A generalization of the Oort conjecture

Andrew Obus*

Abstract. The Oort conjecture (now a theorem of Obus–Wewers and Pop) states that if k is an algebraically closed field of characteristic p, then any cyclic branched cover of smooth projective k-curves lifts to characteristic zero. This is equivalent to the local Oort conjecture, which states that all cyclic extensions of $k[\![t]\!]$ lift to characteristic zero. We generalize the local Oort conjecture to the case of Galois extensions with cyclic p-Sylow subgroups, reduce the conjecture to a pure characteristic p statement, and prove it in several cases. In particular, we show that D_9 is a so-called *local Oort group*.

1. Introduction

This paper concerns the *local lifting problem* about lifting Galois extensions of power series rings from characteristic p to characteristic zero. In particular, in Conjecture 1.9 we state a generalization of the *Oort conjecture* on lifting of cyclic extensions, now a theorem of Obus–Wewers [24] and Pop [28]. Our main result reduces the generalized conjecture to an easy-to-understand, pure characteristic p assertion about existence of certain meromorphic differential forms on $\mathbb{P}^1$. We prove this assertion in several cases, exhibiting the first positive cases of the local lifting problem for a nonabelian group with cyclic p-Sylow subgroup of order greater than p. In particular, we show that D_9 is a so-called *local Oort group*, and we completely solve the "inverse Galois problem" for the local lifting problem for groups with cyclic p-Sylow subgroups. See §1.5 for specifics.

1.1. The local lifting problem. For our purposes, a finite extension B/A of rings is called Γ-*Galois* (or a Γ-*extension*) if A and B are integrally closed integral domains and $\operatorname{Frac}(B)/\operatorname{Frac}(A)$ is Γ-Galois.

Problem 1.1 (The local lifting problem). Let k be an algebraically closed field of characteristic p and Γ a finite group. Let $k[\![z]\!]/k[\![s]\!]$ be a Γ-Galois extension. Does this extension lift to characteristic zero? That is, does there exist a DVR R of characteristic zero with residue field k and a Γ-Galois extension $R[\![Z]\!]/R[\![S]\!]$ that reduces to $k[\![z]\!]/k[\![s]\!]$? In other words, does the Γ-action on $R[\![Z]\!]$ reduce to that on $k[\![z]\!]$, if we assume that Z reduces to z?

We will refer to a Γ-Galois extension $k[\![z]\!]/k[\![s]\!]$ as a *local Γ-extension*.

Remark 1.2. Suppose k is an algebraically closed field, and B/A is any Γ-Galois extension of k-algebras with the Galois group acting by k-automorphisms. Then, if *either* B or A is isomorphic to a power series ring in one variable over k, the other is as well. That is, B/A is a local Γ-extension.

Remark 1.3. Basic ramification theory shows that any group Γ that occurs as the Galois group of a local extension is of the form $P \rtimes \mathbb{Z}/m$, with P a p-group and $p \nmid m$.

The main motivation for the local lifting problem is the following *global lifting problem*, about deformation of curves with an action of a finite group (or equivalently, deformation of Galois branched covers of curves).

Problem 1.4 (The global lifting problem). Let X/k be a smooth, connected, projective curve over an algebraically closed field of characteristic p. Suppose a finite group Γ acts on X. Does (X, Γ) lift to characteristic zero? That is, does there exist a DVR R of characteristic zero with residue field k and a relative projective curve X_R/R with Γ-action such that X_R, along with its Γ-action, reduces to X?

It is a major result of Grothendieck [1, XIII, Corollaire 2.12] that the global lifting problem can be solved whenever Γ acts with tame (prime-to-p) inertia groups, and R can be taken to be the Witt ring $W(k)$. In particular, it holds when Γ is trivial. The wild case is much more subtle, and cannot always be solved. For instance, the group $\mathbb{Z}/p \times \mathbb{Z}/p$ acts faithfully on $\mathbb{P}^1_k$ whenever k is algebraically closed of characteristic p, but there can be no lifting of this action to a genus zero curve when p is odd. However, the *local-global principle* states that the global lifting problem holds for (X, Γ) (and a complete DVR R) if and only if the local lifting problem holds (over R) for each point of X with nontrivial stabilizer in Γ. Specifically, if x is such a point, then its complete local ring is isomorphic to $k[\![z]\!]$. The stabilizer $I_x \subseteq \Gamma$ acts on $k[\![z]\!]$ by k-automorphisms, and we check the local lifting problem for the local I_x-extension $k[\![z]\!]/k[\![z]\!]^{I_x}$. Thus, the global lifting problem is reduced to the local lifting problem.

A proof of the local-global principle for abelian Γ is already implicit in [31]. Proofs for arbitrary Γ have been given by Bertin and Mézard [4], Green and Matignon [16], and Garuti [13].

The author's paper [21] is a detailed exposition of many aspects of the local lifting problem.

1.2. Local Oort groups and the KGB obstruction. The *Oort conjecture* (as mentioned above, now a theorem), states that the local lifting problem holds for all *cyclic* extensions. In [11], Chinburg, Guralnick, and Harbater ask which finite groups Γ of the form $P \rtimes \mathbb{Z}/m$, with P a p-group and $p \nmid m$, have this same property. That is, given a prime p, for which groups Γ is it true that all local Γ-actions (over all algebraically closed fields of characteristic p) lift to characteristic zero? Such a group is called a *local Oort group* (for p). The paper [11] also investigates the notion of a *weak local Oort group* (for p), which is a group Γ for which there exists *at least one* local Γ-extension that lifts to characteristic zero. Characterizing the weak local Oort groups has been referred to as the "inverse Galois problem" for the local lifting problem in [20], where it was proved that elementary abelian p-groups are weak local Oort.

In [11], Chinburg, Guralnick, and Harbater introduce the so-called *KGB obstruction* to local lifting (this is related to the earlier *Bertin obstruction* from [3]). Roughly, given a local Γ-extension, the KGB obstruction vanishes if there is a Γ-extension of certain characteristic zero power series rings for which the different behaves in the same way as for the original local Γ-extension. A lift to characteristic zero gives such an extension, and thus causes the KGB obstruction to vanish. Using this obstruction, Chinburg, Guralnick, and Harbater were able to greatly restrict the possible local Oort groups.

Theorem 1.5 ([11, Theorem 1.2]). *If a group Γ is a local Oort group for p, then Γ is either cyclic, dihedral of order $2p^n$, the alternating group A_4 ($p = 2$), or a generalized quaternion group ($p = 2$).*

In fact, Brewis and Wewers [9] showed that the generalized quaternion groups are not local Oort, so the list of possible local Oort groups consists only of the cyclic groups, D_{p^n}, and A_4. The cyclic case is the Oort conjecture, and the A_4 case has been claimed by Bouw (see [6]) and written up by the author [22]. Thus only the D_{p^n} have unknown "local Oort status," and showing that the local lifting problem holds for these dihedral groups has been referred to as the "strong Oort conjecture" [10]. We propose a somewhat different generalization (Conjecture 1.9) below.

1.2.1. Cyclic p-Sylow groups. Recall that a Γ-extension $L_n/k[\![s]\!]$ gives rise to a higher ramification filtration $\Gamma^i_{i \geq 0}$ for the upper numbering on the group Γ [32, IV]. If $\Gamma = \mathbb{Z}/p^n$, then the breaks in this filtration (i.e. the values i for which $\Gamma^i \supsetneq \Gamma^j$ for all $j > i$) will be denoted by $(u_1, u_2, \ldots, u_n)$. One knows that $u_i \in \mathbb{N}$ and

$$u_i \geq pu_{i-1},$$

for $i = 2, \ldots, n$ (see e.g. [14]).

The higher ramification filtration gives us all the data we need to check the KGB obstruction in the case where Γ has a cyclic p-Sylow subgroup.

Proposition 1.6 ([21, Proposition 5.9]). *Let Γ be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with $p \nmid m$. Suppose Γ is not cyclic (thus not abelian). Let $L_n/k[\![s]\!]$ be a Γ-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u_1, \ldots, u_n)$. Then the KGB obstruction vanishes for $L_n/k[\![s]\!]$ if and only if $u_1 \equiv -1 \pmod{m}$.*

Remark 1.7. By [23, Theorem 1.1], knowing that $u_1 \equiv -1 \pmod{m}$ and Γ is non-abelian implies that Γ is center-free (in particular, $m \mid (p-1)$) and $u_i \equiv -1 \pmod{m}$ for all i.

Remark 1.8. One can also phrase the KGB obstruction in terms of the higher ramification breaks for the lower numbering. In this case, the criterion for vanishing is the same — that the first break for the lower numbering is congruent to $-1 \pmod{m}$.

Our generalization of the Oort conjecture is the following:

Conjecture 1.9. *For local Γ-extensions where Γ has a cyclic p-Sylow subgroup (that is, Γ is of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$), the KGB obstruction is the only obstruction to lifting.*

Remark 1.10. Note that, if $\Gamma = D_{p^n}$ with p an *odd* prime, then all u_i as above are odd (see, e.g. [23, Theorem 1.1]). Thus Conjecture 1.9 (combined with Proposition 1.6) implies that Γ is a local Oort group for p. So for p odd, Conjecture 1.9 is somewhat stronger than the "strong Oort conjecture" mentioned above. However, Conjecture 1.9 says nothing about D_{2^n}. We currently have no opinion as to whether D_{2^n} is a local Oort group. The only results toward this end are that $D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is a local Oort group ([26], or [27] for a special case) and D_4 is a weak local Oort group [8].

1.3. Some history leading to Conjecture 1.9.

The first major result on the local lifting problem was the 1989 paper [31] of Sekiguchi–Oort–Suwa, which showed that $\mathbb{Z}/pm$ is a local Oort group when $p \nmid m$. That $\mathbb{Z}/p^2m$ is a local Oort group was proven in 1998 by Green–Matignon [16]. The full Oort conjecture was proven in 2014 by Obus–Wewers and Pop [24,28].

The local lifting problem for Γ is much more difficult when Γ is non-abelian, even if we assume its p-Sylow subgroup is cyclic. Indeed, it was not until a 2006 paper that D_p (for odd p) was proved to be local Oort by Bouw–Wewers [6], and this proof is significantly more intricate than the $\mathbb{Z}/p$ case. In fact, it was proven in the two papers [6] and [7] that Conjecture 1.9 holds when p *exactly divides the order of* Γ. However, other than this, up until this paper, there was essentially nothing known when Γ is non-abelian. Namely, if the p-Sylow subgroup of Γ is cyclic of order greater than p and Γ is non-abelian, then there was no local Γ-extension with vanishing KGB obstruction that was known *either to lift or not to lift* to characteristic zero. In particular, it was not known if such Γ were weak local Oort groups. We show that they in fact are (Corollary 1.20). Furthermore, our main result (Theorem 1.14) brings the full solution to the local lifting problem for such Γ within reach.

1.4. The (isolated) differential data criterion. While we are not yet able to present a full proof of Conjecture 1.9, we are able to prove it conditionally on certain meromorphic differential forms on $\mathbb{P}^1_k$ existing with special properties. We describe this condition briefly now (for more details, see §7.2).

Consider quadruples $(p, m, \tilde{u}, N_1)$ of natural numbers where:

- p is a prime number.

- $m > 1$ divides $p - 1$.

- $\tilde{u} \equiv -1 \pmod{m}$.

- N_1 is divisible by m.

Write $\tilde{u} = up^\nu$ with u prime to p. Let k be an algebraically closed field of characteristic p. We say that $(p, m, \tilde{u}, N_1)$ *satisfies the differential data criterion* (with respect to k) if there exists a polynomial $f(t) \in k[t^m]$ of degree exactly N_1 in t, such that the meromorphic differential form

$$\omega := \frac{dt}{f(t) t^{\tilde{u}+1}} \in \Omega^1_{k(t)/k}$$

satisfies

$$\mathcal{C}(\omega) = \omega + u t^{-\tilde{u}-1} dt.$$

Here $\mathcal{C}$ is the Cartier operator on differential forms. Note that ω has a zero of order $N_1 + \tilde{u} - 1$ at $t = \infty$.

If $(p, m, \tilde{u}, N_1)$ satisfies the differential data criterion with notation as above, the basic properties of the Cartier operator imply that

$$\omega = dg/g - u \sum_{i=0}^{\nu} t^{-up^i - 1} dt,$$

for some $g \in k(t)$, well-defined up to multiplication by pth powers. We say that $(p, m, \tilde{u}, N_1)$ satisfying the differential data criterion *satisfies the isolated differential data criterion* if there are f and ω as above such that no infinitesimal deformation $\tilde{g}$ of g gives rise to a differential form $\tilde{\omega} := d\tilde{g}/\tilde{g} - u \sum_{i=0}^{\nu} t^{-up^i - 1} dt$ having a zero of order at least $N_1 + \tilde{u} - 1$ at $t = \infty$ (as will be seen in §7.2, this is equivalent to invertibility of a "Vandermonde-like" matrix constructed from the roots of f). This is readily seen to be independent of the choice of g, once f is chosen.

1.5. Main results. Throughout this section, $m \in \mathbb{N}$ is not divisible by p. First we adapt an argument of Pop [28] to reduce Conjecture 1.9 to the case where the successive upper jumps do not grow too quickly.

Proposition 1.11. *Let $L'_n / k[\![s]\!]$ be a non-abelian $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u'_1, \ldots, u'_n)$. For $1 \leq i \leq n$,*

define u_i inductively to be the unique integer such that $u_i \equiv u_i' \pmod{mp}$ and $pu_{i-1} \leq u_i < pu_{i-1} + mp$ (by convention, set $u_0 = 0$). If, for every algebraically closed field κ of characteristic p, every Γ-extension $L_n/\kappa[\![s]\!]$ whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u_1, \ldots, u_n)$ lifts to characteristic zero, then so does $L_n'/k[\![s]\!]$.

Thus we need only consider Γ-extensions whose upper ramification breaks satisfy $u_i < pu_{i-1} + mp$. We say that these extensions have *no essential ramification*.

Example 1.12. For instance, if we have a $\mathbb{Z}/5^4 \rtimes \mathbb{Z}/2$-extension with $(u_1', \ldots, u_4') = (11, 79, 433, 2165)$, then we would have $(u_1, \ldots, u_4) = (1, 9, 53, 265)$.

Remark 1.13. Note the similarity between this definition and [28, Rmk./Defn. 3.1(2)]. In fact, if we consider the upper jumps for the entire G-extension, as opposed to just the $\mathbb{Z}/p^n$-part, then our assumption is exactly that of "no essential ramification" from [28]. Indeed, Proposition 1.11 in the abelian case is equivalent to the main result of [28].

Our main result is the following:

Theorem 1.14. *Let $L_n/k[\![s]\!]$ be a non-abelian $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u_1, \ldots, u_n)$. Suppose that $L_n/k[\![s]\!]$ has vanishing KGB obstruction and no essential ramification. Suppose further that for all $1 < i \leq n$, the quadruple $(p, m, u_{i-1}, N_{i,1})$ satisfies the isolated differential data criterion, where $N_{i,1} = (p-1)u_{i-1}$ if $u_i = pu_{i-1}$ and $N_{i,1} = (p-1)u_{i-1} - m$ otherwise. Then the extension $L_n/k[\![s]\!]$ lifts to characteristic zero.*

Remark 1.15. Our lifts correspond to certain covers of the non-archimedian open disk. We discuss the geometry of the branch locus of these covers in §7.9.

Remark 1.16. Proposition 1.11 and Theorem 1.14 reduce Conjecture 1.9 for the group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ (nonabelian) to realizing the isolated differential data criterion for quadruples $(p, m, \tilde{u}, (p-1)\tilde{u})$ and $(p, m, \tilde{u}, (p-1)\tilde{u} - m)$ such that $\tilde{u} \equiv -1 \pmod{m}$, that $p^{n-1} \nmid \tilde{u}$, and that $\tilde{u} < m(p^{n-1} + p^{n-2} + \cdots + p)$. Thus, once the group is fixed, one need only realize the isolated differential data criterion for finitely many quadruples. Our proof of Corollary 1.18 below proceeds by this method.

If one believes, for a particular group $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, that there is a particular finite field $\mathbb{F}_q$ such that the isolated differential data criterion in the above cases can always be realized using a polynomial $f(t) \in \mathbb{F}_q[t]$, then proving Conjecture 1.9 for Γ is reduced to a finite search.

Example 1.17. In order to show that all extensions as in Example 1.12 lift to characteristic 0, we would have to realize the isolated differential data criterion for $(5, 2, 1, 2)$, $(5, 2, 9, 34)$, and $(5, 2, 53, 212)$.

By realizing various instances of the isolated differential data criterion, we are able to prove the following corollaries, which are special cases of Conjecture 1.9.

Corollary 1.18 (Theorem 8.6). *The dihedral group D_9 is a local Oort group for $p = 3$.*

Corollary 1.19 (Theorem 8.7). *If p is an odd prime, and $L/k[\![s]\!]$ is a D_{p^2}-extension whose $\mathbb{Z}/p^2$-subextension has first upper ramification break $u_1 \equiv 1 \pmod{p}$, then $L/k[\![s]\!]$ lifts to characteristic zero.*

Corollary 1.20 (Theorem 8.8). *If $L/k[\![s]\!]$ is a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks congruent to*

$$(m - 1, p(m - 1), \dots, p^{n-1}(m - 1)) \pmod{mp},$$

then $L/k[\![s]\!]$ lifts to characteristic zero. In particular, $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ is a weak local Oort group whenever the conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful.

Remark 1.21. For each non-abelian $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, Corollary 1.20 includes the case with the smallest possible ramification breaks causing the KGB obstruction to vanish (these breaks are in fact $(m - 1, p(m - 1), \dots, p^{n-1}(m - 1))$).

Remark 1.22. By Proposition 1.6 and Remark 1.7, the action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ must be faithful for Γ to be a weak local Oort group (unless Γ is cyclic). Corollary 1.20 says that this condition suffices as well, and thus solves the "inverse Galois problem" for the local lifting problem for groups with cyclic p-Sylow subgroups.

Remark 1.23. The proof of Theorem 1.14 follows the same basic outline as the analogous assertion for cyclic groups in [24]. However, we never invoke the Oort conjecture itself in the proof. To emphasize this point, note that any lift of a local non-abelian $\Gamma := \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension necessarily yields an "equivariant" lift of its unique local $\mathbb{Z}/p^n$-subextension (see §6.2.2). However, *none* of the cyclic lifts constructed in [24] are equivariant, so they cannot possibly occur inside a lift of a local Γ-extension. Thus the lifts from [24] are "useless" for constructing non-abelian lifts as in Theorem 1.14.

1.6. Outline of the paper. In §2, we recall the explicit parameterization of local $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extensions, and the relationship between the parameterization and the higher ramification filtration. In §3, we prove Proposition 1.11, which allows us to consider only extensions with no essential ramification. Then, §4–§7 are devoted to the proof of Theorem 1.14. In §4, we set up the induction on n that we will use (which is essentially the same framework used in [24]), and in §5, we prove the base case $n = 1$. In §6, we recall the language of characters that was used in [24], and adapt it to our new situation of non-abelian groups. The main part of the proof is in §7, and we give a further, more detailed outline in §7.3. We remark that, although the basic idea of the proof is the same as in [24], the execution is quite different and

more complicated. To enhance the flow of the paper and clarify the main argument, we postpone the proofs of two particularly technical results to §9.

In §8, we give some examples of when the isolated differential data criterion (§1.4) is realized, and derive consequences for the local lifting problem.

1.7. Conventions. The letter K will always be a field of characteristic zero that is complete with respect to a discrete valuation $v : K^\times \to \mathbb{Q}$. We assume that the residue field k of K is algebraically closed of characteristic some fixed *odd* prime p. We also assume that the valuation v is normalized such that $v(p) = 1$. We let $|\cdot|$ be an absolute value on K corresponding to v (it does not matter how it is normalized). The ring of integers of K will be denoted R. The maximal ideal of R will be denoted $\mathfrak{m}$. The notation $R\{T\}$ refers to the ring of power series $\sum_{i=0}^{\infty} c_i T^i$ such that $\lim_{i \to \infty} |c_i| = 0$. We write $\mathfrak{m}\{T\}$ to refer to the subset of $R\{T\}$ for which all c_i lie in $\mathfrak{m}$.

We fix an algebraic closure $\bar{K}$ of K, and whenever necessary, we will replace K by a suitable finite extension within $\bar{K}$, without changing the above notation. Furthermore, we fix once and for all a compatible system of elements $p^r \in \bar{K}$ for $r \in \mathbb{Q}$, such that $p^{r_1} p^{r_2} = p^{r_1 + r_2}$. The letter m will always refer to a prime-to-p integer. The symbol ζ_n denotes a primitive nth root of unity. A curve is always (geometrically) connected.

These are the same conventions used in [24].

2. $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extensions in characteristic p

In this section, we recall the cyclic theory of local extensions in characteristic p, and then show how to adapt it to the metacyclic case considered in this paper. Let $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$.

If $L/k[\![s]\!]$ is a Γ-extension, then, after a possible change of variables, we may assume that the subextension corresponding to the normal subgroup $\mathbb{Z}/p^n \subseteq \Gamma$ can be written as $k[\![t]\!]/k[\![s]\!]$, with $t^m = s$. Let $M = \mathrm{Frac}(L)$. Since $\mathrm{Gal}(M/k(\!(t)\!)) \cong \mathbb{Z}/p^n$, Artin–Schreier–Witt theory states that $M/k(\!(t)\!)$ is given by an Artin–Schreier–Witt equation

$$\wp(y_1, \ldots, y_n) = (f_1, \ldots, f_n),$$

where $(f_1, \ldots, f_n)$ lies in the ring $W_n(k((t)))$ of truncated Witt vectors, F is the Frobenius morphism on $W_n(k((t)))$, and $\wp(y) := F(y) - y$ is the Artin–Schreier–Witt isogeny. Then L is the integral closure of $k[\![t]\!]$ in M. Adding a truncated Witt vector of the form $\wp((g_1, \ldots, g_n))$ to $(f_1, \ldots, f_n)$ does not change the extension, and adjusting by such Witt vectors, we may assume that the f_i are polynomials in t^{-1}, all of whose terms have prime-to-p degree (in this case, we say the Witt vector is in *standard form*). If

$$u_i := \max\{\, p^{i-j} \deg_{t^{-1}}(f_j) \mid j = 1, \ldots, i \,\}, \tag{2.1}$$

then the u_i are exactly the breaks in the higher ramification filtration of $M/k((t))$ [14, Theorem 1.1]. From this, one sees that $p \nmid u_1$, that $u_i \geq pu_{i-1}$ for $2 \leq i \leq n$, and that if $p | u_i$, then $u_i = pu_{i-1}$.

Proposition 2.1. *The extension $L/k[\![s]\!]$ is Γ-Galois if and only if the degrees (in t^{-1}) of all terms appearing in the polynomials f_i are in the same congruence class (mod m).*

Proof. This follows from [23, Proposition 4.3]. □

Thus we can, and will think of Γ-Galois extensions $L/k[\![s]\!]$ as corresponding to Witt vectors $(f_1, \ldots, f_n) \in W_n(k((t)))$ such that the f_i are polynomials in t^{-1} with all degrees of all terms of all f_i congruent to each other (mod m). By (2.1), this implies that all u_i belong to this congruence class.

Recall Proposition 1.6, which states that, for non-abelian Γ, the KGB obstruction vanishes for $L/k[\![s]\!]$ if and only if $u_1 \equiv -1 \pmod{m}$. By Remark 1.7, this is true for all i, and this implies that Γ is center-free. For the rest of the paper, we only consider local Γ-extensions of this form.

3. Reduction to the "no essential ramification" case

In this section, we prove Proposition 1.11. Recall that $L'_n/k[\![s]\!]$ is a non-abelian $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u'_1, \ldots, u'_n)$, and u_i is defined inductively to be the unique integer such that $u_i \equiv u'_i \pmod{mp}$, with $u_1 < mp$ and $pu_{i-1} \leq u_i < pu_{i-1} + mp$ for $i > 1$. We may, and do, assume that $L'_n/k[\![s]\!]$ has vanishing KGB obstruction, i.e. that all u_i and u'_i are -1 (mod m). Write $L'_n = k[\![z]\!]$, and write $M = k[\![t]\!] \subseteq L'_n$, where $t^m = s$, so that M is the subextension of $L'_n/k[\![s]\!]$ corresponding to $\mathbb{Z}/p^n \subseteq \Gamma$. Our proof follows [28]. The key is to make a deformation in characteristic p so that the generic fiber has no non-abelian essential ramification, in some sense (cf. [28, Key Lemma 3.2]).

Proposition 3.1 (Generalized characteristic p Oort conjecture). *Let $\mathcal{A} = k[\![\varpi, s]\!] \supseteq k[\![s]\!]$, and let $\mathcal{K} = \mathrm{Frac}(\mathcal{A})$. There exists a Γ-extension $\mathcal{L}/\mathcal{K}$, with $\mathcal{L} \supseteq L'_n$, having the following properties:*

(i) *The $\mathbb{Z}/m$-subextension $\mathcal{M}/\mathcal{K}$ corresponding to the subgroup $\mathbb{Z}/p^n \subseteq \Gamma$ is given by $\mathcal{M} = \mathcal{K}[t] \subseteq \mathcal{L}$.*

(ii) *If $\mathcal{B}$ is the integral closure of $\mathcal{A}$ in $\mathcal{L}$, we have $\mathcal{B} \cong k[\![\varpi, z]\!]$. In particular, $(\mathcal{B}/(\varpi))/(\mathcal{A}/(\varpi))$ is Γ-isomorphic to the original extension $L'_n/k[\![s]\!]$.*

(iii) *Let $\mathcal{C} = \mathcal{A}[t] \subseteq \mathcal{M}$. Let $\mathcal{R} = \mathcal{A}[\varpi^{-1}]$, let $\mathcal{S} = \mathcal{B}[\varpi^{-1}]$, and let $\mathcal{T} = \mathcal{C}[\varpi^{-1}]$. Then $\mathcal{S}/\mathcal{T}$ is a $\mathbb{Z}/p^n$-extension of Dedekind rings, branched at $m+1$ maximal ideals. Above the ideal (t), the inertia group is $\mathbb{Z}/p^n$, and the upper jumps are $(u_1, \ldots, u_n)$. The other m branched ideals are of the form $(\zeta_m^\alpha t - \mu)$, where μ can be chosen arbitrarily in $\varpi^{p^{\delta_0}} k[\![\varpi^{p^{\delta_0}}]\!]$ for some high enough δ_0, and α ranges from 1 to m.*

(iv) *The only branched ideal of $\mathcal{S}/\mathcal{R}$ with noncyclic inertia group is (s).*

Proof. As in [28], we will prove Proposition 3.1 by deforming a standard form classifying Witt vector $(f_1, \ldots, f_n)$ of the extension $L'_n/k[\![s]\!]$. We must take care to do everything equivariantly. By Proposition 2.1, each f_i can be written as $t^{1-m} g_i(t^{-m})$, where g_i is a polynomial of degree $\le (u'_i + 1 - m)/m$ over k. Equality holds if $p \nmid u'_i$. Choose a factoring

$$f_i = t^{1-m} p_i(t^{-m}) q_i(t^{-m}),$$

where

$$\deg p_i \le \frac{u_i + 1 - m}{m} \quad \text{and} \quad \deg q_i \le \frac{u'_i - u_i}{m}.$$

If $u_i = u'_i$, then take $q_i = 1$. Note that if $p \nmid u_i$, then we must have equality in both inequalities above. Factoring, we can write

$$q_i(t^{-m}) = c \prod_{\alpha=1}^m j_i(\zeta_m^\alpha t^{-1}),$$

where $c \in k$ and the j_i are monic polynomials of degree $\deg(q_i)$. Lastly, factor j_i completely to write

$$j_i(t^{-1}) = \prod_{\nu=1}^{\deg q_i} (t^{-1} - r_{\nu,i}).$$

Now, let $\mu \in \varpi k[\![\varpi]\!] \setminus \{0\}$. We lift the Witt vector $(f_1, \ldots, f_n) \in W_n(k(\!(t)\!))$ to a Witt vector $(F_1, \ldots, F_n) \in W_n(\mathcal{M})$. We choose

$$F_i = c t^{1-m} p_i(t^{-m}) \prod_{\alpha=1}^m \prod_{\nu=1}^{\deg q_i} \left((\zeta_m^\alpha t - \mu)^{-1} - r_{\nu,i} \right).$$

Let us make some observations:

- F_i (viewed as an element of $\mathcal{A}_{(\varpi)}$) reduces to f_i modulo ϖ.

- All terms in F_i are of degree $-1 \pmod{m}$ in t^{-1}.

- F_i has a pole of order $\leq u_i$ at $t = 0$, and, for each $\alpha \in \{1, \ldots, m\}$, a pole of order $\leq (u_i' - u_i)/m$ at $t = \zeta_m^{-\alpha}\mu$.

Let $\mathcal{L}/\mathcal{M}$ be the $\mathbb{Z}/p^n$-extension classified by $(F_1, \ldots, F_n)$. By the second observation above and the discussion in §2, this extends to a Γ-extension $\mathcal{L}/\mathcal{K}$, which will be the extension we seek. In order to prove this, we must show that the degree $\delta_{\mathcal{S}/\mathcal{T}}$ of the different of $\mathcal{S}/\mathcal{T}$ is bounded above by the degree $\delta_{L_n'/k[\![t]\!]}$ of the different of $L_n'/k[\![t]\!]$. Then (i), (ii), and (iii) follow exactly as in the proof of [28, Key Lemma 3.2] (in fact, the argument is marginally easier, as our Witt vectors have no constant terms, so there is no need for Pop's notion of "quasi standard form"). And (iv) follows immediately from (iii), since (s) is the only branched ideal of $\mathcal{T}/\mathcal{R}$.

