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Ergodic components of partially hyperbolic systems

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Abstract. This paper gives a complete classification of the possible ergodic decompositions for certain open families of volume-preserving partially hyperbolic diffeomorphisms. These families include systems with compact center leaves and perturbations of Anosov flows under conditions on the dimensions of the invariant subbundles. The paper further shows that the non-open accessibility classes form a C^1 lamination and gives results about the accessibility classes of non-volume-preserving systems.

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1. Introduction

Invariant measures are important objects in the study of dynamical systems. Often, these measures are ergodic, allowing a single orbit to express the global behaviour of the system. However, this is not always the case. For instance, a Hamiltonian system always possesses a smooth invariant measure, but a generic smooth Hamiltonian yields level sets on which the dynamics are not ergodic [30]. Any invariant measure may be expressed as a linear combination of ergodic measures and while such a decomposition always exists, it is not, in general, tractable to find it. For partially hyperbolic systems, there is a natural candidate for the ergodic decomposition given by the accessibility classes of the system. This paper analyzes certain families of partially hyperbolic systems, characterizing the possible accessibility classes and showing that these coincide with the ergodic components of any smooth invariant measure.

By the classical work of Hopf, the geodesic flow on a surface of negative curvature is ergodic [26]. Further, by the work Anosov and Sinai, the flow is *stably ergodic* meaning that all nearby flows are also ergodic [1, 2]. Based on these techniques, Grayson, Pugh, and Shub showed that the time-one map of this geodesic flow is also stably ergodic as a diffeomorphism [21]. To prove this, they observed two important

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properties. The first property is *partial hyperbolicity*. A diffeomorphism f is partially hyperbolic if there is an invariant splitting of the tangent bundle of the phase space M into three subbundles

$$TM = E^u \oplus E^c \oplus E^s$$

such that vectors in the unstable bundle E^u are expanded by the derivative Tf, vectors in the stable bundle E^s are contracted, and these dominate any expansion and contraction of vectors in the center bundle E^c. (Appendix A gives a precise definition.) The second property is *accessibility*. For a point $x \in M$, the accessibility class $AC(x)$ is the set of all points that can be reached from x by a concatenation of paths, each tangent to either E^s or E^u. A system is called accessible if its phase space consists of a single accessibility class. For the geodesic flow, the phase space M is the unit tangent bundle of the surface, E^c is the direction of the flow, and E^s and E^u are given by the horocycles. Grayson, Pugh, and Shub demonstrated that any diffeomorphism near the time-one map of the flow is both partially hyperbolic and accessible and used this to prove its ergodicity. This breakthrough was followed by a number of papers demonstrating stable ergodicity for specific cases of partially hyperbolic systems (see the surveys [40, 46]) and lead Pugh and Shub to formulate the following conjecture [37].

Conjecture 1. *Ergodicity holds on an open and dense set of volume-preserving partially hyperbolic diffeomorphisms.*

They further split this into two subconjectures.

Conjecture 2. *Accessibility implies ergodicity.*

Conjecture 3. *Accessibility holds on an open and dense set of partially hyperbolic diffeomorphisms (volume-preserving or not).*

The Pugh–Shub conjectures have been established in a number of settings. In particular, they are true when the center bundle E^c is one-dimensional [41]. However, there are a number of partially hyperbolic systems which arise naturally and which are not ergodic, leading to the following questions.

Question. *Is it possible to give an exact description of the set of non-ergodic partially hyperbolic diffeomorphisms?*

Question. *For a non-ergodic partially hyperbolic diffeomorphism, do the ergodic components coincide with the accessibility classes of the system?*

This paper answers these questions in the affirmative under certain assumptions on the system. We first give one example as motivation before introducing more general results. Consider on the 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ a diffeomorphism f defined by

$$f(x, y, z) = (2x + y, x + y, z).$$

The eigenvalues are $\lambda < 1 < \lambda^{-1}$ and f is therefore partially hyperbolic. Arguably, this is the simplest partially hyperbolic example one can find. It preserves Lebesgue measure but is not ergodic. Further, there are several ways to construct nearby diffeomorphisms which are also non-ergodic. With a bit of thought, the following methods come to mind.

(1) Rotate f slightly along the center direction, yielding a diffeomorphism

$$(x, y, z) \mapsto (2x + y, x + y, z + \theta)$$

for some small rational $\theta \in \mathbb{R}/\mathbb{Z}$.

(2) Compose f with a map of the form $(x, y, z) \mapsto (\psi(x, y, z), z)$ for some $\psi : \mathbb{T}^3 \to \mathbb{T}^2$.

(3) Perturb f on a subset of the form $\mathbb{T}^2 \times X$ where $X \subsetneq \mathbb{S}^1$.

(4) Conjugate f with a diffeomorphism close to the identity.

The results of this paper imply that any non-ergodic diffeomorphism in a neighbourhood of f can be constructed by applying these four steps in this order.

Throughout the study of stably ergodic dynamical systems, regularity of the invariant foliations has played a prominent role. One of Anosov's early key contributions was a proof that holonomies along the stable and unstable foliations are absolutely continuous. This allowed him to show that all Anosov systems are stably ergodic. Grayson, Pugh, and Shub adapted this proof in the setting of the perturbation of the time-one map of an Anosov flow to show that the stable holonomy inside of a center-stable leaf is C^1 regular. Determining the exact conditions which imply C^1 regularity lead to the notion of "center bunching" [37, 38]. Roughly speaking, a partially hyperbolic system is center bunched if the derivative in the center direction is sufficiently close to conformal. Further, a qualified case of the second Pugh–Shub conjecture holds: any accessible, center bunched system is ergodic [12].

In the case of one-dimensional center, every partially hyperbolic system is center bunched. Further, F. Rodriguez Hertz, J. Rodriguez Hertz, and R. Ures showed that each accessibility class is either an open subset of the manifold or an immersed codimension one submanifold tangent to $E^u \oplus E^s$ [41]. The submanifolds in the second case form a lamination and are called us-leaves. While each leaf is C^1 regular, it was not previously known if the coordinate charts defining the lamination could taken as C^1. In this paper, we establish this regularity, showing that the us-leaves indeed form a C^1 lamination (see (2.9) below). This then allows us in certain settings to apply Fubini's theorem to the disintegration of the volume into measures on leaves of the lamination and consequently to show that the ergodic components have supports coinciding with the accessibility classes.

The proof of C^1 regularity of the lamination relies on the C^1 regularity of the stable and unstable holonomies inside cs and cu-leaves. In the special case that E^u and E^s are everywhere jointly integrable, these two holonomies commute and together give a well-defined us-holonomy between center leaves. In the case where the us-lamination is defined only on a proper closed subset of the phase space, the holonomies do not commute and so establishing regularity of the lamination is more involved. The basic idea is to define what the derivative of a us-holonomy "should be" at all points and then use Whitney's extension theorem to show that the holonomy defined for points in the lamination extends to a C^1 function in a neighbourhood of these points. For leaves in the us-lamination which are accumulated on by other leaves, the u and s-holonomies inside the cu and cs-foliations provide the candidate derivatives. For isolated us-leaves which accumulate on non-isolated us-leaves, these holonomies cannot be used and a more subtle approach is taken. Section 12 treats all of these issues of regularity in detail.

2. Statement of results

We again refer the reader to the appendix for a list of definitions.

Suppose A and B are automorphisms of a compact nilmanifold N such that A is hyperbolic and $AB = BA$. Then, A and B define a diffeomorphism

$$f_{AB} : M_B \to M_B, \quad (v,t) \mapsto (Av,t)$$

on the manifold

$$M_B = N \times \mathbb{R}/(v,t) \sim (Bv, t-1).$$

Call f_{AB} an *AB-prototype*.

Note that every AB-prototype is an example of a volume-preserving, partially hyperbolic, non-ergodic system. Further, just like the linear example on $\mathbb{T}^3$ given above, every AB-prototype may be perturbed to produce nearby diffeomorphisms which are also non-ergodic.

To consider such perturbations, we use the notion of leaf conjugacy as introduced in [25]. Two partially hyperbolic diffeomorphisms f and g are *leaf conjugate* if there are invariant foliations W^c_f and W^c_g tangent to E^c_f and E^c_g and a homeomorphism h such that for every leaf in L in W^c_f, $h(L)$ is a leaf of W^c_g and $h(f(L)) = g(h(L))$.

We now define a family of diffeomorphisms which will be the focus of the paper. A partially hyperbolic system $f : M \to M$ is an *AB-system* if it preserves an orientation of the center bundle E^c and is leaf conjugate to an AB-prototype.

In order to consider skew-products over infranilmanifolds and systems which do not preserve an orientation of E^c, we also consider the following generalization. A diffeomorphism f_0 is an *infra-AB-system* if an iterate of f_0 lifts to an AB-system on a finite cover. To the best of the author's knowledge, this family of partially hyperbolic diffeomorphisms includes every currently known example of a non-ergodic system with one-dimensional center. Further, there are manifolds on which every conservative partially hyperbolic diffeomorphism is an AB-system.

Question 2.1. *Suppose f is a conservative, non-ergodic, partially hyperbolic C^2 diffeomorphism with one-dimensional center. Is f necessarily an infra-AB-system?*

Skew products with trivial bundles correspond to AB-systems where B is the identity map. The suspensions of Anosov diffeomorphisms correspond to the case $A = B$. These are not the only cases, however. For instance, one could take hyperbolic automorphisms $A, B : \mathbb{T}^3 \to \mathbb{T}^3$ defined by the commuting matrices

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Throughout this paper, the letters A and B will always refer to the maps associated to the AB-system under study, and N and M_B will be the manifolds in the definition. In general, if $f : M \to M$ is an AB-system, M need only be homeomorphic to M_B, not diffeomorphic [14, 15].

We show that every conservative AB-system belongs to one of three cases, each with distinct dynamical and ergodic properties.

Theorem 2.2. *Suppose $f : M \to M$ is a C^2 AB-system which preserves a smooth volume form. Then, one of the following occurs.*

(1) *f is accessible and stably ergodic.*

(2) *E^u and E^s are jointly integrable and f is topologically conjugate to $M_B \to M_B$, $(v, t) \mapsto (Av, t + \theta)$ for some θ. Further, f is (non-stably) ergodic if and only if θ defines an irrational rotation.*

(3) *There are $n \geq 1$, a C^1 surjection $p : M \to \mathbb{S}^1$, and a non-empty open set $U \subsetneq \mathbb{S}^1$ such that*

- *for every connected component I of U, $p^{-1}(I)$ is an f^n-invariant subset homeomorphic to $N \times I$ and the restriction of f^n to this subset is accessible and ergodic, and*
- *for every $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an f^n-invariant submanifold tangent to $(E^u \oplus E^s)$ and homeomorphic to N.*

Note that the first case can be thought of as a degenerate form of the third case with $U = \mathbb{S}^1$. Similarly, the second case with rational rotation corresponds to $U = \varnothing$.

To give the ergodic decomposition of these systems, we decompose the measure and show that each of the resulting measures is ergodic. Suppose μ is a smooth measure on a manifold M and $p : M \to \mathbb{S}^1$ is continuous and surjective such that $p_*\mu = m$ where m is Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. The Rokhlin disintegration theorem [45] implies that μ can be written as

$$\mu = \int_{t \in \mathbb{S}^1} \mu_t \, dm(t)$$

where each μ_t is contained in $p^{-1}(t)$. Moreover, this disintegration is essentially unique; if measures $\{\nu_t\}_{t\in\mathbb{S}^1}$ give another disintegration of μ, then $\nu_t = \mu_t$ for m-a.e. $t \in \mathbb{S}^1$. For an open interval $I \subset \mathbb{S}^1$ define

$$\mu_I := \frac{1}{m(I)} \int_I \mu_t \, dm(t).$$

Note that μ_I is the normalized restriction of μ to $p^{-1}(I)$. Then an open subset $U \subset \mathbb{S}^1$ yields a decomposition

$$\mu = \sum_I m(I)\, \mu_I + \int_{t\in\mathbb{S}^1\setminus U} \mu_t \, dm(t) \tag{2.1}$$

where $\sum_I$ denotes summation over all of the connected components I of U.

Theorem 2.3. *If $f : M \to M$ is a C^2 AB-system and μ is a smooth, invariant, non-ergodic measure with $\mu(M) = 1$, then there are $n \geq 1$, a C^1 surjection $p : M \to \mathbb{S}^1$, and an open set $U \subsetneq \mathbb{S}^1$ such that $p_*\mu = m$ and (2.1) is the ergodic decomposition of (f^n, μ).*

If f is in case (3) of (2.2), then the n, p, and U can be taken to be the same in both theorems. If f is in case (2) and non-ergodic, then θ is rational, and the map p can be defined by composing the topological conjugacy from M to M_B with a projection from M_B to $\mathbb{S}^1$.

As f preserves μ and $p_*\mu = m$, it follows that $p(f(x)) = p(x) + q$ for some rational $q \in \mathbb{S}^1$ and all x with $p(x) \notin U$. Because of this, one can derive the ergodic decomposition of (f, μ) from (2.3). Each component is either of the form $\frac{1}{n}\sum_{j=1}^{n} \mu_{t+jq}$ or $\frac{1}{n}\sum_{j=1}^{n} \mu_{I_{k,j}}$ where if $I_k = (a, b)$ then $I_{k,j} = (a + jq, b + jq)$. In (2.3), the ergodic components of (f^n, μ) are mixing and, in fact, have the Kolmogorov property [12]. The ergodic components of f are mixing if and only if (2.3) holds with $n = 1$.

Using the perturbation techniques of [41], for any AB-prototype f_{AB}, rational number $\theta = \frac{k}{n}$, and open subset $U \subsetneq \mathbb{S}^1$ which satisfies $U + \theta = U$, one can construct an example of a volume-preserving AB-system which satisfies (2.3) with the same n and U. In this sense, the classification given by (2.2) and (2.3) may be thought of as complete. Versions of these theorems for infra-AB-systems are given in Section 14.

Accessibility also has applications beyond the conservative setting. For instance, Brin showed that accessibility and a non-wandering condition imply that the system is (topologically) transitive [8]. Therefore, we state a version of (2.2) which assumes only this non-wandering condition. For a homeomorphism $f : M \to M$, a *wandering domain* is a non-empty open subset U such that $U \cap f^n(U)$ is empty for all $n \geq 1$. Let $NW(f)$ be the non-wandering set, the set of all points $x \in M$ which do not lie in a wandering domain.

Theorem 2.4. *Suppose $f : M \to M$ is an AB-system such that $NW(f) = M$. Then, one of the following occurs.*

(1) *f is accessible and transitive.*

(2) *E^u and E^s are jointly integrable and f is topologically conjugate to $M_B \to M_B$, $(v,t) \mapsto (Av, t + \theta)$ for some θ. Further, f is transitive if and only if θ defines an irrational rotation.*

(3) *There are $n \geq 1$, a continuous surjection $p : M \to \mathbb{S}^1$, and a non-empty open set $U \subsetneq \mathbb{S}^1$ such that*

- *for every connected component I of U, $p^{-1}(I)$ is an f^n-invariant subset homeomorphic to $N \times I$, and*
- *for every $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an f^n-invariant submanifold tangent to $E^u \oplus E^s$ and homeomorphic to N.*

The restriction of f^n to a subset $p^{-1}(t)$ or $p^{-1}(I)$ is transitive.

The non-wandering assumption is used in only a few places in the proof and so certain results may be stated without this assumption. For a partially hyperbolic diffeomorphism with one-dimensional center, a *(us-leaf)* is a complete C^1 submanifold tangent to $E^u \oplus E^s$.

Theorem 2.5. *Every non-accessible AB-system has a compact us-leaf.*

Theorem 2.6. *Suppose $f : M \to M$ is a non-accessible AB-system with at least one compact periodic us-leaf. Then, there are $n \geq 1$, a continuous surjection $p : M \to \mathbb{S}^1$ and an open subset $U \subset \mathbb{S}^1$ with the following properties.*

For $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ is an f^n-invariant compact us-leaf. Moreover, every f-periodic compact us-leaf is of this form.

For every connected component I of U, $p^{-1}(I)$ is f^n-invariant, homeomorphic to $N \times I$ and, letting g denote the restriction of f^n to $p^{-1}(I)$, one of three cases occurs:

(1) *g is accessible,*

(2) *there is an open set $V \subset p^{-1}(I)$ such that*

$$\overline{g(V)} \subset V, \quad \bigcup_{k \in \mathbb{Z}} g^k(V) = p^{-1}(I), \quad \bigcap_{k \in \mathbb{Z}} g^k(V) = \varnothing,$$

and the boundary of V is a compact us-leaf, or

(3) *there are no compact us-leaves in $p^{-1}(I)$, uncountably many non-compact us-leaves in $p^{-1}(I)$, and $\lambda \neq 1$ such that g is semiconjugate to*

$$N \times \mathbb{R} \to N \times \mathbb{R}, \quad (v,t) \mapsto (Av, \lambda t).$$

It is relatively easy to construct examples in the first two cases above. Section 16 gives an example of the third case. It is based on the discovery by Rodriguez Hertz, Rodriguez Hertz, and Ures of a non-dynamically coherent system on the 3-torus [44]. Theorem (2.6) corresponds to a rational rotation on an f-invariant circle. The following two theorems correspond to irrational rotation.

Theorem 2.7. *Suppose $f : M \to M$ is a non-accessible AB-system with no periodic compact us-leaves. Then, there is a continuous surjection $p : M \to \mathbb{S}^1$ and a C^1 diffeomorphism $r : \mathbb{S}^1 \to \mathbb{S}^1$ such that*

- $NW(f) = p^{-1}(NW(r))$,
- *if $t \in NW(r)$ then $p^{-1}(t)$ is a compact us-leaf and $f(p^{-1}(t)) = p^{-1}(r(t))$, and*
- *if I is a connected component of $\mathbb{S}^1 \setminus NW(r)$, then $f(p^{-1}(I)) = p^{-1}(r(I))$. In particular, $p^{-1}(I) \subset M$ is a wandering domain.*

Theorem 2.8. *Suppose $f : M \to M$ is a non-accessible AB-system with no periodic compact us-leaves. Then, f is semiconjugate to*

$$M_B \to M_B, \quad (v, t) \mapsto (Av, t + \theta)$$

for θ defining an irrational rotation.

One can construct C^1 examples of AB-systems satisfying the conditions of (2.7) and with $NW(f) \neq M$. For instance, if r is a Denjoy diffeomorphism of the circle, simply consider a direct product $A \times r$ where A is Anosov.

The diffeomorphism f in (2.4)–(2.8) need only be C^1 in general. If f is a C^2 diffeomorphism, then the surjection $p : M \to \mathbb{S}^1$ may be taken as C^1. This is a consequence of the following regularity result, proven in Section 12.

Theorem 2.9. *For a non-accessible partially hyperbolic C^2 diffeomorphism with one-dimensional center, the us-leaves form a C^1 lamination.*

The existence of a C^0 lamination was shown in [41].

The next sections discuss how this work relates to other results in partially hyperbolic theory, first for three-dimensional systems in Section 3 and for higher dimensions in Section 4. Section 5 gives an outline of the proof and of the organization of the rest of the paper. The appendix gives precise definitions for many of the terms used in these next few sections.

3. Dimension three

The study of partially hyperbolic systems has had its greatest success in dimension three, where $\dim E^u = \dim E^c = \dim E^s = 1$. Still, in this simplest of cases, a number of important questions remain open. Rodiguez Hertz, Rodriguez Hertz, and Ures posed the following conjecture specifically regarding ergodicity.

