

On regular CAT(0) cube complexes and the simplicity of automorphism groups of rank-one CAT(0) cube complexes

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Abstract. We provide a necessary and sufficient condition on a finite flag simplicial complex L for which there exists a unique CAT(0) cube complex whose vertex links are all isomorphic to L . We then find new examples of such CAT(0) cube complexes and prove that their automorphism groups are virtually simple. The latter uses a result, which we prove in the appendix, about the simplicity of certain subgroups of the automorphism group of a rank-one CAT(0) cube complex. This result generalizes previous results by Tits [20] and by Haglund and Paulin [15].

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1. Introduction

Over the past years CAT(0) cube complexes have played a major role in geometric group theory and have provided many examples of interesting group actions on CAT(0) spaces. In the search for highly symmetric CAT(0) cube complexes — just as for their 1-dimensional analogues, trees — it is natural to consider the sub-class of *regular* CAT(0) cube complexes, i.e. cube complexes with the same link at each vertex.

More precisely, recall that a CAT(0) cube complex is a 1-connected cube complex whose vertex links are flag simplicial complexes (see [12]). Let L be a fixed finite flag simplicial complex (throughout the paper we assume that all simplicial complexes are finite and flag). An *L -cube-complex* is a cube complex whose vertex links are all isomorphic to L . For every finite flag simplicial complex L , the Davis complex $D(L)$ of the right-angled Coxeter group W_L associated to L is an example of a CAT(0) L -cube-complex (see Subsection 2.4 for more details).

A crucial difference between general CAT(0) L -cube-complexes and their 1-dimensional analogues — regular trees — is that they are not necessarily unique. This naturally raises the question of determining for a given finite flag simplicial

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complex L whether or not there is a unique CAT(0) L -cube-complex¹. This question can also be viewed as the cube complex analogue of a similar question for polygonal complexes that appeared in the survey paper of Farb, Hruska and Thomas [9]. Regular polygonal complexes have been studied in various works, proving uniqueness for certain links on the one hand, as in [3, 10, 14, 17, 19, 21] but also finding links for which there is a continuum of non-isomorphic complexes on the other, as in [1, 13, 17].

CAT(0) L -cube-complexes are known to be unique for some links L , including:

- Any collection of isolated vertices i.e. the link of a regular tree.
- The simplex Δ^d for all $d \in \mathbb{N}$ i.e. the link of the cube complex consisting of one $(d + 1)$ -dimensional cube.
- The cycle graph C_n for $n \geq 4$ i.e. the link of a regular square tiling of the Euclidean/hyperbolic plane.
- The complete bipartite graph $K_{n,m}$ i.e. the link of a product of two regular trees (see [21]).
- Any trivalent, 3-arc-transitive graph (see [19]).
- The odd graphs O_n (see [17]).

In fact, the question of uniqueness for CAT(0) L -square-complexes (and other polygonal complexes) — i.e. when L is a graph — was answered in a previous paper by the author (see [17]). Moreover, a fuller characterization of the graph condition given there together with more examples of such graphs can be found in the work of Giudici, Li, Seress and Thomas [10].

In this paper we show that the following combinatorial condition on L is necessary and sufficient for uniqueness of CAT(0) L -cube-complexes.

Definition 1.1. A simplicial complex L is *superstar-transitive* if for any two simplices σ, σ' and any isomorphism $\phi: \text{st}_L(\sigma) \rightarrow \text{st}_L(\sigma')$, sending σ to σ' , there exists an automorphism $\Phi: L \rightarrow L$ such that $\Phi|_{\text{st}_L(\sigma)} = \phi$.

The main theorem is thus the following.

Theorem 1.2 (Uniqueness of CAT(0) L -cube-complexes). *Let L be a finite flag simplicial complex. The associated Davis complex $D(L)$ is the unique CAT(0) L -cube-complex if and only if L is superstar-transitive.*

Except for the above examples, we provide in Subsection 5.1.1 a new family of examples of superstar-transitive flag simplicial complexes of arbitrary dimension, the Kneser complexes K_n^d . The Kneser complexes K_n^d generalize the first and last examples in the list above which correspond respectively to $d = 1$ and $d = 2$. The corresponding unique CAT(0) K_n^d -cube-complexes are not Gromov hyperbolic for $d \geq 3$ (see Remark 5.7) and not products.

¹Note that, as for trees, without the CAT(0) assumption one can build many L -cube-complexes for example by taking quotients of Davis complex $D(L)$ by torsion free subgroups of the associated Coxeter group W_L .

As in the case of regular trees, one might expect that these unique CAT(0) L -cube-complexes exhibit rich automorphism group actions. For instance, we prove that one can extend any automorphism of a hyperplane to an automorphism of the whole complex. In fact in Theorem 4.2 we show that the following stronger property holds for any collection of pairwise transverse hyperplanes in a unique CAT(0) L -cube-complex.

Definition 1.3. Let X be a CAT(0) cube complex. A set of pairwise transverse hyperplanes $\hat{h}_1, \dots, \hat{h}_d$ satisfies the *hyperplane automorphism extension property* (HAEP) if for all $\hat{f} \in \text{Aut}(\hat{h}_1 \cup \dots \cup \hat{h}_d)$ there exists an automorphism $f \in \text{Aut } X$ such that f stabilizes the set $\hat{h}_1 \cup \dots \cup \hat{h}_d$ and $f|_{\hat{h}_1 \cup \dots \cup \hat{h}_d} = \hat{f}$.

We then use it to show that certain unique CAT(0) L -cube-complexes have a virtually simple automorphism group. Note that this generalizes the well-known virtual simplicity of the automorphism group of a regular tree proved by Tits [20] and provides new examples of locally compact totally disconnected simple groups. Also note that this is not true in general. For instance, the automorphism group of the unique CAT(0) C_5 -square-complex is discrete and thus contains the Coxeter group W_{C_5} as a finite index subgroup, which in turn virtually maps onto a free group.

The outline of the paper is as follows:

In Section 2 we set the ground for the proof of the main theorem: we define the superstar-transitivity conditions; we introduce an inductive method on the vertices of a CAT(0) cube complex; and recall the definition of the Davis complexes for right-angled Coxeter groups.

In Section 3 we prove the main theorem.

In Section 4 we prove that the *hyperplane automorphism extension property* (HAEP) holds for the unique CAT(0) L -cube-complexes.

In Section 5 we show how the HAEP can be used to prove that certain automorphism groups are virtually simple. We then give some examples of unique CAT(0) L -cube-complexes which have a virtually simple automorphism group.

In the appendix we use the Rank Rigidity theorem of Caprace and Sageev [8] and results of Hamenstädt [16] to generalize previous results by Tits [20] and by Haglund and Paulin [15] about the simplicity of the subgroup of the automorphism group of a CAT(0) cube complex generated by all halfspace fixators. Haglund and Paulin used their result to prove simplicity of the automorphism group of various hyperbolic spaces with walls including the class of “even polyhedral complexes”, which include certain hyperbolic buildings and (k, L) -complexes. Our generalization will be applied to prove the virtual simplicity of the automorphism groups of Kneser complexes (see Corollary 5.6) which can be non-hyperbolic by Remark 5.7. A similar result was proved by Caprace in [5] for the type-preserving automorphism groups of right-angled buildings.

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2. Preliminaries

2.1. Superstar-transitivity and basic definitions. Let X be a CAT(0) cube complex, and let L be a finite flag simplicial complex. Recall the following definitions.

- For a vertex $x \in X^{(0)}$, the link of x in X , $\text{Lk}(x, X)$, is the simplicial complex whose vertices are the edges incident to x and whose simplices are the collections of edges which span cubes in X .
- For $L' \subset L$, the open star $\text{st}_L(L')$ of L' in L is the union of all open simplices of L whose closure intersects that of L' . In particular, the star of a simplex is the union of the stars of its vertices.²
- Let e be the directed edge in X which connects x to y , and let ξ and ζ be the vertices corresponding to e in $\text{Lk}(x, X)$ and $\text{Lk}(y, X)$ respectively. The transfer map along $e \in X^{(1)}$ is the isomorphism $\tau_e: \text{st}_{\text{Lk}(x, X)}(\xi) \rightarrow \text{st}_{\text{Lk}(y, X)}(\zeta)$ which sends a simplex incident to ξ in $\text{Lk}(x, X)$ to the simplex which represent the same cube in $\text{Lk}(y, X)$.

Definition 2.1. For $k \in \mathbb{N} \cup \{0\}$, the simplicial complex L is said to be $\text{st}(\Delta^k)$ -transitive if for any pair of (not necessarily distinct) k -simplices σ, σ' of L and any isomorphism $\phi: \text{st}(\sigma) \rightarrow \text{st}(\sigma')$ there exists an automorphism Φ of L such that $\Phi|_{\text{st}(\sigma)} = \phi$.