Using Hilbert's different formula ([36, p. 311] or [32, IV, Proposition 4]) and the definition of the upper numbering, we obtain

$$\delta_{L_n'/k[\![t]\!]} = \sum_{i=1}^{n}(u_i' + 1)(p^i - p^{i-1}).$$

For $\delta_{\mathcal{S}/\mathcal{T}}$, we add up the contributions from the different branched ideals separately. For the ideal (t), we consider the extension of complete discrete valuation fields given by tensoring $\mathcal{S}/\mathcal{T}$ with $k(\!(\varpi)\!)(\!(t)\!)$ over $\mathcal{T}$. Let $(P_1, \ldots, P_n)$ be the standard form (relative to (t)) of the Witt vector $(F_1, \ldots, F_n)$ classifying this extension. Then the degree of the pole of P_i at $t = 0$ is bounded by u_i, and the upper jumps are bounded by $(u_1, \ldots, u_n)$. Thus the contribution $\delta_{(t)}$ from the ideal (t) to $\delta_{\mathcal{S}/\mathcal{T}}$ satisfies

$$\delta_{(t)} \leq \sum_{i=1}^{n}(u_i + 1)(p^i - p^{i-1}).$$

For each ideal $(\zeta_m^{\alpha} t - \mu)$, we consider the extension of complete discrete valuation fields given by tensoring $\mathcal{S}/\mathcal{T}$ with $k(\!(\varpi)\!)(\!(\zeta_m^{\alpha} t - \mu)\!)$. Let $(P_{1,\alpha}, \ldots, P_{n,\alpha})$ be the standard form of $(F_1, \ldots, F_n)$ relative to $(\zeta_m^{\alpha} t - \mu)$. Then the degree of the pole of $P_{i,\alpha}$ is bounded above by $(u_i' - u_i)/m$. In fact the inequality is strict, because $u_i' - u_i$ is divisible by p. So the contribution δ_{α} from the ideal $(t - \zeta_m^{\alpha}\mu)$ to $\delta_{\mathcal{S}/\mathcal{T}}$ satisfies

$$\delta_{\alpha} \leq \sum_{i=1}^{n}\left(\frac{u_i' - u_i}{m} - 1 + 1\right)(p^i - p^{i-1}).$$

We conclude:

$$\delta_{\mathscr{S}/\mathscr{T}} = \delta_{(t)} + \sum_{\alpha=1}^{m} \delta_\alpha$$

$$\leq \sum_{i=1}^{n}(u_i + 1)(p^i - p^{i-1}) + m\sum_{i=1}^{n}\left(\frac{u_i' - u_i}{m}\right)(p^i - p^{i-1})$$

$$= \sum_{i=1}^{n}(u_i' + 1)(p^i - p^{i-1}) = \delta_{L_n'/k[\![t]\!]} \qquad\qquad \square$$

We omit the proof of the following proposition, which follows from Proposition 3.1 exactly as [28, Theorem 3.6] follows from [28, Key Lemma 3.2].

Proposition 3.2. *Let $Y \to W$ be a branched Γ-cover of projective smooth k-curves. Suppose that the local inertia at each ramification point with non-abelian inertia group has vanishing KGB obstruction. Set $\mathscr{W} = W \times_k k[\![\varpi]\!]$. Then there is a Γ-cover of projective smooth $k[\![\varpi]\!]$-curves $\mathscr{Y} \to \mathscr{W}$ with special fiber the Γ-cover $Y \to W$ such that the ramification points on the generic fiber $\mathscr{Y}_\eta \to \mathscr{W}_\eta$ with non-cyclic inertia have no essential ramification.*

Proof of Proposition 1.11. Let $Y \to W = \mathbb{P}^1$ be the Harbater–Katz–Gabber cover associated to $L_n'/k[\![s]\!]$ (this is called an *HKG-cover* in [28]). This is a Γ-cover that is étale outside $s = 0, \infty$, tamely ramified above $s = \infty$, and totally ramified above $s = 0$ such that the formal completion of $Y \to W$ at $s = 0$ yields the extension $L_n'/k[\![s]\!]$. Let $\mathscr{Y} \to \mathscr{W}$ be the Γ-cover guaranteed by Proposition 3.2, and let $\mathscr{Y}_\eta \to \mathscr{W}_\eta$ be its generic fiber. Recall that we assume that *every* local Γ-extension $L_n/k[\![s]\!]$ with no essential ramification lifts to characteristic zero. Furthermore, by the (standard) Oort conjecture, every cyclic extension of $k[\![s]\!]$ lifts to characteristic zero. So if we base change $\mathscr{Y}_\eta \to \mathscr{W}_\eta$ to the algebraic closure of $k(\!(\varpi)\!)$, the local-global principle tells us that this cover lifts to characteristic zero. Then, [28, Proposition 4.3] tells us that there is a *rank two* characteristic zero valuation ring $\mathcal{O}$ with residue field k such that the Γ-cover $Y \to W$ has a lift over $\mathcal{O}$. Note that this process works starting with any Γ-extension with upper jumps $(u_1', \dots, u_n')$, and that such extensions can be parameterized by some affine space $\mathbb{A}^N$ (with one coordinate corresponding to each possible coefficient in an entry of a classifying Witt vector in standard form).

To conclude, we remark that [28, Proposition 4.7] and its setup carry through exactly in our situation, with our $\mathbb{A}^N$ playing the role of $\mathbb{A}^{|\ll|}$ in [28]. Indeed, we have that the analog of $\Sigma_\ll$ in that proposition contains all closed points, by the paragraph above. Thus we can in fact lift $Y \to W$ over a *discrete* characteristic zero valuation ring. Applying the easy direction of the local-global principle, we obtain a lift of $L_n'/k[\![s]\!]$. This concludes the proof of Proposition 1.11.

4. The induction process

Let $L_n/k[\![s]\!]$ be a $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension, with $k[\![t]\!]/k[\![s]\!]$ the intermediate $\mathbb{Z}/m$-extension, and assume without loss of generality that $t^m = s$. As in [24], the local-global principle thus shows that solvability of the local lifting problem from $L_n/k[\![s]\!]$ is equivalent to the following claim, which will be more convenient to work with:

Claim 4.1. Given a Γ-Galois extension $L_n/k[\![s]\!]$, then after possibly changing the uniformizer s of $k[\![s]\!]$, there exists a Γ-Galois cover $Y_n \to W := \mathbb{P}^1_K$ (where K is the fraction field of some characteristic zero DVR R with residue field k) with the following properties:

(i) The cover $Y_n \to W$ has good reduction with respect to the standard model $\mathbb{P}^1_R$ of W and reduces to a Γ-Galois cover $\bar{Y}_n \to \bar{W} = \mathbb{P}^1_k$ (with s as coordinate on $\bar{W}$) which is totally ramified above $s = 0$, tamely ramified above $s = \infty$, and étale everywhere else. In other words, $\bar{Y}_n \to \bar{W}$ is the Harbater–Katz–Gabber cover for $L_n/k[\![s]\!]$.

(ii) The completion of $\bar{Y}_n \to \bar{W}$ at $s = 0$ yields $L_n/k[\![s]\!]$.

We write $\bar{Y}_n \to \bar{X}$ (resp. $Y_n \to X$) for the unique $\mathbb{Z}/p^n$-subcover of $\bar{Y}_n \to \bar{W}$ (resp. $Y_n \to W$). Then the quotient covers $\bar{X} \to \bar{W}$ and $X \to W$ are both tamely ramified $\mathbb{Z}/m$-covers of $\mathbb{P}^1$'s, and we choose coordinates T on X and S on W such that T (resp. S) reduces to t (resp. s) on $\bar{X}$ (resp. $\bar{W}$), and such that $X \to W$ identifies S with T^m.

If R is a characteristic zero DVR with residue field k and fraction field K, set $D(r) = \{T \in \bar{K} \mid v(T) > r\}$, where v is the unique valuation on $\bar{K}$ (with value group $\mathbb{Q}$) prolonging the valuation on K. We think of this disk as lying in X.

We prove Theorem 1.14 (in the context of Claim 4.1) by induction using the following base case (Lemma 4.2) and induction step (Theorem 4.3).

Lemma 4.2. *Let $L_1/k[\![s]\!]$ be a $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p$-subextension has upper ramification break u_1. Suppose that $L_1/k[\![s]\!]$ has vanishing KGB obstruction. Then there exists a $\mathbb{Z}/p \rtimes \mathbb{Z}/m$-cover $Y \to W$ satisfying Claim 4.1 for $L_1/k[\![s]\!]$, such that $Y \to X = \mathbb{P}^1$ is étale outside the open disk $D(r_1)$, where $r_1 = 1/u_1(p-1)$.*

Theorem 4.3. *Suppose $n > 1$, and let $L_n/k[\![s]\!]$ be a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension with vanishing KGB obstruction whose $\mathbb{Z}/p^n$-subextension $L_n/k[\![t]\!]$ has upper ramification breaks $(u_1, \ldots, u_n)$. Let $L_{n-1}/k[\![s]\!]$ be the unique $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$-subextension. Suppose there exists a $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$-cover*

$$Y_{n-1} \xrightarrow{\mathbb{Z}/p^{n-1}} X \xrightarrow{\mathbb{Z}/m} W$$

satisfying Claim 4.1 for $L_{n-1}/k[\![s]\!]$, such that $Y_{n-1} \to X$ is étale outside the open disk $D(r_{n-1})$, where $r_{n-1} = 1/u_{n-1}(p-1)$. Assume that (p, m, u_{n-1}, N_1) satisfies

the isolated differential data criterion, where $N_1 = (p-1)u_{n-1}$ if $u_n = pu_{n-1}$, and $N_1 = (p-1)u_{n-1} - m$ otherwise. Lastly, assume $u_n < pu_{n-1} + mp$. Then there is a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-cover $Y_n \to W$ satisfying Claim 4.1 for $L_n/k[\![s]\!]$, such that $Y_n \to X$ is étale outside $D(r_n)$, where $r_n = 1/u_n(p-1)$.

Theorem 1.14 now follows immediately from Lemma 4.2 and Theorem 4.3 by induction. After we prove Lemma 4.2 in the next section, we devote most of the rest of the paper to proving Theorem 4.3.

5. The base case

In this section, we prove Lemma 4.2. Maintain the notation of §4, and assume that we are in the situation of Lemma 4.2. Let $\Gamma = \mathrm{Gal}(L_1/k[\![s]\!])$. By [7, Theorem 2.1], the local lifting problem holds for $L_1/k[\![s]\!]$, so there is a Γ-cover $Y \to W$ satisfying Claim 4.1 as desired (the vanishing of the KGB obstruction is exactly the condition in the theorem in [7]). So we need only check that the branch points of the $\mathbb{Z}/p$-subcover $Y \to X = \mathbb{P}^1$ lie in $D(r_1) = D(1/u_1(p-1))$. We start with a lemma.

Lemma 5.1. *In order to prove Lemma 4.2 for $L_1/k[\![s]\!]$, it suffices to prove it for any Γ-extension $L'/k[\![s]\!]$ with the same ramification break.*

Proof. By [29, Lemma 2.1.2], there is a k-automorphism ϕ of $k[\![s]\!]$ giving rise to an isomorphism from L' to L_1 making the diagram below commute:

$$
\begin{array}{ccc}
L' & \xrightarrow{\;\sim\;} & L_1 \\
\uparrow & & \uparrow \\
k[\![s]\!] & \xrightarrow{\;\phi\;} & k[\![s]\!]
\end{array}
$$

Write $\phi(s) = a_1 s + a_2 s^2 + \cdots$, where all $a_i \in k$ and $a_1 \in k^\times$. Now, say $f : Y_1 \to X \to W$ satisfies Lemma 4.2 for $L'/k[\![s]\!]$ with all branch points of $Y_1 \to X$ lying in $D(r_1)$. Consider the cover $f \times_W \mathrm{Spec}\, R[\![S]\!]$. Let $\Phi \in \mathrm{Aut}(R[\![S]\!])$ be any R-automorphism lifting ϕ. Identifying points of $\mathrm{MaxSpec}\, R[\![S]\!]$ with (Galois orbits of) points of $\bar{K}$ of absolute value < 1, we have that Φ^* preserves absolute values, because $|A_1 S + A_2 S^2 + \cdots| = |S|$ whenever all $A_i \in R$ with $A_1 \in R^\times$ and $|S| < 1$. Thus, the branch points of $\Phi^*(f \times_W \mathrm{Spec}\, R[\![S]\!])$ have the same absolute values as those of $f \times_W \mathrm{Spec}\, R[\![S]\!]$, and $\Phi^*(f \times_W \mathrm{Spec}\, R[\![S]\!])$ is a local lifting for $L_1/k[\![s]\!]$. Clearly, if Φ is extended to $\mathrm{Aut}(R[\![T]\!])$, where $T^m = S$, then Φ^* preserves absolute values as well. Applying the local-global principle gives Lemma 4.2. $\square$

We are reduced to showing that, given $u_1 \equiv -1 \pmod m$ and $p \nmid u_1$, Lemma 4.2 holds for some $L'/k[\![s]\!]$ whose $\mathbb{Z}/p$-subextension $L'/k[\![t]\!]$ has ramification break u_1.

We will freely use the terminology of *Hurwitz trees* for the rest of this section (see [6, §3], especially Definition 3.2), as they are the key to the proof of [7, Theorem 2.1].

In particular, for any possible u_1 (called h in [7] and [6]), a Hurwitz tree is constructed in [6] that gives rise to a lift of some $L'/k[\![s]\!]$ whose $\mathbb{Z}/p$-subextension has ramification break u_1. The valuations of the branch points of the lift (in terms of the coordinate T) can be read off from this Hurwitz tree. This is done in the local context in [6], but the local-global principle allows us to conclude the global result of Lemma 4.2. We split the proof up into the two cases $u_1 < p$ and $u_1 > p$.

If $u_1 < p$, then the Hurwitz tree is irreducible [17]. Thus the underlying combinatorial tree consists of two vertices: a root vertex v_0 and a vertex v_1. The points in the set B of [6, Definition 3.2] all lie on v_1. Since the points in B represent the specializations of branch points of $Y \to X$, the valuation of each of these branch points is equal to p times the thickness ϵ of the edge connecting to v_0 and v_1 (the factor of p comes from [30, Proposition 2.3.2]). Since the conductor of the Hurwitz tree is u_1, we see that $|B| = u_1 + 1$. Since the differential form ω_1 on v_1 has simple poles at the points of B and no other zeroes or poles aside from a zero at the point z corresponding to the unique edge e, this zero has order $u_1 - 1$. Then the definition of Hurwitz tree implies that

$$1 = (p-1)u_1\epsilon, \quad \text{or} \quad \epsilon = 1/u_1(p-1).$$

Since $r_1 < p\epsilon = p/u_1(p-1)$, this case is proved.

If $u_1 > p$, then [6, Theorem 4.3] gives a construction of the appropriate Hurwitz tree when $m = 2$, splitting the construction into two cases. In both cases, the underlying combinatorial tree has a root vertex v_0, a vertex v_1, and several other vertices. Furthermore, in both cases, the different δ_{v_1} can be any rational number in $(0, 1)$. Again, the valuation of each of the branch points is $p\epsilon$, where ϵ is the thickness of the edge connecting v_0 to v_1. The definition of Hurwitz tree implies that

$$\delta_{v_1} = (p-1)u_1\epsilon,$$

and taking $\delta_{v_1} > 1/p$ ensures that $p\epsilon > r_1$. As is mentioned in [7, Proof of Theorem 2.1], this can be generalized easily to the case $m > 2$. One has the same freedom for δ_{v_1} and ϵ in this case. This completes the proof of Lemma 4.2.

Remark 5.2. The global context of §4 was simply an encumbrance in this section, but it will be helpful later on.

6. Characters and Swan conductors

In this section we recall the tools of *characters* and *Swan conductors* from [24, §5]. Characters will serve as a substitute for Galois covers, as they are more convenient to manipulate algebraically. We will also relate equivariance of characters to metacyclic extensions.

6.1. Geometric setup. Let $X = \mathbb{P}^1_K$. We write $\mathbb{K} = K(T)$ for the function field of X. Fix a smooth R-model X_R of X, corresponding to the coordinate T. We let $\bar{X} := X_R \times_R k$ denote the special fiber of X_R, and we let X^{an} denote the rigid analytic space associated to X. We write 0 for the K-point $T = 0$ and $\bar{0}$ for its specialization to $\bar{X}$.

Let
$$D :=]\,\bar{0}\,[\,_{X_R} \subset X^{\text{an}},$$

be the open unit disk around 0, that is, the set of points of X^{an} specializing to $\bar{0} \in \bar{X}$ [5]. Then $\hat{\mathcal{O}}_{X_R,\bar{0}} = R[\![T]\!]$, and via T, we make an identification

$$D \cong \{\, x \in (\mathbb{A}^1_K)^{\text{an}} \mid v(x) > 0 \,\}.$$

For $r \in \mathbb{Q}_{\geq 0}$ we define

$$D[r] := \{\, x \in D \mid v(x) \geq r \,\}$$

and, as in §4,

$$D(r) := \{\, x \in D \mid v(x) > r \,\}.$$

We have $D(0) = D$. For $r > 0$ the subset $D[r] \subset D$ is an affinoid subdomain. Let $v_r : \mathbb{K}^\times \to \mathbb{Q}$ denote the "Gauss valuation" with respect to $D[r]$. This is a discrete valuation on $\mathbb{K}$ which extends the valuation v on K and has the property $v_r(T) = r$. It corresponds to the supremum norm on the open subset $D[r] \subset X^{\text{an}}$.

Let κ_r denote the residue field of $\mathbb{K}$ with respect to the valuation v_r. For $r = 0$, we have that κ_0 is naturally identified with the function field of $\bar{X}$. After replacing K by a finite extension (which depends on r) we may assume that $p^r \in K$. Then $D[r]$ is isomorphic to a closed unit disk over K with parameter $T_r := p^{-r}T$. Moreover, the residue field κ_r is the function field of the canonical reduction $\bar{D}[r]$ of the affinoid $D[r]$. In fact, $\bar{D}[r]$ is isomorphic to the affine line over k with function field $\kappa_r = k(t)$, where t is the image of T_r in κ_r. We make this identification of t with the reduction of T_r throughout, whenever it is clear which r we are dealing with.

For a closed point $\bar{x} \in \bar{D}[r]$, we let $\text{ord}_{\bar{x}} : \kappa_r^\times \to \mathbb{Z}$ denote the normalized discrete valuation corresponding to the specialization of $\bar{x}$ on $\bar{D}[r]$. We let ord_∞ denote the unique normalized discrete valuation on κ_r corresponding to the "point at infinity."

Notation 6.1. For $F \in \mathbb{K}^\times$ and $r \in \mathbb{Q}_{\geq 0}$, we let $[F]_r$ denote the image of $p^{-v_r(F)}F$ in the residue field κ_r.

6.2. Characters. We fix $n \geq 1$ and assume that K contains a primitive p^nth root of unity ζ_{p^n} (this is true after a finite extension of K). For an arbitrary field L, we set

$$H^1_{p^n}(L) := H^1(L, \mathbb{Z}/p^n\mathbb{Z}).$$

In the case of $\mathbb{K}$, we have

$$H^1_{p^n}(\mathbb{K}) := H^1(\mathbb{K}, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{K}^\times/(\mathbb{K}^\times)^{p^n}$$

(the latter isomorphism depends on the choice of ζ_{p^n}). Elements of $H^1_{p^n}(\mathbb{K})$ are called *characters* on X. Given an element $F \in \mathbb{K}^\times$, we let $\mathfrak{K}_n(F) \in H^1_{p^n}(\mathbb{K})$ denote the character corresponding to the class of F in $\mathbb{K}^\times/(\mathbb{K}^\times)^{p^n}$.

For $i = 1, \ldots, n$ the homomorphism

$$\mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}, \quad a \mapsto p^{n-i}a,$$

induces an injective homomorphism $H^1_{p^i}(\mathbb{K}) \hookrightarrow H^1_{p^n}(\mathbb{K})$. Its image consists of all characters killed by p^i. We consider $H^1_{p^i}(\mathbb{K})$ as a subgroup of $H^1_{p^n}(\mathbb{K})$ via this embedding.

A character $\chi \in H^1_{p^n}(\mathbb{K})$ gives rise to a branched Galois cover $Y \to X$. If $\chi = \mathfrak{K}_n(F)$ for some $F \in \mathbb{K}^\times$, then Y is a connected component of the smooth projective curve given generically by the Kummer equation $y^{p^n} = F$. If χ has order p^i as element of $H^1_{p^n}(\mathbb{K})$, then the Galois group of $Y \to X$ is the unique subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ of order p^i.

A point $x \in X$ is called a *branch point* for the character $\chi \in H^1_{p^n}(\mathbb{K})$ if it is a branch point for the cover $Y \to X$. The *branching index* of x is the order of the inertia group for some point $y \in Y$ above x. The set of all branch points is called the *branch locus* of χ and is denoted by $\mathbb{B}(\chi)$.

Definition 6.2. A character $\chi \in H^1_{p^n}(\mathbb{K})$ is called *admissible* if its branch locus $\mathbb{B}(\chi)$ is contained in the open disk D.

6.2.1. Reduction of characters. Let $\chi \in H^1_{p^n}(\mathbb{K})$ be an admissible character of order p^n, and let $Y \to X$ be the corresponding cyclic Galois cover. Let Y_R be the normalization of X_R in Y. Then Y_R is a normal R-model of Y and we have $X_R = Y_R/(\mathbb{Z}/p^n)$.

After enlarging our ground field K, we may assume that the character χ is *weakly unramified* with respect to the valuation v_0, see [12]. By definition, this means that for all extensions w of v_0 to the function field of Y the ramification index $e(w/v_0)$ is equal to 1. It then follows that the special fiber $\bar{Y} := Y_R \otimes_R k$ is reduced (see e.g. [2, §2.2]).

Definition 6.3. We say that the character χ has *étale reduction* if the map $\bar{Y} \to \bar{X}$ is generically étale. It has *good reduction* if, in addition, $\bar{Y}$ is smooth.

In terms of Galois cohomology the definition can be rephrased as follows. Let $\hat{\mathbb{K}}_0$ be the completion of $\mathbb{K}$ at v_0. The character χ has étale reduction if and only if the image $\chi|_{\hat{\mathbb{K}}_0}$ of χ in $H^1_{p^n}(\hat{\mathbb{K}}_0)$ under the restriction morphism induced by the inclusion

$\mathrm{Gal}_{\hat{\mathbb{K}}_0} \to \mathrm{Gal}_{\mathbb{K}}$ is *unramified*. The word "unramified" means that $\chi|_{\hat{\mathbb{K}}_0}$ lies in the image of the cospecialization morphism

$$H^1_{p^n}(\kappa_0) \to H^1_{p^n}(\hat{\mathbb{K}}_0)$$

(which is simply the restriction morphism induced by the projection $\mathrm{Gal}_{\hat{\mathbb{K}}_0} \to \mathrm{Gal}_{\kappa_0}$). Since the cospecialization morphism is injective, there exists a unique character $\bar{\chi} \in H^1_{p^n}(\kappa_0)$ whose image in $H^1_{p^n}(\hat{\mathbb{K}}_0)$ is $\chi|_{\hat{\mathbb{K}}_0}$. By construction, the Galois cover of $\bar{X}$ corresponding to $\bar{\chi}$ is isomorphic to an irreducible component of the normalization of $\bar{Y}$.

Definition 6.4. If χ has étale reduction, we call $\bar{\chi}$ the *reduction* of χ, and χ a *lift* of $\bar{\chi}$.

Remark 6.5. Assume that χ is an admissible character with good reduction. The condition that χ is admissible implies that the cover $\bar{Y} \to \bar{X}$ corresponding to the reduction $\bar{\chi}$ is étale over $\bar{X} - \{\bar{0}\}$ (the proof uses Purity of Branch Locus, see e.g. [33, Theorem 5.2.13]). Thus we may speak of the ramification breaks of χ, by which we mean the ramification breaks above the point $\bar{0}$.

6.2.2. Equivariant characters. In the context of §6.1, consider a $\mathbb{Z}/m$-action on $\mathbb{K}$ fixing K, given by $\tau(T) = \zeta_m T$ for τ a generator of $\mathbb{Z}/m$. This gives rise to a $\mathbb{Z}/m$-action on X_R, and we set W_R (resp. W, $\bar{W}$) equal to $X_R/\langle\tau\rangle$ (resp. $X/\langle\tau\rangle$, $\bar{X}/\langle\tau\rangle$). The action of $\mathbb{Z}/m$ on $\mathbb{K}$ naturally gives rise to a $\mathbb{Z}/m$-action on $H^1_{p^n}(\mathbb{K}) \cong \mathbb{K}^\times/(\mathbb{K}^\times)^p$. Let $\psi : \mathbb{Z}/m \to \mathrm{Aut}(\mathbb{Z}/p^n)$ be a homomorphism. Any automorphism of $\mathbb{Z}/p^n$ is given by multiplication by an element of $(\mathbb{Z}/p^n)^\times$, and we use this to identify $\mathrm{Aut}(\mathbb{Z}/p^n)$ with $(\mathbb{Z}/p^n)^\times$.

Definition 6.6. A character $\chi \in H^1_{p^n}(\mathbb{K})$ is called ψ-*equivariant* if $\tau(\chi) = \chi^{\psi(\tau^{-1})}$.

Remark 6.7. Since χ is an element of a p^n-torsion group, the expression $\chi^{\psi(\tau^{-1})}$ is well-defined.

Proposition 6.8. *Let* $\Gamma = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ *via the conjugation action* $\psi : \mathbb{Z}/m \to (\mathbb{Z}/p^n)^\times = \mathrm{Aut}(\mathbb{Z}/p^n)$. *The* $\mathbb{Z}/p^n$-*branched cover* $Y \to X$ *given rise to by* χ *extends to a* Γ-*branched cover* $Y \to W$ *if and only if* χ *is a* ψ-*equivariant character.*

Proof. Letting $S = T^m$, proving the proposition is the same as showing that $K(Y)/K(S)$ is a Γ-extension if and only if χ is ψ-equivariant. Say Γ is generated by τ and σ of orders m and p^n, respectively, with $\tau\sigma = \sigma^{\psi(\tau)}\tau$. Since $K(Y)/\mathbb{K}$ is a Kummer extension, there exists a Kummer generator $f \in K(Y)$ such that $\sigma(f) = \zeta_{p^n} f$. Now, χ being ψ-equivariant is equivalent to $\tau(f^{p^n}) = (f^{p^n})^{(\psi(\tau^{-1}))} g^{p^n}$, for some $g \in \mathbb{K}^\times$ (here we abuse notation and think of $\psi(\tau^{-1})$ as any representative of $\psi(\tau^{-1})$ in $\mathbb{Z}$). This is in turn equivalent to the possibility of extending the action of τ from $\mathbb{K}$ to $K(Y)$ via $\tau(f) = f^{\psi(\tau^{-1})} g$. Suppose this is possible. One calculates

$$\tau(\sigma(f)) = \zeta_{p^n} f^{\psi(\tau^{-1})} g = \sigma^{\psi(\tau)}(\tau(f)).$$

Since $|\mathrm{Aut}(K(Y)/K(S))| \leq mp^n$, we see that the automorphism group is in fact generated by τ and σ subject to $\tau\sigma = \sigma^{\psi(\tau)}\tau$. Thus, $K(Y)/K(S)$ is a Γ-extension.

On the other hand, if $K(Y)/K(S)$ is a Γ-extension, then using $\tau\sigma^{\psi(\tau^{-1})} = \sigma\tau$, we have $\tau(\sigma^{\psi(\tau^{-1})}(f)) = \zeta_{p^n}^{\psi(\tau^{-1})}\tau(f) = \sigma(\tau(f))$, so

$$\frac{\sigma(\tau(f))}{\tau(f)} = \zeta_{p^n}^{\psi(\tau^{-1})}.$$

Kummer theory tells us that $\tau(f) = f^{\psi(\tau^{-1})}$ times an element of $\mathbb{K}^\times$, which is exactly what we need to prove. $\qquad\square$

Note that a $\mathbb{Z}/p^n$-cover of $\mathbb{P}^1_k$, unramified outside $\bar{0}$, is uniquely determined by its germ above the branch point (see, e.g. [19]). Thus, with the above notation, and in light of §2, Claim 4.1 may be reformulated as follows.

Claim 6.9 (cf. [24, Conjecture 5.7]). *Let $\bar{\chi} \in H^1_{p^n}(\kappa_0)$ (note $\kappa_0 \cong k(t)$) be a character of order p^n, unramified outside of $\bar{0}$, such that the corresponding Witt vector $(f_1, \ldots, f_n)$ is given by polynomials in t^{-1} with all degrees of all terms congruent (mod m). Then (after replacing K by a finite extension, if necessary) there exists a homomorphism $\psi : \mathbb{Z}/m \to (\mathbb{Z}/p^n)^\times$ and an admissible, ψ-equivariant character $\chi \in H^1_{p^n}(\mathbb{K})$ with good reduction lifting $\bar{\chi}$.*

By abuse of language, we will say that $\bar{\chi}$ *has vanishing KGB obstruction* if the completion at $\bar{0}$ of the composite cover $\bar{Y} \to \bar{X} \to \bar{W}$ has vanishing KGB obstruction, where $\bar{Y} \to \bar{X}$ is the cover corresponding to $\bar{\chi}$ and $\bar{X} \to \bar{W}$ is the quotient morphism from the beginning of §6.2.2.

Remark 6.10. In the case where $\bar{\chi}$ has vanishing KGB obstruction, the corresponding homomorphism ψ will be injective.

The following lemma follows from an easy calculation, and will be useful in §7.1.

Lemma 6.11. *Under the identification $H^1_{p^n}(\mathbb{K}) \cong \mathbb{K}^\times/(\mathbb{K}^\times)^{p^n}$, a character is ψ-equivariant if and only if it can be identified with*

$$\prod_{i=1}^{m} (\tau^i(g))^{\psi(\tau^i)}$$

for some $g \in \mathbb{K}^\times$.

Lemma 6.12. *If $\bar{\chi}$ has vanishing KGB obstruction, $\tau(T) = \zeta_m T$, and $n = 1$, then $\psi(\tau^\ell) = \zeta_m^{-\ell}$ as elements of $(\mathbb{Z}/p)^\times = \mathbb{F}_p^\times$.*

Proof. This follows from [23, Lemma 4.1(iii)], using the fact that $u_1 \equiv -1 \pmod{m}$. In particular, our ζ_m, u_1, and $\psi_1(\tau)$ are the same as ζ^{-1}, j, and α, respectively, in [23]. $\qquad\square$

It will at times be useful to measure how far an element of $\mathbb{K}$ (in a special form) is from giving rise to a ψ-equivariant character of order p. To this end, we make the following definition:

Definition 6.13. Let $r \in \mathbb{Q}_{\geq 0}$. Recall that $T_r = p^{-r}T$. An element

$$F \in \mathbb{K} \cap (1 + T_r^{-1}\mathfrak{m}\{T_r^{-1}\})$$

has *r-discrepancy valuation* $\geq \sigma$ if there exists

$$F' \in \mathbb{K} \cap (1 + T_r^{-1}\mathfrak{m}\{T_r^{-1}\})$$

such that $\mathfrak{K}_1(F')$ is ψ-equivariant and $v_r(F - F') \geq \sigma$. If $\mathfrak{K}_1(F)$ is ψ-equivariant, we may say that the *r*-discrepancy valuation is ∞.

Definition 6.14. Let $r \in \mathbb{Q}_{\geq 0}$. Suppose $F = \sum_{i=0}^{\infty} \alpha_i T_r^{-i} \in R\{T_r^{-1}\} \otimes_R K$. Then we extend the valuation v_r from $\mathbb{K}$ to $R\{T_r^{-1}\} \otimes_R K$ (and any subring) by setting $v_r(F) = \min_i(v(\alpha_i))$. Furthermore, we write $v_r'(F) = \min_{i \in S}(v(\alpha_i))$, where S is the set of indices either divisible by p or congruent to $-1 \pmod{m}$.

