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Torsion order of smooth projective surfaces
(with an appendix by J.-L. Colliot-Thélène)

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Abstract. To a smooth projective variety X whose Chow group of 0-cycles is **Q**-universally trivial one can associate its torsion order $\mathrm{Tor}(X)$, the smallest multiple of the diagonal appearing in a cycle-theoretic decomposition *à la* Bloch–Srinivas. We show that $\mathrm{Tor}(X)$ is the exponent of the torsion in the Néron–Severi group of X when X is a surface over an algebraically closed field k, up to a power of the exponential characteristic of k.

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1. Introduction

Let X be a smooth projective irreducible variety over a field k. Assume that $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$: this is the strongest case of "decomposition of the diagonal" *à la* Bloch–Srinivas [5]. To X is associated its *torsion order* $\mathrm{Tor}(X)$, the smallest multiple of the diagonal of X appearing in such a decomposition (Definition 2.5). This number is also studied by Chatzistamatiou and Levine in [6].

The integer $\mathrm{Tor}(X)$ kills all normalised motivic birational invariants of smooth projective varieties in the sense of Definition 2.1 (Lemma 2.6). In particular, away from $\mathrm{char}\, k$, the exponent of the torsion subgroup of the geometric Néron–Severi group of X divides $\mathrm{Tor}(X)$ (Corollary 6.4); the main result of this paper is that we have equality when X is a surface and k is algebraically closed: this result was announced in [12, Remark 3.1.5 3)]. In the special case where $\mathrm{Tor}(X) = 1$, it was obtained previously in [17] and [1] (see also Theorem A.1 in the appendix).

The equality follows from a short exact sequence (Corollary 6.4(a)):

$$0 \to CH^2\big(X_{k(X)}\big)_{\mathrm{tors}} \to \mathrm{Tor}\left(H^2(X), H^3(X)\right)^2 \to H^3_{\mathrm{nr}}\big(X \times X, \mathbf{Q}/\mathbf{Z}(2)\big) \to 0 \quad (1.1)$$

where $H^*(X)$ is Betti cohomology of X with integer coefficients in characteristic 0 (for simplicity; in positive characteristic, use l-adic cohomology). It also shows that

$CH^2(X_{k(X)})_{\mathrm{tors}}$ is finite (away from the characteristic of k), with a very explicit bound.[1]

The exact sequence (1.1) is a special case of a more general one appearing in Theorem 6.3, which implies in particular the finiteness of $CH^2(X_{k(Y)})_{\mathrm{tors}}$ for any other smooth projective Y, and an explicit bound on its order. See Theorem A.6 for another proof of this finiteness, and a different bound.

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2. Basic properties of the torsion order

2.1. Review of birational motives. We fix a base field k, and write $\mathbf{Sm}^{\mathrm{proj}} = \mathbf{Sm}^{\mathrm{proj}}(k)$ for the category of smooth projective k-varieties. Recall from [12] the category $\mathbf{Chow}^{\mathrm{o}}(k, A)$ of birational Chow motives with coefficients in a commutative ring A: there is a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{Sm}(k) & \xrightarrow{h} & \mathbf{Chow}^{\mathrm{eff}}(k,A) \\ & \searrow^{h^{\mathrm{o}}} & \downarrow \\ & & \mathbf{Chow}^{\mathrm{o}}(k,A) \end{array}$$

where $\mathbf{Chow}^{\mathrm{eff}}(k, A)$ is the covariant category of effective Chow motives with coefficients in A (opposite to that of [16]), and Hom groups in $\mathbf{Chow}^{\mathrm{o}}(k, A)$ are characterized by the formula

$$\mathbf{Chow}^{\mathrm{o}}(k,A)\big(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(X)\big) = CH_0\big(X_{k(Y)}\big) \otimes A$$

for $X, Y \in \mathbf{Sm}^{\mathrm{proj}}(k)$ (with Y irreducible). When $A = \mathbf{Z}$, we simplify the notation to $\mathbf{Chow}^{\mathrm{eff}}(k)$, $\mathbf{Chow}^{\mathrm{o}}(k)$, or even $\mathbf{Chow}^{\mathrm{eff}}$, $\mathbf{Chow}^{\mathrm{o}}$.

2.2. Motivic birational invariants. Let $X \in \mathbf{Sm}^{\mathrm{proj}}(k)$ be irreducible, with

$$CH_0\big(X_{k(X)}\big) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}:$$

this condition is equivalent to Bloch–Srinivas' decomposition of the diagonal relative to a closed subset of dimension 0 [5]. By [12, Prop. 3.1.1], this means that the birational motive $h^{\mathrm{o}}(X)$ of X in the category $\mathbf{Chow}^{\mathrm{o}}(k, \mathbf{Q})$ is trivial, i.e. that the projection map $h^{\mathrm{o}}(X) \to h^{\mathrm{o}}(\operatorname{Spec} k) =: \mathbf{1}$ is an isomorphism in $\mathbf{Chow}^{\mathrm{o}}(k, \mathbf{Q})$. Then $CH_0(X_K) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$ for any field extension K of k (loc. cit. Condition (vi)).

[1] It would be interesting to completely determine $CH^2(X_{k(X)})_{\mathrm{tors}}$: for example, when X is an Enriques surface and $\operatorname{char} k = 0$, is it $\mathbf{Z}/2$ or $(\mathbf{Z}/2)^2$?

To such an X, we want to associate a numerical invariant. To motivate it, let us introduce a definition:

Definition 2.1. A *motivic invariant* of smooth projective varieties with values in an additive category $\mathcal{A}$ is a functor $F : \mathbf{Sm}^{\mathrm{proj}} \to \mathcal{A}$ which factors through an additive functor $\mathbf{Chow}^{\mathrm{eff}} \to \mathcal{A}$. We say that F is *birational* if it further factors through $\mathbf{Chow}^{\mathrm{o}}$. The invariant F is *normalised* if $F(\operatorname{Spec} k) = 0$.

Remark 2.2. If $X, Y \in \mathbf{Sm}^{\mathrm{proj}}$ are (stably) birationally equivalent, then $h^{\mathrm{o}}(X) \simeq h^{\mathrm{o}}(Y)$ in $\mathbf{Chow}^{\mathrm{o}}$ [12, Prop. 2.3.8]. Hence, to be a motivic birational invariant is stronger than to be a (stable) birational invariant. It is much stronger: $h^{\mathrm{o}}(S) \xrightarrow{\sim} \mathbf{1}$ for S the Barlow surface [2], a complex surface of general type.

Examples 2.3. (a) For any cycle module M_* in the sense of Rost [15], any $K \supseteq k$ and any $n \in \mathbf{Z}$, $X \mapsto A^0(X_K, M_n)$ (resp. $X \mapsto A_0(X_K, M_n)$) defines a contravariant (resp. covariant) motivic birational invariant with values in **Ab**, the category of abelian groups [12, Cor. 6.1.3].

(b) In particular, for $M_* = K_*^M$ (Milnor K-theory), the functor $X \mapsto A_0(X_K, M_0) = CH_0(X_K)$ is a motivic birational invariant. When $K = k(Y)$ for some $Y \in \mathbf{Sm}^{\mathrm{proj}}$, this is also obvious by the intepretation of $CH_0(X_K)$ as $\mathbf{Chow}^{\mathrm{o}}(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(X))$.

(c) Given a contravariant motivic invariant F, we get two (contravariant) normalised invariants by the formulas

$$\underline{F}(X) = \operatorname{Ker}\big(F(k) \to F(X)\big), \quad \bar{F}(X) = \operatorname{Coker}\big(F(k) \to F(X)\big)$$

and similarly for covariant motivic invariants:

$$\underline{F}(X) = \operatorname{Coker}\big(F(X) \to F(k)\big), \quad \bar{F}(X) = \operatorname{Ker}\big(F(X) \to F(k)\big).$$

They are birational if F is birational.