Note that L is superstar-transitive if and only if it is $\text{st}(\Delta^k)$ -transitive for all $k \geq 0$.

2.2. CAT(0) cube complexes terminology. For the definition of CAT(0) cube complexes we refer to Bridson–Haefliger [4]. We recall that CAT(0) cube complexes naturally carry a combinatorial structure coming from the construction of hyperplanes and halfspaces. For the definition and more details see [18].

The hyperplanes of a CAT(0) cube complex X have a natural CAT(0) cube complex structure, and their vertices can be naturally identified with midpoints of

²We follow Ballmann and Brin [1] in our choice of notation.

edges (or simply *mid-edges*) of X . Following the notation in [18], we denote by \hat{h} the bounding hyperplane of the halfspace h , and by h^* its complementary halfspace.

We will say that two points in X are *adjacent* (or *neighbors*) if they are contained in an edge of X . Since we will use this terminology also for mid-edges and vertices, we stress that the mid-edge and an endpoint of an edge are adjacent. We say that a hyperplane *separates* two points if every path between them in X intersects the hyperplane. Again, we stress that this includes the degenerate case that one of the points belongs to the hyperplane. Similarly, a hyperplane *separates* two sets if it does so pointwise. A hyperplane is *adjacent* to a vertex if they cannot be separated by another hyperplane. Or equivalently, if it has a point which is adjacent to the vertex.

2.3. Induction on the vertices of a CAT(0) cube complex. In this subsection we describe the properties of a certain enumeration of the vertices of X .

Definition 2.2. Let X be a CAT(0) cube complex. Let $\{x_n\}_{n < 0}$ be a (possibly empty) set of mid-edges in X . An enumeration $\{x_n\}_{n \geq 0}$ of the vertices of X is *admissible with respect to the mid-edges* $\{x_n\}_{n < 0}$ if the following hold for all $n \geq 0$.

- (1) The elements x_m with $m < n$ which are adjacent to x_n are contained in one cube.
- (2) For every $i < n$ such that there are cubes containing x_n and x_i that share a face C , there exists a neighbor x_m of x_n with $m < n$ that is contained in a cube that contains C .

Lemma 2.3. Let X be a CAT(0) cube complex, and let $\{x_n\}_{n < 0}$ be the set of the vertices of a (possibly empty) transverse collection of hyperplanes $\hat{h}_1, \dots, \hat{h}_d$ viewed as mid-edges in X . Let x_0 be a vertex which is adjacent to all $\hat{h}_1, \dots, \hat{h}_d$, and let $\{x_n\}_{n \geq 0}$ be an enumeration of the vertices of X with non-decreasing distance from x_0 with respect to the shortest path metric d on the 1-skeleton of X . Then, $\{x_n\}_{n \geq 0}$ is admissible with respect to $\{x_n\}_{n < 0}$.

Proof. Let $n \geq 0$. In order to prove Property (1) of admissibility, it is enough to prove that the corresponding hyperplanes that separate x_n from its preceding neighbors pairwise intersect. Let x_m and $x_{m'}$ be two neighbors of x_n with $m, m' < n$. Let \hat{h}, \hat{h}' be the two hyperplanes that separate x_n from $x_m, x_{m'}$ respectively. By the assumption that $d(x_n, x_0)$ is non-decreasing it follows that the hyperplane \hat{h} (resp. \hat{h}') either separates x_n from x_0 , or $m < 0$ (resp. $m' < 0$) in which case x_m (resp. $x_{m'}$) is a mid-edge in the hyperplane \hat{h} (resp. \hat{h}') which belongs to $\{\hat{h}_1, \dots, \hat{h}_d\}$. If they both belong to $\{\hat{h}_1, \dots, \hat{h}_d\}$ then they intersect by assumption. If both separate x_n from x_0 , then they intersect since they are also adjacent to x_n . Without loss of generality we are left with the case that $\hat{h} \in \{\hat{h}_1, \dots, \hat{h}_d\}$ and \hat{h}' separates x_n from x_0 . Since \hat{h} is adjacent to both x_0 and x_n , and \hat{h}' is adjacent to x_n and separates x_0 and x_n it follows that \hat{h}' and \hat{h} intersect, otherwise they separate each other from either x_0 or x_n .

To prove Property (2), let C be a face of cubes that contain x_i and x_n , for $i < n$. Let $\hat{e}_1, \dots, \hat{e}_r$ be the hyperplanes which are adjacent to x_n and either intersect C

or separate x_n and C . Let $\hat{\xi}_{r+1}, \dots, \hat{\xi}_s$ be the hyperplanes which are adjacent to x_n , separate x_n and x_i and are not in $\{\hat{\xi}_1, \dots, \hat{\xi}_r\}$. Since x_i and C are contained in a cube, the hyperplanes $\hat{\xi}_1, \dots, \hat{\xi}_s$ are pairwise transverse. Let us denote by D a cube that contains x_n and intersects all of $\hat{\xi}_1, \dots, \hat{\xi}_s$. By definition D contains C . Therefore it suffices to show that there exists a neighbor $x_m \in D$ of x_n with $m < n$. If there exists $1 \leq t \leq s$ such that $\hat{\xi}_t$ separates x_n and x_0 , then the unique vertex neighbor x_m which is separated by $\hat{\xi}_t$ from x_n has the desired property. Now, assume that all of $\hat{\xi}_1, \dots, \hat{\xi}_s$ do not separate x_n and x_0 . In particular $d(x_i, x_0) > d(x_n, x_0)$ and therefore x_i is a mid-edge (and $i < 0$). Let $\hat{h} \in \{\hat{h}_1, \dots, \hat{h}_d\}$ be the hyperplane that contains x_i . Since x_0 is adjacent to \hat{h} it follows that x_n and \hat{h} are adjacent, because any hyperplane that separates x_n from \hat{h} must also separate x_n from x_0 . The mid-edge $x_m \in \hat{h}$ which is adjacent to x_n has the desired property. \square

Remark 2.4. Let $\{x_n\}_{n \geq 0}$ be an admissible enumeration of the vertices of X with respect to the mid-edges $\{x_n\}_{n < 0}$, and let $X_{\leq n}$ be the subcomplex of all cubes that contain an element of $\{x_i\}_{i \leq n}$. By admissibility we see that for all $n \geq 0$, the cubes that contain x_n and intersect $X_{\leq n-1}$ belong to $X_{\leq n-1}$. Moreover, $\text{Lk}(x_n, X_{\leq n-1}) = \text{st}(\sigma)$ where σ is the simplex corresponding to the smallest cube that contains all the preceding neighbors of x_n .

2.4. Right-angled Coxeter groups and their Davis complex. We recall the construction of the Davis cube complex for the right-angled Coxeter group associated to L .

We first associate to L the right-angled Coxeter group W_L given by the following presentation:

$$W_L = \langle \xi \in L^{(0)} \mid \forall \xi \in L^{(0)}, \xi^2 = 1 \text{ and } \forall \xi \sim \zeta, [\xi, \zeta] = 1 \rangle$$

where $\xi \sim \zeta$ if the vertices ξ and ζ are adjacent in L (note that L is not the Coxeter diagram for W_L).

The *Davis complex* $D(L)$ associated to L is the cube complex obtained by adding cubes to the Cayley graph³ of W_L whenever a 1-skeleton of a cube appears.

Remark 2.5. The complex $D(L)$ is a CAT(0) L -cube-complex [12, pp.131–132], and the identification of $\text{Lk}(x, D(L))$ with L comes canonically from the labeling of the edges of the Cayley graph with the generators $L^{(0)}$. With respect to this identification, the transfer maps along edges of $D(L)$ are the identity maps.

Remark 2.6. Any automorphism Φ of L defines an automorphism of the group W_L . This automorphism extends to an automorphism $F_{1, \Phi}$ of $D(L)$ which fixes the vertex which correspond to $1 \in W_L$ and induces the map Φ on the link of every vertex (considered via the canonical identification with L). By conjugating this

³We assume that in the Cayley graph the bigons corresponding to the involution relation ξ^2 for $\xi \in L^{(0)}$ are identified to one edge.

automorphism with an element of $x \in W_L$ one can obtain an automorphism $F_{x,\Phi}$ which fixes the vertex that corresponds to x and induces the automorphism Φ on the links of $D(L)$.

Remark 2.7. The hyperplane in $D(L)$ transverse to an edge e labeled $\xi \in L^{(0)}$ is isomorphic to $D(\text{Lk}(\xi, L))$. This isomorphism is given by the $W_{\text{Lk}(\xi, L)}$ -equivariant embedding $W_{\text{Lk}(\xi, L)} \hookrightarrow \langle \xi \rangle \setminus W_L$. We note that all the edges which are transverse to a hyperplane have the same label in $L^{(0)}$. Thus the *label* of a hyperplane in $D(L)$ is well-defined.