Lemma 6.15. *Fix* $r \in \mathbb{Q}_{\geq 0}$. *Let* $F \in \mathbb{K} \cap (1 + T_r^{-1}\mathfrak{m}\{T_r^{-1}\})$ *with* $\mathfrak{K}_1(F)$ *ψ-equivariant. Write* $[F - 1]_r = \sum_{i=1}^{\infty} c_i t^{-i}$, *where t is the reduction of T_r in κ_r. If $v_r(F - 1) < p/(p - 1)$, then $c_i = 0$ unless $p|i$ or $i \equiv -1 \pmod{m}$. If $v_r(F - 1) = p/(p - 1)$ and $c_i = 0$ whenever $p|i$, then $c_i = 0$ unless $i \equiv -1 \pmod{m}$.*

Proof. Suppose $v_r(F - 1) = \gamma \leq p/(p - 1)$, and if equality holds, that $c_i = 0$ for $p|i$. Write

$$F = 1 + \sum_{i=1}^{\infty} a_i T_r^{-i} = 1 + A + B,$$

where A consists exactly of the terms $a_i T_r^{-i}$ such that $v(a_i) = \gamma$, and $v_r(B) > \gamma$. Recall that τ is a generator of $\mathbb{Z}/m$ such that $\tau(T) = \zeta_m T$. By equivariance and Lemma 6.12,

$$\tau(F) \equiv (1 + A + B)^{\zeta_m} \equiv 1 + \zeta_m A + B' \pmod{(\mathbb{K}^{\times})^p},$$

where $v_r(B') > \gamma$. On the other hand, $\tau(F) = 1 + \tau(A) + \tau(B)$.

Assume, for a contradiction, that A has some term $a_i T_r^{-i}$ such that i is neither congruent to $-1 \pmod{m}$ nor to $0 \pmod{p}$. In particular, $\tau(A) \neq \zeta_m A$. We must show that

$$Q := (1 + \zeta_m A + B')/(1 + \tau(A) + \tau(B))$$

is not a pth power in $\mathbb{K}$ (in fact, we will show that it is not even a pth power in $1 + T_r^{-1}\mathfrak{m}[\![T_r^{-1}]\!]$). The power series expansion of Q in T_r^{-1} is of the form $1 + \sum_{i=1}^{\infty} d_i T_r^{-i}$, with $v(d_i) \geq \gamma$ for all i. Since $\tau(A) \neq \zeta_m A$, there exists $i \in \mathbb{N}$

such that $p \nmid i$ and $v(d_i) = \gamma$. If $\gamma = v_r(Q - 1) < p/(p-1)$, then Q can only be a pth power if the d_i such that $v(d_i) = \gamma$ all have $p|i$, giving a contradiction. If $\gamma = p/(p-1)$ and A has no terms of degree divisible by p, then Q can only be a pth power if there is *some* i with $p|i$ such that $v(d_i) = \gamma$, again a contradiction. $\square$

The discrepancy valuation of a power series sheds light on the valuation of its coefficients.

Corollary 6.16. *Let $r \in \mathbb{Q}_{\geq 0}$. Suppose $F \in \mathbb{K} \cap (1 + T_r^{-1}\mathfrak{m}\{T_r^{-1}\})$ has discrepancy valuation $\geq \sigma$. Then $v_r(F - 1) \geq \min(\sigma, \ p/(p-1), \ v'_r(F - 1))$.*

Proof. Pick $F' \in \mathbb{K} \cap (1 + T_r^{-1}\mathfrak{m}\{T_r^{-1}\})$ such that $\mathfrak{K}_1(F')$ is ψ-equivariant and $v_r(F - F') \geq \sigma$. It suffices to prove that $v_r(F' - 1) \geq \min(p/(p-1), v'_r(F' - 1))$. But this follows from Lemma 6.15. $\square$

6.3. Swan conductors.

We recall some properties of the depth and differential Swan conductors of characters. For proofs, see [24, §5]. Let $\chi \in H^1_{p^n}(\mathbb{K})$ be a character. As in [24, §5.3], we define the *depth Swan conductor* $\delta_\chi(r)$, which is a continuous, piecewise linear function

$$\delta_\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}.$$

The kinks in $\delta_\chi(r)$ (i.e. non-differentiable points) occur only at rational values of r. As part of the definition, $\delta_\chi(r) = 0$ if and only if χ is unramified with respect to v_r. If this is the case then the reduction $\bar{\chi}_r \in H^1_{p^n}(\kappa_r)$ is well defined.

Let us now assume that $\delta_\chi(r) > 0$, and that $r \in \mathbb{Q}_{\geq 0}$. Then, again as in [24, §5.3], one defines the *differential Swan conductor* of χ with respect to v_r,

$$\omega_\chi(r) \in \Omega^1_{\kappa_r},$$

which we think of as a meromorphic differential on $\mathbb{P}^1_k$ (perhaps more accurately, on $\bar{D}[r]$). The slopes of δ_χ are determined by the orders of zeroes and poles of ω_χ:

Proposition 6.17 ([24, Corollary 5.11]). *If $r \geq 0$ and $\delta_\chi(r) > 0$, then the left and right derivatives of δ_χ at r are given by $\mathrm{ord}_\infty(\omega_\chi(r)) + 1$ and $-\mathrm{ord}_0(\omega_\chi(r)) - 1$, respectively.*

We now recall how the function δ_χ determines whether χ has good reduction. We fix an admissible character $\chi \in H^1_{p^n}(\mathbb{K})$ of order p^n and let $Y \to X$ denote the corresponding Galois cover.

Proposition 6.18. *Let $\chi \in H^1_{p^n}(\mathbb{K})$ be an admissible character of order p^n. Then the character χ has good reduction with upper ramification breaks $(u_1, \ldots, u_n)$ if and only if $\delta_\chi(0) = 0$, the right slope of δ_χ at 0 is u_n, and*

$$|\{x \in \mathbb{B}(\chi) \mid \text{ramification index of } x \text{ is exactly } p^{n-i+1}\}| = u_i - u_{i-1},$$

where we set $u_0 = -1$.

Proof. By definition, χ has étale reduction if and only if $\delta_\chi(0) = 0$. By [24, Corollary 5.13(i) and Proposition 5.10(i)], χ has good reduction if and only if the right slope of δ_χ at 0 is equal to $|\mathbb{B}(\chi)| - 1$, in which case [24, Remark 5.8(i)] shows that this right slope is u_n (note that [24, Proposition 5.10(i)], is not stated as applying to $r = 0$, but from its proof referencing [34], it is clear that the right slope statement does apply). Now the proposition follows from [24, Corollary 5.13(ii)]. $\square$

Proposition 6.19 ([24, Corollary 5.15]). *Let $\chi \in H^1_{p^n}(\mathbb{K})$ be an admissible character of order p^n, let $r \in \mathbb{Q}_{>0}$, and let $\bar{x}$ be a point on the canonical reduction of $D[r]$. Suppose $\delta_\chi(r) > 0$. Then*

$$\mathrm{ord}_{\bar{x}}(\omega_\chi(r)) \geq -|\mathbb{B}(\chi) \cap U(r, \bar{x})|,$$

where $U(r, \bar{x})$ is the residue class of $\bar{x}$ on the affinoid $D[r]$. Equality holds if χ has good reduction.

The depth and differential Swan conductors behave in the following way under addition of characters:

Proposition 6.20 ([24, Proposition 5.9]). *Let $\chi_1, \chi_2 \in H^1_{p^n}(\mathbb{K})$, and let $\chi_3 = \chi_1\chi_2$. For $i \in \{1, 2, 3\}$ and $r \in \mathbb{Q}_{\geq 0}$, set $\delta_i = \delta_{\chi_i}(r)$. If $\delta_i > 0$ then we set $\omega_i := \omega_{\chi_i}(r)$. If $\delta_i = 0$ then $\bar{\chi}_i \in H^1_{p^n}(\kappa_r)$ denotes the reduction of χ_i with respect to v_r.*

(i) *If $\delta_1 \neq \delta_2$ then $\delta_3 = \max(\delta_1, \delta_2)$. If $\delta_1 > \delta_2$ then $\omega_3 = \omega_1$.*

(ii) *Assume $\delta_1 = \delta_2 > 0$. Then*

$$\omega_1 + \omega_2 \neq 0 \quad \Rightarrow \quad \delta_1 = \delta_2 = \delta_3, \ \omega_3 = \omega_1 + \omega_2$$

and

$$\omega_1 + \omega_2 = 0 \quad \Rightarrow \quad \delta_3 < \delta_1.$$

(iii) *Assume $\delta_1 = \delta_2 = 0$. Then $\delta_3 = 0$ and $\bar{\chi}_3 = \bar{\chi}_1\bar{\chi}_2$.*

Lastly, we relate differential Swan conductors with equivariance.

Lemma 6.21. *Let τ and ψ be as in §6.2.2. If $\chi \in H^1_{p^n}(\mathbb{K})$ is ψ-equivariant and $r \in \mathbb{Q}_{\geq 0}$ such that $\delta_\chi(r) > 0$, then $\omega_{\tau(\chi)}(r) = \psi(\tau^{-1})\omega_\chi(r)$, where $\psi(\tau^{-1})$, by abuse of notation, is identified with its image under the "reduction mod p" map $(\mathbb{Z}/p^n)^\times \to \mathbb{F}_p^\times \subseteq k^\times$.*

Proof. Since $\tau(\chi) = \chi^{\psi(\tau^{-1})}$, this follows from Proposition 6.20(ii). $\square$

6.4. Characters of order p. We will now describe in the special case $n = 1$ how to determine the function δ_χ explicitly in terms of a suitable element $F \in \mathbb{K}^\times$ corresponding to the character $\chi \in H^1_p(\mathbb{K}) \cong \mathbb{K}^\times/(\mathbb{K}^\times)^p$.

Proposition 6.22 (cf. [24, Proposition 5.17]). *Let $F \in \mathbb{K}^\times \backslash (\mathbb{K}^\times)^p$, let $\chi := \mathfrak{K}_1(F) \in H^1_p(\mathbb{K})$, and let $r \in \mathbb{Q}_{\geq 0}$. Suppose that $v_r(F) = 0$, and that $g := [F]_r \notin \kappa_r^p$. Suppose, moreover, that χ is weakly unramified with respect to v_r (which is always the case if K is chosen large enough).*

(i) *We have*

$$\delta_\chi(r) = \frac{p}{p-1} - v_r(F).$$

(ii) *If $\delta_\chi(r) > 0$, then*

$$\omega_\chi(r) = \begin{cases} dg/g & \text{if } \delta_\chi(r) = p/(p-1), \\ dg & \text{if } 0 < \delta_\chi(r) < p/(p-1). \end{cases}$$

If, instead, $\delta_\chi(r) = 0$, then $\bar{\chi}$ corresponds to the Artin–Schreier extension given by the equation $y^p - y = g$.

7. Proof of Theorem 4.3

7.1. Plan of the proof. We continue with the notation of §6. Recall that D is the unit disk in $(\mathbb{A}^1_K)^{\text{an}}$ centered at 0, and $D(r)$ and $D[r]$ are, respectively, the open and closed disks of radius $|p|^r$ centered at 0. We are given a character $\bar{\chi}_n \in H^1_{p^n}(\kappa_0)$ of order exactly p^n, unramified outside $\bar{0}$, with upper ramification breaks $(u_1, u_2, \ldots, u_n)$, corresponding to a non-abelian $\Gamma := \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension as in Claim 6.9. We assume that $\bar{\chi}_n$ has vanishing KGB obstruction (see after Claim 6.9). We further assume that $n \geq 2$. For $1 \leq i \leq n$, set $r_i = 1/u_i(p-1)$. Recall that $p \nmid u_1$, that $u_1 \equiv \cdots \equiv u_n \equiv -1 \pmod{m}$, and that

$$pu_{i-1} \leq u_i < pu_{i-1} + mp,$$

for $i = 1, \ldots, n$, where we set $u_0 = 0$. It is automatic that if the first inequality above is strict then $p \nmid u_i$. For $i = 1, \ldots, n$ we set $\bar{\chi}_i := \bar{\chi}_n^{p^{n-i}} \in H^1_{p^i}(\kappa_0)$. By assumption, for each $1 \leq i < n$, there is a compatible sequence of injective homomorphisms $\psi_i : \mathbb{Z}/m \to (\mathbb{Z}/p^i)^\times$ (i.e. ψ_j reduces to ψ_i for $j \geq i$), and a ψ_i-equivariant character χ_i lifting $\bar{\chi}_i$. We assume that $\mathbb{B}(\chi_{n-1})$ lies in the disk $D(r_{n-1})$. Assume that

$$(p, m, u_{n-1}, N_1)$$

satisfies the isolated differential data criterion (§1.4), where $N_1 = (p-1)u_n$ if $u_n = pu_{n-1}$, or $N_1 = (p-1)u_n - m$ if $u_n > pu_{n-1}$. In order to prove Theorem 4.3 (using Claim 6.9 in place of Claim 4.1), we must show that, for the unique $\psi_n : \mathbb{Z}/m \to (\mathbb{Z}/p^n)^\times$ compatible with the ψ_i, there exists an admissible

ψ_n-equivariant character $\chi_n \in H^1_{p^n}(\mathbb{K})$ with (good) reduction $\bar{\chi}_n$. Furthermore, we must show $\mathbb{B}(\chi_n) \subseteq D(r_n)$. We will construct χ_n such that $\chi_n^p = \chi_{n-1}$.

We may assume that χ_{n-1} corresponds to an extension of $\mathbb{K}$ given by a system of Kummer equations
$$y_i^p = y_{i-1}G_i, \quad i = 1, \ldots, n-1$$
with $y_0 := 1$ and $G_i \in \mathbb{K}$. Any $\chi \in H^1_{p^n}(\mathbb{K})$ such that $\chi^p = \chi_{n-1}$ is given by an additional equation
$$y_n^p = y_{n-1}G. \tag{7.1}$$
Since we must have $\mathbb{B}(\chi) \subseteq D$, we will search for G of the form
$$G = \prod_{j=1}^{N}(1 - z_j T^{-1})^{a_j}, \tag{7.2}$$

where $a_j \in \mathbb{Z}$, $(a_j, p) = 1$, and z_j in the maximal ideal $\mathfrak{m}$ of R are pairwise distinct (the ability to restrict our search to rational functions without worrying about missing anything is one benefit of working in the global context). We will say that the rational function G *gives rise* to the character χ.

Remark 7.1. Note that multiplication of G by an element of $(\mathbb{K}^\times)^p$ does not change χ, so when it is convenient, we will think of G as an element of $\mathbb{K}^\times/(\mathbb{K}^\times)^p$.

Lemma 7.2. *It is possible to replace G_{n-1} by its product with an element of $(\mathbb{K}^\times)^p$ so that $G = 1$ gives rise to a ψ_n-equivariant character χ.*

Proof. Identifying $H^1_{p^n}(\mathbb{K})$ with $\mathbb{K}^\times/(\mathbb{K}^\times)^{p^n}$, we have that choosing $G = 1$ corresponds to a character χ given by
$$A := G_1 G_2^p \cdots G_{n-1}^{p^{n-2}}.$$
Let τ be a generator of $\mathbb{Z}/m$. Since χ_{n-1} is assumed to be ψ_{n-1}-equivariant, Lemma 6.11 tells us that
$$A \equiv \prod_{j=1}^{m}(\tau^j(g))^{\psi_{n-1}(\tau^j)} \pmod{(\mathbb{K}^\times)^{p^{n-1}}}$$
for some $g \in \mathbb{K}^\times$. Since ψ_n is compatible with ψ_{n-1}, we have that
$$A \equiv \prod_{j=1}^{m}(\tau^j(g))^{\psi_n(\tau^j)} B^{p^{n-1}} \pmod{(\mathbb{K}^\times)^{p^n}},$$
for some $B \in \mathbb{K}^\times$. Replacing G_{n-1} by its product with $\prod_{j=1}^{m-1}(\tau^j(B))^{p\psi_n(\tau^j)}$ replaces A by
$$\prod_{j=1}^{m}(\tau^j(gB^{p^{n-1}}))^{\psi_n(\tau^j)},$$
which is ψ_n-equivariant by Lemma 6.11. $\qquad\square$

Note that performing the replacement of Lemma 7.2 does not change the character χ_{n-1}. Thus, for the rest of the paper, we assume that G_{n-1} is chosen in accordance with Lemma 7.2.

Proposition 7.3. *Suppose a generator $\tau \in \mathbb{Z}/m$ sends T to $\zeta_m T$. If G in the form of (7.2) gives rise to χ, then in order for χ to be ψ_n-equivariant, it is necessary and sufficient that G be of the form*

$$G = \prod_{j=1}^{N/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} z_j T^{-1})^{\psi_1(\tau^\ell) a_j} \tag{7.3}$$

after a possible reindexing. Here we are viewing $\psi_1(\tau^\ell)$ as an element of $\mathbb{F}_p^\times$, which makes G a well-defined element of $\mathbb{K}^\times/(\mathbb{K}^\times)^p$.

Proof. Identifying $H_{p^n}^1(\mathbb{K})$ with $\mathbb{K}^\times/(\mathbb{K}^\times)^{p^n}$ via $\mathfrak{K}_n$, we have that

$$\chi = \mathfrak{K}_n(G_1 G_2^p \cdots G_{n-1}^{p^{n-2}} G^{p^{n-1}}).$$

Since $\mathfrak{K}_n(G_1 G_2^p \cdots G_{n-1}^{p^{n-2}})$ is assumed to be ψ_n-equivariant, we have that χ being ψ_n-equivariant is equivalent to $\mathfrak{K}_n(G^{p^{n-1}})$ being ψ_n-equivariant, which is equivalent to $\mathfrak{K}_1(G)$ being ψ_1-equivariant. By Lemma 6.11, this is equivalent to G having the desired form. $\square$

Remark 7.4. We say that $G \in \mathbb{K}^\times$ is "of the form (7.3)" if its residue class as an element of $\mathbb{K}^\times/(\mathbb{K}^\times)^p$ is.

Let us assume that none of the z_i is a branch point of χ_{n-1}. If this is the case, then Proposition 6.18 shows that a necessary condition for good reduction of χ is that $N = u_n - u_{n-1}$. We assume this. Note that $N = |\mathbb{B}(\chi) \backslash \mathbb{B}(\chi_{n-1})|$.

We will try to find a choice G_n for G of the form (7.3) giving rise to a character χ_n whose (good) reduction is $\bar{\chi}_n$. In §7.2, we give some contraints that G_n will have to satisfy. In §7.3, we give our strategy in more detail.

7.2. The critical radius. We continue with the setup of §7.1. In particular, recall that χ_i is a lift of $\bar{\chi}_i$ for $1 \leq i < n$, and χ is the character arising from G. The number $r_{n-1} = 1/u_{n-1}(p-1)$ will be of the utmost importance, and we will refer to it as the *critical radius*, or r_{crit}. From [24, Eq. (15)], we know that

$$\delta_\chi(r_{\text{crit}}) = \frac{p}{p-1}, \tag{7.4}$$

regardless of our choice of G (this is, essentially, why the critical radius is "critical." It is the minimal r such that G does not affect $\delta_\chi(r)$). For this section, we let u be the minimal upper ramification break u_i such that u_{n-1} is a power of p times u.

Thus u is prime to p (see §2, just before Proposition 2.1). Set $\nu = n - 1 - i$, so that $u_{n-1} = up^\nu$.

We start with a proposition that is not necessary for the proof of Theorem 4.3, but it guides some of our choices about how we construct G. Namely, we already know that G is of the form (7.3), and that $N = u_n - u_{n-1}$. Our first decision is how many of the branch points of χ to place at the critical radius. That is, how many of the z_j in (7.3) should have valuation r_{crit}?

Proposition 7.5 ([24, Proposition 6.4]). *If χ has good reduction then the following hold.*

(i) *For all j we have $v(z_j) \le r_{\mathrm{crit}}$.*

(ii) *For i, j with $v(z_i) = v(z_j) = r_{\mathrm{crit}}$ we have $\bar{x}_i \ne \bar{x}_j$ (where $\bar{x}_j$ denotes the reduction of $x_j := z_j p^{-r_{\mathrm{crit}}}$).*

(iii) *Write $N = N_1 + N_2$, where N_1/m is the number of z_j in (7.3) with $v(z_j) = r_{\mathrm{crit}}$. If $u_n = pu_{n-1}$ then $N_1 = u_{n-1}(p-1)$ and $N_2 = 0$. Otherwise, $N_1 < u_{n-1}(p-1)$ and $N_2 > 0$.*

Let $t = [T_{r_{\mathrm{crit}}}]_{r_{\mathrm{crit}}}$. Since $\sum_{\ell=1}^m \psi_1(\tau^\ell) = 0$ in $\mathbb{F}_p$, it follows from (7.3) that, up to reordering the z_j and up to a constant factor that we may eliminate by rescaling t, we have

$$[G]_{r_{\mathrm{crit}}} = g = t^{a_0} \prod_{j=1}^{N_1/m} \prod_{\ell=1}^m (1 - \zeta_m^{-\ell} \bar{x}_j t^{-1})^{\psi_1(\tau^\ell)a_j}, \tag{7.5}$$

where $p | a_0$.

Corollary 7.6. *In the notation of Proposition 7.5, if χ has good reduction and $g = [G]_{r_{\mathrm{crit}}}$, then*

$$\omega_\chi(r_{\mathrm{crit}}) = \frac{dg}{g} - u \sum_{s=0}^\nu t^{-up^s-1} dt = \frac{c\,dt}{t^{u_{n-1}+1} \prod_{j=1}^{N_1/m}(t^m - \bar{x}_j^m)}, \tag{7.6}$$

where c is a nonzero constant. In particular, $\mathrm{ord}_\infty(\omega_\chi(r_{\mathrm{crit}})) = N_1 + u_{n-1} - 1$.

Proof. The first equality follows from [24, Lemma 6.3]. This middle expression shows that $\omega_\chi(r_{\mathrm{crit}})$ has no multiple poles outside of $t = 0$, where there is a pole of order $up^\nu + 1 = u_{n-1} + 1$. Furthermore, Proposition 6.19 shows that $\omega_\chi(r_{\mathrm{crit}})$ has a simple pole at each of the N_1 points $\zeta_m^\ell \bar{x}_j$, no zeroes outside of $t = \infty$, and no other poles outside of $t = 0$. It follows that $\omega_\chi(r_{\mathrm{crit}})$ has the form in the third expression, from which $\mathrm{ord}_\infty(\omega_\chi(r_{\mathrm{crit}}))$ can be read off. $\square$

Remark 7.7. Notice that t^{a_0} from (7.5) disappears in the logarithmic derivative.

Recall (§1.4) that (p, m, u_{n-1}, N_1) *satisfies the differential data criterion* with respect to k if there exists a polynomial $f(t) \in k[t^m]$ of degree exactly N_1 in t, such that the meromorphic differential form

$$\omega := \frac{dt}{f(t)t^{u_{n-1}+1}} \in \Omega^1_{k(t)/k}$$

satisfies $\mathcal{C}(\omega) = \omega + ut^{-u_{n-1}-1}dt$, where u is the prime-to-p part of u_{n-1}. Note that this implies $f(0) \neq 0$, otherwise the order of the pole of ω at $t = 0$ will be too large compared to that of $\mathcal{C}(\omega)$ and $t^{-u_{n-1}-1}dt$. We will suppress k when it is understood.

Proposition 7.8. *Suppose p, m, u_{n-1} are as in this section, and N_1 is as in Proposition 7.5. The following are equivalent:*

(i) *There exists G of the form (7.3) such that $g := [G]_{r_{\text{crit}}}$ satisfies (7.6).*

(ii) *The quadruple (p, m, u_{n-1}, N_1) satisfies the differential data criterion.*

(iii) *There exists a solution to the following system of equations:*

$$\sum_{j=1}^{N_1/m} a_j \bar{x}_j^q = \begin{cases} u/m, & q = u, \\ 0, & \text{otherwise,} \end{cases} \tag{7.7}$$

where the $\bar{x}_j \in k$, the $a_j \in \mathbb{F}_p^\times$, and q ranges over those numbers from 1 to $N_1 + u_{n-1} - 1$ that are congruent to $-1 \pmod{m}$ and not divisible by p.

Proof. We first prove (i) implies (ii). Suppose g is a solution to (7.6). Taking $\omega = \omega_\chi(r_{\text{crit}})$ and $f(t) = c^{-1} \prod_{j=1}^{N_1/m}(t^m - \bar{x}_j^m)$, and noting that the Cartier operator preserves logarithmic differential forms, it is clear that $\mathcal{C}(\omega) = \omega + ut^{-u_{n-1}-1}dt$.

Now we prove (ii) implies (i). Suppose (p, m, u_{n-1}, N_1) satisfies the differential data criterion via a differential form $\omega = dt/f(t)t^{u_{n-1}+1}$. By the properties of the Cartier operator, ω is equal to a logarithmic differential form minus $u \sum_{s=0}^{\nu} t^{-up^s-1}dt$. Since t^{-up^s-1} has trivial residues, the residues of ω are the same as those of a logarithmic differential form. In particular, they lie in $\mathbb{F}_p$ ([7, Lemma 1.5]).

Factor f as $c^{-1} \prod_{j=1}^{N_1/m}(t^m - \bar{x}_j^m)$. Let a_j be the residue of ω at $\bar{x}_j$. An easy calculation shows that the residue of ω at $\zeta_m^{-\ell}\bar{x}_j$ is $\zeta_m^{-\ell}a_j$, which is $\psi_1(\tau^\ell)a_j$, by Lemma 6.12. Since $m|(p-1)$ (see Remark 1.7), all of these residues lie in $\mathbb{F}_p$. Now, take

$$g = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell}\bar{x}_j t^{-1})^{\psi_1(\tau^\ell)a_j}, \tag{7.8}$$

where by abuse of notation we take an arbitrary lift of each $\psi_1(\tau^\ell)a_j$ and consider it as an element of $\mathbb{Z}$. Then dg/g has the same residues at the simple poles $\zeta_m^\ell \bar{x}_j$ as ω. So

$$\beta := dg/g - u \sum_{s=0}^\nu t^{-up^s-1}dt - \omega$$

is a logarithmic differential form with no poles outside of 0. Since a nonzero logarithmic differential form has only simple poles, and at least two of them, we conclude that $\beta = 0$. So g is a solution to (7.6). Let $x_1, \ldots, x_{N_1/m}$ be lifts of $\bar{x}_1, \ldots, \bar{x}_{N_1/m}$ to R. Then we take G to be anything in the form (7.3) such that $z_i = p^{r_{crit}}x_i$ for $1 \leq i \leq N_1/m$ and $v(z_i) < r_{crit}$ for $i > N_1/m$.

Lastly, we prove that (i) and (iii) are equivalent (cf. [24, p. 266]). We identify the choices of the a_j and $\bar{x}_j$ in (i) and (iii). If G is of the form (7.3) with $g = [G]_{r_{crit}}$, then differentiating logarithmically, we obtain

$$\frac{dg}{g} = \sum_{j=1}^{N_1/m} \sum_{\ell=1}^m \frac{\psi_1(\tau^\ell)a_j \zeta_m^{-\ell} \bar{x}_j t^{-2}dt}{1 - \zeta_m^{-\ell}\bar{x}_j t^{-1}}.$$

Since $\psi_1(\tau^\ell) = \zeta_m^{-\ell}$ by Lemma 6.12, we obtain

$$\frac{dg}{g} = \sum_{q=1}^\infty \left(\sum_{j=1}^{N_1/m} \sum_{\ell=1}^m \zeta_m^{(-q-1)\ell} a_j \bar{x}_j^q \right) t^{-q-1}dt. \tag{7.9}$$

Thus all terms in the expansion (7.9) disappear unless $q \equiv -1 \pmod{m}$. In particular, $\omega_\chi(r_{crit}) = dg/g - u \sum_{s=0}^\nu t^{-up^s-1}dt$ has a zero of order at least $N_1 + u_{n-1} - 1$ at ∞ if and only if, for all $q \equiv -1 \pmod{m}$ between 1 and $N_1 + u_{n-1} - 1$ inclusive, we have

$$\sum_{j=1}^{N_1/m} a_j \bar{x}_j^q = \begin{cases} u/m, & q = u, up, \ldots, up^\nu, \\ 0, & \text{otherwise.} \end{cases} \tag{7.10}$$

Now, if an equation in (7.10) holds for q, then it also holds for pq, as replacing q with pq simply raises both the left hand and right hand sides of the equation to the pth power. So $\omega_\chi(r_{crit})$ has a zero of order at least $N_1 + u_{n-1} - 1$ at ∞ if and only if (iii) holds. But $\omega_\chi(r_{crit})$ cannot have a zero of order greater than $N_1 + u_{n-1} - 1$ at ∞, as it has at worst a pole of order $u_{n-1} + 1$ at 0 and N_1 simple poles at the $\zeta_m^\ell \bar{x}_j$. So (iii) is equivalent to $\omega_\chi(r_{crit})$ having a zero of order exactly $N_1 + u_{n-1} - 1$ at ∞, poles in the aforementioned places, and no other zeroes. That is, (iii) is equivalent to (i). $\square$

Remark 7.9. From its proof, it is clear that Proposition 7.8 is not just an existence result. In particular, any f realizing the differential data criterion gives rise to a g satisfying (7.6), which in turn gives rise to a solution to the system (7.7) (the $\bar{x}_j$

in (7.7) are representatives from the μ_m-equivalence classes of roots of $f(t)$). The definition of realizing the differential data criterion (Proposition 7.8(ii)) is easier to state than the criterion in Proposition 7.8(iii), and is also usually easier to work with computationally, but it is the criterion of Proposition 7.8(iii) that we will mostly use in our proofs.

Remark 7.10. One checks that (7.7) is a system of N_1/m equations in N_1/m variables if and only if if $u_{n-1}(p-1) - mp < N_1 \leq u_{n-1}(p-1)$.

Remark 7.11. The choice of the $a_j \in \mathbb{F}_p^\times$ in (7.3) is known as the "type" (cf. [6,7,24]). One of the advantages of phrasing the differential data criterion in terms of the Cartier operator, rather than in terms of the equations (7.7), is that this phrasing is "type independent." That is, one does not have to determine the a_j separately — they fall out automatically as the residues of ω, which is determined solely in terms of the roots of f (which correspond to the $\bar{x}_j$). In the papers mentioned above, one of the difficulties is guessing the correct type in an analogous situation.

Furthermore, since the problem is naturally symmetric in the $\bar{x}_j$, it makes sense to "symmetrize" things by thinking in terms of f instead. The coefficients of f will in general lie in smaller fields than the $\bar{x}_j$.

In §7.4, it will become important not only to be able to satisfy the equivalent criteria of Proposition 7.8, but to do so in an "isolated" fashion, that is, to choose the a_j and $\bar{x}_j$ as in Proposition 7.8(iii) such that no infinitesimal deformation of the $\bar{x}_j$ yields a solution to (7.7). For fixed a_j, the Jacobian matrix of (7.7) at a solution $(\bar{x}_j)_j$ is the $N_1/m \times N_1/m$ matrix

$$\left(q a_j \bar{x}_j^{q-1} \right)_{q,j} \tag{7.11}$$

over k, where j ranges from 1 to N_1/m and q ranges over those numbers from 1 to $N_1 + u_{n-1} - 1$ that are congruent to $-1 \pmod{m}$ and not divisible by p. Thus, in light of Remark 7.9, we make a definition (cf. [24, Assumption 7.2]).

Definition 7.12. Suppose p, m, u_{n-1} are as in this section, and N_1 is as in Proposition 7.5. The quadruple (p, m, u_{n-1}, N_1) satisfies the *isolated differential data criterion* if there is a polynomial $f \in k[t^m]$ realizing the differential data criterion for (p, m, u_{n-1}, N_1) (equivalently, a $g \in k(t)$ satisfying (7.6)) that gives rise to a solution to the system of equations (7.7) for which the matrix (7.11) is invertible over k (or is empty).

