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Slow manifolds for infinite-dimensional evolution equations

Felix Hummel and Christian Kuehn

Abstract. We extend classical finite-dimensional Fenichel theory in two directions to infinite dimensions. Under comparably weak assumptions we show that the solution of an infinite-dimensional fast-slow system is approximated well by the corresponding slow flow. After that we construct a two-parameter family of slow manifolds $S_{\varepsilon,\zeta}$ under more restrictive assumptions on the linear part of the slow equation. The second parameter ζ does not appear in the finite-dimensional setting and describes a certain splitting of the slow variable space in a fast decaying part and its complement. The finite-dimensional setting is contained as a special case in which $S_{\varepsilon,\zeta}$ does not depend on ζ. Finally, we apply our new techniques to three examples of fast-slow systems of partial differential equations.

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1. Introduction

In this work, we study infinite-dimensional fast-slow evolution equations of the form

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= Au^\varepsilon + f(u^\varepsilon, v^\varepsilon),\\ \partial_t v^\varepsilon &= Bv^\varepsilon + g(u^\varepsilon, v^\varepsilon), \end{aligned} \tag{1.1}$$

where $\varepsilon \geq 0$ is a small parameter, A and B are linear operators on Banach spaces X and Y, respectively, f, g are sufficiently regular nonlinearities, and $(u^\varepsilon, v^\varepsilon) = (u^\varepsilon(t), v^\varepsilon(t)) \in X \times Y$ are the unknown functions, where the superscript indicates the dependence of the solution on ε. In particular, the class of systems (1.1) are multiscale evolution equations, where the small parameter ε hints at a formal time-scale separation between the variables u^ε and v^ε.

The motivation to study (1.1) is best explained via the finite-dimensional setting, where $(u^\varepsilon, v^\varepsilon) \in \mathbb{R}^m \times \mathbb{R}^n$, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, and one often assumes that f, g are sufficiently smooth. Multiple time scale ordinary differential equations (ODEs) are employed across broad areas of mathematics [21] and form one of the few classes of higher-dimensional dynamical systems, where analytical results about nonlinear dynamics can be obtained due to the time scale separation structure. If we let $\varepsilon \to 0$

in (1.1) we obtain the slow subsystem (or reduced system)

$$\begin{aligned} 0 &= Au^0 + f(u^0, v^0), \\ \partial_t v^0 &= Bv^0 + g(u^0, v^0), \end{aligned} \tag{1.2}$$

which is a differential-algebraic equation defined on the critical set

$$S_0 := \{(u^0, v^0) \in \mathbb{R}^m \times \mathbb{R}^n : 0 = Au^0 + f(u^0, v^0)\},$$

which we shall assume to be a manifold referred to as the critical manifold. If $\mathcal{S}_0 \subseteq S_0$ is compact and normally hyperbolic submanifold, i.e., all eigenvalues of the matrix $A + \mathrm{D}_u f(z) \in \mathbb{R}^{m\times m}$ have nonzero real part for all $z \in \mathcal{S}_0$, then Fenichel–Tikhonov theory [14, 32] guarantees the existence of a locally invariant slow manifold $\mathcal{S}_\varepsilon$. Of course, for practical applications, the case of a critical manifold, which is attracting in the fast directions, is the most frequently encountered. This case occurs when all eigenvalues of $A + \mathrm{D}_u f(z)$ have negative real part and we shall focus on the attracting setting here. For any normally hyperbolic critical manifold, the flow on $\mathcal{S}_\varepsilon$ is approximated well by the slow subsystem flow of (1.1); see also [18, 21, 35] for detailed expositions of Fenichel theory. One reason to intuitively expect such an approximation result in finite dimensions is better visible on the fast time scale $r := t/\varepsilon$, which leads upon substitution in (1.1) to

$$\begin{aligned} \partial_r u^\varepsilon &= Au^\varepsilon + f(u^\varepsilon, v^\varepsilon), \\ \partial_r v^\varepsilon &= \varepsilon\big(Bv^\varepsilon + g(u^\varepsilon, v^\varepsilon)\big). \end{aligned} \tag{1.3}$$

Indeed, sending $\varepsilon \to 0$ in (1.3) yields the fast subsystem (or layer equations)

$$\begin{aligned} \partial_r u^0 &= Au^0 + f(u^0, v^0), \\ \partial_r v^0 &= 0. \end{aligned} \tag{1.4}$$

The full fast-slow system on $\mathbb{R}^m \times \mathbb{R}^n$ can then be treated near $\mathcal{S}_0$ as a bounded perturbation of the fast subsystem since B and g satisfy local bounds due to the assumptions of sufficient regularity on g, so the fast linear hyperbolic dynamics driven by $A + \mathrm{D}_u f(z)$ for $z \in \mathcal{S}_0$ dominates near z. To make this intuition precise is already difficult in the finite-dimensional setting with Fenichel theory providing the comprehensive standard [14], even for multiple time scale dynamical systems, which cannot be written directly [34] in the standard form (1.1).