(d) Suppose that F is a motivic invariant with values in the category of $\mathbf{Z}[1/p]$-modules, where p is the exponential characteristic of k (or, more generally, in a $\mathbf{Z}[1/p]$-linear additive category); assume F contravariant to fix ideas. Then F is birational if and only if, for any $Y \in \mathbf{Sm}^{\mathrm{proj}}$, the map $F(Y) \to F(Y \times \mathbf{P}^1)$ is an isomorphism. This follows from [12, Th. 2.4.2].

Definition 2.4. The category $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}$ is the quotient of $\mathbf{Chow}^{\mathrm{o}}$ by the ideal generated by $\mathbf{1}$.

Thus a motivic birational invariant is normalised if and only if it factors through $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}$.

Let $M, N \in \mathbf{Chow}^{\mathrm{o}}$. By definition, $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}(M, N)$ is the quotient of $\mathbf{Chow}^{\mathrm{o}}(M, N)$ by the group of morphisms $f : M \to N$ which factor through $\mathbf{1}$. If $M = h^{\mathrm{o}}(Y)$ and $N = h^{\mathrm{o}}(X)$, this gives

$$\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}\big(h^{\mathrm{o}}(Y), h^{\mathrm{o}}(X)\big) \simeq \operatorname{Coker}\big(CH_0(X) \to CH_0\big(X_{k(Y)}\big)\big).$$

2.3. The torsion order. If now the birational motive of X is trivial in $\mathbf{Chow}^{\mathrm{o}}(k, \mathbf{Q})$, then the image of $h^{\mathrm{o}}(X)$ in $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}$ is torsion; in other words, there is an integer $n > 0$ such that $n1_{h^{\mathrm{o}}(X)} = 0$ in $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}(h^{\mathrm{o}}(X), h^{\mathrm{o}}(X))$.

Definition 2.5. The smallest such integer n is called the *torsion order* of X, and denoted by $\mathrm{Tor}(X)$. We extend this to arbitrary (connected) X by setting $\mathrm{Tor}(X) = 0$ if $h^{\mathrm{o}}(X)$ is not trivial in $\mathbf{Chow}^{\mathrm{o}}(k, \mathbf{Q})$.

If p is the exponential characteristic of k, we write $\mathrm{Tor}^p(X)$ for the part of $\mathrm{Tor}(X)$ which is prime to p (so $\mathrm{Tor}^p(X) = \mathrm{Tor}(X)$ if $\mathrm{char}\, k = 0$).

In $\mathbf{Chow}^{\mathrm{o}}$, the identity morphism $1_{h^{\mathrm{o}}(X)}$ is given by $\eta_X \in CH_0(X_{k(X)})$, where η_X is the generic point viewed as a closed point of $X_{k(X)}$. This gives a concrete description of the torsion order:

Lemma 2.6. *Suppose that $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$. Then the torsion order of X is the order n of η_X in the group $CH_0(X_{k(X)})/CH_0(X)$ (it is 0 if and only if η_X has infinite order). Moreover, we have $nF(X) = 0$ for any normalised motivic birational invariant F. In particular,*

$$nCH_0\big(X_K\big)_0 = n\frac{CH_0\big(X_K\big)}{CH_0(X)} = 0 \quad \textit{for any } K \supseteq k$$

where $CH_0(X_K)_0 = \mathrm{Ker}(CH_0(X_K) \xrightarrow{\deg} \mathbf{Z})$.

Proof. The first and second statements are tautological; the third follows as a special case of the second. □

2.4. Torsion order and index. Another important invariant is:

Definition 2.7. The *index* of an irreducible $X \in \mathbf{Sm}^{\mathrm{proj}}$ is the positive generator of $\mathrm{Im}(\deg : CH_0(X) \to \mathbf{Z})$. We denote it by $I(X)$.

Proposition 2.8. *Let $X \in \mathbf{Sm}^{\mathrm{proj}}$, irreducible. Write n for its torsion order and d for its index.*

(a) *If F is a motivic invariant and $\underline{F}$ is as in Example 2.3(c), then we have $d\underline{F}(X) = 0$.*

(b) *n is divisible by d.*

(c) *Suppose $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$. If $x \in CH_0(X)$ is an element of degree d, then $m(x_{k(X)} - d\eta_X) = 0$ in $CH_0(X_{k(X)})$ for some $m > 0$, and $n \mid md$.*

(d) *If $d = 1$ and m is minimal in (c), then $n = m$.*

Proof. (a) Suppose F is contravariant. Let $\alpha \in F(k)$ be such that $\pi^*\alpha = 0$, where $\pi : X \to \mathrm{Spec}\, k$ is the structural morphism. If $x \in CH_0(X)$ is an element of degree d, it defines a morphism $x : \mathbf{1} \to h^{\mathrm{o}}(X)$ such that $\pi \circ x = d$. Hence $d\alpha = 0$.

(b) A diagram chase yields an exact sequence

$$CH_0(X)_0 \to CH_0\big(X_{k(X)}\big)_0 \to \frac{CH_0\big(X_{k(X)}\big)}{CH_0(X)} \to \mathbf{Z}/d \to 0 \tag{2.1}$$

where $CH_0(X_K)_0$ was defined in Lemma 2.6 and the last map sends the class of η_X to 1.

(c) The first claim follows from (2.1), and the second follows from pushing this identity in $CH_0(X_{k(X)})/CH_0(X)$.

(d) If $d = 1$, (2.1) yields a surjection

$$CH_0\big(X_{k(X)}\big)_0 \twoheadrightarrow \frac{CH_0\big(X_{k(X)}\big)}{CH_0(X)}.$$

Let $y \in CH_0(X_{k(X)})_0$ mapping to the class of η_X. This means that $\eta_X - y = x_{k(X)}$ for some $x \in CH_0(X)$, and necessarily $\deg(x) = 1$. By Lemma 2.6, we have $ny = 0$ so the conclusion is true for this choice of x. But if $x' \in CH_0(X)$ is of degree 1, then $n(x' - x) = 0$ hence the conclusion remains true when replacing x by x'. □

Remark 2.9. When $d = 1$, we can avoid the recourse to the category $\mathbf{Chow}^{\mathrm{o}}_{\mathrm{norm}}$: in this case, the morphism $h^{\mathrm{o}}(X) \to \mathbf{1}$ is (noncanonically) split, hence we may consider its kernel $h^{\mathrm{o}}(X)_0 \in \mathbf{Chow}^{\mathrm{o}}$. The endomorphism ring of this birational motive is canonically isomorphic to $CH_0(X_{k(X)})/CH_0(X)$.

2.5. Change of base field and products.

Proposition 2.10. *Let K/k be a field extension. Then:*

(a) $\mathrm{Tor}(X_K) \mid \mathrm{Tor}(X)$.

(b) *If k and K are algebraically closed, then* $\mathrm{Tor}(X_K) = \mathrm{Tor}(X)$.

Proof. (a) is obvious, and (b) follows from the rigidity theorem for torsion in Chow groups [13]. □

Proposition 2.11. *For any connected $X, Y \in \mathbf{Sm}^{\mathrm{proj}}$,* $\mathrm{Tor}(X \times Y) \mid \mathrm{Tor}(X)\,\mathrm{Tor}(Y)$.