Conjecture 3.1. *If a conservative partially hyperbolic diffeomorphism in dimension three is not ergodic, then there is a periodic 2-torus tangent to* $E^u \oplus E^s$.

They also showed that the existence of such a torus would have strong dynamical consequences. We state this theorem as follows.

Theorem 3.2 ([43]). *If a partially hyperbolic diffeomorphism on a three dimensional manifold* M *has a periodic 2-torus tangent to* $E^u \oplus E^s$*, then* M *has solvable fundamental group.*

In fact, the theorem may be stated in a much stronger form. See [43] for details.

Work on classifying partially hyperbolic systems has seen some success in recent years, at least for 3-manifolds with "small" fundamental group. This was made possible by the breakthrough results of Brin, Burago, and Ivanov to rule out partially hyperbolic diffeomorphisms on the 3-sphere and prove dynamical coherence on the 3-torus [7, 9]. Building on this work, the author and R. Potrie gave a classification up to leaf conjugacy of all partially hyperbolic systems on 3-manifolds with solvable fundamental group. Using the terminology of the current paper, the conservative version of this classification can be stated as follows.

Theorem 3.3 ([23]). *A conservative partially hyperbolic diffeomorphism on a 3-manifold with solvable fundamental group is (up to finite iterates and finite covers) either*

(a) *an AB-system,*

(b) *a skew-product with a non-trivial fiber bundle, or*

(c) *a system leaf conjugate to an Anosov diffeomorphism.*

Further, the ergodic properties of each of these three cases have been examined in detail. Case (a) is the subject of the current paper. Case (b) was studied in [42], where it was first shown that there are manifolds on which all partially hyperbolic systems are accessible and ergodic. Case (c) was studied in [24], which showed that if such a system is not ergodic then it is topologically conjugate to an Anosov diffeomorphism (not just leaf conjugate). It is an open question if such a non-ergodic system can occur. All of these results can be synthesized into the following statement, similar in form to (2.2).

Theorem 3.4. *Suppose* M *is a 3-manifold with solvable fundamental group and* $f : M \to M$ *is a* C^2 *conservative partially hyperbolic system. Then, (up to finite iterates and finite covers) one of the following occurs.*

(1) f *is accessible and stably ergodic.*

(2) E^u *and* E^s *are jointly integrable and* f *is topologically conjugate either to a linear hyperbolic automorphism of* $\mathbb{T}^3$ *or to*

$$M_B \to M_B, \ (v, t) \mapsto (Av, t + \theta)$$

where $A, B : \mathbb{T}^2 \to \mathbb{T}^2$ *define an AB-prototype and* $\theta \in \mathbb{S}^1$.

(3) *There are* $n \geq 1$, *a* C^1 *surjection* $p : M \to \mathbb{S}^1$, *and a non-empty open set* $U \subsetneq \mathbb{S}^1$ *such that*

- *for every connected component* I *of* U, $p^{-1}(I)$ *is an* f^n*-invariant subset homeomorphic to* $\mathbb{T}^2 \times I$ *and the restriction of* f^n *to this subset is accessible and ergodic,*
- *for every* $t \in \mathbb{S}^1 \setminus U$, $p^{-1}(t)$ *is an* f^n*-invariant 2-torus tangent to* $E^u \oplus E^s$.

If (3.1) is true, then this theorem encapsulates every possible ergodic decomposition for a 3-dimensional partially hyperbolic system.

Question 3.5. *Is the condition "with solvable fundamental group" necessary in* (3.4)*?*

4. Higher dimensions

We next consider the case of skew products in higher dimension. In related work, K. Burns and A. Wilkinson studied stable ergodicity of rotation extensions and of more general group extensions over Anosov diffeomorphisms [11], and M. Field, I. Melbourne, V. Niţică, and A. Török have analyzed group extensions over Axiom A systems, proving results on transitivity, ergodicity, and rates of mixing [16, 17, 31].

In this paper, we use the following definition taken from [20]. Let $\pi : M \to X$ define a fiber bundle on a compact manifold M over a topological manifold X. If a partially hyperbolic diffeomorphism $f : M \to M$ is such that the center direction E^c_f is tangent to the fibers of the bundle and there is a homeomorphism $A : X \to X$ satisfying $\pi f = A\pi$, then f is a *partially hyperbolic skew product*. We call A the *base map* of the skew product. While f must be C^1, π in general will only be continuous.

This definition has the benefit that it is open: any C^1-small perturbation of a partially hyperbolic skew product is again a partially hyperbolic skew product. This can be proven using the results in [25] and the fact that the base map is expansive. The base map also has the property that it is topologically Anosov [3]. As with smooth Anosov systems, it is an open question if all topologically Anosov systems are algebraic in nature.

Question 4.1. *If* A *is a base map of a partially hyperbolic skew product, then is* A *topologically conjugate to a hyperbolic infranilmanifold automorphism?*

We now consider the case where $\dim E^c = 1$ in order to relate skew products to the AB-systems studied in this paper. The following is easily proved.

Proposition 4.2. *Suppose* f *is a partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and* E^c *is one-dimensional and has an orientation preserved by* f. *Then,* f *is an AB-system if and only if the fiber bundle defining the skew product is trivial.*

If we are interested in the ergodic properties of the system, we can further relate accessibility to triviality of the fiber bundle.

Theorem 4.3. *Suppose f is a partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and E^c is one-dimensional and orientable. If f is not accessible, then the fiber bundle defining the skew product is trivial.*

Corollary 4.4. *Suppose f is a conservative C^2 partially hyperbolic skew product where the base map is a hyperbolic nilmanifold automorphism and E^c is one-dimensional and has an orientation preserved by f. Then, f satisfies one of the three cases of* (2.2) *and if f is not ergodic, its ergodic decomposition is given by* (2.3).

Theorem (4.3) is proved in Section 13. A similar statement, (14.5), still holds when "nilmanifold" is replaced by "infranilmanifold" and the condition on orientability is dropped.

Every partially hyperbolic skew product has compact center leaves and an open question, attributed in [40] to C. C. Pugh, asks if some form of converse statement holds.

Question 4.5. *Is every partially hyperbolic diffeomorphism with compact center leaves finitely covered by a partially hyperbolic skew product?*

This question was studied independently by D. Bohnet, P. Carrasco, and A. Gogolev who gave positive answers under certain assumptions [5,6,13,20]. In relation to the systems studied in the current paper, the following results are relevant.

Theorem 4.6 ([20]). *If f is a partially hyperbolic diffeomorphism with compact center leaves, and $\dim E^c = 1$, $\dim E^u \le 2$, and $\dim E^s \le 2$, then f is finitely covered by a skew product.*

Corollary 4.7. *Suppose $f : M \to M$ is a partially hyperbolic diffeomorphism with compact center leaves, $\dim E^c = 1$, and $\dim M = 4$. If f is not accessible, then f is an infra-AB-system.*

A compact foliation is *uniformly compact* if there is a uniform bound on the volume of the leaves.

Theorem 4.8 ([6]). *If f is a partially hyperbolic diffeomorphism with uniformly compact center leaves and $\dim E^u = 1$, then f is finitely covered by a partially hyperbolic skew product where the base map is a hyperbolic toral automorphism.*

Corollary 4.9. *Suppose f is a partially hyperbolic diffeomorphism with uniformly compact center leaves and $\dim E^u = \dim E^c = 1$. If f is not accessible, then f is an infra-AB-system.*

In the conservative setting, we may then invoke the results of the current paper to describe the ergodic properties of these systems.

Question 4.10. *If f is a non-accessible partially hyperbolic diffeomorphism with compact one-dimensional center leaves, then is f an infra-AB-system?*

Positive answers to both (4.1) and (4.5) would give a positive answer to (4.10).

In his study of hyperbolic flows, Anosov established a dichotomy, now known as the "Anosov alternative" which states that every transitive Anosov flow is either topologically mixing or the suspension of an Anosov diffeomorphism with constant roof function [1, 17]. Ergodic variants of the Anosov alternative have also been studied and the following holds.

Theorem 4.11 ([10, 33]). *For an Anosov flow $\phi_t : M \to M$, the following are equivalent:*

- *the time-one map ϕ_1 is not accessible,*
- *the strong stable and unstable foliations are jointly integrable,*

and both imply the flow is topologically conjugate to the suspension of an Anosov diffeomorphism.

Corollary 4.12. *Suppose every Anosov diffeomorphism is topologically conjugate to an infranilmanifold automorphism. Then, every non-accessible time-one map of an Anosov flow is an infra-AB-system.*

Thus, if the conjecture about Anosov diffeomorphisms is true, then the results given in Section 14 will classify the ergodic properties of diffeomorphisms which are perturbations of time-one maps of Anosov flows. This conjecture is true when the Anosov diffeomorphism has a one dimensional stable or unstable bundle [32].

Corollary 4.13. *Suppose f is the time-one map of an Anosov flow with* $\dim E^u_f = 1$. *If f is not accessible, then it is an AB-system.*

5. Outline

Most of the remaining sections focus on proving the results listed in Section 2 and we present here an outline of the main ideas.

A partially hyperbolic system has *global product structure* if it is dynamically coherent and, after lifting the foliations to the universal cover $\tilde{M}$, the following hold for all $x, y \in \tilde{M}$:

(1) $W^u(x)$ and $W^{cs}(y)$ intersect exactly once,

(2) $W^s(x)$ and $W^{cu}(y)$ intersect exactly once,

(3) if $x \in W^{cs}(y)$, then $W^c(x)$ and $W^s(y)$ intersect exactly once, and

(4) if $x \in W^{cu}(y)$, then $W^c(x)$ and $W^u(y)$ intersect exactly once.

Theorem 5.1. *Every AB-system has global product structure.*

This proof of this theorem is left to Section 15. That section also proves the following.

Theorem 5.2. *AB-systems form a C^1-open subset of the space of diffeomorphisms.*

Now assume f is a non-accessible AB-system. There is a lamination consisting of us-leaves [41], and this lamination lifts to the universal cover. Global product structure implies that for a center leaf L on the cover, every leaf of the lifted us-lamination intersects L exactly once. Each deck transformation maps the lamination to itself and this leads to an action of the fundamental group on a closed subset of L as depicted in Figure 1.

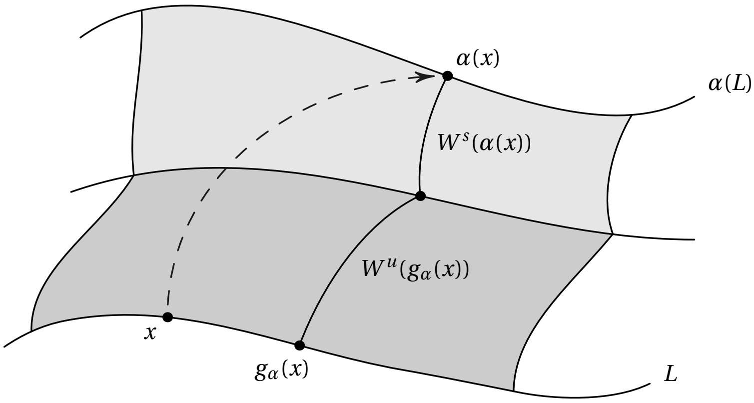

Figure 1. After lifting to the universal cover, an AB-system has a center leaf L invariant under the lifted dynamics f. Each deck transformation α then defines a function $g_\alpha : L \to L$ where $g_\alpha(x)$ is the unique point for which $W^s(\alpha(x))$ intersects $W^u(g_\alpha(x))$. These functions together with f define a solvable action on a closed subset of L and this action is semiconjugate to an affine action on $\mathbb{R}$.

In Section 6, we consider an order-preserving action of a nilpotent group G on a closed subset $\Gamma \subset \mathbb{R}$. We also assume there is f acting on Γ such that $fGf^{-1} = G$. Then, f and G generate a solvable group. Solvable groups acting on the line were studied by Plante [35]. By adapting his results, we prove (6.5) which (omitting some details for now) states that either $\operatorname{Fix}(G)$ is non-empty or, up to a common semiconjugacy from Γ to $\mathbb{R}$, each $g \in G$ gives a translation $x \mapsto x + \tau(g)$ and f gives a scaling $x \mapsto \lambda x$.

Instead of applying this result immediately to AB-systems, Section 7 introduces the notion of an "AI-system" which can be thought of as the lift of an AB-system to a covering space homeomorphic to $N \times \mathbb{R}$ where, as always, N is a nilmanifold. Using (6.5), Section 7 gives a classification result, (7.1), for the accessibility classes of AI-systems. Section 8 applies the results for AI-systems to give results about AB-systems and gives a proof of (2.5). The higher dimensional dynamics of the AB-system depend on the one-dimensional dynamics on an invariant circle. Sections 9 and 10 consider the cases of rational and irrational rotation respectively and prove Theorems (2.6)–(2.8).

Section 11 gives the proofs of (2.2), (2.3), and (2.4) based on the other results. In order to establish the ergodic decomposition, the lamination of us-leaves must be C^1. By (2.9), this holds if the diffeomorphism is C^2. The proof requires a highly technical application of Whitney's extension theorem and is given in Section 12. The specific version of this regularity result for AB-systems can be stated as follows.

Proposition 5.3. *Let $f : M \to M$ be a C^2 AB-system. Then, there is a C^1 surjection $p : M \to \mathbb{S}^1$ and $U \subset \mathbb{S}^1$ such that the compact us-leaves of f are exactly the sets $p^{-1}(t)$ for $t \in \mathbb{S}^1 \setminus U$.*

If S is a center leaf which intersects each compact us-leaf exactly once, then p may be defined so that its restriction to S is a C^1-diffeomorphism.

If μ is a probability measure given by a C^1 volume form on M, then p may be chosen so that $p_*\mu$ is Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$.

Section 13 proves (4.3) concerning the triviality of non-accessible skew products. Infra-AB-systems are treated in Section 14.

6. Actions on subsets of the line

Notation. To avoid excessive parentheses, if f and g are composable functions, we simply write fg for the composition. In this section, μ is a measure on the real line and $\mu[x, y)$ denotes the measure of the half-open interval $[x, y)$.

Let $\mathrm{Homeo}^+(\mathbb{R})$ denote the group of orientation-preserving homeomorphisms of the line. If Γ is a non-empty closed subset of $\mathbb{R}$, let $\mathrm{Homeo}^+(\Gamma)$ denote the group of all homeomorphisms of Γ which are restrictions of elements of $\mathrm{Homeo}^+(\mathbb{R})$. That is, g is in $\mathrm{Homeo}^+(\Gamma)$ if it is a homeomorphism of Γ and $g(x) < g(y)$ for $x < y$.

We now adapt results of Plante to this setting.

Proposition 6.1. *Suppose Γ is a non-empty closed subset of $\mathbb{R}$ and G is a subgroup of* $\mathrm{Homeo}^+(\Gamma)$ *with non-exponential growth. Then, there is a measure μ on $\mathbb{R}$ such that*

- $\operatorname{supp} \mu \subset \Gamma$,
- *$\mu(X) = \mu(g(X))$ for all $g \in G$ and Borel sets $X \subset \mathbb{R}$, and*
- *if $X \subset \mathbb{R}$ is compact, then $\mu(X) < \infty$.*

Proof. In the case $\Gamma = \mathbb{R}$, this is a restatement of (1.3) in [35]. One can check that the techniques in [35] and [34] extend immediately to the case $\Gamma \neq \mathbb{R}$. □

Proposition 6.2. *Let Γ, G, and μ be as in (6.1) and suppose* $\mathrm{Fix}(G)$ *is empty. Then there is a non-zero homomorphism $\tau : G \to \mathbb{R}$ such that for all $x \in \mathbb{R}$*

$$\tau(g) = \begin{cases} \mu[x, g(x)) & \text{if } x < g(x), \\ 0 & \text{if } x = g(x), \\ -\mu[g(x), x) & \text{if } g(x) < x. \end{cases}$$

Proof. Choose any $x \in \mathbb{R}$ and define τ as above. One can then show that τ is a non-zero homomorphism and independent of the choice of x. See (5.3) of [34] for details. □

Proposition 6.3. *Let Γ, G, μ, τ be as in (6.2) and suppose $f \in \text{Homeo}^+(\mathbb{R})$ is such that $F : G \to G$ defined by $F(g)(x) = fgf^{-1}(x)$ is a group automorphism. Then, there is $\lambda > 0$ such that $\tau(F(g)) = \lambda\tau(g)$ for all $g \in G$.*

Moreover, if $\lambda \neq 1$, then $f_*\mu = \lambda\mu$ and any homeomorphism of $\mathbb{R}$ which commutes with f has a fixed point.

Proof. The first half of the statement follows as an adaptation of §4 of [35]. Further, if $\lambda \neq 1$, then $f_*\mu = \lambda\mu$ by (4.2) of [35]. To prove the final claim, we first show that if $\lambda \neq 1$ then f has a fixed point. Consider $x \in \Gamma$. As $\text{Fix}(G)$ is empty by assumption, there is $g \in G$ such that $x < g(x)$. Then,

$$\mu[x, +\infty) \geq \mu[x, g^k(x)) = k\,\tau(g)$$

for all $k \geq 1$. This shows that $\mu[x, +\infty) = \infty$ for any $x \in \mathbb{R}$.

Assume, without loss of generality, that $\lambda < 1$ and $x < f(x)$ for some $x \in \mathbb{R}$. Then,

$$\mu[x, \sup_{k\geq 0} f^k(x)) = \sum_{k=0}^{\infty} \lambda^k \mu[x, f(x)) < \infty$$

and therefore, $x_0 := \sup_{k\geq 0} f^k(x) < \infty$ is a fixed point for f. If $h \in \text{Homeo}^+(\mathbb{R})$ commutes with f then for all $k \in \mathbb{Z}$

$$\mu[x_0, h^k(x_0)) = \mu[f(x_0), fh^k(x_0)) = \lambda\,\mu[x_0, h^k(x_0))$$

which is possible only if $\mu[x_0, h^k(x_0)) = 0$. Then $\mu[x_0, \sup_{k\in\mathbb{Z}} h^k(x_0)) = 0$ and so $\sup h^k(x_0) < \infty$ is a fixed point for h. □

We now consider the case where G is a fundamental group of a nilmanifold.

Proposition 6.4. *Let G be a torsion-free, finitely-generated, nilpotent group and suppose $\phi \in \text{Aut}(G)$ is such that $\phi(g) \neq g$ for all non-trivial $g \in G$. If H is a ϕ-invariant subgroup, then $\phi(gH) \neq gH$ for all non-trivial cosets $gH \neq H$.*

Proof. First, we show that the function $\psi : G \to G$ defined by $\psi(g) = g^{-1}\phi(g)$ is a bijection. If G is abelian, then G is isomorphic to $\mathbb{Z}^d$ for some d and ψ is an invertible linear map, and hence bijective. Suppose now that G is non-abelian and let Z be its group-theoretic center. Pick some element $g_0 \in G$. As G/Z is of smaller nilpotency class, by induction there is $g \in G$ such that $\psi(gZ) = g_0Z$ or equivalently $\psi(g)z_0 = g_0$ for some $z_0 \in Z$. As $\psi|_Z$ is an automorphism of Z, there is $z \in Z$ such that $\psi(gz) = \psi(g)\psi(z) = \psi(g)z_0 = g_0$. As g_0 was arbitrary, this shows ψ is onto.