3. Uniqueness of L -cube-complexes

In this section we prove Theorem 1.2. We begin by introducing the notion of admissible maps.

Definition 3.1. Let $\{x_n\}_{n \geq 0}$ be an admissible enumeration of the vertices of a CAT(0) cube complex X with respect to a set of mid-edges $\{x_n\}_{n < 0}$. Let Y be another CAT(0) cube complex. For $-1 \leq n \leq \infty$, let $X_{\leq n}$ be the subcomplex of X consisting of all the closed cubes that contain an element of $\{x_i\}_{i \leq n}$. An n -admissible map between X and Y is a combinatorial map $F_n: X_{\leq n} \rightarrow Y$ which is a local isomorphism at each x_i , $i \leq n$. That is, the map F_n induces an isomorphism $\text{Lk}(x_i, X) \rightarrow \text{Lk}(F_n(x_i), Y)$.

We will prove the following useful extension lemma.

Lemma 3.2. Let L be a superstar-transitive simplicial complex. Let X, Y be CAT(0) L -cube complexes, let $\{x_n\}_{n \geq 0}$ be an admissible enumeration of the vertices of X with respect to a set of mid-edges $\{x_n\}_{n < 0}$. Let $0 \leq n < \infty$ and let F_{n-1} be an $(n-1)$ -admissible map between X and Y . Then there exists an isomorphism of cube complexes $F_\infty: X \rightarrow Y$ extending F_{n-1} .

Proof. We first show that one can extend F_{n-1} to an n -admissible map F_n . If x_n does not have neighbors⁴ in $X_{\leq n-1}$ then by Property (2), F_{n-1} is not defined on any of the faces of cubes that contain x_n . Thus, one can extend F_{n-1} by mapping x_n to an arbitrary vertex $F(x_n) \in Y$ and the cubes around x_n by an arbitrary isomorphism $(F_n)_{x_n}: \text{Lk}(x_n, X) \rightarrow \text{Lk}(F(x_n), Y)$.

Now, assume that x_n has neighbors in $X_{\leq n-1}$. In order to define F_n on the cubes that contain x_n , we need to find an isomorphism $(F_n)_{x_n}: \text{Lk}(x_n, X) \rightarrow \text{Lk}(F(x_n), Y)$ that agrees with F_{n-1} on the cubes that intersect $X_{\leq n-1}$. By Remark 2.4, we see that $(F_n)_{x_n}$ is already defined on a subcomplex of $\text{Lk}(x_n, X)$ of the form $\text{st}(\sigma)$, where σ is the simplex corresponding to the smallest cube that contains all the neighbors x_m with $m < n$ of x_n . The map $(F_{n-1})_{x_n}$ defines an isomorphic embedding of $\text{st}(\sigma)$ into $\text{Lk}(F(x_n), X)$. By superstar-transitivity this can be extended to an isomorphism $\text{Lk}(x_n, X_{\leq n}) = \text{Lk}(x_n, X) \rightarrow \text{Lk}(F(x_n), Y)$.

⁴In fact, this case can only happen if $n = 0$ and the set of midedges is empty.

Using the above extension procedure inductively one can extend F_{n-1} to an ∞ -admissible map $F_\infty: X \rightarrow Y$. By admissibility, F_∞ is a local isomorphism. Since X and Y are connected and simply connected, F_∞ is an isomorphism. \square

We are now ready to prove Theorem 1.2.

Proof. We begin by proving that if L is superstar-transitive then there is a unique CAT(0) L -cube-complex. Let X and Y be two CAT(0) L -cube-complexes. Let $\{x_n\}_{n \geq 0}$ be an enumeration of $X^{(0)}$ as in Lemma 2.3. Starting from an empty (-1) -admissible map F_{-1} and applying Lemma 3.2, we obtain an isomorphism $F: X \rightarrow Y$ between the two cube complexes. This completes the proof of the first implication.

We shall now prove that if L is not superstar-transitive for all k then there exist more than one CAT(0) L -cube-complexes. Assume that k is the minimal non-negative integer such that L is not $\text{st}(\Delta^k)$ -transitive.

If $k = 0$, let $\xi, \zeta \in L^{(0)}$ and let $\phi: \text{st}(\xi) \rightarrow \text{st}(\zeta)$ be an isomorphism such that there is no automorphism of L extending ϕ . Let X be the following cube complex. Let \mathfrak{h}_ξ (resp. \mathfrak{h}_ζ) be a halfspace in $D(L)$ defined by a hyperplane labeled by ξ (resp. ζ). By Remark 2.7, the hyperplane $\hat{\mathfrak{h}}_\xi$ (resp. $\hat{\mathfrak{h}}_\zeta$) can be identified with $D(\text{Lk}(\xi, L))$ (resp. $D(\text{Lk}(\zeta, L))$) and thus the map ϕ defines an isomorphism $F_\phi: \hat{\mathfrak{h}}_\xi \rightarrow \hat{\mathfrak{h}}_\zeta$ by Remark 2.5. Form the space $X = \mathfrak{h}_\xi \sqcup_{F_\phi} \mathfrak{h}_\zeta$, see Figure 1. The space X is a CAT(0) L -cube-complex.

To see that $X \not\cong D(L)$ note that in $D(L)$ for each hyperplane $\hat{\mathfrak{h}}$ of $D(L)$ there is a reflection fixing this hyperplane and exchanging \mathfrak{h} and \mathfrak{h}^* , while in X the distinguished hyperplane $\hat{\mathfrak{h}}_\xi = \hat{\mathfrak{h}}_\zeta$ does not satisfy this property since such a reflection would imply that the induced maps on the links extend the transfer maps ϕ to an isomorphism of the links - contradicting the assumption on ϕ .

If $k \geq 1$, let $\phi: \text{st}(\sigma) \rightarrow \text{st}(\sigma')$ be an isomorphism between the stars of the k -simplices $\sigma, \sigma' \subset L$ that cannot be extended to an automorphism of L . Let $\sigma = [\xi_0, \dots, \xi_k]$ and let $\tilde{\sigma} = [\xi_1, \dots, \xi_k]$ be one of its codimension-1 faces. By the minimality of k , the complex L is $\text{st}(\Delta^{k-1})$ -transitive, and therefore there exists an automorphism $\tilde{\Phi}$ of L extending $(\phi|_{\text{st}(\tilde{\sigma})})^{-1}$ (which, in particular, sends σ' to σ). Hence, by post-composition with $\tilde{\Phi}$, we may assume that $\sigma = \sigma'$, i.e. $\phi: \text{st}(\sigma) \rightarrow \text{st}(\sigma)$, and moreover $\phi|_{\text{st}(\tilde{\sigma})} = \text{id}_{\text{st}(\tilde{\sigma})}$. We denote by $\phi_i := \phi|_{\text{st}(\xi_i)}$, which by our assumption is the identity map for all $i > 0$.

Let $\mathfrak{h}_0, \dots, \mathfrak{h}_k$ be a collection of halfspace in $D(L)$ whose bounding hyperplanes are pairwise transverse and labeled by ξ_0, \dots, ξ_k respectively. Let $\Sigma = \mathfrak{h}_0 \cap \dots \cap \mathfrak{h}_k$ be the sector they define. Each ϕ_i defines an automorphism F_i of $\hat{\mathfrak{h}}_i$. These automorphisms coincide on $\hat{\mathfrak{h}}_i \cap \hat{\mathfrak{h}}_j$ thus define an automorphism F of the boundary of this sector, i.e.

$$\partial\Sigma = \bigcup_{i=1}^k \left(\hat{\mathfrak{h}}_i \cap \left(\bigcap_{j \neq i} \mathfrak{h}_j \right) \right).$$

