Equidistribution, ergodicity and irreducibility associated with Gibbs measures

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Abstract. We generalize an equidistribution theorem *à la* Bader–Muchnik for operator-valued measures constructed from a family of boundary representations associated with Gibbs measures in the context of convex cocompact discrete group of isometries of a simply connected connected Riemannian manifold with pinched negative curvature. We combine a functional analytic tool, namely the property RD of hyperbolic groups, together with a dynamical tool: an equidistribution theorem of Paulin, Pollicott and Schapira inspired by a result of Roblin. In particular, we deduce irreducibility of these new classes of boundary representations.

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1. Introduction

Viewing the group $SL(2, \mathbb{R})$ as a group acting by isometries of the hyperbolic plane we have an induced action on the geometric boundary of the hyperbolic plane which is identified with the circle. The Lebesgue measure is quasi-invariant under this boundary action (i.e. the sets of Lebesgue measure zero are preserved under the action) and so there is a naturally associated unitary representation of $SL(2, \mathbb{R})$ on $L^2(\mathbb{S}^1)$ called the quasi-regular representation. The quasi-regular representation is irreducible and is part of a family of irreducible unitary representations of $SL(2, \mathbb{R})$ on $L^2(\mathbb{S}^1)$ called the principal series which forms one of the families composing the unitary dual. For a general locally compact group G, especially when G is a discrete countable group, there is no hope of computing its unitary dual so we will restrict ourselves to the problem of determining when the associated quasi-regular representation of a G quasi-invariant action is irreducible. From a dynamical

viewpoint the associated quasi-regular representation is interesting because it reflects the ergodic theoretic properties of the action such as ergodicity and mixing.

Early on, Furstenberg [24] showed that when G is a semisimple Lie group the space G/P equipped with Haar measure where P is a minimal parabolic subgroup, nowadays called the Poisson-Furstenberg boundary, can be realized as the Poisson boundary of a random walk on a lattice in G. Motivated by these results we further restrict the problem which we state as the following conjecture of Bader and Muchnik [10]:

Conjecture 1.1. *For a locally compact group G and a spread-out probability measure μ on G, the quasi-regular representation associated to a μ-boundary of G is irreducible.*

For the rest of the paper we will restrict ourselves to the case when G is a discrete countable group. Analogously to the case of $SL(2, \mathbb{R})$ the action of the free group $\mathbb{F}_n$ on its boundary is quasi-invariant with respect to the Patterson–Sullivan measure class and thus there is the associated quasi-regular representation. Figà-Talamanca and Picardello (see [25] and [27]) construct the analog of the principal series which are unitary representations of $\mathbb{F}_n$ on $L^2(\partial \mathbb{F}_n)$ and show they are all irreducible. For homogeneous trees Figà-Talamanca and Steger [26] show similar irreducibility results for lattices in the automorphism group. Kuhn and Steger [36] have also constructed different examples of irreducible representations of the free group. The conjecture has also been solved for some actions of simple algebraic groups by Bekka and Cowling in [11]. When G is a lattice in a Lie group Cowling and Steger [21] showed that the irreducible representations of the ambient semisimple Lie group restricted to G remain irreducible. In particular the quasi-regular representation of $SL(2, \mathbb{R})$ on $L^2(\mathbb{S}^1)$ restricted to lattices is irreducible. Later on in the context of CAT(-1) spaces for which a discrete group of isometries G acts cocompactly, Connell and Muchnik (see [18] and [19]) proved when the geometric boundary is equipped with a certain class of Gibbs measures that it can be realized as the Poisson boundary of a random walk in G. This result led Bader and Muchnik [10] to prove the conjecture for the action of the fundamental group of a compact negatively curved manifold on the geometric boundary of the universal cover of the manifold, endowed with the Patterson–Sullivan measure class. Recently the first named author has also generalized the main theorem of Bader and Muchnik in [10] to the context of CAT(-1) spaces and so irreducibility of boundary representations associated with Patterson–Sullivan measures. Moreover Garncarek [29] has generalized the irreducibility result of [10] for the action of a Gromov-Hyperbolic group on its geometric boundary endowed with the Patterson–Sullivan measure class. He has also deduced thanks to the work of [13] that if a symmetric random walk on a Gromov-Hyperbolic group has finite exponential moment with respect to a word metric and such that the associated Green metric satisfies the *Ancona inequality* then the action on the Poisson boundary with respect to the harmonic measure is irreducible thanks to the work of [13]. It is

not clear at all to us if any of the measures constructed in [19] have finite exponential moment and satisfies the *Ancona inequality* and therefore it is not clear at all that our result of irreducibility would follow from an application of these two results. Hence it legitimates our dynamical approach to prove irreducibility of quasi-regular representations associated with the class of measure arising as conditional measures of the *Gibbs measures*, called *Gibbs streams* in [19] or also called *Patterson densities* in [43]; generalizing the Patterson–Sullivan measures class.

Bader and Muchnik prove in [10, Theorem 3] an equidistribution theorem for some operator-valued measures associated with Patterson–Sullivan measures. This theorem can be thought of as a generalization of von Neumann's ergodic theorem for quasi-invariant measures for fundamental groups acting on the geometric boundary of universal covers of compact negatively curved manifolds endowed with the Patterson–Sullivan measures. These quasi-regular representations are called *boundary representations*. It turns out that the irreducibility of boundary representations follows from this generalization of von Neumann's ergodic theorem.

We generalize the results of Bader and Muchnik to the action of a convex cocompact discrete subgroup of isometries of a pinched negatively curved manifold on its boundary endowed with the *Gibbs streams* measure class rather called in this paper *Patterson densities* measure class. The Patterson densities are constructed by first assigning a weight to each element of the orbit and then proceeding as in the construction of the Patterson–Sullivan measures which we think of as the unweighted case.

Historically it was Sinai who first merged the field of equilibrium statistical mechanics from which the concept of Patterson density is imported from with the field of hyperbolic smooth dynamical systems. Given a Hölder-continuous potential F on the unit tangent bundle of a compact negatively curved manifold, the pressure of F associated with the geodesic flow is given by

$$P(F) = \sup_m \left\{ h_m - \int F \, dm \right\},$$

where the supremum is taken over all measures on the unit tangent bundle which are invariant under the fundamental group of the manifold and the geodesic flow and h_m is the metric entropy of m associated with the geodesic flow. Bowen [7] proved for negatively curved manifolds that there exists a unique measure called the Gibbs measure which achieves the supremum and is in fact the eigenmeasure associated to the transfer operator of F. As we said, the *Patterson densities* arise as conditional measures of the Gibbs measure and when $F = 0$ the Patterson densities are the Patterson–Sullivan measures and the Gibbs measure is the Bowen–Margulis–Sullivan measure that maximizes the entropy.

The main tools of this paper are the property RD (Rapid Decay) that hyperbolic groups satisfy (see [32] and [33]) combined with a spectral characterization of the amenability of the action on the boundary (see [35] and [5]) together

with an equidistribution theorem of Paulin–Pollicott–Schapira inspired by Roblin's equidistribution theorem which is itself based on the mixing property of the geodesic flow. Indeed this idea of using the mixing property of the geodesic flow goes back to Margulis [38] who used it in order to count the closed geodesics on compact negatively curved manifolds. However, the first object to understand is the *Harish-Chandra function* associated with Patterson densities. This function plays a major role in harmonic analysis of spherical functions and in the theory of irreducible representations of semismple Lie groups, see for example [28].

Notation. Let M be a complete connected Riemannian manifold with pinched negative curvature. Let $X = \widetilde{M}$, let $q : X \to M$ be a universal Riemannian covering map with a covering group Γ viewed as a non-elementary discrete group of isometries of X, denote the sphere at infinity by ∂X and endow $\overline{X} = X \cup \partial X$ with the cone topology.

The limit set of Γ denoted by Λ_Γ is the set of all accumulation points in ∂X of an orbit. Namely $\Lambda_\Gamma := \overline{\Gamma x} \cap \partial X$, with the closure in $\overline{X}$. Notice that the limit set does not depend on the choice of $x \in X$. We denote by $\Omega\Gamma$ the subset of $T^1 X$ of tangent vectors to the geodesic lines in X whose endpoints both lie in Λ_Γ. Following the notation in [18], define the geodesic hull $GH(\Lambda_\Gamma)$ as the union of all geodesics in X with both endpoints in Λ_Γ. The convex hull of Λ_Γ denoted by $CH(\Lambda_\Gamma)$, is the smallest convex subset of X containing $GH(\Lambda_\Gamma)$. In CAT(-1) spaces we always have $CH(\Lambda_\Gamma) = GH(\Lambda_\Gamma)$. We say that Γ is *convex cocompact* if it acts cocompactly on $CH(\Lambda_\Gamma)$.

Let $p : T^1 X \to X$ be the base point projection map from the unit tangent bundle to X. Let $g = (g_t)_{t \in \mathbb{R}}$ be the geodesic flow on $T^1 M$ and $\widetilde{g} = (\widetilde{g}_t)_{t \in \mathbb{R}}$ the one on $T^1 X$ and equip the unit tangent bundle with the following metric

$$d_{T^1 X}(\mathrm{v}, \mathrm{w}) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} d\big(p(g_t(\mathrm{v})), p(g_t(\mathrm{w}))\big) e^{-t^2/2} dt,$$

where we use the notation v for an element in $T^1 X$ (and v for an element of ∂X).

Let $F : T^1 M \to \mathbb{R}$ be a Hölder-continuous map, called a *potential*, and let $\widetilde{F} = F \circ q$ be the Γ-invariant potential associated on $T^1 X$. In this work, as it has been suggested by Kaimanovich, we assume that $\widetilde{F}$ *is symmetric*, that is $\widetilde{F}$ is invariant by the antipodal map

$$\iota : \mathrm{v} \in T^1 X \mapsto -\mathrm{v} \in T^1 X. \tag{1.1}$$

For all $x, y \in X$, let us define

$$\int_x^y \widetilde{F} := \int_0^{d(x,y)} \widetilde{F}\big(\widetilde{g}_t(\mathrm{v})\big) dt$$

where $\mathrm{v} = (x, \vec{v}_{x,y}) \in T^1 X$ and $\vec{v}_{x,y}$ is the unit tangent vector at x to a geodesic

from x through y. Set:

$$d^F(x, y) := \int_x^y \widetilde{F}. \tag{1.2}$$

A priori d^F is not non-negative and is far to be a distance, nevertheless the symmetry of $\widetilde{F}$ implies

$$d^F(x, y) = d^F(y, x). \tag{1.3}$$

Define the Gibbs cocycle as

$$C_v^F(x, y) := \lim_{t \to +\infty} \int_y^{v_t} \widetilde{F} - \int_x^{v_t} \widetilde{F} = \lim_{t \to +\infty} d^F(y, v_t) - d^F(x, v_t), \tag{1.4}$$

where v_t is any geodesic ray ending at a point v in ∂X. Observe that if $\widetilde{F} = -1$ the Gibbs cocycle is nothing else than the Busemann cocycle, that is the horospherical distance from x to y relative to v.

The foundations of Patterson–Sullivan measures theory are in the important papers [42,50]. See [14,16], and [44] for more general results in the context of CAT(-1) spaces. These measures are also called *conformal densities*. In this paper we are dealing with the *Patterson density of* (Γ, F) where F is the potential function defined above and Γ a discrete group of isometries of X.

Recall that $\gamma_* \nu$ means $\gamma_* \nu(B) = \nu(\gamma^{-1} B)$ where γ is in Γ and B is a Borel subset of some measure space. More specifically we say that ν^F *is a Patterson density of dimension* $\sigma \in \mathbb{R}$ *for* (Γ, F) if ν^F is a map which satisfies the following conditions:

- ν^F is a map from $x \in X \mapsto \nu_x^F \in M(\overline{X})$, i.e. ν_x^F is a positive finite measure.
- For all x and y in X, ν_x^F and ν_y^F are equivalent, and we have

$$\frac{d\nu_y^F}{d\nu_x^F}(v) = e^{C_v^{F-\sigma}(x, y)}.$$

- For all $\gamma \in \Gamma$, and for all $x \in X$ we have $\gamma_* \nu_x^F = \nu_{\gamma x}^F$.

In this context define the critical exponent of (Γ, F) for $c > 0$ large enough as

$$\sigma_{\Gamma, F} := \limsup \frac{1}{n} \sum_{n-c \leq d(\gamma x, x) \leq n} e^{d^F(x, \gamma x)}.$$

Even if the construction of a Patterson density was not done in this general context with a potential function, the technic is exactly the same. We attribute the following proposition to Patterson in his seminal paper [42], ensuring the existence of a Patterson density:

Proposition 1.2 (S.-J. Patterson). *If* $\sigma_{\Gamma, F} < \infty$, *then there exists at least one Patterson density of dimension* $\sigma_{\Gamma, F}$ *with support exactly equal to* Λ_Γ.

A Patterson density ν^F of dimension σ gives rise to a unitary representations $(\pi_{\nu_x^F})_{x \in X}$ defined for $x \in X$ as:

$$\pi_{\nu_x^F} : \Gamma \to \mathcal{U}\big(L^2(\partial X, \nu_x^F)\big)$$

$$\big(\pi_{\nu_x^F}(\gamma)\xi\big)(v) = e^{\frac{1}{2}C_v^{F-\sigma}(x, \gamma x)}\xi(\gamma^{-1}v), \tag{1.5}$$

where $\xi \in L^2(\partial X, \nu_x^F)$ and $v \in \partial X$.

The representations $(\pi_{\nu_x^F})_{x \in X}$ are unitarily equivalent. Let x be in X and denote $\pi_{\nu_x^F}$ by π_x. The matrix coefficient

$$\phi_x : \Gamma \to \langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X} \rangle \in \mathbb{R}^+, \tag{1.6}$$

is called the *Harish-Chandra function*, where $\mathbf{1}_{\partial X}$ denotes the characteristic function of ∂X.