Transferring the finite-dimensional situation to general evolution equations on Banach spaces turns out to be challenging. At first sight, one may hope that the classical Fenichel approach to show the existence of $\mathcal{S}_\varepsilon$ via a Lyapunov–Perron method or via a Hadamard graph transform [14, 35] can still be applied utilizing variants/extensions of infinite-dimensional center manifold theory [33]. So far, the best available results in this direction are due to Bates et al. [4, 5], who cover the case of semiflows, when the perturbation induced by the slow dynamics is bounded. In particular, this includes the case of partially dissipative systems, where $A = \Delta$ is the Laplacian and $B = 0$ so that the slow variable dynamics is an ODE. Yet, even for quite standard reaction-diffusion systems [15, 16, 22] with $A = \Delta$ and $B = \Delta$ on bounded domains, there has been no major progress to generalize Fenichel's theory from the 1970s. The main problem is that on the fast time scale we can never view $\varepsilon B v^\varepsilon$ as a bounded perturbation if B is a differential operator (this statement will be made precise below); indeed, for differential operators we encounter the formal limit $0 \cdot \infty$ since B is an unbounded operator. Furthermore, the classical concept of normal hyperbolicity is problematic since $\varepsilon B v^\varepsilon$ is not necessarily "small" in any norm compared to the linear part of the u^ε-variable. For example, when $B = \Delta$ on a bounded domain, a spectral Galerkin decomposition shows that the v^ε-variable may have fast decaying components in its linear part. This implies that the case of hyperbolic operators for B (which we include here as well) is somewhat easier. In fact, a very special case of fast-slow invariant manifold theory was carried out for the Maxwell–Bloch equations in [25], where u^ε is governed by an ODE and B is a first-order partial derivative.

Another hope might be that one can adapt the theory of inertial manifolds [28,31], which has been used to constructed low-dimensional attracting invariant manifolds for several classes of partial differential equations (PDEs). Yet, inertial manifold theory is based on global dissipation and compact embeddings to construct reduced lower-dimensional invariant manifolds. For the fast-slow evolution system (1.1), we are not interested in global reduction but local persistence/perturbation of manifolds. In fact, we shall see below that our slow manifold can even grow upon perturbation in a suitable sense in comparison to the critical manifold.

In this work, we provide a quite general fast-slow invariant manifold theory for the evolution equations (1.1). We briefly outline our results in a non-technical form:

- We identify the key problems with Fenichel theory in infinite dimensions via several explicit examples including the problems with unbounded and differential operators B as well as with the notion of normal hyperbolicity; see Section 3.

- We assume that A is the generator of a C_0 semigroup having zero in its resolvent and that the nonlinearity is (locally) Lipschitz. Then we prove an approximation result that the flow of the full evolution equation for sufficiently small $\varepsilon > 0$ is, near S_0, well-approximated by the flow of the slow subsystem on S_0; see Theorem 4.13.

- Under suitable regularity assumptions on B and g, we prove the existence of a two-parameter family of slow manifolds $S_{\varepsilon,\zeta}$. The second small parameter $\zeta > 0$ controls additional "fast" contributions of the v^ε-dynamics. We also prove differentiability of $S_{\varepsilon,\zeta}$ if f is C^1, we show estimates on the distance of $S_{\varepsilon,\zeta}$ to the critical manifold, and a result regarding local attraction of trajectories near $S_{\varepsilon,\zeta}$; see Section 5.

In the proofs, there are several important new technical steps. The approximation result given in Theorem 4.13 does not provide a slow manifold, and is hence weaker than classical Fenichel theory but it also uses weaker assumptions. It shows that there exists a very general result that the slow subsystem can be used to approximate the full dynamics in a suitable sense near S_0. In fact, the proof of this result seems to be difficult to achieve on the fast time scale, or even directly with the original full evolution equations (1.1) on the slow time scale. We use an intermediate approximating evolution equation (see also the calculations starting from equation (4.2)), which changes the right-hand side of the fast component as follows

$$\begin{aligned} \varepsilon\partial_t u^{\varepsilon,0} &= Au^{\varepsilon,0} + f(u^{\varepsilon,0}, v^0) - \varepsilon\partial_t A^{-1} f\big(h^0(v^0), v^0\big), \\ \partial_t v^0 &= Bv^0 + g\big(h^0(v^0), v^0\big), \end{aligned} \tag{1.5}$$

where $h^0\colon Y \to X$ is a local parametrization of the critical manifold. On the finite-dimensional level, when $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ one can nicely see, why this choice might be helpful. Looking formally at different orders of $\mathcal{O}(\varepsilon^k)$ one has for $k = 0, 1$ from the first equation