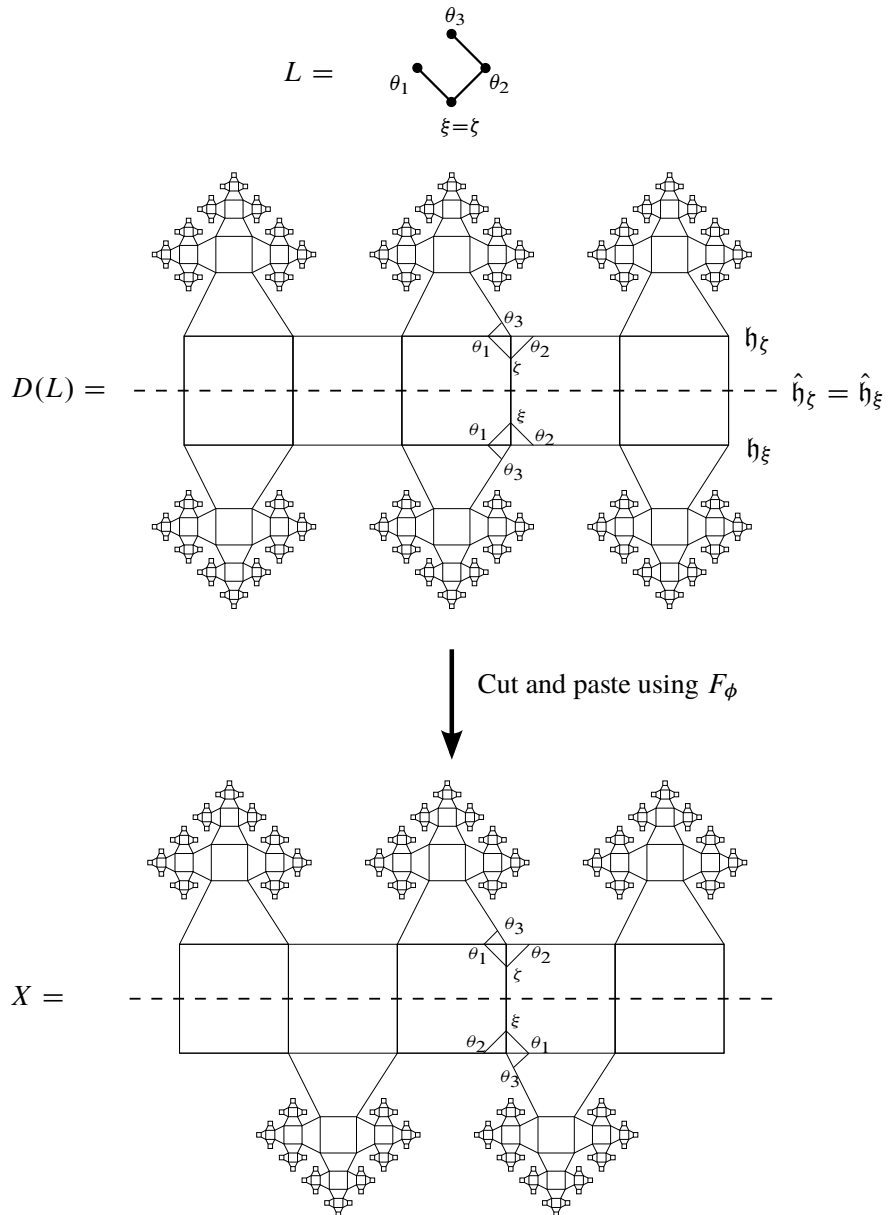


Figure 1. An example of the complex X for a non-st(Δ^0)-transitive link L . Let L be the 3-edge path graph. Let ξ and ζ be the same vertex, shown in the figure, and let $\phi: st(\xi) \rightarrow st(\zeta)$ be the non-extendible isomorphism which exchanges θ_1 and θ_2 . Since $\xi = \zeta$, the hyperplanes \hat{h}_ξ and \hat{h}_ζ may be chosen the same, and are isomorphic to the real line (thought of as the Davis complex, $D(Lk(\xi, L))$, of the infinite dihedral group generated by the reflections θ_1 and θ_2). The induced map $F_\phi: \hat{h}_\xi \rightarrow \hat{h}_\zeta$ is the reflection around some vertex of \hat{h}_ξ . The space X in the figure is obtained by gluing the halfspaces h_ξ and h_ζ using F_ϕ .

Let X be the space obtained by gluing Σ to $D(L) \setminus \Sigma^\circ$ along Σ with respect to F . The space X is a CAT(0) L -cube-complex.

Assume for contradiction that $X \cong D(L)$. Consider the $k + 1$ commuting reflections, r_0, \dots, r_k , with respect to the hyperplanes $\hat{h}_0, \dots, \hat{h}_k$. Let C be a cube that intersects $\bigcap_i \hat{h}_i$ and corresponds to σ . Let C be identified with $[0, 1]^{k+1}$ such that $v_0 := (0, \dots, 0)$ is the vertex in the sector Σ , and its adjacent vertices $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ correspond to the vertices ξ_0, \dots, ξ_k in the link of v_0 .

The reflections r_i determine automorphisms $r_{i,v}: \text{Lk}(v, X) \rightarrow \text{Lk}(r_i(v), X)$ for $i = 0, \dots, k$ and $v \in \{0, 1\}^{\{0, \dots, k\}}$. These reflections have the following properties:

- (1) The maps $r_{i,v}$ fix σ (after identifying each link with L using the natural identification).
- (2) The restriction $r_{i,v}|_{st(\xi_i)} = \tau_{e_{i,v}}$ where $e_{i,v}$ is the edge connecting v and $r_i(v)$ and $\tau_{e_{i,v}}$ is the transfer map along $e_{i,v}$. Note also that by the construction all the transfer maps $\tau_{e_{i,v}}$ are the identity maps except for $\tau_{e_{0,v_0}} = \phi_0$ (and $\tau_{e_{0,r_0(v_0)}} = \phi_0^{-1}$).
- (3) For all $i, j \in \{0, \dots, k\}, i \neq j$ and $v \in \{0, 1\}^{\{0, \dots, k\}}$

$$r_{j,r_i(v)} \circ r_{i,v} = r_{i,r_j(v)} \circ r_{j,v}.$$

Let us denote $[n] := \{1, \dots, n\}$ for all $n \in \mathbb{N}$. Let $v \mapsto \tilde{v}$ denote the embedding

$$\{0, 1\}^{[k]} \hookrightarrow \{0\} \times \{0, 1\}^{[k]} \subset \{0, 1\}^{\{0\} \cup [k]},$$

and let $R_v := r_{0,\tilde{v}}$ for $v \in \{0, 1\}^{[k]}$.

For an injective map $\pi: [m] \rightarrow [k]$ and $v \in \{0, 1\}^{[k]}$ we define the automorphism Φ_π^v of L by induction on m in the following way:

- If $m = 1$, $\Phi_\pi^v = R_{r_{\pi(m)}v}^{-1} \circ R_v$.
- If $m > 1$, $\Phi_\pi^v = \Phi_{\pi|_{[m-1]}}^{r_{\pi(m-1)}r_{\pi(m)}v} \circ \Phi_{\pi|_{[m-1]}}^v$

Consider the automorphism defined by $m = k, \pi = \text{id}_{[k]}, v_0 = (0, \dots, 0) \in \{0, 1\}^{[k]}$

$$\Phi := \Phi_{\text{id}}^{v_0}.$$

We complete the proof by contradicting the assumption that ϕ is not extendible, using the following claim.

Claim. The restriction $\Phi|_{st(\sigma)}$ is ϕ .

Proof. After expanding Φ using the inductive definition we get

$$\Phi = R_{v_{2^k-1}}^{-1} \circ \dots \circ R_{v_1}^{-1} \circ R_{v_0}$$

where $\{v_i\}_{i=0}^{2^k-1} = \{0, 1\}^{[k]}$ (in fact, the sequence $\{v_i\}_{i=0}^{2^k-1}$ form a Hamiltonian cycle of the 1-skeleton of the cube $[0, 1]^{[k]}$). Thus, using the second property of the maps $r_{i,v}$ and the construction of X , we get

$$\Phi|_{\text{st}(\xi_0)} = R_{v_{2^k-1}}^{-1}|_{\text{st}(\xi_0)} \circ \cdots \circ R_{v_1}^{-1}|_{\text{st}(\xi_0)} \circ R_{v_0}|_{\text{st}(\xi_0)} = \text{id}_{\text{st}(\xi_0)} \circ \cdots \circ \text{id}_{\text{st}(\xi_0)} \circ \phi_0 = \phi_0.$$

We are left to prove that $\Phi|_{\text{st}(\xi_i)} = \phi_i = \text{id}_{\text{st}(\xi_i)}$ for all $i \in [k]$. We do so by proving by induction on m that for all injective maps $\pi: [m] \rightarrow [k]$, for all $v \in \{0, 1\}^{[k]}$ and for all $i \in \pi([m])$ we have $\Phi_\pi^v|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}$. In particular, we get $\Phi|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}$ for all $i \in [k]$.