Construction of ergodic operator-valued measures. The Banach space of bounded linear operators from the Banach space of continuous functions on a locally compact space Z to the Banach space of bounded operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{L}\big(C(Z), \mathcal{B}(\mathcal{H})\big)$. The Banach space $\mathcal{L}\big(C(Z), \mathcal{B}(\mathcal{H})\big)$ is naturally isomorphic to the dual of the Banach space $C(Z)\widehat{\otimes}\mathcal{H}\widehat{\otimes}\overline{\mathcal{H}}$ where $\widehat{\otimes}$ denotes the projective tensor product: Thus $\mathcal{L}\big(C(Z), \mathcal{B}(\mathcal{H})\big)$ will be called the space of *operator-valued measures*.

Pick x in X, and a positive real number ρ and define for all integers $n \geq 1$ the annulus

$$C_n(x) = \big\{\gamma \in \Gamma \mid n - 1 \leq d(\gamma x, x) < n\big\}.$$

Let D_y be the unit Dirac mass centered at a point $y \in X$. Consider the sequence of operator-valued measures defined for all integers $n \geq 1$ as:

$$\mathcal{M}_x^n : f \in C(\overline{X}) \mapsto c_{\Gamma,F}\, e^{-\sigma_{\Gamma,F}n} \sum_{\gamma \in C_n(x)} e^{d^F(x, \gamma x)} D_{\gamma x}(f)\frac{\pi_x(\gamma)}{\phi_x(\gamma)} \in \mathcal{B}\big(L^2(\partial X, \nu_x^F)\big), \tag{1.7}$$

with the normalization constant

$$c_{\Gamma,F} = \frac{\sigma_{\Gamma,F}\|m_F\|}{1 - e^{-\sigma_{\Gamma,F}}}, \tag{1.8}$$

where $\|m_F\|$ is the mass of the so-called *Gibbs* measure associated with ν^F. We refer to Section 2.3 for definitions and properties of Gibbs measures. The normalization constant $c_{\Gamma,F}$ ensures that for any x in X

$$c_{\Gamma,F}\, e^{-\sigma_{\Gamma,F}n} \sum_{\gamma \in C_n(x)} e^{d^F(x, \gamma x)} D_{\gamma x} \otimes D_{\gamma^{-1}y} \rightharpoonup \nu_x^F \otimes \nu_y^F,$$

as n goes to $+\infty$ with respect to the weak* convergence on $C(\overline{X})^*$.

If $f \in C(\overline{X})$, we denote by $f_{|\partial X}$ its continuous restriction to the space ∂X.

Let $m(f)$ be the operator in $\mathcal{B}(L^2(\partial X, \nu_x^F))$ acting on $L^2(\partial X, \nu_x^F)$ by multiplication and define the operator-valued measure $\mathcal{M}_x$ as:

$$\mathcal{M}_x : f \in C(\overline{X}) \mapsto m(f_{|\partial X}) P_{\mathbf{1}_{\partial X}} \in \mathcal{B}(L^2(\partial X, \nu_x^F)). \qquad (1.9)$$

where $P_{\mathbf{1}_{\partial X}}$ denotes the orthogonal projection on the space of constant functions.

Main results. The main result of this paper is the following theorem:

Theorem 1.3 (Equidistribution *à la* Bader–Muchnik). *Let Γ be a convex cocompact discrete group of isometries of a complete connected Riemannian manifold with pinched negative curvature X. Let $\widetilde{F} : T^1 X \to \mathbb{R}$ be a Hölder-continuous Γ-invariant potential and let ν^F be a Patterson density for (Γ, F) of dimension $\sigma_{\Gamma,F}$.*

Assume that $\widetilde{F}$ is symmetric *and assume that the Gibbs measure associated with ν^F is* mixing *with respect to the geodesic flow. Then for each x in $CH(\Lambda_\Gamma)$ we have*

$$\mathcal{M}_x^n \rightharpoonup \mathcal{M}_x$$

as $n \to +\infty$ with respect to the weak* topology of the Banach space

$$\mathcal{L}(C(\overline{X}), \mathcal{B}(L^2(\partial X, \nu_x^F))).$$

In other words we have for all $f \in C(\overline{X})$ and all $\xi, \eta \in L^2(\partial X, \nu_x^F)$:

$$\lim_{n \to +\infty} \langle \mathcal{M}_x^n(f)\xi, \eta \rangle = \left(\int_{\partial X} \xi \, d\nu_x^F \right) \left(\int_{\partial X} f_{|\partial X} \overline{\eta} \, d\nu_x^F \right).$$

With the same hypotheses of the above theorem, we deduce immediately an ergodic theorem *à la* von Neumann for the Patterson density (Γ, F) associated with ν_x^F on ∂X.

Corollary 1.4 (Ergodicity *à la* von Neumann). *For all $x \in CH(\Lambda_\Gamma)$*

$$c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} \frac{\pi_x(\gamma)}{\phi_x(\gamma)} \to P_{I_{\partial X}}$$

as $n \to +\infty$ with respect to the weak operator topology on $\mathcal{B}(L^2(\partial X, \nu_x^F))$.

In the same setting of Theorem 1.3 we have:

Corollary 1.5 (Irreducibility). *Assume that $\widetilde{F}$ is cohomologous to a symmetric potential and assume that the Gibbs measure is* mixing. *For all $x \in X$, the representations $\pi_x : \Gamma \to \mathcal{U}(L^2(\partial X, \nu_x^F))$ are irreducible.*

Remark 1.6. The assumption of mixing of Gibbs measures with respect to the geodesic flow is automatic in the case of constant curvature, hence all boundary representations of convex cocompact groups associated with a Patterson density with a Hölder-continuous potential $\widetilde{F}$ cohomologous to a symmetric potential is irreducible. Note that this property does not depend on the base point.

We obtain also the following theorem which classifies the unitary representations associated with a Patterson density. We refer to Subsection 2.2 for the definitions concerning items (3) and (4).

Theorem 1.7. *Let Γ be a convex cocompact discrete group of isometries of X, pick a point x in X and let ν_x^F and ν_x^G be Patterson densities associated with Hölder-continuous Γ-invariant* symmetric *potentials $\widetilde{F}$ and $\widetilde{G}$ on $T^1 X$. Assume that the Gibbs measure is* mixing *with respect to the geodesic flow. Then the following assertions are equivalent:*

(1) *The unitary $\pi_{\nu_x^F}$ and $\pi_{\nu_x^G}$ are equivalent as unitary representations.*

(2) *The measures ν_x^F and ν_x^G are in the same class.*

(3) *The potentials $\widetilde{F}$ and $\widetilde{G}$ have the same periods.*

(4) *The Gibbs cocycles associated with F and G are cohomologous in restriction to $\Omega\Gamma$.*

The method of the proof of Theorem 1.3 consists of two steps: given a sequence of functionals of the dual of a separable Banach space, we shall prove:

Step 1: The sequence is uniformly bounded: existence of accumulation points (by the Banach–Alaoglu theorem).

Step 2: Identification of the limit using equidistribution theorems (only one accumulation point).

Structure of the paper. In Section 2 we remind the reader of some standard facts about the geometry in negative curvature, Gibbs cocycles and about Gibbs measures generalizing the Bowen–Margulis–Sullivan measures on the unit tangent bundle to provide the generalization of Roblin's equidistribution theorem by Paulin, Pollicott and Schapira. In Section 3 we prove fundamental estimates on the Harish-Chandra function. In Section 4 we prove uniform boundedness for the sequences of operators using property RD of de la Harpe et Jolissaint and the amenability of the action on the boundary, thus concluding *Step 1* of the proof of Theorem 1.3. In Section 5 we use Paulin–Pollicott–Schapira's equidistribution theorem to achieve *Step 2* of the proof of Theorem 1.3. In Section 6 we prove our Theorem 1.3 and its corollaries as well as Theorem 1.7.

2. Preliminaries

2.1. Geometry of negative curvature and potential functions. Recall that X is a complete simply connected Riemannian manifold with dimension at least 2 and pinched sectional curvature $-b^2 \leq K \leq -1$ with $b \geq 1$, equipped with its Riemannian distance denoted by d. The geometric boundary or the boundary at infinity, also called Gromov boundary is denoted by ∂X. We consider Γ a non-elementary discrete group of isometries of X.

2.1.1. Busemann functions, Bourdon's metric. Let x be in X, let r be a geodesic ray and define the Busemann function associated with the geodesic ray r as

$$b_r(x) = \lim_{t \to \infty} d(x, r(t)) - t.$$

Let x and y be in X and consider the unique semi-infinite geodesic $[xy)$ passing through x and y, starting at x. Define w_x^y as the unique point at the boundary so that

$$w_x^y := [xy) \cap \partial X. \tag{2.1}$$

The limit $\lim_{t \to \infty} d(x, r(t)) - d(y, r(t))$ exists, is equal to $b_r(x) - b_r(y)$, and is independent of the choice of r. The horospherical distance from x to y relative to v is defined as

$$\beta_v(x, y) = \lim_{t \to \infty} d(x, r(t)) - d(y, r(t)). \tag{2.2}$$

Recall that the Gromov product of two points $a, b \in X$ relative to $x \in X$ is

$$(a, b)_x = \frac{1}{2}(d(x, a) + d(x, b) - d(a, b)).$$

Let v, w be in ∂X such that $v \neq w$. If $a_n \to v \in \partial X$, $b_n \to w \in \partial X$, then

$$(v, w)_x = \lim_{n \to \infty} (a_n, b_n)_x$$

exists and does not depend on the sequences a_n and b_n. If r is a geodesic ray representing v we have:

$$(v, y)_x = \lim_{t \to +\infty} \frac{1}{2}(d(x, r(t)) + d(x, y) - d(r(t), y)),$$

then we obtain:

$$\beta_v(x, y) = 2(v, y)_x - d(x, y). \tag{2.3}$$

Thus, if $z \in X$ is a point on the geodesic connecting v and w, then

$$(v, w)_x = \frac{1}{2}(\beta_v(x, z) + \beta_w(x, z)).$$

The geometric boundary is endowed with the Bourdon metric which defines the same topology on the boundary as the cone topology (see [17, Chapitre III.H, Proposition 3.7 and Proposition 3.21]. Indeed the formula

$$d_x(v, w) = e^{-(v,w)_x} \tag{2.4}$$

defines a metric on ∂X when we set $d_x(v, v) = 0$. This is due to Bourdon and we refer to [14, Théorème 2.5.1] for more details. We have the following comparison formula:

$$d_y(v, w) = e^{\frac{1}{2}(\beta_v(x, y) + \beta_w(x, y))} d_x(v, w). \tag{2.5}$$

If x and y are points of X and R is a positive real number, we define the shadow $\mathcal{O}_R(x, y)$ to be the set of v in ∂X such that the geodesic ray issued from x with limit point v hits the closed ball of center y with radius $R > 0$.

The Sullivan shadow lemma is a very useful tool in ergodic theory of discrete groups, and it has been generalized to the context of Gibbs measure by Moshen in [39], see also [43, Proposition 11.1].

Lemma 2.1. *Let Γ be a discrete group of isometries of X and ν^F be a Patterson density of dimension σ for (Γ, F). For all $\sigma \geq \sigma_{\Gamma,F}$ and for any compact subset $K \subset X$ there exists a positive contant $C > 0$ such that for all x and y in $\Gamma K \subset X$:*

$$\frac{1}{C} e^{d^F(x, y) - \sigma d(x, y)} \leq \nu_x^F(O_R(x, y)) \leq C e^{d^F(x, y) - \sigma d(x, y)}.$$

Assuming that Γ is convex cocompact we will use the above lemma with $K \subset CH(\Lambda_\Gamma)$ being the closure of a fundamental domain of the action of Γ acting on $CH(\Lambda_\Gamma)$. If Γ is cocompact, then the limit set is the entire geometric boundary and the shadow lemma holds everywhere on X.

We say that X is a δ-hyperbolic space if we have the following inequality: for all $x, y, z, t \in \overline{X}$

$$(x, z)_t \geq \min\{(x, y)_t, (y, z)_t\} - \delta, \tag{2.6}$$

see [17, 3.17 Remarks (4), p. 433]. Using the Bourdon metric on the boundary we can compare a shadow to certain balls. More precisely we have the following proposition. This lemma, rather easy and well known, will be very useful since the boundary admits the structure of a metric space.

Lemma 2.2.

(1) *Let $R \geq 4\delta$. Then*

$$B(w_x^y, e^{-d(x,y)}) \subset O_R(x, y).$$

(2) *Let any $R > 0$, and set $C = e^{2\delta + R}$. Then*

$$O_R(x, y) \subset B\left(w_x^y, C e^{-d(x,y)}\right).$$

Proof. We first prove the first inclusion. Let v such that $(v, w_x^y)_x > d(x, y)$. We let z be on $[xv)$ such that $d(x, z) = d(x, y)$. We have

$$d(y, z) = d(x, y) + d(x, z) - 2(y, z)_x.$$

We have

$$
\begin{aligned}
(y, z)_x &\geq \min\{(y, w_x^y)_x, (z, w_x^y)_x\} - \delta \\
&= \min\{d(x, y), (z, w_x^y)_x\} - \delta \\
&\geq \min\{d(x, y), (z, v)_x, (v, w_x^y)_x\} - 2\delta \\
&\geq \min\{d(x, y), d(x, z), d(x, y)\} - 2\delta \\
&= d(x, y) - 2\delta,
\end{aligned}
$$

it follows that $d(y, z) \leq 4\delta$.