Proof. If $\mathrm{Tor}(X) = 0$ or $\mathrm{Tor}(Y) = 0$, this is obvious. Otherwise, let $m > 0$ (resp. $n > 0$) be such that $m1_{h^{\mathrm{o}}(X)}$ (resp. $n1_{h^{\mathrm{o}}(Y)}$) factors through $\mathbf{1}$. Then $mn1_{h^{\mathrm{o}}(X\times Y)} = m1_{h^{\mathrm{o}}(X)} \otimes n1_{h^{\mathrm{o}}(Y)}$ factors through $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$. □

3. Torsion order for cycle modules

For any abelian group A, write

$$\exp(A) = \inf\{m > 0 \mid mA = 0\}$$

and, by convention, $\exp(A) = 0$ if no such integer m exists. Also write

$$\exp^p(A) = \exp\left(A[1/p]\right).$$

3.1. General case.

We refine the notion of torsion order as follows:

Definition 3.1. Let M be a cycle module. For $X \in \mathbf{Sm}^{\mathrm{proj}}$, $K \supseteq k$ and $n \in \mathbf{Z}$, write $F_n(X_K) = \mathrm{Coker}(M_n(K) \to A^0(X_K, M_n))$: then $X \mapsto F_n(X_K)$ is a normalised motivic birational invariant in the sense of Definition 2.1. We set

$$\begin{aligned} \mathrm{Tor}_K\left(X, M_n\right) &= \exp\left(F_n\left(X_K\right)\right), \\ \mathrm{Tor}\left(X, M_n\right) &= \mathrm{lcm}_{K \supseteq k} \mathrm{Tor}_K\left(X, M_n\right), \\ \mathrm{Tor}\left(X, M\right) &= \mathrm{lcm}_n \mathrm{Tor}\left(X, M_n\right). \end{aligned}$$

where lcm means lower common multiple.

By Lemma 2.6, $\mathrm{Tor}_K(X, M_n) \mid \mathrm{Tor}(X, M_n) \mid \mathrm{Tor}(X)$. Moreover,

Lemma 3.2. $\mathrm{Tor}(X, M_{n-1}) \mid \mathrm{Tor}(X, M_n)$.

Proof. Let K/k be an extension. We have a naturally split exact sequence ([15, Prop. 2.2] and its proof):

$$0 \to A^0\left(X_K, M_n\right) \to A^0\left(X_{K(t)}, M_n\right) \to \bigoplus_{x \in (\mathbf{A}^1_K)^{(1)}} A^0\left(X_{K(x)}, M_{n-1}\right) \to 0.$$

Indeed, $K \mapsto A^0(X_K, M_n)$ defines a cycle module. Comparing with the same exact sequence for $X = \mathrm{Spec}\, k$, we get the conclusion. □

3.2. Unramified cohomology of degree ≤ 2.

For $K \supseteq k$, we write $\bar{K}$ for an algebraic closure of K and $G_K = \mathrm{Gal}(\bar{K}/K)$. Let p be the exponential characteristic of k. We compute $\mathrm{Tor}(X, \mathcal{H}_n)$ for low values of n, where $\mathcal{H}_n$ is the cycle module

$$K \mapsto H^n_{\text{ét}}\left(K, (\mathbf{Q}/\mathbf{Z})'(n-1)\right)$$

with

$$(\mathbf{Q}/\mathbf{Z})'(n-1) := \varinjlim_{(m,p)=1} \mu_m^{\otimes n-1}.$$

As is well known, we have

$$A^0(X_K, \mathcal{H}_n) = \begin{cases} H^0\big(K, (\mathbf{Q}/\mathbf{Z})'(-1)\big) & \text{for } n = 0, \\ H^1\big(X_K, (\mathbf{Q}/\mathbf{Z})'\big) & \text{for } n = 1, \\ \mathrm{Br}\,\big(X_K\big)[1/p] & \text{for } n = 2. \end{cases}$$

Let X be such that $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$; then $b^1(X) = 0$ and $b^2(X) = \rho(X)$ where $b^i(X)$ (resp. $\rho(X)$) denotes the ith Betti number (resp. the Picard number) of X [12, Prop. 3.1.4 3)]. In particular, we have $\mathrm{Pic}^0_{X/k} = 0$ and for any $K \supseteq k$, $H^1(X_{\bar{K}}, (\mathbf{Q}/\mathbf{Z})') \xrightarrow{\sim} \mathrm{NS}(X_{\bar{k}})_{\mathrm{tors}}[1/p]$ and similarly $\mathrm{Br}(X_{\bar{K}})\{l\} \xrightarrow{\sim} H^3_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_l)_{\mathrm{tors}}$ for $l \neq p$, so $\mathrm{Br}(X_{\bar{k}})[1/p] \xrightarrow{\sim} \mathrm{Br}(X_{\bar{K}})[1/p]$. (We neglected Tate twists in these computations.)

In the sequel, we abbreviate $X_{\bar{k}}$ to $\bar{X}$; for simplicity, we assume $I(X) = 1$ so that $H^i(K, (\mathbf{Q}/\mathbf{Z})'(j)) \to H^i(X_K, (\mathbf{Q}/\mathbf{Z})'(j))$ is split injective for any K, i, j. The Hochschild–Serre spectral sequence then gives isomorphisms (see Definition 3.1 for the notation F_n):

$$F_0\big(X_K\big) = 0, \quad F_1\big(X_K\big) = \big(\mathrm{NS}(\bar{X})_{\mathrm{tors}}[1/p]\big)^{G_K}$$

and an exact sequence

$$0 \to H^1\big(K, \mathrm{NS}(\bar{X})\big)[1/p] \to F_2(X_K) \to \big(\mathrm{Br}(\bar{X})[1/p]\big)^{G_K}. \tag{3.1}$$

For $K \supseteq \bar{k}$, G_K acts trivially on $\mathrm{NS}(\bar{X})$ and $\mathrm{Br}(\bar{X})$. Then $H^1(K, \mathrm{NS}(\bar{X})) = \mathrm{Hom}(G_K, \mathrm{NS}(\bar{X})_{\mathrm{tors}})$ and the last map in (3.1) is split surjective: indeed, $\mathrm{Br}(\bar{X})[1/p]$ maps to $F_2(X_K)$ by functoriality. This yields:

Proposition 3.3. *Let X be such that $I(X) = 1$ and $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$. Then*

$$\begin{aligned} &\mathrm{Tor}\,\big(X, \mathcal{H}_0\big) = 1 \\ &\mathrm{Tor}\,\big(X, \mathcal{H}_1\big) = \exp^p\,\big(\mathrm{NS}(\bar{X})_{\mathrm{tors}}\big) \\ &\mathrm{Tor}\,\big(X, \mathcal{H}_2\big) = \mathrm{lcm}\,\big(\exp^p\,\big(\mathrm{NS}(\bar{X})_{\mathrm{tors}}\big), \exp^p\,\big(\mathrm{Br}(\bar{X})\big). \end{aligned}$$

In particular, $\mathrm{Tor}(X)$ *is divisible by* $\exp^p(\mathrm{NS}(\bar{X})_{\mathrm{tors}})$ *and* $\exp^p(\mathrm{Br}(\bar{X}))$. □

(Of course, one could recover this conclusion directly by considering the normalised motivic birational functors $X \mapsto \mathrm{NS}(\bar{X})_{\mathrm{tors}}[1/p]$ and $X \mapsto \mathrm{Br}(\bar{X})[1/p]$.)

Remark 3.4. When k is algebraically closed, the above computation yields $\mathrm{Tor}_k(X, \mathcal{H}_1) = \exp^p(\mathrm{NS}(X)_{\mathrm{tors}})$ and $\mathrm{Tor}_k(X, \mathcal{H}_2) = \exp^p(\mathrm{Br}(X))$.

When $\dim X = 2$, $\exp^p(\mathrm{NS}(\bar{X})_{\mathrm{tors}}) = \exp^p(\mathrm{Br}(\bar{X}))$ by Poincaré duality. We shall see in Corollary 6.4 that, then, $\mathrm{Tor}^p(\bar{X}) = \exp^p(\mathrm{NS}(\bar{X})_{\mathrm{tors}}) = \exp^p(\mathrm{Br}(\bar{X}))$. In view of Proposition 3.3, this also yields

$$\mathrm{Tor}^p(X) = \mathrm{Tor}(X, \mathcal{H}) \quad \text{if } \dim X \leq 2. \tag{3.2}$$

Question 3.5. Is the equality (3.2) true in general? In other words, does the cycle module $\mathcal{H}_*$ always compute the torsion index?