To prove injectivity, suppose $\psi(g) = \psi(g')$. By induction, $g' = gz$ for some $z \in Z$. Then,

$$\psi(g) = \psi(g') = \psi(g)\psi(z) \quad \Rightarrow \quad \psi(z) = 1 \quad \Rightarrow \quad z = 1 \quad \Rightarrow \quad g' = g.$$

If H is a ϕ-invariant subgroup, then $\psi(H) = H$ and the bijectivity of ψ implies that $\psi(gH) \neq H$ for any non-trivial coset. □

The results of J. Franks and A. Manning [18, 19, 29] show that for any Anosov diffeomorphism on a nilmanifold, the resulting automorphism on the fundamental group satisfies the hypotheses of (6.4).

Lemma 6.5. *Suppose* $\Gamma \subset \mathbb{R}$, $G < \operatorname{Homeo}^+(\Gamma)$, *and* $f \in \operatorname{Homeo}^+(\mathbb{R})$ *are such that*

- Γ *is closed and non-empty,*
- G *is finitely generated and nilpotent,*
- $F : G \to G$ *defined by* $F(g)(x) = fgf^{-1}(x)$
  is a group automorphism with no non-trivial fixed points, and
- $\operatorname{Fix}(G)$ *is empty.*

Then, there are

- *a closed non-empty subset* $\Gamma_0 \subset \Gamma$,
- *a continuous surjection* $P : \mathbb{R} \to \mathbb{R}$,
- *a non-zero homeomorphism* $\tau : G \to \mathbb{R}$, *and*
- $0 < \lambda \neq 1$

such that for $x, y \in \mathbb{R}$ *and* $g \in G$

- $x \leq y$ *implies* $P(x) \leq P(y)$,
- $Pg(x) = P(x) + \tau(g)$,
- $Pf(x) = \lambda P(x)$,
- $\Gamma_0 = \{x \in \Gamma : g(x) = x \text{ for all } g \in \ker \tau\}$, *and*
- *for each* $t \in \mathbb{R}$, $P^{-1}(t)$ *is either a point* $z \in \Gamma_0$ *or an interval* $[a, b]$ *with* $a, b \in \Gamma_0$.

Moreover, any homeomorphism which commutes with f *has a fixed point in* $P^{-1}(0)$.

Proof. The conditions on G imply that it has non-exponential growth [22]. Therefore, we are in the setting of the previous propositions. In particular, there are μ, τ, and λ as above.

First, suppose that the image $\tau(G)$ is a cyclic subgroup of $\mathbb{R}$ in order to derive a contradiction. In this case, the condition $\tau F = \lambda\tau$ in (6.3) implies that $\lambda\tau(G) = \tau(G)$ and therefore $\lambda = 1$. Then, F maps a coset of $\ker \tau$ to itself. As

$\text{Homeo}^+(\Gamma)$ is torsion free, so is G, and by (6.4), F has a non-trivial fixed point, in contradiction to the hypotheses of the lemma being proved. Therefore, $\tau(G)$ is non-cyclic.

Consequently, $\tau(G)$ is a dense subgroup of $\mathbb{R}$. Further $\lambda \neq 1$, as otherwise, one could derive a contradiction exactly as above. By (6.3), f has at least one fixed point, say $x_0 \in \mathbb{R}$. Define a function $P : \mathbb{R} \to \mathbb{R}$ by

$$P(x) = \begin{cases} \mu[x_0, x) & \text{if } x > x_0, \\ 0 & \text{if } x = x_0, \\ -\mu[x, x_0) & \text{if } x < x_0. \end{cases}$$

By definition, P is (non-strictly) increasing. The density of $\tau(G)$ implies that $P(\mathbb{R})$ is dense. Then, as a monotonic function without jumps, P is continuous and therefore surjective. For each $t \in \mathbb{R}$, the pre-image $P^{-1}(t)$ is either a point or a closed interval, In either case, one can verify that $g(P^{-1}(t)) = P^{-1}(t)$ for all $g \in \ker \tau$ and therefore the boundary of $P^{-1}(t)$ is in Γ_0. The other properties of P listed in the lemma are easily verified.

The statement for homeomorphisms commuting with f follows by adapting the proof of (6.3). □

7. AI-systems

We now consider partially hyperbolic systems on non-compact manifolds. Suppose M is compact and $f : M \to M$ is partially hyperbolic. Then, any lift of f to a covering space of M is also considered to be partially hyperbolic. Also, any restriction of a partially hyperbolic diffeomorphism to an open invariant subset is still considered to be partially hyperbolic.

Let A be a hyperbolic automorphism of the compact nilmanifold N and $I \subset \mathbb{R}$ an open interval. The *AI-prototype* is defined as

$$f_{AI} : N \times I \to N \times I, \quad (v, t) \to (Av, t).$$

A partially hyperbolic diffeomorphism f on a (non-compact) manifold $\hat{M}$ is an *AI-system* if it has global product structure, preserves the orientation of its center direction, and is leaf conjugate to an AI-prototype.

Theorem 7.1. *Suppose $f : \hat{M} \to \hat{M}$ is an AI-system with no invariant compact us-leaves. Then, either*

(1) *f is accessible,*

(2) *there is an open set $V \subset \hat{M}$ such that*

$$\overline{f(V)} \subset V, \quad \bigcup_{k \in \mathbb{Z}} f^k(V) = \hat{M}, \quad \bigcap_{k \in \mathbb{Z}} f^k(V) = \varnothing,$$

and the boundary of V is a compact us-leaf, or

(3) *there are no compact us-leaves in* $\hat{M}$*, uncountably many non-compact us-leaves in* $\hat{M}$ *and there is* $\lambda \neq 1$ *such that* f *is semiconjugate to*

$$N \times \mathbb{R} \to N \times \mathbb{R}, \quad (v, t) \mapsto (Av, \lambda t).$$

Notation. For a point x on a manifold supporting a partially hyperbolic system, let $W^s(x)$ be the stable manifold through x, and $W^u(x)$ the unstable manifold. Then $AC(x)$, the accessibility class of x, is the smallest set containing x which satisfies

$$W^s(y) \cup W^u(y) \subset AC(x)$$

for all $y \in AC(x)$. For an arbitrary subset X of the manifold, define

$$W^s(X) = \bigcup_{x \in X} W^s(x), \quad W^u(X) = \bigcup_{x \in X} W^u(x), \quad \text{and} \quad AC(X) = \bigcup_{x \in X} AC(x).$$

Note that $AC(X)$ may or may not be a single accessibility class.

Proposition 7.2 ([41]). *Suppose* f *is a partially hyperbolic system with one-dimensional center on a (not necessarily compact) manifold* M*. For* $x \in M$*, the following are equivalent:*

- $AC(x)$ *is not open.*
- $AC(x)$ *has empty interior.*
- $AC(x)$ *is a complete* C^1 *codimension one submanifold.*

If L *is a curve through* x *tangent to the center direction, then the following are also equivalent to the above:*

- $AC(x) \cap L$ *is not open in* L.
- $AC(x) \cap L$ *has empty interior in* L.

If f *is non-accessible, the set of non-open accessibility classes form a lamination.*

Assumption 7.3. *For the remainder of the section, assume* $f : \hat{M} \to \hat{M}$ *is a non-accessible AI-system.*

All of the analysis of this section will be on the universal cover. Let $\tilde{M}$ and $\tilde{N}$ be the universal covers of M and N. Then, f and the leaf conjugacy h lift to functions $f : \tilde{M} \to \tilde{M}$, and $h : \tilde{M} \to \tilde{N} \times I$ still denoted by the same letters. Every lifted center leaf of the lifted f is of the form $h^{-1}(v \times I)$ for some $v \in \tilde{N}$. In general, the choice of the lifts of f and h are not unique. They may be chosen, however, so that $hfh^{-1}(v \times I) = Av \times I$ where $A : \tilde{N} \to \tilde{N}$ is a hyperbolic Lie group automorphism. As A fixes the identity element of the Lie group, there is a center leaf mapped to itself by f. Let L denote this leaf. As L is homeomorphic to $\mathbb{R}$, assume there is an ordering on the points of L and define open intervals $(a, b) \subset L$ for $a, b \in L$ and suprema $\sup X$ for subsets $X \subset L$ exactly as for $\mathbb{R}$.

Define a closed subset

$$\Lambda = \{t \in L : AC(t) \text{ is not open}\}.$$

Lemma 7.4. Λ *is non-empty.*

Proof. As $\tilde{M}$ is connected, if all accessibility classes were open, f would be accessible (both on $\tilde{M}$ and $\hat{M}$). Therefore, there is at least one non-open accessibility class. By global product structure, this class intersects L. □

Lemma 7.5. *If* $t \in \Lambda$*, then* $AC(t) = W^s W^u(t) = W^u W^s(t)$.

This is an adaptation to the case of global product structure of local arguments used in the proof of (7.2).

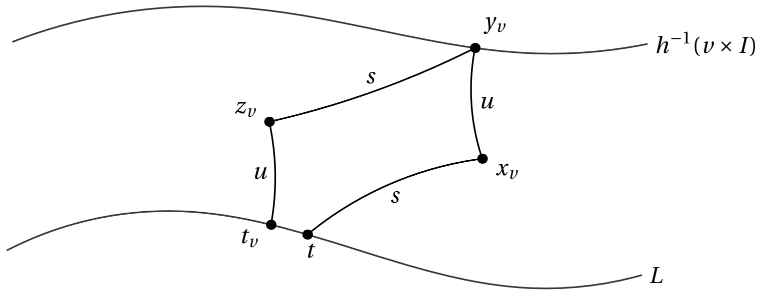

Figure 2. A "bracket" of points defined by global product structure. The proof of (7.5) shows that if $t \in \Lambda$, then $t_v = t$.

Proof. Each center leaf in $\tilde{M}$ is of the form $h^{-1}(v \times I)$ for some $v \in \tilde{N}$. By global product structure, for each $v \in \tilde{N}$, there exist unique points $x_v, y_v, z_v, t_v \in \tilde{M}$ such that

$$x_v \in W^s(t), \quad y_v \in W^u(x_v) \cap h^{-1}(v \times I), \quad z_v \in W^s(y_v), \quad t_v \in W^u(z_v) \cap L.$$

See Figure 2. These points depend continuously on v. As $\tilde{N}$ is connected, the set

$$\{t_v : v \in \tilde{N}\} \subset L \cap AC(t)$$

is connected and, by (7.2), has empty interior as a subset of L. Therefore, it consists of the single point t. This shows that both $W^s W^u(t)$ and $W^u W^s(t)$ intersect each center leaf $h^{-1}(v \times I)$ in the same unique point y_v and so the two sets are identical. This set is both s-saturated and u-saturated and so contains $AC(t)$. □

By global product structure, for any $x \in \tilde{M}$, there is a unique point $R(x) \in L$ such that $W^u(x)$ intersects $W^s(R(x))$. This defines a retraction, $R : \tilde{M} \to L$. By the previous lemma, if $t \in \Lambda$, then $R^{-1}(t) = AC(t)$.

Let $\alpha : \tilde{M} \to \tilde{M}$ be a deck transformation of the covering $\tilde{M} \to \hat{M}$. Then, as depicted in Figure 1, α defines a map $g_\alpha \in \text{Homeo}^+(\Lambda)$ given by the restriction of $R \circ \alpha$ to Λ. Define

$$G = \{g_\alpha : \alpha \in \pi_1(\tilde{M})\}.$$

Lemma 7.6. *G is a finitely generated, nilpotent subgroup of* $\text{Homeo}^+(\Lambda)$.

Proof. For $\alpha \in \pi_1(\hat{M})$ and $t \in \Lambda$, $g_\alpha(t)$ is given by the unique intersection of $\alpha(AC(t))$ and L. Then,

$$AC(g_\alpha(g_\beta(t))) = \alpha(AC(g_\beta(t))) = \alpha\beta(AC(t)) = AC(g_{\alpha\beta}(t))$$

shows that $\pi_1(\hat{M}) \to \text{Homeo}^+(\Lambda)$, $\alpha \mapsto g_\alpha$ is a group homomorphism. As $\hat{M}$ is homotopy equivalent to the nilmanifold N, its fundamental group is finitely generated and nilpotent. □

It is necessary to define G with elements in $\text{Homeo}^+(\Lambda)$ as, in general, the same construction on L will define a subset of $\text{Homeo}^+(L)$ but not a subgroup.

Lemma 7.7. *For a point $t \in \Lambda$, $AC(t) \subset \tilde{M}$ projects to a compact us-leaf in $\hat{M}$ if and only if $t \in$* $\text{Fix}(G)$.

Proof. Consider $t \in \Lambda$ and let $\hat{X} \subset \hat{M}$ be the image of $AC(t)$ by the covering $\tilde{M} \to \hat{M}$. First, suppose $t \in \text{Fix}(G)$. By global product structure, there is a unique map $\sigma : \tilde{N} \to AC(t)$ such that $h\sigma(v) \in v \times I$ for every $v \in \tilde{N}$. For any deck transformation $\alpha \in \pi_1(\hat{M})$,

$$\alpha(AC(t)) = AC(g_\alpha(t)) = AC(t)$$

which implies that $\alpha\sigma = \sigma\alpha_N$ where α_N is the corresponding deck transformation for the covering $\tilde{N} \to N$. It follows that σ quotients to a homeomorphism from the compact nilmanifold N to $\hat{X}$ and therefore $\hat{X}$ is compact.

To prove the converse, suppose $\hat{X}$ is compact. From the definition of an AI-system, one can see that every center leaf on $\hat{M}$ is properly embedded. Therefore, $\hat{X}$ intersects each center leaf in a compact set. If $\tilde{X}$ is the pre-image of $\hat{X}$ by covering $\tilde{M} \to \hat{M}$, then $\tilde{X}$ intersects each center leaf on $\tilde{M}$ in a compact set. In particular, $\tilde{X} \cap L$ is compact. Note that $\tilde{X} \cap L$ is exactly equal to the orbit $Gt = \{g(t) : g \in G\}$. Define $s = \sup Gt$. Then, $s \in Gt$ by compactness and $g(Gt) = Gt$ implies $g(s) = s$ for each $g \in G$. This shows that $\{s\} = Gs = Gt$ and therefore $t = s \in \text{Fix}(G)$. □

Lemma 7.8. *Suppose $J \subset L$ is an open interval such that $\partial J \subset$* $\text{Fix}(f) \cap \text{Fix}(G)$. *Let X be the image of $AC(J)$ by the covering $\tilde{M} \to \hat{M}$. Then, $f|_X$ is an AI-system.*

This lemma is the justification for assuming there are no invariant, compact leaves in (7.1). If such leaves exist, the AI-system can be decomposed into smaller systems.

Proof. Assume the subinterval J in the hypothesis is of the form $J = (a, b)$ with $a, b \in L$. Unbounded subintervals of the form $(a, +\infty)$ and $(-\infty, b)$ are handled similarly.

For every center leaf $h^{-1}(v \times I)$, let $a_v, b_v \in I$ be such that $v \times a_v \in h(AC(a))$ and $v \times b_v \in h(AC(b))$. The set $\tilde{X} = \bigcup_{v \in \tilde{N}} h^{-1}(v \times (a_v, b_v))$ is s-saturated,

u-saturated, and contains J. Therefore, $AC(J) \subset \tilde{X}$. By global product structure, one can show that $\tilde{X} \subset AC(J)$, so the two sets are equal. By its construction $\tilde{X}$ is simply connected, and invariant under deck transformations. Therefore, it is the universal cover for X. Global product structure is inherited from $\tilde{M}$. For instance, for $x, y \in AC(J)$, there is a unique point $z \in \tilde{M}$ such that $z \in W^s(x) \cap W^{cu}(y)$. Since, $W^s(x) \subset \tilde{X}$, z is in $\tilde{X}$.

Compose h with a homeomorphism which maps each $v \times (a_v, b_v)$ to $v \times (0,1)$ by rescaling the second coordinate. This results in a leaf conjugacy between f on $\tilde{X}$ and $A \times \mathrm{id}$ on $\tilde{N} \times (0,1)$ which quotients down to a leaf conjugacy from X to $N \times (0,1)$. □

We now show that if the AI-system has no fixed compact us-leaves, then it satisfies either case (2) or case (3) of (7.1) depending on whether it has any (non-fixed) compact us-leaves.

Lemma 7.9. *If* $\mathrm{Fix}(G)$ *is non-empty and* $\mathrm{Fix}(f) \cap \mathrm{Fix}(G)$ *is empty, then* f *satisfies case (2) of* (7.1).

Proof. We first show that f restricted to L is fixed-point free. Suppose, instead, that $f(t) = t \in L$. By assumption $t \notin \mathrm{Fix}(G)$, so let J be the connected component of $L \setminus \mathrm{Fix}(G)$ containing t. As $\mathrm{Fix}(G)$ is f-invariant, $f(J) = J$ and each $s \in \partial J$ is then an element of $\mathrm{Fix}(f) \cap \mathrm{Fix}(G)$, a contradiction.

Without loss of generality, assume $t < f(t)$ for all $t \in L$. Choose some $t_0 \in \mathrm{Fix}(G)$ and define $L^+ = \{t \in L : t > t_0\}$. Then,

$$\overline{f(L^+)} \subset L^+, \quad \bigcup_{k \in \mathbb{Z}} f^k(L^+) = L, \quad \text{and} \quad \bigcap_{k \in \mathbb{Z}} f^k(L^+) = \varnothing.$$

One can then show that the covering $\tilde{M} \to \hat{M}$ takes $AC(L^+)$ to an open set $V \subset \hat{M}$ which satisfies the second case of (7.1). □

Lemma 7.10. *If* $\mathrm{Fix}(G)$ *is empty, then* f *satisfies case (3) of* (7.1).

Proof. In this case, the hypotheses of (6.5) hold with $\Gamma = \Lambda$. Let $P : L \to \mathbb{R}$ and $\tau : G \to \mathbb{R}$ be as in (6.5).

If $\alpha \in \pi_1(\hat{M})$ is a deck transformation $\tilde{M} \to \tilde{M}$, then $h \alpha h^{-1}$ is equal to $\alpha_N \times \mathrm{id}$ on $\tilde{N} \times I$ for some deck transformation $\alpha_N \in \pi_1(N)$. As N is a nilmanifold, any homomorphism from $\pi_1(N)$ to $\mathbb{R}$ defines a unique homomorphism from the nilpotent Lie group $\tilde{N}$ to $\mathbb{R}$ [28]. This implies that there is a unique Lie group homomorphism $T : \tilde{N} \to \mathbb{R}$ such that $T \alpha_N(v) = T(v) + \tau(g_\alpha)$ for all $v \in \tilde{N}$ and $\alpha \in \pi_1(\hat{M})$.

Let $R : \tilde{M} \to L$ be the retraction defined earlier in this section and let $H : \tilde{M} \to \tilde{N}$ be the composition of the leaf conjugacy $h : \tilde{M} \to \tilde{N} \times I$ with projection onto the first coordinate. Define

$$Q : \tilde{M} \to \mathbb{R}, \quad x \mapsto PR(x) - TH(x).$$

We will show that Q quotients to a function $\hat{M} \to \mathbb{R}$ and use this to construct the semiconjugacy in the last case of (7.1).

First, consider a point $x \in \tilde{M}$ which has a non-open accessibility class. Then, $R(x) \in \Lambda$ and, for $\alpha \in \pi_1(\hat{M})$,

$$PR(\alpha(x)) = Pg_\alpha R(x) = PR(x) + \tau(g_\alpha)$$

and

$$TH\alpha(x) = T\alpha_N H(x) = TH(x) + \tau(g_\alpha)$$

which together show $Q\alpha(x) = Q(x)$.