$$Au^{0,0} + f(u^{0,0}, v^0) = 0 \quad \text{and} \quad u^{0,0} + A^{-1} f\big(h^0(v^0), v^0\big) = \text{constant},$$

so upon using an initial condition with $h^0(v^0) = u^{0,0}$ one just obtains the condition of the critical manifold twice, to leading-order and to first order in ε. This means that our intermediate system (1.5) is likely to be a locally better approximation to the full fast-slow dynamics near S_0 and it is a regularization of the slow subsystem. Other important ingredients to obtain the approximation result are the use of interpolation-extrapolation scales and suitably adapted Gronwall-type arguments involving mild solutions.

For the construction of the slow manifold family $S_{\varepsilon,\zeta}$, we use a re-partitioning the slow dynamics into two parts, which can formally be expressed as

$$Y = Y_F^\zeta \oplus Y_S^\zeta.$$

The part Y_S^ζ comes from modes/directions, where εB yields a sufficiently small perturbation so that these modes are slow. Moreover, the linear part of the dynamics on Y_S^ζ is supposed to exist also backwards in time. The other part Y_F^ζ comes from modes, which are fast as B dominates the small parameter ε. The parameter ζ describes which parts of the linear dynamics in the slow variable space are considered as fast and which ones are considered as slow. This is quantified in terms of an exponential dichotomy. In a simplified special case, which already covers many examples, this dichotomy is given by estimates of the form

$$\begin{aligned} \|e^{tB} y_F\|_{Y_1} &\le C_B e^{(N_F^\zeta + \zeta^{-1}\omega_A)t} \|y_F\|_{Y_1} \qquad (t \ge 0,\ y_F \in Y_F^\zeta \cap Y_1), \\ \|e^{-tB} y_S\|_{Y_1} &\le M_B e^{-(N_S^\zeta + \zeta^{-1}\omega_A)t} \|y_S\|_{Y_1} \quad (t \ge 0,\ y_F \in Y_S^\zeta \cap Y_1). \end{aligned}$$

Here, $Y_1 = D(B)$ denotes the domain of B, the parameter ω_A is essentially the growth bound of the C_0-semigroup generated by the operator A, the two constants C_B, $M_B > 0$ are independent of ζ and the two constants $N_S^\zeta > N_F^\zeta > 0$ do depend on ζ. The difference of N_S^ζ and N_F^ζ is an important quantity in our theory. It describes how well the splitting separates fast and slow parts of the linearized slow dynamics. It is related to gaps in the spectrum of the operator B. For the construction of our slow manifolds, we need the spectral gaps to be of a certain size in relation to the Lipschitz constants of the nonlinearities. In certain situations, especially if the equation in the slow variable is a parabolic equation on a bounded domain, this leads to similar conditions as the ones for the existence of inertial manifolds for dissipative equations. However, since the slow manifolds we construct do not have to be finite-dimensional, our spectral assumptions are not as restrictive as those for inertial manifolds. In particular, unbounded spectrum in the imaginary direction will usually not be a problem for our theory, so that hyperbolic or dissipative equations in the slow variable are in a certain sense easier for our theory. The precise conditions for the existence of slow manifolds will be given later in Section 5.1.

Having this splitting available, we then proceed to set up a Lyapunov–Perron functional iteration to obtain the existence of $S_{\varepsilon,\zeta}$. The dynamical properties of $S_{\varepsilon,\zeta}$ can be established using relatively long estimates in combination with mild solution representations, time differentiation of the manifold parametrization along solutions, and contraction mapping arguments.

The paper is structured as follows: In Section 2 we collect technical background results regarding interpolation-extrapolation scales of Banach spaces and operators on these spaces, as well as suitable variants of Gronwall-type lemmas. In Section 3, we illustrate the difficulties of the classical Fenichel viewpoint and the barriers to generalize the bounded perturbation results for semiflows. In Section 4, we prove the general result on slow flow approximation for semiflows, while in Section 5 we obtain the slow manifold family and its precise dynamic properties. We present three illustrating examples in Section 6 and conclude with an outlook in Section 7.

2. Preliminaries

2.1. Interpolation-extrapolation scales. We briefly introduce some required notions and results in connection with interpolation-extrapolation scales. As a general reference, we would like to mention [1, Chapter V].