4. Extension of functors

Definition 4.1. If F is a contravariant functor from smooth k-schemes of finite type to abelian groups, we extend it to smooth k-schemes essentially of finite type by the formula

$$\widetilde{F}(X) = \varinjlim_{\mathcal{X}} F(\mathcal{X}) \tag{4.1}$$

where $\mathcal{X}$ runs through the smooth models of finite type of X/k.

Note that if $F(X) = A^n_{\mathrm{alg}}(X)$, then F is defined on all smooth k-schemes (not necessarily of finite type), but does not commute with filtering colimits; so the natural map

$$\widetilde{A}^n_{\mathrm{alg}}(X) \to A^n_{\mathrm{alg}}(X)$$

is not an isomorphism in general, see [12, Rk. 2.3.10 2)]. By contrast, we have:

Lemma 4.2. *For any cycle module M, the functors $A^p(-, M_q)$ of §5 below commute with filtering colimits of smooth schemes.*

Proof. This is obvious, since the same is true for the cycle complexes of [15]. □

As a special case, one recovers the commutation of Chow groups with filtering colimits [3, Lemma 1A.1].

5. The Rost spectral sequence

Let M be a cycle module. For any smooth X/k, recall its *cycle cohomology with coefficients in M*:

$$A^p(X, M_q) = H^p\Big(\cdots \to \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x)) \to \cdots\Big)$$

where the differentials are defined through Rost's axioms. We assume:

(i) $M_n = 0$ for $n < 0$;

(ii) $M_0(K) = A$ for any K/k, where A is a torsion-free abelian group.

By Rost's axioms [15], there is then a canonical homomorphism of cycle modules

$$K^M \otimes A \to M$$

where K^M is the cycle module given by Milnor's K-theory. For any $n \geq 0$, this yields a *surjective homomorphism*

$$CH^n(X) \otimes A = A^n\big(X, K^M_n \otimes A\big) \longrightarrow\!\!\!\!\!\rightarrow A^n\big(X, M_n\big) =: A^n_M(X). \tag{5.1}$$

We may thus think of $A^n_M(X)$ as the group of cycles of codimension n modulo "M-equivalence".

Examples 5.1. (1) For $M = K^M \otimes A$, we get $A^n_M(X) = CH^n(X) \otimes A$.

(2) Let H be Betti cohomology (in characteristic 0) or l-adic cohomology (in characteristic $\neq l$): in the first case, let $A = \mathbf{Z}$ and in the second case let $A = \mathbf{Z}_l$. For a function field K/k, set

$$\mathbf{H}_n(K) := \tilde{H}^n\big(\operatorname{Spec} K, A(n)\big)$$

see Definition 4.1. (This is not the cycle module $\mathscr{H}$ considered in Subsection 3.2.) By [4, Th. 7.3] and [11, proof of Prop. 4.5], one has

$$A^n_{\mathbf{H}}(X) = A^n_{\mathrm{alg}}(X) \otimes A$$

where $A^n_{\mathrm{alg}}(X)$ is the group of cycles of codimension n on X, modulo algebraic equivalence.

We now take two smooth k-varieties X, Y, and study the Rost spectral sequence [15, Cor. 8.2] attached to the first projection $\pi : Y \times X \to Y$:

$$E_1^{p,q}(r) = \bigoplus_{y \in Y^{(p)}} A^q\big(X_{k(y)}, M_{r-p}\big) \Rightarrow A^{p+q}\big(Y \times X, M_r\big) \tag{5.2}$$

abutting to the coniveau filtration on $A^{p+q}(Y \times X, M_r)$ with respect to Y. Note that $A^q(X_{k(y)}, M_{r-p}) = 0$ for $p + q > r$ by Condition (i) on M, hence $E_1^{p,q}(r) = 0$ in that range.

Take $r = 2$: we only have to consider $p + q \leq 2$. By definition, we have for a function field K/k (see (5.1) for the notation A^q_M):

$$A^q(X_K, M_q) = \varinjlim_U A^q_M(X \times U) =: \tilde{A}^q_M(X_K)$$

where U runs through smooth models of K as above (see Lemma 4.2). This yields

$$E_2^{0,2}(2) = \tilde{A}^2_M\big(X_{k(Y)}\big)$$

$$E_2^{1,1}(2) = \operatorname{Coker}\Big(A^1\big(X_{k(Y)}, M_2\big) \to \bigoplus_{y \in Y^{(1)}} \tilde{A}^1_M\big(X_{k(y)}\big)\Big)$$

$$E_2^{2,0}(2) = \operatorname{Coker}\Big(\bigoplus_{y \in Y^{(1)}} A^0\big(X_{k(y)}, M_1\big) \to Z^2(Y) \otimes A\Big).$$

The latter group is a quotient of $A^2_M(Y)$ (consider the maps $M_1(k(y)) \to A^0(X_{k(y)}, M_1)$). If X has a 0-cycle of degree 1, the map $A^2_M(Y) \to A^2_M(Y \times X)$ is split, hence $\pi^* : A^2_M(Y) \to E_2^{2,0}(2)$ is an isomorphism. Thus $E_2 = E_\infty$ in the Rost spectral sequence. We summarise this discussion:

Proposition 5.2. *Let* $\mathrm{gr}^*_Y A^2_M(X \times Y)$ *be the associated graded to the coniveau filtration relative to* Y*. Assume that* X *has a* 0*-cycle of degree* 1*. Then we have isomorphisms*

$$\mathrm{gr}^0_Y A^2_M(X \times Y) = \tilde{A}^2_M\big(X_{k(Y)}\big)$$
$$\mathrm{gr}^1_Y A^2_M(X \times Y) = \operatorname{Coker}\Big(A^1\big(X_{k(Y)}, M_2\big) \to \bigoplus_{y \in Y^{(1)}} \tilde{A}^1_M\big(X_{k(y)}\big)\Big)$$
$$\mathrm{gr}^2_Y A^2_M(X \times Y) = A^2_M(Y).$$

Moreover, we have an exact sequence:

$$\begin{aligned} 0 \to A^1\big(Y, M_2\big) \to A^1\big(Y \times X, M_2\big) \to A^1\big(X_{k(Y)}, M_2\big) \\ \to \bigoplus_{y \in Y^{(1)}} \tilde{A}^1_M\big(X_{k(y)}\big) \to A^2_M(Y \times X)/A^2_M(Y). \end{aligned} \tag{5.3}$$

6. Trivial birational motives of surfaces

We start with a special case of Proposition 5.2:

Theorem 6.1. *Suppose* k *algebraically closed, and let* X/k *be a smooth projective variety such that* $\mathrm{Pic}^0_{X/k} = 0$*. Then for any smooth* Y*, there is an exact sequence*

$$\begin{aligned} CH^2(Y) \oplus \mathrm{Pic}(Y) \otimes \mathrm{NS}(X) \oplus CH^2(X) \\ \to CH^2(Y \times X) \to CH^2\big(X_{k(Y)}\big)/CH^2(X) \to 0 \end{aligned} \tag{6.1}$$

where the maps $CH^2(X), CH^2(Y) \to CH^2(Y \times X)$ *are induced by the two projections, and the map* $\mathrm{Pic}(Y) \otimes \mathrm{NS}(X) \to CH^2(Y \times X)$ *is given by the cross-product of cycles.*

A version of this theorem is found in Merkurjev's appendix [14]; I thank J.-L. Colliot-Thélène for pointing out this reference.