Remark 7.13. Dividing by (nonzero) constants, one sees that the isolated differential data criterion holds if the matrix $\left(\bar{x}_j^{q-1} \right)_{q,j}$ (for the same q and j as in (7.11)) is invertible.

Remark 7.14. The differential data criterion is analogous to [24, Assumption 7.1] in the cyclic case, and the isolated differential data criterion is analogous to [24, Assumption 7.2].

Definition 7.15. If $g \in k(t)$ is a solution to (7.6) realizing the isolated differential data criterion, then we define $\mathcal{G}_{\mathrm{crit},g} \subseteq \mathbb{K}$ to be the set of all G of the form (7.3) (but with N_1 replacing N) with $[G]_{r_{\mathrm{crit}}} = g$.

To sum up, we have shown that a G of the form (7.3) can only give rise to a character with good reduction if it lies in $\mathcal{G}_{\mathrm{crit},g}$, for some g solving (7.6).

7.3. Plan of the proof, part II. Maintain the notation of §7.1 and §7.2. Recall that we are searching for G_n of the form (7.3) giving rise to a character χ_n with good reduction $\bar{\chi}_n$. The proposition below follows immediately from Proposition 6.18 and the discussion at the beginning of §7.1.

Proposition 7.16. *If G is of the form (7.3) such that all z_j satisfy $v(z_j) > r_n = 1/u_n(p-1)$, such that $\delta_\chi(0) = 0$, and such that the right slope of δ_χ at 0 is u_n, then G gives rise to a ψ_n-equivariant character χ with good reduction $\bar{\chi}$ having ramification breaks $(u_1, \ldots, u_n)$, and such that $\mathbb{B}(\chi) \subseteq D(r_n)$.*

The argument outlined in the remainder of this section is the most important difference between this paper and [24].

Recall that $pu_{n-1} \leq u_n < pu_{n-1} + mp$ (in fact, since all u_i are congruent to $-1 \pmod{m}$, we have $u_n \leq pu_{n-1} + m(p-1)$). As was mentioned before Proposition 7.5, we must decide how many of the z_j to choose such that $v(z_j) = r_{\mathrm{crit}} = r_{n-1} = 1/(p-1)u_{n-1}$. Recall that there are N/m z_j in total. Let N_1 and N_2 be two multiples of m such that $N_1 + N_2 = N = u_n - u_{n-1}$. If $u_n = pu_{n-1}$, we choose $N_1 = u_n - u_{n-1}$ and $N_2 = 0$. Otherwise, we take some N_1 such that $(p-1)u_{n-1} - mp < N_1 < (p-1)u_{n-1}$. This gives $0 \leq N_2 \leq 2mp - 2m$, with the first equality holding if and only if $u_n = pu_{n-1}$. Note that $N_1 + u_{n-1}$ is divisible by p if and only if $u_n = pu_{n-1}$. We will construct G and a rational number r_{hub} such that N_1/m of the z_j satisfy $v(z_j) = r_{\mathrm{crit}}$ and the other N_2/m of the z_j satisfy $v(z_j) = r_{\mathrm{hub}}$. If $u_n = pu_{n-1}$, we declare $r_{\mathrm{hub}} = 0$. Otherwise, $0 < r_{\mathrm{hub}} < r_{\mathrm{crit}}$ is defined by the following proposition.

Proposition 7.17. *In the notation above, suppose $v(z_j) = r_{\mathrm{crit}}$ for N_1/m of the z_j and $v(z_j) = r_{\mathrm{hub}}$ for N_2/m of the z_j. Suppose further that $N_2 > 0$ and χ has good reduction. Then*

$$r_{\mathrm{hub}} = \frac{1}{N_2} - \frac{N_1}{(p-1)u_{n-1}N_2}.$$

Furthermore, $r_n < r_{\mathrm{hub}} < r_{\mathrm{crit}}$.

Proof. Under the assumptions in the proposition, $\mathbb{B}(\chi)$ has N_2 points with valuation r_{hub} and another N_1 points with valuation r_{crit}. Using Proposition 6.18 along with the fact that χ_{n-1} has good reduction, $\mathbb{B}(\chi)$ has exactly $u_{n-1} + 1$ other points, all of which have valuation greater than r_{crit}. Since χ has good reduction, Propositions 6.17 and 6.19 imply that the right slope of δ_χ is $N_1 + N_2 + u_{n-1} = u_n$

for $0 \leq r < r_{\text{hub}}$ and $N_1 + u_{n-1}$ for $r_{\text{hub}} \leq r < r_{\text{crit}}$. Furthermore, $\delta_\chi(r_{\text{crit}}) = p/(p-1)$ by (7.4), and $\delta_\chi(0) = 0$. Thus we obtain the equation

$$(N_1 + N_2 + u_{n-1})r_{\text{hub}} + (N_1 + u_{n-1})(r_{\text{crit}} - r_{\text{hub}}) = \frac{p}{p-1}.$$

This yields $r_{\text{hub}} = 1/N_2 - N_1/(p-1)u_{n-1}N_2$, proving the first part of the proposition.

Since $(p-1)u_{n-1} - N_1 < u_n - u_{n-1} - N_1 = N_2$, it follows easily that $r_{\text{hub}} < r_{\text{crit}}$. On the other hand, we know

$$N_2 = u_n - u_{n-1} - N_1 \leq (p-1)u_{n-1} - N_1 + m(p-1).$$

So

$$r_{\text{hub}} \geq \frac{(p-1)u_{n-1} - N_1}{(p-1)u_{n-1}((p-1)u_{n-1} - N_1 + m(p-1))}.$$

Now, since $(p-1)u_{n-1} - N_1 \geq m$, we have

$$\frac{(p-1)u_{n-1} - N_1}{(p-1)u_{n-1} - N_1 + m(p-1)} \geq \frac{1}{p}.$$

Putting everything together, we obtain

$$r_{\text{hub}} \geq \frac{1}{(p-1)u_{n-1}p} > \frac{1}{(p-1)u_n} = r_n. \qquad \square$$

From the proof of Proposition 7.17, it is clear that if χ has good reduction, then

$$\delta_\chi(r_{\text{hub}}) = u_n r_{\text{hub}} = \frac{p}{p-1} - (N_1 + u_{n-1})(r_{\text{crit}} - r_{\text{hub}}) =: \delta_{\text{hub}}, \qquad (7.12)$$

regardless of whether $u_n = pu_{n-1}$.

We will work under the running assumption that (p, m, u_{n-1}, N_1) satisfies the isolated differential data criterion. Thus, we let g be a solution to (7.6) realizing the isolated differential data criterion, and we define $\mathcal{G}_{\text{crit},g}$ as in Definition 7.15. Our first step, to which §7.4 is devoted (and which parallels [24] very closely), is to find $G_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ such that G_{crit} gives rise to a character χ_{crit} with $\delta_{\chi_{\text{crit}}}(r_{\text{hub}}) = \delta_{\text{hub}}$.

If $u_n = pu_{n-1}$, then $N_1 = N$, so G_{crit} is already of the form (7.3). In this case, we set $G = G_{\text{crit}}$, from which $\chi = \chi_{\text{crit}}$ satisfies $\delta_\chi(0) = 0$, and the right slope of δ_χ at 0 is u_n. Since G is already in the form (7.3), Proposition 7.16 shows that χ has good reduction $\bar{\chi}$ with upper ramification breaks $(u_1, \ldots, u_n)$. We then show quite easily that G can be replaced by some $G_n \in \mathcal{G}_{\text{crit},g}$ (and thus still of the form (7.3)) such that G_n gives rise to a character χ_n with good reduction $\bar{\chi}_n$ and $\mathbb{B}(\chi_n) \subseteq D(r_n)$. By Proposition 7.3 and the discussion at the beginning of §7.1, this proves Theorem 4.3 (using Claim 6.9 instead of Claim 4.1) when $u_n = pu_{n-1}$.

If $u_n > pu_{n-1}$, our next step (§7.5) is to construct a space $\mathcal{G}_{\text{hub}} \subseteq \mathbb{K}$ consisting of certain functions whose images in $\mathbb{K}^\times / (\mathbb{K}^\times)^p$ have the form

$$\prod_{j=N_1/m+1}^{N/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} z_j T^{-1})^{\psi_1(\tau^\ell) a_j},$$

where $v(z_j) = r_{\text{hub}}$ for all j. To do this, we will need to assume $N_2 \leq mp$. This assumption will always be satisfied if $N_1 = (p-1)u_{n-1} - m$. Now, $\mathcal{G}_{\text{hub}}$ will have the property that if $G_{\text{hub}} \in \mathcal{G}_{\text{hub}}$, then the character χ given rise to by $G_{\text{crit}}G_{\text{hub}}$ satisfies $\delta_\chi(r_{\text{hub}}) = \delta_{\text{hub}}$, and the left slope of δ_χ at r_{hub} is u_n. This puts us on the right track for having the right-slope of δ_χ at 0 be u_n. Furthermore, $G_{\text{crit}}G_{\text{hub}}$ will be of the form (7.3).

In §7.6, in the case $u_n > pu_{n-1}$, we will construct a particular function $G_{\text{hub}} \in \mathcal{G}_{\text{hub}}$ and modify our original choice of $G_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ such that if $G = G_{\text{hub}}G_{\text{crit}}$ gives rise to χ, then $\delta_\chi(0) = 0$, and δ_χ is linear of slope u_n on the interval $[0, r_{\text{hub}}]$. Since G is in the form (7.3), Proposition 7.16 shows that χ has good reduction $\bar{\chi}$ with upper ramification breaks $(u_1, \ldots, u_n)$.

In §7.7, still in the case $u_n > pu_{n-1}$, we replace G with G_n, where G_n is still a product of an element of $\mathcal{G}_{\text{crit},g}$ and one of $\mathcal{G}_{\text{hub}}$ (and thus still of the form (7.3)), such that G_n gives rise to a ψ_n-equivariant character χ_n with good reduction $\bar{\chi}_n$ and $\mathbb{B}(\chi_n) \subseteq D(r_n)$ (recall that having good reduction specifically equal to $\bar{\chi}_n$ is what we seek, whereas §7.6 only gives us *some* good reduction). This is analogous to what happens in the case $u_n = pu_{n-1}$, but a little more difficult. In particular, it is tricky to deal with the coefficient of t^{-u_n} in the last component of the standard form Witt vector corresponding to $\bar{\chi}_n$ (no such issue arises in the $u_n = pu_{n-1}$ case because this coefficient is always zero). The underlying calculations concerning this coefficient are deferred to §7.8.

By Proposition 7.3 and the discussion at the beginning of §7.1, we obtain a proof of Theorem 4.3 (using Claim 6.9 instead of Claim 4.1) in the case $u_n > pu_{n-1}$. In §7.9, we summarize the geometry of the branch locus for the lifts we construct.

Remark 7.18. In [24], the construction in the case $u_n = pu_{n-1}$ is used to obtain the proof of lifting in the case $u_n > pu_{n-1}$. Our technique here is different, in that it proves both cases independently. In fact, our method here can be used to give an alternate proof of [24, Theorem 1.4], and thus (combining with [28]) of the Oort conjecture.

7.4. Controlling δ_χ between r_{crit} and r_{hub}. Maintain the previous notation. In particular, g is a solution to (7.6) realizing the isolated differential data criterion, and $\mathcal{G}_{\text{crit},g}$ is defined as in Definition 7.15.

Recall that any $G \in \mathcal{G}_{\text{crit},g}$ gives rise to a character χ of order p^n lifting χ_{n-1}. by adjoining the equation $y_n^p = y_{n-1}G$. By (7.6) and Proposition 6.17, we know

that the left derivative of δ_χ at r_{crit} is $N_1 + u_{n-1}$. Recall also from (7.4) that $\delta_\chi(r_{\text{crit}}) = p/(p-1)$.

Let $\lambda(G)$ be the minimal λ in the interval $[r_{\text{hub}}, r_{\text{crit}}]$ such that

$$\delta_\chi(r) = p/(p-1) - (N_1 + u_{n-1})(r_{\text{crit}} - r)$$

for all $r \in [\lambda, r_{\text{crit}}]$. In other words, $\lambda(G)$ is the largest element in $[r_{\text{hub}}, r_{\text{crit}}]$ where δ_χ has left slope less than $N_1 + u_{n-1}$ (or is r_{hub} if there is no such point). Since $G \in G_{\text{crit},g}$, we have $\lambda(G) < r_{\text{crit}}$. Note that

$$\delta_\chi(\lambda(G)) = \frac{p}{p-1} - (r_{\text{crit}} - \lambda)(N_1 + u_{n-1}) < \frac{p}{p-1}.$$

Lemma 7.19. *Suppose $G \in \mathcal{G}_{\text{crit},g}$ with $\lambda := \lambda(G) > r_{\text{hub}}$. Identify κ_λ with $k(t)$. Then $\omega_\chi(\lambda)$ can be written in the form*

$$\omega_\chi(\lambda) = \frac{c\, dt}{t^{N_1 + u_{n-1} + 1}} + df, \tag{7.13}$$

where $c \in k^\times$ and $f \in t^{1-m}k[t^{-m}]$ has degree less than $N_1 + u_{n-1}$ in t^{-1}.

Proof. If $u_n = pu_{n-1}$ then the same argument as in the proof of [24, Proposition 6.13] shows that $\omega_\chi(\lambda)$ is as in (7.13) for some $c \in k^\times$ and $f \in \kappa_\lambda$, with f a polynomial in t^{-1} of degree $< N_1 + u_{n-1}$ and without constant term.

If $u_n > pu_{n-1}$, so that $N_1 < (p-1)u_{n-1}$ (see the beginning of §7.3), then $\delta_\chi(\lambda) > p\delta_{\chi_{n-1}}(\lambda) = \lambda pu_{n-1}$ (the equality is due to [24, Lemma 6.1]). Thus $\mathcal{C}(\omega_\chi(\lambda)) = 0$ (also from [35, Proposition 4.3(ii)]) and $p \nmid N_1 + u_{n-1}$. Since the differential form in (7.13) is exact in this case, (7.13) holds as well, with the same conditions on the terms of f.

By Lemmas 6.12 and 6.21, the τ-equivariance of χ implies that $\tau(df) = \zeta_m df$, where τ acts on t and dt by multiplication by ζ_m. That is, we may assume that f only has terms of degree t^{-q} where $q \equiv -1 \pmod{m}$. $\qquad \square$

The following proposition is crucial, and will be proved in §9.

Proposition 7.20 (cf. [24, Corollary 7.5]). *Let $G \in \mathcal{G}_{\text{crit},g}$, let $r \in [r_{\text{hub}}, r_{\text{crit}}) \cap \mathbb{Q}$, and let $f \in t^{1-m}k[t^{-m}]$ be a polynomial of degree less than $N_1 + u_{n-1}$ in t^{-1}, which we regard as the reduction of T_r in κ_r (§6.1). Assume f has no terms of degree divisible by p. Let $\beta = (N_1 + u_{n-1})(r_{\text{crit}} - r)$. After a possible finite extension of K, there exist $G' \in \mathcal{G}_{\text{crit},g}$ and $F \in \mathbb{K}$ with $v_r(F) = 0$ and $[F]_r = f$ such that*

$$\frac{G'}{G} \equiv 1 - p^\beta F \pmod{(\mathbb{K}^\times)^p}.$$

We now show that λ can be reduced.

Proposition 7.21 (cf. [24, Proposition 6.13]). *Suppose $G \in \mathcal{G}_{\text{crit},g}$ with $\lambda(G) > r_{\text{hub}}$. Then there exists $G' \in \mathcal{G}_{\text{crit},g}$ with $\lambda(G') < \lambda(G)$.*

Proof. This follows from Lemma 7.19 and Proposition 7.20 exactly as in the proof of [24, Proposition 6.13] with $N_1 + u_{n-1}$ playing the role of m_n there.   □

Proposition 7.22 (cf. [24, Proposition 6.15]). *The function $G \mapsto \lambda(G)$ takes a minimal value on $\mathcal{G}_{\mathrm{crit},g}$.*

Proof. Recall that $g = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} \bar{x}_j t^{-1})^{\psi_1(\tau^\ell) a_j}$, with the $\psi_1(\tau^\ell) a_j$ viewed as lying in $\mathbb{Z}$ (7.8). Let $U' \subseteq \mathbb{A}^{N_1/m}(k)$ be the open subset consisting of those $(\bar{y}_1, \ldots, \bar{y}_{N_1/m})$ such $\bar{y}_i^m \neq \bar{y}_j^m$ if $i \neq j$, and let $V \subset \mathbb{A}_k^{N_1/m}$ be the subvariety such that the $\bar{y}_j$ and a_j give a solution to (7.6). Since g realizes the isolated differential data criterion, the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{N_1/m})$ is an isolated point of V. In particular, $V' := V \backslash \{\bar{x}\} \subseteq \mathbb{A}_k^{N_1/m}$ is closed and $U = U' \backslash V' \subseteq \mathbb{A}_k^{N_1/m}$ is open.

Let $\mathcal{G}'_{\mathrm{crit},g} \supseteq \mathcal{G}_{\mathrm{crit},g}$ be the set of $G \in \mathbb{K}$ such that

$$G = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} z_j T^{-1})^{\psi_1(\tau^\ell) a_j},$$

where if $y_j = z_j p^{-r_{\mathrm{crit}}}$, then the reductions $\bar{y}_j$ give a point $(\bar{y}_1, \ldots, \bar{y}_{N_1/m}) \in U$. By identifying each $G \in \mathcal{G}'_{\mathrm{crit},g}$ with $(y_1, \ldots, y_{N_1/m})$, we identify $\mathcal{G}'_{\mathrm{crit},g}$ with the rigid analytic space

$$U^{\mathrm{rig}} := \{y \in (\mathbb{A}^{N_1/m})^{\mathrm{an}} \mid \bar{y} \in U\},$$

where $\bar{y}$ is the canonical reduction of y. Since U is open, U^{rig} is a finite union of open affinoid subdomains of $(\mathbb{A}^{N_1/m})^{\mathrm{an}}$. In particular, it is quasi-compact and quasi-separated.

Extend the domain of λ from $\mathcal{G}_{\mathrm{crit},g}$ to $\mathcal{G}'_{\mathrm{crit},g}$, keeping the definition the same. The family of $\mathbb{Z}/p^n$-covers of $\mathbb{P}_K^1$ parameterized by U via taking the Kummer extensions given rise to by points in U is a good relative Galois cover in the language of [25, §5]. By [25, Corollary 5.3(ii)] (taking $r_0 = r_{\mathrm{crit}}$ and $m_{\mathrm{Swan}} = N_1 + u_{n-1} + 1$ in the notation of that corollary), λ ($= \lambda_{\mathrm{Swan}}$) achieves its minimal value on $\mathcal{G}'_{\mathrm{crit},g}$ after a possible extension of K. On the other hand, our construction of $\mathcal{G}'_{\mathrm{crit},g}$ shows that $\lambda(G) = r_{\mathrm{crit}}$ if $G \in \mathcal{G}'_{\mathrm{crit},g} \backslash \mathcal{G}_{\mathrm{crit},g}$ and $\lambda(G) < r_{\mathrm{crit}}$ if $G \in \mathcal{G}_{\mathrm{crit},g}$. Thus our minimal value must be achieved on $\mathcal{G}_{\mathrm{crit},g}$.   □

Corollary 7.23.

  (i) *There exists $G_{\mathrm{crit}} \in \mathcal{G}_{\mathrm{crit},g}$ giving rise to a character χ_{crit} such that*

$$\delta_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}.$$

  (ii) *If $u_n = p u_{n-1}$, then χ_{crit} has good reduction.*

Proof. Proposition 7.21, combined with Proposition 7.22, shows that there exists $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$ giving rise to a character χ_{crit} such that $\lambda(\chi_{\mathrm{crit}}) = r_{\mathrm{hub}}$. In other words, $\delta_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$. This proves (i).

If $u_n = pu_{n-1}$, then $r_{\mathrm{hub}} = 0$, and $\lambda(G_{\mathrm{crit}}) = 0$. That is, $\delta_{\chi_{\mathrm{crit}}}$ is linear of slope $N_1 + u_{n-1} = pu_{n-1}$ on the interval $[0, r_{\mathrm{crit}}]$, with $\delta_{\chi_{\mathrm{crit}}}(0) = 0$. Part (ii) then follows from Proposition 7.16. $\qquad\square$

Definition 7.24. We let $C \in k^{\times}$ be the coefficient of $t^{-(N_1+u_{n-1}+1)} dt$ in $\omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$.

In fact, C is independent of the choice of $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$. This is the statement of Proposition 7.47, whose proof will be deferred to §7.8.

7.5. Controlling ω_{χ} at r_{hub}. The material in this section is only necessary if $u_n > pu_{n-1}$. So we now assume that $pu_{n-1} < u_n < pu_{n-1}+mp$ (this is the assumption of no essential ramification). Recall that this means that $p \nmid u_n$, that $(p-1)u_{n-1}-mp < N_1 < (p-1)u_{n-1}$, and that $N_2 = u_n - u_{n-1} - N_1 \leq 2mp - 2m$. Throughout this section, let $s = (N_1 + u_{n-1})(r_{\mathrm{crit}} - r_{\mathrm{hub}})$, and let $t = [T]_{r_{\mathrm{hub}}}$. We have constructed a rational function $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$ (Corollary 7.23) giving rise to a character χ_{crit} such that (see (7.12))

$$\delta_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}} = u_n r_{\mathrm{hub}} = \frac{p}{p-1} - (N_1 + u_{n-1})(r_{\mathrm{crit}} - r_{\mathrm{hub}}) = \frac{p}{p-1} - s. \tag{7.14}$$

Let $C \in k^{\times}$ be the constant from Definition 7.24.

For the rest of this section, we will make a further assumption.

Assumption 7.25. $N_2 \leq mp$.

Note that Assumption 7.25 is always satisfied when $N_1 = (p-1)u_{n-1} - m$ (and that for any other choice of N_1, there will be values of u_n leading to a violation of Assumption 7.25).

As mentioned in §7.3, our eventual goal is (after possibly modifying G_{crit}), to construct a rational function $G_{\mathrm{hub}} \in \mathbb{K}$ with N_2 zeroes and poles away from $T = 0$, all of which have valuation r_{hub}, so that if we let $G_n = G_{\mathrm{hub}}G_{\mathrm{crit}}$, then G_n gives rise to a character χ_n with good reduction $\bar{\chi}_n$. In this case, δ_{χ_n} would be linear of slope u_n on the interval $[0, r_{\mathrm{hub}}]$ and linear of slope $N_1 + u_{n-1}$ on the interval $[r_{\mathrm{hub}}, r_{\mathrm{crit}}]$. In particular, the differential form $\omega_{\chi_n}(r_{\mathrm{hub}})$ would have to have a zero of order $u_n - 1$ at $t = \infty$ and a pole of order $N_1 + u_{n-1} + 1$ at $t = 0$ (Proposition 6.17). By Proposition 6.19, there can be no zeroes away from ∞. By Lemma 6.21, $\omega_{\chi_n}(r_{\mathrm{hub}})$ must transform equivariantly under $t \mapsto \zeta_m t$. So we will search for G_{hub} such that $G_n := G_{\mathrm{hub}}G_{\mathrm{crit}}$ can give rise to a character χ_n such that $\delta_{\chi_n}(r_{\mathrm{hub}})$ is still δ_{hub} and

$$\omega_{\chi_n}(r_{\mathrm{hub}}) = \frac{c\,dt}{(t^m - \bar{\alpha}^m)^{N_2/m} t^{N_1+u_{n-1}+1}}, \tag{7.15}$$

where c and $\bar{\alpha}$ are in $k^{\times}$. Proposition 7.33 will show how a valid G_{hub} arises.

Lemma 7.26. *The differential form on the right hand side of* (7.15) *has a zero of order $u_n - 1$ at ∞, a pole of order $N_1 + u_{n-1} + 1$ at 0, no zeroes away from ∞, transforms as in Lemma* 6.21, *and is exact for all choices of c and $\bar\alpha$.*

Proof. Once we note that $m | (N_1 + u_{n-1} + 1)$, all assertions become trivial except the last one. Multiplying a differential form by a pth power does not change its exactness, so it suffices to show that

$$\frac{(t^m - \bar\alpha^m)^{p - N_2/m} dt}{t^{N_1 + u_{n-1} + 1}}$$

is exact. By Assumption 7.25, the numerator is a polynomial in t. Expanding everything out, t occurs to degrees $-(N_1 + u_{n-1} + 1)$ through $mp - u_n - 1$, counting by m's. Since $-(N_1 + u_{n-1} + 1) > -pu_{n-1} - 1$ and $mp - u_n - 1 < -pu_{n-1} + mp - 1$, and since all the degrees in question are divisible by m, we see that none of the above degrees is congruent to $-1 \pmod{p}$. This means that the differential form is exact. $\qquad\square$

Lemma 7.27. *Let ω be the differential form on the right hand side of* (7.15). *Then $\omega - \omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$ is exact. Furthermore, we can write*

$$\omega - \omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \left(\frac{a(t)}{(t^m - \bar\alpha^m)^{N_2/m}} + \frac{b(t)}{t^{N_1 + u_{n-1} + 1}} \right) dt, \qquad (7.16)$$

where both fractions are proper, $a(t)$ and $b(t)$ are in $k[t^m]$, and each of the two summands is exact. Lastly, choosing $c = C(-\bar\alpha^m)^{N_2/m}$ on the right hand side of (7.15) *results in $b(t)$ having no constant term.*

Proof. The exactness of $\omega - \omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$ follows from Lemmas 7.19 and 7.26, noting that $p \nmid N_1 + u_{n-1}$. Since $\omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})/dt$ is a proper fraction in t with denominator $t^{N_1 + u_{n-1} + 1}$ and $\bar\alpha \neq 0$, the theory of partial fractions gives the desired decomposition into the two summands. The polynomials $a(t)$ and $b(t)$ lie in $k[t^m]$ by Lemma 6.21, combined with the fact that χ_{crit} is τ-equivariant. Each of the two summands is exact because their sum is, and a sum of two proper fractions (times dt) with relatively prime denominators can only be exact if each one is. Lastly, in order for $b(t)$ not to have a constant term, we need only ensure that when $\omega - \omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$ is written as $f(t)dt / (t^m - \bar\alpha^m)^{N_2/m} t^{N_1 + u_{n-1} + 1}$, that $f(t)$ has no constant term. This is accomplished by taking $c = C(-\bar\alpha^m)^{N_2/m}$, where C is the coefficient of the $dt / t^{N_1 + u_{n-1} + 1}$ term of $\omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$. $\qquad\square$

Proposition 7.28. *Let ω be the right hand side of* (7.15), *with c chosen as in Lemma* 7.27 *and C chosen as in Definition* 7.24. *By modifying G_{crit} within $\mathscr{G}_{\mathrm{crit},g}$, we can ensure*

$$\omega - \omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \frac{a(t)dt}{(t^m - \bar\alpha^m)^{N_2/m}},$$

with $a(t)$ as in Lemma 7.27.

Proof. By our choice of c, we may assume that $b(t)$ has no constant term in the notation of Lemma 7.27. Thus we can write

$$\frac{b(t)dt}{t^{N_1+u_{n-1}+1}} = df,$$

where $f \in t^{1-m}[t^{-m}]$ has degree less than $N_1 + u_{n-1}$ in t^{-1}. By Proposition 7.20, there exists $G'_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ such that

$$\frac{G'_{\text{crit}}}{G_{\text{crit}}} \equiv 1 + p^s F \pmod{(\mathbb{K}^\times)^p},$$

where $v_\lambda(F) = 0$, where $[F]_\lambda = f$. As in the proof of [24, Proposition 6.13], replacing G_{crit} by G'_{crit} has the effect of adding df to $\omega_{\chi_{\text{crit}}}(r_{\text{hub}})$, which in turn has the effect of subtracting df from the right hand side of (7.16). This proves the proposition. $\qquad\square$

By Proposition 7.28 we may, and do, assume that $b(t) = 0$ in (7.16). We do a further partial fractions decomposition on the other term to obtain

$$\frac{a(t)dt}{(t^m - \bar{\alpha}^m)^{N_2/m}} = \sum_{\ell=0}^{m-1} \frac{\beta_\ell(t)dt}{(\zeta_m^\ell t - \bar{\alpha})^{N_2/m}},$$

where $\beta_\ell(t)$ is a polynomial of degree less than N_2/m. Using equivariance under $t \mapsto \zeta_m t$, it is not hard to check that $\beta_\ell(t) = \beta_0(\zeta_m^\ell t)$ for all ℓ.

The following definition is the key idea of §7.5.

Definition 7.29. Let $\bar{\alpha} \in k^\times$. Let $T_{\text{hub}} = p^{-r_{\text{hub}}}T$ and $s = p/(p-1) - \delta_{\text{hub}}$. For any lift α of $\bar{\alpha}$ to R and fixed lifts of ζ_m^ℓ from $\mathbb{F}_p$ to $\mathbb{Z}$ (denoted again by ζ_m^ℓ by abuse of notation), let $\mathcal{G}_{\text{hub},\alpha} \subseteq \mathbb{K}^\times$ be the set of all rational functions of the form

$$\prod_{\ell=0}^{m-1} \left(1 + p^s A(\zeta_m^\ell T_{\text{hub}})\right)^{\zeta_m^{-\ell}},$$

such that

$$A(T_{\text{hub}}) \text{ is of the form } \frac{B(T_{\text{hub}})}{(T_{\text{hub}} - \alpha)^{N_2/m-1}}, \tag{7.17}$$

where $B(T_{\text{hub}})$ is a polynomial of degree at most $N_2/m - 2$ and $v_{r_{\text{hub}}}(A) = 0$ with $d([A]_{r_{\text{hub}}})/dt = \beta_0(t)/(t - \bar{\alpha})^{N_2/m}$.

Definition 7.30. Let $\mathcal{G}_{\text{hub}} = \bigcup_{\alpha \in R^\times} \mathcal{G}_{\text{hub},\alpha}$.

Remark 7.31. By Lemma 7.27, $a(t)dt/(t^m - \bar{\alpha}^m)^{N_2/m}$ is exact. It is straightforward then to show that $\beta_0(t)dt/(t - \bar{\alpha})^{N_2/m}$ is also exact. In particular, $\mathcal{G}_{\text{hub},\alpha}$ is nonempty for all $\alpha \in R^\times$.

The following definition will be useful in the proof of Proposition 7.39.

Definition 7.32. For $\alpha \in R^\times$, define $\mathscr{G}'_{\mathrm{hub},\alpha} \supseteq \mathscr{G}_{\mathrm{hub},\alpha}$ exactly as in Definition 7.29, except that we impose the condition $v_{r_{\mathrm{hub}}}(A) \geq 0$ instead of $v_{r_{\mathrm{hub}}}(A) = 0$, and we place no condition on $d([A]_{r_{\mathrm{hub}}})/dt$.

We prove the major result of this section.