Let $T: X \supset D(T) \to X$ be a densely defined closed linear operator on a Banach space X with $0 \in \rho(T)$. Moreover, for $\theta \in (0,1)$ let $(\cdot,\cdot)_\theta$ be an exact admissible interpolation functor, i.e., an exact interpolation functor such that X_1 is dense in $(X_0, X_1)_\theta$, whenever

$$X_1 \overset{d}{\hookrightarrow} X_0.$$

We define a family of Banach spaces $(X_\alpha)_{\alpha\in[-1,\infty)}$ and a family of operators $(T_\alpha)_{\alpha\in[-1,\infty)} \in \mathcal{B}(X_{\alpha+1}, X_\alpha)$ as follows:

- For $k \in \mathbb{N}_0$ we choose $X_k := D(T^k)$ endowed with $\|x\|_{X_k} := \|T^k x\|_X$ $(x \in D(T^k))$. In particular, $X_0 = D(T^0) = D(\mathrm{id}_X) = X$. Moreover, $T_k := T|_{X_{k+1}}$.
- X_{-1} is defined as the completion of $X = X_0$ with respect to the norm $\|x\|_{X_{-1}} = \|T^{-1}x\|_{X_0}$. The operator $T_0 = T$ is then closable on X_{-1} and T_{-1} is defined to be the closure. One can also define (X_{-k}, T_{-k}) for $k \in \mathbb{N}$ by iteration, but we do not go beyond $k = -1$ in this paper.
- For $k \in \mathbb{N}_0 \cup \{-1\}$, $\theta \in (0,1)$ and $\alpha = k + \theta$ we define $X_\alpha := (X_k, X_{k+1})_\theta$ and $T_\alpha = T_k|_{D(T_\alpha)}$, where

$$D(T_\alpha) = \{x \in X_{k+1} : T_k x \in X_\alpha\}.$$

The family $(X_\alpha, T_\alpha)_{\alpha\in[-1,\infty)}$ is a densely injected Banach scale in the sense that

$$X_\alpha \overset{d}{\hookrightarrow} X_\beta,$$

whenever $\alpha \geq \beta$ (i.e., the injection is continuous with dense range), and

$$T_\alpha : X_{\alpha+1} \to X_\alpha$$

is an isomorphism for all $\alpha \in \mathbb{R}$. Moreover, $T_\alpha : X_\alpha \supset X_{\alpha+1} \to X_\alpha$ is a densely defined closed linear operator with $0 \in \rho(T_\alpha)$ for all $\alpha \in \mathbb{R}$. The family $(X_\alpha, T_\alpha)_{\alpha\in\mathbb{R}}$ is an interpolation-extrapolation scale.

One of the nice things about interpolation-extrapolation scales is that semigroups can be shifted along these scales. More precisely, we have the following (cf. [1, Chapter V, Theorem 2.1.3]):

Theorem 2.1. *Let T be the generator of a C_0-semigroup $(S(t))_{t\geq 0}$ and let $\omega_S \in \mathbb{R}$ be the growth bound of S, i.e.,*

$$\omega_S := \inf\{\omega \in \mathbb{R} \mid \exists M > 0\ \forall t \geq 0 : \|S(t)\|_{\mathcal{B}(X)} \leq M e^{\omega t}\}.$$

Then $T_\alpha : X_\alpha \supset X_{\alpha+1} \to X_\alpha$ also generates a C_0 semigroup $(S_\alpha(t))_{t\geq 0}$ with the same growth bound and for all $\alpha, \beta \in [-1,\infty)$, $\alpha \geq \beta$, the following diagram commutes:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{S_\alpha(t)} & X_\alpha \\ \downarrow & & \downarrow \\ X_\beta & \xrightarrow{S_\beta(t)} & X_\beta. \end{array}$$

Moreover, if $(S(t))_{t\geq 0}$ is holomorphic then the same holds for $(S_\beta(t))_{t\geq 0}$ and for all $\omega > \omega_S$ there is a constant C also depending on α and β such that

$$\|S_\beta(t)\|_{\mathcal{B}(E_\beta, E_\alpha)} \leq C t^{\beta-\alpha} e^{-\omega t} \quad (t > 0).$$

2.2. Estimates for the incomplete gamma function. In this paper we frequently encounter terms of the form

$$\int_0^t \frac{e^{\varepsilon^{-1}\omega(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\, ds$$

with $\gamma \in (0,1]$, $\omega < 0$ and $\varepsilon > 0$. In the following, we derive certain elementary estimates which we use several times. They might not be of great importance on their own, but being able to refer to them will be useful at some places. Note that the substitution $r = -\varepsilon^{-1}\omega(t-s)$ yields

$$\int_0^t \frac{e^{\varepsilon^{-1}\omega(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\, ds = \frac{1}{|\omega|^\gamma}\int_0^{\varepsilon^{-1}|\omega|t} \frac{e^{-r}}{r^{1-\gamma}}\, dr = \frac{\widetilde{\Gamma}(\gamma, \varepsilon^{-1}|\omega|t)}{|\omega|^\gamma},$$

where $\widetilde{\Gamma}(\gamma, t) := \int_0^t \frac{e^{-r}}{r^{1-\gamma}}\, dr$ denotes the incomplete gamma function.

