V_{i+1} such that

$$p^{C'_{i+1}} N_j \subset M_{i+1}|_{U_j} \subset p^{C_{i+1}} N_j \tag{4.1}$$

for $j = 1, \dots, i+1$ for some constants C_{i+1} and C'_{i+1}. The proposition then follows after rescaling M_n by p^{-C_n}.

The claim is obvious for $i = 1$ and for the induction step it is sufficient to extend $M_i|_{V_i \cap U_{i+1}}$ to U_{i+1} such that this extension satisfies (4.1). By Lemma 4.11 below it is sufficient to check that there are C_{i+1} and C'_{i+1} such that

$$p^{C'_{i+1}} N_{i+1}|_{V_i \cap U_{i+1}} \subset M_i|_{V_i \cap U_{i+1}} \subset p^{C_{i+1}} N_{i+1}|_{V_i \cap U_{i+1}}.$$

However this may be checked on the open covering $U_j \cap U_{i+1}$ for $j \in \{1, \dots, i\}$ of $V_i \cap U_{i+1}$ and hence by induction hypothesis it is enough to show that

$$p^{C'_{i+1}} N_{i+1}|_{U_j \cap U_{i+1}} \subset p^{C'_j} N_j|_{U_j \cap U_{i+1}} \subset p^{C_j} N_j|_{U_j \cap U_{i+1}} \subset p^{C_{i+1}} N_{i+1}|_{U_j \cap U_{i+1}}.$$

But by [17, Prop 6.5] an étale lattice in $\mathcal{N}|_{U_i\cap U_j}$ is unique up to p-isogeny and hence the required constants C_{i+1} and C'_{i+1} do exist. □

Lemma 4.11. *Let $X = \mathrm{Spa}(A, A^+)$ be a reduced affinoid adic space and $U \subset X$ an quasi-compact open subset. Let $\mathcal{N}_X = (\mathscr{B}^{\dagger,r}_{X,\mathrm{rig}})^d$ and let N_U be a finitely generated $\mathscr{A}_U^{[r,\infty)}$-submodule of $\mathcal{N}_U = \mathcal{N}_X|_U$.*

Let $N', N'' \subset \mathcal{N}_X$ be $\mathscr{A}_X^{[r,\infty)}$-lattices such that

$$N''|_U \subset N_U \subset N'|_U\,.$$

Then there exists a quasi-coherent $\mathscr{A}_X^{[r,\infty)}$-module N_X such that $N_X|_U = N_U$ and

$$N'' \subset N_X \subset N'.$$

Proof. After localizing we may assume that N' is free. Denote by $j : U \hookrightarrow X$ the open embedding of U. We define N_X by

$$N_X = \ker(N' \longrightarrow j_*(N'_U/N_U))$$

and claim that N_X is a coherent $\mathscr{A}_X^{[r,\infty)}$-module containing (locally on X) a basis of $\mathcal{N}_X$.

It is obvious that $N'' \subset N_X \subset N'$ and hence N_X contains a basis of $\mathcal{N}_X$. It remains to check that this sheaf is quasi-coherent. Let $U = \bigcup U_i$ be a finite covering by open affinoids such that N_U is associated to a finitely generated $\Gamma(U_i, \mathscr{A}_X^{[r,\infty)})$-module. Choose a covering $X = \bigcup V_j$ by open affinoids such that $V_j \cap U \subset U_{i_j}$ for some index i_j. Then N_X is associated to the $\Gamma(V_j, \mathscr{A}_X^{[r,\infty)})$-module

$$\ker\big(\Gamma(V_j, N') \longrightarrow \Gamma(U_{i_j}, N'_U/N_U) \otimes_{\Gamma(U_{i_j}, \mathscr{A}_X^{[r,\infty)})} \Gamma(V_j \cap U, \mathscr{A}_X^{[r,\infty)})\big).$$

Especially N_X is quasi-coherent. □

Proof of Proposition 4.8. As X is quasi-compact, we can choose a locally free model $(\mathcal{N}_r, \Phi_r)$ of $(\mathcal{N}, \Phi)$ over $\mathscr{B}^{\dagger,r}_{X,\mathrm{rig}}$ for some $r \gg 0$. After enlarging r if necessary, we can assume that there exists a finite covering $X = \bigcup U_i$ and étale lattices $M_i \subset \mathcal{N}_r|_{U_i}$. Using Proposition 4.10 we find that there exist a quasi-coherent $\mathscr{A}_X^{[r,\infty)}$-module $N_0 \subset \mathcal{N}$ such that

$$N_0|_{U_i} \subset M_i$$

and such that N_0 generates $\mathcal{N}$ as a $\mathscr{B}^{\dagger,r}_{\mathrm{rig}}$-module. As N_0 is quasi-coherent, we may assume (after eventually refining the covering) that U_i is affine and that $N_0|_{U_i}$ is associated to a module over $\Gamma(U_i, \mathscr{A}_X^{[r,\infty)})$.