We now prove the second inclusion. Let $v \in O_R(x, y)$ such that $[xv) \cap B_X(y, R) \neq \emptyset$ and let $z \in [xv)$ so that $d(y, z) < R$. We have

$$
\begin{aligned}
(v, w_x^y) &\geq \min\{(v, y)_x, (y, w_x^y)_x\} - \delta \\
&= \min\{(v, y), d(x, y)\} - \delta \\
&\geq \min\{(v, z)_x, (z, y)_x, d(x, y)\} - 2\delta \\
&= \min\{d(x, z), (z, y)_x, d(x, y)\} - 2\delta \\
&\geq \min\{d(x, y) - R, d(x, y) - R, d(x, y)\} - 2\delta \\
&= d(x, y) - R - 2\delta. \qquad \square
\end{aligned}
$$

2.1.2. Gibbs Cocycle and some geometric properties.

Given $\widetilde{F} : T^1 X \to \mathbb{R}$ a Γ-invariant Hölder-continuous potential we define, as in 1.4 from the Introduction, the Gibbs cocycle $C_v^F(x, y)$ where $v \in \partial X$ and $x, y \in X$. We shall give some properties of the Gibbs cocycle but first of all note that if $\widetilde{F} = -1$ then

$$C_v^F(x, y) = \beta_v(x, v).$$

Hence, for every $s \in \mathbb{R}$ we have:

$$C_v^{F-s}(x, y) = C_v^F(x, y) + s\beta_v(x, y). \tag{2.7}$$

Observe that if x belongs to the geodesic ray from y to v then

$$C_v^F(x, y) = \int_x^y \widetilde{F}.$$

The Gibbs cocycle satisfies the following cocycle property: for all $x, y, z \in X$ and $v \in \partial X$ we have

$$C_v^F(x, z) = C_v^F(x, y) + C_v^F(y, z) \quad \text{and} \quad C_v^F(y, x) = -C_v^F(x, y), \tag{2.8}$$

and the following Γ-invariance property: for all $\gamma \in \Gamma$, all $x, y \in X$ and $v \in \partial X$:

$$C_{\gamma v}^F(\gamma x, \gamma y) = C_v^F(x, y). \tag{2.9}$$

We now provide a lemma stating some useful properties and local estimates of the Gibbs cocycle.

Lemma 2.3. *Fix $R > 0$ and assume that $\widetilde{F}$ is bounded on $p^{-1}(CH(\Lambda_\Gamma)) \subset T^1X$. There exists positive constants $C(R)$, $D(R)$ and $E(R)$ so that:*

(1) *For all $x \in CH(\Lambda_\Gamma)$ and for all $y \in X$ such that $d(x, y) \leq R$ and for all $v \in \partial X$ we have*

$$|C_v^F(x, y)| \leq C(R).$$

(2) *For all x in X, for all $y \in CH(\Lambda_\Gamma)$ and for all $v \in O_R(x, y)$ we have*

$$\left| C_v^F(x, y) + d^F(x, y) \right| \leq D(R).$$

(3) *For all $x \in CH(\Lambda_\Gamma)$ and for all y, z such that $d(y, z) \leq R$ we have*

$$|d^F(x, y) - d^F(x, z)| \leq E(R).$$

For a proof of items (1) and (2) see [43, Lemma 3.4] and for item (3) see [43, Lemma 3.2].

2.2. Gibbs Cocycles, cohomology, periods and unitary representations.
Following [37], we recall some fundamental correspondences between potential functions, Hölderian cocycles, and periods. We complete theses fundamental observations by adding a correspondence dealing with unitary boundary representations.

We say that a function defined on the boundary ∂X is Hölder-continuous if it is Hölder-continuous with respect to Bourdon's metric associated with some base boint in X. Note that this definition does not depend on the choice of the base point. We say that a cocycle $C : \Gamma \times \partial X \to \mathbb{R}$ is a *Hölderian cocycle* if for all γ the map $C(\gamma, \cdot)$ is Hölder-continuous and if it satisfies the cocycle equality

$$C(\gamma_1\gamma_2, v) = C(\gamma_1, \gamma_2 v) + C(\gamma_2, v),$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and for all $v \in \partial X$. We say that two cocycles C and C' are *cohomologous* if there exists a function $H : \partial X \to \mathbb{R}$ such that

$$C(\gamma, \xi) - C'(\gamma, \xi) = H(\gamma\xi) - H(\xi). \tag{2.10}$$

Let γ be a hyperbolic isometry, also called a *loxodromic* element, and denote by γ^+ its attractive fixed point. Observe that the quantity

$$C(\gamma, \gamma^+)$$

depends only on the conjugacy class of γ. Let x be on the axis of γ and consider the cocycle

$$C : (\gamma, v) \in \Gamma \times \partial X \mapsto C_v^F(x, \gamma x).$$

Observe that

$$C(\gamma, \gamma_+) = \int_{\gamma x}^x \widetilde{F}$$

and by assumption on the symmetry of $\widetilde{F}$ we have also

$$C(\gamma, \gamma_+) = \int_x^{\gamma x} \widetilde{F}.$$

We call the quantity $\int_x^{\gamma x} \widetilde{F}$ the period of γ and we denote it by $\mathrm{Per}(,)$. The set

$$\mathrm{Per}(\widetilde{F}) := \{\mathrm{Per}(,) \text{ with } \gamma \text{ a loxodromic element}\}, \qquad (2.11)$$

is called *the periods of* $\widetilde{F}$. Observe that if $\widetilde{F} = 1$ then $\mathrm{Per}(,)$ is nothing but the translation length of γ and $\mathrm{Per}(\widetilde{F})$ is the length spectrum of M.

Remark 2.4. Observe also that this definition of *periods of* $\widetilde{F}$ coincides with the definition of *periods of a Hölderian cocycle* in [37].

Let $\widetilde{F}^* = T^1 X \to \mathbb{R}$ be another Hölder-continuous Γ-invariant function. We say that $\widetilde{F}^*$ *is cohomologous to* $\widetilde{F}$ if there exists a function differentiable along every flow line $G : T^1 X \to \mathbb{R}$ such that

$$\widetilde{F}^*(v) - \widetilde{F}(v) = \frac{d}{dt}_{|t=0} G(g_t v). \qquad (2.12)$$

Consider the cocycle

$$C^F : (\gamma, v) \mapsto C_v^F(x, \gamma x).$$

First note that if $\widetilde{F}$ is bounded then C^F is Hölder-continuous . Then observe that if $\widetilde{F}^*$ is cohomologous to $\widetilde{F}$ then C^F and C^{F^*} are cohomologous (see [43, §3.3 Remarks and Proposition 3.5] for more details). The periods are an invariant of the cohomology class of potentials and also of cocycles. We have

Proposition 2.5. *Let $\widetilde{F}$ and $\widetilde{G}$ be two Hölder continuous Γ-invariant functions on the unit tangent bundle of X. Pick $x \in X$. The following assertions are equivalent*

(1) v_x^F *and* v_x^G *are in the same class.*

(2) *The functions $\widetilde{F}$ and $\widetilde{G}$ have the same periods.*

(3) *The Gibbs cocycles C^F and C^G associated with $\widetilde{F}$ and $\widetilde{G}$ are cohomologous in restriction to $\Omega\Gamma$.*

Proof. Thanks to Remark 2.4 we refer to [37, §III, Proposition 1] for the equivalence between (1) and (2) and between (1) and (3). For (2) implies (3) we refer to [43, Remark 3.1]. $\square$

At the level of unitary representations we say that $\pi_{\nu_x^F}$ and $\pi_{\nu_x^G}$ are equivalent if there exists a unitary operator $U : L^2(\partial X, \nu_x^F) \to L^2(\partial X, \nu_x^G)$ such that:

$$U\pi_{\nu_x^F} = \pi_{\nu_x^G} U. \tag{2.13}$$

Lemma 2.6. *Let ν^F and ν^{F^*} be two Patterson densities of dimension σ. Pick x in X and consider the unitary representations $\pi_{\nu_x^F}$ and $\pi_{\nu_x^{F*}}$. If F and F^* are cohomologous then $\pi_{\nu_x^F}$ and $\pi_{\nu_x^{F*}}$ are equivalent.*

Proof. Since $\widetilde{F}$ and $\widetilde{F}^*$ are cohomologous then the cocycles

$$C^F : (\gamma, v) \mapsto C_v^F(x, \gamma x) \quad \text{and} \quad C^{F^*} : (\gamma, v) \mapsto C_v^{F^*}(x, \gamma x)$$

are cohomologous. Thus the multiplication operator by $e^{\frac{1}{2}H}$ from $L^2(\partial X, \nu_x^F)$ to $L^2(\partial X, \nu_x^{F^*})$ intertwines the unitary representations $\pi_{\nu_x^F}$ and $\pi_{\nu_x^{F^*}}$ where $H : \partial X \to \mathbb{R}$ satisfies the identity (2.10). $\square$

2.3. Gibbs measures and Roblin–Paulin–Pollicott–Schapira's equidistribution theorem.

2.3.1. Hopf parametrization.
Let us now recall a parametrization of $T^1 X$ in terms of the boundary at infinity of X.

If $\mathrm{v} = (x, \vec{v})$ is an element of $T^1 X$, consider the unique geodesic defined by $\vec{v}$ represented by an isometry $r : \mathbb{R} \to X$ such that $r(0) = q(\mathrm{v})$ and $\frac{d}{dt}_{|t=0} r = \vec{v}$. We denote by v_- and v_+ the endpoints of the geodesic such that $r(-\infty) = v_-$ and $r(+\infty) = v_+$.

Let us define $\partial^2 X = \partial X \times \partial X - \Delta$, where Δ is the diagonal of $\partial X \times \partial X$. For every base point x_0 in X, the space $T^1 X$ may be identified with $\partial^2 X \times \mathbb{R}$, by the map which maps a unit tangent vector v to the triple (v_-, v_+, t) where t represents the algebraic distance on the image of the geodesic represented by r between $r(0)$ and the closest point of the geodesic to x_0. This parametrization, depending a priori on x_0, differs from the one defined by another base point x_0' only by an additive term on the third factor (independent of the time t).

2.3.2. The potential gap.
For all x in X and for all $v, w \in \partial X$ define the *gap map* as

$$D_{x,F}(v, w) := \exp\left(\frac{1}{2}\left(\int_x^{w_t} \widetilde{F} - \int_{v_t}^{w_t} \widetilde{F} + \int_{v_t}^x \widetilde{F}\right)\right).$$

We observe that $D_{x,F}$ generalizes Bourdon's metric d_x since for $F = -1$ we obtain $D_{x,F} = d_x$. Note the Γ invariance property $D_{\gamma x, F}(\gamma v, \gamma w) = D_{x,F}(v, w)$ for all $\gamma \in \Gamma$ and for all $v, w \in \partial X$.

2.3.3. The Gibbs states of (Γ, F). Let σ be a real number and let $(v_x^F)_{x \in X}$ be a Patterson density of dimension σ for (Γ, F). Once we have fixed a base point $x_0 \in X$ and used the Hopf parametrization, define *the Gibbs measures on $T^1 X$ associated with $(v_x^F)_{x \in X}$* as

$$dm(v) = \frac{dv_{x_0}(v_-)dv_{x_0}(v_+)dt}{D^2_{F-\sigma, x_0}(v_-, v_+)}. \tag{2.14}$$

The groups Γ and $\mathbb{R}$ act on $\partial^2 X \times \mathbb{R}$ via $\gamma(v_-, v_+, t) = (\gamma v_-, \gamma v_+, t)$ and via the goedesic flow $s(v_-, v_+, t) = (v_-, v_+, t + s)$. Observe that both actions commute. Thus define m_F on $\Gamma \backslash T^1 X = T^1 M$, and we call m_F *the Gibbs measures on $T^1 M$ associated with $(v_x^F)_{x \in X}$*. If $\|m_F\| < \infty$ we say that m_F is finite. The finiteness of the Gibbs measures will always be satisfied when we consider convex cocompact groups.

2.3.4. Mixing property of Gibbs measures. We say that g_t is mixing on $\Gamma \backslash T^1 X$ with respect to m_F if for all bounded Borel subsets $A, B \subset \Gamma \backslash T^1 X$ we have $\lim_{t \to +\infty} m_F(A \cap g_t(B)) = m_F(A)m_F(B)$.

There exists a condition which guarantees that the geodesic flow on $T^1 X$ is mixing: it is related to the non-arithmeticity of the spectrum of Γ, see [9]. More precisely: the translation length of an element $\gamma \in \Gamma$ is defined as $t(\gamma) := \inf\{d(x, \gamma x), x \in X\}$. The spectrum of Γ is defined as the subgroup of $\mathbb{R}$ generated by $t(\gamma)$ where γ ranges over the hyperbolic isometries in Γ. We say that Γ has an *arithmetic spectrum* if its spectrum is a discrete subgroup of $\mathbb{R}$. We refer to [43, Theorem 8.1] for a proof of the fact that the non-arithmeticity of the spectrum implies the mixing property of the geodesic flow with respect the to the Gibbs measures. The non-arithmeticity condition is verified in the following cases: for isometries group of Riemannian surfaces, hyperbolic spaces and isometries groups of CAT(-1) spaces with a non-trivial component in their limit set. We refer to [22] and to [44, Proposition 1.6, Chapitre 1] for more details.

We have finished the preparations to state Theorem 1.3 and its corollary which will be one of our main tools. The main idea of these equidistribution results goes back to the pioneering work of Margulis [38] who made a connection between the mixing property of the geodesic flow with the counting of closed geodesics on a compact negatively curved manifold. The form of the following equidistribution results, due to Paulin, Pollicott and Schapira [43, Theorem 9.1], is inspired by the results of T. Roblin in [44, Théorème 4.1.1]. We refer also to [8] for an introduction to Roblin's equidistribution theorem.