Lemma 2.2. *For all $t \geq 0$, $\varepsilon > 0$, $\gamma \in (0,1]$ and $\omega < 0$ it holds that*

$$\int_0^t \frac{e^{\varepsilon^{-1}\omega(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\, ds \leq \min\left\{\frac{t^\gamma}{\gamma\varepsilon^\gamma}, \frac{\Gamma(\gamma)}{|\omega|^\gamma}\right\}.$$

Here, Γ denotes the gamma function.

Proof. Hölder's inequality yields

$$\int_0^t \frac{e^{\varepsilon^{-1}\omega(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}} \leq \frac{1}{\varepsilon^\gamma}\int_0^t \frac{1}{(t-s)^{1-\gamma}}\, ds = \frac{t^\gamma}{\gamma\varepsilon^\gamma}.$$

On the other hand, since $\widetilde{\Gamma}(\gamma, t)$ is increasing in t, it follows that

$$\int_0^t \frac{e^{\varepsilon^{-1}\omega(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\, ds = \frac{\widetilde{\Gamma}(\gamma, \varepsilon^{-1}|\omega|t)}{|\omega|^\gamma} \leq \lim_{t\to\infty} \frac{\widetilde{\Gamma}(\gamma, \varepsilon^{-1}|\omega|t)}{|\omega|^\gamma} = \frac{\Gamma(\gamma)}{|\omega|^\gamma},$$

which completes the proof. □

Lemma 2.3. *For all $t \geq 0$, $\varepsilon > 0$, $\gamma \in (0,1]$ and $\omega < \widetilde{\omega}$ it holds that*

$$e^{\varepsilon^{-1}\widetilde{\omega}t}\int_0^t \frac{e^{\varepsilon^{-1}(\omega-\widetilde{\omega})s}}{\varepsilon^\gamma s^{1-\gamma}}\, ds \leq \frac{e^\gamma}{\gamma^{1-\gamma}|\widetilde{\omega}|^\gamma}.$$

Proof. By Lemma 2.2 it holds that

$$e^{\varepsilon^{-1}\widetilde{\omega}t}\int_0^t \frac{e^{\varepsilon^{-1}(\omega-\widetilde{\omega})s}}{\varepsilon^\gamma s^{1-\gamma}}\, ds \leq e^{\varepsilon^{-1}\widetilde{\omega}t}\frac{t^\gamma}{\gamma\varepsilon^\gamma}.$$

The right-hand side attains its maximum at $t = |\gamma\varepsilon\widetilde{\omega}^{-1}|$. This yields the assertion. □

Lemma 2.4. *For all $t \geq 0$, $\varepsilon > 0$, $\gamma \in (0,1]$ and $\omega < \tilde{\omega} < 0$ it holds that*

$$\int_0^t \varepsilon^{-1}|\omega| e^{\varepsilon^{-1}\tilde{\omega}(t-s)} \int_0^s \frac{e^{\varepsilon^{-1}\omega r}}{\varepsilon^\gamma r^{1-\gamma}} \, dr \, ds \leq \frac{\Gamma(\gamma)|\omega|^{1-\gamma}}{\tilde{\omega}}.$$

Proof. Using Lemma 2.2 we obtain

$$\begin{aligned}\int_0^t \varepsilon^{-1}|\omega| e^{\varepsilon^{-1}\tilde{\omega}(t-s)} \int_0^s \frac{e^{\varepsilon^{-1}\omega r}}{\varepsilon^\gamma r^{1-\gamma}} \, dr \, ds &\leq \varepsilon^{-1}|\omega|^{1-\gamma}\Gamma(\gamma) \int_0^t e^{\varepsilon^{-1}\tilde{\omega}(t-s)} \, ds \\ &\leq \frac{\Gamma(\gamma)|\omega|^{1-\gamma}}{\tilde{\omega}}.\end{aligned}$$

□

Corollary 2.5. *For all $t \geq 0$, $\varepsilon > 0$, $\gamma \in (0,1]$ and $\omega < \tilde{\omega} < 0$ it holds that*

$$\int_0^t \left(\frac{e^{\varepsilon^{-1}\omega s}}{\varepsilon^\gamma s^{1-\gamma}} + \varepsilon^{-1}|\omega| \int_0^s \frac{e^{\varepsilon^{-1}\omega r}}{\varepsilon^\gamma r^{1-\gamma}} \, dr \right) e^{\varepsilon^{-1}\tilde{\omega}(t-s)} \, ds \leq \left(\frac{e^\gamma}{\gamma^{1-\gamma}} + \Gamma(\gamma) \left| \frac{\omega}{\tilde{\omega}} \right|^{1-\gamma} \right) \frac{1}{\tilde{\omega}^\gamma}.$$