Let $\mathcal{N}_{r_i}$ denote the restriction of $\mathcal{N}_r$ to $\mathscr{B}^{\dagger,r_i}_{X,\mathrm{rig}}$, where we write $r_i = p^i r$. Then we inductively define quasi-coherent $\mathscr{A}_X^{[r_i,\infty)}$-modules $N_i \subset \mathcal{N}_{r_i}$ by setting

$$N_{i+1} = N_i \otimes_{\mathscr{A}_X^{[r_i,\infty)}} \mathscr{A}_X^{[r_{i+1},\infty)} + \Phi(\varphi^* N_i).$$

By assumption, we always have

$$N_j|_{U_i} \subset M_i \otimes_{\mathscr{A}_{U_i}^{[r,\infty)}} \mathscr{A}_{U_i}^{[r_j,\infty)}.$$

We define an $\mathscr{A}_X^\dagger$-submodule $N \subset \mathcal{N}$, by setting

$$N = \left(\varinjlim_{i\in\mathbb{N}} N_i\right) \otimes \mathscr{A}_X^\dagger,$$

where the direct limit again is the direct limit in the category of sheaves. Further we define $\hat{N}$ to be the image of the canonical morphism

$$N \otimes_{\mathscr{A}_X^\dagger} \mathscr{A}_{X,K} \longrightarrow \hat{\mathcal{N}},$$

where $\hat{\mathcal{N}}$ is the $\mathscr{B}_{X,K}$-module associated to $\mathcal{N}$. We claim that $\hat{N}$ is coherent. This is a local claim and may be checked on affinoid open subsets $U = \mathrm{Spa}(A, A^+) \subset X$ such that $U \subset U_i$ for some i and such that $N_0|_U$ is associated to a $\Gamma(U, \mathscr{A}_X^{[r,\infty)})$-module. It follows from the construction that $\hat{N}|_U$ is the sheaf associated to the $\Gamma(U, \mathscr{A}_{X,K}) = (A^+ \otimes_{\mathbb{Z}_p} W')((T)))^\wedge$-module

$$\Gamma(U, \hat{N}) \subset \Gamma(U, M_i \otimes_{\mathscr{A}_{U_i}^\dagger} \mathscr{A}_{X,K}). \tag{4.2}$$

We point out that the ring $(A^+ \otimes_{\mathbb{Z}_p} W')((T))^\wedge$ is noetherian. Indeed, the ring A^+ is the ring of power bounded elements in a reduced Tate-algebra and hence noetherian, compare Remark 3.2. The ring $(A^+ \otimes_{\mathbb{Z}_p} W')((T))$ is a localization of the noetherian ring $(A^+ \otimes_{\mathbb{Z}_p} W')[[T]]$ (the power-series ring over a noetherian ring is noetherian) and hence itself noetherian. Finally the p-adic completion of the noetherian $\mathbb{Z}_p$-algebra $(A^+ \otimes_{\mathbb{Z}_p} W')((T))$ is still noetherian.

As the right hand side of (4.2)is finitely generated, so is the left hand side and it follows that $\hat{N}$ is coherent.

Moreover $\hat{N}$ can be defined over a formal model $\mathcal{X}$ of X: indeed we may take a formal model such that there is an open covering of $\mathcal{X}$ realizing a covering of X by open subsets of the form U as above. This formal model clearly does the job.

Further the construction implies that

$$\hat{\Phi}(\varphi^* \hat{N}) \subset \hat{N},$$
$$\hat{N} \otimes_{\mathscr{A}_{X,K}} \mathscr{B}_{X,K} = \hat{\mathcal{N}}.$$

It is left to show that $\hat{\Phi}(\varphi^*\hat{N}) \to \hat{N}$ is an isomorphism. In order to do so, we may work locally on X and hence assume that X is affinoid and $\hat{N}$ is contained in an étale $\mathscr{A}_{X,K}$-lattice $\hat{M} \subset \hat{\mathcal{N}}$. Moreover we may assume that $\hat{N}$ is the coherent sheaf associated to its global sections (that we also denote by $\hat{N}$ by abuse of notation) and that these global sections are finitely generated over $\Gamma(X, \mathscr{A}_{X,K})$.

Given a maximal ideal $\mathfrak{m} \subset A^+$ we denote by $k_\mathfrak{m} = A^+/\mathfrak{m}$ the residue field of $\mathfrak{m}$. By Nakayama's lemma we are reduced to show that for all maximal ideals $\mathfrak{m} \subset A^+$ the canonical map of finite free $(k_\mathfrak{m} \otimes_{\mathbb{F}_p} k')((T))$-modules (here k' denotes the residue field of $\mathcal{O}_{K_0'}$)

$$\hat{\Phi} : \varphi^*\hat{N} \otimes_{A^+} k_\mathfrak{m} \longrightarrow \hat{N} \otimes_{A^+} k_\mathfrak{m}, \tag{4.3}$$

is an isomorphism. As both, source and target, have the same dimension as $k_\mathfrak{m}((T))$-vector spaces (one is just a "twist" of the other) it is enough to show that the map is surjective.

For a rigid analytic point $x \in X$ we write $\mathfrak{m}_x \subset A$ for the maximal ideal defining x and $\mathfrak{m}_x^+ = \mathfrak{m}_x \cap A^+ \subset A^+$. Then x specializes to $\mathfrak{m} \in \operatorname{Spec} A^+/pA^+$ if and only if $\mathfrak{m}_x^+ \subset \mathfrak{m}$.

Given x, the fiber $\hat{N} \otimes k(x)^+$ is a finitely generated module over the ring $\mathscr{A}_{X,K} \otimes k(x)^+$ which is (a product of) complete discrete valuation rings. Write

$$\big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}} \subset \hat{N} \otimes k(x)^+$$

for the submodule which is ϖ_x-torsion free. This submodule has to be free and

$$\begin{aligned}\big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}}[\tfrac{1}{p}] &= (\hat{N} \otimes k(x)^+)[\tfrac{1}{p}] \\ &= (\hat{M} \otimes k(x)^+)[\tfrac{1}{p}] = \hat{\mathcal{N}} \otimes k(x).\end{aligned}$$

It follows from Lemma 4.12 below that $\big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}}$ is an étale φ-module, i.e. $\hat{\Phi}$ is surjective.