Now, consider a point $x \in \tilde{M}$ which has an open accessibility class, and let $J \subset \tilde{M}$ be the connected component of $W^c(x) \cap AC(x)$ which contains x. The set Γ_0 from (6.5) is a subset of $\Gamma = \Lambda$ and therefore P is constant on $L \setminus \Lambda$. Then, PR is constant on J and, by continuity, constant on the closure of J as well. As H is constant on center leaves, $Q = PR - TH$ is also constant on the closure of J. Let y be a point on the boundary of J. Then, as $AC(y)$ is non-open, $Q(x) = Q(y) = Q\alpha(y) = Q\alpha(x)$. This shows that Q quotients down to a function $\hat{Q} : \hat{M} \to \mathbb{R}$. A much simpler argument shows that $H : \tilde{M} \to \tilde{N}$ quotients down to a function $\hat{H} : \hat{M} \to N$.

The properties of F and P in (6.5) imply that $TA = \lambda T$ and therefore $THf = TAH = \lambda TH$. As $PRf = PfR = \lambda PR$, this shows that $Qf = \lambda Q$. Then, $\hat{H} \times \hat{Q}$ is the desired semiconjugacy in (7.1). By (6.5), $P(\Lambda) = \mathbb{R}$ and so Λ is uncountable. Each G-orbit of Λ corresponds to a distinct us-leaf, and so there are uncountably many. □

This concludes the proof of (7.1). We note one additional fact which will be used in the next section.

Corollary 7.11. *If* $\mathrm{Fix}(G)$ *is empty, any homeomorphism of L which commutes with f has a fixed point.*

Proof. This follows from the use of (6.5) in the previous proof. □

8. AB-systems

Assumption 8.1. *In this section, assume $f : M \to M$ is a non-accessible AB-system.*

The AB-prototype f_{AB} has an invariant center leaf which is a circle. By the leaf conjugacy, f also has an invariant center leaf. Call this leaf S. Note that f lifts to an AI-system. This is because the AB-prototype f_{AB} lifts to the AI-prototype $A \times \mathrm{id}$ on $N \times \mathbb{R}$. If $h : M \to M_B$ is the leaf conjugacy, then hfh^{-1} is homotopic to f_{AB} and therefore also lifts to $N \times \mathbb{R}$.

Let $\pi : \tilde{M} \to M$ be the universal covering, and choose a lift $\tilde{f} : \tilde{M} \to \tilde{M}$ and $\tilde{S}$ a connected component of $\pi^{-1}(S)$ such that $\tilde{f}(\tilde{S}) = \tilde{S}$. The universal cover $\tilde{N} \times \mathbb{R}$ of the manifold M_B has a deck transformation of the form $(v, t) \mapsto (Bv, t-1)$. Conjugating this by the lifted leaf conjugacy gives a deck transformation $\beta : \tilde{M} \to \tilde{M}$ and one can assume that $\beta(\tilde{S}) = \tilde{S}$. Then, $\tilde{S}$ plays the role of L in the previous section. Define $\Lambda = \{t \in \tilde{S} : AC(t) \text{ is not open}\}$ and G as a subgroup of $\text{Homeo}^+(\Lambda)$ as in the previous section.

Lemma 8.2. $\text{Fix}(G)$ *is non-empty.*

Proof. This follows from (7.11) since β and $\tilde{f}$ are commuting diffeomorphisms when restricted to $\tilde{S}$ and β is fixed-point free. □

Lemma 8.3. *For $t \in \Lambda$, $AC(\pi(t)) \subset M$ is compact if and only if $t \in \text{Fix}(G)$.*

Proof. If $t \in \text{Fix}(G)$, then, by (7.7), $AC(\pi(t))$ is covered by a compact us-leaf of the AI-system and is therefore compact itself.

Conversely, suppose $t \in \Lambda$ is such that $AC(\pi(t)) \subset M$ is a compact us-leaf. Note that as $\beta(\text{Fix}(G)) = \text{Fix}(G)$ there are $a, b \in \text{Fix}(G)$ such that $a < t < b$ in the ordering on $\tilde{S}$. Then, Gt is contained in (a, b), a bounded subset of $\tilde{S}$. Considering the supremum as in (7.7), one shows that $s := \sup Gt$ is in $\text{Fix}(G)$. Consequently, $AC(\pi(t))$ accumulates on $\pi(s)$ which, as $AC(\pi(t))$ is compact, implies $\pi(s) \in AC(\pi(t))$ and so there is a deck transformation $\alpha : \tilde{M} \to \tilde{M}$ such that $\alpha(s) \in AC(t)$. This implies there is $k \in \mathbb{Z}$ and $g \in G$ such that $t = \beta^k g(s) = \beta^k(s) \in \text{Fix}(G)$. □

In this, and the next two sections, define

$$K = \{x \in S : AC(x) \subset M \text{ is compact}\}.$$

The last lemma shows that $K = \pi(\text{Fix}(G))$.

Corollary 8.4. *K is closed and non-empty.* □

This also completes the proof of (2.5).

Corollary 8.5. *$K \cap NW(f|_S)$ is non-empty.*

Proof. K is non-empty, f-invariant, and closed. □

Corollary 8.6. *f has a compact periodic us-leaf if and only if $f|_S$ has rational rotation number.*

Proof. As a consequence of (8.3), any compact us-leaf X in M intersects S in a unique point t. If $f^n(X) = X$ then $f^n(t) = t$ and $f|_S$ has rational rotation number. If, conversely, $f|_S$ has rational rotation number, its non-wandering set consists of periodic points, and a compact periodic leaf exists by (8.5). □

The following is also from the last proof.

Corollary 8.7. *All compact periodic us-leaves have the same period.* □

Lemma 8.8. *If $K = S$, then f on M is topologically conjugate to a function $(v, x) \mapsto (Av, \tilde{r}(x))$ defined on the manifold*

$$M_B = N \times \mathbb{R}/(Bv, t) \sim (v, t + 1)$$

where $\tilde{r} : \mathbb{R} \to \mathbb{R}$ is a lift of a homeomorphism $r : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ topologically conjugate to $f|_S$.

Proof. Let $\phi : \tilde{S} \to \mathbb{R}$ be any homeomorphism such that $\phi\beta(t) = \phi(t) + 1$ for all t. Define $\tilde{r}$ as $\phi \tilde{f} \phi^{-1}$. Extend ϕ to all of $\tilde{M}$ by making it constant on accessibility classes. As in the proof of (7.10), let $H : \tilde{M} \to \tilde{N}$ be the first coordinate of the lifted leaf conjugacy $h : \tilde{M} \to \tilde{N} \times \mathbb{R}$. Then, the function $H \times \phi : \tilde{M} \to \tilde{N} \times \mathbb{R}$ gives a topological conjugacy between $\tilde{f}$ on $\tilde{M}$ and $A \times \tilde{r}$.

The fundamental group of M_B is generated by deck transformations of the form $(v, t) \mapsto (\alpha_N(v), t)$ or $(v, t) \mapsto (Bv, t - 1)$. Using the fact that $\text{Fix}(G) = \tilde{S}$ and the definition of $\tilde{r}$, one can then show that $H \times \phi$ quotients down to a topological conjugacy defined from M to M_B. □

Lemma 8.9. *Suppose $J \subset S$ is an open interval such that $\partial J \subset \text{Fix}(f) \cap K$. Then, $f|_{AC(J)}$ is an AI-system.*

Proof. Let $\tilde{J}$ be a lift of J to $\tilde{S}$. Then, as $f(J) = J$, $\tilde{f}(\tilde{J}) = \beta^k(\tilde{J})$ for some $k \in \mathbb{Z}$. By replacing the lift $\tilde{f}$ by $\tilde{f}\beta^k$, assume, without loss of generality that $\tilde{f}(\tilde{J}) = \tilde{J}$. As $K = \pi(\text{Fix}(G))$, $\partial\tilde{J} \subset \text{Fix}(\tilde{f}) \cap \text{Fix}(G)$, and so by (7.8), $AC(\tilde{J})$ projects to X on $\hat{M}$ such that the dynamics on X is an AI-system. As $\tilde{J}$ is contained in a fundamental domain of the covering $\tilde{S} \to S$, one can show that X is contained in a fundamental domain of the covering $\hat{M} \to M$. Therefore, the dynamics on $\pi(AC(\tilde{J})) = AC(J)$ is an AI-system. □

We now give a C^0 version of (5.3).

Lemma 8.10. *There is a continuous surjection $p : M \to \mathbb{S}^1$ such that $p|_S$ is a homeomorphism, $p|_{W^c(x)}$ is a covering for any center leaf $W^c(x)$ ($x \in M$) and p is constant on each compact accessibility class.*

Proof. Define p on S so that $p|_S$ maps S to $\mathbb{S}^1$ with constant speed along S. Extend p to $AC(K) \cup S$ by making p constant on accessibility classes. Then, for any center leaf $W^c(x)$, let J be a connected component of $W^c(x) \setminus AC(K)$ and define p on J so that J is mapped at constant speed to $\mathbb{S}^1$ and extends continuously to the boundary $\partial J \subset AC(K)$. Transversality of the center foliation and us-lamination implies that p is continuous. The other properties are easily verified. □

Compare this short C^0 proof to the C^1 proof in Section 12.

We now consider the cases of rational and irrational rotation of $f|_S$ separately in the next two sections.

9. Rational rotation

This section proves (2.6).

Assumption 9.1. *Assume f is a non-accessible AB-system with at least one periodic compact us-leaf.*

Let S, K, and other objects be defined as in Section 8. By (8.7), all compact periodic leaves have the same period. Call this period n. Define $K_n = K \cap \mathrm{Fix}(f^n) \subset S$. By (8.4), K_n is closed. Let $p : M \to \mathbb{S}^1$ be the projection given by (8.10) and define $U \subset \mathbb{S}^1$ as $U = \mathbb{S}^1 \setminus p(K_n)$.

Note that if $t \notin U$, then $p^{-1}(t)$ is an f^n-invariant compact us-leaf. Moreover, every such leaf is of this form. This proves the first part of (2.6).

To prove the rest of the theorem, replace f by its iterate f^n and assume $n = 1$. The new f is still an AB-system, albeit with a different "A" than before. Now $K_n = \mathrm{Fix}(f) \cap K \subset S$. If I is a connected component of $U \subset \mathbb{S}^1$, then $p^{-1}(I) \cap S$ is a connected component of $S \setminus K_1$ and (8.9) implies that f restricted to $p^{-1}(I) = AC(\pi(J))$ is an AI-system. Since $J \cap K_n$ is empty, $AC(J)$ contains no invariant compact us-leaves. Therefore, the AI-system falls into one of the cases given in (7.1). As these cases correspond exactly to those given in (2.6), this concludes the proof.

10. Irrational rotation

This section proves (2.7) and (2.8).

Assumption 10.1. *Assume f is a non-accessible AB-system with no periodic compact us-leaves.*

Let S, K and other objects be defined as in Section 8. By (8.6), $f|_S$ has irrational rotation number.

Lemma 10.2. *$NW(f|_S) \subset K$.*

Proof. For any C^1 circle diffeomorphism with irrational rotation, the non-wandering set is minimal. The result then follows from (8.5). □

Lemma 10.3. *If I is a connected component of $S \setminus NW(f|_S)$, then $AC(I)$ is a wandering domain. That is, the sets $f^k(AC(I)) = AC(f^k(I))$ are pairwise disjoint for all $k \in \mathbb{Z}$.*

Proof. Let J be the closure of I. Note that any compact leaf in $AC(J)$ must be of the form $AC(t)$ for some $t \in J$. By the properties of circle diffeomorphisms, the sets $f^k(J)$ are pairwise disjoint. By the last lemma, $\partial J \subset K$. If $AC(J)$ intersects $AC(f^k(J))$, then this intersection has a boundary consisting of compact us-leaves. Such a compact leaf would intersect S in a point $t \in J \cap f^k(J)$, a contradiction. □

Lemma 10.4. $NW(f) = AC(NW(f|_S))$.

Proof. The last lemma shows $NW(f) \subset AC(NW(f|_S))$.

To prove the other inclusion, suppose $t \in NW(f|_S)$, $x \in AC(t)$ and $V \subset M$ is a neighbourhood of x. There is a sequence $\{n_k\}$ such that $f^{n_k}(t)$ converges to t. By taking a further subsequence, assume $f^{n_k}(x)$ converges to some point $y \in AC(t)$. Let $D \subset V$ be a small unstable plaque containing x. Then $f^{n_k}(D)$ is a sequence of ever larger unstable plaques, and

$$W^u(y) \subset \overline{\bigcup_k f^{n_k}(D)}.$$

Unstable leaves of the Anosov diffeomorphism A are dense in N [19]. Therefore, by the leaf conjugacy, $W^u(y)$ is dense in $AC(t)$. This shows that some iterate $f^{n_k}(V)$ intersects V. □

Now, let $p : M \to \mathbb{S}^1$ be as in (8.10). We may assume $p|_S$ is a C^1-diffeomorphism. Define $r : \mathbb{S}^1 \to \mathbb{S}^1$ by $rp(t) = pf(t)$ for all $t \in S$. Then, (2.7) can be proved from the above lemmas. As r has irrational rotation number, it is semiconjugate to a rigid rotation $t \mapsto t + \theta$. Using this and the leaf conjugacy, one can prove (2.8) using an argument similar to the proof of (8.8).

11. Proving theorems (2.2), (2.3), and (2.4)

This section gives the proofs of several of the theorems stated in Section 2 based on results proved in other sections.

The proof of (2.4) makes use of a result of Brin regarding transitivity [8]. The following is an extension of this result to the non-compact case, though the proof is in essence the same.

Proposition 11.1 (Brin). *Suppose f is a partially hyperbolic diffeomorphism of a (not necessarily compact) manifold M. If V is open and $f(V) = V \subset NW(f)$, then $\overline{V} = AC(\overline{V})$.*

In particular, if f is accessible and $NW(f) = M$, then f is transitive.

Proof. For $\epsilon > 0$ and $y \in M$, let $W^u_\epsilon(y)$ be the set of all points reachable from y by a path tangent to E^u of length less than ϵ.

If $x \in V$, then $x \in NW(f)$ implies there are sequences $\{x_k\}$ and $\{y_k\}$ both converging to x and such that $y_k = f^{j_k}(x_k)$ for some non-zero $j_k \in \mathbb{Z}$. By swapping x_k with y_k if necessary, assume every j_k is positive. If j_k is bounded, then x is periodic, so we may freely assume that $j_k \to +\infty$. As V is open, there is $\epsilon > 0$ such that $W^u_\epsilon(x_k) \subset V$ for all large k. The uniform expansion of E^u implies there is $r_k \to \infty$ such that $W^u_{r_k}(y_k) \subset f^{j_k}(W^u_\epsilon(x_k)) \subset f^{j_k}(V) = V$ and therefore the entire unstable manifold $W^u(x)$ lies in the closure of V. This proves $W^u(\overline{V}) = \overline{V}$. Similarly, $W^s(\overline{V}) = \overline{V}$ and so $AC(\overline{V}) = \overline{V}$. □

Proof of (2.4). By (11.1), any accessible f satisfies case (1) of (2.4). Therefore, assume that f is non-accessible.

For now, assume f has no periodic compact us-leaves, so that (2.7) holds. That theorem, with the assumption $NW(f) = M$, implies that $NW(r) = \mathbb{S}^1$ and that every point in M lies in a compact us-leaf. This shows that (8.8) holds and the r in that lemma can be taken as the same r in (2.7). As $NW(r) = \mathbb{S}^1$, r is topologically conjugate to a rigid rotation $t \mapsto t + \theta$ and therefore f satisfies case (2) of (2.4).

For the remainder of the proof, assume f has a periodic compact us-leaf, so that (2.6) holds. Let I be a connected component of U and $g : p^{-1}(I) \to p^{-1}(I)$ be as in (2.6). The condition $NW(f) = M$ implies $NW(g) = p^{-1}(I)$. This is only possible in the first of the three cases in (2.6), where g is accessible. Then, g is transitive by (11.1).

If $t \in \mathbb{S}^1 \setminus U$, then f^n restricted to $p^{-1}(I)$ is topologically conjugate to a hyperbolic nilmanifold automorphism and is therefore transitive [19]. Hence, if U is non-empty, the third case of (2.4) is satisfied.

If U is empty, then every $p^{-1}(t)$ is an f^n-invariant compact us-leaf and (8.8) holds with $r : \mathbb{S}^1 \to \mathbb{S}^1$ topologically conjugate to a rigid rational rotation $t \mapsto t + \theta$. This shows that f is in case (2) of (2.4). □

To prove ergodicity of the components of the decomposition given in (2.3), we use results given in [12], [41], and in the classical work of Birkhoff and Hopf. These results were formulated for systems on compact manifolds, but the proofs are local in nature, involving short holonomies along stable and unstable manifolds. The results, therefore, generalize to the non-compact case so long as the measure is still finite.

Proposition 11.2. *Let f be a homeomorphism of a (not necessarily compact) manifold M and let $C_0(M)$ be the space of continuous functions $M \to \mathbb{R}$ with compact support. Suppose μ is an invariant measure with $\mu(M) = 1$ and there is an invariant closed submanifold S such that μ is equivalent to Lebesgue measure on S.*

(1) *For $\phi \in C_0(M)$ the limits*

$$\phi^s(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi f^k(x) \quad \textit{and} \quad \phi^u(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi f^{-k}(x)$$

exist and are equal μ-almost everywhere.

(2) *There is a countable set* $\{\phi_j\}_{j=1}^{\infty} \subset C_0(M)$ *(depending only on* M*) such that* (f, μ) *is ergodic if and only if* ϕ_j^s *and* ϕ_j^u *are constant* μ*-almost everywhere for every* j*.*

Further, suppose f *is a* C^2 *partially hyperbolic diffeomorphism with one dimensional center.*

(3) *If* $\phi \in C_0(M)$*, then* ϕ^s *is constant on stable leaves and* ϕ^u *is constant on unstable leaves.*

(4) *If* $S = M$*,* $X^s, X^u \subset M$ *are measurable, and*

$$W^s(X^s) = X^s, \quad W^u(X^u) = X^u \quad \textit{and} \quad \mu(X^s \triangle X^u) = 0,$$

then there is $X \subset M$ *measurable such that*

$$AC(X) = X \quad \textit{and} \quad \mu(X^s \triangle X) = 0 = \mu(X^u \triangle X).$$

(5) *If* $S = M$ *and* f *is accessible, then* (f, μ) *is ergodic.*

Proof. Item (1) is a re-statement of the classic Birkhoff Ergodic Theorem.

To prove (2), let $\{\phi_j\}$ be a countable set whose linear span is dense in $C_0(M)$ with respect to the supremum norm. As any function in $C_0(S)$ may be extended to a function in $C_0(M)$, the linear span of $\{\phi_j\}$ is dense in $L^1(\mu)$. Suppose the bounded linear operator $\phi \mapsto \phi^s$ on $L^1(\mu)$ takes every element of $\{\phi_j\}$ to the subspace of constant functions. By density, every $\phi \in L^1(\mu)$ is mapped to the same subspace. Therefore (f, μ) is ergodic. The converse statement in (2) follows directly from the properties of ergodicity.

Proofs of (3)–(5) can be found in both [12] and [41]. □

Proof of (2.3). As μ is a finite, f-invariant measure which is equivalent to Lebesgue, $NW(f) = M$ by Poincaré recurrence. Let p, n, and U then be given as in (2.4). By (5.3), assume $p_*\mu = m$ where m is Lebesgue measure on $\mathbb{S}^1$. Without loss of generality, assume $n = 1$.