For the base case, $m = 1$, let $i = \pi(1)$. Property (3) provides the following relation

$$R_{r_i(v)} \circ r_{i,\tilde{v}} = r_{0,r_i(v)} \circ r_{i,\tilde{v}} = r_{i,r_0(\tilde{v})} \circ r_{0,\tilde{v}} = r_{i,r_0(\tilde{v})} \circ R_v.$$

When restricted to $\text{st}(\xi_i)$ we obtain

$$R_{r_i(v)}|_{\text{st}(\xi_i)} = R_{r_i(v)}|_{\text{st}(\xi_i)} \circ r_{i,\tilde{v}}|_{\text{st}(\xi_i)} = r_{i,r_0(\tilde{v})}|_{\text{st}(\xi_i)} \circ R_v|_{\text{st}(\xi_i)} = R_v|_{\text{st}(\xi_i)}$$

since $r_{i,\tilde{v}}|_{\text{st}(\xi_i)} = r_{i,r_0(\tilde{v})}|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}$ by Property (2). Thus,

$$\Phi_\pi^v|_{\text{st}(\xi_i)} = R_{r_i(v)}^{-1}|_{\text{st}(\xi_i)} \circ R_v|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}.$$

Now assume $m > 1$. We divide the proof of the inductive step into 3 cases:

Case 1. If $i \in \pi([m-1])$, then by the induction hypothesis

$$\Phi_\pi^v|_{\text{st}(\xi_i)} = \Phi_{\pi|_{[m-1]}}^{r_{\pi(m-1)}r_{\pi(m)}v}|_{\text{st}(\xi_i)} \circ \Phi_{\pi|_{[m-1]}}^v|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}.$$

Case 2. If $i = \pi(m)$ and $m > 2$,

$$\begin{aligned} \Phi_\pi^v &= \Phi_{\pi|_{[m-1]}}^{r_{\pi(m-1)}r_{\pi(m)}v} \circ \Phi_{\pi|_{[m-1]}}^v \\ &= \left(\Phi_{\pi|_{[m-2]}}^{r_{\pi(m-2)}r_{\pi(m)}v} \circ \Phi_{\pi|_{[m-2]}}^{r_{\pi(m-1)}r_{\pi(m)}v} \right) \circ \left(\Phi_{\pi|_{[m-2]}}^{r_{\pi(m-2)}r_{\pi(m-1)}v} \circ \Phi_{\pi|_{[m-2]}}^v \right). \end{aligned}$$

If we denote $J := \Phi_{\pi|_{[m-2]}}^{r_{\pi(m-2)}r_{\pi(m)}v}$, then

$$\Phi_\pi^v = J \left(\Phi_{\pi|_{[m-2]}}^{r_{\pi(m-1)}r_{\pi(m)}v} \circ \Phi_{\pi|_{[m-2]}}^{r_{\pi(m-2)}r_{\pi(m-1)}v} \right) \circ \left(\Phi_{\pi|_{[m-2]}}^v \circ \Phi_{\pi|_{[m-2]}}^{r_{\pi(m-2)}r_{\pi(m)}v} \right) J^{-1}.$$

Let $\pi': [m] \rightarrow [k]$ be the injective map defined by

$$\pi'(j) = \begin{cases} \pi(j) & j \leq m-2, \\ \pi(m) & j = m-1, \\ \pi(m-1) & j = m. \end{cases}$$

Then,

$$\begin{aligned}\Phi_\pi^v &= J \left(\Phi_{\pi'|_{[m-1]}}^{r_{\pi(m-2)}r_{\pi(m-1)}v} \Phi_{\pi'|_{[m-1]}}^{r_{\pi(m-2)}r_{\pi(m)}v} \right) J^{-1} \\ &= J \left(\Phi_{\pi'}^{r_{\pi(m-2)}r_{\pi(m)}v} \right) J^{-1}.\end{aligned}$$

Since now $i = \pi(m) \in \pi'([m-1])$, we can deduce from the previous case that

$$\Phi_{\pi'}^{r_{\pi(m-2)}r_{\pi(m)}v} \Big|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}$$

and since Φ_π^v and $\Phi_{\pi'}^{r_{\pi(m-2)}r_{\pi(m)}v}$ are conjugates we get $\Phi_\pi^v \Big|_{\text{st}(\xi_i)} = \text{id}_{\text{st}(\xi_i)}$.

Case 3. If $i = \pi(m)$ and $m = 2$,

$$\begin{aligned}\Phi_\pi^v &= \Phi_{\pi|_{[m-1]}}^{r_{\pi(m-1)}r_{\pi(m)}v} \circ \Phi_{\pi|_{[m-1]}}^v \\ &= \left(R_{r_{\pi(m)}v}^{-1} \circ R_{r_{\pi(m-1)}r_{\pi(m)}v} \right) \circ \left(R_{r_{\pi(m-1)}v}^{-1} \circ R_v \right) \\ &= J \left(\left(R_{r_{\pi(m-1)}r_{\pi(m)}v} \circ R_{r_{\pi(m-1)}v}^{-1} \right) \circ \left(R_v \circ R_{r_{\pi(m)}v}^{-1} \right) \right) J^{-1}.\end{aligned}$$

When $J = R_{r_{\pi(m)}v}^{-1}$, the proof proceeds similarly to the proof of Case 2 and the proof for $m = 1$. \square

4. Hyperplane automorphism extension property

Recall from the introduction the following.

Definition 4.1. Let X be a CAT(0) cube complex. A set of pairwise transverse hyperplanes $\hat{h}_1, \dots, \hat{h}_d$ satisfies the *hyperplane automorphism extension property* (HAEP) if for all $\hat{f} \in \text{Aut}(\hat{h}_1 \cup \dots \cup \hat{h}_d)$ there exists an automorphism $f \in \text{Aut} X$ such that f stabilizes $\hat{h}_1 \cup \dots \cup \hat{h}_d$ and $f|_{\hat{h}_1 \cup \dots \cup \hat{h}_d} = \hat{f}$.

Theorem 4.2. Let L be a superstar-transitive simplicial complex, and let X be the unique CAT(0) L -cube-complex. Every transverse set of hyperplanes $\hat{h}_1, \dots, \hat{h}_d$ in X satisfies the HAEP.

Proof. Let $\hat{f} \in \text{Aut}(\hat{h}_1 \cup \dots \cup \hat{h}_d)$. Let $\{x_n\}_{n < 0}$ be the set of the vertices of the hyperplanes $\hat{h}_1, \dots, \hat{h}_d$ viewed as mid-edges in X . Let $\{x_n\}_{n \geq 0}$ be an admissible enumeration of the vertices of X with respect to $\{x_n\}_{n < 0}$, as obtained by Lemma 2.3. The automorphism $\hat{f} \in \text{Aut}(\hat{h}_1 \cup \dots \cup \hat{h}_d)$ defines a (-1) -admissible map F_{-1} . Applying Lemma 3.2 to F_{-1} we get an isomorphism $F: X \rightarrow Y$ that extends \hat{f} . \square

Recall from [15] that for a group G of automorphisms of a CAT(0) cube complex X we denote by G^+ the subgroup of G generated by all the elements that fix some halfspace of X , i.e. $G^+ = \langle \text{Fix}_G(\mathfrak{h}) \mid \mathfrak{h} \text{ a halfspace of } X \rangle$.

Lemma 4.3. *Let \hat{h}, \hat{k} be a transverse pair of hyperplanes which satisfies the HAEP, then each automorphism f of \hat{h} which fixes the halfspace $\mathfrak{k} \cap \hat{h}$ of \hat{h} can be extended to an automorphism F of X which fixes \mathfrak{k} .*

In particular, if the hyperplane \hat{h} satisfies the HAEP for any transverse hyperplane \hat{k} then any element of $\text{Aut}^+ \hat{h}$ can be extended to an element of $\text{Aut}^+ X$.

Proof. Let $\hat{f} \in \text{Aut}(\hat{h} \cup \hat{k})$ be the automorphism defined by $\hat{f}|_{\hat{h}} = f, \hat{f}|_{\hat{k}} = \text{id}_{\hat{k}}$. By the HAEP, we can extend \hat{f} to an automorphism, F' , of X . Finally, define F to be the automorphism defined by $F|_{\mathfrak{k}^*} = F'|_{\mathfrak{k}^*}, F|_{\mathfrak{k}} = \text{id}_{\mathfrak{k}}$. \square

5. Virtual simplicity of automorphism groups

Lemma 5.1. *Let G act transitively on a set S , and let H be a subgroup of G . If S/H is finite and for some $x \in S, H_x = \text{Stab}_H(x)$ has finite index in $G_x = \text{Stab}_G(x)$, then H has finite index in G .*

Proof. It follows from the following inequality $|G/H| \leq |S/H| \cdot |G_x/H_x| < \infty$. \square

Proposition 5.2. *Let X be a proper finite-dimensional CAT(0) cube complex. Let $G = \text{Aut } X$. Assume the following properties hold:*

- (1) *There exists a hyperplane orbit $G\hat{h}$ such that G^+ has finitely many orbits in $G\hat{h}$.*
- (2) *The group $\text{Aut}^+ \hat{h}$ has finite index in $\text{Aut} \hat{h}$.*
- (3) *The hyperplane \hat{h} satisfies the HAEP.*
- (4) *For every hyperplane \hat{k} transverse to \hat{h} the pair $\hat{h} \cup \hat{k}$ satisfies the HAEP.*

Then, G^+ has finite index in G .