Now we consider the morphism (4.3) and assume $\bar{f} \in \hat{N} \otimes_{A^+} k_\mathfrak{m}$. As $\hat{N}$ is p-torsion free, there exists some $x \in X$ such that x is in the tube of $\mathfrak{m}$ and $f \in \big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}}$ such that $\bar{f} = f \mod \varpi_x$, where ϖ_x is a uniformizer of $k(x)^+$, i.e. $\mathfrak{m}/\mathfrak{m}_x^+ = (\varpi_x)$ as ideals in $A^+/\mathfrak{m}_x^+ = k(x)^+$. As $\big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}}$ is étale, there exists an $f' \in \varphi^*\big(\hat{N} \otimes k(x)^+\big)^{\text{tors-free}}$ such that $\hat{\Phi}(f') = f$. Reducing modulo ϖ_x it follows that $(f' \mod \varpi_x)$ maps to $\bar{f}$. We have shown that (4.3) is surjective as claimed. □

Lemma 4.12. *Let F be a finite extension of $\mathbb{Q}_p$ and $(\hat{N}, \hat{\Phi})$ be a free étale φ-module over $\mathscr{A}_{F,K}$. Let $\hat{N}_1 \subset \hat{N}$ be a finitely generated submodule such that $\hat{N}_1[1/p] = \hat{N}[1/p]$ and $\hat{\Phi}(\varphi^*\hat{N}_1) \subset \hat{N}_1$. Then $(\hat{N}_1, \hat{\Phi})$ is an étale φ-module, i.e.*

$$\hat{\Phi}(\varphi^*\hat{N}_1) = \hat{N}_1.$$

Proof. As $\mathscr{A}_{F,K}$ is (a product of) discrete valuation rings, it is clear that $\hat{N}_1$ is free on d generators, where d is the $\mathscr{A}_{F,K}$-rank of $\hat{N}$. Let $b_1,\dots,b_d$ be a basis of $\hat{N}$ and $e_1,\dots,e_d$ be a basis of $\hat{N}_1$. Let A denote the change of basis matrix from $\underline{b}$ to $\underline{e}$ and denote by $\mathrm{Mat}_{\underline{b}}(\hat{\Phi})$ resp. $\mathrm{Mat}_{\underline{e}}(\hat{\Phi})$ the matrix of $\hat{\Phi}$ in the basis $\underline{b}$ resp. $\underline{e}$ of $\hat{N}[1/p] = \hat{N}_1[1/p]$. Then our assumptions imply that

$$\mathrm{Mat}_{\underline{e}}(\hat{\Phi}) \in \mathrm{Mat}_{d\times d}(\mathscr{A}_{F,K}).$$

On the other hand

$$\mathrm{Mat}_{\underline{e}}(\hat{\Phi}) = A^{-1}\mathrm{Mat}_{\underline{b}}(\hat{\Phi})\varphi(A)$$

and hence $\det\big(\mathrm{Mat}_{\underline{e}}(\hat{\Phi})\big) \in \mathscr{A}_{F,K}^{\times}$, as $\hat{N}$ is étale, and

$$\mathrm{val}_p(\det A) = \mathrm{val}_p(\det\varphi(A)). \qquad \square$$

5. Families of p-adic Galois representations

In this section we study the relation between Galois representations and (φ,Γ)-modules in families. This problem was first considered by Dee in [9] for families parametrized by a complete local noetherian $\mathbb{Z}_p$-algebra. Later the problem was considered by Berger and Colmez in [3] and Kedlaya and Liu in [17], where they define a functor from p-adic families of G_K-representations to p-adic families of overconvergent (φ,Γ)-modules.

Definition 5.1. Let G a topological group and X an adic space locally of finite type over $\mathbb{Q}_p$. A family of G-representations over X is a vector bundle $\mathcal{V}$ over X endowed with a continuous G-action.

We write $\mathrm{Rep}_X G$ for the category of families of G-representations over X. Recall that we write $G_K = \mathrm{Gal}(\bar{K}/K)$ for the absolute Galois group of a fixed local field K. In this case Berger and Colmez define the functor

$$\mathbf{D}^{\dagger} : \mathrm{Rep}_X G_K \longrightarrow \big\{\text{étale } (\varphi,\Gamma)\text{-modules over } \mathscr{B}_X^{\dagger}\big\},$$

which maps a family $\mathcal{V}$ of G_K-representations on X to the étale (φ,Γ)-module

$$\mathbf{D}^{\dagger}(\mathcal{V}) = \big(\mathcal{V}\otimes_{\mathcal{O}_X}\tilde{\mathscr{B}}_X^{\dagger}\big)^{H_K}.$$

More precisely they construct this functor if X is a reduced affinoid adic space of finite type. As the functor $\mathbf{D}^{\dagger}$ is fully faithful in this case and maps $\mathcal{V}$ to a free $\mathscr{B}_X^{\dagger}$-module it follows that we can consider $\mathbf{D}^{\dagger}$ on the full category $\mathrm{Rep}_X G_K$, whenever X is reduced.