Proof. Consider the Rost spectral sequence (5.2) for the cycle module $M = K^M$. Since $\mathrm{Pic}^0_{X/k} = 0$, we have $\mathrm{NS}(X) \xrightarrow{\sim} \mathrm{Pic}(X_{k(y)})$ for any $y \in Y^{(1)}$, hence

$$E_2^{1,1} = \operatorname{Coker}\big(A^1\big(X_{k(Y)}, K_2\big) \to Z^1(Y) \otimes \mathrm{NS}(X)\big).$$

Then the natural map $k(Y)^* \otimes \mathrm{NS}(X) \to A^1(X_{k(Y)}), K_2)$ realises $E_2^{1,1}$ as a quotient of $\mathrm{Pic}(Y) \otimes \mathrm{NS}(X)$. We conclude by applying Proposition 5.2. □

Theorem 6.1 may be compared with a computation of the cohomology of $Y \times X$. We use l-adic cohomology, neglecting Tate twists: so $H^i(X) := \prod_{l \neq p} H^i_{\text{ét}}(X, \mathbf{Z}_l)$, where p is the exponential characteristic of k ($p = 1$ if $\operatorname{char} k = 0$). If $k = \mathbf{C}$, we have $H^i(X) \simeq H^i_B(X) \otimes \prod_l \mathbf{Z}_l$, by M. Artin's comparison theorem. We note that the choice of a rational point of X gives a retraction of the map $F(Y) \to F(Y \times X)$ for any contravariant functor $F : \mathbf{Sm}^{\text{proj}} \to \mathbf{Ab}$; the quotient $F(Y \times X, Y)$ is therefore a direct summand of $F(Y \times X)$. Then the Künneth formula gives split exact sequences

$$0 \to H^3(X) \to H^3(Y \times X, Y) \to \operatorname{Tor}\big(H^2(Y), H^2(X)\big) \to 0 \qquad (6.2)$$

and

$$\begin{aligned} 0 \to H^2(Y) \otimes H^2(X) \oplus H^1(Y) \otimes H^3(X) \oplus H^4(X) \\ \to H^4(Y \times X, Y) \\ \to \operatorname{Tor}\big(H^2(Y), H^3(X)\big) \oplus \operatorname{Tor}\big(H^3(Y), H^2(X)\big) \to 0. \end{aligned} \qquad (6.3)$$

We now make the following

Assumption 6.2. k is algebraically closed, Y is projective and X is a surface such that $CH_0(X_{k(X)}) \otimes \mathbf{Q} \xrightarrow{\sim} \mathbf{Q}$.

Recall that, then, $\operatorname{Alb}(X) = \operatorname{Pic}^0_{X/k} = 0$ and $CH^2(X) = \mathbf{Z}$ (Roĭtman's theorem), so that Theorem 6.1 applies. Recall also that

$$\begin{aligned} H^1(X) &= 0 \\ \operatorname{NS}(X) \otimes \hat{\mathbf{Z}}' &\xrightarrow{\sim} H^2(X) \\ H^3(X) &\simeq \operatorname{Hom}\big(\operatorname{NS}(X)_{\text{tors}}, (\mathbf{Q}/\mathbf{Z})'\big) \\ H^4(X) &= \hat{\mathbf{Z}}' \end{aligned}$$

where $\hat{\mathbf{Z}}' = \prod_{l \neq p} \mathbf{Z}_l$. Thus (6.1) and (6.3) yield a commutative diagram

$$\begin{array}{ccccccc} & (\operatorname{Pic}(Y) \otimes \operatorname{NS}(X) \oplus \mathbf{Z}) \otimes \hat{\mathbf{Z}}' & \to & CH^2(Y \times X, Y) \otimes \hat{\mathbf{Z}}' & \longrightarrow & \frac{CH^2(X_{k(Y)})}{CH^2(X)} \otimes \hat{\mathbf{Z}}' & \longrightarrow 0 \\ & \downarrow \psi & & \downarrow \operatorname{cl}^2_{Y \times X, Y} & & \downarrow \varphi & \\ 0 \to & \begin{matrix} H^2(Y) \otimes H^2(X) \\ \oplus H^1(Y) \otimes H^3(X) \oplus \hat{\mathbf{Z}}' \end{matrix} & \xrightarrow{\theta} & H^4(Y \times X, Y) & \longrightarrow & \begin{matrix} \operatorname{Tor}(H^2(Y), H^3(X)) \\ \oplus \operatorname{Tor}(H^3(Y), H^2(X)) \end{matrix} & \to 0. \end{array} \qquad (6.4)$$

An obvious generalisation of the exact sequence (2.1) boils down to an isomorphism

$$CH_0\big(X_{k(Y)}\big)_0 \xrightarrow{\sim} CH_0\big(X_{k(Y)}\big)/CH_0(X).$$

In (6.4), the left vertical map ψ is diagonal; its cokernel is

$$\operatorname{Coker}\psi = H^2_{\mathrm{tr}}(Y)\otimes \mathrm{NS}(X)\oplus H^1(Y)\otimes H^3(X)$$

where $H^2_{\mathrm{tr}}(Y) := \operatorname{Coker}\mathrm{cl}^1_Y$, and its kernel is $\mathrm{Pic}^0(Y)\otimes\mathrm{NS}(X)\otimes\hat{\mathbf{Z}}'$ (we use here that $H^2_{\mathrm{tr}}(Y)$ is torsion-free). The snake lemma thus yields an exact sequence

$$\begin{aligned}&\mathrm{Pic}^0(Y)\otimes\mathrm{NS}(X)\otimes\hat{\mathbf{Z}}' \xrightarrow{\alpha} \operatorname{Ker}\mathrm{cl}^2_{Y\times X,Y} \xrightarrow{\beta} \operatorname{Ker}\varphi\\ &\to H^2_{\mathrm{tr}}(Y)\otimes\mathrm{NS}(X)\oplus H^1(Y)\otimes H^3(X)\xrightarrow{\gamma}\operatorname{Coker}\mathrm{cl}^2_{Y\times X,Y}\to\operatorname{Coker}\varphi\to 0.\end{aligned}\tag{6.5}$$

To go further, we use étale motivic cohomology as in [11]; the cycle class map $\mathrm{cl}^2_{X\times X}$ extends to an étale cycle class map [11, (3-1)]:

$$\widetilde{\mathrm{cl}}^2_{Y\times X,Y} : H^4_{\text{ét}}\big(Y\times X, Y, \mathbf{Z}(2)\big)\otimes\hat{\mathbf{Z}}' \to H^4(Y\times X, Y).$$

Theorem 6.3. *Under Assumption 6.2,* $\operatorname{Ker}\mathrm{cl}^2_{Y\times X,Y}$ *and* $\operatorname{Ker}\widetilde{\mathrm{cl}}^2_{Y\times X,Y}$ *are torsion-free; the exact sequence (6.5) yields a surjection*

$$\mathrm{Pic}^0(Y)\otimes\mathrm{NS}(X)\otimes\hat{\mathbf{Z}}' \twoheadrightarrow \operatorname{Ker}\mathrm{cl}^2_{Y\times X,Y}$$

and an exact sequence of finite groups

$$\begin{aligned}0\to\operatorname{Ker}\varphi\to H^2_{\mathrm{tr}}(Y)\otimes\mathrm{NS}(X)_{\mathrm{tors}}\oplus H^1(Y)\otimes H^3(X)&\\ \to H^3_{\mathrm{nr}}\big(Y\times X, Y;(\mathbf{Q}/\mathbf{Z})'(2)\big)\to\operatorname{Coker}\varphi\to 0&\end{aligned}\tag{6.6}$$

where $H^3_{\mathrm{nr}}(Y\times X, Y;(\mathbf{Q}/\mathbf{Z})'(2)) := \varinjlim_{(m,p)=1} H^3_{\mathrm{nr}}(Y\times X, Y;\mu_m^{\otimes 2})$. *In particular,* $CH_0(X_{k(Y)})/CH_0(X)[1/p]\simeq CH_0(X_{k(Y)})_{\mathrm{tors}}[1/p]$ *is finite.*

Proof. This proof is ugly, mainly because the Leray spectral sequence for étale motivic cohomology relative to the projection $(Y\times X, Y)\to Y$ does not behave as well as the spectral sequence (5.2). So, instead of comparing directly étale motivic and l-adic cohomology, we have to wiggle through.