Proof. Let $S = G\hat{h}$ and let $H = G^+$. The proposition will follow from the previous lemma once we show that $G_{\hat{h}}^+ = \text{Stab}_{G^+}(\hat{h})$ has finite index in $G_{\hat{h}} = \text{Stab}_G(\hat{h})$.

By Condition (3), we have that the restriction $G_{\hat{h}}|_{\hat{h}}$ is exactly $\text{Aut} \hat{h}$. Similarly, by Condition (4) and Lemma 4.3 we deduce that $G_{\hat{h}}^+|_{\hat{h}}$ is exactly $\text{Aut}^+ \hat{h}$. Thus, by Condition (2), we get $[G_{\hat{h}} : G_{\hat{h}}^+] = [\text{Aut} \hat{h} : \text{Aut}^+ \hat{h}] < \infty$. \square

5.1. Examples of virtually simple automorphism groups.

5.1.1. The Kneser complex and the associated Davis complex. We define the Kneser complex, $K(n, S)$, to be the simplicial complex whose vertices are all subsets of size n of a finite set S with at least $2n$ elements, and a collection of vertices span a simplex if they represent pairwise disjoint subsets.

Let $n, d \in \mathbb{N}$, and let $S = \{1, \dots, nd + 1\}$. The complex $L := K_n^d = K(n, S)$ is a $(d - 1)$ -dimensional flag simplicial complex. Since $|S| \neq 2n$, the Erdős–Kő–Rado theorem tells us that any automorphism of L is induced from a permutation of S (see Corollary 7.8.2 in [11]).

Proposition 5.3. *The complex L is superstar-transitive.*

Proof. Let k be a non-negative integer, let σ, σ' be two k -simplices of L and let $\phi: \text{st}(\sigma) \rightarrow \text{st}(\sigma')$ be an isomorphism. Since L is clearly Δ^k -transitive (i.e. any isomorphism from one simplex to another can be extended to an automorphism of L) we may assume without loss of generality that $\sigma = \sigma' = [v_0, \dots, v_k]$ and ϕ fixes σ .

The map ϕ induces automorphisms on the links $\phi_i: \text{Lk}(v_i, L) \rightarrow \text{Lk}(v_i, L)$ for $i = 0, \dots, k$. For all $1 \leq i \leq k$, the link $\text{Lk}(v_i, L)$ is naturally identified with $K(n, S \setminus v_i)$ and thus ϕ_i is given by a permutation $\pi_i \in \text{Sym}(S \setminus v_i)$.

If $k = 0$, let π be the permutation which fixes the set v_i and restricts to π_i on $S \setminus v_i$.

If $k > 0$, the maps ϕ_i and ϕ_j coincide along $\text{Lk}(e_{i,j}, L)$ where $e_{i,j}$ is the edge connecting v_i and v_j . As before $\text{Lk}(e_{i,j}, L)$ is naturally identified with $K(n, S \setminus (v_i \cup v_j))$ and thus $\pi_i|_{S \setminus (v_i \cup v_j)} = \pi_j|_{S \setminus (v_i \cup v_j)}$. Hence, the maps π_i define a unique permutation $\pi \in \text{Sym}(S)$ whose restriction to $S \setminus v_i$ is π_i for all $0 \leq i \leq k$.

In both cases the permutation π defines an automorphism Φ of L whose restriction to $\text{st}(\sigma)$ is ϕ . \square

Lemma 5.4. *If $n \geq 2$ then the group $\text{Aut } L$ is generated by all elements which fix the star of a vertex in L .*

Proof. Since $\text{Aut } L = \text{Sym}(S)$ and since $\text{Sym}(S)$ is generated by transpositions, it suffices to show that all transpositions fix a star of a vertex. But this is true since the transposition exchanging $a, b \in S$ fixes the star of a vertex which contains a and b . \square

Lemma 5.5. *The group $\text{Aut}^+ D(L)$ acts transitively on the hyperplanes of $D(L)$.*

Proof. Recall from Remark 2.6 that for every vertex $x \in D(L)$ and for every $\Phi \in \text{Aut } L$ there is an automorphism $F_{x, \Phi}$ of $D(L)$ that fixes x and induces the automorphism Φ on the links of $D(L)$. Moreover, if Φ fixes the star of the vertex ξ then $F_{x, \Phi}$ fixes the carrier of the hyperplane labeled by ξ adjacent to x , and thus $F_{x, \Phi}$

is in $\text{Aut}^+ D(L)$ (for example, by Remark A.2). Now, since $\text{Aut} L$ is generated by all the elements that fix some star, it follows that for all $\Phi \in \text{Aut} L$ and vertex $x \in D(L)$, $F_{x,\Phi}$ is in $\text{Aut}^+ D(L)$. Since $\text{Aut} L$ acts transitively on the vertices of L , its image in $\text{Stab}_{\text{Aut}^+(X)}(x)$ under the isomorphism $\Phi \mapsto F_{x,\Phi}$ acts transitively on the hyperplanes adjacent to x . This holds for all $x \in D(L)$, and any two adjacent vertices are contained in the carrier of a hyperplane, and therefore $\text{Aut}^+ D(L)$ acts transitively on hyperplanes. \square

Corollary 5.6. *Let $n \geq 2, d \geq 1$ and let $L = K(n, \{1, \dots, nd + 1\})$. The group $G = \text{Aut} D(L)$ is virtually simple.*

Proof. We begin by showing that G^+ is simple by verifying the assumptions of Corollary A.4 and Claim A.8.

For all finite flag simplicial L the right-angled Davis complexes $D(L)$ is proper, finite dimensional and cocompact (since the Coxeter group W_L acts cocompactly on $D(L)$). The complex $D(L)$ is essential since L is not the star of any of its vertices. Similarly every sector $\mathfrak{h} \cap \mathfrak{k}$ in $D(L)$ contains a hyperplane because the vertices of L are not contained in the star of any of its edges (thus, in the link of a vertex in $\mathfrak{h} \cap \mathfrak{k}$ which is adjacent to both $\hat{\mathfrak{h}}, \hat{\mathfrak{k}}$ there is at least one hyperplane which does not intersect $\hat{\mathfrak{h}}$ nor $\hat{\mathfrak{k}}$). The complex $D(L)$ is irreducible since L is not a join of two subcomplexes, because the complement graph (the graph of non-empty intersections of subsets of size $n \geq 2$ in $\{1, \dots, nd + 1\}$) is connected. The group G is non-elementary because $\partial D(L) = \Lambda G$ contains more than 2 points (because L contains an independent set of vertices of size 3) and G acts without a fixed point at infinity (the Coxeter group acts with inversions along hyperplanes, thus does not fix a point in $\partial D(L)$).

In order to prove the corollary it suffices to show that G^+ is of finite index in G . We prove it by induction on $d \geq 0$. The base case $d = 0$, is trivial since the complex $D(L)$ is a single vertex, thus G is trivial.

The conditions of Proposition 5.2 hold: Condition (1) by Lemma 5.5, Condition (2) by the induction hypothesis (the hyperplane are isomorphic to the Davis complex associated to $K(n, \{1, \dots, n(d-1) + 1\})$), and Conditions (3) and (4) by Theorem 4.2 and Proposition 5.3. \square

Remark 5.7. We note that for $n \geq 2, d \geq 3$ the Kneser complex K_n^d has embedded full 4-cycles. For example, the vertices corresponding to the subsets $\{1, \dots, n\}$, $\{n+1, \dots, 2n\}$, $\{1, \dots, n-1, 2n+1\}$, $\{n+1, \dots, 2n-1, 2n+2\}$ form an embedded full 4-cycle. This implies that the corresponding Davis complex is not hyperbolic, because the Coxeter group $W_{K_n^d}$ contains a direct product of two infinite dihedral groups.

5.1.2. Superstar-transitive graphs and unique square complexes.

Lemma 5.8. *Let L be a finite, connected, flag, simplicial, superstar-transitive graph. If all the vertices in L have degree ≥ 3 , then the subgroup $\text{Aut}^+(L)$ of $\text{Aut}(L)$ generated by the automorphisms which fix the star of a vertex has at most two vertex orbits. Moreover, these orbits form a partition of the graph.*

Proof. If v_1, v_2, w are adjacent to v then, by $\text{st}(\Delta^1)$ -transitivity, there exists an automorphism exchanging v_1, v_2 and fixing the star of w . Thus all the adjacent vertices of v are in the same orbit of $\text{Aut}^+(L)$. This holds for all v , and by connectivity we get the desired conclusion. \square

Lemma 5.9. *The group $\text{Aut}^+ D(L)$ has at most two orbits of hyperplanes.*

Proof. As in the proof of Lemma 5.5, for all $x \in D(L)$, $\text{Aut}^+ D(L)$ has at most two orbits of hyperplanes whose carrier contains x . Using the fact that any two adjacent vertices are in the carrier of two transverse hyperplanes we deduce that $\text{Aut}^+ D(L)$ has at most two orbits of hyperplanes in $D(L)$. \square

This lemma allows us to deduce, as in the proof of Corollary 5.6 the following.