In [17] Kedlaya and Liu consider the variant

$$\mathbf{D}_{\mathrm{rig}}^{\dagger} : \mathcal{V} \longmapsto \big(\mathcal{V}\otimes_{\mathcal{O}_X}\tilde{\mathscr{B}}_{X,\mathrm{rig}}^{\dagger}\big)^{H_K} = \mathbf{D}^{\dagger}(\mathcal{V})\otimes_{\mathscr{B}_X^{\dagger}}\mathscr{B}_{X,\mathrm{rig}}^{\dagger}$$

which we will also consider here. Note that for an adic space X of finite type over $\mathbb{Q}_p$, the (φ,Γ)-module $\mathbf{D}^\dagger(\mathcal{V})$ is always defined over some $\mathscr{B}_X^{\dagger,s}\subset\mathscr{B}_X^\dagger$, for $s\gg 0$. Especially an étale lattice can be defined over $\mathscr{A}_X^{[s,\infty)}$ for $s\gg 0$.

5.1. The admissible locus. In this section we will always assume that our adic spaces are reduced.

It is known that the functors $\mathbf{D}^\dagger$ and $\mathbf{D}^\dagger_{\mathrm{rig}}$ are not essentially surjective. In [17], Kedlaya and Liu construct a *local inverse* to this functor. More precisely, they show that if $\mathcal{N}$ is a family of (φ,Γ)-modules over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$, then every rigid analytic point at which $\mathcal{N}$ is étale has an affinoid neighborhood on which the family $\mathcal{N}$ is the image of a family of G_K-representations. However, we need to extend this result to the setup of adic spaces in order to define a natural *subspace* over which such a family $\mathcal{N}$ is induced by a family of G_K-representations.

Theorem 5.2. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $\mathcal{N}$ be a family of (φ,Γ)-modules of rank d over $\mathscr{B}^\dagger_{X,\mathrm{rig}}$.*

(i) *The subset*

$$X^{\mathrm{adm}}=\left\{x\in X\;\middle|\;\begin{array}{c}\dim_{k(x)}\big((\mathcal{N}\otimes_{\mathscr{B}^\dagger_{X,\mathrm{rig}}}\tilde{\mathscr{B}}^\dagger_{X,\mathrm{rig}})\otimes k(x)\big)^{\Phi=\mathrm{id}}=d\\ \textit{and this } k(x)\textit{-vector space generates}\\ (\mathcal{N}\otimes_{\mathscr{B}^\dagger_{X,\mathrm{rig}}}\tilde{\mathscr{B}}^\dagger_{X,\mathrm{rig}})\otimes k(x)\end{array}\right\}$$

is open.

(ii) *There exists a family of G_K-representations $\mathcal{V}$ on X^{adm} such that there is a canonical and functorial isomorphism*

$$\boldsymbol{D}^\dagger_{\mathrm{rig}}(\mathcal{V})\cong\mathcal{N}|_{X^{\mathrm{adm}}}.$$

(iii) *Let $\mathcal{V}$ be a family G_K-representations on X such that $\boldsymbol{D}^\dagger_{\mathrm{rig}}(\mathcal{V})=\mathcal{N}$. Then $X^{\mathrm{adm}}=X$.*

Remark 5.3. Note that (iii) is not contained in [12] (in the context of a different semi-linear operator): there the claim is only made if we assume $X^{\mathrm{int}}=X$.

Let A be a complete topological $\mathbb{Q}_p$-algebra and let $A^+\subset A$ be a ring of integral elements. Assume that the completed tensor products $A^+\widehat{\otimes}\tilde{\mathbf{A}}^\dagger$ and $A\widehat{\otimes}\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ are defined[5]. In this case the following approximation Lemma of Kedlaya and Liu applies.

[5]The examples we consider here, are $\Gamma(X,\mathcal{O}_X)$ for an affinoid adic space of finite type and the completions of $k(x)$ for a point $x\in X$. In the latter case the completed tensor product is the completion of the fiber of $\tilde{\mathscr{A}}^\dagger$ resp. $\tilde{\mathscr{B}}^\dagger_{\mathrm{rig}}$ at the point x.

Lemma 5.4. *Let $\tilde{\mathcal{N}}$ be a free (φ,Γ)-module over $A\widehat{\otimes}\tilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$ such that there exists a basis on which Φ acts via* $\mathrm{id}+B$ *with*

$$B \in p\mathrm{Mat}_{d\times d}(A^{+}\widehat{\otimes}\tilde{\boldsymbol{A}}^{\dagger}).$$

Then $\tilde{\mathcal{N}}^{\Phi=\mathrm{id}}$ is free of rank d as an A-module. Moreover, an A-module basis of $\tilde{\mathcal{N}}^{\Phi=\mathrm{id}}$ is an $A\widehat{\otimes}\tilde{\boldsymbol{B}}^{\dagger}_{\mathrm{rig}}$-module basis of $\tilde{\mathcal{N}}$.

Proof. This is [17, Theorem 5.2]. □

Corollary 5.5. *Let X be an adic space locally of finite type over $\mathbb{Q}_p$ and $\tilde{\mathcal{N}}$ be a family of (φ,Γ)-modules over $\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}}$. Let x in X, then*

$$\dim_{\widehat{k(x)}}(\iota_x^*\tilde{\mathcal{N}})^{\dot{\mathrm{L}}=\mathrm{id}} = d \iff \dim_{k(x)}\big((\mathcal{N}\otimes_{\mathscr{B}^{\dagger}_{X,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}})\otimes k(x)\big)^{\Phi=\mathrm{id}} = d.$$