Theorem 2.7 (Paulin, Pollicott and Schapira). *Let Γ be a discrete group of isometries of X and assume that $\sigma_{\Gamma, F}$ is finite and positive. Assume that m_F is finite and mixing*

under the geodesic flow on $T^1 M$. Then for all $x, y \in X$ and for all $c > 0$:

$$\frac{\sigma_{\Gamma,F} \|m_F\|}{1 - e^{-c\sigma_{\Gamma,F}}} e^{-\sigma_{\Gamma,F} n} \sum_{\{\gamma \in \Gamma \mid n-c < d(x,\gamma y) \leq n\}} e^{d^F(x,\gamma y)} D_{\gamma^{-1}x} \otimes D_{\gamma y} \rightharpoonup \nu_x^F \otimes \nu_y^F$$

as $n \to +\infty$ with respect to the weak* convergence of $C(\overline{X} \times \overline{X})^*$.

As a corollary we obtain the following result that we shall use in the Step 2 of the computation of the limit in Section 5.

For a subset A in ∂X with a vertex x, denote by $\mathcal{C}_x(A)$ that is the union of the geodesic rays or lines starting from x and ending at A, and this a subset of $\mathcal{C}_x(A) \subset \overline{X}$ so that $\mathcal{C}_x(A) \cap \partial X = A$.

Corollary 2.8. *Let Γ be a discrete group of isometries of X. Assume that m_F is finite and mixing under the geodesic flow on $T^1 M$. If U and V are two Borel sets, then for all $x, y \in X$ and for all $c > 0$:*

$$\limsup_{n \to +\infty} \frac{\sigma_{\Gamma,F} \|m_F\|}{1 - e^{-c\sigma_{\Gamma,F}}} e^{-\sigma_{\Gamma,F} n} \sum_{C_n(x)} e^{d^F(x,\gamma x)} (D_{\gamma x} \otimes D_{\gamma^{-1}x})(\chi_{\mathcal{C}_x(U)} \otimes \chi_{\mathcal{C}_x(V)})$$

$$\leq \nu_x^F(\overline{U})\nu_x^F(\overline{V}).$$

We recall that we have defined the normalization constant $c_{\Gamma,F}$ as

$$c_{\Gamma,F} := \frac{\sigma_{\Gamma,F} \|m_F\|}{1 - e^{-\sigma_{\Gamma,F}}}.$$

3. The Harish-Chandra function

The goal of this section is to prove the following estimate on the Harish-Chandra function.

Proposition 3.1 (Harish-Chandra's estimate). *Let $\nu^F = (\nu_x^F)_{x \in X}$ be a Patterson density of dimension $\sigma_{\Gamma,F}$. There exists a constant $R > 0$ and a constant $C > 0$ (depending on R) such that for all $\gamma \in \Gamma$ satisfying $d(x, \gamma x) \geq R$ with x in $CH(\Lambda_\Gamma)$ we have*

$$C^{-1}d(x, \gamma x)e^{\frac{1}{2}d^F(x, \gamma x) - \frac{1}{2}\sigma_{\Gamma,F} d(x, \gamma x)}$$

$$\leq \phi_x(\gamma) \leq Cd(x, \gamma x)e^{\frac{1}{2}d^F(x, \gamma x) - \frac{1}{2}\sigma_{\Gamma,F} d(x, \gamma x)}.$$

Remark 3.2. It would be probably more appropriate to call these estimates *Harish-Chandra Anker's estimates* because Anker has improved estimates established by Harish-Chandra in the setting of semisimple Lie groups. He improved notably the lower bound by adding a polynomial, see [6].

3.1. Some technical lemmas. The following lemma is due to S. Alvarez in [3]. Since our methods are rather analytical and since our conventions are different, we give another shorter proof.

Lemma 3.3. *For any $R \geq 4\delta$, there exits $r > 0$ such that for all x, y, z aligned in this order we have for all $v \in \partial X \setminus O_R(x, y)$:*

$$\beta_v(y, z) \leq r - d(y, z).$$

Proof. We have $\beta_v(y, z) = 2(v, z)_y - d(y, z)$. The hyperbolic inequality (2.6) implies that $(v, z)_y \leq (v, w_y^z)_y + \delta$. An upper bound of the quantity $(v, w_y^z)_y$ is equivalent to a lower bound of $d_y(v, w_y^z)$. We have $\beta_{w_y^z}(x, y) = d(x, y)$ (because $w_y^z = w_x^y$) and $\beta_v(x, y) = 2(v, y)_x - d(x, y)$. Thus the hyperbolic inequality (2.6) implies

$$\beta_v(x, y) + \beta_{w_y^z}(x, y) = 2(v, y)_x \geq 2 \min\{(v, w_x^y)_x, (w_x^y, y)_x\} - 2\delta$$
$$= 2 \min\{(v, w_x^y)_x, d(x, y)\} - 2\delta.$$

Since $v \in \partial X \setminus O_R(x, y)$ we have $d_x(v, w_x^y) \geq e^{-d(x,y)}$ by Lemma 2.2, equivalently $(v, w_x^y)_x \leq d(x, y)$. Thus

$$e^{\frac{1}{2}(\beta_v(x, y) + \beta_{w_y^z}(x, y))} \geq \frac{e^{-\delta}}{d_x(v, w_x^y)} = \frac{e^{-\delta}}{d_x(v, w_y^z)}.$$

By the conformal equivalence of the metric on the boundary we have:

$$d_y(v, w_y^z) = d_x(v, w_y^z) e^{\frac{1}{2}(\beta_v(x, y) + \beta_{w_y^z}(x, y))},$$

hence

$$e^{-\delta} \leq d_y(v, w_y^z),$$

and we set $r = 4\delta$ to conclude the proof. $\qquad\square$

Before proceeding we will need to set up some notation. We follow the decomposition used by Alvarez in [3].

Definition of $A_{i,R}(\gamma)$. Fix $R > 0$ such that Lemma 2.2 is available. Let γ be in Γ such that $d(x, \gamma x) \geq R$ and consider the geodesic $[xw_x^{\gamma x})$ starting at x and passing through γx and ending at $w_x^{\gamma x} \in \partial X$. Let z_i for $i = 0, \ldots, N$ be a finite sequence of points belonging to $[xw_x^{\gamma x})$ aligned in the following order: $z_N, \ldots, z_0$, with $z_0 = \gamma x$ and so that the choice of z_N satisfies $d(x, z_N) < R/2$, and $d(z_i, z_{i+1}) = R/2$. Observe that $d(x, \gamma x) = d(x, z_N) + N\frac{R}{2}$.

For $i = 1, \ldots, N$ notice that $O_R(x, z_{i-1}) \subset O_R(x, z_i)$. Indeed, let ξ be in $O_R(x, z_{i-1})$ and thus $(\xi, z_{i-1})_x \geq d(x, z_{i-1}) - R$. We have

$$
\begin{aligned}
(\xi, z_i)_x &\geq \min\{(\xi, z_{i-1})_x, (z_i, z_{i-1})_x\} - \delta \\
&= \min\{(\xi, z_{i-1})_x, d(x, z_i)\} - \delta \\
&\geq \min\{d(x, z_{i-1}) - R, d(x, z_i)\} - \delta \\
&= \min\{d(x, z_i) - R/2, d(x, z_i)\} - \delta \\
&= d(x, z_i) - R/2 - \delta \\
&\geq d(x, z_i) - R,
\end{aligned}
$$

where the last inequality follows from the fact that $R \geq 2\delta$.

We set

$$
A_{i,R}(\gamma) := O_R(x, z_i) \backslash O_R(x, z_{i-1}).
$$

Observe that $A_{N,R} = \partial X \backslash O_R(x, z_{N-1})$.

We can decompose the boundary as the following disjoint union

$$
\partial X := \sqcup_{i=1}^{N} A_{i,R}(\gamma) \sqcup O_R(x, \gamma x). \tag{3.1}
$$

Proposition 3.4. *We suppose here that $\widetilde{F}$ is symmetric. Let ν^F be a Patterson density of dimension $\sigma \geq \sigma_{\Gamma,F}$ so that the estimates in Moshen's Shadow lemma hold (Lemma 2.1). There exists a constant $C > 0$ such that for all $\gamma \in \Gamma$, $v \in A_{i,R}(\gamma)$, and $1 \leq i \leq N$ we have that*

$$
C^{-1} e^{d^F(z_i, \gamma x) - \sigma d(z_i, \gamma x)} \leq e^{C_v^{F-\sigma}(z_i, \gamma x)} \leq C e^{d^F(z_i, \gamma x) - \sigma d(z_i, \gamma x)}.
$$

Proof. Recall that $C_v^{F-\sigma}(z_i, \gamma x) = C_v^F(z_i, \gamma x) + \sigma \beta_v(z_i, \gamma x)$.

If $\beta_v(z_i, \gamma x) \geq 0$ then Lemma 3.3 implies that $d(z_i, \gamma x) \leq r$ for some positive real number r. Therefore the estimates follow from Lemma 2.3(2). Now we call z' the point of the intersection of the horosphere centered at v passing through γx and the geodesic passing through v and γx. If $\beta_v(z_i, \gamma x) < 0$, then $\gamma x, z'$ and v are aligned in this order. Thus we can write

$$
\lim_{t \to +\infty} \int_{\gamma x}^{v_t} F = \int_{\gamma x}^{z'} F + \lim_{t \to +\infty} \int_{z'}^{v_t} F.
$$

Since $\beta_v(\gamma x, z') = d(\gamma x, z')$ we have

$$
C_v^{F-\sigma}(z_i, \gamma x) = C_v^{F-\sigma}(z_i, z') + \int_{\gamma x}^{z'} F - \sigma,
$$

besides, the symmetry of F implies

$$
C_v^{F-\sigma}(z_i, \gamma x) = C_v^{F-\sigma}(z_i, z') + \left(d^F(z', \gamma x) - \sigma d(z', \gamma x) \right). \tag{3.2}
$$

Notice that

$$d(z_i, z') \leq d(z_i, \gamma x) + d(\gamma x, z')$$
$$= d(z_i, \gamma x) + \beta_v(\gamma x, z')$$
$$= d(z_i, \gamma x) + \beta_v(\gamma x, z_i)$$
$$\leq d(z_i, \gamma x) + r - d(\gamma x, z_i)$$
$$= r.$$

Thus, the fist term on the right hand side equality (3.2) is bounded by Lemma 2.3(1). In the second term on the right hand side equality (3.2) the quantity $d^F(z', \gamma x) \leq d^F(z_i, \gamma x) + E(r)$, for some positive constant $E(r)$ by Lemma 2.3(3); and the triangle inequality implying $d(z', \gamma x) \leq d(z_i, \gamma x) + r$ completes the proof. □

Proposition 3.5. *There exists a positive constant $C > 0$ such that for all $i = 1, \ldots, N$ and for all $v \in A_{i,R}(\gamma)$ we have:*

$$C^{-1} e^{-\frac{1}{2}\sigma d(x, z_i)} e^{\sigma d(x, z_i)} \leq e^{\frac{\sigma}{2}\beta_v(x, z_i)} \leq C e^{-\frac{1}{2}\sigma d(x, z_i)} e^{\sigma d(x, z_i)}.$$

Proof. The proof is based on the hyperbolic inequality (2.6).

Let us prove the right hand side inequality. We have

$$\beta_v(x, z_i) = 2(v, z_i)_x - d(x, z_i).$$

We shall just control the Gromov product $(v, z_i)_x$ for $v \in A_{i,R}$.

For all i, for all v we have:

$$(v, w_x^{\gamma x}) \geq \min\{(v, z_i)_x, (z_i, w_x^{\gamma x})_x\} - \delta$$
$$= \min\{(v, z_i)_x, d(x, z_i)\} - \delta$$
$$= (v, z_i)_x - \delta.$$

Therefore,

$$e^{\frac{\sigma}{2}\beta_v(x, z_i)} \leq e^{\sigma\delta} \frac{e^{-\frac{1}{2}\sigma d(x, z_i)}}{d_x^\sigma(v, w_x^{\gamma x})}.$$

If $v \in A_{i,R}$ then v is not in $O_R(x, z_{i-1})$, and thus v is not in $B(w_x^{\gamma x}, e^{-d(x, z_{i-1})})$ hence by Lemma 2.2 we have $d_x(v, w_x^\gamma) \geq e^{-d(x, z_{i-1})} = e^{-d(x, z_i) - \frac{R}{2}}$. We deduce

$$e^{\frac{\sigma}{2}\beta_v(x, z_i)} \leq e^{\sigma(\delta + R/2)} e^{-\frac{1}{2}\sigma d(x, z_i)} e^{\sigma d(x, z_i)}.$$

We prove now the left hand side inequality. To do so, write $\beta_v(x, z_i) = \beta_v(x, z_{i-1}) + \beta_v(z_{i-1}, z_i)$. Note that for all $v \in \partial X$ we have $\beta_v(z_{i-1}, z_i) \geq -R/2$. Now write again

$$\beta_v(x, z_{i-1}) = 2(v, z_{i-1})_x - d(x, z_{i-1}).$$

We have

$$(v, z_{i-1})_x \geq \min\{(v, w_x^{\gamma x})_x, (w_x^{\gamma x}, z_{i-1})_x\} - \delta$$
$$= \min\{(v, w_x^{\gamma x})_x, d(x, z_{i-1})\} - \delta.$$

If v is in $A_{i,R}$, then v is not in $B(w_x^{\gamma x}, e^{-d(x, z_{i-1})})$. Hence $(v, w_x^{\gamma x})_x \leq d(x, z_{i-1})$. It follows that

$$(v, z_i)_x \geq d(x, z_{i-1}) - \delta.$$

We deduce that

$$e^{\frac{\sigma}{2}\beta_v(x, z_i)} \geq e^{\sigma(\delta - R/4)} e^{-\frac{1}{2}\sigma d(x, z_{i-1})} e^{\sigma d(x, z_{i-1})}.$$

Since we have $d(x, z_{i-1}) = d(x, z_i) + R/2$ it follows that

$$e^{\frac{\sigma}{2}\beta_v(x, z_i)} \geq e^{\sigma\delta} e^{-\frac{1}{2}\sigma d(x, z_i)} e^{\sigma d(x, z_i)}.$$

Hence, we set $C = e^{\sigma(\delta + R/2)}$ to conclude the proof. $\qquad\square$

3.2. Proof of estimates.
We are ready to establish the Harish-Chandra estimates.

Proof. We only prove the upper bound, the lower bound follows by the same method.