For each connected component I of U, the set $p^{-1}(I)$ is an accessibility class and therefore (f, μ_I) is ergodic by (11.2) where μ_I is as in (2.1).

Let $\{\phi_j\}_{j=1}^{\infty}$ be as in (11.2) and for $j \in \mathbb{N}$ and $q \in \mathbb{Q}$ define $X_{j,q}^s = \{x \in M : \phi_j^s(x) < q\}$. Define $X_{j,q}^u$ similarly. By items (3) and (4) of (11.2), there is $X_{j,q} = AC(X_{j,q})$ equal mod zero to both $X_{j,q}^s$ and $X_{j,q}^u$. Define a "bad" set Y by

$$Y = \bigcup_{j,q} \big(X_{j,q}^s \triangle X_{j,q} \cup X_{j,q}^u \triangle X_{j,q} \big)$$

and note that $\mu(Y) = 0$. Equation (2.1) implies that there is a "good" set $Z \subset \mathbb{S}^1 \setminus U$ such that $U \cup Z$ has full measure in $\mathbb{S}^1$ and $\mu_t(Y \cap p^{-1}(t)) = 0$ for all $t \in Z$ where μ_t

is given by the decomposition in (2.1). By (5.3), we may further assume that μ_t is equivalent to Lebesgue measure on $p^{-1}(t)$ for all $t \in Z$.

As $p^{-1}(t)$ is an accessibility class, every $X_{j,q} \cap p^{-1}(t)$ is either empty or all of $p^{-1}(t)$. Therefore for $t \in Z$, every $X^s_{j,q}$ and $X^u_{j,q}$ either has μ_t-measure equal to zero or one, and item (2) of (11.2) implies that (f, μ_t) is ergodic. Thus, modulo a set of measures whose combined support has μ-measure zero, every measure in (2.1) is ergodic. This shows that (2.1) is the ergodic decomposition of μ. □

One might be tempted to prove (2.3) by arguing that for $t \notin U$, f restricted to $p^{-1}(t)$ is an Anosov diffeomorphism and therefore the invariant measure μ_t is ergodic. The problem is that we have only shown that $p^{-1}(t)$ is a C^1 submanifold of M, which is not enough regularity to conclude ergodicity for an Anosov system. Hence, the above proof.

Proof of (2.2). If f is in case (1) or (3) of (2.4), it is fairly easy to show that f is also in the corresponding case of (2.2). Therefore, assume f is in case (2) of (2.4).

If θ is rational, then $(v, t) \mapsto (Av, t + \theta)$ is non-transitive and therefore f is not ergodic.

Suppose θ is irrational and f is not ergodic. Then there are $j \in \mathbb{N}$ and $q \in \mathbb{Q}$ such that the sets $X^s_{j,q}$, $X^u_{j,q}$, and $X_{j,q}$, defined as in the last proof, have neither zero measure nor full measure with respect to the f-invariant measure μ. Write $X = X_{j,q}$. As $X = AC(X)$, there is $Y \subset \mathbb{S}^1$ such that $X = p^{-1}(Y)$ and $p_*\mu = m$ implies that $m(Y)$ is neither zero nor one. The condition $p_*\mu = m$ further implies that p gives a semiconjugacy from f to a rigid irrational rotation $R_\theta(x) = x + \theta$ on $\mathbb{S}^1$. Then, $f(X) = X$ implies $R_\theta(Y) = Y$ which contradicts the ergodicity of (R_θ, m). □

12. Regularity

This section proves (2.9), showing that the us-lamination of a partially hyperbolic diffeomorphism is C^1 if the center is one-dimensional and the diffeomorphism is C^2.

We first give a general idea of the method of proof before providing all the technical details. Let Λ denote the lamination of us-leaves. Suppose L is a compact segment of a center leaf and C is a tubular neighbourhood of L. If $x \in C \cap \Lambda$, then there is a unique point $y \in L$ so that x and y are connected by a short path inside a single us-leaf. This defines a map, the us-holonomy, from $C \cap \Lambda$ to L that we wish to show is C^1 regular in the sense of Whitney. Equivalently, we wish to show that this map extends to a C^1 function from C to L. By local product structure of the splitting $E^u \oplus E^c \oplus E^s$, the function $C \cap \Lambda \to L$ may be written either as an unstable holonomy composed with a stable holonomy or vice versa, and on $C \cap \Lambda$ these holonomies commute. That is, with x and y as above, $y = h^s(h^u(x)) = h^u(h^s(x))$. Further, h^s and h^u are known to be C^1. If other leaves of Λ accumulate on the leaf through x, then $h^s(h^u(x_n)) = h^u(h^s(x_n))$ holds for a sequence of points x_n on distinct leaves where the x_n converge to x. From this, it follows that the derivatives commute as well: $D(h^s \circ h^u) = D(h^u \circ h^s)$ at x. For such points x, we use this as the candidate for the derivative of the us-holonomy $C \cap \Lambda \to L$ when applying Whitney's extension theorem. If x lies on an isolated leaf of Λ, then $D(h^s \circ h^u)$ and $D(h^u \circ h^s)$ may differ at x and neither can be used as the candidate derivative. Further, a sequence of isolated leaves of Λ might accumulate on a non-isolated leaf. To handle this, we first restrict $D(h^s \circ h^u)$ to a function defined only for points on non-isolated leaves, and then take any continuous extension of this restricted function to all of $C \cap \Lambda$. This extended function is then used as the candidate derivative in Whitney's extension theorem.

We now give the full proof, starting with a known result on the regularity of the stable and unstable holonomies.

Proposition 12.1. *Suppose $f : M \to M$ is a C^2 dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center. Then any unstable holonomy h^u inside a cu-leaf is C^1. Moreover, the derivative of h^u tends uniformly to one as the unstable distance between the point x and its image $h^u(x)$ tends to zero.*

Proof. That such a holonomy is C^1 is proved in an erratum [39] to the paper [38]. If $y \in W^u(x)$ and h^u is the holonomy taking x to y, then adapting the argument in §3 of [36] one can show that the norm of the derivative of h^u at x is given by

$$J_{xy} = \prod_{n=0}^{\infty} \frac{\|T^c_{f^{-n}(y)} f\|}{\|T^c_{f^{-n}(x)} f\|}$$

where $T^c_z f : E^c_z \to E^c_{f(z)}$ is the restriction of the derivative $T_z f : T_z M \to T_{f(z)} M$. As f is C^2, the derivative $T_z f$ is Lipschitz in z and the center bundle E^c is Hölder by [25]. Therefore,

$$\log J_{xy} \le \sum_{n=0}^{\infty} L\big[\operatorname{dist}(f^{-n}(x), f^{-n}(y))\big]^\theta \le \sum_{n=0}^{\infty} L\big[C\mu^{-n}\big]^\theta \big[\operatorname{dist}(x, y)\big]^\theta$$

for appropriate constants $L, C, \mu > 1$ and $0 < \theta < 1$. This shows that J_{xy} tends uniformly to one as $\operatorname{dist}(x, y)$ tends to zero. □

Proposition 12.2. *Suppose $f : M \to M$ is a C^2 dynamically coherent partially hyperbolic diffeomorphism with one-dimensional center. Suppose $L_0 \subset M$ is a compact interval inside a center leaf and $g : L_0 \to \mathbb{R}$ is C^1. Then g extends to a C^1 function defined on a neighbourhood of L_0 which is constant on us-leaves.*

Proof. Without loss of generality, assume g is defined so that $|g(x) - g(y)|$ is the arc length of the center segment between x and y. Any other C^1 function on L_0 can be constructed by composition with this specific g.

By local product structure and the compactness of L_0, one may construct a compact set $C \subset M$ containing L_0 with the following properties:

- The interior of C contains the (one-dimensional) interior of L_0.
- If $W^c(x)$ is a center leaf, then every connected component of $W^c(x) \cap C$ is a compact interval, called a "center segment."
- If $AC(y)$ is a us-leaf, then every connected component of $AC(y) \cap C$ is a compact set homeomorphic to a closed ball and called a "us-plaque."
- Each center segment intersects each us-plaque in exactly one point.
- L_0 is a center segment.

By a C^1 change of coordinates, assume that $C \subset \mathbb{R}^d$.

Let $\Sigma \subset C$ be the union of all us-plaques, and $\Sigma' \subset \Sigma$ the union of all us-plaques which are accumulated on by other us-plaques. If $x \in \Sigma'$, define

$$D(x) = \lim_{n\to\infty} \frac{\|\sigma_n \cap L_0 - \sigma \cap L_0\|}{\|\sigma_n \cap L - \sigma \cap L\|}$$

where L is the center segment through x, σ is the us-plaque through x, and σ_n are us-plaques converging to σ. By (12.1), this limit exists, is independent of the sequence σ_n tending to σ, and is non-zero. The C^1 regularity of the holonomies also implies that if ρ_n is another sequence of us-plaques converging to σ, then

$$D(x) = \lim_{n\to\infty} \frac{\|\sigma_n \cap L_0 - \rho_n \cap L_0\|}{\|\sigma_n \cap L - \rho_n \cap L\|}$$

so long as $\sigma_n \neq \rho_n$ for large n. Further, by (12.1), the ratio $D(L_1 \cap \sigma)/D(L_2 \cap \sigma)$ tends uniformly to one as $\operatorname{dist}(L_1, L_2)$ tends to zero. As D is continuous when restricted to each center segment and uniformly continuous on each us-plaque σ, it is therefore continuous on all of Σ'. Define $D(x) = 1$ for all $x \in L_0$ and note that this agrees with the above definition on the intersection $\Sigma' \cap L_0$. Then, choose a continuous positive extension $D : \Sigma \cup L_0 \to \mathbb{R}$.

Also extend $g : L_0 \to \mathbb{R}$ to a function $g : \Sigma \cup L_0 \to \mathbb{R}$ by making it constant on each us-plaque. To further extend g to a C^1 function on all of C, we will define for each point $x \in \Sigma \cup L_0$ a candidate derivative $dg_x : \mathbb{R}^d \to \mathbb{R}$ and show that Whitney's extension theorem applies. Choose an orientation for E^c and for each $x \in \Sigma \cup L_0$, let v_x^c be the unique oriented unit vector in E_x^c. Define dg_x as the unique linear map such that $dg_x(v_x^c) = D(x)$ and $\ker dg_x = E_x^u \oplus E_x^s$. As both $D(x)$ and the splitting $E_x^u \oplus E_x^c \oplus E_x^s$ are continuous in x, the linear map dg_x is continuous in x.

Define the function $R : C \times C \to \mathbb{R}$ by

$$R(x_n, y_n) = \frac{1}{\|y_n - x_n\|}\Big(g(y_n) - g(x_n) - dg_{x_n}(y_n - x_n)\Big).$$

To apply Whitney's extension theorem, one needs to show that for any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $\|x_n - y_n\|$ converging to zero, the sequence $R(x_n, y_n)$ also converges to zero. If this does not hold, there are sequences $\{x_n\}$ and $\{y_n\}$ so that $R(x_n, y_n)$ is bounded away from zero. Therefore, without loss of generality, one may replace these sequences by subsequences and assume x_n and y_n both converge to a point $q \in C$. We will also restrict to further subsequences as necessary later in the proof.

We prove the convergence in progressively more general cases.

Case 1. First, assume x_n, y_n, and q are all on the same center segment $L \neq L_0$. Let σ_n, ρ_n and σ be such that

$$\sigma_n \cap L = x_n, \quad \rho_n \cap L = y_n, \quad \text{and} \quad \sigma \cap L = q.$$

If $\sigma \notin \Sigma'$, then $x_n = y_n = q$ for large n. Therefore, assume $\sigma \in \Sigma'$. Then,

$$\lim_{n\to\infty} \frac{g(y_n) - g(x_n)}{\|x_n - y_n\|} = \lim_{n\to\infty} \frac{\|\sigma_n \cap L_0 - \rho_n \cap L_0\|}{\|\sigma_n \cap L - \rho_n \cap L\|} = D(q).$$

As both the candidate derivative dg_x and the center direction v_x^c are continuous in x,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{\|y_n - x_n\|} dg_{x_n}(y_n - x_n) &= \big(\lim_{n\to\infty} dg_{x_n}\big)\big(\lim_{n\to\infty} \frac{y_n - x_n}{\|y_n - x_n\|}\big) \\ &= dg_q(v_q^c) = D(q).\end{aligned}$$

Therefore, $\lim_{n\to\infty} R(x_n, y_n) = D(q) - D(q) = 0$.

Case 2. Now, consider the case where x_n and y_n are on the same center segment L_n for each n. Define x_n^c to be on the same us-plaque as x_n and the same center segment as q. Define y_n^c similarly. Then,

$$g(x_n) - g(y_n) = g(x_n^c) - g(y_n^c).$$

By (12.1),

$$\lim_{n\to\infty} \frac{\|y_n - x_n\|}{\|y_n^c - x_n^c\|} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{g(y_n) - g(x_n)}{\|y_n - x_n\|} = \lim_{n\to\infty} \frac{g(y_n^c) - g(x_n^c)}{\|y_n^c - x_n^c\|} = D(q)$$

where the last equality is by the previous case. As before,

$$\lim_{n\to\infty} \frac{1}{\|y_n - x_n\|} dg_{x_n}(y_n - x_n) = dg_q(v_q^c) = D(q)$$

and therefore $\lim_{n\to\infty} R(x_n, y_n) = 0$.

Case 3. Now consider x_n and z_n as general sequences in Σ converging to q. Define y_n as the unique point lying on the same center segment as x_n and the same us-plaque as z_n. By taking subsequences, assume

$$\lim_{n\to\infty} \frac{z_n - y_n}{\|z_n - y_n\|}$$

exists. By continuity of the partially hyperbolic splitting, this limit is in $E^u_q \oplus E^s_q$. Therefore,

$$\lim_{n\to\infty} \frac{1}{\|z_n - y_n\|}\, dg_{x_n}(z_n - y_n) = \big(\lim_{n\to\infty} dg_{x_n}\big)\big(\lim_{n\to\infty} \frac{z_n - y_n}{\|z_n - y_n\|}\big) = 0$$

implying, with $g(z_n) = g(y_n)$, that

$$\lim_{n\to\infty} \frac{1}{\|z_n - y_n\|}\big(g(z_n) - g(y_n) - dg_{x_n}(z_n - y_n)\big) = 0.$$

By transversality of the foliations, there is a constant $c_1 > 0$ such that $\|z_n - x_n\| \geq c_1\|z_n - y_n\|$ and therefore

$$\lim_{n\to\infty} \frac{1}{\|z_n - x_n\|}\big(g(z_n) - g(y_n) - dg_{x_n}(z_n - y_n)\big) = 0$$

as well. Again by transversality, there is $c_2 > 0$ such that $\|z_n - x_n\| \geq c_2\|y_n - x_n\|$ and therefore by the previous case

$$\lim_{n\to\infty} \frac{1}{\|z_n - x_n\|}\big(g(y_n) - g(x_n) - dg_{x_n}(y_n - x_n)\big) = 0.$$

Added together, these limits show that $\lim_{n\to\infty} R(x_n, z_n) = 0$.

Case 4. Now consider the case where $x_n \in L_0$ and $z_n \in \Sigma$ for all n. Define y_n from x_n and z_n exactly as in the last case. Then,

$$\begin{aligned} R(x_n, z_n) = {} & \frac{1}{\|z_n - x_n\|}\big(g(z_n) - g(y_n) - dg_{x_n}(z_n - y_n)\big) \\ & + \frac{1}{\|z_n - x_n\|}\big(g(y_n) - g(x_n) - dg_{x_n}(y_n - x_n)\big) \end{aligned}$$

and, similar to the previous case, both summands can be shown to converge to zero. The case $x_n \in \Sigma$ and $z_n \in L_0$ is almost identical.

Case 5. If both $\{x_n\}$ and $\{z_n\}$ are in L_0, then $\lim_{n\to\infty} R(x_n, z_n) = 0$ simply by the fact that g is C^1 when restricted to L_0.

The general case. The final case to consider is where $\{x_n\}$ and $\{z_n\}$ are general sequences in $X = \Sigma \cup L_0$. By taking subsequences, one can assume each sequence lies either entirely in L_0 or entirely in Σ and therefore reduce to a previous case. $\square$

We now prove the following restatement of (2.9).

Corollary 12.3. *If $f : M \to M$ is a non-accessible, partially hyperbolic C^2 diffeomorphism with one-dimensional center, the non-open accessibility classes form a C^1 lamination. That is, around any point $x \in M$ there is a neighbourhood V and functions $g : V \to \mathbb{R}$ and $\psi : V \to \mathbb{R}^{d-1}$ such that $g \times \psi$ is a C^1 embedding and if $AC(y)$ is a us-leaf and σ a connected component of $AC(y) \cap V$, then $\sigma = g^{-1}(t)$ for some $t \in \mathbb{R}$.*

Proof. Define a coordinate chart $\phi \times \psi : V \to \mathbb{R} \times \mathbb{R}^{d-1}$ such that the kernel of the derivative $d\phi : T_x M \to \mathbb{R}$ at x is equal to $E^u_x \oplus E^s_x$. By (12.2), after replacing V by a subset, there is a C^1 function $g : V \to \mathbb{R}$ constant on us-plaques and such that g and ϕ are equal on a center segment through x. Then, the derivative of $g \times \psi$ is invertible at x and so, after again replacing V by a subset, $g \times \psi$ is the desired C^1 embedding. $\square$

We now proceed to prove (5.3). Recall the definition of an AI-system from Section 7.

Proposition 12.4. *Let $f : \hat{M} \to \hat{M}$ be a C^2 AI-system and $X \subset \hat{M}$ a compact us-leaf. Then, there is a neighbourhood V of X, an open subset $U \subset (0,1)$ and functions $p : V \to (0,1)$ and $\psi : V \to X$ such that $p \times \psi$ is a C^1 diffeomorphism and the compact us-leaves in V are exactly of the form $p^{-1}(t)$ for $t \notin U$.*

Moreover, p restricted to each center segment $L \subset V$ is a C^1 diffeomorphism.

In this context, a center segment is a connected component of the intersection of V with a center leaf.

Proof. There is a neighbourhood V of X such that inside V each center segment intersects each compact us-leaf in a unique point. Therefore, the proofs of the previous results of this section hold as before with compact us-leaves now filling the role of us-plaques. This gives the existence of p and ψ.

As the function D is positive in the proof of (12.2), for $x \in X$ and unit vector $v^c \in E^c_x$ the derivative dp_x of p satisfies $dp_x(v^c) \neq 0$. By continuity, this property holds for all x in a neighbourhood of X and so, by replacing V by a subset, the restriction of p to any center segment L has non-zero derivative along all of L. $\square$

As it is a local result, (12.4) also holds for a compact us-leaf in an AB-system instead of an AI-system. To go from the local to the global requires a technical lemma which "fills in the gaps" between compact us-leaves.