Theorem 5.10. *Let L be a finite, connected, flag, simplicial, superstar-transitive graph all of whose vertices have degree ≥ 3 and which is not a complete bi-partite graph. Let $X = D(L)$ be the unique CAT(0) square complex whose vertex links are isomorphic to L . Then $\text{Aut}(X)$ is virtually simple.*

A. Simplicity of automorphism groups of rank one cube complexes

Let T be a tree, and let $\text{Aut}(T)$ be the automorphism group of T . In [20], J. Tits showed that under certain conditions the subgroup $\text{Aut}^+(T)$ of $\text{Aut}(T)$ generated by the fixators of halfspaces of T is simple. In [15], F. Paulin and F. Haglund generalized Tits' result for Gromov hyperbolic cube complexes. In this appendix, we further generalize these results to rank one CAT(0) cube complexes. A similar result for (not necessarily locally finite) thick right-angled buildings was established by P.-E. Caprace in [5].

Let X be a CAT(0) metric space. A rank one isometry is a hyperbolic isometry $g \in \text{Isom}(X)$ none of whose axes bounds a flat halfplane (i.e. a subspace which is isometric to the Euclidean half plane). Any hyperbolic element in a Gromov hyperbolic cube complex is such since there are no flat halfspaces in a hyperbolic cube complex. In general, rank one elements in locally compact CAT(0) cube complexes act on the boundary ∂X with a north-south dynamics similarly to the action of hyperbolic elements in hyperbolic spaces. For $G \subset \text{Isom}(X)$ we denote by $\Lambda(G)$ the *limit set* of G in ∂X , i.e. the set of accumulation points in ∂X of an orbit of G . The group G is called *elementary* if either $|\Lambda(G)| \leq 2$ or G fixes a point at

infinity. Let X be a proper CAT(0) space, and let $G \leq \text{Isom}(X)$ be a non-elementary subgroup which contains a rank one element, then the set of pairs of fixed points in ∂X of rank one elements is dense in the complement of the diagonal Δ of $\Lambda(G) \times \Lambda(G)$ (see [16]).

Let X be a CAT(0) cube complex. Let \mathcal{H} be the set of all halfspaces of X , and let $\hat{\mathcal{H}}$ be the set of corresponding hyperplanes. We denote by $\hat{\cdot}: \mathcal{H} \rightarrow \hat{\mathcal{H}}$ the natural map mapping each halfspace to its bounding hyperplane, and by $*$: $\mathcal{H} \rightarrow \mathcal{H}$ the map sending a halfspace to its complementary halfspace. Recall the following definitions from [8]. We say that X is *irreducible* if it cannot be expressed as a (non-trivial) product. We say that X is *essential* if every halfspace $h \in \mathcal{H}$ contains points arbitrarily far from \hat{h} . Let $G \leq \text{Aut}(X)$ a group of automorphisms of X . We say that X is *G-essential* if every halfspace of X contains G -orbit points arbitrarily far from its bounding hyperplane. In [8], Caprace and Sageev proved the following rank rigidity result.

Theorem (Rank Rigidity for CAT(0) cube complexes [8]). *If X is a finite-dimensional irreducible CAT(0) cube complex, and $G \leq \text{Aut}(X)$ acts essentially on X without fixed points at infinity, then G contains a rank one isometry.*

Definition A.1. Let X and G be as above. We denote by G^+ the subgroup of G generated by the fixators of halfspaces of X , i.e.

$$G^+ = \langle g \in \text{Fix}_G(h) \mid h \in \mathcal{H}(X) \rangle$$

We recall from [15, 20] that the action of G on X satisfies *property (P)* if for every nested sequence of halfspaces $(h_n)_{n \in \mathbb{Z}} \subset \mathcal{H}$, $h_{n+1} \subset h_n$, the following map is an isomorphism:

$$\text{Fix}_G \left(\bigcup_n \hat{h}_n \right) \rightarrow \prod_n \text{Fix}_G(\hat{h}_n \cup \hat{h}_{n+1})|_{h_n \cap h_{n+1}^*} \tag{A.1}$$

Where $\text{Fix}_G(\hat{h}_n \cup \hat{h}_{n+1})|_{h_n \cap h_{n+1}^*}$ is the image of $\text{Fix}_G(\hat{h}_n \cup \hat{h}_{n+1})$ in the group $\text{Aut}(h_n \cap h_{n+1}^*)$ under the restriction map.

Remark A.2. Note that $G = \text{Aut}(X)$ satisfies property (P), and in this case G^+ is generated by the fixators of the carriers of hyperplanes.

We prove the following.

Theorem A.3. *Let X be a proper finite-dimensional irreducible CAT(0) cube complex, and let $G \leq \text{Aut}(X)$ be a non-elementary group acting essentially on X with property (P) and $\Lambda(G) = \partial X$. Then for all $N \triangleleft G^+$ either $N = G^+$ or N acts trivially on ∂X . In particular, if we further assume that G acts faithfully on ∂X , then G^+ is either simple or trivial.*

By the remark above, we deduce:

Corollary A.4. *Let X be a proper finite-dimensional irreducible essential CAT(0) cube complex with co-compact, non-elementary $\text{Aut}(X)$ action. Then for all $N \triangleleft \text{Aut}^+(X)$ either $N = \text{Aut}^+(X)$ or N acts trivially on ∂X . In particular, if we further assume that $\text{Aut}(X)$ acts faithfully on ∂X , then $\text{Aut}^+(X)$ is either simple or trivial.*

We remark that under mild assumptions the action of $\text{Aut}(X)$ on ∂X is faithful. For example the following proposition follows from works of Caprace and Monod, and gives a criterion for having a faithful action of the isometry group of a CAT(0) space. Recall that a CAT(0) space is *geodesically complete* (or *has extendable geodesics*) if every geodesic segment can be extended to a bi-infinite geodesic.

Proposition A.5 ([6, Proposition 1.5], [7, Lemma 2.18]). *Let X be a geodesically complete proper CAT(0) space with trivial Euclidean de Rahm factor and with a cocompact isometry group, then $\text{Isom}(X)$ acts faithfully on ∂X .*

In the applications of Theorem A.3 in Section 5 the above proposition is sufficient for deducing the faithfulness of the action, since all the CAT(0) cube complexes which we consider there are geodesically complete. However, we will use the criterion for faithful action on the boundary of a CAT(0) cube complexes stated in Claim A.8 since it is easier to verify. For the proof of this criterion and of Theorem A.3 we will need the following.

Definition A.6. For $\mathfrak{h} \in \mathcal{H}(X)$ let

$$\mathfrak{h}_\infty := \{\xi \in \partial X \mid r \cap \mathfrak{h} \neq \emptyset, \forall \text{geodesic rays } r \text{ such that } r(\infty) = \xi\}.$$

It can also be defined as the collection of points in ∂X that are not accumulation of \mathfrak{h}^* . And for $\hat{\mathfrak{h}} \in \mathcal{H}(X)$ let $\hat{\mathfrak{h}}_\infty$ be the set of accumulation points of $\hat{\mathfrak{h}}$ in ∂X .

Recall from [2] that two hyperplanes $\hat{\mathfrak{h}}, \hat{\mathfrak{k}}$ are *strongly separated* if there is no hyperplane which intersects both of them.

Proposition A.7 ([8, Proposition 5.1]). *Under the same assumptions as in Theorem A.3, for all $\mathfrak{h} \in \mathcal{H}$ there exists a halfspace $\mathfrak{k} \subset \mathfrak{h}$ such that $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{k}}$ are strongly separated. In particular, the open set \mathfrak{h}_∞ is non-empty.*

This enables us to study the action on the boundary by studying the action on halfspaces, as in the following claim.

Claim A.8. Under the same assumptions as in Theorem A.3, assume moreover that the intersection of any pair of crossing halfspaces $\mathfrak{h}_1, \mathfrak{h}_2$ contains a halfspace $\mathfrak{k} \subset \mathfrak{h}_1 \cap \mathfrak{h}_2$. Then $\text{Aut}(X)$ acts faithfully on ∂X .

Proof. Let $1 \neq g \in \text{Aut}(X)$, and let $\mathfrak{h} \in \mathcal{H}(X)$ be a halfspace such that $g\mathfrak{h} \neq \mathfrak{h}$. Then g and \mathfrak{h} satisfy one of the following cases:

Case 1. $g\mathfrak{h} \subset \mathfrak{h}^*$. Then g sends the corresponding \mathfrak{h}_∞ into \mathfrak{h}_∞^* which are non-empty and disjoint.