Proof. The proof is the same as the proof of [12, Proposition 8.20 (i)]. □

Proof of Theorem 5.2. Let $x\in X^{\mathrm{adm}}$ and denote by Z the Zariski-closure of x, that is, the subspace defined by the ideal of all functions vanishing at x. This is an reduced adic space locally of finite type and we have $k(x)=\mathcal{O}_{Z,x}$, as the ideal of functions on Z that vanish at x is trivial by definition. Then

$$\begin{aligned}\big((\mathcal{N}\otimes_{\mathscr{B}^{\dagger}_{X,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}})\otimes k(x)\big) &= \big((\mathcal{N}|_Z\otimes_{\mathscr{B}^{\dagger}_{Z,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{Z,\mathrm{rig}})\otimes k(x)\big)\\ &= \varinjlim_{x\in U\subset Z}\Gamma(U,\mathcal{N}|_Z\otimes_{\mathscr{B}^{\dagger}_{Z,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{Z,\mathrm{rig}}).\end{aligned}$$

By this identification we may choose an affinoid neighborhood $U\subset Z$ of x in Z such that a basis of the Φ-invariants extends to U and forms a basis of $\mathcal{N}|_U$. Then

$$\mathcal{V}_U = \big(\mathcal{N}|_Z\otimes_{\mathscr{B}^{\dagger}_{Z,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{U,\mathrm{rig}}\big)^{\Phi=\mathrm{id}}$$

is free of rank d over $\mathcal{O}_U$ and

$$\mathcal{V}_U\otimes_{\mathcal{O}_U}\tilde{\mathscr{B}}^{\dagger}_{U,\mathrm{rig}} = \mathcal{N}|_Z.$$

On $\mathcal{V}_U$ we have the diagonal G_K-action given by the natural action on $\tilde{\mathscr{B}}^{\dagger}_{U,\mathrm{rig}}$ and the Γ-action on $\mathcal{N}$. It is a direct consequence of the construction that

$$\mathbf{D}^{\dagger}_{\mathrm{rig}}(\mathcal{V}_U) = \mathcal{N}|_U.$$

Especially it follows that $\mathcal{N}$ is étale at x. It follows that we already have $X^{\mathrm{adm}}\subset X^{\mathrm{int}}$. Replacing X by X^{int} we may assume that $\mathcal{N}$ is étale everywhere.

Now let $x\in X^{\mathrm{adm}}$ and let U denote a neighborhood of x to which we can lift a basis of Φ-invariants. As $\mathcal{N}$ is known to be étale, we can shrink U such that we are in the situation of Lemma 5.4.

It follows that X^{adm} is open and that

$$\big(\mathcal{N}\otimes_{\mathscr{B}^{\dagger}_{X,\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{X,\mathrm{rig}}\big)^{\Phi=\mathrm{id}}$$

gives a vector bundle $\mathcal{V}$ on X^{adm}. Again, we have the diagonal action of G_K. As above we find that

$$\mathbf{D}^{\dagger}_{\mathrm{rig}}(\mathcal{V})=\mathcal{N}|_{X^{\mathrm{adm}}\cap X^{\mathrm{int}}}=\mathcal{N}|_{X^{\mathrm{adm}}}.$$

Finally (iii) is obvious by the construction of [3]. □

Theorem 5.6. *Let $f : X\to Y$ be a morphism of adic spaces locally of finite type over $\mathbb{Q}_p$ with Y reduced. Further let $\mathcal{N}_Y$ be a family of (φ,Γ)-modules over $\mathscr{B}^{\dagger}_{Y,\mathrm{rig}}$ and write $\mathcal{N}_X$ for the pullback of $\mathcal{N}_Y$ to X. Then $f^{-1}(Y^{\mathrm{adm}})=X^{\mathrm{adm}}$ and $f^*\mathcal{V}_Y=\mathcal{V}_X$ on X^{adm}.*

Proof. Using the discussion above, the proof is the same as the proof of [12, Proposition 8.22]. □

Proposition 5.7. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $\mathcal{N}$ be a family of (φ,Γ)-modules over $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$. Then the inclusion*

$$f : X^{\mathrm{adm}}\longrightarrow X$$

is open and partially proper.

Proof. We have already shown that f is open. Especially it is quasi-separated and hence we may apply the valuative criterion for partial properness, see [14, 1.3]. Let (x,A) be a valuation ring of X with $x\in X^{\mathrm{adm}}$ and let $y\in X$ be a center of (A,x). We need to show that $y\in X^{\mathrm{adm}}$. As y is a specialization of x, the inclusion $i : k(y)\hookrightarrow k(x)$ identifies $k(y)$ with a dense subfield of $k(x)$. Especially

$$\tilde{\mathcal{N}}_y:=\mathcal{N}\otimes_{\mathscr{B}^{\dagger}_{k(y),\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{k(y),\mathrm{rig}}\longrightarrow\mathcal{N}\otimes_{\mathscr{B}^{\dagger}_{k(x),\mathrm{rig}}}\tilde{\mathscr{B}}^{\dagger}_{k(x),\mathrm{rig}}=:\tilde{\mathcal{N}}_x$$

is dense. Let $e_1,\dots,e_d$ be a basis of $\tilde{\mathcal{N}}_x$ on which Φ acts as the identity. We may approximate this basis by a basis of $\tilde{\mathcal{N}}_y$. Thus we can choose a basis of $\tilde{\mathcal{N}}_y$ on which Φ acts by $\mathrm{id}+A$ with

$$A\in\mathrm{Mat}_{d\times d}(\tilde{\mathscr{B}}^{\dagger}_{k(y),\mathrm{rig}})$$

sufficently small. For example we can choose

$$A\in p\,\mathrm{Mat}_{d\times d}(\tilde{\mathscr{A}}^{\dagger}_{k(y)}).$$

By Lemma 5.4 and Corollary 5.5 it follows that $y\in X^{\mathrm{adm}}$. □

5.2. Existence of Galois representations. In this section we link deformations of Galois representations and deformations of étale φ-modules.