Lemma 12.5. *Let N be a C^1 manifold, and for $0 < \epsilon < \frac{1}{2}$ define*

$$V_\epsilon = N \times ([0,\epsilon) \cup (1-\epsilon, 1]) \subset N \times [0,1].$$

If there are $\epsilon > 0$ and a C^1 function $g : V_\epsilon \to [0,1]$ such that

- $\frac{\partial g}{\partial t}\big|_{(x,t)} > 0$ *for all $(x,t) \in V_\epsilon$, and*
- $g(x,0) = 0$ *and $g(x,1) = 1$ for all $x \in N$*

then there are $\delta > 0$ and a C^1 function $h : N \times [0,1] \to [0,1]$ such that

- $h(x,t) = g(x,t)$ *for all $(x,t) \in V_\delta$,*
- $(x,t) \mapsto (x, h(x,t))$ *is a C^1 diffeomorphism of $N \times [0,1]$, and*
- *if $x \in N$ satisfies $g(x,t) = t$ for all $(x,t) \in V_\delta$, then $h(x,t) = t$ for all $t \in [0,1]$.*

Proof. Pick $\delta > 0$ small enough that there is a continuous function $h_0 : N \times [0,1] \to [0,1]$ which for each $x \in N$ satisfies the following properties:

- $t \mapsto h_0(x,t)$ is strictly increasing and linear on each of the intervals $[\delta, 3\delta]$, $[3\delta, 1-3\delta]$, and $[1-3\delta, 1-\delta]$; and
- h_0 agrees with g and $\frac{\partial h_0}{\partial t}$ agrees with $\frac{\partial g}{\partial t}$ at the points of the form (x,δ) and $(x, 1-\delta)$.

Then, define h by $h(x,t) = g(x,t)$ for $(x,t) \in V_\delta$, $h(x,t) = h_0(x,t)$ for $(x,t) \in V_{2\delta} \setminus V_\delta$, and $h(x,t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} h_0(x,s)ds$ otherwise. □

Proposition 12.6. *Let $f : \hat{M} \to \hat{M}$ be a C^2 AI-system, and J a compact interval inside a center leaf such that its endpoints x_0 and x_1 lie inside compact us-leaves. Then there are $r : AC(J) \to AC(x_0)$ and $p : AC(J) \to [0,1]$ such that $r \times p$ is a C^1 diffeomorphism and every compact us-leaf in $AC(J)$ is of the form $p^{-1}(t)$ for some $t \in [0,1]$.*

Proof. By approximating the center bundle E^c by a C^1 vector field v, one may define a C^1 flow taking points in $AC(x_0)$ to points in $AC(x_1)$. By rescaling v, assume the flow takes each point in $AC(x_0)$ to a point in $AC(x_1)$ in exactly one unit of time. This flow then defines a C^1 diffeomorphism between $AC(J)$ and $AC(x_0) \times [0,1]$. Therefore, we may assume our system is defined on a space of the form $N \times [0,1]$ where N is a manifold C^1-diffeomorphic to $AC(x_0)$ and that $r : N \times [0,1] \to N$ is given by projection onto the first coordinate. Further assume that the flow v is tangent to E^c on the center leaf containing J. Then, when viewed as a subset of $N \times [0,1]$, J is of the form $J = \{x_0\} \times [0,1]$.

By adapting the arguments in the proofs of (12.2) and (12.4), there is a C^1 function $g : N \times [0,1] \to [0,1]$ which is constant on compact us-leaves and such that $g(x_0, t) = t$ for all $t \in [0,1]$.

Let $\Sigma \subset N \times [0,1]$ be the union of all compact us-leaves. For a point $z \in N \times [0,1]$, let v^c_z be the oriented unit vector in E^c_z. Then, due to the construction of g as in the proof of (12.2), $dg_z(v^c_z)$ is positive for all $z \in \Sigma$. As dg is continuous, there is a C^1 vector field $\hat{v}$ approximating v^c such that $dg_z(\hat{v}(z))$ is positive for all $z \in \Sigma$. By another C^1 change of coordinates, assume v is equal to $\hat{v}$ and therefore $\frac{\partial g}{\partial t}\big|_{(x,t)} = dg_{(x,t)}(v(x,t))$ for all $(x,t) \in N \times [0,1]$. By uniform continuity, there is $\epsilon > 0$ such that $dg_z(v(z)) > 0$ for all z at distance at most ϵ from Σ. Hence, there are at most a finite number of regions $X_i \subset N \times [0,1]$ such that

- the boundary of X_i is given by two compact us-leaves,
- there are no compact leaves in the interior of X_i, and
- $\frac{\partial g}{\partial t}\big|_{(x,t)} \le 0$ for some $(x,t) \in X_i$.

By (12.5), define a C^1 function $p : N \times [0,1] \to [0,1]$ which is equal to g everywhere outside of $\cup_i X_i$ and such that $\frac{\partial p}{\partial t}\big|_{(x,t)} > 0$ for all $(x,t) \in N \times [0,1]$.

Since both r and p are submersions, $r \times p$ has an invertible derivative at every point and is therefore a C^1 diffeomorphism. □

Corollary 12.7. *In the setting of* (12.6)*, if $L \subset \hat{M}$ is a center leaf, then p and r may be chosen so that p restricted to $L \cap AC(J)$ is a C^1 diffeomorphism onto* $[0,1]$.

Proof. Take $J \subset L$ in the previous proof. □

Corollary 12.8. *In the setting of* (12.6)*, if μ is a probability measure given by a continuous volume form on $AC(J)$, then p may be chosen so that $p_*\mu$ is Lebesgue measure on* $[0,1]$.

Proof. Assume $\rho : N \times [0,1] \to \mathbb{R}$ is a positive density function such that

$$\mu(X) = \int_X \rho \, dm_N \times dm$$

where $m_N \times m$ is the product of the Lebesgue measures on N and $[0,1]$.

If $h : [0,1] \to [0,1]$ is defined by $h(t) = \mu\big(p^{-1}([0,t])\big)$, then

$$\frac{dh}{dt} = \int_{N \times \{t\}} \rho \, dm_N$$

is continuous and positive, showing that h is a C^1 diffeomorphism. Replacing p with the composition hp, the result is proved. □

Proof of (5.3). As noted in Section 8, every AB-system $f : M \to M$ lifts to an AI-system $\hat{f} : \hat{M} \to \hat{M}$. Moreover, if the AB-system has a compact us-leaf, the covering $\hat{M} \to M$ has a fundamental domain which is bounded between two compact leaves $AC(x)$ and $\beta(AC(x))$ where β is the deck transformation defined in Section 8.

Then, (12.6) applies where the region $AC(J)$ is exactly this fundamental domain and therefore, there is a C^1 surjection $p : AC(J) \to [0,1]$. Moreover, the candidate derivative in the application of Whitney's extension theorem may be chosen so that it agrees on $AC(x)$ and $\beta(AC(x))$. Then, p quotients down to a C^1 function $M \to \mathbb{S}^1$ as desired.

The other statements in (5.3) follow from the above two corollaries. □

13. Skew products

This sections proves (4.3) showing that non-accessible skew products have trivial fiber bundles.

Proof of (4.3). As the base map A has a fixed point, there is a fiber S such that $f(S) = S$. By replacing f by f^2 if necessary, assume f preserves the orientation of S. As $\pi_2(N)$ is trivial (see, for instance, [19]), the long exact sequence of fiber bundles gives a short exact sequence $0 \to Z \to G \to H \to 0$ where $Z = \pi_1(S)$, $G = \pi_1(M)$, and $H = \pi_1(N)$. By naturality, f induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle A_*} & & \\ 0 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0. \end{array}$$

As can be shown for any circle bundle with oriented fibers, the subgroup Z is contained in the center of G. In this case, as $H = G/Z$ is nilpotent, G is then also nilpotent.

Skew products have global product structure. The proof is similar to that given for AB-systems in Section 15 and we leave the details to the reader. Similar to the case for AB-systems, we may then consider the universal cover $\tilde{M}$ of M, a topological line $\tilde{S} \subset \tilde{M}$ which covers S, and a lift $\tilde{f} : \tilde{M} \to \tilde{M}$ such that $\tilde{f}(\tilde{S}) = \tilde{S}$. Let $\Lambda \subset \tilde{S}$ be the set of all points $t \in \tilde{M}$ such that $AC(t)$ is not open. Then G induces an action on Λ.

Let z be a non-trivial element of Z. Then z may be regarded as a fixed-point free homeomorphism of $\tilde{S}$. By (6.1) and (6.2), there is a homomorphism $\tau : G \to \mathbb{R}$ such that $\tau(z)$ is non-zero. By (6.3), there is $\lambda > 0$ such that $\tau f_*(g) = \lambda\, \tau(g)$ for all $g \in G$. Since, $f_*(z) = z$, this implies that λ equals one. By rescaling τ, assume $\tau(Z) = \mathbb{Z}$. Then, $\tau : G \to \mathbb{R}$ quotients to a homomorphism $\hat{\tau} : H \to \mathbb{R}/\mathbb{Z}$ and $\hat{\tau} A_* = \hat{\tau}$.

As A is hyperbolic, A_* has no non-trivial fixed points and, by (6.4), no non-trivial fixed cosets. As all of the cosets of $\ker \hat{\tau}$ are fixed by A_*, it follows that $\hat{\tau} = 0$. That is, $\tau(G) = \mathbb{Z}$. One can then define a map which takes each $g \in G$

to the unique $z \in Z$ such that $\tau(g) = \tau(z)$. This shows that the exact sequence $0 - \to Z - \to G - \to H - \to 0$ splits. Then, G is isomorphic to $H \times Z$ and the bundle is trivial.

In fact, one can find a compact us-leaf directly. Viewing H now as a subgroup of G equal to the kernel of τ, choose a point $x \in \tilde{S}$ and define $y = \sup_{g \in H} g(x)$. Then, with μ as in (6.1), $\mu[x, y) = 0$ which implies $y < +\infty$. In other words, y is a well-defined point in $\tilde{S}$. Since y is in $\mathrm{Fix}(H)$ it projects to a point in M contained in a compact us-leaf. □

14. Infra-AB-systems

We now consider infra-AB-systems as defined in Section 2.

First, recall the definition of an infranilmanifold. Let $\tilde{N}$ be a simply connected nilpotent Lie group. A diffeomorphism $\phi : \tilde{N} \to \tilde{N}$ is a *(right* translation) if there is $v \in \tilde{N}$ such that $\phi(u) = u \cdot v$ for all $u \in \tilde{N}$. Let $\mathrm{Trans}(\tilde{N})$ be the group of all translations (which is canonically isomorphic to $\tilde{N}$ itself). Let $\mathrm{Aut}(\tilde{N})$ be the group of all automorphisms of $\tilde{N}$. Then the group of *affine* diffeomorphisms, $\mathrm{Aff}(\tilde{N})$, is the smallest group containing both $\mathrm{Trans}(\tilde{N})$ and $\mathrm{Aut}(\tilde{N})$. Equivalently, $\psi \in \mathrm{Aff}(\tilde{N})$ if and only if there is $\phi \in \mathrm{Aut}(\tilde{N})$ and $v \in \tilde{N}$ such that $\psi(u) = \phi(u) \cdot v$ for all $u \in \tilde{N}$.

If a subgroup $\Gamma < \mathrm{Aff}(\tilde{N})$ is such that $\Gamma \cap \mathrm{Trans}(\tilde{N})$ has finite index in Γ and $N_0 := \tilde{N}/\Gamma$ is a compact manifold, then N_0 is a (compact) *infranilmanifold.* If $A \in \mathrm{Aff}(\tilde{N})$ quotients to a function $A_0 : N_0 \to N_0$ then A_0 is also called *affine*.

Theorem 14.1. *Suppose f_0 is a conservative C^2 infra-AB-system. Then, either*

(1) *f_0 is accessible and stably ergodic,*

(2) *E^u and E^s are jointly integrable and f_0 is topologically conjugate to an algebraic map, or*

(3) *there are $n \geq 1$, a C^1 surjection p_0 from M_0 to either $\mathbb{S}^1$ or $\mathbb{S}^1/\mathbb{Z}_2$, and a non-empty open subset $U \subsetneq p_0(M_0)$ with the following properties.*

- *If $t \notin U$ then $p_0^{-1}(t)$ is an f_0^n-invariant compact us-leaf homeomorphic to an infranilmanifold. Moreover, every f_0-periodic compact us-leaf is of this form.*
- *If I is a connected component of U, then $p_0^{-1}(I)$ is f_0^n-invariant and homeomorphic to a (possibly twisted) I-bundle over an infranilmanifold.*

This theorem is proved at the end of the section and the exact nature of the "algebraic map" in case (2) is given in the proof. Also, as will be evident from the proof, if E^c is orientable then $p_0(M_0) = \mathbb{S}^1$. Otherwise, $p_0(M_0) = \mathbb{S}^1/\mathbb{Z}_2$ which is the 1-dimensional orbifold constructed by quotienting $\mathbb{R}$ by both $\mathbb{Z}$ and the involution $t \mapsto -t$. This orbifold is homeomorphic to a compact interval. A set $p^{-1}(I)$ will be twisted (as an I-bundle) if and only if I is homeomorphic to a half-open interval.

The ergodic decomposition given in (2.3) also generalizes.

Theorem 14.2. *Let $f_0 : M_0 \to M_0$ be a C^2 infra-AB-system and suppose there is a smooth, f_0-invariant, non-ergodic measure ζ supported on M_0. Then, there are $n \geq 1$, a C^1 surjection p_0 from M_0 to either $\mathbb{S}^1$ or $\mathbb{S}^1/\mathbb{Z}_2$, and an open subset $U \subsetneq p_0(M_0)$ such that*

$$\zeta = \sum_I m(I)\,\zeta_I + \int_{t \notin U} \zeta_t \, dm(t) \tag{14.1}$$

is the ergodic decomposition for (f_0^n, ζ).

Here, the components ζ_I and ζ_t of the decomposition are defined analogously to (2.1).

Proof. Let $\pi : M_0 \to M$ be the finite covering and f an AB-system such that $\pi f = f_0^m \pi$ for some $m \geq 1$. Then, ζ lifts to a measure μ on M which (up to rescaling the measure so that $\mu(M) = 1$) satisfies the hypotheses of (2.3). If ζ_t is a component of the decomposition (14.1), then its support is a single accessibility class X_0. If X is a connected component of $\pi^{-1}(X_0) \subset M$, then there is an ergodic component (f^n, μ_t) of (f^n, μ) where μ_t is supported on X and such that $\pi^* \mu_t$ (up to rescaling) is equal to ζ_t. Ergodicity of (f_0^{mn}, ζ_t) then follows from the ergodicity of (f^n, μ_t). Ergodicity of components of the form ζ_I can be proven similarly. □

The theorems in Section 2 concerning non-conservative AB-systems may also be generalized using techniques similar to those in the proof of (14.1) below. In the interests of brevity, we leave the statements and proofs of these other results to the reader. The following two known results about functions on infranilmanifolds will be useful.

Lemma 14.3. *If N_0 is an infranilmanifold, there is a nilmanifold N finitely covering N_0 such that every homeomorphism of N_0 lifts to N.*

Proof. This follows from the fact that $\Gamma \cap \mathrm{Trans}(\tilde{N})$ is the unique maximal normal nilpotent subgroup of $\pi_1(M)$. A proof of this is given in [4], a paper which also contains an infamously incorrect result about maps between infranilmanifolds. (See the discussion in [27].) However, the proof of the above fact about $\Gamma \cap \mathrm{Trans}(\tilde{N})$ is widely held to be correct. □

Lemma 14.4. *If a homeomorphism B on a compact infranilmanifold N_0 commutes with a hyperbolic affine diffeomorphism A, then B itself is affine.*

Proof. This follows by a combination of the results of Mal'cev and Franks. First, consider the case where $N = N_0$ is a nilmanifold. Let x be a fixed point of A. Then

$y := B(x)$ is also a fixed point of A. Using the standard definition of the fundamental group for based spaces, the diagram

$$\begin{array}{ccc} \pi_1(N,x) & \xrightarrow{B_*} & \pi_1(N,y) \\ \downarrow{\scriptstyle A_*} & & \downarrow{\scriptstyle A_*} \\ \pi_1(N,x) & \xrightarrow{B_*} & \pi_1(N,y) \end{array}$$

commutes. By [28], there is a unique affine map $\phi : (N,x) \to (N,y)$ such that $\phi_* = B_*$. (If $x \neq y$ one shows this by considering two distinct lattices of the form $\tilde{x}\Gamma\tilde{x}^{-1}$ and $\tilde{y}\Gamma\tilde{y}^{-1}$ on the Lie group $\tilde{N}$ in order to construct a Lie group homomorphism which quotients down to ϕ.)

As $\phi_* A_* = A_* \phi_*$, the uniqueness given in [28] entails that $\phi A = A\phi$ as functions on N. As N is aspherical, ϕ is homotopic to B. Then, using that A is a π_1-diffeomorphism as defined in [19], it follows that ϕ and B are equal.

Now suppose N_0 is an infranilmanifold. By (14.3), there is a nilmanifold N and a normal finite covering $N \to N_0$ such that both A and B lift to functions $N \to N$. By abuse of notation, we still call these functions A and B. As the covering is finite, there is $j \geq 1$ such that $A^j\gamma = \gamma A^j$ for every deck transformation γ. In particular, there is a deck transformation $\gamma : N \to N$ such that $A^j B = BA^j\gamma$. Then, $A^{jk}B = B(A^j\gamma)^k = BA^{jk}\gamma^k$ for all $k \in \mathbb{Z}$, and, taking $k \geq 1$ such that γ^k is the identity, A^{jk} commutes with B and the problem reduces to the previous case. □

Proposition 14.5. *Suppose f_0 is a partially hyperbolic skew product where the base map is a hyperbolic infranilmanifold automorphism and E^c is one-dimensional. If f_0 is not accessible, it is an infra-AB-system.*

Proof. Lift the fiber bundle projection $\pi : M_0 \to N_0$ to $\tilde{\pi} : \tilde{M} \to \tilde{N}$ where $\tilde{M}$ and $\tilde{N}$ are the universal covers. Let G consist of those deck transformations $\alpha \in \pi_1(M_0)$ which preserve the orientation of the lifted center bundle and for which $\tilde{\pi}\alpha = \gamma\tilde{\pi}$ for some $\gamma \in \mathrm{Trans}(\tilde{N})$. Then, G is a finite index subgroup of $\pi_1(M_0)$ defining a finite cover $M = \tilde{M}/G$ and one can show that $f_0 : M_0 \to M_0$ lifts to $f : M \to M$ where the base map $A_0 : N_0 \to N_0$ lifts to the nilmanifold $\tilde{N}/\tilde{\pi}(G)$. If f_0 is not accessible, then f is not accessible. The fiber bundle on M is then trivial by (4.3), implying that f^2, which preserves the orientation of E^c, is an AB-system. □

We now prove (14.1).

Assumption 14.6. *For the remainder of the section, assume $f : M \to M$ is a non-accessible conservative C^2 AB-system, $\pi : M \to M_0$ is a (not-necessarily normal) finite covering map and that $f_0 : M_0 \to M_0$ and $m \geq 1$ are such that $\pi f = f_0^m \pi$.*

Note this implies that f_0 is partially hyperbolic and the splitting on the tangent bundle TM_0 lifts to the splitting for f on TM.

For now, make the following additional assumptions, which will be removed later.

Assumption 14.7. *Assume until the end of the proof of* (14.9) *that*

- E^c *on* M_0 *is orientable;*
- f_0 *preserves the orientation of* E^c*; and*
- $m = 1$*, that is,* $\pi f = f_0 \pi$.

By the assumption $m = 1$, both f_0 and f can be lifted to the same map $\tilde{f}$ on the universal cover $\tilde{M}$.

As f is an AB-system defined by nilmanifold automorphisms $A, B : N \to N$, there is a map $H : \tilde{M} \to \tilde{N}$ whose fibers are the center leaves of f and where $\tilde{N}$ is the universal cover of N and therefore a nilpotent Lie group. Further, A lifts to a hyperbolic automorphism of $\tilde{N}$, which we also denote by A, and the leaf conjugacy implies that $H\tilde{f} = AH$.