Pick $x \in CH(\Lambda_\Gamma)$ and write the Harish-Chandra function as a sum of integrals over the partition (3.1) as follows:

$$\phi_x(\gamma) = \sum_{i=1}^N \int_{A_{i,R}} e^{\frac{1}{2}C_v^{F-\sigma}(x, \gamma x)} dv_x^F(v) + \int_{O_R(x, \gamma x)} e^{\frac{1}{2}C_v^{F-\sigma}(x, \gamma x)} dv_x^F(v).$$

To prove the proposition we will show that each integral is comparable to

$$e^{\frac{1}{2}d^F(x, \gamma x) - \frac{1}{2}\sigma d(x, \gamma x)}.$$

The upper bound over $O_R(x, \gamma x)$.

$$\int_{O_R(x, \gamma x)} e^{\frac{1}{2}C_v^{F-\sigma}(x, \gamma x)} dv_x^F(v)$$

$$\leq C v_x^F(O_R(x, \gamma x)) e^{-\frac{1}{2}d^F(x, \gamma x) + \frac{1}{2}\sigma d(x, \gamma x)}$$

$$\leq C e^{\frac{1}{2}d^F(x, \gamma x) - \frac{1}{2}\sigma d(x, \gamma x)},$$

where the first inequality follows from Lemma 2.3(2) since γx is in $CH(\Lambda_\Gamma)$, and the second inequality follows from the upper bound of Mohsen's shadow Lemma (Lemma 2.1), the compact K being the closure of a fundamental domain of the action of Γ on $CH(\Lambda_\Gamma)$.

The upper bound over $A_{i,R}(\gamma)$. We have established two useful inequalities dealing with the terms we shall control: the first one follows from Proposition 3.4. There exists $C > 0$ such that we have for all i, for all $v \in A_{i,R}(\gamma)$:

$$e^{\frac{1}{2}C_v^{F-\sigma}(z_i, \gamma x)} \le Ce^{\frac{1}{2}d^F(z_i, \gamma x) - \frac{1}{2}\sigma d(z_i, \gamma x)}. \tag{3.3}$$

The second one is from Proposition 3.5. There exists $C > 0$ so that

$$e^{\frac{\sigma}{2}\beta_v(x, z_i)} \le Ce^{-\frac{1}{2}\sigma d(x, z_i)}e^{\sigma d(x, z_i)}. \tag{3.4}$$

Combining these two estimates will yield the bound over $A_{i,R}(\gamma)$. We will use a constant C which absorbs the other constants. Now estimating over $A_{i,R}(\gamma)$ we get,

$$\int_{A_{i,R}(\gamma)} e^{\frac{1}{2}C_v^{F-\sigma}(x, \gamma x)}d\nu_x^F(v) = \int_{A_{i,R}(\gamma)} e^{\frac{1}{2}C_v^{F-\sigma}(z_i, \gamma x)}e^{\frac{1}{2}C_v^F(x, z_i)}$$
$$\cdot e^{\frac{1}{2}\sigma\beta_v(x, z_i)}d\nu_x^F(v)$$

$$\text{Inequality (3.3)} \le Ce^{\frac{1}{2}\left(d^F(z_i, \gamma x) - \sigma d(z_i, \gamma x)\right)}$$
$$\cdot \int_{A_{i,R}(\gamma)} e^{\frac{1}{2}C_v^F(x, z_i)}e^{\frac{1}{2}\sigma\beta_v(x, z_i)}d\nu_x^F(v)$$

$$\text{Lemma 2.3(2)} \le Ce^{\frac{1}{2}\left(d^F(z_i, \gamma x) - \sigma d(z_i, \gamma x)\right)}e^{-\frac{1}{2}d^F(x, z_i)}$$
$$\cdot \int_{A_{i,R}(\gamma)} e^{\frac{1}{2}\sigma\beta_v(x, z_i)}d\nu_x^F(v)$$

$$\text{Inequality (3.4)} \le Ce^{\frac{1}{2}\left(d^F(z_i, \gamma x) - \sigma d(z_i, \gamma x)\right)}e^{-\frac{1}{2}d^F(x, z_i)}$$
$$\cdot e^{-\frac{1}{2}\sigma d(x, z_i)}e^{\sigma d(x, z_i)}\nu_x^F(A_{i,R}(\gamma))$$

$$= Ce^{\frac{1}{2}d^F(z_i, \gamma x)}e^{-\frac{1}{2}\sigma d(x, \gamma x)}e^{-\frac{1}{2}d^F(x, z_i)}$$
$$\cdot e^{\sigma d(x, z_i)}\nu_x^F(A_{i,R}(\gamma))$$

$$\text{Moshen's shadow Lemma} \le Ce^{\frac{1}{2}d^F(z_i, \gamma x) - \frac{1}{2}\sigma d(x, \gamma x)}e^{-\frac{1}{2}d^F(x, z_i)}$$
$$\cdot e^{\sigma d(x, z_i)}e^{d^F(x, z_i) - \sigma d(x, z_i)}$$

$$= Ce^{\frac{1}{2}d^F(x, \gamma x) - \frac{\sigma}{2}d(x, \gamma x)}.$$

Combining the *upper bound over* $O_R(x, \gamma x)$ with the *upper bound over* $A_{i,R}(\gamma)$ for all $i = 1, \ldots, N$ leads to

$$\phi_x(\gamma) \le C(N + 1)e^{\frac{1}{2}d^F(x, \gamma x) - \frac{\sigma}{2}d(x, \gamma x)}.$$

Since $N\frac{R}{2} = d(z_N, \gamma x) \le d(x, \gamma x)$ we obtain the left hand side inequality of Harish-Chandra's estimates. $\qquad\square$

Remark 3.6. In particular we prove that there exists $C > 0$ such that for all $i = 1, \ldots, N$:

$$\int_{A_{i,R}(\gamma)} e^{\frac{1}{2} C_v^{F-\sigma}(x, \gamma x)} \, d\nu_x^F(v) \leq C e^{\frac{1}{2} d^F(x, \gamma x) - \frac{1}{2}\sigma d(x, \gamma x)}. \tag{3.5}$$

4. Uniform boundedness via RD

4.1. Quasi-regular representations. Let Γ be a discrete countable group acting on a measure space (S, ν) with a Γ-quasi-invariant measure ν. This action gives rise to a unitary representation after correction by the square root of the Radon Nikodym derivative of the action:

$$\pi_\nu : \Gamma \to \mathcal{U}(L^2(S, \nu))$$

defined for $\xi \in U(L^2(S, \nu))$ and for $s \in S$ as

$$\big(\pi_\nu(\gamma)\xi\big)(s) = \left(\frac{d\gamma_*\nu}{d\nu}\right)^{\frac{1}{2}}(s)\xi(\gamma^{-1}s). \tag{4.1}$$

This unitary representation is called the quasi-regular representation associated with $\Gamma \curvearrowright (S, \nu)$ (also called *Koopman representation*).

In the following we will denote by $\lambda_\Gamma : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ the left regular representation.

Recall that a unitary representation π is *weakly contained* in a unitary representation ρ if for all functions $f \in \ell^1(\Gamma)$ we have

$$\|\pi(f)\| \leq \|\rho(f)\|. \tag{4.2}$$

We refer to [12, Appendix F] and to [23, Section 18] for more details.

Let ρ be a unitary representation of Γ and let μ be a bounded measure on Γ and define the operator $\rho(\mu)$ as:

$$\rho(\mu) := \sum_{\gamma \in \Gamma} \mu(\gamma)\rho(\gamma),$$

and observe $\rho(\mu) \in \mathcal{B}(L^2(S, \nu))$.

4.2. Spectral characterization of amenable action. The ideas of this subsection are related to the ideas of Nevo in [40] where we can find that for hyperbolic groups it is possible to bound operator norms of unitary representations which are weakly contained in (a tensor power of) the regular representation using property RD, and the resulting effective ergodic theorem. Moreover in [41], it is shown that the operator norm of any probability measure on a group, acting in the unitary representation

associated with any of the Poisson boundaries, is equal to the convolution norm in the regular representation. The same results holds for the operator norm in the unitary representation associated with any quasi-invariant measure on the boundary.

It is well known that the amenability of a discrete group can be characterized by the fact that the trivial representation is weakly contained in the left regular representation. Kuhn was probably inspired by this property to prove an analog result for quasi-regular representations associated with ergodic *amenable actions in Zimmer's sense* in [35]. We describe briefly which notion of amenable action we shall consider.

We know since Spatzier in [47] that the action of $\pi_1(M)$, the fundamental group of a compact manifold with negative sectional curvature M, on the geometric boundary of the universal cover of M is amenable in Zimmer's sense with respect to the standard measure class. Eventually, Spatzier and Zimmer showed in [48, Theorem 3.1] that this action is amenable with respect to *any quasi-invariant measure*. Later, after the work of Adams [1], Kaimanovich [34] proved that the action of a closed subgroup of isometries of a hyperbolic space with a finite critical exponent (-critical exponent- in the usual sense without a potential function) is *topologically amenable*. In this paper, we consider the action of a discrete group of isometries on the geometric boundary as a topological space. The notion of topological amenability is the more appropriate notion we shall consider since the space appears naturally as a topological space rather than only as a measurable space.

Definition 4.1. An action $\Gamma \curvearrowright S$ on a topological space S is topologically amenable if there exists a sequence of continuous maps

$$\mu^n : s \in S \mapsto \mu_s^n \in \mathrm{Prob}(\Gamma)$$

of probabilities on Γ such that

$$\lim_{n \to +\infty} \sup_{s \in S} \|\gamma_* \mu_s^n - \mu_{\gamma s}^n\| \to 0$$

as $n \to \infty$.

It turns out that in the case of a topological space *topologically amenable* and *amenable in Zimmer's sense* are equivalent, see [5]. Therefore we will not have to pay attention to any quasi-invariant measure on the geometric boundary.

It is shown in [2] that for a locally compact group G acting on (S, μ) that the definition of amenable action in Zimmer's sense is equivalent to the existence of a G-equivariant conditional expectation from $L^\infty(G \times S)$ to $L^\infty(S)$. Hence if Γ is a discrete group of isometries of a complete simply connected pinched negatively curved Riemannian manifold X, with a finite critical exponent, we have that $(\Gamma \times \partial X, \partial X)$ is a Γ-*pair* in the sense of [4].

We deduce from [4, Corollary 3.2.2] the following.

Proposition 4.2. *Let Γ be a discrete group of isometries of X a complete simply connected Riemmanian manifold with pinched curvature, with a finite critical exponent. For any quasi-invariant measures ν on the geometric boundary ∂X we have for any bounded μ measure on Γ*

$$\|\pi_\nu(\mu)\| \leq \|\lambda_\Gamma(\mu)\|.$$

Remark 4.3. Indeed, by [46, Lemma 2.3] due to Shalom with the same hypothesis we have the other inequality and thus we obtain for any bounded μ measure on Γ an equality

$$\|\pi_\nu(\mu)\| = \|\lambda_\Gamma(\mu)\|.$$

4.3. Property RD. The property RD comes from the theory of C*-algebras and has been introduced in the important paper [31] by Haagerup.

A length function $|\cdot|$ on a discrete countable group Γ is a function $|\cdot| : \Gamma \to \mathbb{R}^+$, satisfying $|e| = 0$ where e is the neutral element of Γ, $|\gamma^{-1}| = |\gamma|$ and $|\gamma_1\gamma_2| \leq |\gamma_1| + |\gamma_2|$. Let $s > 0$ and define the Sobolev space associated with Γ denoted by $H^s(\Gamma)$ as the space

$$H^s(\Gamma) := \left\{ f : \Gamma \to \mathbb{C} \text{ such that } \|f\|_{H^s}^2 := \sum_\Gamma |f(\gamma)|^2 (1 + |\gamma|)^{2s} < \infty \right\}.$$

Given a discrete countable group equipped with a length function $|\cdot|$ we say that Γ *satisfies property RD with respect to* $|\cdot|$ if the space H^s convolves $H^s(\Gamma) * \ell^2(\Gamma) \subset \ell^2(\Gamma)$ in the following way:

$\exists C, s > 0$ such that for all $f \in H^s(\Gamma), \xi \in \ell^2(\Gamma)$,
$$\text{we have } \|f * \xi\|_2 \leq C \|f\|_{H^s} \|\xi\|_2.$$

In terms of operator norm, property RD means that there exist two positive constants C and $s > 0$ such that the multiplication operator by convolution by a function in $H^s(\Gamma)$ is continuous:

$\exists C, s > 0$ such that for all $f \in H^s(\Gamma)$, we have $\|\lambda_\Gamma(f)\| \leq C \|f\|_{H^s}$.

This inequality means, in operator algebraic terms that we have the continuous inclusion

$$H^s(\Gamma) \hookrightarrow C_r^*(\Gamma). \tag{4.3}$$

Remark 4.4. If we specialize the property RD to the abelian group $\mathbb{Z}$ (with its standard word length function) we obtain the well known fact, using the Fourier transform, that an element in $L^2(\mathbb{S}^1)$ with Fourier coefficients "Rapidly Decreasing" to 0 define a continous function on the circle $\mathbb{S}^1$.

It is obvious that convex cocompact groups in CAT(-1) spaces are Gromov hyperbolic. Hence they have RD and the next proposition follows. We extract the following inequality expressed in norm of convolution operators established in [33].