In the following $(R, \mathfrak{m})$ will denote a complete local noetherian ring, topologically of finite type over $\mathbb{Z}_p$. Again we have a notion of an étale φ-module over

$$R\widehat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K = \varprojlim_n \big((R/\mathfrak{m}^n)\otimes_{\mathbb{Z}_p}\mathbf{A}_K\big).$$

By this we mean an $R\hat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K$-module D of finite type together with an isomorphism $\Phi : \varphi^* D \to D$. Note that D is not required to be locally free.

A Galois representation with coefficients in R (or a family of Galois representations on Spf R) is a continuous representation

$$G \longrightarrow \mathrm{GL}_d(R),$$

where G is the absolute Galois group of some field L. The relation between Galois representations and étale φ-modules with coefficients in local rings was first considered by Dee, see [9, 2].

Theorem 5.8. *Let X be a reduced adic space of finite type over $\mathbb{Q}_p$ and let $(\mathcal{N}, \Phi)$ be a family of étale φ-modules over $\mathscr{B}^{\dagger}_{X,\mathrm{rig}}$. Then there exists a formal model $\mathcal{X}$ of X and an étale $\mathscr{A}_{X,K}$-module $N \subset \hat{\mathcal{N}}$ generating $\hat{\mathcal{N}}$ that admits a model over $\mathcal{X}$. Let $x_0 \in \bar{\mathcal{X}}$ be a closed point in the special fiber of $\mathcal{X}$ of X and let $Y \subset X$ denote the tube over x_0. Then $(\mathcal{N}, \Phi)|_Y$ is associated to a family of H_K-representations on the open subspace Y.*

Proof. Let us write $\hat{\mathcal{N}}$ for the $\mathscr{B}_{X,K}$-module associated to $\mathcal{N}$. It follows from Proposition 4.8 that there exist an étale φ-module $\hat{N}$ over $\mathscr{A}_{X,K}$ such that $\hat{N} \subset \hat{\mathcal{N}}$ as φ-modules and such that $\hat{N}$ contains a basis of $\hat{\mathcal{N}}$. Moreover there is some formal model $\mathcal{X}$ of X such that $\hat{N}$ is defined over $\mathcal{X}$. Choose a formal affine neighborhood $\mathcal{U} = \mathrm{Spf}(A^+)$ of x_0 and write U for its generic fiber. We write $\mathfrak{m} \subset A^+$ for the maximal ideal defining x_0 and write R for the $\mathfrak{m}$-adic completion of A^+. Then Y is the generic fiber of Spf R in the sense of Berthelot. Write $\mathfrak{N} = \Gamma(U, \hat{N})$. This is a $\Gamma(U, \mathscr{A}_{X,K})$-module on which $\hat{\Phi}$ induces a semi-linear isomorphism.

It follows that $\hat{\mathfrak{N}} = \mathfrak{N}\widehat{\otimes}_{A^+}R$ is a finitely generated étale φ-module over $\Gamma(Y, \mathscr{A}_{X,K}) = R\widehat{\otimes}_{\mathbb{Z}_p}\mathbf{A}_K$. Hence, by [9], there is a finitely generated R-module E with continuous H_K action associated with $\hat{\mathfrak{N}}$. Then

$$Y \supset V \mapsto E \otimes_R \Gamma(V, \mathcal{O}_X)$$

defines the desired family of Galois representations[6] on Y . □

[6]Note that we do not claim that locally on Y the integral representation E is associated with an étale lattice in $(\mathcal{N}, \Phi)$. This is only true up to p-isogeny.

Corollary 5.9. *Let X be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and $\mathcal{N}$ be a family of étale (φ, Γ)-modules on X. Then there exists a formal model $\mathcal{X}$ of X and an étale $\mathscr{A}_{X,K}$-module $N \subset \hat{\mathcal{N}}$ generating $\hat{\mathcal{N}}$ which admits a model over $\mathcal{X}$. Let $x_0 \in \bar{\mathcal{X}}$ be a closed point in the special fiber $\mathcal{X}$ of X and let $Y \subset X$ denote the tube of x_0. Then $\mathcal{N}|_Y$ is associated to a family of G_K-representations on the open subspace Y.*

Proof. By the above theorem it follows that $Y = Y^{\mathrm{adm}}$. The claim follows from Theorem 5.2. □

Conjecture 5.10. *The claim of the theorem (and the corollary) also holds true if we replace x_0 by a (locally) closed subscheme of the special fiber over which there exists a Galois representation that is associated with the reduction of the étale submodule.*

5.3. Local constancy of the reduction modulo p.

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$, uniformizer ϖ_L and residue field k_L. Let V be a d-dimensional L-vector space with a continuous action of a compact group G. We choose a G-stable $\mathcal{O}_L$-lattice $\Lambda \subset V$ and write $\bar{\Lambda} = \Lambda/\varpi_L\Lambda$ for the reduction modulo the maximal ideal of $\mathcal{O}_L$. Then $\bar{\Lambda}$ is a (continuous) representation of G on a d-dimensional $k_L = \mathcal{O}_L/\varpi_L\mathcal{O}_L$-vector space. The representation $\bar{\Lambda}$ depends on the choice of a G-stable lattice $\Lambda \subset V$, however it is well known that its semisimplification $\bar{\Lambda}^{\mathrm{ss}}$(i.e. the direct sum of its Jordan-Hölder constituents) is independent of Λ and hence only depends on the representation V. In the following we will write $\bar{V}$ for this representation and refer to it as the reduction modulo ϖ_L of the representation V.