Define $\tilde{S} = H^{-1}(\{0\})$ where 0 is the identity element of the Lie group. Then $\tilde{S}$ is an $\tilde{f}$-invariant center leaf which covers a circle $S \subset M$ and S further covers a circle $S_0 \subset M_0$. By (2.2), there is a C^1 surjection $p : M \to \mathbb{S}^1$ and a constant $\theta \in \mathbb{S}^1$ such that if $x \in M$ has non-open accessibility class $AC(x)$ then p is constant on $AC(x)$ and $pf(x) = p(x) + \theta$. By (5.3), assume p restricted to S is a C^1 diffeomorphism. Using p and the covering $\pi : M \to M_0$, define a map

$$q : M_0 \to \mathbb{S}^1, \quad x_0 \mapsto \sum_{y \in \pi^{-1}(x_0)} p(y).$$

It follows that if $x_0 \in M_0$ has non-open accessibility class $AC(x_0)$ then q is constant on $AC(x_0)$ and $qf_0(x_0) = q(x_0) + \theta d$ where d is degree of the covering. Further, q restricted to S_0 is a C^1 covering from S_0 to $\mathbb{S}^1$ (though not necessarily of degree d). After lifting q to a map $\tilde{q} : \tilde{M} \to \mathbb{R}$, there is a homomorphism $q_* : \pi_1(M_0) \to \mathbb{Z}$ such that $\tilde{q}\gamma(\tilde{x}) = \tilde{q}(\tilde{x}) + q_*(\gamma)$ for every $\tilde{x} \in \tilde{M}$ and deck transformation $\gamma \in \pi_1(M_0)$.

As the deck transformations preserve the lifted center foliation, for each $\gamma \in \pi_1(M_0)$, there is a unique homeomorphism $B_\gamma : \tilde{N} \to \tilde{N}$ such that $H\gamma = B_\gamma H$.

Lemma 14.8. $B_\gamma \in \operatorname{Aff}(\tilde{N})$ *for all* $\gamma \in \pi_1(M_0)$.

Proof. We may view $\pi_1(M)$ as a finite index subgroup of $\pi_1(M_0)$. The definition of an AB-system implies that $B_\gamma \in \operatorname{Aff}(\tilde{N})$ for all $\gamma \in \pi_1(M)$.

Now consider the subgroups $K_3 < K_2 < K_1 < \pi_1(M_0)$ defined as follows:

$$\begin{aligned} &K_1 \text{ is the kernel of } q_*, \\ &K_2 = K_1 \cap \pi_1(M), \text{ and} \\ &K_3 = \{\alpha \in K_2 : \alpha\beta K_2 = \beta K_2 \text{ for all } \beta \in K_1\}. \end{aligned}$$

By its definition, K_3 is a normal finite index subgroup of K_1. The lift $\tilde{f}$ of f_0 induces a homomorphism $f_* : \pi_1(M_0) \to \pi_1(M_0)$ given by $f_*(\gamma) = \tilde{f}\gamma\tilde{f}^{-1}$. There is a constant $c \in \mathbb{R}$ such that

$$\tilde{q}\tilde{f}(\tilde{x}) = \tilde{q}(\tilde{x}) + c$$

for all $\tilde{x} \in \tilde{M}$ with non-open accessibility class. This implies that $f_*(K_1) = K_1$. From this, one can show that $f_*(K_2) = K_2$ and therefore $f_*(K_3) = K_3$.

Note that $N_3 := \tilde{N}/\{B_\gamma : \gamma \in K_3\}$ is a nilmanifold (which finitely covers the original nilmanifold N), and the hyperbolic Lie group automorphism $A : \tilde{N} \to \tilde{N}$ descends to an Anosov diffeomorphism on N_3.

Suppose $\gamma \in K_1$. As f_* permutes the cosets of K_3, there is $j \geq 1$ such that $f_*^j(\gamma)K_3 = \gamma K_3$. This implies that A^j and B_γ descend to commuting diffeomorphisms on N_3. Then, by (14.4), B_γ is affine. Thus, we have established the desired result for all $\gamma \in K_1$, and further shown that $N_1 := \tilde{N}/\{B_\gamma : \gamma \in K_1\}$ is an infranilmanifold (finitely covered by the original nilmanifold N).

Now suppose $\gamma \in \pi_1(M_0)$ is an arbitrary deck transformation. Then

$$\tilde{q}\tilde{f}\gamma\tilde{f}^{-1}\gamma^{-1}(\tilde{x}) = \tilde{q}(\tilde{x})$$

for all $\tilde{x} \in \tilde{M}$ with non-open accessibility class. This implies that $f_*(\gamma)K_1 = \gamma K_1$. and so A and B_γ descend to commuting diffeomorphisms on N_1. As A is hyperbolic, $B_\gamma \in \operatorname{Aff}(\tilde{N})$ by (14.4). □

If f is accessible, then clearly f_0 is accessible. Therefore to prove (14.1), it is enough to consider f in cases (2) and (3) of (2.2).

Proposition 14.9. *If f is in case (3) of (2.2) and f_0 satisfies assumption (14.7), then f_0 is in case (3) of (14.1).*

Proof. By replacing f_0, f, and $\tilde{f}$ by iterates, assume $n = 1$ in (2.2) and that the lift $\tilde{f}$ was chosen so that $\tilde{f}(\tilde{X}) = \tilde{X}$ for every accessibility class $\tilde{X} \subset \tilde{M}$.

The image of q_* is equal to $\ell\mathbb{Z}$ for some $\ell \geq 1$. Then $\tilde{p}_0 := \frac{1}{\ell}\tilde{q}$ quotients to a function $p_0 : M_0 \to \mathbb{S}^1$. As the original $p : M \to \mathbb{S}^1$ was C^1, the functions q, $\tilde{q}$, $\tilde{p}_0$, and p are also C^1. Also, p_0 is constant on compact us-leaves and its restriction to S_0 is a C^1 covering. If, for some $t \in \mathbb{S}^1$, X_0 and Y_0 are compact us-leaves in the pre-image $p_0^{-1}(t)$, then they lift to closed us-leaves $\tilde{X}, \tilde{Y} \subset \tilde{M}$ such that $\tilde{p}_0(\tilde{X}) - \tilde{p}_0(\tilde{Y})$ is an integer. By the definition of $\tilde{p}_0$, there is then a deck transformation taking $\tilde{X}$ to $\tilde{Y}$ and so $X_0 = Y_0$. This shows that every compact us-leaf in M_0 is of the form $p_0^{-1}(t)$ for some t.

If X_0 is instead an open accessibility class, then its boundary consists of two compact us-leaves and from this one can show that $p_0^{-1}(p_0(X_0)) = X_0$.

Note that every accessibility class X_0 on M_0 is the projection of an accessibility class $\tilde{X}$ on $\tilde{M}$. As $\tilde{f}$ fixes accessibility classes, so does f_0. Further, using K_1 and N_1 as in the proof of the lemma above, X_0 is homeomorphic to $\tilde{X}/K_1$. If $\tilde{X}$ is non-open, then $\tilde{X}/K_1$ is homeomorphic to the infranilmanifold N_1. If $\tilde{X}$ is open, then $\tilde{X}/K_1$ is an I-bundle over N_1 where the fibers of the I-bundle are segments of center leaves.

This shows that f_0 satisfies case (3) of (14.1). □

We now remove the additional assumptions above and show that this result still holds.

Proposition 14.10. *If f is in case (3) of (2.2) and f_0 does not satisfy assumption (14.7), then f_0 is in case (3) of (14.1).*

Proof. In case (3) of (14.1), we are free to replace f_0 by an iterate. By replacing f_0 by f_0^m, one can assume $m = 1$. That is, $\pi f = f_0 \pi$. By replacing f_0 by f_0^2, one can assume f_0 preserves the orientation of any orientable bundle. Thus, the only condition to test is when E^c is non-orientable.

Any non-orientable bundle on a manifold lifts to an orientable bundle on a double cover and any bundle-preserving diffeomorphism lifts as well. Therefore, we are free to consider the following situation. As before, E^c is orientable and f_0 preserves the orientation, but now there is an involution $\tau : M_0 \to M_0$, such that τ reverses the orientation of E^c and τ commutes with f_0. As a consequence of this commutativity, τ preserves the partially hyperbolic splitting of f_0. Choose a continuous function $p_1 : M_0 \to \mathbb{S}^1$ which satisfies $2p_1(x) = p_0(x) - p_0\tau(x)$. As τ^2 is the identity, $p_1\tau(x) = -p_1(x)$ and so p_1 descends to a function $p_2 : M_0/\tau \to \mathbb{S}^1/\mathbb{Z}_2$.

Since $\mathbb{S}^1 \to \mathbb{S}^1$, $x \mapsto -x$ has two fixed points, one can show that τ fixes exactly two accessibility classes on M_0. Let X_0 be one of these two classes, and lift τ and X_0 to the universal cover to get $\tilde{X}$ and $\tilde{\tau}$ such that $\tilde{\tau}(\tilde{X}) = \tilde{X}$. As f and τ commute, it follows from an adaptation of (14.8) that $B_{\tilde{\tau}} \in \operatorname{Aff}(\tilde{N})$. If X_0 is compact, then X_0/τ is homeomorphic to an infranilmanifold. If instead X_0 is open, then X_0 is an I-bundle over N_0 where the fibers are center segments, and τ reverses the orientation of these fibers. Therefore, X_0/τ is a twisted I-bundle over an infranilmanifold.

This shows that case (3) holds for the quotient of f_0 to M_0/τ where p_0 and $U \subset \mathbb{S}^1$ are replaced by p_2 and $U/\mathbb{Z}_2 \subset \mathbb{S}^1/\mathbb{Z}_2$. □

Now consider the situation where f is in case (2) of (2.2). The following proposition shows that f_0 is "algebraic" as stated in case (2) and concludes the proof of (14.1).

Proposition 14.11. *Suppose f_0 is an infra-AB-system and $E^u \oplus E^s$ is integrable. Then there is a lift $\tilde{f}_0$ of f_0 to the universal cover $\tilde{M}$ and a homeomorphism $h : \tilde{M} \to \tilde{N} \times \mathbb{R}$ such that*

$$h \tilde{f}_0 h^{-1} \in \operatorname{Aff}(\tilde{N}) \times \operatorname{Isom}(\mathbb{R})$$

and

$$h \gamma h^{-1} \in \operatorname{Aff}(\tilde{N}) \times \operatorname{Isom}(\mathbb{R})$$

for every deck transformation $\gamma \in \pi_1(M_0)$.

Here, $\operatorname{Isom}(\mathbb{R})$ is the group of functions of the form $t \mapsto t + c$ or $t \mapsto -t + c$.

Proof. First consider the case where f_0 satisfies assumption (14.7) and recall the functions $H : \tilde{M} \to \tilde{N}$ and $\tilde{q} : \tilde{M} \to \mathbb{R}$ defined earlier in this section. By global product structure and the integrability of $E^u \oplus E^s$, $H \times \tilde{q}$ is a homeomorphism.

The results already given in this section then show that $h = H \times \tilde{q}$ satisfies the conclusions of the lemma.

If f_0 does not satisfy (14.7) and E^c is orientable on M_0, then there is $m > 1$ such that f_0^m satisfies (14.7). Let H and $\tilde{q}$ be given for f_0^m. Define $a = +1$ if $\tilde{f_0}$ preserves the orientation of E^c and $a = -1$ if $\tilde{f_0}$ reverses the orientation. Define $r : \tilde{M} \to \mathbb{R}$ by $r(x) = \sum_{k=0}^{m-1} a^k \, \tilde{q} \, \tilde{f}_0^k(x)$ and take $h = H \times r$.

If E^c is non-orientable on M_0, then f_0 lifts to a double cover on which E^c is orientable. Then, let H and r be defined as in the previous case. Choose a deck transformation $\tilde{\tau} : \tilde{M} \to \tilde{M}$ which reverses the orientation of E^c on $\tilde{M}$ and define a function $s : \tilde{M} \to \mathbb{R}$ by $s(x) = r(x) - r\tilde{\tau}(x)$ and take $h = H \times s$. □

15. Openness

This section establishes that AB-systems have global product structure and form an open subset of the space of C^1 diffeomorphisms.

Lemma 15.1. *Suppose G is a simply connected nilpotent Lie group. For any distinct $u, v \in G$, there is a unique one-dimensional Lie subgroup $G_{u,v}$ such that $v^{-1}u \in G_{u,v}$. (That is, u lies in the coset $vG_{u,v}$.)*

Proof. This follows from the fact that for such groups, the exponential map from the Lie algebra to the Lie group is surjective [28]. □

A right-invariant metric on such a group G is a metric $d : G \times G \to [0, \infty)$ such that $d(u, v) = d(u \cdot w, v \cdot w)$ for all $u, v, w \in G$. For such a metric, we define a function $d_1 : G \times G \to [0, \infty)$ where $d_1(u, v)$ is the length of the path from u to v which lies in the coset $vG_{u,v}$ given by (15.1). Clearly, $d(u, v) \le d_1(u, v)$ for all $u, v \in G$. Further, d_1 is continuous and the ratio $d_1(u, v)/d(u, v)$ tends uniformly to one as $d(u, v)$ tends to zero. Note that d_1 is not a metric on G in general. (If $G = \mathbb{R}^d$ is abelian, however, the coset uG_1 is simply the line through u and v and $d = d_1$.)

If $\phi : G \to G$ is an automorphism and G_1 is a one dimensional subgroup, then there is λ such that $d_1(\phi(u), \phi(v)) = \lambda d_1(u, v)$ for all $u, v \in G$ with $u \in vG_1$. This follows because both G_1 and $\phi(G_1)$ are Lie groups isomorphic to $\mathbb{R}$ and d_1 restricts to a right-invariant metric on either of G_1 or $\phi(G_1)$.

Lemma 15.2. *Suppose G is a simply connected nilpotent Lie group, d is a right-invariant metric, $\{\phi_k\}$ is a sequence of Lie group automorphisms of G, $G_1 \subset G$ is a one-dimensional Lie subgroup, $u_0 \in G$, and $v_0 \in u_0 G_1$ with $u_0 \neq v_0$.*

(1) *If $\lim_{k\to\infty} d(\phi_k(u_0), \phi_k(v_0)) = 0$, then*

$$\lim_{k\to\infty} d(\phi_k(u), \phi_k(v)) = 0$$

for all $u \in G$ and $v \in uG_1$.

(2) *If* $a \geq 1$ *and* $\lim_{k\to\infty} a^k d(\phi_k(u_0), \phi_k(v_0)) = 0$*, then*

$$\lim_{k\to\infty} a^k d(\phi_k(u), \phi_k(v)) = 0$$

for all $u \in G$ *and* $v \in uG_1$.

(3) *If* $\sup_k d(\phi_k(u_0), \phi_k(v_0)) < \infty$*, then*

$$\sup_k d_1(\phi_k(u_0), \phi_k(\hat{v})) = 1$$

for some $\hat{v} \in u_0 G_1$.

Proof. Let λ_k be such that $d_1(\phi^k(u), \phi^k(v)) = \lambda_k d_1(u, v)$ when $u \in vG_1$. Then in the first item, the two limits hold if and only if $\lambda_k \to 0$ and so one implies the other. For the second item, consider $a^k \lambda_k$. For the final item, if the first supremum is finite, then $\Lambda := \sup_k \lambda_k < \infty$ and one can take $\hat{v} \in v_0 G_1$ such that $d_1(\hat{v}, v_0) = 1/\Lambda$. □

We now show that every AB-system has global product structure.

Proof of (5.1). Let $f : \tilde{M} \to \tilde{M}$ be the lift of the AB-system to the universal cover and $h : \tilde{M} \to \tilde{M}_B$ the lifted leaf conjugacy to the AB-prototype. The functions f and h are written without tildes as all the analysis will be on the universal covers.

Measuring distances on the manifold $\tilde{M}_B$ requires care. The metric $d_{\tilde{M}_B}$ on $\tilde{M}_B$ is defined by lifting a metric from M_B. If $p_k = (u_k, s_k)$, and $q_k = (v_k, t_k)$ are sequences in $\tilde{M}_B = \tilde{N} \times \mathbb{R}$, then $d_{\tilde{M}_B}(p_k, q_k)$ may not converge to zero, even if both $d_{\tilde{N}}(u_k, v_k) \to 0$ on $\tilde{N}$ and $|s_k - t_k| \to 0$ on $\mathbb{R}$. The convergence depends on the exact nature of the automorphism B. If s_k and t_k are *bounded* sequences in $\mathbb{R}$, however, then one can show in this special case that $d_{\tilde{M}_B}(p_k, q_k) \to 0$ if and only if both $d_{\tilde{N}}(u_k, v_k) \to 0$ on $\tilde{N}$ and $|s_k - t_k| \to 0$ on $\mathbb{R}$.

There is a deck transformation $\beta : \tilde{M}_B \to \tilde{M}_B$ defined by $\beta(v, t) = (Bv, t - 1)$ which is an isometry with respect to $d_{\tilde{M}_B}$. For general $\{p_k\}$ and $\{q_k\}$, let $\{n_k\}$ be the unique sequence of integers such that $0 \leq |s_k - n_k| < 1$ for all k. Then, $\beta^{n_k}(p_k) \in \tilde{N} \times [0, 1)$ for all k and

$$d_{\tilde{M}_B}(p_k, q_k) = d_{\tilde{M}_B}(\beta^{n_k}(p_k), \beta^{n_k}(q_k)) \to 0$$

if and only if both

$$d_{\tilde{N}}(B^{n_k}(u_k), B^{n_k}(v_k)) \to 0 \quad \text{and} \quad |s_k - t_k| \to 0.$$

In what follows, we write d without a subscript for the metrics on $\tilde{M}$, $\tilde{M}_B$, and $\tilde{N}$. There is no ambiguity as they are all treated as distinct manifolds. If Y is a subset of one of these three manifolds, then

$$\operatorname{dist}(x, Y) := \inf_{y \in Y} d(x, y).$$

Also let $d_s(x, y)$ denote distance measured along the corresponding stable foliation: W^s_f if $x, y \in \tilde{M}$, W^s_A if $x, y \in \tilde{N}$, and $W^s_{A\times\mathrm{id}}$ if $x, y \in \tilde{N} \times \mathbb{R}$. Similarly for d_u and d_c.

The leaf conjugacy implies that every cs-leaf of f intersects a cu-leaf in a unique center leaf. Therefore, establishing global product structure reduces to showing existence and uniqueness of intersections inside the cs and cu leaves.

Uniqueness. Suppose $x \in \tilde{M}$ and $x \neq y \in W^c_f(x) \cap W^s_f(x)$. Then as $k \to \infty$,

$$d_s(f^k(x), f^k(y)) \to 0 \quad \text{and} \quad d_c(f^k(x), f^k(y)) \nrightarrow 0$$

since if both sequences tended to zero, local product structure would imply that x and y were equal. Define $p_k = hf^k(x)$ and $q_k = hf^k(y)$. As the leaf conjugacy is uniformly continuous, $d(p_k, q_k) \to 0$ and $d_c(p_k, q_k) \nrightarrow 0$. If $p_k = (u_k, s_k)$ and $q_k = (v_k, t_k)$, then, as noted above,

$$d(p_k, q_k) \to 0 \quad \Rightarrow \quad |s_k - t_k| \to 0 \quad \Rightarrow \quad d_c(p_k, q_k) \to 0,$$

a contradiction.