Proposition 4.5. *Let X be a complete simply connected Riemannian pinched negatively curved manifold and let Γ be a convex cocompact discrete group of isometries of X. Pick a point x in X and recall the definition of an annulus $C_n := C_n(x)$. Let χ_n be the characteristic function of C_n, then*

$$\|\lambda_\Gamma(f\chi_n)\| \leq C n \|f\|_2.$$

Proof. In [33, Proposition 3.2.4], Jolissaint proves that there exists a positive constant c, depending only on the action of Γ on X, with the following property:

Let $k, l, m \in \mathbb{N}$. If k, l and m satisfy $|k - l| \leq m \leq k + l$ with f, g are in the group algebra $\mathbb{C}\Gamma$ are supported in C_k and C_l respectively, then

$$\|(f * g)\chi_m\|_2 \leq c\|f\|_2\|g\|_2.$$

If k, l and m satisfy $|k - l| > m$ or $m > k + l$, then

$$\|(f * g)\chi_m\|_2 = 0.$$

Following the techniques in [31, Lemma 1.3, Lemma 1.4] and in [33, Proposition 1.2.6] we have: for f supported in C_k and for all g supported in C_l that

$$\|(f * g)\chi_m\|_2 \leq \sum_{l>0} \|(f * g\chi_l)\chi_m\|_2$$

$$\leq C\|f\|_2 \sum_{l=|k-m|}^{k+m} \|g\chi_l\|_2$$

$$\leq C\|f\|_2 \sum_{l>0}^{2\min(k,m)} \|g\chi_{m+k-l}\|_2$$

$$\leq C\|f\|_2 k^{\frac{1}{2}} \Big(\sum_{l>0}^{2\min(k,m)} \|g\chi_{m+k-l}\|_2^2\Big)^{\frac{1}{2}}.$$

Thus

$$\|f * g\|_2^2 = \sum_{m>0} \|(f * g)\chi_m\|_2^2$$

$$\leq C^2 k \|f\|_2 \sum_m \Big(\sum_{l>0}^{2\min(k,m)} \|g\chi_{m+k-l}\|_2^2\Big)$$

$$\leq C^2 k^2 \|f\|_2^2\|g\|_2^2.$$

Thus we obtain for f supported in C_k and for all $g \in \ell^2(\Gamma)$ that

$$\sup_{\|g\|_2 \leq 1} \|f * g\| \leq C k \|f\|_2,$$

with some positive constant $C > 0$. $\hfill \square$

We use the equality on operators norms given by the amenability of the action on the boundary and we express the inequality of norm operators given in Proposition 4.5 in its dual form with the matrix coefficients associated with the boundary representation. We obtain:

Proposition 4.6. *Let X be a complete simply connected Riemannian pinched negatively curved manifold and let Γ be a convex cocompact discrete group of isometries of X. Let ν be a Γ-quasi-invariant measure on ∂X and consider π_ν its associated quasi-regular representation. There exists $C > 0$ such that for all unit verctors $\xi, \eta \in L^2(\partial X, \nu)$ we have*

$$\sum_{\gamma \in C_n} |\langle \pi_\nu(\gamma)\xi, \eta \rangle|^2 \leq C n^2.$$

Proof. Observe that it is sufficient to prove the above inequality only for positive vectors ξ, η in $L^2(\partial X, \nu)$.

Using Proposition 4.2 and Proposition 4.5 we have for a positive function f supported in the annulus C_n

$$\|\pi_\nu(f)\| = \|\lambda_\Gamma(f)\| \leq C n \|f\|_2.$$

Consider

$$f(\cdot) = \chi_{C_n}(\cdot)\langle \pi_\nu(\cdot)\xi, \eta \rangle,$$

with ξ and η two nonzero unit positive vectors in L^2, and notice that f is a positive function on Γ supported on C_n. We have

$$\begin{aligned}
0 \leq \sum_{\gamma \in C_n} \langle \pi_\nu(\gamma)\xi, \eta \rangle^2 &= \langle \pi_\nu(f)\xi, \eta \rangle \\
&\leq \|\pi_\nu(f)\| \\
&\leq C n \|f\|_2 \\
&\leq C n \Big(\sum_{\gamma \in C_n} \langle \pi_\nu(\gamma)\xi, \eta \rangle^2 \Big)^{1/2}.
\end{aligned}$$

Divide each term of the above inequality by $\big(\sum_{\gamma \in C_n} \langle \pi_\nu(\gamma)\xi, \eta \rangle^2 \big)^{1/2}$ and take the square to obtain:

$$\sum_{\gamma \in C_n} \langle \pi_\nu(\gamma)\xi, \eta \rangle^2 \leq C^2 n^2. \hfill \square$$

4.4. Uniform boundedness. We shall consider the operator:

$$T_x^n := c_{F,\Gamma} e^{-\sigma n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} \frac{\pi_x(\gamma)}{\phi_x(\gamma)}. \tag{4.4}$$

with $c_{\Gamma,F}$ given in (1.8) and recall that $C_n(x) = \{n - 1 \le d(x, \gamma x) < n\}$. Observe that T_x^n is nothing else than

$$T_x^n = \mathcal{M}_x^n(\mathbf{1}_{\overline{X}}),$$

where $\mathbf{1}_{\overline{X}}$ denotes the unit function on the compact set $\overline{X}$.

The Harish-Chandra estimates are fundamental to prove the uniform boundedness of the sequence of operators defined above. The potential function F is always assumed to be symmetric.

Proposition 4.7. *We have* $\sup_n \|T_x^n\| < +\infty$.

Proof. Pick $x \in X$, let $\nu := \nu_x^F$ be a Patterson density of (Γ, F) of dimension σ and consider π_ν the quasi-regular representation associated. Then Proposition 4.6 implies for all unit vectors $\xi, \eta \in L^2(\partial X, \nu)$ we have

$$\sum_{\gamma \in C_n(x)} |\langle \pi_\nu(\gamma)\xi, \eta \rangle|^2 \le C n^2.$$

Observe that Cauchy–Schwarz inequality implies that for all unit vectors $\xi, \eta, \xi', \eta' \in L^2(\partial X, \nu)$ we have

$$\sum_{\gamma \in C_n(x)} |\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle| \le C n^2.$$

Therefore for all unit vectors ξ, η, ξ', η' we have

$$\begin{aligned}
C n^2 &\ge \sum_{\gamma \in C_n(x)} |\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle| \\
&= \sum_{\gamma \in C_n(x)} \frac{|\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle|}{\phi_x^2(\gamma)} \phi_x^2(\gamma) \\
\text{Proposition 3.1} \ &\ge C' \sum_{\gamma \in C_n(x)} d^2(x, \gamma x) e^{d^F(x,\gamma x)} e^{-\sigma|\gamma|} \frac{|\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle|}{\phi_x^2(\gamma)} \\
&\ge n^2 C' e^{-\sigma n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} \frac{|\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle|}{\phi_x^2(\gamma)} \\
&\ge \frac{C'}{c_{\Gamma,F}} n^2 \left(c_{\Gamma,F} e^{-\sigma n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} \frac{|\langle \pi_\nu(\gamma)\xi, \eta \rangle \langle \pi_\nu(\gamma)\xi', \eta' \rangle|}{\phi_x^2(\gamma)} \right).
\end{aligned}$$

Applying the above inequality for $\xi' = \eta' = 1$ we obtain for all unit vectors ξ and η in $L^2(\partial X, \nu)$:

$$Cn^2 \geq \frac{C'}{c_{\Gamma, F}} n^2 |\langle T_x^n \xi, \eta \rangle|.$$

Hence

$$\sup_n \|T_x^n\| < \infty,$$

and the proof is done.   □

Remark 4.8. Notice that Bader and Muchnik in [10] use a different method to prove uniform boundedness of the sequence of operators. Our method combining the property RD with the equality concerning the spectral radius gives another short proof of the uniform boundedness when the quasi-invariant measure is the Patterson–Sullivan measure class.

Remark 4.9. Notice also that this uniform bound for the Patterson–Sullivan measure class gives a sharp estimate of the spectral gap of $\lambda_\Gamma(\mu_n)$ where μ_n is the probability measure on the groups supported over an annulus C_n

$$\mu_n = \frac{1}{|C_n|} \chi_{C_n}.$$

More specifically we obtain

$$C^{-1} n e^{-\frac{1}{2}\sigma_\Gamma n} \leq \|\lambda(\mu_n)\| \leq C n e^{-\frac{1}{2}\sigma_\Gamma n},$$

for some positive constant $C > 0$ and where σ_Γ is the usual critical exponent in the Patterson–Sullivan theory, with a potential $F = 0$.

5. Analysis of matrix coefficients

5.1. Technical tools. Let Γ be a discrete group of isometries of X and let ν^F be a Patterson density of dimension σ. Let $(d_x)_{x \in X}$ be a family of visual metrics.

Let U be a subset of ∂X and $a > 0$ be a positive real number and define $U_x(a)$ the subset of ∂X as

$$U_x(a) = \{v| \inf_{w \in U} d_x(v, w) < e^{-a}\}. \tag{5.1}$$

We will write $U(a)$ instead of $U_x(a)$ once x has been fixed. Recall that

$$\cap_{a>0} U(a) = \overline{U}.$$

In order to have Harish-Chandra's estimates available we pick $x \in CH(\Lambda_\Gamma)$ for the rest of this section. The following lemma generalizes Lemma 5.2 of [10].

Lemma 5.1. *Let $a > 0$ be a positive real number, let γ be in Γ and let $w_x^{\gamma x} \in \partial X$. Consider the ball ∂X defined as $B_a = B(w_x^{\gamma x}, e^{-a})$ and let U be a Borel subset of $\partial X \backslash B_a$. There exists C_a such that we have*

$$\frac{\langle \pi_x(\gamma) \mathbf{1}_{\partial X}, \chi_U \rangle}{\phi_x(\gamma)} \leq \frac{C_a}{d(\gamma x, x)}.$$

Proof. Define the following sets of indices

$$I = \{i \text{ such that } A_{i,R}(\gamma) \cap \partial X \backslash B_a \neq \varnothing\},$$

and

$$J = \{i \text{ such that } e^{-d(x, z_{i-1})} \geq e^{-a}\}.$$

If i is in I, then $A_{i,R}(\gamma)$ is not included in B_a. Since $A_{i,R}(\gamma) \subset B(w_x^{\gamma x}, C e^{-d(x, z_{i-1})})$ then $B(w_x^{\gamma x}, C e^{-d(x, z_{i-1})})$ cannot be included in B_a where $C = e^{2\delta + R}$ (see Lemma 2.2). This means that i satisfies $C e^{-d(x, z_{i-1})} \geq e^{-a}$.

There is only a finite number of i such that $d(x, z_{i-1}) \leq a + \log(C) = a + 2\delta + R$. Hence by denoting $N_a := |J|$ the cardinal of J, we obtain $|I| = |\{i \text{ such that } A_{i,R}(\gamma) \cap \partial X \backslash B_\epsilon \neq \varnothing\}| \leq |J| = N_a$.

Since U is in $\partial X \backslash B_a$ we have:

$$\langle \pi(\gamma) \mathbf{1}_{\partial X}, \chi_U \rangle \leq \sum_{i=1}^{N} \int_{A_{i,R}(\gamma) \cap \partial X \backslash B_\epsilon} e^{C_{F-\sigma, v}(x, \gamma x)} d\nu_x^F(v)$$

$$\leq \sum_{i \in I} \int_{A_{i,R}(\gamma) \cap \partial X \backslash B_\epsilon} e^{C_{F-\sigma, v}(x, \gamma x)} d\nu_x^F(v)$$

$$\text{Remark 3.6} \leq C N_a e^{\frac{\sigma}{2}(d^F(x, \gamma x) - d(x, \gamma x))}$$

$$\begin{array}{l} \text{Left hand side inequality of} \\ \text{Proposition 3.1} \end{array} \leq \frac{C_a}{d(x, \gamma x)} \phi_x(\gamma). \qquad \square$$

It turns out that the following results are very close to the results of [10, Section 5]. We shall indicate all the minor modifications that we need to do to achieve Step 2.

Recall the notation of a cone of basis $A \subset \partial X$ of vertex x in X:

$$\mathcal{C}_x(A).$$

Proposition 5.2. *Pick $x \in CH(\Lambda_\Gamma)$ and let $\mu_n \in \ell^1(\Gamma)$ such that*

$$\sup_n \|\mu_n\|_{\ell^1} < +\infty,$$

and which satisfies

$$\lim_{n \to +\infty} \mu_n(\gamma) = 0,$$

for all $\gamma \in \Gamma$.

Then for every Borel subset $U \subset \partial X$ we have for all $a > 0$

$$\limsup_{n \to +\infty} \sum_{\gamma \in \Gamma} \mu_n(\gamma) \frac{\langle \pi_x(\gamma) \mathbf{1}, \chi_U \rangle}{\phi_x(\gamma)} \le \limsup_{n \to +\infty} \sum_{\gamma \in \Gamma} \mu_n(\gamma) D_{\gamma x}(\chi_{\mathcal{C}_x(U(a))}).$$

Proof. Let U be a Borel subset of ∂X and let a be a positive number and consider $U(a)$ (see Definition (5.1)). Let N_0 be nonnegative integer. Consider the following partition of Γ:

$$\Gamma = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_2$$

with

$$\Gamma_1 = \{\gamma \in \Gamma | d(x, \gamma x) \le N_0\},$$
$$\Gamma_2 = \{\gamma \in \Gamma | \gamma x \in \mathcal{C}_x(U(a))\} \cap \Gamma_1^c,$$
and
$$\Gamma_3 = \{\gamma \in \Gamma | \gamma x \notin \mathcal{C}_x(U(a))\} \cap \Gamma_1^c.$$

Note that $\gamma x \notin \mathcal{C}_x(U(a))$ is equivalent to $w_x^{\gamma x} \notin U(a)$. Therefore

$$U \cap B(w_x^{\gamma x}, e^{-a}) = \varnothing$$

so that Lemma 5.1 is available. The proof follows now exactly the proof of [10, Proposition 5.1] and [15, Proposition 5.1]. $\square$

5.2. Application of Paulin–Pollicott–Schapira's equidistribution theorem.

The purpose of this section is to use Corollary 2.8 for computing the limit of the sequence of operator-valued measures $(\mathcal{M}_x^n)_{n \in \mathbb{N}^*}$.