Proposition 7.33. *If $G_{\mathrm{hub}} \in \mathscr{G}_{\mathrm{hub},\alpha}$ and G_{crit} is chosen as in Proposition 7.28, then $G_{\mathrm{crit}}G_{\mathrm{hub}}$ gives rise to a character χ such that $\delta_\chi(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$, and $\omega := \omega_\chi(r_{\mathrm{hub}})$ is the right hand side of* (7.15) *with c chosen as in Lemma 7.27. Consequently, the left slope of δ_χ at r_{hub} is u_n.*

Furthermore, $G_{\mathrm{crit}}G_{\mathrm{hub}}$ is of the form (7.3).

Proof. The product

$$\prod_{\ell=0}^{m-1} \left(1 + p^s A(\zeta_m^\ell T_{\mathrm{hub}})\right)^{\zeta_m^{-\ell}}$$

can be written as

$$1 + p^s \sum_{\ell=0}^{m-1} \zeta_m^{-\ell} A(\zeta_m^\ell T_{\mathrm{hub}}) + D,$$

where $v_{r_{\mathrm{hub}}}(D) > s$. By the definition of A, the derivative of $\sum_{\ell=0}^{m-1} [\zeta_m^{-\ell} A(\zeta_m^\ell T_{\mathrm{hub}})]_{r_{\mathrm{hub}}}$ is

$$\sum_{\ell=0}^{m-1} \frac{\beta_0(\zeta_m^\ell t)}{(\zeta_m^\ell t - \bar\alpha)^{N_2/m}} = \sum_{\ell=0}^{m-1} \frac{\beta_\ell(t)}{(\zeta_m^\ell t - \bar\alpha)^{N_2/m}} = \frac{a(t)dt}{(t^m - \bar\alpha^m)^{N_2/m}}.$$

By Propositions 6.20 and 7.28, we get that $\delta_\chi(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$ and $\omega_\chi(r_{\mathrm{hub}}) = \omega$.

Since ω has a zero of order $N_2 + N_1 + u_{n-1} - 1 = u_n - 1$ at ∞, Proposition 6.17 shows that the left slope of δ_χ at r_{hub} is u_n.

Showing that $G_{\mathrm{hub}}G_{\mathrm{crit}}$ is of the form (7.3) is equivalent to showing that $1 + p^s A(T_{\mathrm{hub}})$ is, up to multiplication by a pth power, a polynomial in T^{-1} with constant term 1 and at most N_2/m distinct roots. In order to do this, we multiply $1 + p^s A(T_{\mathrm{hub}})$ by $(T_{\mathrm{hub}} - \alpha)^p/T_{\mathrm{hub}}^p$, and we leave it to the reader to verify that everything works (the roots will be the $N_2/m - 1$ roots of $(T_{\mathrm{hub}} - \alpha)^{N_2/m-1} + p^s B(T_{\mathrm{hub}})$, along with α). $\qquad\square$

Remark 7.34. Let $G_{\mathrm{hub}} \in \mathscr{G}'_{\mathrm{hub},\alpha} \setminus \mathscr{G}_{\mathrm{hub},\alpha}$ and $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$. By the discussion above, if $G_{\mathrm{crit}}G_{\mathrm{hub}}$ gives rise to a character χ, then it is *not* the case that both $\delta_\chi(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$ and the left slope of δ_χ at r_{hub} is u_n.

7.6. Controlling δ_χ between r_{hub} and 0. We maintain the assumption of §7.5 that $pu_{n-1} < u_n < pu_{n-1} + mp$, as well as all the notation so far. Fix $\alpha \in R^\times$ and g a solution to (7.6) realizing the isolated differential data criterion. By Proposition 7.33, there exists $G_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ such that, for any $G_{\text{hub}} \in \mathcal{G}_{\text{hub},\alpha}$, the character χ given rise to by $G_{\text{hub}}G_{\text{crit}}$ has $\delta_\chi(r_{\text{hub}}) = \delta_{\text{hub}}$ and δ_χ has a left slope of u_n at r_{hub}. The goal of this section is to find a particular $\tilde{G}_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ and $\tilde{G}_{\text{hub}} \in \mathcal{G}_{\text{hub},\alpha}$ such that $\tilde{G}_{\text{hub}}\tilde{G}_{\text{crit}}$ gives rise to a character χ with $\delta_\chi(0) = 0$. Since $\delta_{\text{hub}} = u_n r_{\text{hub}}$ by definition, one can test this by seeing if δ_χ is linear of slope u_n on the interval $[0, r_{\text{hub}}]$. Let $\mathcal{G}_{g,\alpha}$ be the (nonempty) subset of $\mathcal{G}_{\text{hub},\alpha}\mathcal{G}_{\text{crit},g}$ consisting of elements giving rise to characters χ with $\delta_\chi(r_{\text{hub}}) = \delta_{\text{hub}}$ and such that the left slope of δ_χ at r_{hub} is u_n. Note that every $G \in \mathcal{G}_{g,\alpha}$ is of the form (7.3). If $G \in \mathcal{G}_{g,\alpha}$ gives rise to χ, then we define $\mu(G)$ to be the minimal element of $[0, r_{\text{hub}}]$ such that $\delta_\chi(\mu(G)) = u_n\mu(G)$ (that is, $\mu(G)$ is the largest element of $[0, r_{\text{hub}}]$ where δ_χ has left slope less than u_n, or 0, if no such element exists). This is analogous to the definition of $\lambda(G)$ in §7.4. For any $G \in G_{g,\alpha}$, we have $\mu(G) < r_{\text{hub}}$. The goal of this section is to prove the existence of $G \in \mathcal{G}_{g,\alpha}$ such that $\mu(G) = 0$. Then G will give rise to a character with good reduction and upper jumps $(u_1, \ldots, u_n)$. The argument is parallel to that of §7.4.

Lemma 7.35. *Suppose $G \in \mathcal{G}_{g,\alpha}$ with $\mu := \mu(G) > 0$. Identify κ_μ with $k(t)$. Then $\omega_\chi(\mu)$ can be written in the form*

$$\frac{c\, dt}{t^{u_n+1}} + df,$$

where $c \in k^\times$ and $f \in t^{1-m}k[t^{-m}]$ has degree less than u_n in t^{-1}.

Proof. After noting that

$$\delta_\chi(\mu) = p/(p-1) - s - u_n(r_{\text{hub}} - \mu) = \mu u_n > \mu p u_{n-1},$$

where s is as in Definition 7.29, the proof is exactly the same as the $u_n > pu_{n-1}$ case of Lemma 7.19. $\qquad\square$

As in §7.4, we also postpone the proof of the following crucial result to §9.

Proposition 7.36. *Suppose $N_1 = (p-1)u_{n-1} - m$ (this is consistent with Assumption 7.25). Let $G_{\text{crit}}, G_{\text{hub}} \in G_{\text{crit},g}, \mathcal{G}_{\text{hub},\alpha}$, respectively. Let $r \in [0, r_{\text{hub}}) \cap \mathbb{Q}$, and let $f \in t^{1-m}k[t^{-m}]$ have degree less than u_n in t^{-1}, which we regard as the reduction of T_r in κ_r (§6.1). Assume f has no terms of degree divisible by p. Let $\beta = p/(p-1) - u_n r$. After a possible finite extension of K, there exist $G'_{\text{crit}}, G'_{\text{hub}} \in \mathcal{G}_{\text{crit},g}, \mathcal{G}_{\text{hub},\alpha}$ respectively, and $F \in \mathbb{K}$ with $v_r(F) = 0$ and $[F]_r = f$ such that*

$$\frac{G'_{\text{crit}}G'_{\text{hub}}}{G_{\text{crit}}G_{\text{hub}}} \equiv 1 - p^\beta F \pmod{(\mathbb{K}^\times)^p}.$$

Remark 7.37. The proofs of Propositions 7.20 and 7.36 are the only places where the *isolatedness* in the isolated differential data criterion is used.

This has the following consequence:

Proposition 7.38. *Suppose $G \in \mathcal{G}_{g,\alpha}$ with $\mu(G) > 0$. Then there exists $G' \in \mathcal{G}_{g,\alpha}$ with $\mu(G') < \mu(G)$.*

Proof. The proof is exactly the same as the proof of Proposition 7.21, using Lemma 7.35 and Proposition 7.36 in place of Lemma 7.19 and Proposition 7.20, and taking $\beta = p/(p-1) - u_n\mu$ instead of $\beta = (N_1 + u_{n-1})(r_{crit} - \lambda)$. $\qquad\square$

Proposition 7.39. *The function $G \mapsto \mu(G)$ takes a minimal value on $\mathcal{G}_{g,\alpha}$.*

Proof. We identify $\mathcal{G}'_{hub,\alpha}$ with the rigid $(N_2/m-1)$-dimensional closed unit polydisc corresponding to the coefficients of A in Definition 7.32. This is an affinoid space. Let $\mathcal{G}'_{crit,g}$ be as in the proof of Proposition 7.22, and let $\mathcal{G}'_{g,\alpha} = \mathcal{G}'_{crit,g}\mathcal{G}'_{hub,\alpha} \subseteq \mathbb{K}$. It is easy to see that $\mathcal{G}'_{g,\alpha} \cong \mathcal{G}'_{crit,g} \times \mathcal{G}'_{hub,\alpha}$, and is thus identified with a quasi-compact, quasi-separated rigid-analytic space.

Extend the domain of μ from $\mathcal{G}_{g,\alpha}$ to $\mathcal{G}'_{g,\alpha}$, keeping the definition the same. The family of $\mathbb{Z}/p^n$-covers of $\mathbb{P}^1_K$ parameterized by $\mathcal{G}'_{g,\alpha}$ via taking the Kummer extensions given rise to by points in $\mathcal{G}_{g,\alpha}$ is a good relative Galois cover in the language of [25, §5]. By [25, Corollary 5.3(ii)], μ achieves its minimal value on $\mathcal{G}'_{g,\alpha}$, after a possible extension of K. On the other hand, suppose $G = G_{crit}G_{hub} \in \mathcal{G}'_{g,\alpha} \backslash \mathcal{G}_{g,\alpha}$ with $G_{crit} \in \mathcal{G}'_{crit,g}$ and $G_{hub} \in \mathcal{G}'_{hub,\alpha}$. We claim that the left-slope of δ_χ at r_{hub} is less than u_n, which means that $\mu(G) = r_{hub}$. Since $\mu(G) < r_{hub}$ when $G \in \mathcal{G}_{g,\alpha}$, this means that the minimal value of μ on $G'_{g,\alpha}$ must be achieved on $\mathcal{G}_{g,\alpha}$, thus completing the proof.

To prove the claim, first assume that $\delta_\chi(r_{hub}) \neq \delta_{hub}$. Then $\delta_\chi(r_{hub}) > \delta_{hub}$. Since δ_χ is concave up on $[r_{hub}, r_{crit}]$ (combine Propositions 6.17 and 6.19 with the fact that G has no zeroes or poles with valuation in (r_{hub}, r_{crit})), we have that the right-slope of δ_χ at r_{hub} is less than $N_1 + u_{n-1}$. Since G_{hub} has at most N_2 zeroes and poles with valuation r_{hub}, the left-slope of δ_χ at r_{hub} is less than $N_1 + N_2 + u_{n-1} = u_n$ at r_{hub} (again, combine Propositions 6.17 and 6.19).

Now, assume $\delta_\chi(r_{hub}) = \delta_{hub}$. Then the left-slope of δ_χ at r_{crit} is $N_1 + u_{n-1}$, and $G_{crit} \in \mathcal{G}_{crit,g}$. Remark 7.34 shows that, if $G_{hub} \in \mathcal{G}'_{hub,\alpha} \backslash \mathcal{G}_{hub,\alpha}$, then the left-slope of δ_χ at r_{hub} is less than u_n. So assume $G_{hub} \in \mathcal{G}_{hub,\alpha}$. The definition of $G_{g,\alpha}$ shows that if $G \in G'_{g,\alpha} \backslash G_{g,\alpha}$, then the left-slope of δ_χ at r_{hub} is less than u_n. The claim, and thus the proposition, is proved. $\qquad\square$

Corollary 7.40. *Suppose $u_n > pu_{n-1}$. Then there exists $G \in \mathcal{G}_{g,\alpha}$ giving rise to a character χ having good reduction.*

Proof. Proposition 7.38, combined with Proposition 7.39, shows that there exists $G \in \mathcal{G}_{g,\alpha}$ giving rise to a character χ such that $\mu(\chi) = 0$. That is, $\delta_\chi(0) = 0$. The corollary then follows from Proposition 7.16. $\qquad\square$

7.7. Ensuring the correct reduction on the boundary. In this section, we prove Theorem 7.43, which will complete the proof of Theorem 4.3. Maintain all notation from the previous sections, including C as the constant from Definition 7.24. First, we prove two lemmas.

Lemma 7.41. *If $G_1 \in \mathcal{G}_{g,\alpha_1}$ gives rise to a character χ with good reduction $\bar{\chi}$ that corresponds (after completion at $t = 0$) to the Witt vector $(f_1, \ldots, f_\chi) \in W_n(k((t)))$, and $G_2 \in \mathbb{K}$ is such that $G_2/G_1 \equiv 1 + p^{p/(p-1)} F \pmod{(\mathbb{K}^\times)^p}$, with $v_0(F) = 0$ and $[F]_0 = f$ in $k(t) \subseteq k((t))$, then G_2 gives rise to a character χ' with étale reduction $\bar{\chi}'$ that corresponds (after completion at $t = 0$) to the Witt vector $(f_1, \ldots, f_{n-1}, f_\chi + f) \in W_n(k((t)))$.*

Proof. Replacing G_1 by G_2 has the effect of multiplying χ by

$$\psi_n := \mathfrak{K}_n\big((G_2/G_1)^{p^{n-1}}\big) \in H^1_{p^n}(\mathbb{K}).$$

This is just the image of $\psi_1 := \mathfrak{K}_1(G_2/G_1) \in H^1_p(\mathbb{K})$. Proposition 6.22 shows that $\delta_{\psi_1}(0) = 0$ and the reduction $\bar{\psi}_1$ corresponds to the Artin–Schreier extension given by $y^p - y = f$. Consequently, $\delta_{\psi_n}(0) = 0$ and its reduction $\bar{\psi}_n$ corresponds to the extension encoded by the Witt vector $(0, \ldots, 0, f)$.

By Proposition 6.20(iii), we conclude that the reduction of $\bar{\chi}' = \bar{\chi}\bar{\psi}_n$ corresponds to the sum of the Witt vectors $(f_1, \ldots, f_\chi)$ and $(0, \ldots, 0, f)$. This is $(f_1, \ldots, f_{n-1}, f_\chi + f)$, as desired. $\qquad\square$

The proof of the following lemma relies on Lemma 7.48, proven in §7.8.

Lemma 7.42. *Suppose $u_n > pu_{n-1}$, and let $\alpha_1 \in R^\times$ with reduction $\bar{\alpha}_1 \in k^\times$. Suppose $G_1 \in \mathcal{G}_{g,\alpha_1}$ gives rise to a character χ with good reduction $\bar{\chi}$. Then if $(f_1, \ldots, f_{n-1}, f_\chi)$ is the Witt vector in standard form corresponding to $\bar{\chi}$ (after completion at $t = 0$), the t^{-u_n} coefficient of f_χ is $-Cu_n^{-1}(-\bar{\alpha}_1^m)^{N_2/m}$.*

Proof. Let γ be the (nonzero) t^{-u_n} coefficient of f_χ, and assume for a contradiction that $\gamma \neq -Cu_n^{-1}(-\bar{\alpha}_1^m)^{N_2/m}$. Let $\bar{\alpha}_2$ be such that

$$Cu_n^{-1}\big((-\bar{\alpha}_1^m)^{N_2/m} - (-\bar{\alpha}_2^m)^{N_2/m}\big) = -\gamma.$$

By our assumption, $\bar{\alpha}_2 \neq 0$. Let $\alpha_2 \in R$ be a lift of $\bar{\alpha}_2$. Using Corollary 7.40, choose $G_2 \in \mathcal{G}_{\alpha_2}$ giving rise to a character χ' with good reduction $\bar{\chi}'$. Since $\gamma \neq 0$, we have $\alpha_1^{N_2} \neq \alpha_2^{N_2}$, so Lemma 7.48 applies. In particular, Lemma 7.48 implies that

$$G_2/G_1 \equiv 1 + p^{p/(p-1)} F \pmod{(\mathbb{K}^\times)^p},$$

where $F \in \mathbb{K}$ satisfies $v_0(F) = 0$ and $[F]_0$ is a polynomial in $t^{-1} = [T^{-1}]_0$ of degree u_n with leading term $-\gamma t^{-u_n}$. By Lemma 7.41, replacing G_1 with G_2 replaces f_χ in the Witt vector for $\bar{\chi}$ with $f_\chi + [F]_0$, which has degree less than u_n in t^{-1}. This means that the nth higher ramification jump for the upper numbering of $\bar{\chi}'$ is less than u_n (§2), which contradicts Proposition 6.18. $\qquad\square$

If $u_n = pu_{n-1}$, let $\mathcal{G} = \mathcal{G}_{\text{crit},g}$. Otherwise, let $\mathcal{G} = \bigcup_{\alpha \in R^\times} \mathcal{G}_{g,\alpha}$. Note that all elements of $\mathcal{G}$ are of the form (7.3). Recall that $\bar{\chi}_n$ is our original character, with upper ramification breaks $(u_1, \ldots, u_n)$. Furthermore, we saw in §2 that $\bar{\chi}_n$ corresponds (upon completion at $t = 0$) to a (truncated) Witt vector $w_n := (f_1, \ldots, f_n) \in W_n(k((t)))$, and we may assume that each $f_i \in t^{1-m}k[t^{-m}]$, and all terms of f_i have prime-to-p degree.

If $u_n = pu_{n-1}$, Corollary 7.23 shows that there exists $G \in \mathcal{G}$ giving rise to a character χ with good reduction $\bar{\chi}$ corresponding (after completion at $t = 0$) to the Witt vector $w_\chi := (f_1, \ldots, f_{n-1}, f_\chi)$, where $f_\chi \in t^{1-m}k[t^{-m}]$ has degree less than u_n in t^{-1}, and all terms of prime-to-p degree. If $u_n > pu_{n-1}$, Corollary 7.40 and Lemma 7.42 guarantee (after a possible finite extension of R) the existence of $\alpha \in R^\times$ and $G \in \mathcal{G}_{g,\alpha}$ such that G gives rise to a character χ with good reduction $\bar{\chi}$ corresponding (after completion at $t = 0$) to the Witt vector $w_\chi := (f_1, \ldots, f_{n-1}, f_\chi)$, where $f_\chi \in t^{1-m}k[t^{-m}]$ has degree u_n in t^{-1}, all terms of prime-to-p degree, and the coefficient of t^{-u_n} in f_χ is the same as that in f_n. In both cases, f_χ and f_n differ by a polynomial of degree less than u_n in t^{-1}.

Theorem 7.43. *There exists $G_n \in \mathcal{G}$ giving rise to a (ψ_n-equivariant) character χ_n with good reduction $\bar{\chi}_n$.*

Proof. Let $f = f_n - f_\chi \in t^{1-m}k[t^{-m}]$, which has degree less than u_n in t^{-1}. By Proposition 7.20 (in the case $u_n = pu_{n-1}$) or 7.36 (in the case $u_n > pu_{n-1}$), there exists $G_n \in \mathcal{G}$ such that

$$\frac{G_n}{G} \equiv 1 + p^{p/(p-1)}F \pmod{(\mathbb{K}^\times)^p},$$

for some $F \in \mathbb{K}$ satisfying $v_0(F) = 0$ and $[F]_0 = f$. By Lemma 7.41, replacing G by G_n gives rise to a character χ_n whose reduction corresponds to the Witt vector $(f_1, \ldots, f_{n-1}, f_\chi + f) = (f_1, \ldots, f_{n-1}, f_n)$. In other words, the reduction of χ_n is $\bar{\chi}_n$. Since G_n, by virtue of being in $\mathcal{G}$, is of the form of (7.3), Proposition 7.16 shows that χ_n has good reduction and is ψ_n-equivariant. $\square$

Since the G_n guaranteed by Theorem 7.43 lies in $\mathcal{G}$, all the zeroes and poles of G_n have valuation r_{crit} or (in the case $u_n > pu_{n-1}$), valuation r_{hub}. By Proposition 7.17, we conclude that $\mathbb{B}(\chi_n) \subseteq D(r_n)$. In particular, χ_n is admissible. Thus, we have proven Theorem 4.3, using Claim 6.9 in place of Claim 4.1. Since Lemma 4.2 was proven in §5, we obtain Theorem 1.14.

7.8. Calculations. Maintain the notation of the previous sections. The purpose of this section is twofold: to prove Lemma 7.48, which is used in the proof of Lemma 7.42, and to prove Proposition 7.47, which shows that C as defined in Definition 7.24 depends only on g. Propositions 7.20 and 7.36 have much more complicated proofs, and are deferred to §9.

Lemma 7.44. *Suppose* $\bar{\alpha}_1, \bar{\alpha}_2, C \in k^\times$. *For* $i = 1, 2$, *write*

$$\omega_i = \frac{C(-\bar{\alpha}_i^m)^{N_2/m}\, dt}{(t^m - \bar{\alpha}^m)^{N_2/m} t^{N_1 + u_{n-1} + 1}}.$$

Then, when expanded out as a power series in t^{-1}, *one obtains*

$$\omega_2 - \omega_1 = (C(-\bar{\alpha}_2^m)^{N_2/m} - C(-\bar{\alpha}_1^m)^{N_2/m}) t^{-(u_n + 1)} dt + \text{higher order terms.}$$

Proof. This is a straightforward computation, using the fact that $u_n = N_1 + N_2 + u_{n-1}$. $\square$

Corollary 7.45. *Let* $\bar{\alpha}_1, \bar{\alpha}_2 \in k^\times$ *with* $\bar{\alpha}_1^{N_2} \neq \bar{\alpha}_2^{N_2}$. *Choose lifts* α_i *of the* $\bar{\alpha}_i$ *in* R. *If* $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$ *and* $G_{\mathrm{hub},i} \in \mathscr{G}_{\mathrm{hub},\alpha_i}$ $(i = 1, 2)$ *are chosen as in Proposition 7.28 such that* $G_{\mathrm{crit}} G_{\mathrm{hub},i}$ *gives rise to a character* χ_i *such that* $\omega_i := \omega_{\chi_i}(r_{\mathrm{hub}})$ $(i = 1, 2)$, *then* $\delta_{\chi_i}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$ *and the expansion of* $\omega_{\chi_2 \chi_1^{-1}}(r_{\mathrm{hub}})$ *as a power series in* t^{-1} *is*

$$\omega_2 - \omega_1 = (C(-\bar{\alpha}_2^m)^{N_2/m} - C(-\bar{\alpha}_1^m)^{N_2/m}) t^{-(u_n + 1)} dt + \text{higher order terms.}$$

Proof. This follows from Lemma 7.44, using Propositions 6.20 and 7.33. $\square$

Lemma 7.46. *Suppose* $F_1 = 1 + \sum_{i=1}^\infty a_i T^{-i}$ *and* $F_2 = 1 + \sum_{i=1}^\infty b_i T^{-i}$ *lie in* $\mathbb{K} \cap (R\{T^{-1}\} \otimes K)$. *Suppose that for some* $0 < \alpha \leq p/(p-1)$ *and* $M > 0$ *not divisible by* p, *we have* $v(a_i) \geq \alpha$ *for all* $i > 0$ *(with strict inequality holding when* $p|i$), *and that* $v(b_i) \geq \alpha$ *for all* $i \geq M$ *(with strict inequality holding for* $i > M$). *Lastly, suppose that* $F_1 H^p = F_2$ *for some* $H \in \mathbb{K}^\times$. *Then* $v(a_i - b_i) > \alpha$ *for all* $i \geq M$.

Proof. We may assume $H = 1 + \sum_{i=1}^\infty c_i T^{-i}$ and $H^p = 1 + \sum_{i=1}^\infty d_i T^{-i}$ as power series expansions. It suffices to show that $v(d_i) > \alpha$ for all $i \geq M$. If $v(c_i) > \alpha/p$ for all i, then we are done. If not, let i_0 be the maximal i such that $v(c_i) \leq \alpha/p$ (such an i_0 must exist). If $i_0 < M/p$, then $v(d_i) > \alpha$ for all $i \geq M$. If $i_0 > M/p$, then $v(d_{pi_0}) \leq \alpha$, and $F_1 H^p = F_2$ then shows that $v(b_{pi_0}) \leq \alpha$, contradicting our assumptions on F_2. $\square$

We now show that the value of C from Definition 7.24 only depends on g. Recall from (7.14) that $s = p/(p-1) - \delta_{\mathrm{hub}} = u_n r_{\mathrm{hub}}$.

Proposition 7.47. *Assume* $u_n > p u_{n-1}$. *When* $\omega_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}})$ *is expanded as a Laurent series in* t, *the coefficient of* $t^{-(N_1 + u_{n-1} + 1)} dt$ *does not depend on the choice of* $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$, *so long as* $\delta_{\chi_{\mathrm{crit}}}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$.

Proof. For $i = 1, 2$, suppose $G_{\mathrm{crit},i} \in \mathscr{G}_{\mathrm{crit},g}$ gives rise to a character $\chi_{\mathrm{crit},i}$ with $\delta_{r_{\mathrm{hub}}}(\chi_{\mathrm{crit},i}) = \delta_{\mathrm{hub}}$. Proposition 6.20 shows that $\delta_{\chi_{\mathrm{crit},2} \chi_{\mathrm{crit},1}^{-1}}(r_{\mathrm{hub}}) \geq \delta_{\mathrm{hub}}$. If $\delta_{\chi_{\mathrm{crit},2} \chi_{\mathrm{crit},1}^{-1}} > \delta_{\mathrm{hub}}$, then by Proposition 6.20, we have $\omega_{\chi_{\mathrm{crit},1}}(r_{\mathrm{hub}}) = \omega_{\chi_{\mathrm{crit},2}}(r_{\mathrm{hub}})$,

and we are done. If $\delta_{\chi_{\mathrm{crit},2}\chi_{\mathrm{crit},1}^{-1}}(r_{\mathrm{hub}}) = \delta_{\mathrm{hub}}$, then Proposition 6.20 shows that we must prove that $\omega_{\chi_{\mathrm{crit},2}\chi_{\mathrm{crit},1}^{-1}}(r_{\mathrm{hub}})$ does not have a nontrivial term of the form $ct^{-(N_1+u_{n-1}+1)}dt$.

Now,

$$\chi_{\mathrm{crit},2}\chi_{\mathrm{crit},1}^{-1} = \mathfrak{K}_n\big((G_{\mathrm{crit},2}/G_{\mathrm{crit},1})^{p^{n-1}}\big),$$

which is identified with $\mathfrak{K}_1(G_{\mathrm{crit},2}/G_{\mathrm{crit},1})$. By Proposition 6.22, we can write

$$G_{\mathrm{crit},2}/G_{\mathrm{crit},1} \equiv 1 + p^s F \quad (\mathrm{mod}\ (\mathbb{K}^\times)^p)$$

for some $F \in \mathbb{K}$ with $v_{r_{\mathrm{hub}}}(F) = 0$. If we write $1 + p^s F = 1 + \sum_{i=1}^\infty b_i T_{\mathrm{hub}}^{-i}$, then by multiplying by a pth power, we may assume that $v(b_i) > s$ whenever $p|i$. According to Proposition 6.22, we must show that either $p|(N_1 + u_{n-1})$ or $v(b_{N_1+u_{n-1}}) > s$. So we assume $p \nmid (N_1 + u_{n-1})$.

On the other hand, since both $G_{\mathrm{crit},i}$ lie in $\mathcal{G}_{\mathrm{crit},g}$, we may assume that the $G_{\mathrm{crit},i}$ are chosen in $\mathbb{K}$ such that the quotient $G_{\mathrm{crit},2}/G_{\mathrm{crit},1}$ lies in $1 + T_{\mathrm{crit}}^{-1}\mathfrak{m}\{T_{\mathrm{crit}}^{-1}\}$. If we write $G_{\mathrm{crit},2}/G_{\mathrm{crit},1} = 1 + \sum_{i=1}^\infty a_i T_{\mathrm{hub}}^{-i}$, then

$$v(a_i) > (N_1 + u_{n-1})(r_{\mathrm{crit}} - r_{\mathrm{hub}}) = s \tag{7.18}$$

for all $i \geq N_1 + u_{n-1}$.

The proposition now follows from Lemma 7.46, taking M, α, F_1, and F_2 to be $N_1 + u_{n-1}$, s, $G_{\mathrm{crit},2}/G_{\mathrm{crit},1}$, and $1 + p^s F$, respectively. $\qquad\square$

Let $C \in k^\times$ be the coefficient of $t^{-(N_1+u_{n-1}+1)}dt$ from Proposition 7.47 (equivalently, Definition 7.24).

Lemma 7.48. *Suppose $u_n > pu_{n-1}$. Let $\bar\alpha_1, \bar\alpha_2 \in k^\times$ with $\bar\alpha_1^{N_2} \neq \bar\alpha_2^{N_2}$. Choose lifts α_i of the $\bar\alpha_i$ to R. For $i = 1, 2$, suppose $G_i \in \mathcal{G}_{g,\alpha_i}$ gives rise to a character with good reduction. Then we can write*

$$G_2/G_1 \equiv 1 + p^{p/(p-1)}F \quad (\mathrm{mod}\ (\mathbb{K}^\times)^p),$$

where $F \in \mathbb{K}$ satisfies $v_0(F) = 0$ and $[F]_0$ is a polynomial in $t^{-1} = [T^{-1}]_0$ of degree u_n with leading term

$$Cu_n^{-1}\big((-\bar\alpha_1^m)^{N_2/m} - (-\bar\alpha_2^m)^{N_2/m}\big)t^{-u_n}.$$

Proof. Since G_1 and G_2 both give rise to characters with étale reduction, Proposition 6.20 shows that $\mathfrak{K}_n((G_2/G_1)^{p^{n-1}})$ has étale reduction as well. In particular, $\mathfrak{K}_1(G_2/G_1)$ has étale reduction. By Proposition 6.22, we can write

$$G_2/G_1 \equiv 1 + p^{p/(p-1)}F \quad (\mathrm{mod}\ (\mathbb{K}^\times)^p)$$

for some $F \in \mathbb{K}$ with $v_0(F) \geq 0$. Furthermore, by multiplying by a pth power, we may assume that when F is expanded out as a power series in T^{-1}, no terms with valuation 0 have degree divisible by p.

On the other hand, Corollary 7.45 and Proposition 6.22 show that we can write

$$G_2/G_1 \equiv 1 + p^s \Phi \quad (\mathrm{mod} \ (\mathbb{K}^\times)^p),$$

where $v_{r_{\mathrm{hub}}}(\Phi) = 0$ and $[\Phi]_{r_{\mathrm{hub}}}$ has derivative

$$C\left((-\bar\alpha_2^m)^{N_2/m} - (-\bar\alpha_1^m)^{N_2/m}\right)t^{-(u_n+1)} + \text{higher order terms})dt$$

when expanded out as a power series in $t^{-1} = [T^{-1}]_{r_{\mathrm{hub}}}$.