We have a commutative diagram

$$\begin{array}{ccc} H^3_{\text{ét}}\big(Y\times X, Y, (\mathbf{Q}/\mathbf{Z})'(2)\big) & \xrightarrow{\sim} & \varinjlim_{(m,p)=1} H^3_{\text{ét}}\big(Y\times X, Y, \mu_m^{\otimes 2}\big)\\ \downarrow & & \downarrow\\ H^4_{\text{ét}}\big(Y\times X, Y, \mathbf{Z}(2)\big)\otimes\hat{\mathbf{Z}}' & \xrightarrow{\widetilde{\mathrm{cl}}^2_{Y\times X,Y}} & H^4(Y\times X, Y)\end{array}\tag{6.7}$$

in which the right vertical map is injective, because $H^3(Y \times X, Y)$ is torsion by (6.2). Thus $\operatorname{Ker} \widetilde{\mathrm{cl}}^2_{Y\times X,Y}$ is torsion-free, and so is its subgroup $\operatorname{Ker} \mathrm{cl}^2_{Y\times X,Y}$. But the image of α in (6.5) is divisible, hence a direct summand. Therefore the image of β is torsion-free, hence 0. So we get the surjection promised in the theorem, and an exact sequence

$$0 \to \operatorname{Ker}\varphi \to H^2_{\mathrm{tr}}(Y) \otimes \mathrm{NS}(X) \oplus H^1(Y) \otimes H^3(X) \\ \to \operatorname{Coker} \mathrm{cl}^2_{Y\times X,Y} \to \operatorname{Coker}\varphi \to 0. \quad (6.8)$$

As a consequence, $\operatorname{Ker}\varphi$ is finitely generated; since it is torsion it must be finite, hence $CH_0(X_{k(Y)})/CH_0(X)[1/p]$ is finite.

We now deduce from [11, Th. 1.1] the following surjection:

$$H^3_{\mathrm{nr}}\big(Y \times X, Y; (\mathbf{Q}/\mathbf{Z})'(2)\big) \longrightarrow\!\!\!\!\!\rightarrow \big(\operatorname{Coker} \mathrm{cl}^2_{Y\times X,Y}\big)_{\mathrm{tors}} \quad (6.9)$$

(if $k = \mathbf{C}$, this is due to Colliot-Thélène–Voisin [10, Th. 3.7], with Betti cohomology instead of l-adic cohomology). This map has divisible kernel; however, $Z \mapsto H^3_{\mathrm{nr}}(Y \times Z, Y; (\mathbf{Q}/\mathbf{Z})'(2))$ is a normalised motivic birational invariant, hence $H^3_{\mathrm{nr}}(Y \times X, Y; (\mathbf{Q}/\mathbf{Z})'(2))$ is killed by $\operatorname{Tor}(X)$ and therefore finite; so (6.9) is an isomorphism.

Let $M = \operatorname{Coker} \mathrm{cl}^2_{Y\times X,Y}/\mathrm{tors}$; by [11, Cor. 3.5], this is actually $\operatorname{Coker} \widetilde{\mathrm{cl}}^2_{Y\times X,Y}$, although we won't use it. The composition of the map γ of (6.5) with the projection $p : \operatorname{Coker} \mathrm{cl}^2_{Y\times X,Y} \to M$ has image isomorphic to $H^2_{\mathrm{tr}}(Y) \otimes (\mathrm{NS}(X)/\mathrm{tors})$.

I claim that $p \circ \gamma$ is surjective. To see this, choose a retraction ρ of the map θ in Diagram (6.4); composing $\rho \circ \mathrm{cl}^2_{Y\times X,Y}$ with the projection to $\operatorname{Coker}\psi$, we get an induced map

$$CH^2\big(X_{k(Y)}\big)/CH^2(X) \otimes \hat{\mathbf{Z}}' \to \operatorname{Coker}\psi$$

whose composition with

$$\operatorname{Coker}\psi \to H^2_{\mathrm{tr}}(Y) \otimes \big(\mathrm{NS}(X)/\mathrm{tors}\big)$$

is 0 since $CH^2(X_{k(Y)})/CH^2(X)$ is torsion. This shows that ρ induces a map

$$\bar{\rho} : \operatorname{Coker} \mathrm{cl}^2_{Y\times X,Y} \to H^2_{\mathrm{tr}}(Y) \otimes \big(\mathrm{NS}(X)/\mathrm{tors}\big)$$

factoring through a left inverse of the inclusion $H^2_{\mathrm{tr}}(Y) \otimes (\mathrm{NS}(X)/\mathrm{tors}) \hookrightarrow M$ induced by γ. But $\gamma \otimes \mathbf{Q}$ is an isomorphism, since $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are torsion; therefore $p \circ \gamma$ is surjective as claimed.

Chasing in (6.8) with this information and using the isomorphism (6.9) now yields the exact sequence (6.6). □

Corollary 6.4. (a) *Suppose $Y = X$. Then we have a commutative diagram of short exact sequences:*

$$\begin{array}{ccccc} 0\to & CH^2(X\times X)\otimes\hat{\mathbf{Z}}' & \xrightarrow{\mathrm{cl}^2_{X\times X}} & H^4(X\times X) & \to H^3_{\mathrm{nr}}(X\times X,(\mathbf{Q}/\mathbf{Z})'(2))\to 0\\ & \downarrow & & \downarrow & \| \\ 0\to & CH^2(X_{k(X)})/CH^2(X)[1/p] & \xrightarrow{\varphi} & \mathrm{Tor}(H^2(X),H^3(X))^2 & \to H^3_{\mathrm{nr}}(X\times X,(\mathbf{Q}/\mathbf{Z})'(2))\to 0.\end{array}$$

(b) *In particular, the first map of* (6.1) *(for $Y = X$) has p-primary torsion kernel, and* $\mathrm{Tor}^p(X) = \exp^p(\mathrm{NS}(X)_{\mathrm{tors}})$.

Proof. Indeed, we have $\mathrm{Pic}^0(X) = H^1(X) = H^2_{\mathrm{tr}}(X) = 0$, and Theorem 6.3 boils down to the injectivity of $\mathrm{cl}^2_{X\times X,X}$ and φ, plus an isomorphism

$$H^3_{\mathrm{nr}}\big(X\times X, X; (\mathbf{Q}/\mathbf{Z})'(2)\big) \xrightarrow{\sim} \mathrm{Coker}\,\varphi.$$

But $H^3_{\mathrm{nr}}(X,(\mathbf{Q}/\mathbf{Z})'(2)) = 0$, hence

$$H^3_{\mathrm{nr}}\big(X\times X,(\mathbf{Q}/\mathbf{Z})'(2)\big) \xrightarrow{\sim} H^3_{\mathrm{nr}}\big(X\times X, X;(\mathbf{Q}/\mathbf{Z})'(2)\big). \qquad \square$$

Corollary 6.5. *If Y is a curve, we have a short exact sequence*

$$0\to CH^2\big(X_{k(Y)}\big)_{\mathrm{tors}}[1/p]\to H^1(Y)\otimes H^3(X)\to \mathrm{Coker}\,\mathrm{cl}^2_{Y\times X}\to 0.$$

Proof. In this case, $H^2_{\mathrm{tr}}(Y) = 0$ and the target of φ is 0. $\square$

Remarks 6.6. (a) The special case $\mathrm{NS}(X)_{\mathrm{tors}} = 0$ and $\mathrm{char}\,k = 0$ of Corollary 6.4(b) was proven in [1, Cor. 1.10] and [17, Prop. 2.2]. As Colliot-Thélène points out, the methods of [8] imply that for any smooth projective k-variety X with $b^1 = 0$ and $b^2 = \rho$, $\mathrm{Ker}(CH^2(X_K)\to CH^2(X_{\bar K}))$ is killed by $\exp(\mathrm{NS}(X)_{\mathrm{tors}})\cdot\exp(\mathrm{Br}(X))$ (see Theorem A.1).