We assume here that ν^F is a Patterson density of dimension $\sigma_{\Gamma, F}$ and that the Gibbs measure is mixing with respect to the geodesic flow. The following proposition generalizes Proposition 5.5 of [10].

Proposition 5.3. *Let $U, V, W \subset \partial X$ be Borel subsets such that $\nu_x^F(\partial U) = \nu_x^F(\partial V) = \nu_x^F(\partial W) = 0$. Then we have:*

$$\lim_{n \to +\infty} \langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)}) \chi_V, \chi_W \rangle = \nu_x^F(U \cap W) \nu_x^F(V).$$

We need some lemmas to prepare the proof of this proposition.

Lemma 5.4. *Let U be a Borel subset of ∂X with $\nu_x^F(\partial U) = 0$ and let W be a Borel subset of ∂X such that $\nu_x^F(\partial W) = 0$, satisfying $U \cap W = \varnothing$. Then we have*

$$\limsup_{n \to +\infty} \langle \mathcal{M}_{x,\rho}^n(\chi_{\mathcal{C}_x(U)}) \mathbf{1}_{\partial X}, \chi_W \rangle = 0.$$

Proof. For all integers n we have:

$$\langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\mathbf{1}_{\partial X}, \chi_W\rangle$$

$$= c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} D_{\gamma x}(\chi_{\mathcal{C}_x(U)}) \frac{\langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \chi_W\rangle}{\phi_x(\gamma)}$$

$$= \sum_{\gamma \in \Gamma} \mu_n(\gamma) \frac{\langle \pi_x(\gamma)\mathbf{1}_{\partial X}, \chi_W\rangle}{\phi_x(\gamma)},$$

where the inequality follows from the fact that π_x preserves the cone of positive functions, and where

$$\mu_n(\gamma) := c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} e^{d^F(x,\gamma x)} \chi_{C_n(x)}(\gamma) D_{\gamma x}(\chi_{\mathcal{C}_x(U)}).$$

Observe that Corollary 2.8 implies that

$$\sup_n \|\mu_n\|_{\ell^1} < +\infty.$$

Proposition 5.2 implies for $b > 0$:

$$\limsup_{n \to +\infty} \langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\mathbf{1}_{\partial X}, \chi_W\rangle$$

$$\leq \limsup_{n \to +\infty} \sum_{\gamma \in \Gamma} \mu_n(\gamma) D_{\gamma x}(\chi_{\mathcal{C}_x(W(b))})$$

$$= \limsup_{n \to +\infty} c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} D_{\gamma x}(\chi_{\mathcal{C}_x(U) \cap \mathcal{C}_x(W(b))})$$

$$= \limsup_{n \to +\infty} c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} D_{\gamma x}(\chi_{\mathcal{C}_x(U \cap W(b))}).$$

Note the general fact $\partial(A \cap B) \subset \partial A \cup \partial B$. Since all, but at most countably many of the sets $W(b)$ have zero measure boundary Corollary 2.8 implies that

$$\limsup_{n \to +\infty} \langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\mathbf{1}_{\partial X}, \chi_W\rangle \leq \nu_x^F(U \cap W(b)).$$

With the hypothesis $U \cap W = \varnothing$, we have by letting $b \to +\infty$

$$\limsup_{n \to +\infty} \langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\mathbf{1}_{\partial X}, \chi_W\rangle = 0. \qquad \square$$

Lemma 5.5. *Let U be a Borel subset of ∂X and let V be a Borel subset of ∂X. For $a > 0$ we have*

$$\limsup_{n \to +\infty} \langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V, \mathbf{1}_{\partial X}\rangle$$

$$\leq \limsup_{n \to +\infty} c_{\Gamma,F} e^{-\sigma_{\Gamma,F} n} \sum_{\gamma \in C_n(x)} e^{d^F(x,\gamma x)} D_{\gamma^{-1} x}(\chi_{\mathcal{C}_x(U)}) D_{\gamma x}(\chi_{\mathcal{C}_x(V(a))}).$$

Proof. We have for all integer n:

$$\langle \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V, \mathbf{1}_{\partial X}\rangle$$

$$= \langle \chi_V, \mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})^*\mathbf{1}_{\partial X}\rangle$$

$$= c_{\Gamma,F}\,\mathrm{e}^{-\sigma_{\Gamma,F}n}\sum_{\gamma\in C_n(x)}\mathrm{e}^{d^F(x,\gamma x)}D_{\gamma^{-1}x}(\chi_{\mathcal{C}_x(U)})\frac{\langle\pi_x(\gamma)\mathbf{1}_{\partial X},\chi_V\rangle}{\phi_x(\gamma)}$$

$$\leq \sum_{\gamma\in\Gamma}\mu_n(\gamma)\frac{\langle\pi_x(\gamma)\mathbf{1}_{\partial X},\chi_V\rangle}{\phi_x(\gamma)},$$

with

$$\mu_n(\gamma) = c_{\Gamma,F}\,\mathrm{e}^{-\sigma_{\Gamma,F}n}\mathrm{e}^{d^F(x,\gamma x)}\chi_{C_n(x)}(\gamma)D_{\gamma^{-1}x}(\chi_{\mathcal{C}_x(U)}).$$

Applying Proposition 5.2 to μ_n defined above we obtain for all $a > 0$:

$$\limsup_{n\to+\infty}\langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V,\mathbf{1}_{\partial X}\rangle$$

$$\leq \limsup_{n\to+\infty}c_{\Gamma,F}\,\mathrm{e}^{-\sigma_{\Gamma,F}n}\sum_{\gamma\in C_n(x)}D_{\gamma^{-1}x}(\chi_{\mathcal{C}_x(U)})D_{\gamma x}(\chi_{\mathcal{C}_x(V(a))}). \qquad \square$$

Lemma 5.6. *Let $U, V, W \subset \partial X$ be Borel subsets such that*

$$\nu_x^F(\partial U) = \nu_x^F(\partial V) = \nu_x^F(\partial W) = 0.$$

Then

$$\limsup_{n\to+\infty}\langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V,\chi_W\rangle \leq \nu_x^F(U\cap W)\nu_x^F(V).$$

Proof. Let $a > 0$ and $b > 0$, and consider $V(a)$ and $W(b)$ such that $\nu_x^F(\partial W(b)) = 0 = \nu_x^F(\partial V(a))$. Let $W(b)^c = \partial X\backslash W(b)$. Set $U_1 = U\cap W(b)$ and $U_2 = U\cap W(b)^c$. Observe that $U_1\cap W(b)^c = \varnothing = U_2\cap W(b)$. It is easy to see that we can extend U_1 and U_2 to $\overline{X}$ by $\mathcal{C}_x(U_1)$ and $\mathcal{C}_x(U_2)$ such that $\mathcal{C}_x(U) = \mathcal{C}_x(U_1)\sqcup\mathcal{C}_x(U_2)$ since $U = U_1\sqcup U_2$. We have:

$$\langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V,\chi_W\rangle = \langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U_1)})\chi_V,\chi_W\rangle + \langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U_2)})\chi_V,\chi_W\rangle$$

$$\leq \langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U_1)})\chi_V,\mathbf{1}_{\partial X}\rangle + \langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U_2)})\mathbf{1}_{\partial X},\chi_W\rangle.$$

Applying Lemma 5.4 to the second term and Lemma 5.5 to the first term of the right hand side inequality above, we obtain:

$$\limsup_{n\to+\infty}\langle\mathcal{M}_x^n(\chi_{\mathcal{C}_x(U)})\chi_V,\chi_W\rangle$$

$$\leq \limsup_{n\to+\infty}c_{\Gamma,F}\,\mathrm{e}^{-\sigma_{\Gamma,F}n}\sum_{\gamma\in C_n(x)}D_{\gamma^{-1}x}(\chi_{\mathcal{C}_x(U_1)})D_{\gamma x}(\chi_{\mathcal{C}_x(V(a))}).$$

Then, since $\nu_x^F(\partial U_1) = 0 = \nu_x^F(\partial V(a))$, Corollary 2.8 leads to

$$\limsup_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle \le \nu_x^F(U\cap W(b))\nu_x^F(V(a)).$$

Because the above inequality holds for all but at most countably many values of a and b, by letting them go to $+\infty$ we obtain the required inequality. $\qquad\square$

Proof of Proposition 5.3. By Lemma 5.6 it is sufficient to prove that

$$\liminf_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle = \nu_x^F(U\cap W)\nu_x^F(V).$$

If B is a Borel subset of ∂X, we set $B^0 = B$ and $B^1 = \partial X\setminus B$. We have

$$
\begin{aligned}
\langle \mathcal{M}_x^n(\mathbf{1}_{\overline{X}})\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle &= \langle \mathcal{M}_x^n(\chi_{e_x(U^0)} + \chi_{e_x(U^1)})\chi_{V^0} + \chi_{V^1}, \chi_{W^0} + \chi_{W^1}\rangle \\
&= \sum_{i,j,k}\langle \mathcal{M}_x^n(\chi_{e_x(U^i)})\chi_{V^j}, \chi_{W^k}\rangle \\
&= \langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle + \sum_{i,j,k\ne(0,0,0)}\langle \mathcal{M}_x^n(\chi_{e_x(U^i)})\chi_{V^j}, \chi_{W^k}\rangle.
\end{aligned}
$$

Since $\liminf_{n\to+\infty}\langle \mathcal{M}_x^n(\mathbf{1}_{\overline{X}})\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle = \lim_{n\to+\infty}\langle \mathcal{M}_x^n(\mathbf{1}_{\overline{X}})\mathbf{1}_{\partial X}, \mathbf{1}_{\partial X}\rangle = \|\nu_x^F\|^2$ we have:

$$
\begin{aligned}
\|\nu_x^F\|^2 &\le \liminf_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle + \sum_{i,j,k\ne(0,0,0)}\limsup_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U^i)})\chi_{V^j}, \chi_{W^k}\rangle \\
&\le \limsup_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle + \sum_{i,j,k\ne(0,0,0)}\limsup_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U^i)})\chi_{V^j}, \chi_{W^k}\rangle \\
&\le \sum_{i,j,k}\nu_x^F(U^i\cap W^k)\nu_x^F(V^j) \\
&= \|\nu_x^F\|^2,
\end{aligned}
$$

where the last inequality comes from Lemma 5.6. Hence the inequalities of the above computation are equalities, so

$$
\begin{aligned}
\liminf_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle &= \nu_x^F(U\cap W)\nu_x^F(V) \\
&= \limsup_{n\to+\infty}\langle \mathcal{M}_x^n(\chi_{e_x(U)})\chi_V, \chi_W\rangle
\end{aligned}
$$

and the proof is done. $\qquad\square$

6. Conclusion

6.1. Standard facts about Borel subsets of measure zero frontier. Recall two standard facts about measure theory:

Lemma 6.1. *Assume that (Z, d, μ) is a metric measure space. Then the σ-algebra generated by the Borel subsets with measure zero frontier generates the Borel σ-algebra.*

Let χ_A be the characteristic function of a Borel subset A of ∂X.

Lemma 6.2. *Assume that (Z, d, μ) is a metric measure space such that μ is a finite Borel measure. Then the closure of the subspace spanned by the characteristic functions of Borel subsets having zero measure frontier is*

$$\overline{\text{Span}\{\chi_A | \mu(\partial A) = 0\}}^{L^2} = L^2(Z, \mu).$$

6.2. Proofs.

Proof of Theorem 1.3. Let ν^F be a Γ-invariant Gibbs conformal density of dimension $\sigma_{\Gamma, F}$ with $\widetilde{F}$ a symmetric potential function and Γ convex cocompact. Let x be in the $CH(\Lambda_\Gamma)$ and consider π_x associated with ν_x^F. There are two steps.

Step 1: $(\mathcal{M}_x^n)_{n \geq N}$ is uniformly bounded. Note that the norm of operators of $\mathcal{M}_x^n$ is less or equal than the norm of $\mathcal{M}_x^n(\mathbf{1}_{\overline{X}})$. Recall that

$$\mathcal{M}_x^n(\mathbf{1}_{\overline{X}}) = T_x^n$$

where T_x^n is the sequence of operators defined in (4.4). Proposition 4.7 completes the first step.