Case 2. $g\mathfrak{h} \subset \mathfrak{h}$. Let $\hat{l} \in \hat{\mathcal{H}}(X)$ be a hyperplane transversal to \hat{h} , (we may assume without loss of generality that $\forall n \in \mathbb{N} \ g^n\mathfrak{h} \not\subset \hat{l}$, by determining the orientation of \hat{l} so that $g^m\mathfrak{h} \not\subset \hat{l}$ for the minimal m such that $g^n\hat{h} \cap \hat{l} = \emptyset$). By our assumption, let $\mathfrak{k} \subset \hat{l} \cap \mathfrak{h}$. Then either $\forall n \in \mathbb{N}, g^n\hat{h} \subset \mathfrak{k}^*$ or $\exists n \in \mathbb{N}, g^n\hat{h} \cap \mathfrak{k} \neq \emptyset$. If $\forall n, g^n\hat{h} \subset \mathfrak{k}^*$ then there exists r such that $\mathfrak{k} \subset g^{r-1}\mathfrak{h} \cap g^r\mathfrak{h}^*$, hence g takes \mathfrak{k}_∞ into $g\mathfrak{k}_\infty$ (which are disjoint since $\mathfrak{k}_\infty \subset g^r\mathfrak{h}_\infty^*$ and $g\mathfrak{k}_\infty \subset g^r\mathfrak{h}_\infty$). If $\exists n, g^n\hat{h} \cap \mathfrak{k} \neq \emptyset$ then we may assume that n is minimal. By our assumption there exists $\mathfrak{k}_1 \subset \mathfrak{k} \cap g^n\mathfrak{h}^*$. Therefore $\mathfrak{k}_1 \subset g^{n-1}\mathfrak{h} \cap g^n\mathfrak{h}^*$, and g acts non-trivially on ∂X as before.

Case 3. $g\hat{h} \cap \hat{h} \neq \emptyset$. By our assumption let $\mathfrak{k} \subset \mathfrak{h}^* \cap g\mathfrak{h}$, then g sends \mathfrak{k}_∞ into $g\mathfrak{h}_\infty^*$. □

For completeness we include the proof of the following.

Lemma A.9 ([15, Lemme 6.4]). *Let $g \in \text{Aut}(X)$, and $\mathfrak{h} \in \mathcal{H}(X)$ such that $g\mathfrak{h} \subset \mathfrak{h}$, and let $F = \text{Fix}_G(\cup_{n \in \mathbb{Z}} g^n\hat{h})$, then $F = [g, F]$.*

Proof. Clearly $F \supset [g, F]$. Now, let $f \in F$. we will show that there exist $f' \in F$ such that $f = f'^{-1}g^{-1}f'g$. By Property (P) we can define f' by its restrictions to the sets $g^n\mathfrak{h} \cap g^{n+1}\mathfrak{h}^*$. We do so by induction on n .

For $n = 0$, define

$$f'|_{\mathfrak{h} \cap g\mathfrak{h}^*} = \text{id}_{\mathfrak{h} \cap g\mathfrak{h}^*}$$

and for $n > 0$, define

$$f'|_{g^n\mathfrak{h} \cap g^{n+1}\mathfrak{h}^*} = gf'fg^{-1}.$$

Similarly define f' for $n < 0$.

The automorphism f' is well-defined. Indeed, if $x \in g^n\mathfrak{h} \cap g^{n+1}\mathfrak{h}^*$, then $fg^{-1}(x) \in g^{n-1}\mathfrak{h} \cap g^n\mathfrak{h}^*$, and therefore $f'(f(g(x)))$ is defined by the induction hypothesis. Now observe that f' has the desired property. □

Proposition A.10. *Let X be a proper CAT(0) space, and let $H \leq \text{Isom}(X)$ be non-elementary. Assume H contains a rank one isometry and acts non-elementary on X with $\Lambda(H) = \partial X$, and let $N \triangleleft H$, then either N acts trivially on ∂X , or $\Lambda(N) = \Lambda(H) = \partial X$ and N is non-elementary.*

Proof. Assume that N does not act trivially on ∂X . First we show that the limit set $\Lambda(N)$ is either $\Lambda(H)$ or empty.

Let $\xi \in \Lambda(N)$ and $h \in H$. There exists a sequence $n_k \in N$ such that for all $x \in X, n_k x \rightarrow \xi$. Apply $h \in H$ to ξ . By normality of N , we get a sequence $n'_k = hn_k h^{-1} \in N$.

$$h\xi \leftarrow hn_k.x = n'_k.(hx)$$

Therefore $h\xi \in \Lambda(N)$; hence $\Lambda(N)$ is H -invariant. By minimality of the action of H on $\Lambda(H)$ (see [16]) we get $\Lambda(N) = \Lambda(H)$ or \emptyset .

To show that $\Lambda(N) = \Lambda(H)$ assume for contradiction that $\Lambda(N) = \emptyset$. Then the action of N on X is bounded; hence has a fixed point x_0 . Since N acts non-trivially on ∂X , there exists $\xi \in \partial X$ and $n \in N$ such that $n.\xi \neq \xi$. By $\partial X = \Lambda(H)$, there exists a sequence $h_k \in H$ such that $h_k.x_0 \rightarrow \xi$. By applying n we get $nh_k.x_0 \rightarrow n.\xi$. On the other hand, by normality, we have $nh_k = h_k n'_k$ (for some $n'_k \in N$). Thus, $nh_k.x_0 = h_k n'_k.x_0 = h_k.x_0 \rightarrow \xi$. Therefore, we get $\xi = n.\xi$ which contradicts our assumption.

To see that N is non-elementary, we are left to show that N does not fix a point at infinity. Assume by contradiction that $\xi \in \partial X$ is N -fixed, then by normality $g\xi$ is N -fixed for all $g \in H$. By minimality we get that $\Lambda(H) = \partial X$ is N -fixed. Hence N acts trivially on ∂X . But we assumed that N acts non-trivially on ∂X . \square

Remark. Without assuming $\partial X = \Lambda(H)$, the same argument shows that for every normal subgroup $N \triangleleft H$ either N acts trivially on $\Lambda(H)$ or $\Lambda(N) = \Lambda(H)$.

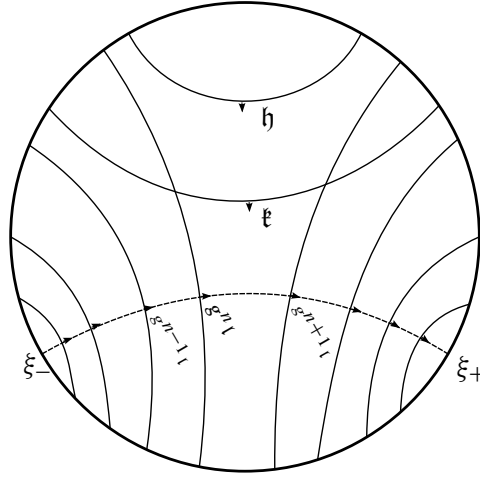


Figure A.1. The action of g on X and the halfspaces h , k and l

We shall now prove Theorem A.3.

Proof. Let $N \triangleleft G^+$ and assume N acts non-trivially on ∂X . In order to prove the theorem, it suffices to show that $\text{Fix}_G(h) \subset N$ for all $h \in \mathcal{H}(X)$. Let $h \in \mathcal{H}(X)$. Apply Proposition A.10 and the Rank Rigidity theorem first on $G^+ \triangleleft G$ and then on $N \triangleleft G^+$, to obtain that $\Lambda(N) = \partial X$, N is non-elementary and contains a rank one isometry. By Proposition A.7 there exists a halfspace $k \subset h$ such that \hat{h} and \hat{k} are strongly separated. The set $\partial^2 X \cap (k_\infty \times k_\infty)$ is a non-empty open set in $\partial^2 X$, hence, by Theorem 1.1(2) of [16], there exists a rank one isometry g whose two fixed points (ξ_+, ξ_-) in ∂X are in $\partial^2 X \cap (k_\infty \times k_\infty)$. By passing to a power of g we may further assume that there exists $l \in \mathcal{H}(X)$ such that $gl \subset l$. See Figure A.1.

Let $F = \text{Fix}_G(\bigcup_{n \in \mathbb{Z}} g^n \hat{l})$. By the above we see that $\bigcup_{n \in \mathbb{Z}} g^n \hat{l} \subset \mathfrak{h}$. Therefore, by Lemma A.9 we have:

$$\text{Fix}_G(\mathfrak{h}) \subset F = [g, F] \subset N \quad \square$$

Remark. In fact, one can assume a weaker version of property (P). For example, assuming that for every element $g \in G$ and $l \in \mathcal{H}(X)$ such that $gl \subset l$, the map (A.1) is an isomorphism for the collection $\{g^n \hat{l}\}_{n \in \mathbb{Z}}$.

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