Proof. This follows from summing up the estimates of Lemmas 2.3 and 2.4. □

Lemma 2.6. *Let $\omega < 0$ and $\gamma \in (0,1]$. Then it holds that*

$$\int_0^t \frac{e^{\omega s}}{(t-s)^{1-\gamma}} \, ds \leq \frac{e^{1+\omega t} + \gamma}{\gamma |\omega|^\gamma}.$$

Proof. This follows from

$$\begin{aligned}\int_0^t \frac{e^{\omega s}}{(t-s)^{1-\gamma}} ds &= e^{\omega t} \int_0^t \frac{e^{-\omega s}}{s^{1-\gamma}} ds = \frac{e^{\omega t}}{|\omega|^\gamma} \int_0^{|\omega| t} \frac{e^r}{r^{1-\gamma}} dr \\ &\leq \frac{e^{\omega t}}{|\omega|^\gamma} \left(\int_0^1 \frac{e^r}{r^{1-\gamma}} dr + \int_1^{\max\{1, |\omega| t\}} \frac{e^r}{r^{1-\gamma}} dr \right) \\ &\leq \frac{e^{\omega t}}{|\omega|^\gamma} \left(\frac{e}{\gamma} + e^{-\omega t} \right) = \frac{e^{1+\omega t} + \gamma}{\gamma |\omega|^\gamma}.\end{aligned}$$

□

2.3. Some Gronwall type inequalities. In most of the proofs of this paper, Gronwall type inequalities are essential ingredients. Here, we collect the versions which we use throughout this work.

Lemma 2.7. *Let $T > 0$, $u, v, c \colon [0,T] \to [0,\infty)$ be continuous and suppose that c' is locally integrable. If $v(t) \leq c(t) + \int_0^t u(s)v(s) \, ds$ for all $t \in [0,T]$, then*

$$v(t) \leq c(0) \exp\left(\int_0^t u(s) \, ds \right) + \int_0^t c'(s) \exp\left(\int_s^t u(r) \, dr \right) ds \quad (t \in [0,T]).$$

Proof. This is a well-known version of Gronwall's inequality. A proof can, for example, be found in [9, Corollary 2]. The statement therein is formulated for c being differentiable, but the argument relies on integration by parts and thus, also the asserted version holds true. □

Lemma 2.8. *Let $x \in \mathbb{R}$, $\varepsilon, N, T > 0$, $\gamma \in (0,1]$, $p \in (1,\infty)$ and let $p' = p/(p-1)$ be the conjugated Hölder index. Let further $v, c\colon [0,T] \to [0,\infty)$ be continuous. Suppose that c' is locally integrable and that $[t \mapsto \mathrm{e}^{-\varepsilon^{-1}xt}c(t)]$ is non-decreasing. If*

$$v(t) \le c(t) + N\int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}x(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}} v(s)\,\mathrm{d}s$$

for all $t \in [0,T]$, then we have the estimate

$$v(t) \le pc(0)\mathrm{e}^{\varepsilon^{-1}\tilde{x}t} + p\int_0^t \big(c'(s) - \varepsilon^{-1}xc(s)\big)\mathrm{e}^{\varepsilon^{-1}\tilde{x}(t-s)}\,\mathrm{d}s \quad (t \in [0,T]),$$

where $\tilde{x} := x + pN^{1/\gamma}(p'/\gamma)^{(1-\gamma)/\gamma}$.

Proof. Let $\theta(t) := \sup_{0\le s\le t} \mathrm{e}^{-\varepsilon^{-1}xs}v(s)$. Then we have the estimate

$$\mathrm{e}^{-\varepsilon^{-1}xt}v(t) \le c(t)\mathrm{e}^{-\varepsilon^{-1}xt} + N\int_0^t \frac{1}{\varepsilon^\gamma (t-s)^{1-\gamma}}\theta(s)\,\mathrm{d}s.$$

If we choose $\sigma = (\gamma/p'N)^{1/\gamma}\varepsilon$, then we obtain

$$\begin{aligned}
\mathrm{e}^{-\varepsilon^{-1}xt}v(t) &\le c(t)\mathrm{e}^{-\varepsilon^{-1}xt} + N\int_0^{[t-\sigma]_+} \frac{1}{\varepsilon^\gamma (t-s)^{1-\gamma}}\theta(s)\,\mathrm{d}s \\
&\quad + N\int_{[t-\sigma]_+}^t \frac{1}{\varepsilon^\gamma (t-s)^{1-\gamma}}\theta(t)\,\mathrm{d}s \\
&\le c(t)\mathrm{e}^{-\varepsilon^{-1}xt} + \frac{N}{\varepsilon^\gamma\sigma^{1-\gamma}}\int_0^t \theta(s)\,\mathrm{d}s - \frac{N}{\gamma\varepsilon^\gamma}\big[(t-s)^\gamma\big]_{s=[t-\sigma]_+}^t \theta(t) \\
&\le c(t)\mathrm{e}^{-\varepsilon^{-1}xt} + \frac{N}{\varepsilon^\gamma\sigma^{1-\gamma}}\int_0^t \theta(s)\,\mathrm{d}s + \frac{1}{p'}\theta(t).
\end{aligned}$$