(b) In the first version of this paper, I had proven Corollaries 6.4 and 6.5 but had doubts on the finiteness of $CH_0(X_{k(Y)})_{\mathrm{tors}}$ in general. Colliot-Thélène provided a proof based on his 1991 CIME course [9], see Theorem A.6. This encouraged me to find a proof in the spirit of this note, and Theorem 6.3 is the result. Note that the group Θ appearing in [9, Th. 7.1] coincides with $H^4_{\text{ét}}(X,\mathbf{Z}(2))_{\mathrm{tors}}$. In this spirit, a weaker analogue of [9, Th. 7.3] is the following fact: for any field F, the functor

$$\mathbf{Sm}^{\mathrm{proj}}(F)\ni Z\mapsto \mathrm{Ker}\,\big(H^4_{\text{ét}}(Z,\mathbf{Z}(2))\big)\to H^4_{\text{ét}}\big(Z_{\bar F},\mathbf{Z}(2)\big)$$

is a normalised motivic birational invariant (indeed, the map $H^2_{\text{ét}}(Y,\mathbf{Z}(1))\to H^2_{\text{ét}}(Y_{\bar F},\mathbf{Z}(1))$ is injective for any smooth projective Y). As a consequence, $\mathrm{Ker}(H^4_{\text{ét}}(X,\mathbf{Z}(2))\to H^4_{\text{ét}}(X_{\bar F},\mathbf{Z}(2))$ is killed by $\mathrm{Tor}(X)$ if X has a trivial birational motive.

A. Cycles de codimension deux, complément à deux anciens articles

par Jean-Louis Colliot-Thélène

A.1. Introduction. On donne des conséquences faciles de résultats établis dans [8] (avec W. Raskind) et dans le rapport de synthèse [9], en particulier dans une section où je développais des arguments de S. Saito et de P. Salberger.

A.2. Notations et rappels. Pour simplifier les énoncés, on se limite ici aux variétés définies sur un corps de caractéristique nulle. On note $\bar{k}$ une clôture algébrique de k. Pour une telle k-variété X, supposée projective, lisse, géométriquement connexe sur le corps k, on note $\bar{X} = X \times_k \bar{k}$. On note b_i le i-ième nombre de Betti l-adique de $\bar{X}$. On sait qu'il est indépendant du nombre premier l. On note ρ le rang du groupe de Néron–Severi géométrique $\mathrm{NS}(\bar{X})$. Pour tout entier i, on note ici

$$H^i\big(\bar{X}, \hat{\mathbf{Z}}(j)\big) := \prod_l H^i_{\text{ét}}\big(\bar{X}, \mathbf{Z}_l(j)\big).$$

Le sous-groupe de torsion $H^i(\bar{X}, \hat{\mathbf{Z}}(j))_{\text{tors}}$ est fini. On note e_i son exposant. Pour $k = \mathbf{C}$ le corps des complexes,

$$H^i_{\text{Betti}}\big(X(\mathbf{C}), \mathbf{Z}\big) \otimes \mathbf{Z}_l \simeq H^i_{\text{ét}}\big(X, \mathbf{Z}_l\big).$$

On sait que l'on a un isomorphisme de groupes finis $\mathrm{NS}(\bar{X})_{\text{tors}} = H^2(\bar{X}, \hat{\mathbf{Z}}(1))_{\text{tors}}$. Le groupe de Brauer $\mathrm{Br}(\bar{X})$ de $\bar{X}$ est extension du groupe fini $H^3(\bar{X}, \hat{\mathbf{Z}}(1))_{\text{tors}}$ par $(\mathbf{Q}/\mathbf{Z})^{b_2-\rho}$. La condition $H^1(X, O_X) = 0$ équivaut à $b_1 = 0$. La condition $H^2(X, O_X) = 0$ équivaut (théorie de Hodge) à $\rho = b_2$, c'est-à-dire à la finitude du groupe de Brauer de $\bar{X}$. Pour X une variété lisse, on note $CH^i(X)$ le groupe de Chow des cycles de codimension i de X. Pour X une variété projective, on note $CH_i(X)$ le groupe de Chow des cycles de dimension i de X.

A.3. Exposant de torsion. L'énoncé suivant aurait pu être inclus dans [8]. Comme indiqué formellement ci-dessus, l'entier e_i est l'annulateur de la torsion du i-ème groupe de cohomologie entière.

Théorème A.1. *Soit k un corps de caractéristique zéro. Soit X une k-variété projective, lisse, connexe, satisfaisant $X(k) \neq \emptyset$. Supposons que le réseau* $\mathrm{NS}(\bar{X})/\mathrm{tors}$ *admet une base globalement respectée par le groupe de Galois absolu de k.*

(a) *Supposons $b_1 = 0$ et $\rho = b_2$. Alors le groupe de torsion*

$$\mathrm{Ker}\big[CH^2(X) \to CH^2(\bar{X})\big]$$

est annulé par le produit $e_2.e_3$, qui est aussi le produit de l'exposant de $\mathrm{NS}(\bar{X})_{\text{tors}}$ *et de l'exposant du groupe* $\mathrm{Br}(\bar{X})$.

(b) *Si de plus $b_3 = 0$, alors $CH^2(X)_{\text{tors}}$ est annulé par $e_2.e_3.e_4$.*

Démonstration. Il suffit de suivre les démonstrations du §3 de [8]. On note $H^i(k,\bullet)$ les groupes de cohomologie galoisienne.

Sous l'hypothèse $H^1(X,O_X)=0$, le théorème 1.8 de [8] donne une suite exacte de modules galoisiens

$$0\to D_0\to H^0\big(\bar X,\mathcal{K}_2\big)\to H^2\big(\bar X,\hat{\mathbf{Z}}(1)\big)_{\mathrm{tors}}\to 0$$

où D_0 est uniquement divisible. Le groupe $K_2\bar k$ est uniquement divisible. On a la suite exacte

$$0\to H^0\big(\bar X,\mathcal{K}_2\big)/K_2\bar k\to K_2\bar k(X)/K_2\bar k\to K_2\bar k(X)/H^0\big(\bar X,\mathcal{K}_2\big)\to 0.$$

Comme on a supposé $X(k)\neq\emptyset$, on a $H^1(k,K_2\bar k(X)/K_2\bar k)=0$ [7, Theorem 1]. On voit alors que le groupe $H^1(k,K_2\bar k(X)/H^0(\bar X,\mathcal{K}_2))$ est un sous-groupe de $H^2(k,H^2(\bar X,\hat{\mathbf{Z}}(1))_{\mathrm{tors}})$ et donc est annulé par e_2.

Sous les deux hypothèses $H^2(X,O_X)=0$ et $H^1(X,O_X)=0$ (cette dernière garantissant $\mathrm{Pic}(\bar X)=\mathrm{NS}(\bar X)$), le théorème 2.12 de [8] donne une suite exacte de modules galoisiens

$$0\to D_1\to \mathrm{NS}(\bar X)\otimes\bar k^*\to H^1\big(\bar X,\mathcal{K}_2\big)\to\big[D_2\oplus H^3\big(\bar X,\hat{\mathbf{Z}}(2)\big)_{\mathrm{tors}}\big]\to 0,$$

où D_1 et D_2 sont uniquement divisibles. L'hypothèse que l'action du groupe de Galois sur $\mathrm{NS}(\bar X)/\mathrm{tors}$ est triviale assure via le théorème 90 de Hilbert que l'on a $H^1(k,\mathrm{NS}(\bar X)\otimes\bar k^*)=0$. De la suite exacte ci-dessus on déduit que $H^1(k,H^1(\bar X,\mathcal{K}_2))$ est un sous-groupe de $H^1(k,H^3(\bar X,\hat{\mathbf{Z}}(2))_{\mathrm{tors}})$ et donc est annulé par e_3.