Step 2: Computation of the limit of $(\mathcal{M}_x^n)_{n \in \mathbb{N}*}$. As in [10] and in [15], the sequence $(\mathcal{M}_x^n)_{n \in \mathbb{N}^*}$ has actually one accumulation point that we denote by $\mathcal{M}_x^\infty$. We shall compute it:

Since we assume that the Gibbs measure is mixing it follows from Proposition 5.3 and from the definition (1.9) of $\mathcal{M}_x$ that for all Borel subsets $U, V, W \subset \partial X$ satisfying $\nu_x^F(\partial U) = \nu_x^F(\partial V) = \nu_x^F(\partial W) = 0$ we have

$$\langle \mathcal{M}_x^\infty(\chi_{\mathcal{C}_x(U)}) \chi_V, \chi_W \rangle = \nu_x^F(U \cap W) \nu_x^F(V) = \langle \mathcal{M}_x(\chi_{\mathcal{C}_x(U)}) \chi_V, \chi_W \rangle.$$

Observe also that the above equality holds for all balls of the space X instead of $C_x(U)$ and everything is null in this case. Since $\{\mathcal{C}_x(U) | U \subset \partial X$ such that $\nu_x^F(\partial U) = 0\}$ together with the balls of X generate the Borel σ-algebra of $\overline{X}$ and since the equality holds for all Borel subsets having zero measure boundary Lemma 6.2 completes the proof. $\square$

Proof of Corollary 1.4. Observe that $\mathcal{M}_x(\mathbf{1}_{\overline{X}})$ is the orthogonal projection onto the space of constant functions and apply the definition of weak* convergence to the triple $(\mathbf{1}_{\overline{X}}, \xi, \eta)$ for $\xi, \eta \in L^2(\partial X, \nu_x^F)$. $\qquad \square$

Proof of Corollary 1.5. Since $(\pi_{\nu_x^F})_{x \in X}$ are unitarily equivalent, it suffices to prove irreducibility for some $\pi_{\nu_x^F}$ with x in X. We pick x in $CH(\Lambda_\Gamma)$. Since $\widetilde{F}$ is cohomologuous to a symmetric potential by Lemma 2.6 we can assume that $\widetilde{F}$ itself is symmetric. Therefore Theorem 1.3 shows that the vector $\mathbf{1}_{\partial X}$ is cyclic for the representation $\pi_{\nu_x^F}$ by applying the weak* convergence to the triple $(f, \mathbf{1}_{\partial X}, \eta)$. Moreover, Corollary 1.4 shows that the orthogonal projection onto the space of constant functions is in the von Neumann algebra associated with $\pi_{\nu_x^F}$. Thus, a classical argument [29, Lemma 6.1] completes the proof. $\qquad \square$

Before giving the proof of Theorem 1.7 we say that an operator $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = L^2(X, m)$ is a Hilbert space for some measure space (X, m), is a *positive operator* if it preserves $\mathcal{H}^+$ the cone of positive functions. For example, any quasi-regular representation is a positive operator as well as the operators we consider in (4.4).

Proof of Theorem 1.7. The implications: $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ follow from Proposition 2.5. We only have to prove $(1) \Rightarrow (2)$. We follow a standard method, see for example [29, Lemma 7.3]:

Let $\pi_F := \pi_{\nu_x^F}$ and $\pi_G := \pi_{\nu_x^G}$ be equivalent unitary representations associated with ν_x^F and ν_x^G, with $\widetilde{F}$ and $\widetilde{G}$ two symmetric potentials. There exists U a unitary operator from $L^2(\partial X, \nu_x^F)$ to $L^2(\partial X, \nu_x^G)$ satisfying

$$U\pi_F = \pi_G U.$$

The map
$$\Phi : T \in W^*_{\pi_G}(\Gamma) \mapsto U^* T U \in W^*_{\pi_F}(\Gamma)$$

is a spatial isomorphism of von Neumann algebras. It follows from the irreducibility of these representations (Corollary 1.5) that the von Neumann algebras

$$W^*_{\pi_F}(\Gamma) = \mathcal{B}(L^2(\partial X, \nu_x^F)) \quad \text{and} \quad W^*_{\pi_G}(\Gamma) = \mathcal{B}(L^2(\partial X, \nu_x^G)).$$

Consider now the maximal abelian von Neumann algebras

$$L^\infty(\partial X, \nu^F) \subset \mathcal{B}(L^2(\partial X, \nu_x^F)) \quad \text{and} \quad L^\infty(\partial X, \nu^G) \subset \mathcal{B}(L^2(\partial X, \nu_x^G))$$

acting on L^2 by multiplication. Now observe that the set of projections

$$\{p \in \mathcal{B}(L^2(\partial X, \nu_x^F)) \text{ such that } p \text{ and } 1 - p \text{ are orthogonal positive projections}\}$$

is equal to the set
$$\{\chi_B \text{ where } B \text{ is a Borel subset of } \partial X\}.$$

Since the isomorphism Φ preserves the cone of positive operators and since $L^\infty(\partial X, \nu^F)$ is generated by its projections χ_B with B Borel subsets, the automorphism Φ restricts to an algebra isomorphism from

$$\Phi : L^\infty(\partial X, \nu^G) \to L^\infty(\partial X, \nu^F).$$

It is well known that there exists $\varphi : (\partial X, \nu^F) \to (\partial X, \nu^G)$ a measure class preserving Borel isomorphism such that

$$\Phi(f) = f \circ \varphi$$

for all $f \in L^\infty(\partial X, \nu^G)$. Therefore ν^G and ν^F are in the same class. $\square$

References

[1] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups, *Topology*, **33** (1994), no. 4, 765–783. Zbl 0838.20042 MR 1293309

[2] S. Adams, G. A. Elliott, and T. Giordano, Amenable actions of groups, *Trans. Amer. Math. Soc.*, **344** (1994), no. 2, 803–822. Zbl 0814.47009 MR 1250814

[3] S. Alvarez, A Fatou theorem for F-harmonic functions, 2014. arXiv:1407.0679

[4] C. Anantharaman-Delaroche, On spectral characterizations of amenability, *Israel J. Math.*, **137** (2003), 1–33. Zbl 1274.37003 MR 2013348

[5] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, in *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, 35–46, Contemp. Math., 282, Amer. Math. Soc., Providence, RI, 2001. Zbl 1016.46042 MR 1855241

[6] J.-P. Anker, La forme exacte de l'estimation fondamentale de Harish-Chandra. (French) [The exact form of Harish-Chandra's fundamental estimate], *C. R. Acad. Sci. Paris Sér. I Math.*, **305** (1987), no. 9, 371–374. Zbl 0636.22005 MR 0910372

[7] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics, 470, Springer-Verlag, Berlin, 1975. Zbl 0308.28010 MR 0442989

[8] M. Babillot, Points entiers et groupes discrets: de l'analyse aux systèmes dynamiques, in *Rigidité, groupe fondamental et dynamique*, 1–119, Panor. Synthèses, 13, Soc. Math. France, Paris, 2002. Zbl 1077.11071 MR 1993148

[9] M. Babillot, On the mixing property for hyperbolic systems, *Israel J. Math.*, **129** (2002), 61–76. Zbl 1053.37001 MR 1910932

[10] U. Bader and R. Muchnik, Boundary unitary representations, irreducibility and rigidity, *J. Mod. Dyn.*, **5** (2011), no. 1, 49–69. Zbl 1259.46051 MR 2787597

[11] M.-E. B. Bekka and M. Cowling, Some irreducible unitary representations of G(K) for a simple algebraic group G over an algebraic number field K, *Math. Zeitschrift*, **241** (2002), no. 4, 731–741. Zbl 1022.22018 MR 1942238

[12] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, 11, Cambridge University Press, Cambridge, 2008. Zbl 1146.22009 MR 2415834

[13] S. Blachère, P. Haïssinsky, and P. Mathieu, Harmonic measures versus quasiconformal measures for hyperbolic groups, *Ann. Sci. Éc. Norm. Sup. (4)*, **44** (2011), no. 4, 683–721. Zbl 1243.60005 MR 2919980

[14] M. Bourdon, *Structure conforme et flot géodésique d'un* CAT(-1)*-espace*, *Enseign. Math. (2)*, **41** (1995), no. 1-2, 63–102. Zbl 0871.58069 MR 1341941

[15] A. Boyer, Equidistribution, Ergodicity, Irreducibility in CAT(-1) spaces, 2016. arXiv:1412.8229

[16] M. Burger and S. Mozes, CAT(-1)-spaces, divergence groups and their commensurators, *J. Amer. Math. Soc.*, **9** (1996), no. 1, 57–93. Zbl 0847.22004 MR 1325797

[17] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319, Springer-Verlag, Berlin, 1999. Zbl 0988.53001 MR 1744486

[18] C. Connell and R. Muchnik, Harmonicity of quasiconformal measures and Poisson boundaries of hyperbolic spaces, *Geom. Funct. Anal.*, **17** (2007), no. 3, 707–769. Zbl 1166.60323 MR 2346273

[19] C. Connell and R. Muchnik, Harmonicity of Gibbs measures, *Duke Math. J.*, **137** (2007), no. 3, 461–509. Zbl 1133.60032 MR 2309151

[20] M. Coornaert, Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. (French) [Patterson–Sullivan measures on the boundary of a hyperbolic space in the sense of Gromov], *Pacific J. Math.*, **159** (1993), no. 2, 241–270. Zbl 0797.20029 MR 1214072

[21] M. Cowling and T. Steger, The irreducibility of restrictions of unitary representations of lattices, *J. für die reine und Angew. Math.*, **420** (1991), 85–98. Zbl 0760.22014 MR 1124567

[22] F. Dal'bo, Remarques sur le spectre des longueurs d'une surface et comptages. (French) [Remarks on the length spectrum of a surface, and counting], *Bol. Soc. Brasil. Mat. (N.S.)*, **30** (1999), no. 2, 199–221. Zbl 1058.53063 MR 1703039

[23] J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, 1969. Zbl 0174.18601 MR 246136

[24] H. Furstenberg, Poisson formula for semi-simple Lie groups, *Ann. of Math. (2)*, **77** (1963), 335–386. Zbl 0192.12704 MR 146298

[25] A. Figà-Talamanca and M.-A. Picardello, Spherical functions and harmonic analysis on free groups, *J. Funct. Anal.*, **47** (1982), no. 3, 281–304. Zbl 0489.43008 MR 0665019

[26] A. Figà-Talamanca and T. Steger, Harmonic analysis for anisotropic random walks on homogeneous trees, *Mem. Amer. Math. Soc.*, **110** (1994), no. 531, xii+68pp. Zbl 0836.43019 MR 1219707

[27] A. Figà-Talamanca and M.-A. Picardello, *Harmonic analysis on free groups*, Lecture Notes in Pure and Applied Mathematics, 87, Marcel Dekker, Inc., New York, 1983. Zbl 0536.43001 MR 0710827

[28] R. Gangolli and V.-S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Springer-Verlag, New-York, 1988. Zbl 0675.43004 MR 954385

[29] L. Garncarek, Boundary representations of hyperbolic groups, 2014. arXiv:1404.0903

[30] A. Gorodnik and A. Nevo, *The ergodic theory of lattice subgroups*, Annals of Mathematics Studies, 172, Princeton University Press, Princeton, NJ, 2010. Zbl 1186.37004 MR 2573139

[31] U. Haagerup, An example of a nonnuclear C^*-algebra which has the metric approximation property, *Invent. Math.*, **50** (1978/79), no. 3, 279–293. Zbl 0408.46046 MR 520930

[32] P. de la Harpe, Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint. (French) [Hyperbolic groups, operator algebras and Jolissaint's theorem], *C. R. Acad. Sci. Paris Sér. I Math.*, **307** (1988), no. 14, 771–774. Zbl 0653.46059 MR 0972078

[33] P. Jolissaint, Rapidly decreasing functions in reduced C^*-algebras of groups, *Trans. Amer. Math. Soc.*, **317** (1990), no. 1, 167–196. Zbl 0711.46054 MR 943303

[34] V.-A. Kaimanovich, Boundary amenability of hyperbolic spaces, in *Discrete geometric analysis*, 83–111, Contemp. Math., 347, Amer. Math. Soc., Providence, RI, 2004. Zbl 1061.43003 MR 2077032

[35] M.-G. Kuhn, Amenable actions and weak containment of certain representations of discrete groups, *Proc. Amer. Math. Soc.*, **122** (1994), no. 3, 751–757. Zbl 0829.43003 MR 1209424

[36] G. Kuhn and T. Steger, More irreducible boundary representations of free groups, *Duke Math. J.*, **82** (1996), no. 2, 381–436. Zbl 0851.22005 MR 1387235

[37] F. Ledrappier, Structure au bord des variétés à courbure négative. (French) [Boundary structure of negatively curved manifolds], in *Séminaire de Théorie Spectrale et Géométrie, 13, Année 1994–1995*, 97–122, Univ. Grenoble I, Saint-Martin-d'Hères, 1995. Zbl 0931.53005 MR 1715960

[38] G.-A. Margulis, *On some aspects of the theory of Anosov systems*. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows. Translated from the Russian by Valentina Vladimirovna Szulikowska, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. Zbl 1140.37010 MR 2035655

[39] O. Moshen, Le bas du spectre d'une variété hyperbolique est un point selle, *Ann. Sci. École Norm. Sup.*, **40** (2007), 191–207. Zbl 1128.58008 MR 2339284

[40] A. Nevo, Spectral transfer and pointwise ergodic theorems for semi-simple Kazhdan groups, *Math. Res. Lett.*, **5** (1998), no. 3, 305–325. Zbl 0942.22007 MR 1637840

[41] A. Nevo, The spectral theory of amenable actions and invariants of discrete groups, *Geom. Dedicata*, **100** (2003), 187–218. Zbl 1033.22001 MR 2011122

[42] S.-J. Patterson, The limit set of a fuchsian group, *Acta mathematica*, **136** (1976), 241–273. Zbl 0336.30005 MR 450547

[43] F. Paulin, M. Pollicott, and B. Schapira, Equilibrium states in negative curvature, *Astérisque*, **373** (2015), viii+281pp. Zbl 1347.37001 MR 3444431

[44] T. Roblin, *Ergodicité et équidistribution en courbure négative*, *Mém. Soc. Math. Fr. (N.S.)*, **95** (2003), vi+96pp. Zbl 1056.37034 MR 2057305

[45] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag London, 2002. Zbl 1090.46001 MR 1888309

[46] Y. Shalom, Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group, *Ann. of Math. (2)*, **152** (2000), no. 1, 113–182. Zbl 0970.22011 MR 1792293

[47] R.-J. Spatzier, An example of an amenable action from geometry, *Ergodic Theory Dynam. Systems*, **7** (1987), no. 2, 289–293. Zbl 0615.58028 MR 89679

[48] R.-J. Spatzier and R.-J. Zimmer, Fundamental groups of negatively curved manifolds and actions of semisimple groups, *Topology*, **30** (1991), no. 4, 591–601. Zbl 0744.57022 MR 1133874

[49] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*. With the assistance of Timothy S. Murphy, Princeton Mathematical Series, 43, Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993. Zbl 0821.42001 MR 1232192

[50] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, *Publications mathématiques de l'IHES*, **50** (1979), 171–202. Zbl 0439.30034 MR 0556586

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