The aim of this section is to show that the reduction modulo ϖ_L is locally constant in a family[7] of p-adic representations of G. In the context of families of Galois representations this was shown by Berger for families of 2-dimensional crystalline representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ in a weaker sense: Berger showed that every rigid analytic point has a neighborhood on which the reduction is constant, see [2].

Let X be an adic space locally of finite type over $\mathbb{Q}_p$ and E a vector bundle on X endowed with a continuous G-action. If $x \in X$, then we write

$$\big(E \otimes \overline{k(x)}\big) = \big(\overline{E \otimes k(x)}\big)^{\mathrm{ss}}$$

for the semisimplification of the G-representation in the special fiber

$$\overline{k(x)} = k(x)^+/(\varpi_x)$$

of $k(x)$.

[7]This seems to be a well known fact, at least in the context of pseudo-characters. As we do not want to assume $p > d$ here, we give a different proof.

Proposition 5.11. *Let X be an adic space locally of finite type and let E be a vector bundle on X endowed with a continuous action of a compact group G. Then the semi-simplification of the reduction $E \otimes \overline{k(x)}$ is locally constant.*

Proof. As the claim remains the same once we replace X by its reduced underlying subspace, we may assume that X is reduced. Moreover, we may assume that $X = \mathrm{Spa}(A, A^+)$ is affinoid. For $g \in G$ we consider the map

$$f_g : x \longmapsto \mathrm{charpoly}\big(g|E \otimes k(x)\big)$$

Let us write $f_{g,i}(x)$ for the i-th coefficient of $f_g(x)$. As $E \otimes k(x)$ admits an $k(x)^+$-lattice stable under the action of G, we find that $f_{g,i}(x) \in k(x)^+$, and hence $f_{g,i}$ defines a map

$$f_i : G \longrightarrow \Gamma(X, \mathcal{O}_X^+) = A^+.$$

By construction this map is continuous and hence so is the induced map

$$\bar{f}_i : G \longrightarrow \bar{A} = A^+/A^{++},$$

where $A^{++} \subset A$ denotes the ideal of topologically nilpotent elements. However, as $\bar{A}$ is endowed with the discrete topology this morphism has to be constant. On the other hand $\bar{f}_{g,i}(x)$ is the i-th coefficient of the characteristic polynomial of g acting on $\mathcal{E}_x/\varpi_x\mathcal{E}_x$, where $\mathcal{E}_x \subset E \otimes k(x)$ is a G-stable $k(x)^+$-lattice and $\varpi_x \in k(x)^+$ is a uniformizer. Now [8, Theorem 30.16] implies the claim[8]. □

6. An example

In this section we give an example in order to show how the condition on the reduction modulo p to be locally constant obstructs the existence of a global étlale lattice.

For this section we use different notations. Let K be a totally ramified quadratic extension of $\mathbb{Q}_p$. Fix a uniformizer $\pi \in \mathcal{O}_K$ and a compatible system $\pi_n \in \bar{K}$ of p^n-th roots of π. Let us write $K_\infty = \bigcup K(\pi_n)$ and $G_{K_\infty} = \mathrm{Gal}(\bar{K}/K_\infty)$ for this section. Further let $E(u) \in \mathbb{Z}_p[u]$ denote the minimal polynomial of π. Finally we adapt the notation from [12] and write

$$\mathscr{B}_X^R = \mathscr{B}_{X,\mathrm{rig}}^\dagger \quad \text{and} \quad \mathscr{B}_X^{[0,1)} = \mathrm{pr}_{X,*}\,\mathcal{O}_{X\times\mathbb{U}}.$$

We consider the following family $(D, \Phi, \mathcal{F}^\bullet)$ of filtered φ-modules on

$$X = \mathbb{P}_K^1 \times \mathbb{P}_K^1.$$

[8]After this paper was written, we noticed that the idea to consider all coefficients of the characteristic polynomial is used in [7] to generalize pseudo-characters.

Let $D = \mathcal{O}_X^2 = \mathcal{O}_X e_1 \oplus \mathcal{O}_X e_2$ and $\Phi = \operatorname{diag}(\varpi_1, \varpi_2)$, where ϖ_1 and ϖ_2 are the zeros of $E(u)$. We consider a filtration $\mathcal{F}^\bullet$ of $D_K = D \otimes_{\mathbb{Q}_p} K$ such that $\mathcal{F}^0 = D_K$ and $\mathcal{F}^2 = 0$. Fix an isomorphism $D \otimes_{\mathbb{Q}_p} K \cong \mathcal{O}_X^2 \oplus \mathcal{O}_X^2$ and let the filtration step $\mathcal{F}^1$ be the universal subspace on X. This is a family of filtered φ-modules in the sense of [12]. One easily computes that

$$X^{\mathrm{wa}} = X \backslash \{(0,0), (\infty, \infty)\},$$

where $X^{\mathrm{wa}} \subset X$ is the weakly admissible locus defined in [12, 4.2]. Generalizing a construction of Kisin [16] the family $(D, \Phi, \mathcal{F}^\bullet)$ defines a family $(\mathcal{M}, \Phi)$ consisting of a vector bundle on $X^{\mathrm{wa}} \times \mathbb{U}$ and an injection $\Phi : \varphi^*\mathcal{M} \to \mathcal{M}$ such that $E(u) \operatorname{coker} \Phi = 0$ (see [12, Theorem 5.4]).