Write $1 + p^s\Phi = 1 + \sum_{i=1}^\infty a_i T^{-i}$. Then $v(a_{u_n}) = s + u_n r_{\mathrm{hub}} = p/(p-1)$ and $v(a_i) > p/(p-1)$ when $i > u_n$. Also,

$$[a_{u_n} T^{-u_n}]_0 = C u_n^{-1}\left((-\bar\alpha_1^m)^{N_2/m} - (-\bar\alpha_2^m)^{N_2/m}\right)t^{-u_n}.$$

The lemma now follows from Lemma 7.46, taking M, α, F_1, and F_2 to be u_n, $p/(p-1)$, $1 + p^s\Phi$, and $1 + p^{p/(p-1)}F$, respectively. $\square$

7.9. Geometry of the branch locus. In this section, we briefly summarize the geometry of the branch locus of the lifts our method gives for extensions as in Theorem 1.14. We only sketch the arguments. Recall that we start with a Γ-extension $k[\![z]\!]/k[\![s]\!]$ whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(u_1, \ldots, u_n)$ and no essential ramification. We have shown that we can lift this to a Γ-extension $R[\![Z]\!]/R[\![S]\!]$. Let $k[\![t]\!]/k[\![s]\!]$ and $R[\![T]\!]/R[\![S]\!]$ be the respective intermediate subextensions. The generic fiber of $\mathrm{Spec}\, R[\![Z]\!] \to \mathrm{Spec}\, R[\![T]\!] \to \mathrm{Spec}\, R[\![S]\!]$ corresponds to a tower of branched covers of non-archimedean disks. Since $\mathrm{Spec}\, R[\![T]\!] \to \mathrm{Spec}\, R[\![S]\!]$ is simply a $\mathbb{Z}/m$-cover totally branched at $S = 0$, we describe the branch locus of the $\mathbb{Z}/p^n$-cover $\mathrm{Spec}\, R[\![Z]\!] \to \mathrm{Spec}\, R[\![T]\!]$.

For each $1 \le j \le n$, let $i = n + 1 - j$, and let $N_{i,1}$ be as in Theorem 1.14. For $j < n$, there are $u_i - u_{i-1}$ branch points of index p^j arranged as follows: $N_{i,1}$ of these branch points are equidistant from each other and from the origin, at a mutual distance of $|p|^{r_{i,\mathrm{crit}}}$, where $r_{i,\mathrm{crit}} = 1/u_{i-1}(p-1)$. If we let $N_{i,2} = u_i - u_{i-1} - N_{i,1}$, then the other $N_{i,2}$ branch points lie at a distance of $p^{r_{i,\mathrm{hub}}}$ from the origin, where

$$r_{i,\mathrm{hub}} = \frac{1}{N_{i,2}} - \frac{N_{i,1}}{(p-1)u_{n-1}N_{i,2}}$$

(assuming $N_{i,2} > 0$). This all follows from the discussion before Proposition 7.17 and the inductive nature of the proof of Theorem 1.14. The $N_{i,2}$ branch points at a distance of $p^{r_{i,\mathrm{hub}}}$ from the origin come in m families, with any two points in distinct families at a distance of exactly $|p|^{r_{i,\mathrm{hub}}}$ from each other. Within a family, the distance between any two of them is not easy to calculate exactly, but it cannot be more than $|p|^{r_{i,\mathrm{hub}}+s_i/(N_{i,2}/m-1)}$, where $s_i = p/(p-1) - u_i r_{i,\mathrm{hub}}$. This follows from (7.14) and Definition 7.29 via a Newton polygon argument, along with the inductive nature of the proof of Theorem 1.14.

The $u_1 + 1$ branch points of index p^n are arranged as in §5. Specifically, if $u_1 < p$, then all these points are equidistant from the origin and from each other, at a mutual distance of $|p|^{1/u_1(p-1)}$. If $u_1 > p$, then these points all lie at a distance $|p|^{p\epsilon}$ from the origin, where ϵ can be chosen in the interval $(1/pu_1(p-1), 1/u_1(p-1))$. We refer the reader to [6, Theorem 4.3] for the finer geometry of this situation when $m = 2$, and leave the generalization to $m > 2$ as an exercise.

Since any two branch points of distinct indices lie at distinct distances from the origin, the ultrametric inequality determines their distance from each other uniquely.

8. Examples of lifting

In this section, we write down several examples where the isolated differential data criterion holds, and we derive consequences for the local lifting problem.

8.1. Instances of the isolated differential data criterion. Because of Theorem 4.3, the quadruples for which the isolated differential data criterion (Definition 7.12) is of interest to us are those of the form $(p, m, \tilde{u}, N_1)$, where $N_1 = (p-1)\tilde{u}$ or $N_1 = (p-1)\tilde{u} - m$. Recall that we always assume $m|(p-1)$ and $\tilde{u} \equiv -1 \pmod{m}$.

We begin with some small examples:

Proposition 8.1. *The quadruples* $(3, 2, 1, 2)$, $(3, 2, 1, 0)$, $(3, 2, 5, 8)$, *and* $(3, 2, 5, 10)$ *satisfy the isolated differential data criterion.*

Proof. The first two cases are covered under Proposition 8.2, which does not depend on this proposition, so suppose we are in the third or fourth cases.

Let $f_8 = t^8 + t^6 + 1$, and let $f_{10} = 2t^{10} + t^8 + t^6 + 1$. We claim that f_{N_1} realizes the isolated differential data criterion for $(3, 2, 5, N_1)$ with $N_1 \in \{8, 10\}$. Let $\omega_{N_1} = dt/f_{N_1}t^6$. The assertion that

$$\mathcal{C}(\omega) = \omega - \frac{dt}{t^6}$$

can be checked using a computer algebra system, for instance SAGE (it is easier to verify that

$$\mathcal{C}(f_{N_1}^3 \omega) = f_{N_1}\omega - \frac{f_{N_1}dt}{t^6},$$

as no power series are necessary — indeed, the computation is small enough to be checked by hand). Thus f_{N_1} realizes the differential data criterion for $(3, 2, 5, N_1)$.

Let $\bar{x}_1, \ldots, \bar{x}_{N_1/2}$ be a complete set of representatives of the μ_2-equivalence classes of the roots of f_{N_1}. We note that the $\bar{x}_j^2$ are pairwise distinct. Then f_{N_1} realizes the isolated differential data criterion if the matrix in Remark 7.13 is invertible over k. That is, we must show that the matrix

$$A_{N_1} = \left((\bar{x}_j^2)^i\right)_{i,j}$$

with $j \in \{1, \ldots, N_1/2\}$ and $i \in \{0, 2, 3, 5\}$ ($N_1 = 8$) or $i \in \{0, 2, 3, 5, 6\}$ ($N_1 = 10$) is invertible. Heinemann's formula for generalized Vandermonde determinants ([18, Theorem IV] — take $n = 4$ and $s = 3$ in the formula if $N_1 = 8$ and $n = 5$ and $s = 4$ if $N_1 = 10$) shows that

$$\det(A_{N_1}) = \begin{cases} D \det \begin{pmatrix} e_3 & e_4 \\ e_0 & e_1 \end{pmatrix} & N_1 = 8 \\[2ex] D \det \begin{pmatrix} e_4 & e_5 \\ e_1 & e_2 \end{pmatrix} & N_1 = 10 \end{cases}$$

where

- D is an integral power of the (standard) Vandermonde determinant corresponding to $\bar{x}_1^2, \ldots, \bar{x}_{N_1/2}^2$.

- For all s, the number e_s is the sth elementary symmetric polynomial in the $\bar{x}_j^2$'s.

Since the standard Vandermonde determinants are invertible, we need only show that $e_3 e_1 - e_4 e_0$ is invertible when $N_1 = 8$, and that $e_4 e_2 - e_5 e_1$ is invertible when $N_1 = 10$. If $N_1 = 8$, then $\prod_{j=1}^4 (t - \bar{x}_j^2) = t^4 + t^3 + 1$, and thus $e_3 e_1 - e_4 e_0 = -1$. If $N_1 = 10$, then $\prod_{j=1}^5 (t - \bar{x}_j^2) = t^5 + 2t^4 + 2t^3 + 2$, and $e_4 e_2 - e_5 e_1 = -1$. We are done. $\qquad\square$

Proposition 8.2. *For any (odd) prime p, the isolated differential data criterion holds for $(p, 2, 1, N_1)$, when $N_1 = p - 1$ or $N_1 = p - 3$.*

Proof. Using Proposition 7.8, we may realize the differential data criterion by solving the system of equations (7.7). If $N_1 = 0$ there is nothing to do, so assume otherwise. Let $\bar{x}_j = j$ for all $j \in \{1, 2, \ldots, N_1/2\}$. The square matrix $(\bar{x}_j^q)_{q,j}$ for $j \in \{1, 2, \ldots, N_1/2\}$ and $q \in \{1, 3, \ldots, N_1 - 1\}$ is Vandermonde (up to multiplication of each column by a nonzero scalar). Since $N_1 < p$, the $\bar{x}_j$ all have distinct squares and thus the columns of this Vandermonde matrix are all distinct. So the system (7.7) has a unique solution for the a_j with the $a_j \in k$. Since the $\bar{x}_j^q$ in fact lie in $\mathbb{F}_p$, so do the a_j. We must show that no a_j is zero.

For a contradiction, assume, after possibly renumbering the $\bar{x}_j$, that $a_{N_1/2} = 0$. If $N_1 = 2$, this is clearly a contradiction, and we are done. Assume otherwise. Since $u = 1$ in (7.7), we must have

$$\sum_{j=1}^{N_1/2-1} a_j \bar{x}_j^q = 0$$

for all $q \in \{3, 5, \ldots, N_1 - 1\}$. Since $\bar{x}_j \neq 0$, we once again have (up to rescaling) a Vandermonde system of linear equations for the a_j, $j \in \{1, 2, \ldots, N_1/2 - 1\}$. Thus all a_j are zero, which is a contradiction. This gives the differential data criterion.

To prove isolatedness, we remark that the matrix $\left(\bar{x}_j^{q-1} \right)_{q,j}$ for $q \in \{1, 3, \ldots, N_1 - 1\}$ and $j \in \{1, 2, \ldots, N_1/2\}$ in Remark 7.13 is, up to scaling, Vandermonde with distinct columns. So it is invertible.   □

Lemma 8.3. *The quadruple* $(p, m, \tilde{u}, (p-1)\tilde{u})$ *satisfies the differential data criterion for all odd primes* p*, all* $m \mid (p - 1)$*, and all* $\tilde{u} \equiv -1 \pmod{m}$*.*

Proof. As in Proposition 8.2, we will realize the differential data criterion by solving the system of equations (7.7). Write $\tilde{u} = up^{\nu}$, with $p \nmid u$. Note that the set S of $u(p^{\nu+1} - 1)$th roots of unity whose $-u$th powers have trace zero (under $\mathrm{Tr}_{\mathbb{F}_{p^{\nu+1}}/\mathbb{F}_p}$) has cardinality $u(p^{\nu} - 1)$. Thus, we have

$$|\mu_{u(p^{\nu+1}-1)} \setminus S| = u(p^{\nu+1} - p^{\nu}).$$

Furthermore, multiplication by mth roots of unity (which all lie in $\mathbb{F}_p$) preserves S and $\mu_{u(p^{\nu+1}-1)} \setminus S$. We take the $\bar{x}_i$ to be any complete set of orbit representatives for the multiplicative action of μ_m on $\mu_{u(p^{\nu+1}-1)} \setminus S$. Note that there are

$$u(p^{\nu+1} - p^{\nu})/m = (p - 1)\tilde{u}/m$$

of these orbits, so we have the correct number of $\bar{x}_j$. Furthermore, for each $\bar{x}_j$, let the associated a_j be given by the formula

$$a_j = -\mathrm{Tr}(\bar{x}_j^{-u}) = -\sum_{i=0}^{\nu} \bar{x}_j^{-up^i},$$

where for simplicity, we write Tr for $\mathrm{Tr}_{\mathbb{F}_{p^{\nu+1}}/\mathbb{F}_p}$. This is a nonzero element of $\mathbb{F}_p$.

We then have, for any $q \equiv -1 \pmod{m}$:

$$\sum_{j=1}^{N/m} a_j \bar{x}_j^q = \sum_{j=1}^{N/m} -\mathrm{Tr}(\bar{x}_j^{-u})\bar{x}_j^q$$

$$= \frac{1}{m} \sum_{x \in \mu_{u(p^{\nu+1}-1)} \setminus S} -\mathrm{Tr}(x^{-u})x^q = \frac{1}{m} \sum_{x \in \mu_{u(p^{\nu+1}-1)}} -\mathrm{Tr}(x^{-u})x^q$$

$$= \frac{1}{m} \sum_{x \in \mu_{u(p^{\nu+1}-1)}} -\left(x^{q-u} + x^{q-up} + \cdots + x^{q-up^{\nu}}\right)$$

$$= \begin{cases} u/m, & q \equiv u, up, \ldots, up^{\nu} \pmod{u(p^{\nu+1} - 1)}, \\ 0, & \text{otherwise.} \end{cases}$$

The second equality above comes from the fact that $u \equiv -1 \pmod{m}$ and $q \equiv -1 \pmod{m}$, so multiplying any $\bar{x}_j$ by any mth root of unity leaves $\mathrm{Tr}(\bar{x}_j^{-u})\bar{x}_j^q$ invariant. This solves system (7.7) when we restrict to the case $1 \leq q \leq (p - 1)\tilde{u} + \tilde{u} - 1 = up^{\nu+1} - 1$ and $p \nmid q$.   □

Proposition 8.4. *In the situation of Lemma* 8.3, *if* $\tilde{u} = (m-1)p^\nu$ *for some* $\nu \geq 0$, *then* $(p, m, \tilde{u}, (p-1)\tilde{u})$ *satisfies the isolated differential data criterion.*

Proof. Let the a_j and $\bar{x}_j$ be as in Lemma 8.3. Recall that each $\bar{x}_j$ is in $\mu_{u(p^{\nu+1}-1)}$.

For any set $\Sigma \subseteq \mathbb{Z}$, let $\overline{\Sigma}$ be its image as a subset of $\mathbb{Z}/u(p^{\nu+1}-1)$. Write $u = m - 1$. Let

$$A = \left(\bar{x}_j^{q-1} \right)_{q,j}$$

with j ranging from 1 to $(p-1)up^\nu/m$ and q ranging from 1 to $up^{\nu+1} - 1$ over those numbers congruent to $-1 \pmod{m}$ and not divisible by p. By Remark 7.13, it suffices to show that A is invertible.

We first claim that the set

$$B := \{ q \mid q \text{ corresponds to a row of } A \}$$

and the set

$$C := \{ m - 1 + imp \}_{0 \leq i < \frac{N}{m}}$$

satisfy $\overline{B} = \overline{C}$. To prove the claim, note that if

$$C' = \{ m - 1 + imp \}_{0 \leq i < \frac{u(p^{\nu+1}-1)}{m}},$$

then $\overline{C'}$ is exactly the set of elements of $\mathbb{Z}/u(p^{\nu+1}-1)$ congruent to $-1 \pmod{m}$ (this abuse of language is justified since $m \mid (p-1) \mid u(p^{\nu+1}-1)$). Furthermore, a straightforward computation shows that

$$\overline{C'}\backslash\overline{C} = \{ up, (u+m)p, \ldots, up^{\nu+1} - mp \}.$$

Now, by the Chinese Remainder Theorem, the set

$$\{ up - 1, (u+m)p - 1, \ldots, up^{\nu+1} - mp - 1 \},$$

viewed as a subset of $\mathbb{Z}$, is exactly the set of integers between 1 and $up^{\nu+1} - 1$ which are congruent to $-1 \pmod{m}$ and to $0 \pmod{p}$ (this is where we use $u = m - 1$). Thus $\overline{C'}\backslash\overline{C} = \overline{C'}\backslash\overline{B}$. Since $\overline{B} \subseteq \overline{C'}$ is clear, we have $\overline{B} = \overline{C}$, proving the claim.

The claim shows that the elements of $\overline{B}$, arranged appropriately, form an arithmetic progression with common difference mp. If A' is the matrix obtained by rearranging the rows of A to correspond to this ordering, then the definition of A shows that the jth column of A' is a geometric progression with common ratio $\bar{x}_j^{mp}$. The common ratios of the columns are pairwise distinct, as the x_j are all $u(p^{\nu+1}-1)$th roots of unity lying in pairwise distinct multiplicative μ_m-orbits, and $p \nmid u(p^{\nu+1}-1)$. We can scale each column to make a new matrix A'' where the first entry in each column is equal to 1. Then A'' is a Vandermonde matrix with pairwise distinct column ratios. So A'' is invertible, which means A is invertible. $\quad\square$

Remark 8.5. It is not hard to show, in the context of Lemma 8.3, that if $\tilde{u}$ is *not* a pth power times $(m-1)$, then the proposed solution in Lemma 8.3 will *never* realize the isolated differential data criterion. Indeed, the matrix A from the proof of Proposition 8.4 can be shown to have at least two identical rows.

8.2. Affirmative local lifting results.

Theorem 8.6. *The dihedral group D_9 is a local Oort group for $p = 3$.*

Proof. By Proposition 1.11, we need only consider D_9-extensions whose $\mathbb{Z}/9$-subextension has upper jumps $(1, 3)$, $(1, 5)$, $(1, 7)$, $(5, 15)$, $(5, 17)$, or $(5, 19)$. By Theorem 1.14, it suffices to show that the isolated differential data criterion holds for $(3, 2, 1, 2)$, $(3, 2, 1, 0)$, $(3, 2, 5, 10)$, and $(3, 2, 5, 8)$. This follows from Proposition 8.1. $\square$

Theorem 8.7. *If p is an odd prime, and $L/k[\![s]\!]$ is a D_{p^2}-extension whose $\mathbb{Z}/p^2$-subextension has first upper ramification break $u_1 \equiv 1 \pmod{p}$, then $L/k[\![s]\!]$ lifts to characteristic zero.*

Proof. Since u_1 is odd, we have that $u_1 \equiv 1 \pmod{2p}$. By Proposition 1.11, we need only consider D_{p^2}-extensions whose $\mathbb{Z}/p^2$-subextension has first upper jump 1. By Theorem 1.14, it suffices to show that the isolated differential data criterion holds for $(p, 2, 1, p-1)$ and $(p, 2, 1, p-3)$ This follows from Proposition 8.2. $\square$

Theorem 8.8. *If $L/k[\![s]\!]$ is a $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks congruent to*

$$(m - 1, p(m - 1), \ldots, p^{n-1}(m - 1)) \pmod{mp},$$

then $L/k[\![s]\!]$ lifts to characteristic zero. In particular, $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ is a weak local Oort group whenever the conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful.

Proof. By Proposition 1.11, we need only consider $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extensions whose $\mathbb{Z}/p^n$-subextension has upper ramification breaks $(m-1, p(m-1), \ldots, p^{n-1}(m-1))$ (such extensions exist by [23, Theorem 1.1]). By Theorem 1.14, it suffices to show that the isolated differential data criterion holds for

$$(p, m, (m - 1)p^{\nu-1}, (p - 1)(m - 1)p^{\nu-1})$$

for $0 \leq \nu < n$. This follows from Proposition 8.4. $\square$

9. Proof of Propositions 7.20 and 7.36

We use the notation of §6.1, §6.2 and §7 throughout. In particular, recall that

- $pu_{n-1} \le u_n < pu_{n-1} + mp$ (no essential ramification).

- $N = N_1 + N_2 = u_n - u_{n-1}$, and both N_1 and N_2 are divisible by m (Proposition 7.5).

- $N_1 \le (p-1)u_{n-1}$ with strict inequality unless $u_n = pu_{n-1}$ (Proposition 7.5).

- $N_2 \le mp$ (Assumption 7.25).

- $r_{\mathrm{crit}} = 1/(p-1)u_{n-1}$ (beginning of §7.2).

- $r_{\mathrm{hub}} = 1/N_2 - N_1/(p-1)u_{n-1}N_2$, or $r_{\mathrm{hub}} = 0$ if $N_2 = 0$ (Proposition 7.17).

- $s = (N_1 + u_{n-1})(r_{\mathrm{crit}} - r_{\mathrm{hub}}) = p/(p-1) - u_n r_{\mathrm{hub}} = p/(p-1) - \delta_{\mathrm{hub}}$ (Equation (7.14)).

- If $r \in \mathbb{Q}_{\ge 0}$, then $T_r = p^{-r}T$. For short, $T_{\mathrm{crit}} = p^{-r_{\mathrm{crit}}}T$ and $T_{\mathrm{hub}} = p^{-r_{\mathrm{hub}}}T$ (beginning of §6.2).

- v_r and v_r' are defined as in Definition 6.14 (and v_r is a valuation on $R\{T_r^{-1}\} \otimes_R K$).

- $\mathscr{G}_{\mathrm{crit},g}$ and $\mathscr{G}_{\mathrm{hub},\alpha}$ are defined as in Definitions 7.15, 7.29, respectively. Here g is a solution to (7.6), corresponding to an f realizing the isolated differential data criterion (Remark 7.9).

As a matter of notation, in the context of a congruence between two power series or polynomials in T_r^{-1}, the symbol $\equiv'$ (resp. $='$) means that the congruence (resp. equality) need only hold for terms of degree congruent to $-1 \pmod{m}$ or $0 \pmod{p}$ in T_r^{-1}.

While neither Proposition 7.20 nor Proposition 7.36 follows directly from the other, their proofs are very similar, and we will prove them simultaneously. Essentially, the proof of Proposition 7.20 is an easier version of the proof of Proposition 7.36.

9.1. Preliminaries. We start by defining the ch (think "crit-hub") "valuation," which is not actually a valuation, but has many similar properties.

Definition 9.1. For a power series

$$F = \sum_{q=0}^{N_1+u_{n-1}-1} c_q T_{\mathrm{crit}}^{-q} + \sum_{q=N_1+u_{n-1}}^{\infty} p^s c_q T_{\mathrm{hub}}^{-q} \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K,$$

we write $v_{\mathrm{ch}}(F) = \min_q v(c_q)$. We define $v_{\mathrm{ch}}'(F)$ in the same way, except we only take the minimum over q that are congruent either to $-1 \pmod{m}$ or $0 \pmod{p}$.

Definition 9.2.

(i) An element $f \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$ is called *hub-negligible* if, for all $r \in [0, r_{\mathrm{hub}}] \cap \mathbb{Q}$, we have $v_r(f) > p/(p-1) - u_n r$. If $f \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$, then making a *hub-negligible adjustment* to f means replacing it with some $f' \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$ where $f' - f$ is hub-negligible.

(ii) An element $f \in R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K$ is called *crit-negligible* if, for all $r \in [r_{\mathrm{hub}}, r_{\mathrm{crit}}] \cap \mathbb{Q}$, we have $v_r(f) > (N_1 + u_{n-1})(r_{\mathrm{crit}} - r)$. If $f \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$, then making a *crit-negligible adjustment* to f means replacing it with some $f' \in R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K$ where $f' - f$ is crit-negligible.

Lemma 9.3.

(i) *Let*

$$f = \sum_{i=0}^{\infty} c_i T_{\mathrm{hub}}^{-i} \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K.$$

If $v_0(c_i T_{\mathrm{hub}}^{-i}) > p/(p-1)$ for $i < u_n$ and $v_{\mathrm{ch}}(f) > 0$, then f is hub-negligible.

(ii) *Let*

$$f = \sum_{i=0}^{\infty} c_i T_{\mathrm{crit}}^{-i} \in R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K.$$

If $v_0(c_i T_{\mathrm{crit}}^{-i}) > p/(p-1)$ for $i < N_1 + u_{n-1}$ and $v_{r_{\mathrm{crit}}}(f) > 0$, then f is crit-negligible.

Proof. It suffices to check each monomial in f. In case (i), if $i < u_n$, the definition of hub-negligibility yields $v_r(c_i T_{\mathrm{hub}}^{-i}) > p/(p-1) - ir$, proving the lemma for these terms. When $i \geq u_n$, the fact that $v_{\mathrm{ch}}(c_i T_{\mathrm{hub}}^{-i}) > 0$ implies that $v_{r_{\mathrm{hub}}}(c_i T_{\mathrm{hub}}^{-i}) = v(c_i) > s$. Now, $s = p/(p-1) - u_n r_{\mathrm{hub}}$. Thus, for $r \leq r_{\mathrm{hub}}$,

$$v_r(c_i T_{\mathrm{hub}}^{-i}) > s + i(r_{\mathrm{hub}} - r) \geq s + u_n(r_{\mathrm{hub}} - r) = p/(p-1) - u_n r.$$

In case (ii), if $i < N_1 + u_{n-1}$, then

$$v_r(c_i T_{\mathrm{crit}}^{-i}) > p/(p-1) - ir > p/(p-1) - (N_1 + u_{n-1})r \geq (N_1 + u_{n-1})(r_{\mathrm{crit}} - r),$$

because $N_1 + u_{n-1} \leq p u_{n-1}$. This proves the lemma for these terms. If $i \geq N_1 + u_{n-1}$, then

$$v_r(c_i T_{\mathrm{crit}}^{-i}) > i(r_{\mathrm{crit}} - r) \geq (N_1 + u_{n-1})(r_{\mathrm{crit}} - r),$$

and we are done. $\square$

Remark 9.4. Lemma 9.3 shows that if $f \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$ and $v_{\mathrm{ch}}(f) > 0$, then removing the terms of f of degree at least u_n in T^{-1} is a hub-negligible adjustment. Likewise, if $f \in R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K$ and $v_{r_{\mathrm{crit}}}(f) > 0$, then removing the terms of f of degree at least $N_1 + u_{n-1}$ in T^{-1} is a crit-negligible adjustment.

Lemma 9.5. *Suppose f_1 and f_2 are in $R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$.*

(i) *We have $v_{\mathrm{ch}}(f_1 + f_2) \geq \min(v_{\mathrm{ch}}(f_1), v_{\mathrm{ch}}(f_2))$, with equality if $v_{\mathrm{ch}}(f_1) \neq v_{\mathrm{ch}}(f_2)$, and the same holds for v'_{ch}.*

(ii) *We have $v_{\mathrm{ch}}(f_1 f_2) \geq v_{\mathrm{ch}}(f_1) + v_{\mathrm{ch}}(f_2)$.*

Proof. Part (i) is obvious, and reduces part (ii) to the case of monomials. The only non-obvious case is if $f_1 = a T_{\mathrm{crit}}^{-b}$ and $f_2 = c p^s T_{\mathrm{hub}}^{-d}$, where $b < N_1 + u_{n-1}$ and $b + d \geq N_1 + u_{n-1}$. Then $v_{\mathrm{ch}}(f_1) = v(a)$ and $v_{\mathrm{ch}}(f_1 f_2) = v(a) + v(c) + b(r_{\mathrm{crit}} - r_{\mathrm{hub}})$. If $d \geq N_1 + u_{n-1}$, then $v_{\mathrm{ch}}(f_2) = v(c)$, which proves part (ii) since $r_{\mathrm{crit}} > r_{\mathrm{hub}}$. If $d < N_1 + u_{n-1}$, then $v_{\mathrm{ch}}(f_2) = v(c) + s - d(r_{\mathrm{crit}} - r_{\mathrm{hub}})$. So $v_{\mathrm{ch}}(f_1 f_2) - v_{\mathrm{ch}}(f_1) - v_{\mathrm{ch}}(f_2) = (b + d)(r_{\mathrm{crit}} - r_{\mathrm{hub}}) - s$. This is nonnegative, since $b + d \geq N_1 + u_{n-1}$. This proves part (ii). $\square$

Corollary 9.6. *If f_1 and f_2 are in $R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$ with $v_{\mathrm{ch}}(f_1)$ and $v_{\mathrm{ch}}(f_2) \geq 0$, then $v_{\mathrm{ch}}(f_1 f_2 - 1) \geq \min(v_{\mathrm{ch}}(f_1 - 1), v_{\mathrm{ch}}(f_2 - 1))$.*

Proof. Since $f_1 f_2 - 1 = (f_1 - 1)(f_2 - 1) + (f_1 - 1) + (f_2 - 1)$, the corollary follows from Lemma 9.5. $\square$

Remark 9.7. Of course, since $v_{r_{\mathrm{crit}}}$ is a valuation, Lemma 9.5 and Corollary 9.6 are also true when applied to $R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K$, with $v_{r_{\mathrm{crit}}}$ and $v'_{r_{\mathrm{crit}}}$ replacing v_{ch} and v'_{ch}, respectively.

Lemma 9.8. *If $f \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$, then $v_{\mathrm{ch}}(f) \geq \max(v_{r_{\mathrm{crit}}}(f), v_{r_{\mathrm{hub}}}(f) - s)$, and the same is true when v_{ch}, $v_{r_{\mathrm{crit}}}$, and $v_{r_{\mathrm{hub}}}$ are replaced by v'_{ch}, $v'_{r_{\mathrm{crit}}}$, and $v'_{r_{\mathrm{hub}}}$, respectively. Furthermore, $v_{r_{\mathrm{hub}}}(f) \geq v_{\mathrm{ch}}(f)$ and $v'_{r_{\mathrm{hub}}}(f) \geq v'_{\mathrm{ch}}(f)$.*

Proof. It suffices to prove the statements for v_{ch} applied to monomials $f = T_{r_{\mathrm{hub}}}^{-i}$. Then $v_{r_{\mathrm{hub}}}(f) = 0$ and $v_{r_{\mathrm{crit}}}(f) = i(r_{\mathrm{hub}} - r_{\mathrm{crit}})$. If $i \geq N_1 + u_{n-1}$, then $v_{\mathrm{ch}}(f) = -s$, which is greater than $i(r_{\mathrm{hub}} - r_{\mathrm{crit}})$ and nonpositive. If $i < N_1 + u_{n-1}$, then $v_{\mathrm{ch}}(f) = i(r_{\mathrm{hub}} - r_{\mathrm{crit}})$, which is greater than or equal to $-s$ and nonpositive. $\square$

Lemma 9.9.