By the monotonicity of the right-hand side, it follows that we can replace $\mathrm{e}^{-\varepsilon^{-1}xt}v(t)$ by $\theta(t)$ on the left-hand side. Therefore, we obtain

$$\theta(t) \le pc(t)\mathrm{e}^{-\varepsilon^{-1}xt} + \frac{pN}{\varepsilon^\gamma\sigma^{1-\gamma}}\int_0^t \theta(s)\,\mathrm{d}s,$$

so that Lemma 2.7 implies

$$\begin{aligned}
\theta(t) &\le pc(0)\exp\Big(\frac{pN}{\varepsilon^\gamma\sigma^{1-\gamma}}t\Big) \\
&\qquad + p\int_0^t (c'(s) - \varepsilon^{-1}xc(s))\exp\Big(-\varepsilon^{-1}xs + \frac{pN}{\varepsilon^\gamma\sigma^{1-\gamma}}(t-s)\Big)\,\mathrm{d}s,
\end{aligned}$$

and therefore

$$\begin{aligned} v(t) &\le pc(0)\exp\bigg(\Big(\varepsilon^{-1}x+\frac{pN}{\varepsilon^\gamma\sigma^{1-\gamma}}\Big)t\bigg) \\ &\quad + p\int_0^t (c'(s)-\varepsilon^{-1}xc(s))\exp\bigg(\Big(\varepsilon^{-1}x+\frac{pN}{\varepsilon^\gamma\sigma^{1-\gamma}}\Big)(t-s)\bigg)\,\mathrm{d}s \\ &= pc(0)\mathrm{e}^{\varepsilon^{-1}\tilde{x}t} + p\int_0^t \big(c'(s)-\varepsilon^{-1}xc(s)\big)\mathrm{e}^{\varepsilon^{-1}\tilde{x}(t-s)}\,\mathrm{d}s. \end{aligned}$$ □

Remark 2.9. For the sake of simplicity, we will apply Lemma 2.8 with $p=2$ most of the time. However, this is not optimal in many cases. In particular, if $\gamma=1$ then it is actually better to take p close to 1. This way, we may actually take $\omega_f>\omega_A+C_AL_f$ instead of $\omega_f=\omega+(2C_AL_f)^{1/\gamma}(1/\gamma)^{(1-\gamma)/\gamma}$ later in this paper. This might be of importance if one wants ω_f to be as small as possible.

Lemma 2.10. *Let $x,y\in\mathbb{R}$, $\varepsilon,N,M,T>0$ as well as $\gamma,\delta\in(0,1]$. Let further $v,c\colon[0,T]\to[0,\infty)$ be continuous. Suppose that c' is locally integrable and that $[t\mapsto \mathrm{e}^{-yt}c(t)]$ is non-decreasing. If $0<N\Gamma(\gamma)/(\varepsilon y-x)^\gamma<1$ and if*

$$v(t)\le c(t)+N\int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}x(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}v(s)\,\mathrm{d}s + M\int_0^t\frac{\mathrm{e}^{y(t-s)}}{(t-s)^{1-\delta}}v(s)\,\mathrm{d}s$$

for all $t\in[0,T]$, then for all $\mu\in(0,1-(N\Gamma(\gamma)/(\varepsilon y-x)^\gamma))$, we have the estimate

$$v(t)\le\frac{1}{1-\mu-(N\Gamma(\gamma)/(\varepsilon y-x)^\gamma)}\bigg[c(0)\mathrm{e}^{\tilde{y}t}+\int_0^t(c'(s)-yc(s))\mathrm{e}^{\tilde{y}(t-s)}\,\mathrm{d}s\bigg] \qquad (t\in[0,T]),$$

where $\tilde{y}:=y+M^{1/\delta}(\delta\mu)^{(\delta-1)/\delta}(1-\mu-(N\Gamma(\gamma)/(\varepsilon y-x)^\gamma))^{-1}$.