We define the family $(\mathcal{N}, \Phi)$ over $\mathscr{B}^R_{X^{\mathrm{wa}}}$ as

$$(\mathcal{N}, \Phi) = (\mathcal{M}, \Phi) \otimes_{\mathscr{B}^{[0,1)}_{X^{\mathrm{wa}}}} \mathscr{B}^R_{X^{\mathrm{wa}}}. \tag{6.1}$$

This is obviously a family of φ-modules over the Robba ring which is étale at all rigid analytic points. We can cover the weakly admissible set $X^{\mathrm{wa}} = X_1 \cup X_2 \cup X_3 \cup X_4$, where

$$\begin{aligned}
X_1 &= \big((\mathbb{P}^1\backslash\{\infty\}) \times (\mathbb{P}^1\backslash\{\infty\})\big)\backslash\{(0,0)\} \cong \mathbb{A}^2\backslash\{0\},\\
X_2 &= (\mathbb{P}^1\backslash\{\infty\}) \times (\mathbb{P}^1\backslash\{0\}) \cong \mathbb{A}^2,\\
X_3 &= (\mathbb{P}^1\backslash\{0\}) \times (\mathbb{P}^1\backslash\{\infty\}) \cong \mathbb{A}^2,\\
X_4 &= \big((\mathbb{P}^1\backslash\{0\}) \times (\mathbb{P}^1\backslash\{0\})\big)\backslash\{(\infty,\infty)\} \cong \mathbb{A}^2\backslash\{0\}.
\end{aligned}$$

Now the space X^{wa} contains K-valued points x_1, x_2 and x_3 such that

$$\begin{aligned}
(\mathcal{M}, \Phi) \otimes k(x_1) &\cong \left(\mathcal{O}^2_{\mathbb{U}_K}, \begin{pmatrix} 0 & -E(u) \\ 1 & \varpi_1 + \varpi_2 \end{pmatrix}\right)\\
(\mathcal{M}, \Phi) \otimes k(x_2) &\cong \left(\mathcal{O}^2_{\mathbb{U}_K}, \begin{pmatrix} 0 & -(u - \varpi_1) \\ (u - \varpi_2) & \varpi_1 + \varpi_2 \end{pmatrix}\right)\\
(\mathcal{M}, \Phi) \otimes k(x_3) &\cong \left(\mathcal{O}^2_{\mathbb{U}_K}, \begin{pmatrix} -(u - \varpi_1) & 0 \\ 0 & -(u - \varpi_2) \end{pmatrix}\right).
\end{aligned}$$

The semi-simplifications of the reduction modulo π of the obvious Φ-stable $W[[u]]$-lattices in these φ-modules are

$$\begin{aligned}
(\mathfrak{M}, \Phi) \otimes \overline{k(x_1)} &\cong \left(\mathbb{F}_p[[u]]^2, \begin{pmatrix} 0 & -u^2 \\ 1 & 0 \end{pmatrix}\right),\\
(\mathfrak{M}, \Phi) \otimes \overline{k(x_2)} &\cong \left(\mathbb{F}_p[[u]]^2, \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}\right),\\
(\mathfrak{M}, \Phi) \otimes \overline{k(x_3)} &\cong \left(\mathbb{F}_p[[u]]^2, \begin{pmatrix} -u & 0 \\ 0 & -u \end{pmatrix}\right).
\end{aligned}$$

Using Caruso's classification [6, Corollary 8] of those φ-modules we find that they are all non-isomorphic and in fact even stay non-ismomorphic after inverting u. After inverting u these φ-modules correspond (up to twist) under Fontaine's equivalence of categories to the restriction to G_{K_∞} of the reduction modulo π of the constructed Galois representations $\mathcal{E} \otimes k(x_i)$. By [5, Theorem 3.4.3] this restriction is fully faithful and hence we find that

$$\overline{\mathcal{E} \otimes k(x_i)} \ncong \overline{\mathcal{E} \otimes k(x_j)}$$

as G_K-representations for $i \neq j$.

As $(\mathcal{M}, \Phi)$ is admissible in a neighborhood of each of the x_i, we can find some y_i such that

$$\overline{\mathcal{E} \otimes k(x_i)} \cong \overline{\mathcal{E} \otimes k(y_i)}$$

for $i = 1, 2, 3$ and such that in addition $y_i \in X_2$ for all i for example. Let us fix a covering $X_2 \cong \mathbb{A}^2 = \bigcup U_i$ by an increasing sequence of closed discs around the origin and let $V_i \subset U_i$ be the corresponding open disc.

Assume that there exists an étale $\mathscr{A}_X^\dagger$-lattice in $(\mathcal{N}, \Phi)$ over all the U_i defined above. Then it follows from Corollary 5.9 that there exists a family of G_K-representations associated to $(\mathcal{M}, \Phi)$ on all the V_i.

By the construction in [12] this family is naturally contained in

$$D \otimes_{\mathcal{O}_{V_i}} (\mathcal{O}_{V_i} \widehat{\otimes} B_{\mathrm{cris}})$$

and in fact identified with

$$\mathrm{Fil}^0 \left(D \otimes_{\mathcal{O}_{V_i}} (\mathcal{O}_{V_i} \widehat{\otimes} B_{\mathrm{cris}})\right)^{\Phi=\mathrm{id}}.$$

However, if this assumption is true for all i, we easily can find some i such that $y_1, y_2, y_3 \in U_i$ map to the origin in the special fiber, i.e. $y_1, y_2, y_3 \in V_i$. By Proposition 5.11, we know that the reduction modulo p of the G_K-representation on the fibers of the family $\mathcal{E}$ has to be constant, contradicting the choice of the y_i.

Hence we see that a formal model of U_i over which we have an integral étale structure as in Proposition 4.8 must be a blow up that separates the specializations of the points y_1, y_2 and y_3.

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