(i) *Let $f, g \in R\{T_{\mathrm{hub}}^{-1}\} \otimes_R K$ such that f is hub-negligible and $v_{\mathrm{ch}}(g) \geq 0$. Then fg is hub-negligible.*

(ii) *Let $f, g \in R\{T_{\mathrm{crit}}^{-1}\} \otimes_R K$ such that f is crit-negligible and $v_{r_{\mathrm{crit}}}(g) \geq 0$. Then fg is crit-negligible.*

Proof. For part (i), since $v_{\mathrm{ch}}(g) \geq 0$, we have $v_r(g) \geq 0$ for all $r \leq r_{\mathrm{hub}}$. Thus

$$v_r(fg) = v_r(f) + v_r(g) \geq v_r(f) > p/(p-1) - u_n r.$$

Part (ii) is similar and just as easy. $\square$

Lemma 9.10. *If $f = 1 + h \in 1 + T_{\text{hub}}^{-1} R\{T_{\text{hub}}^{-1}\} \otimes_R K$ with $v_{\text{ch}}(h) = \beta > 0$, then $v_{\text{ch}}(f^{-1} - 1) = \beta$.*

Proof. We have $f^{-1} - 1 = 1/(1 + h) - 1 = -h + h^2 - h^3 + \cdots$. Now the result follows from Lemma 9.5. $\square$

Lemma 9.11. *Assume that $N_1 = (p - 1)u_{n-1} - m$, so that $r_{\text{hub}} > 0$. Let*

$$I := 1 + \sum_{l=1}^{\lfloor (N_1 + u_{n-1} - 1)/p \rfloor} b_l T_{\text{crit}}^{-pl} + \sum_{l=\lceil (N_1 + u_{n-1})/p \rceil}^{\lfloor (u_n - 1)/p \rfloor} p^s b_l T_{\text{hub}}^{-pl},$$

with all $b_l \in \mathfrak{m}$. Let

$$H := 1 + \sum_{l=1}^{\lfloor (N_1 + u_{n-1} - 1)/p \rfloor} b_l^{1/p} T_{\text{crit}}^{-l} + \sum_{l=\lceil (N_1 + u_{n-1})/p \rceil}^{\lfloor (u_n - 1)/p \rfloor} (p^s b_l)^{1/p} T_{\text{hub}}^{-l},$$

for any choice of pth roots of the coefficients. Then each term $c_i T^{-i}$ in $I - H^p$ for $i \geq 1$ satisfies

$$v_{\text{ch}}(c_i T^{-i}) > \theta_i + \frac{v_{\text{ch}}(I)}{p},$$

where

$$\theta_i = \begin{cases} \frac{p-1}{p}\left(\frac{p}{p-1} - i\, r_{\text{crit}}\right), & i < N_1 + u_{n-1}, \\ \frac{p-1}{p}\left(\frac{p}{p-1} - i\, r_{\text{hub}} - s\right), & i \geq N_1 + u_{n-1}. \end{cases}$$

The same holds when, instead of taking $I - H^p$, we expand out $I/H^p - 1$ as a power series in T^{-1}.

Proof. The terms in $H^p - I$ are the cross-terms in H^p. We consider the two cases separately. Note that the multinomial coefficient in any cross-term of H^p has valuation at least 1.

Suppose $i < N_1 + u_{n-1}$. Then we must show that

$$v_{r_{\text{crit}}}(c_i T^{-i}) \geq 1 + v_{\text{ch}}(I)/p - i/pu_{n-1}.$$

Each term in H can be written either as $b_l^{1/p} T_{\text{crit}}^{-l}$ or as $b_l^{1/p} p^{s/p} p^{-l(r_{\text{crit}} - r_{\text{hub}})} T_{\text{crit}}^{-l}$. Note that $v(b_l^{1/p}) \geq v_{\text{ch}}(I)/p$. If no terms of the second form factor into the given $c_i T^{-i}$, then the result is obvious. If at least one such term factors in, then

$$v_{r_{\text{crit}}}(c_i T^{-i}) \geq 1 + \frac{v_{\text{ch}}(I)}{p} + \frac{s}{p} - i(r_{\text{crit}} - r_{\text{hub}}).$$

Since $s = (N_1 + u_{n-1})(r_{\text{crit}} - r_{\text{hub}})$, it suffices to show that

$$((N_1 + u_{n-1})/p - i)(r_{\text{crit}} - r_{\text{hub}}) > -i/pu_{n-1}.$$

Substituting in $r_{\mathrm{crit}} = 1/(p-1)u_{n-1}$ and $r_{\mathrm{hub}} = m/N_2(p-1)u_{n-1}$, and multiplying both sides by $-pu_{n-1}$, we are reduced to showing that

$$i > (pi - N_1 - u_{n-1})\left(\frac{N_2 - m}{N_2(p-1)}\right).$$

Since $N_2 \leq mp$, the right hand side is at most $i - (N_1 + u_{n-1})/p$ (if it is positive), from which the result follows.

Now suppose $i \geq N_1 + u_{n-1}$. The we must show that

$$v_{r_{\mathrm{hub}}}(c_i T^{-i}) > 1 + v_{\mathrm{ch}}(I)/p - \frac{p-1}{p}(i r_{\mathrm{hub}} + s) + s.$$

At least one term of the form $b_l^{1/p} p^{s/p} T_{\mathrm{hub}}^{-l}$ factors into $c_i T^{-i}$ and all terms factoring in have nonnegative valuation at r_{hub}. So $v_{r_{\mathrm{hub}}}(c_i T^{-i}) \geq 1 + v_{\mathrm{ch}}(I)/p + s/p$. The desired inequality follows immediately.

To prove the statement for $I/H^p - 1$, note that $I/H^p - 1 = (I - H^p)H^{-p}$. By Lemma 9.10, we have $v_{\mathrm{ch}}(H^{-p}) = 0$. Write $(I/H^p - 1) = \sum_{i=1}^{\infty} d_i T^{-i}$. By Lemma 9.5,

$$v_{\mathrm{ch}}(d_i T^{-i}) \geq \min_{j \leq i} v_{\mathrm{ch}}(c_j T^{-j}) > \min_{j \leq i} \theta_j + \frac{v_{\mathrm{ch}}(I)}{p} = \theta_i + \frac{v_{\mathrm{ch}}(I)}{p}. \qquad \square$$

Remark 9.12. If we take θ_i to be any number less than 1, then Lemma 9.11 also holds for $I \in 1 + T_{\mathrm{crit}}^{-p}\mathfrak{m}[T_{\mathrm{crit}}^{-p}]$ and $v_{r_{\mathrm{crit}}}$ replacing v_{ch}. In particular, if we assume I has degree less than $N_1 + u_{n-1}$ in T^{-1}, then we may define θ_i (for $i < N_1 + u_{n-1}$) as in Lemma 9.11.

9.2. The underlying Hensel's lemma calculation.

For each of Propositions 7.20 and 7.36, we get most of the way to a proof via an application of Hensel's lemma. For Proposition 7.20, the necessary result is as follows.

Lemma 9.13 (cf. [24, Lemma 7.4(i)]). *Let $G \in \mathcal{G}_{\mathrm{crit},g}$, and let $J \in 1 + T_{\mathrm{crit}}^{-1}\mathfrak{m}\{T_{\mathrm{crit}}^{-1}\}$. There exists a unique $G' \in \mathcal{G}_{\mathrm{crit},g}$ and a unique polynomial $I \in 1 + T_{\mathrm{crit}}^{-p}\mathfrak{m}[T_{\mathrm{crit}}^{-p}]$ of degree $< N_1 + u_{n-1}$ in T_{crit}^{-1} such that*

$$\frac{G'}{G}I \equiv' J \pmod{T_{\mathrm{crit}}^{-(N_1+u_{n-1})}}.$$

If $J \equiv' 1 \pmod{p^\beta, T_{\mathrm{crit}}^{-(N_1+u_{n-1})}}$ for $\beta \in \mathbb{Q}_{>0}$, then $v_{r_{\mathrm{crit}}}(G'/G - 1) \geq \beta$ and $v_{r_{\mathrm{crit}}}(I - 1) \geq \beta$.

Proof. By assumption we have

$$G = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m}(1 - \zeta_m^{-\ell} x_j T_{\mathrm{crit}}^{-1})^{\psi_1(\tau^\ell)a_j},$$

where $x_j \in R$ is a lift of $\bar{x}_j$, where the $\bar{x}_j$ are a solution to (7.7) corresponding to g.

We set

$$G' = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} x_j' T_{\text{crit}}^{-1})^{\psi_1(\tau^\ell) a_j}, \quad x_j' := x_j + \epsilon_j,$$

and where the ϵ_j are for the moment considered as indeterminates. We also set

$$I := 1 + \sum_{l=1}^{\lfloor (N_1+u_{n-1}-1)/p \rfloor} b_l T_{\text{crit}}^{-pl}$$

for another system of indeterminates b_l. Write

$$\frac{G'}{G} I = 1 + \sum_{q=1}^{\infty} c_q T_{\text{crit}}^{-q},$$

where c_q is a formal power series in (ϵ_j, b_l). One computes, using $\psi_1(\tau^\ell) = \zeta_m^{-\ell}$ (Lemma 6.12), that

$$\frac{\partial c_q}{\partial \epsilon_j}\Big|_{\epsilon_j=b_l=0} = \sum_{\ell=1}^{m} \zeta_m^{-(q+1)\ell} a_j x_j^{q-1}$$

$$\frac{\partial c_q}{\partial b_l}\Big|_{\epsilon_j=b_l=0} = \begin{cases} 1, & q = pl, \\ 0, & q \neq pl. \end{cases}$$

(9.1)

In particular, when $q \equiv -1 \pmod{m}$, we have

$$\frac{\partial c_q}{\partial \epsilon_j}\Big|_{\epsilon_j=b_l=0} = m a_j x_j^{q-1},$$

and otherwise $\partial c_q / \partial \epsilon_j = 0$. The congruence

$$\frac{G'}{G} I \equiv' J \pmod{T_{\text{crit}}^{-(N_1+u_{n-1})}}$$

(9.2)

corresponds to a system of equations in the indeterminates (ϵ_j, b_l), one equation for each $c_q T_{\text{crit}}^{-q}$ for $q \equiv -1 \pmod{m}$ or $q \equiv 0 \pmod{p}$, with $1 \leq q < N_1 + u_{n-1}$. The Jacobian matrix M_{crit} of this system of equations is invertible over R if and only if its reduction is invertible over k. From (9.1) it is easy to see that this is true iff the matrix from (7.11) is invertible (One obtains the matrix in (7.11) from the Jacobian matrix by eliminating all of the columns corresponding to the b_l, which are standard basis vectors, along with the rows corresponding to equations for which $p|q$). The matrix from (7.11) is invertible because we are assuming that g realizes the isolated differential data criterion for (p, m, u_{n-1}, N_1). By Hensel's lemma, we conclude that (9.2) has a (unique) solution with $\epsilon_j, b_l \in \mathfrak{m}$, proving the first statement of the lemma. In fact, by the effective Hensel's Lemma, the second statement holds as well. $\square$

The analogous result toward Proposition 7.36 is the following:

Lemma 9.14. *Suppose $u_n > pu_{n-1}$. Let $G_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$, and $G_{\mathrm{hub}} \in \mathscr{G}_{\mathrm{hub},\alpha}$, and let*

$$J = 1 + \sum_{q=1}^{N_1+u_{n-1}-1} c_q T_{\mathrm{crit}}^{-q} + \sum_{q=N_1+u_{n-1}}^{\infty} p^s c_q T_{\mathrm{hub}}^{-q}$$

with all $c_q \in \mathfrak{m}$ and $\lim_{q\to\infty} c_q = 0$. There exists a unique $G'_{\mathrm{crit}} \in \mathscr{G}_{\mathrm{crit},g}$, a unique $G'_{\mathrm{hub}} \in \mathscr{G}_{\mathrm{hub},\alpha}$ and a unique polynomial

$$I := 1 + \sum_{l=1}^{\lfloor (N_1+u_{n-1}-1)/p \rfloor} b_l T_{\mathrm{crit}}^{-pl} + \sum_{l=\lceil (N_1+u_{n-1})/p \rceil}^{\lfloor (u_n-1)/p \rfloor} p^s b_l T_{\mathrm{hub}}^{-pl},$$

such that

$$\frac{G'_{\mathrm{crit}} G'_{\mathrm{hub}}}{G_{\mathrm{crit}} G_{\mathrm{hub}}} I \equiv' J \quad (\mathrm{mod}\ T_{\mathrm{hub}}^{-u_n}). \tag{9.3}$$

If for some $\beta \in \mathbb{Q}_{>0}$, we have $v(c_q) \geq \beta$ for all $q < u_n$, then

$$v_{r_{\mathrm{crit}}}(G'_{\mathrm{crit}}/G_{\mathrm{crit}} - 1) \geq \beta, \quad v_{\mathrm{ch}}(I - 1) \geq \beta, \quad \text{and} \quad v_{r_{\mathrm{hub}}}(G'_{\mathrm{hub}}/G_{\mathrm{hub}} - 1) \geq s + \beta.$$

Proof. As in Lemma 9.13, we have

$$G_{\mathrm{crit}} = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} x_j T_{\mathrm{crit}}^{-1})^{\zeta_m^{-\ell} a_j}.$$

Furthermore, by Definition 7.29, we have

$$G_{\mathrm{hub}} = \prod_{\ell=0}^{m-1} \left(1 + p^s \frac{\sum_{j=1}^{N_2/m-1} y_j \zeta_m^{-j\ell} T_{\mathrm{hub}}^{-j}}{(1 - \alpha \zeta_m^{-\ell} T_{\mathrm{hub}}^{-1})^{N_2/m-1}} \right)^{\zeta_m^{-\ell}}$$

with $y_j \in R$ (divide the numerator and denominator in (7.17) by $T_{\mathrm{hub}}^{N_2/m-1}$). Here, we are thinking of ζ_m as an integer given by taking some arbitrary lift of $\zeta_m \in \mathbb{F}_p^\times$ to $\mathbb{Z}$.

We look for potential solutions for G'_{crit} and G'_{hub} in the forms

$$G'_{\mathrm{crit}} = \prod_{j=1}^{N_1/m} \prod_{\ell=1}^{m} (1 - \zeta_m^{-\ell} x'_j T_{\mathrm{crit}}^{-1})^{\zeta_m^{-\ell} a_j}, \quad x'_j := x_j + \epsilon_j \tag{9.4}$$

and

$$G'_{\mathrm{hub}} = \prod_{\ell=0}^{m-1} \left(1 + p^s \frac{\sum_{j=1}^{N_2/m-1} y'_j \zeta_m^{-j\ell} T_{\mathrm{hub}}^{-j}}{(1 - \alpha \zeta_m^{-\ell} T_{\mathrm{hub}}^{-1})^{N_2/m-1}} \right)^{\zeta_m^{-\ell}}, \quad y'_j := y_j + \gamma_j, \tag{9.5}$$

where the ϵ_j and γ_j are considered as indeterminates.

Write

$$\frac{G'_{\text{hub}}G'_{\text{crit}}}{G_{\text{hub}}G_{\text{crit}}}I = 1 + \sum_{q=1}^{N_1+u_{n-1}-1} c_q T_{\text{crit}}^{-q} + \sum_{q=N_1+u_{n-1}}^{\infty} p^s c_q T_{\text{hub}}^{-q}, \tag{9.6}$$

where c_q is a formal power series in $(\epsilon_j, \gamma_j, b_l)$. By (9.6), the congruence (9.3) expresses the c_q relevant to $\equiv'$-congruence for $q < u_n$ in terms of formal power series in the indeterminates $(\epsilon_j, \gamma_j, b_l)$. We take M to be the Jacobian of this system of equations at 0. More specifically, let M be the Jacobian matrix (at $\epsilon_j = \gamma_j = b_l = 0$ for all j, l) of the following outputs and inputs: For the outputs, we take the variables c_q for $q < u_n$, where either $q \equiv -1 \pmod{m}$ or $p|q$. For the input variables, we take the ϵ_j, the b_l for $pl < N_1 + u_{n-1}$, the γ_j, and the b_l for $pl \geq N_1 + u_{n-1}$, in that order. The matrix M will be shown to be invertible over R in Proposition 9.16. We conclude by Hensel's lemma that (9.3) has a (unique) solution with $\epsilon_j, \gamma_j, b_l \in \mathfrak{m}$. In fact, by the effective Hensel's Lemma, $(\epsilon_j), v(\gamma_j)$, and $v(b_l)$ are all at least as large as $\min_{q<u_n} v(c_q)$. Given the forms in (9.4) and (9.5), this proves the lemma. $\qquad\square$

Remark 9.15. The reason we rescale some of the c_q and the b_l by p^s is to force M to be invertible. Our scaling of the c_q motivates the definition of v_{ch} in §9.1.

The rest of this section is dedicated to proving that the matrix M in the proof of Lemma 9.14 has entries in R and is invertible over R.

Let us calculate the entries of M, using the notation of the proof of Lemma 9.14. To do this, we calculate the partial derivatives of the c_q with respect to the ϵ_j, γ_j, and b_l at the point $\epsilon_j = \gamma_j = b_l = 0$ (all partials calculated below are evaluated at this point, and we suppress the point in the notation). For $q \equiv -1 \pmod{m}$, as in (9.1), we have

$$\frac{\partial c_q}{\partial \epsilon_j} = \begin{cases} ma_j x_j^{q-1}, & q < N_1 + u_{n-1}, \\ ma_j x_j^{q-1} p^{(r_{\text{crit}}-r_{\text{hub}})q-s}, & q \geq N_1 + u_{n-1}. \end{cases} \tag{9.7}$$

Also, we have

$$\frac{\partial c_q}{\partial b_l} = \begin{cases} 1, & q = pl \\ 0, & q \neq pl. \end{cases} \tag{9.8}$$

To calculate $\partial c_q/\partial\gamma_j$, first set $G_{\text{hub},0}$ and $G'_{\text{hub},0}$ equal to the $\ell = 0$ factors of G_{hub} and G'_{hub}, respectively. Then

$$\frac{G'_{\text{hub},0}}{G_{\text{hub},0}} = 1 + p^s \frac{\sum_{j=1}^{N_2/m-1} \gamma_j T_{\text{hub}}^{-j}}{(1 - \alpha T_{\text{hub}}^{-1})^{N_2/m-1} + p^s C(T_{\text{hub}})}.$$

When this is expanded out as a power series in T_{hub}^{-1}, the coefficient of T_{hub}^{-q} is

$$p^s \left(\sum_{j=1}^{N_2/m-1} \gamma_j \alpha^{q-j} \binom{q-j+N_2/m-2}{N_2/m-2} + O(p^s) \right),$$

where $O(p^s)$ represents terms with valuation at least s.

A computation now yields that

$$
\frac{\partial c_q}{\partial \gamma_j} =
\begin{cases}
p^{s-(r_{\mathrm{crit}}-r_{\mathrm{hub}})q} \sum_{\ell=0}^{m-1} \left(\zeta_m^{-(q+1)\ell} \alpha^{q-j} \binom{q-j+N_2/m-2}{N_2/m-2} + O(p^s) \right), \\
\hspace{8cm} q < N_1 + u_{n-1}, \\[2mm]
\sum_{\ell=0}^{m-1} \left(\zeta_m^{-(q+1)\ell} \alpha^{q-j} \binom{q-j+N_2/m-2}{N_2/m-2} + O(p^s) \right), \hspace{1cm} q \geq N_1 + u_{n-1}.
\end{cases}
\tag{9.9}
$$

In particular, when $q \equiv -1 \pmod{m}$, we have

$$
\frac{\partial c_q}{\partial \gamma_j} =
\begin{cases}
m p^{s-(r_{\mathrm{crit}}-r_{\mathrm{hub}})q} \left(\alpha^{q-j} \binom{q-j+N_2/m-2}{N_2/m-2} + O(p^s) \right), & q < N_1 + u_{n-1}, \\[2mm]
m \alpha^{q-j} \binom{q-j+N_2/m-2}{N_2/m-2} + O(p^s), & q \geq N_1 + u_{n-1}.
\end{cases}
\tag{9.10}
$$

It is clear from the above formulas that the entries of M lie in R. Write

$$
M = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array} \right)
\tag{9.11}
$$

as a block matrix, with the columns of M_1 corresponding to the variables x_j and b_l for $pl < N_1 + u_{n-1}$, and the rows of M_1 corresponding to the c_q for $q < N_1 + u_{n-1}$. Then one checks that M_1 is a square matrix of size $N_1/m + \lfloor (N_1 + u_{n-1} - 1)/p \rfloor$ (cf. Remark 7.10), and M_4 is square as well as will be seen in the proof of Proposition 9.16 below. In particular, M is a square matrix.

Proposition 9.16. *The matrix M is invertible over R.*

Proof. It suffices to show that the reduction $\bar{M} = \left(\begin{array}{c|c} \bar{M}_1 & \bar{M}_2 \\ \hline \bar{M}_3 & \bar{M}_4 \end{array} \right)$ of M has nonzero determinant. From (9.9), the valuation of $\partial c_q/\partial \gamma_j$ for $q < N_1 + u_{n-1}$ is at least $s - (r_{\mathrm{crit}} - r_{\mathrm{hub}})q$, which is $(r_{\mathrm{crit}} - r_{\mathrm{hub}})(N_1 + u_{n-1} - q) > 0$. Also, $\partial c_q/\partial b_l$ for $pl \geq N_1 + u_{n-1}$ and $q < N_1 + u_{n-1}$ is 0 by (9.8). Thus $\bar{M}_2 = 0$. So $\bar{M}$ is block lower triangular, and $\det(\bar{M}) = \det(\bar{M}_1) \det(\bar{M}_4)$. But M_1 is just the Jacobian matrix for the system in (9.2), where it was shown that M_1 is invertible over R. Thus $\det(\bar{M}_1) \neq 0$. So we are reduced to showing that $\bar{M}_4$ is invertible.

Each column of $\bar{M}_4$ corresponding to a variable b_l has a 1 in the row corresponding to $q = pl$ and a 0 in each other position. Eliminating these columns and the rows where 1's appear, we are left with an $(N_2/m - 1) \times (N_2/m - 1)$ matrix $\bar{M}_4'$. The entries of $\bar{M}_4'$ are the reductions of $\partial c_q/\partial \gamma_j$, where $1 \leq j \leq N_2 - 1$, and q ranges from $N_1 + u_{n-1}$ to $u_n - 1$ over those numbers congruent to $-1 \pmod{m}$ and not divisible by p. By (9.10), after multiplying rows and columns by units, the entry of $\bar{M}_4'$ corresponding to (q, j) is $\binom{q-j+N_2/m-2}{N_2/m-2}$, thought of as an element of $\mathbb{F}_p \subseteq k$. We will view the binomial coefficients as integers, and show that the determinant is not divisible by p.

We will modify $\bar{M}'_4$, without changing its determinant. For the first modification, moving from left to right, we subtract the $j = 2$ column from the $j = 1$ column. Then we subtract the $j = 3$ column from the $j = 2$ column. We continue until we subtract the $j = N_2/m - 1$ column from the $j = N_2/m - 2$ column. This gives a matrix whose entry in the (q, j) slot is $\binom{q-j+N_2/m-3}{N_2/m-3}$, except in the last column, where the entries are $\binom{q-j+N_2/m-2}{N_2/m-2}$. For the second modification, we repeat this process once more, except that we stop after subtracting the $j = N_2/m - 2$ column from the $j = N_2/m - 3$ column. For the third modification, we repeat again, stopping after subtracting the $j = N_2/m - 3$ column from the $j = N_2/m - 4$ column. We continue repeating until the $(N_2/m - 2)$nd modification, which consists only of subtracting the $j = 2$ column from the $j = 1$ column. All in all, the jth column gets modified $N_2/m - j - 1$ times. This leaves us with a matrix whose entry in the (q, j)-slot is $\binom{q-1}{j-1}$. We apply the formula given on [15, p. 308] (the "alternate expression" when $b = 0$) to get that the determinant of this matrix is

$$\frac{\prod_{1 \le i < j \le N_2/m-1}(b_i - b_j)}{1!2!\cdots(N_2/m - 1)!}, \tag{9.12}$$

where the b_i are the values of q corresponding to our c_q.

It suffices to check that the numerator in (9.12) is not divisible by p (in any case, the denominator is not divisible by p because $N_2 \le mp$ by Assumption 7.25). The expression $b_i - b_j$ can only take on values $m, 2m, \ldots, N_2 - m$, as $(u_n - m) - (N_1 + u_{n-1}) = N_2 - m$, and $u_n - m$ and $N_1 + u_{n-1}$ are the least and greatest values of q, respectively. By Assumption 7.25, we have $N_2 \le mp$, so the expression $b_i - b_j$ is never divisible by p. We are done. $\qquad\square$

9.3. Completion of the proofs. The main task in completing the proofs is to turn the I that occurs in Lemmas 9.13 and 9.14, and that is very close to a pth power, into an actual pth power. This will be done through a series of results. In each case, we will state and prove the result relevant to Lemma 9.14. Then we will state the analogous result relevant to Lemma 9.13, and mention which modifications are necessary for the proof to carry through. As a matter of fact, there are more straightforward proofs of most of the "Lemma 9.13 versions," but since we must write the more complicated versions anyway, we omit the simpler versions to save space.

Lemma 9.17. *Suppose $u_n > pu_{n-1}$. Let $G_{\text{crit}}, G_{\text{hub}} \in \mathcal{G}_{\text{crit},g}, \mathcal{G}_{\text{hub},\alpha}$ respectively. Let $J \in 1 + T^{-1}\mathfrak{m}\{T^{-1}\}$ such that $v_{\text{ch}}(J - 1) > 0$. Let θ_i be as in Lemma 9.11. After a possible finite extension of K and hub-negligible adjustment to J, there exist $G'_{\text{crit}}, G'_{\text{hub}} \in \mathcal{G}_{\text{crit},g}, \mathcal{G}_{\text{hub},\alpha}$ respectively, and a polynomial $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ such that if*

$$\frac{J}{(G'_{\text{crit}}G'_{\text{hub}}/G_{\text{crit}}G_{\text{hub}})H^p} = 1 + \sum_{i=1}^{\infty} c_i T^{-i},$$

then for $0 < i < u_n$, there exists $\epsilon > 0$ such that

$$
v_{\mathrm{ch}}(c_i T^{-i}) \geq
\begin{cases}
\min\left(v_{\mathrm{ch}}(J-1) + v'_{\mathrm{ch}}(J-1), \theta_i + \epsilon + \frac{v'_{\mathrm{ch}}(J-1)}{p}\right), \\
\hspace{5cm} p|i \text{ or } i \equiv -1 \pmod{m}, \\
\min\left(v_{\mathrm{ch}}(J-1), \theta_i + \epsilon + \frac{v'_{\mathrm{ch}}(J-1)}{p}\right), \quad \text{otherwise.}
\end{cases}
$$

If $v'_{\mathrm{ch}}(J-1) \geq \beta$ for some $0 < \beta < p/(p-1)$, then we can choose G'_{crit}, G'_{hub}, and H above such that $v_{r_{\mathrm{crit}}}(G'_{\mathrm{crit}}/G_{\mathrm{crit}} - 1) \geq \beta$, that $v_{r_{\mathrm{hub}}}(G'_{\mathrm{hub}}/G_{\mathrm{hub}} - 1) \geq s + \beta$, and that $v_{\mathrm{ch}}(H^p - 1) \geq \min(\beta, (p-1)r_{\mathrm{hub}}/p)$.

Proof. Let G'_{crit}, G'_{hub}, and

$$
I := 1 + \sum_{l=1}^{\lfloor (N_1 + u_{n-1} - 1)/p \rfloor} b_l T_{\mathrm{crit}}^{-pl} + \sum_{l=\lceil (N_1 + u_{n-1})/p \rceil}^{\lfloor (u_n - 1)/p \rfloor} p^s b_l T_{\mathrm{hub}}^{-pl},
$$

be the unique solution guaranteed by Lemma 9.14. So $(G'_{\mathrm{crit}} G_{\mathrm{hub}}/G_{\mathrm{crit}} G_{\mathrm{hub}})I \equiv' J$ (mod $T_{\mathrm{hub}}^{-u_n}$). Set

$$
H := 1 + \sum_{l=1}^{\lfloor (N_1 + u_{n-1} - 1)/p \rfloor} b_l^{1/p} T_{\mathrm{crit}}^{-l} + \sum_{l=\lceil (N_1 + u_{n-1})/p \rceil}^{\lfloor (u_n - 1)/p \rfloor} (p^s b_l)^{1/p} T_{\mathrm{hub}}^{-l},
$$

for any choice of pth roots. Let $L = (G'_{\mathrm{crit}} G'_{\mathrm{hub}}/G_{\mathrm{crit}} G_{\mathrm{hub}})I$. Then

$$
\frac{J}{(G'_{\mathrm{crit}} G'_{\mathrm{hub}}/G_{\mathrm{crit}} G_{\mathrm{hub}})H^p} = \left(\frac{J}{L}\right)\left(\frac{I}{H^p}\right).
$$

Now, Lemma 9.14 gives us that $v_{\mathrm{ch}}(I-1) \geq v'_{\mathrm{ch}}(J-1)$. Lemma 9.11 shows that, if

$$
\frac{I}{H^p} - 1 = \sum_{i=1}^{u_n - 1} d_i T^{-i},
$$

then for all $0 < i < u_n$

$$
v_{\mathrm{ch}}(d_i T^{-i}) > \theta_i + v'_{\mathrm{ch}}(J-1)/p. \tag{9.13}
$$

Additionally, Lemma 9.14, combined with Corollary 9.6 and Lemma 9.8, gives us that $v_{\mathrm{ch}}(L-1) \geq v'_{\mathrm{ch}}(J-1) > 0$. Now, $J/L - 1 = (J-L)/L = (J-L)(1+(L-1))^{-1}$ and $v_{\mathrm{ch}}(J-L) = v_{\mathrm{ch}}(J-1-(L-1)) \geq v_{\mathrm{ch}}(J-1)$ by Lemma 9.5(i). Using Lemmas 9.10 and 9.5(ii), we have

$$
v_{\mathrm{ch}}\left(\frac{J}{L} - 1\right) \geq v_{\mathrm{ch}}(J-1). \tag{9.14}
$$

On the other hand, by construction, $J - L$ has no T^{-i} term if $i \equiv -1 \pmod{m}$ or $p|i$ and $i < u_n$. Expanding $L = 1 + (L - 1)$ out as a power series, and again using Lemma 9.5, this implies

$$v_{\mathrm{ch}}'\left(\frac{J}{L} - 1\right) \geq v_{\mathrm{ch}}(J - L) + v_{\mathrm{ch}}(L - 1) \geq v_{\mathrm{ch}}(J - 1) + v_{\mathrm{ch}}'(J - 1). \quad (9.15)$$

Now, we note that θ_i is a decreasing function of i. Letting c_i be as in the lemma, and using (9.13), this has the consequence that, for $i < u_n$,

$$v_{\mathrm{ch}}(c_i T^{-i}) \geq \begin{cases} \min\left(v_{\mathrm{ch}}'\left(\frac{J}{L} - 1\right), \theta_i + \epsilon + \frac{v_{\mathrm{ch}}'(J-1)}{p}\right), & p|i \text{ or } i \equiv -1 \pmod{m}, \\ \\ \min\left(v_{\mathrm{ch}}\left(\frac{J}{L} - 1\right), \theta_i + \epsilon + \frac{v_{\mathrm{ch}}'(J-1)}{p}\right), & \text{otherwise.} \end{cases}$$

$$(9.16)$$

for some $\epsilon > 0$.