La proposition 3.6 de [8] fournit une suite exacte

$$H^1\big(k,K_2\bar k(X)/H^0\big(\bar X,\mathcal{K}_2\big)\big)\to\mathrm{Ker}\big[CH^2(X)\to CH^2(\bar X)\big]\\ \to H^1\big(k,H^1\big(\bar X,\mathcal{K}_2\big)\big).$$

On voit donc que $\mathrm{Ker}[CH^2(X)\to CH^2(\bar X)]$ est annulé par le produit $e_2.e_3$. Par Bloch et Merkurjev–Suslin, $CH^2(\bar X)_{\mathrm{tors}}$ est un sous-quotient de $H^3_{\text{ét}}(\bar X,\mathbf{Q}/\mathbf{Z}(2))$ [9, Théorème 3.3.2]. Si $b_3=0$, alors $CH^2(\bar X)_{\mathrm{tors}}$, est un sous-quotient de $H^4(\bar X,\hat{\mathbf{Z}}(2))_{\mathrm{tors}}$, d'exposant e_4. Sous les hypothèses du théorème, on obtient alors que $CH^2(X)_{\mathrm{tors}}$ est annulé par $e_2.e_3.e_4$. □

Remarques A.2. (1) Soit Y une variété projective et lisse sur le corps des complexes $\mathbf{C}$ satisfaisant les hypothèses du théorème. Pour tout corps k contenant $\mathbf{C}$, le théorème s'applique à la k-variété $X=Y\times_{\mathbf{C}}k$. L'hypothèse sur l'action galoisienne est alors automatiquement satisfaite pour la k-variété X, car on a $\mathrm{NS}(Y)=\mathrm{NS}(\bar X)$.

(2) Lorque $e_2=1=e_3$, l'énoncé (a) est le théorème 3.10 b) de [8].

(3) Si X est une surface, $e_4=1$, et $b_1=b_3$. En outre, $e_2=e_3$. Sous les hypothèses du théorème, on trouve que le groupe $CH^2(X)_{\mathrm{tors}}=CH_0(X)_{\mathrm{tors}}$ est annulé par le carré de l'exposant de la torsion de $\mathrm{NS}(\bar X)$.

A.4. Finitude. On utilise ici les notations et résultats du §7 de [9].

Théorème A.3. *Soient k un corps de caractéristique zéro et $\overline{k}$ une clôture algébrique. Soit X une k-variété projective et lisse, géométriquement intègre. Notons $\overline{X} = X \times_k \overline{k}$. Notons $b_i \in \mathbf{N}$ les nombres de Betti l-adiques de $\overline{X}$ et $\rho = \mathrm{rang}(\mathrm{NS}(\overline{X}))$. Supposons $H^1(X, O_X)$=0, ce qui équivaut à $b_1 = 0$. Supposons aussi $H^2(X, O_X) = 0$, ce qui équivaut à $\rho = b_2$. Supposons $b_3 = 0$. Alors le conoyau de l'application*

$$H^3_{\text{ét}}\big(k, \mathbf{Q}/\mathbf{Z}(2)\big) \oplus \big[H^1\big(X, \mathcal{K}_2\big) \otimes \mathbf{Q}/\mathbf{Z}\big] \to H^3_{\text{ét}}\big(X, \mathbf{Q}/\mathbf{Z}(2)\big)$$

est d'exposant fini.

Démonstration. L'hypothèse $b_3 = 0$ implique que le groupe $H^3_{\text{ét}}(\overline{X}, \mathbf{Q}/\mathbf{Z}(2))$ s'identifie au groupe fini $H^4_{\text{ét}}(\overline{X}, \hat{\mathbf{Z}}(2))_{\text{tors}}$. L'énoncé est alors une conséquence immédiate du Théorème 7.3 de [9], auquel je renvoie pour les notations. □

Théorème A.4. *Soient k un corps de caractéristique zéro et $\overline{k}$ une clôture algébrique. Soit X une k-variété projective et lisse, géométriquement intègre. Notons $\overline{X} = X \times_k \overline{k}$. Supposons que chacun des entiers b_1, $b_2 - \rho$ et b_3 associés à $\overline{X}$ est nul. Supposons $X(k) \neq \emptyset$. Alors il existe un entier $N > 0$ annulant le groupe $CH^2(X)_{\text{tors}}$ et tel que pour tout entier $n > 0$ multiple de N, l'application*

$$CH^2(X)_{\text{tors}} \to CH^2(X)/n \to H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big)$$

composée de la projection naturelle et de l'application classe de cycle en cohomologie étale est injective.

Démonstration. Il suffit de combiner le théorème A.3 avec le théorème 7.2 de [9]. □

Remarque A.5. Si X est une surface, l'hypothèse $b_3 = 0$ est impliquée par $b_1 = 0$.

On dit qu'un corps k de caractéristique zéro est à cohomologie galoisienne finie si pour tout module fini galoisien M sur k, tous les groupes de cohomologie galoisienne $H^i(k, M)$ sont finis. Parmi les corps de caractéristique zéro satisfaisant cette propriété, on trouve : les corps algébriquement clos, les corps réels clos, les corps p-adiques, les corps de séries formelles itérées sur un des corps précédents.

Théorème A.6. *Soit k un corps de caractéristique zéro à cohomologie galoisienne finie. Soit K un corps de type fini sur k. Soit X une K-variété projective et lisse satisfaisant $X(K) \neq \emptyset$. Notons $\overline{X} = X \times_K \overline{K}$. Supposons que chacun des entiers b_1, $b_2 - \rho$ et b_3 associés à $\overline{X}$ est nul. Alors le groupe $CH^2(X)_{\text{tors}}$ est fini.*

Démonstration. D'après le théorème A.4, il existe un entier $n > 0$ tel que le groupe $CH^2(X)_{\text{tors}}$ s'identifie à un sous-groupe de l'image de l'application classe de cycle

$$CH^2(X)/n \to H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big).$$

Soit Y une k-variété intègre de corps des fonctions K. Quitte à restreindre la k-variété Y à un ouvert non vide convenable, il existe un Y-schéma intègre, projectif et lisse $\mathcal{X} \to Y$ dont la fibre générique est la K-variété X. L'application de restriction $CH^2(\mathcal{X}) \to CH^2(X)$ est surjective, et les applications classe de cycle

$$CH^2(X)/n \to H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big) \quad \text{et} \quad CH^2(\mathcal{X})/n \to H^4\big(\mathcal{X}, \mu_n^{\otimes 2}\big)$$

sont compatibles. L'image de

$$CH^2(X)/n \to H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big)$$

est donc dans l'image de la restriction

$$H^4(\mathcal{X}, \mu_n^{\otimes 2}) \to H^4_{\text{ét}}\big(X, \mu_n^{\otimes 2}\big).$$

Sous les hypothèses du théorème, les groupes $H^i(W, \mu_n^{\otimes j})$ sont finis pour toute variété W de type fini sur k, en particulier $H^4(\mathcal{X}, \mu_n^{\otimes 2})$ est fini. On conclut que $CH^2(X)_{\text{tors}}$ est fini. □

Remarque A.7. Si X est une K-surface, $b_1 = b_3$ et l'hypothèse est simplement que $b_1 = 0$ et $b_2 - \rho = 0$, et la conclusion est que $CH_0(X)_{\text{tors}}$ est fini.

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