Existence. Suppose $x \in \tilde{M}$ lies on a center leaf L_0 and $L_1 \subset W^{cs}_f(x)$ is a distinct center leaf. Then $h(L_0) = \{v_0\} \times \mathbb{R}$ and $h(L_1) = \{v_1\} \times \mathbb{R}$ for distinct points $v_0, v_1 \in \tilde{N}$. As L_0 and L_1 are subsets of the same cs-leaf of f, v_0 and v_1 lie on the same stable leaf of A. By (15.1), there is a one-dimensional subgroup $\tilde{N}_1 \subset \tilde{N}$ such that $v_0^{-1} \cdot v_1 \in \tilde{N}_1$. By item (2) of (15.2), the coset $v_0\tilde{N}_1$ is a subset of $W^s_A(v_0)$.

If U^s_f is a small neighbourhood of x in $W^s_f(x)$, then $h(U^s_f) \subset W^s_A(v_0) \times \mathbb{R}$ and the set $h(W^c_f(U^s_f)) = W^c_{A\times\mathrm{id}}(h(U^s_f))$ is a neighbourhood of $h(x)$ in $W^s_A(v_0) \times \mathbb{R}$. Therefore, if $v \in W^s_A(v_0)$ is sufficiently close to v_0, then there is $y \in W^s_f(x)$ such that $h(y) \in \{v\} \times \mathbb{R}$.

In particular, let v be such that $v \in v_0\tilde{N}_1$ and fix such a point y. See Figure 3. Let $\{n_k\}$ be such that $\beta^{n_k} hf^k(x) \in \tilde{N} \times [0, 1)$ for all k. Then,

$$\begin{aligned} & d(f^k(x), f^k(y)) \to 0 \\ \Rightarrow\ & d(\beta^{n_k} hf^k(x), \beta^{n_k} hf^k(y)) \to 0 \\ \Rightarrow\ & d(B^{n_k} A^k(v_0), B^{n_k} A^k(v)) \to 0 \end{aligned}$$

which by (15.2) implies $d(B^{n_k} A^k(v_0), B^{n_k} A^k(v_1)) \to 0$.

Then, as $hf^k(L_1) = \{A^k(v_1)\} \times \mathbb{R}$,

$$\begin{aligned} & \mathrm{dist}(\beta^{n_k} hf^k(x), \beta^{n_k} hf^k(L_1)) \to 0 \\ \Rightarrow\ & \mathrm{dist}(hf^k(x), hf^k(L_1)) \to 0 \\ \Rightarrow\ & \mathrm{dist}(f^k(x), f^k(L_1)) \to 0. \end{aligned}$$

Thus, for sufficiently large k, $W^s_f(f^k(x))$ intersects $f^k(L_1)$ showing that $W^s_f(x)$ intersects L_1. □

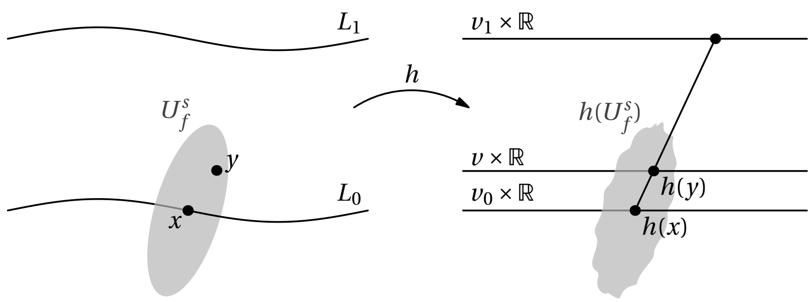

Figure 3. A depiction of points and leaves occuring in the proof of global product structure. In this figure, the stable direction E^s_f is shown as if it were two-dimensional and U^s_f is drawn as a small plaque tangent to E^s_f. The entire left side of the figure lies inside a three-dimensional cs-leaf of f and the right side lies inside a cs-leaf of $A \times \mathrm{id}$.

A sequence $\{x_k\}$ is an ϵ-c-pseudoorbit if for each $k \in \mathbb{Z}$ the points $f(x_k)$ and x_{k+1} lie ϵ-close on the same center leaf. A partially hyperbolic system is *plaque expansive* if there is $\epsilon > 0$ such that if $\{x_k\}$ and $\{y_k\}$ are ϵ-c-pseudoorbits and $d(x_k, y_k) < \epsilon$ for all $k \in \mathbb{Z}$, then x_0 and y_0 are on the same local center leaf.

Theorem 15.3. *Every AB-system is plaque expansive.*

Since plaque expansive systems are open in the C^1 topology [25], this also proves (5.2).

Proof. Let $f : M \to M$ be an AB-system. Let $C > 1$ be a constant to be defined shortly. Since f expands in the unstable direction, there is $\epsilon_0 > 0$ such that if points $x, y, x', y' \in M$ satisfy

$$\frac{1}{C} \le d_u(x, y) \le C, \quad d_c(f(x), x') < \epsilon_0, \quad \text{and} \quad y' \in W^c(f(y)) \cap W^u(x')$$

then $d_u(x, y) < (1 - \epsilon_0) d_u(x', y')$. This result then also holds for points on the universal cover $\tilde{M}$ where f for the remainder of the proof denotes the lift $f : \tilde{M} \to \tilde{M}$.

Let $h : \tilde{M} \to \tilde{N} \times \mathbb{R}$ be the lifted leaf conjugacy. Define sets

$$X = \{(v, w) \in \tilde{N} \times \tilde{N} : \ v \in W^u_A(w), \ d(v, w) \le 1\}$$

and

$$X_1 = \{(v, w) \in \tilde{N} \times \tilde{N} : \ v \in W^u_A(w), \ \tfrac{1}{2} \le d_1(v, w) \le 1\}.$$

and a function

$$D : X \times [-1, 1] \to \mathbb{R}, \quad (v, w, t) \mapsto d_u(h^{-1}(v \times \mathbb{R}), h^{-1}(w \times t)).$$

That is, $D(v,w,t)$ is the distance, measured along an unstable leaf of f, between the center leaf $h^{-1}(v \times \mathbb{R})$ and the point $h^{-1}(w \times t)$. Such a function is well-defined and continuous by global product structure.

If $\alpha : \tilde{N} \to \tilde{N}$ is a deck transformation for the covering $\tilde{N} \to N$, then $\alpha \times \mathrm{id}$ is a deck transformation for the covering $\tilde{M}_B \to M_B$ and one can verify that $D(\alpha(v), \alpha(w), t) = D(v,w,t)$. Using the compactness of N and $[-1,1]$, there is $C > 1$ such that

$$D(X \times [-1,1]) \subset [0,C] \quad \text{and} \quad D(X_1 \times [-1,1]) \subset [\frac{1}{C}, C].$$

This defines the constant C used above.

For some $\epsilon > 0$ let $\{x_k\}$ and $\{z_k\}$ be ϵ-c-pseudoorbits such that $d(x_k, z_k) < \epsilon$. By increasing ϵ and by sliding the points z_k along center leaves, assume, without loss of generality, that there is a point y_k for each k such that x_k and y_k are connected by a short unstable segment and y_k and z_k are connected by a short stable segment. By again increasing ϵ, one can show that $\{y_k\}$ is a ϵ-c-pseudoorbit. We may freely assume that the original ϵ was chosen small enough that $d_c(f(x_k), x_{k+1}) < \epsilon_0$ for all k. We will show that x_0 and y_0 lie on the same center leaf. An analogous argument holds for y_0 and z_0 which will complete the proof.

Suppose x_0 and y_0 lie on distinct center leaves. Then, using β as in the previous proof, there are $v_x \neq v_y \in \tilde{N}$ and $\{n_k\}$ such that $\beta^{n_k} h(x_k) \in \{B^{n_k} A^k v_x\} \times (-1,1)$ and $\beta^{n_k} h(y_k) \in \{B^{n_k} A^k v_y\} \times (-1,1)$ for all $k \in \mathbb{Z}$. This implies that

$$\sup_k d(B^{n_k} A^k v_x, B^{n_k} A^k v_y) < \infty.$$

Let $\tilde{N}_1 \subset \tilde{N}$ be a one-dimensional subgroup such that $v_y \in v_x \tilde{N}_1$. By (15.2), there is $\hat{v} \in v_x \tilde{N}_1$ such that

$$\sup_{k \in \mathbb{Z}} d_1(B^{n_k} A^k v_x,\ B^{n_k} A^k \hat{v}) = 1.$$

By the global product structure of f, there is a unique sequence $\{\hat{y}_k\}$ in $\tilde{M}$ such that $h(\hat{y}_k) \in \{A^k \hat{v}\} \times \mathbb{R}$ and $\hat{y}_k \in W^u_f(x_k)$. Then, $S = \sup_{k \in \mathbb{Z}} d_u(x_k, \hat{y}_k)$ satisfies $\frac{1}{C} \leq S \leq C$. Let $k \in \mathbb{Z}$ be such that $d_u(x_k, \hat{y}_k) > (1-\epsilon_0)S$. The definition of ϵ_0 implies that $d_u(x_{k+1}, \hat{y}_{k+1}) > S$, a contradiction. □

16. The dynamically-incoherent example

This section gives a construction of the example due to Rodriguez Hertz, Rodriguez Hertz, and Ures of a partially hyperbolic system on the 3-torus having an invariant cs-torus [44]. For this specific construction, E^u and E^s are jointly integrable and the tangent foliation has exactly one compact leaf. The system therefore gives an example of case (3) of (2.6). This version of the example given here was written before the version in [44] was made publicly available, and it was not clear at the time what the accessibility classes of the latter would be.

We use the following to prove the example is partially hyperbolic.

Proposition 16.1. *Suppose f is a diffeomorphism of a compact manifold M, $TM = E^s \oplus E^c \oplus E^u$ is an invariant splitting, and there is $k > 1$ such that*

$$\|Tf^k v_x^s\| < \|Tf^k v_x^c\| < \|Tf^k v_x^u\| \quad \text{and} \quad \|Tf^k v_x^s\| < 1 < \|Tf^k v_x^u\|$$

for all $x \in NW(f)$ and unit vectors $v_x^* \in E_x^*$ $(* = s, c, u)$. Then, f is partially hyperbolic.

To prove this, note that if the above inequalities hold on $NW(f)$, they also hold on a neighbourhood U of $NW(f)$ and any orbit of f has a uniformly bounded number of points which lie outside of U. The details are left to the reader.

Now, we return to constructing the example on $\mathbb{T}^3$. The example has a linear stable bundle, so we first consider dynamics in dimension two. Define $\lambda = \frac{1}{2}(1 + \sqrt{5})$ and functions

$$\psi : \mathbb{R} \to \mathbb{R},\ x \mapsto x + \tfrac{2}{3}\sin x \quad \text{and} \quad g : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (\psi(x), \lambda y + \cos x).$$

The derivative of g is

$$Dg = \begin{pmatrix} \psi'(x) & 0 \\ -\sin x & \lambda \end{pmatrix}.$$

On the vertical line $x = 0$, there is an expanding fixed point for g. Through this point is an invariant one-dimensional unstable manifold associated to the larger eigenvalue of Dg. One can show that this unstable manifold may be expressed as the graph of a function $u : (-\pi, \pi) \to \mathbb{R}$. For now, only consider u on $[0, \pi)$. By an invariant cone argument, one can show that $u'(x) < 0$ for all $x \in (0, \pi)$. Using that $\psi'(x) < \lambda$ when x is close to π and that

$$\frac{|\lambda t - \sin x|}{|\psi'(x)|} > \frac{\lambda}{|\psi'(x)|}|t| > |t|,$$

for $t < 0$, one can show that $\lim_{x \nearrow \pi} u'(x) = -\infty$.

Define a foliation W^u on $[0, \pi) \times \mathbb{R}$ by all graphs of functions of the form $x \mapsto u(x) + b$ for $b \in \mathbb{R}$. This foliation is g-invariant. Reflecting about the y-axis, extend this to a foliation on $(-\pi, \pi) \times \mathbb{R}$. By including the vertical lines on the boundary, extend this foliation to $[-\pi, \pi] \times \mathbb{R}$ and then, by 2π-periodicity in x, to all of $\mathbb{R}^2$. Call this foliation W^u and let E^u be the C^0 line field tangent to it.

Now consider the hyperbolic fixed point of g on the line $x = \pi$. Part of the stable manifold of this point is given by the graph of a function $c : (0, \pi] \to \mathbb{R}$. One can show that $c'(x) > 0$ for all $x \in (0, \pi)$ and, since $\psi'(0) > \lambda$, that $\lim_{x\searrow 0} c'(x) = +\infty$. From the definition of g, there is a constant $C > 1$ such that g^{-1} maps the region $[-C, C] \times [0, \pi]$ into itself. The stable manifold given by graph(c) must therefore be contained in this region, showing that c is a bounded function and can be continuously extended to all of $[0, \pi]$. By reflection and periodicity, further extend c to a continuous function $\mathbb{R} \to \mathbb{R}$ which is differentiable except at $2\pi\mathbb{Z}$ and such that $g(\text{graph}(c)) = \text{graph}(c)$. By considering translates, $x \mapsto c(x) + b$, define a foliation W^c on $\mathbb{R}^2$ and let E^c be the unique continuous line field on $\mathbb{R}^2$ which is tangent to W^c on $(\mathbb{R} \setminus 2\pi\mathbb{Z}) \times \mathbb{R}$. As $u' < 0 < c'$ on $(0, \pi)$, E^u and E^c are transverse.

The matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda = \frac{1}{2}(1 + \sqrt{5})$ and $-\lambda^{-1}$. Therefore, there is a lattice $\Lambda \subset \mathbb{Z}^2$ such that $(y, z) \mapsto (\lambda y, -\lambda^{-1} z)$ quotients to an Anosov diffeomorphism on the 2-torus $\mathbb{R}^2/\Lambda$. Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(x, y, z) = \big(x + \tfrac{2}{3} \sin x,\ \lambda y + \cos x,\ -\lambda^{-1} z\big)$$

and a splitting $E^c \oplus E^u \oplus E^s$ by $E^s = \frac{\partial}{\partial z}$ and where $E^c \oplus E^u$ on each xy-plane is given by the earlier splitting constructed for g. This splitting is f-invariant and there is a foliation tangent to $E^u \oplus E^s$. Define $M = (\mathbb{R} \times \mathbb{R}^2)/(2\pi\mathbb{Z} \times \Lambda)$. Both f and the splitting descend to M. Here, $NW(f) \subset M$ consists of two tori, one tangent to $E^c \oplus E^s$ and the other tangent to $E^u \oplus E^s$. Using (16.1), one can verify that f is partially hyperbolic. It has a foliation tangent to $E^u \oplus E^s$ with one compact leaf and all other leaves are planes.

This is not an example of an AB-system as there is no invariant foliation tangent to E^c. In the above analysis, the crucial properties needed for the term $\cos x$ in the formula $\lambda y + \cos x$ for the second coordinate of g were that $\cos' < 0$ on $(0, \pi)$ and $\cos'(\pi) \le 0 = \cos'(0)$. Therefore, replace $\lambda y + \cos x$ by $\lambda y + \sin x - x$ in all of the above analysis. As $\sin x - x$ is an odd function, the resulting function $c : \mathbb{R} \to \mathbb{R}$ is odd and its graph is a C^1 submanifold in $\mathbb{R}^2$. Defining $f : \mathbb{R}^3 \to \mathbb{R}^3$ now by

$$f(x, y, z) = \big(x + \tfrac{2}{3} \sin x,\ \lambda y + \sin x - x,\ -\lambda^{-1} z\big)$$

and quotienting by the lattice in $\mathbb{R}^3$ generated by $\{0\} \times \Lambda$ and $(2\pi, \frac{2\pi}{\lambda - 1}, 0)$ one constructs a skew product on $\mathbb{T}^3$ having a foliation tangent to $E^u \oplus E^s$ with exactly one compact leaf.

A. Definitions

This appendix defines a number of notions in smooth dynamical theory.

All manifolds considered in this paper are Riemannian manifolds without boundary. Suppose f is a C^1 diffeomorphism on a compact manifold and there is a Tf-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ of the tangent bundle and $k \geq 1$ such that $\|Tf^k v^s\| < 1 < \|Tf^k v^u\|$ for all unit vectors $v^s \in E^s$ and $v^u \in E^u$. If E^c is the zero bundle, then f is an *Anosov* diffeomorphism. If E^u, E^c, and E^s are all non-zero and $\|Tf^k v^s\| < \|Tf^k v^c\| < \|Tf^k v^u\|$ for all $p \in M$ and unit vectors $v^s \in E^s_p$, $v^c \in E^c_p$, and $v^u \in E^u_p$ then f is a *partially hyperbolic* diffeomorphism. The notion of partially hyperbolicity is also extended to certain non-compact manifolds in Section 7.

A C^1 flow is an *Anosov* flow if its time-one map is a partially hyperbolic diffeomorphism with a center bundle given by the direction of the flow.

A partially hyperbolic diffeomorphism f is *dynamically coherent* if there are invariant foliations W^{cu} and W^{cs} tangent to $E^c \oplus E^u$ and $E^c \oplus E^s$. As a consequence, there is also an invariant center foliation W^c tangent to E^c. Global product structure is defined in Section 5.

For homeomorphisms $f : X \to X$ and $g : Y \to Y$, a *topological semiconjugacy* is a continuous surjection $h : X \to Y$ such that $hf = gh$. If h is a homeomorphism, it is a *topological conjugacy*.

Partially hyperbolic diffeomorphisms f and g are *leaf conjugate* if they are dynamically coherent and there is a homeomorphism h such that for every center leaf L of f, $h(L)$ is a center leaf of g and $hf(L) = gh(L)$.

A homeomorphism $f : M \to M$ is *(topologically) transitive* if every non-empty open f-invariant subset of M is dense in M.

For a homeomorphism $f : M \to M$, a Borel measure μ is *invariant* if $\mu(X) = \mu(f(X))$ for every measurable set $X \subset M$. The pair (f, μ) is *ergodic* if μ is f-invariant and either $\mu(X) = 0$ or $\mu(X) = 1$ for every f-invariant measurable $X \subset M$. We often write that f is ergodic or μ is ergodic if the context is clear. For brevity, we sometimes say that a system f with a finite non-probability measure μ is ergodic when, to be precise, we should actually say that the pair $(f, \frac{1}{\mu(M)}\mu)$ is ergodic. A homeomorphism f is *conservative* if it has an invariant measure given by a smooth volume form on M. A conservative C^2 diffeomorphism is *stably ergodic* if it has a neighbourhood $\mathcal{U}$ in the C^1 topology of C^1 diffeomorphisms such that every conservative C^2 diffeomorphism in $\mathcal{U}$ is also ergodic. For a discussion of why the quirky combination of C^1 and C^2 regularity is necessary, see [46].

If $\tilde{N}$ is a simply connected nilpotent Lie group and Γ is a discrete subgroup such that $N := \tilde{N}/\Gamma$ is a compact manifold, then N is called a (compact) *nilmanifold* [28]. If $\tilde{A} : \tilde{N} \to \tilde{N}$ is a Lie group automorphism which descends to $A : N \to N$, then A is a *nilmanifold automorphism* (also called a toral automorphism when $N = \mathbb{T}^d$). If A is Anosov, it is called *hyperbolic*. Infranilmanifolds and their automorphisms are defined in Section 14.

If $f : M \to N$ is a continuous function and $\pi_M : \hat{M} \to M$ and $\pi_N : \hat{N} \to N$ are covering maps, then a *lift* of f is a function $\hat{f} : \hat{M} \to \hat{N}$ such that $\pi_N \hat{f} = f \pi_M$. Note that if π_M and π_N are universal covering maps, then at least one such lift exists, but is not unique in general.

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