Combining (9.14), (9.15), and (9.16) proves the first part of the lemma. The last statement about G_{crit}' and G_{hub}' follows easily from Lemma 9.14. Lemma 9.14 also shows that $v_{\mathrm{ch}}(I - 1) \geq \beta$. Since θ_i is nonincreasing in i, and H^p and I have degree less than u_n in T^{-1}, Lemma 9.11 (along with Lemma 9.5(i)) shows that $v_{\mathrm{ch}}(H^p - 1) > \min(\beta, \theta_{u_n-1})$. One calculates that $\theta_{u_n-1} = (p-1)r_{\mathrm{hub}}/p$, and this completes the proof. $\qquad\square$

Lemma 9.18. *Let $G_{\mathrm{crit}} \in \mathcal{G}_{\mathrm{crit},g}$. Let $J \in 1 + T_{\mathrm{crit}}^{-1}\mathfrak{m}\{T_{\mathrm{crit}}^{-1}\}$. Let θ_i be as in Remark 9.12. After a possible finite extension of K and crit-negligible adjustment to J, there exist $G_{\mathrm{crit}}' \in \mathcal{G}_{\mathrm{crit},g}$ and a polynomial $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ such that if*

$$\frac{J}{(G_{\mathrm{crit}}'/G_{\mathrm{crit}})H^p} = 1 + \sum_{i=1}^{\infty} c_i T^{-i},$$

then for $0 < i < N_1 + u_{n+1}$, there exists $\epsilon > 0$ such that

$$v_{r_{\mathrm{crit}}}(c_i T^{-i}) \geq \begin{cases} \min\left(v_{r_{\mathrm{crit}}}(J - 1) + v_{r_{\mathrm{crit}}}'(J - 1), \theta_i + \epsilon + \frac{v_{r_{\mathrm{crit}}}'(J-1)}{p}\right), \\ \hspace{5cm} p|i \text{ or } i \equiv -1 \pmod{m}, \\ \min\left(v_{r_{\mathrm{crit}}}(J - 1), \theta_i + \epsilon + \frac{v_{r_{\mathrm{crit}}}'(J-1)}{p}\right), \text{ otherwise.} \end{cases}$$

If $v_{r_{\mathrm{crit}}}'(J - 1) \geq \beta$ for some $0 < \beta < p/(p-1)$, then we can choose G_{crit}' and H above such that

$$v_{r_{\mathrm{crit}}}(G_{\mathrm{crit}}'/G_{\mathrm{crit}} - 1) \geq \beta,$$

and that

$$v_{r_{\mathrm{crit}}}(H^p - 1) \geq \min(\beta, (p-1)r_{\mathrm{crit}}/p).$$

Proof. The proof is the same as that of Lemma 9.17, replacing Lemma 9.14 by
Lemma 9.13, Lemma 9.11 by Remark 9.12, Lemma 9.5 and Corollary 9.6 by
Remark 9.7, v_{ch} and v'_{ch} by $v_{r_{\mathrm{crit}}}$ and $v'_{r_{\mathrm{crit}}}$, "hub-negligible" by "crit-negligible,"
omitting the second summations in I and H, replacing all u_n's by $N_1 + u_{n-1}$'s, and
omitting all mentions of G_{hub} and G'_{hub}. $\square$

Lemma 9.19. *Let $0 < \sigma < p/(p-1)$. Let $G_{\mathrm{crit}}, G_{\mathrm{hub}} \in G_{\mathrm{crit},g}, \mathscr{G}_{\mathrm{hub},\alpha}$, respectively.
Let $J \in 1 + T^{-1}\mathfrak{m}\{T^{-1}\}$ such that $v_{\mathrm{ch}}(J-1) > 0$. After a possible finite extension
of K and hub-negligible adjustment to J, there exist $G'_{\mathrm{crit}}, G'_{\mathrm{hub}} \in \mathscr{G}_{\mathrm{crit},g}, \mathscr{G}_{\mathrm{hub},\alpha}$
respectively, and a polynomial $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$, such that*

$$v'_{\mathrm{ch}}\left(\frac{J}{(G'_{\mathrm{crit}}G'_{\mathrm{hub}}/G_{\mathrm{crit}}G_{\mathrm{hub}})H^p} - 1\right) \geq \sigma.$$

We can choose $G'_{\mathrm{crit}}, G'_{\mathrm{hub}}$, and H above such that

$$v_{\mathrm{ch}}\big((G'_{\mathrm{crit}}G'_{\mathrm{hub}}/G_{\mathrm{crit}}G_{\mathrm{hub}})H^p - 1\big) > \min\big(v'_{\mathrm{ch}}(J-1), (p-1)r_{\mathrm{hub}}/p\big).$$

Proof. We will build $G'_{\mathrm{crit}}, G'_{\mathrm{hub}}$, and H through successive approximation. Let θ_i be
as in Lemma 9.11, and let $\eta_i = (p/(p-1))\theta_i$. We make the following observations.
First, $\theta_{u_n} = \eta_{u_n} = 0$. Second, the η_i form a decreasing sequence. Third,
if $v_{\mathrm{ch}}(c_i T^{-i}) > \eta_i$ for some $c_i \in K$, then $v_0(c_i T^{-i}) = v(c_i) > p/(p-1)$.
Fourth, if $x > \eta_i$, then $\theta_i + x/p > \eta_i$.

By the first observation above, we know that $v'_{\mathrm{ch}}(J-1) > \eta_j$ for some
$0 < j \leq u_n$. Let $G_{\mathrm{crit},1}, G_{\mathrm{hub},1}$, and H_1 be the $G'_{\mathrm{crit}}, G'_{\mathrm{hub}}$, and H guaranteed
by Lemma 9.17 (after making a hub-negligible adjustment to J), and set $J_1 :=
(G'_{\mathrm{crit},1}G'_{\mathrm{hub},1}/G_{\mathrm{crit}}G_{\mathrm{hub}})H_1^p$. It follows from Lemma 9.17, Corollary 9.6, and
Lemma 9.8 that $v_{\mathrm{ch}}(J_1 - 1) > 0$. Thus $v_{\mathrm{ch}}(J_1) = 0$. Also, $v_{\mathrm{ch}}(J_1^{-1}) = 0$ as
a consequence of Lemma 9.10, so $v_{\mathrm{ch}}(J/J_1 - 1) = v_{\mathrm{ch}}((J - J_1)(J_1^{-1})) > 0$ by
Lemma 9.5(ii). Write $J/J_1 = 1 + \sum_{i=1}^{\infty} d_i T^{-i}$. For $i \geq j$ and either $p|i$
or $i \equiv -1 \pmod{m}$, Lemma 9.17 and the second, third, and fourth observations
above show that $v_0(d_i T^{-i}) > p/(p-1)$. For $i < j$, there exists $\epsilon > 0$ such that

$$v_{\mathrm{ch}}(d_i T^{-i}) \geq \begin{cases} \min\left(v_{\mathrm{ch}}(J-1) + v'_{\mathrm{ch}}(J-1), \theta_i + \epsilon + \frac{v'_{\mathrm{ch}}(J-1)}{p}\right), \\ \qquad\qquad\qquad\qquad\qquad p|i \text{ or } i \equiv -1 \pmod{m}, \\ \min\left(v_{\mathrm{ch}}(J-1), \theta_i + \epsilon + \frac{v'_{\mathrm{ch}}(J-1)}{p}\right), \quad \text{otherwise.} \end{cases}$$

If $A_1 := \sum_{i \in I} d_i T^{-i}$ where $I \subseteq [j, \infty)$ consists of those integers congruent
to $0 \pmod{p}$ or $-1 \pmod{m}$, or greater than or equal to u_n, then A_1 is hub-
negligible by Lemma 9.3(i) and Remark 9.4. Since $v_{\mathrm{ch}}(J_1) = 0$, Lemma 9.9(i)
shows that $J_1 A_1$ is hub-negligible. So we may (and do) replace J with $J - J_1 A_1$,
and we assume that $d_i = 0$ for $i \in I$.

Since the θ_i form a decreasing sequence, we have

$$v'_{\text{ch}}\left(\frac{J}{J_1}-1\right) \geq \min\left(v_{\text{ch}}(J-1) + v'_{\text{ch}}(J-1), \theta_{j-1} + \epsilon + \frac{v'_{\text{ch}}(J-1)}{p}\right)$$

and

$$v_{\text{ch}}\left(\frac{J}{J_1}-1\right) \geq \min\left(v_{\text{ch}}(J-1), \theta_{u_n-1} + \epsilon + \frac{v'_{\text{ch}}(J-1)}{p}\right).$$

For $l > 1$, define $G'_{\text{crit},l}$, $G'_{\text{hub},l}$, and H_l inductively as the G'_{crit}, G'_{hub}, and H guaranteed by Lemma 9.17 with J/J_{l-1} in place of J and $G_{\text{crit},l-1}$ and $G_{\text{hub},l-1}$ in place of G_{crit} and G_{hub} (note that, since $v_{\text{ch}}(J_{l-1}) = 0$ for the same reason that $v_{\text{ch}}(J_1) = 0$, Lemma 9.3(i) shows that the hub-negligible adjustment to J/J_{l-1} required for Lemma 9.17 can be achieved by making a hub-negligible adjustment to J). Define

$$J_l = \frac{G'_{\text{crit},l} G'_{\text{hub},l}}{G_{\text{crit}} G_{\text{hub}}}(H_1 \cdots H_l)^p$$

so that

$$\frac{J}{J_l} = \frac{J/J_{l-1}}{(G'_{\text{crit},l} G'_{\text{hub},l}/G'_{\text{crit},l-1} G'_{\text{hub},l-1})H_i^p}.$$

At each stage, we replace J with $J - J_l A_l$, where A_l is the part of J/J_l consisting of terms of degree i in T^{-1}, where $j \leq i \leq u_n - 1$ and either $p|i$ or $i \equiv -1 \pmod{m}$. As before, this is a hub-negligible adjustment. By Lemma 9.17, there exists $\epsilon > 0$ such that

$$v'_{\text{ch}}\left(\frac{J}{J_l}-1\right) \geq \min\left(v_{\text{ch}}\left(\frac{J}{J_{l-1}}-1\right) + v'_{\text{ch}}\left(\frac{J}{J_{l-1}}-1\right),\right.$$
$$\left.\theta_{j-1} + \epsilon + \frac{v'_{\text{ch}}(J/J_{l-1}-1)}{p}\right)$$

and

$$v_{\text{ch}}\left(\frac{J}{J_l}-1\right) \geq \min\left(v_{\text{ch}}\left(\frac{J}{J_{l-1}}-1\right), \theta_{u_n-1} + \epsilon + \frac{v'_{\text{ch}}(J/J_{l-1}-1)}{p}\right).$$

Since $\eta_{j-1} = (p/(p-1))\theta_{j-1}$, there exists some l_{j-1} for which $v'_{\text{ch}}(J/J_{l_{j-1}}-1) > \eta_{j-1}$.

Replacing j by $j - 1$, we can repeat the entire process again. Induction now shows that, after further hub-negligible adjustments to J, we get down to the case $j = 1$. That is, there exists l_1 such that $v'_{\text{ch}}(J/J_{l_1} - 1) > \eta_1$. Replacing J with $J - J_{l_1} A_{l_1}$ as above, we obtain that $v'_{\text{ch}}(J/J_{l_1} - 1) = \infty$. In particular, setting G'_{crit} and G'_{hub} equal to G'_{crit,l_1} and G'_{hub,l_1} respectively, and setting $H = (H_1 \cdots H_{l_1})$, gives the desired solution.

To prove the last statement, note that Lemma 9.17 shows that all H_i satisfy $v_{\text{ch}}(H_i - 1) > \min(v'_{\text{ch}}(J - 1), (p - 1)r_{\text{hub}}/p)$. By Corollary 9.6, $v_{\text{ch}}(H - 1)$ has the same property. Lemma 9.17 and Lemma 9.8 imply that $v_{\text{ch}}(G'_{\text{crit}}/G_{\text{crit}} - 1)$ and $v_{\text{ch}}(G'_{\text{hub}}/G_{\text{hub}})$ also have this property. Combining all this with Corollary 9.6 proves the last statement of the lemma. $\qquad\square$

Lemma 9.20. *Let* $0 < \sigma < p/(p-1)$. *Let* $G_{\text{crit}} \in G_{\text{crit},g}$. *Let* $J \in 1 + T_{\text{crit}}^{-1}\mathfrak{m}\{T_{\text{crit}}^{-1}\}$. *After a possible finite extension of K and crit-negligible adjustment to J, there exist* $G'_{\text{crit}} \in \mathcal{G}_{\text{crit},g}$ *and a polynomial* $H \in 1 + T_{\text{crit}}^{-1}\mathfrak{m}[T_{\text{crit}}^{-1}]$, *such that*

$$v'_{r_{\text{crit}}}\left(\frac{J}{(G'_{\text{crit}}/G_{\text{crit}})H^p} - 1\right) \geq \sigma.$$

We can choose G'_{crit} *and* H *above such that* $v_{r_{\text{crit}}}((G'_{\text{crit}}/G_{\text{crit}})H^p - 1) > \min(v'_{r_{\text{crit}}}(J - 1), (p - 1)r_{\text{crit}}/p)$.

Proof. The proof is the same as that of Lemma 9.19, replacing Lemma 9.17 by Lemma 9.18, Lemma 9.11 by Remark 9.12, Corollary 9.6 by Remark 9.7, Lemma 9.3(i) by Lemma 9.3(ii), Lemma 9.9(i) by Lemma 9.9(ii), v_{ch} and v'_{ch} by $v_{r_{\text{crit}}}$ and $v'_{r_{\text{crit}}}$, "hub-negligible" by "crit-negligible," u_n by $N_1 + u_{n-1}$, and omitting all mentions of G_{hub} and G'_{hub}. $\qquad\square$

Lemma 9.21. *Let* $0 < \sigma < p/(p-1)$. *Let* $J \in 1 + T^{-1}\mathfrak{m}\{T^{-1}\}$ *with* $v_{\text{ch}}(J-1) > 0$. *After a possible finite extension of K and hub-negligible adjustment to J, there exists* $J' \in 1 + T^{-1}\mathfrak{m}\{T^{-1}\} \cap \mathbb{K}$ *such* $J' =' J$, *and* J' *has* r_{hub}-*discrepancy valuation at least* σ. *Furthermore,* $v_{\text{ch}}(J' - 1) > 0$.

Proof. Suppose the r_{hub}-discrepancy valuation of J is at least $\sigma_0 \geq 0$. By Lemma 9.19, after making a hub-negligible adjustment to J, there exist G_{crit}, G'_{crit}, G_{hub}, G'_{hub}, and H (as in that lemma) such that

$$v'_{\text{ch}}\left(\frac{J}{(G'_{\text{crit}}G'_{\text{hub}}/G_{\text{crit}}G_{\text{hub}})H^p} - 1\right) \geq \sigma. \tag{9.17}$$

For shorthand, write $G' = G'_{\text{crit}}G'_{\text{hub}}$ and $G = G_{\text{crit}}G_{\text{hub}}$. Also, by Lemma 9.19,

$$v_{\text{ch}}\left(\frac{G'}{G}H^p - 1\right) \geq \beta := \min\left(v'_{\text{ch}}(J - 1), (p-1)r_{\text{hub}}/p\right). \tag{9.18}$$

Equations (9.17) and (9.18) also hold with $v_{r_{\text{hub}}}$ and $v'_{r_{\text{hub}}}$ in place of v_{ch} and v'_{ch} by Lemma 9.8.

By Corollary 6.16, since $(G'/G)H^p$ has infinite r_{hub}-discrepancy valuation, we have $v_{r_{\text{hub}}}(J/(G'/G)H^p - 1) \geq \min(\sigma, \sigma_0)$. Thus

$$v'_{r_{\text{hub}}}\left(J - \frac{G'}{G}H^p\right) = v'_{r_{\text{hub}}}\left(\left(\frac{J}{(G'/G)H^p} - 1\right)\frac{G'}{G}H^p\right) \geq \min(\sigma, \sigma_0 + \beta). \tag{9.19}$$

Now, replace all terms of J of degree not congruent to $-1 \pmod{m}$ or $0 \pmod{p}$ in T^{-1} with the corresponding terms of $(G'/G)H^p$. Since $\mathfrak{K}_1((G'/G)H^p)$ is ψ-equivariant, (9.19) shows that our new J has r_{hub}-discrepancy valuation at least $\min(\sigma, \sigma_0 + \beta)$ and lies in $\mathbb{K}$. By (9.18), we still have $v_{\text{ch}}(J - 1) \geq \beta$. Repeating this process, we eventually obtain J with r_{hub}-discrepancy valuation at least σ. This is the J' we seek. $\qquad\square$

Lemma 9.22. *Let* $0 < \sigma < p/(p-1)$. *Let* $J \in 1 + T_{\text{crit}}^{-1}\mathfrak{m}\{T_{\text{crit}}^{-1}\}$. *After a possible finite extension of K and crit-negligible adjustment to J, there exists* $J' \in 1 + T^{-1}\mathfrak{m}\{T^{-1}\} \cap \mathbb{K}$ *such* $J' =' J$, *and* J' *has* r_{crit}-*discrepancy valuation at least σ. Furthermore,* $v_{r_{\text{crit}}}(J' - 1) > 0$.

Proof. The proof is the same as that of Lemma 9.21, replacing Lemma 9.19 by Lemma 9.20, v_{ch}, $v_{r_{\text{hub}}}$, v'_{ch}, and $v'_{r_{\text{hub}}}$ by $v_{r_{\text{crit}}}$, $v_{r_{\text{crit}}}$, $v'_{r_{\text{crit}}}$, and $v'_{r_{\text{crit}}}$, respectively, "hub-negligible" by "crit-negligible," and omitting all mentions of G_{hub} and G'_{hub}. $\qquad\square$

We recall the main proposition to be proved:

Proposition 9.23 (Proposition 7.36). *Suppose* $N_1 = (p-1)u_{n-1} - m$ *(this is consistent with Assumption 7.25). Let* $G_{\text{crit}}, G_{\text{hub}} \in G_{\text{crit},g}, \mathscr{G}_{\text{hub},\alpha}$, *respectively. Let* $r \in [0, r_{\text{hub}}) \cap \mathbb{Q}$, *and let* $f \in t^{1-m}k[t^{-m}]$ *have degree less than u_n in t^{-1}, which we regard as the reduction of T_r in κ_r (§6.1). Assume f has no terms of degree divisible by p. Let $\beta = p/(p-1) - u_n r$. After a possible finite extension of K, there exist* $G'_{\text{crit}}, G'_{\text{hub}} \in \mathscr{G}_{\text{crit},g}, \mathscr{G}_{\text{hub},\alpha}$ *respectively, and* $F \in \mathbb{K}$ *with* $v_r(F) = 0$ *and* $[F]_r = f$ *such that*

$$\frac{G'_{\text{crit}}G'_{\text{hub}}}{G_{\text{crit}}G_{\text{hub}}} \equiv 1 - p^\beta F \pmod{(\mathbb{K}^\times)^p}.$$

Proof. We first remark that if $A \in \mathbb{K}$ such that $v_r(A) = 0$, then $[p^\beta A]_r$ is unaffected by hub-negligible adjustments to A. Essentially, this is the reason for defining hub-negligible as we do.

Let F' be a polynomial in T^{-1} of the same degree as f such that $v_r(F') = 0$, that $[F']_r = f$, and that F' has no terms of degree divisible by p. Now,

$$v_{\text{ch}}(p^\beta F') \geq \min\left(\beta - \deg(f)(r_{\text{hub}} - r) - s, \ \beta - (N_1 + u_{n-1})(r_{\text{crit}} - r)\right).$$

Since $\deg(f) < u_n$ and $r \geq 0$, one calculates that $v_{\text{ch}}(p^\beta F')$ is positive. Choose σ such that $\beta - (r_{\text{hub}} - r) < \sigma < p/(p-1)$. By Lemma 9.21, after making an adjustment to F' resulting in a hub-negligible adjustment to $p^\beta F'$, there exists $F'' \in p^{-\beta}T^{-1}\mathfrak{m}\{T^{-1}\}$ such that $p^\beta F'' =' p^\beta F'$, that $1 - p^\beta F''$ has r_{hub}-discrepancy valuation at least σ, and that $v_{\text{ch}}(1 - p^\beta F'') > 0$.

Now, since $1 - p^\beta F''$ has r_{hub}-discrepancy valuation at least σ, it has r-discrepancy valuation at least $\sigma + (r - r_{\text{hub}}) > \beta$. By Lemma 6.15 (noting that $[p^\beta F']_r$, and thus $[p^\beta F'']_r$, has no terms of degree divisible by p), we have that $[p^\beta F'']_r$ contains

only terms of degree congruent to $-1 \pmod{m}$ in t^{-1}. Since the same is true by construction for $[p^\beta F']_r$, we have $[F'']_r = [p^\beta F'']_r = [p^\beta F']_r = f$.

Furthermore, Lemma 9.19 yields $G'_{\text{crit}}, G'_{\text{hub}} \in \mathscr{G}_{\text{crit},g}, \mathscr{G}_{\text{hub},\alpha}$, respectively, and $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ such that after making an adjustment to F'' resulting in a hub-negligible adjustment to $p^\beta F''$ (which does not change $[p^\beta F'']_r = f$ or $v_r(p^\beta F'') = \beta$), we obtain

$$v'_{\text{ch}}\left(\frac{1 - p^\beta F''}{(G'_{\text{crit}}G'_{\text{hub}}/G_{\text{crit}}G_{\text{hub}})H^p} - 1\right) \geq \sigma. \tag{9.20}$$

By Lemma 9.8, (9.20) also holds when v'_{ch} is replaced by $v'_{r_{\text{hub}}}$. Since the fraction in (9.20) has r_{hub}-discrepancy valuation $\geq \sigma$, Corollary 6.16 shows that $v'_{r_{\text{hub}}}$ can even be replaced by $v_{r_{\text{hub}}}$ in (9.20). We conclude that

$$v_{r_{\text{hub}}}\left(\frac{G'_{\text{crit}}G'_{\text{hub}}}{G_{\text{crit}}G_{\text{hub}}}H^p - (1 - p^\beta F'')\right) \geq \sigma.$$

In particular,

$$v_r\left(\frac{G'_{\text{crit}}G'_{\text{hub}}}{G_{\text{crit}}G_{\text{hub}}}H^p - (1 - p^\beta F'')\right) \geq \sigma + (r_{\text{hub}} - r) > \beta. \tag{9.21}$$

Finally, let F be such that $(G'_{\text{crit}}G'_{\text{hub}}/G_{\text{crit}}G_{\text{hub}})H^p = 1 - p^\beta F$. Since $[F'']_r = f$, we need only show that $v_r(F) = 0$ and $[F]_r = [F'']_r$. This follows from (9.21) and the fact that $v_r(p^\beta F'') = \beta$. $\qquad\square$

Remark 9.24. Being able to replace $v'_{r_{\text{hub}}}$ by $v_{r_{\text{hub}}}$ in (9.20) in order to clear denominators is the essential reason why we need the concept of discrepancy valuation.

Proposition 7.20, which we recall below, now follows easily.

Proposition 9.25 (Proposition 7.20). *Let $G \in \mathscr{G}_{\text{crit},g}$, let $r \in [r_{\text{hub}}, r_{\text{crit}}) \cap \mathbb{Q}$, and let $f \in t^{1-m}k[t^{-m}]$ be a polynomial of degree less than $N_1 + u_{n-1}$ in t^{-1}, which we regard as the reduction of T_r in κ_r (§6.1). Assume f has no terms of degree divisible by p. Let $\beta = (N_1 + u_{n-1})(r_{\text{crit}} - r)$. After a possible finite extension of K, there exist $G' \in \mathscr{G}_{\text{crit},g}$ and $F \in \mathbb{K}$ with $v_r(F) = 0$ and $[F]_r = f$ such that*

$$\frac{G'}{G} \equiv 1 - p^\beta F \pmod{(\mathbb{K}^\times)^p}.$$

Proof. The proof is the same as that of Proposition 9.23, replacing Lemma 9.21 by Lemma 9.22, Lemma 9.19 by Lemma 9.20, $v_{\text{ch}}, v_{r_{\text{hub}}}, v'_{\text{ch}}$, and $v'_{r_{\text{hub}}}$ by $v_{r_{\text{crit}}}$, $v_{r_{\text{crit}}}, v'_{r_{\text{crit}}}$, and $v'_{r_{\text{crit}}}$, respectively, "hub-negligible" by "crit-negligible," choosing $\beta - (r_{\text{crit}} - r) < \sigma < p/(p-1)$, and omitting all mentions of G_{hub} and G'_{hub}. $\qquad\square$

References

[1] *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin]. Zbl 01992950 MR 2017446

[2] K. Arzdorf and S. Wewers, A local proof of the semistable reduction theorem, 2011. arXiv:1211.4624

[3] J. Bertin, Obstructions locales au relèvement de revêtements galoisiens de courbes lisses, *C. R. Acad. Sci. Paris Sér. I Math.*, **326** (1988). no. 1, 55–58. Zbl 0952.14018 MR 1649485

[4] J. Bertin and A. Mézard, Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques, *Invent. Math.*, **141** (2000), no. 1, 195–238. Zbl 0993.14014 MR 1767273

[5] S. Bosch and W. Lütkebohmert, Stable reduction and uniformization of abelian varieties. II, *Invent. Math.*, **78** (1984), no. 2, 257–297. Zbl 0554.14015 MR 0767194

[6] I. I. Bouw and S. Wewers, The local lifting problem for dihedral groups, *Duke Math. J.*, **134** (2006), no. 3, 421–452. Zbl 1108.14025 MR 2254623

[7] I. I. Bouw, S. Wewers, and L. Zapponi, Deformation data, Belyi maps, and the local lifting problem, *Trans. Amer. Math. Soc.*, **361** (2009, no. 12, 6645–6659. Zbl 1244.11065 MR 2538609

[8] L. H. Brewis, Liftable D_4-covers, *Manuscripta Math.*, **126** (2008), no. 3, 293–313. Zbl 1158.14027 MR 2411230

[9] L. H. Brewis and S. Wewers, Artin characters, Hurwitz trees and the lifting problem, *Math. Ann.*, **345** (2009), no. 3, 711–730. Zbl 1222.14045 MR 2534115

[10] T. Chinburg, R. Guralnick, and D. Harbater, Oort groups and lifting problems, *Compos. Math.*, **144** (2008), no. 4, 849–866. Zbl 1158.12003 MR 2441248

[11] T. Chinburg, R. Guralnick, and D. Harbater, The local lifting problem for actions of finite groups on curves, *Ann. Sci. Éc. Norm. Supér. (4)*, **44** (2011), no. 4, 537–605. Zbl 1239.14024 MR 2919977

[12] H. P. Epp, Eliminating wild ramification, *Invent. Math.*, **19** (1973), 235–249. Zbl 0254.13008 MR 321929

[13] M. A. Garuti, Prolongement de revêtements galoisiens en géométrie rigide, *Compos. Math.*, **104** (1996), no. 3, 305–331. Zbl 0885.14011 MR 1424559

[14] M. A. Garuti, Linear systems attached to cyclic inertia, in *Arithmetic fundamental groups and noncommutative algebra (Berkeley, CA, 1999)*, 377–386, Proc. Sympos. Pure Math., 70, Amer. Math. Soc., Providence, RI, 2002. Zbl 1072.14017 MR 1935414

[15] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.*, **58** (1985), no. 3, 300–321. Zbl 0579.05004 MR 815360

[16] B. Green and M. Matignon, Liftings of Galois covers of smooth curves, *Compos. Math.*, **113** (1998), no. 3, 237–272. Zbl 0923.14006 MR 1645000

[17] B. Green and M. Matignon, Order p automorphisms of the open disc of a p-adic field, *J. Amer. Math. Soc.*, **12** (1999), no. 1, 269–303. Zbl 0923.14007 MR 1630112

[18] E. R. Heineman, Generalized Vandermonde determinants, *Trans. Amer. Math. Soc.*, **31** (1929), no. 3, 464–476. Zbl 55.0064.02 MR 1501494

[19] N. M. Katz, Local-to-global extensions of representations of fundamental groups, *Ann. Inst. Fourier (Grenoble)*, **36** (1986), no. 4, 69–106. Zbl 0564.14013 MR 0867916

[20] M. Matignon, p-groupes abéliens de type $(p, \cdots, p)$ et disques ouverts p-adiques, *Manuscripta Math.*, **99** (1999), no. 1, 93–109. Zbl 0953.12004 MR 1697205

[21] A. Obus, The (local) lifting problem for curves, in *Galois-Teichmüller theory and arithmetic geometry*, 359–412, Adv. Stud. Pure Math., 63, Math. Soc. Japan, Tokyo, 2012. Zbl 1321.14028 MR 3051249

[22] A. Obus, The local lifting problem for A_4, *Algebra Number Theory*, **10** (2016), no. 8, 1683–1693. Zbl 1352.14019 MR 3556796

[23] A. Obus and R. Pries, Wild tame-by-cyclic extensions, *J. Pure Appl. Algebra*, **214** (2010), no. 5, 565–573. Zbl 1233.11125 MR 2577662

[24] A. Obus and S. Wewers, Cyclic extensions and the local lifting problem, *Ann. of Math. (2)*, **180** (2014), no. 1, 233–284. Zbl 1307.14042 MR 3194815

[25] A. Obus and S. Wewers, Wild ramification kinks, *Res. Math. Sci.*, **3** (2016), Paper No. 21, 27pp (electronic). Zbl 06646664 MR 3552986

[26] G. Pagot, Relèvement en caractéristique zéro d'actions de groupesabéliens de type $(p, \ldots, p)$, Ph. D. Thesis. Available at: https://www.math.u-bordeaux.fr/~mmatigno/preprints.html

[27] G. Pagot, $\mathbb{F}_p$-espaces vectoriels de formes différentielles logarithmiques sur la droite projective, *J. Number Theory*, **97** (2002), no. 1, 58–94. Zbl 1076.14504 MR 1939137

[28] F. Pop, The Oort conjecture on lifting covers of curves, *Ann. of Math. (2)*, **180** (2014), no. 1, 285–322. Zbl 1311.12003 MR 3194816

[29] R. J. Pries, Conductors of wildly ramified covers. III, *Pacific J. Math.*, **211** (2003), no. 1, 163–182. Zbl 1058.14051 MR 2016596

[30] M. Raynaud, Spécialisation des revêtements en caractéristique $p > 0$, *Ann. Sci. École Norm. Sup. (4)*, **32** (1999), no. 1, 87–126. Zbl 0999.14004 MR 1670532

[31] T. Sekiguchi, F. Oort, and N. Suwa, On the deformation of Artin-Schreier to Kummer, *Ann. Sci. École Norm. Sup. (4)*, **22** (1989), no. 3, 345–375. Zbl 0714.14024 MR 1011987

[32] J.-P. Serre, *Corps locaux*, Deuxième édition, Publications de l'Université de Nancago, VIII, Hermann, Paris, 1968. Zbl 0137.02601 MR 354618

[33] T. Szamuely, *Galois groups and fundamental groups*, Cambridge Studies in Advanced Mathematics, 117, Cambridge University Press, Cambridge, 2009. Zbl 1189.14002 MR 2548205

[34] S. Wewers, Swan conductors on the boundary of Lubin-Tate spaces, 2005. arXiv:math.NT/0511434

[35] S. Wewers, Fiercely ramified cyclic extensions of p-adic fields with imperfect residue field, *Manuscripta Math.*, **143** (2014), no. 3-4, 445–472. Zbl 1304.11132 MR 3167623

[36] O. Zariski and P. Samuel, *Commutative algebra. Vol. 1*. With the cooperation of I. S. Cohen. Corrected reprinting of the 1958 edition, Graduate Texts in Mathematics, 28, Springer-Verlag, New York-Heidelberg-Berlin, 1975. Zbl 0313.13001 MR 384768

A. Obus, Department of Mathematics, University of Virginia,
Charlottesville, VA 22904, USA
E-mail: andrewobus@gmail.com