Proof. The proof is similar to the one of Lemma 2.8. We define

$$\theta(t):=\sup_{0\le s\le t}\mathrm{e}^{-ys}v(s),$$

so that we obtain

$$\begin{aligned} \mathrm{e}^{-yt}v(t) &\le \mathrm{e}^{-yt}c(t)+N\int_0^t\frac{\mathrm{e}^{(\varepsilon^{-1}x-y)(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\theta(s)\,\mathrm{d}s + M\int_0^t\frac{1}{(t-s)^{1-\delta}}\theta(s)\,\mathrm{d}s \\ &\le \mathrm{e}^{-yt}c(t)+N\int_0^t\frac{\mathrm{e}^{(\varepsilon^{-1}x-y)(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\,\mathrm{d}s\;\theta(t) + M\int_0^t\frac{1}{(t-s)^{1-\delta}}\theta(s)\,\mathrm{d}s \\ &\le \mathrm{e}^{-yt}c(t)+\frac{N\Gamma(\gamma)}{(\varepsilon y-x)^\gamma}\theta(t)+M\int_0^t\frac{1}{(t-s)^{1-\delta}}\theta(s)\,\mathrm{d}s, \end{aligned}$$

where we used Lemma 2.2. For some $\sigma \geq 0$ we split again

$$\begin{aligned} e^{-yt}v(t) &\leq e^{-yt}c(t) + \frac{N\Gamma(\gamma)}{(\varepsilon y - x)^{\gamma}}\theta(t) + \int_0^{[t-\sigma]_+} M\frac{1}{(t-s)^{1-\delta}}\theta(s)\,ds \\ &\quad + \int_{[t-\sigma]_+}^{t} M\frac{1}{(t-s)^{1-\delta}}\,ds\,\theta(t) \\ &\leq e^{-yt}c(t) + \left(\frac{N\Gamma(\gamma)}{(\varepsilon y - x)^{\gamma}} + \frac{M\sigma^{\delta}}{\delta}\right)\theta(t) + \frac{M}{\sigma^{1-\delta}}\int_0^t \theta(s)\,ds. \end{aligned}$$

Now we choose $\mu \in (0, 1 - (N\Gamma(\gamma)/(\varepsilon y - x)^{\gamma}))$ and $\sigma = (\delta\mu/M)^{1/\delta}$. If we also use the monotonicity of the right-hand side, then we obtain

$$\theta(t) \leq e^{-yt}c(t) + \left(\frac{N\Gamma(\gamma)}{(\varepsilon y - x)^{\gamma}} + \mu\right)\theta(t) + M^{1/\delta}(\delta\mu)^{(\delta-1)/\delta}\int_0^t \theta(s)\,ds.$$

Since $0 < (N\Gamma(\gamma)/(\varepsilon y - x)^{\gamma}) + \mu < 1$ this yields

$$\begin{aligned} \theta(t) \leq{}& \frac{1}{1 - \mu - (N\Gamma(\gamma)/(\varepsilon y - x)^{\gamma})} e^{-yt}c(t) \\ &+ \frac{M^{1/\delta}(\delta\mu)^{(\delta-1)/\delta}}{1 - \mu - (N\Gamma(\gamma)/(\varepsilon y - x)^{\gamma})}\int_0^t \theta(s)\,ds. \end{aligned}$$

Hence, the assertion follows from Lemma 2.7. □

3. Problems with fast-slow systems in infinite dimensions

Here we give some reasons why it is difficult to apply perturbation theorems for normally hyperbolic invariant manifolds in infinite dimensions such as the ones in [4,5] to infinite-dimensional fast-slow systems.

3.1. Problems with small perturbations. In finite dimensions, the usual approach to show the existence of slow manifolds is to show that the flow of the fast-slow system on the fast time scale is a small perturbation of the flow generated by the fast subsystem. Then the existence of slow manifolds follows from the persistence of normally hyperbolic invariant manifolds under small perturbation. But even though such persistence results are also available in infinite dimensions (see, for example, [4,5]), this approach does not work directly for many interesting infinite-dimensional examples. Consider for instance the following situation: Let X, Y be Banach spaces. Suppose that

$$A: X \supset D(A) \to X \quad \text{and} \quad B: Y \supset D(B) \to Y$$

are generators of C_0-semigroups

$$(T_A(t))_{t\geq 0} \subset \mathcal{B}(X) \quad \text{and} \quad (T_B(t))_{t\geq 0} \subset \mathcal{B}(Y),$$

respectively. Let further $L_1 \in \mathcal{B}(Y, X)$ and $L_2 \in \mathcal{B}(X, Y)$ be bounded linear operators. Then the operator

$$\begin{pmatrix} A & L_1 \\ \varepsilon L_2 & \varepsilon B \end{pmatrix} : X \times Y \supset D(A) \times D(B) \to X \times Y$$

generates a C_0-semigroup $(T_\varepsilon(t))_{t\geq 0}$ for all $\varepsilon \geq 0$. Hence, for all $u_0 \in X$, $v_0 \in Y$ and all $\varepsilon \geq 0$ there is a unique solution to the fast-slow system

$$\begin{aligned} \partial_t u^\varepsilon &= Au^\varepsilon + L_1 v^\varepsilon, \\ \partial_t v^\varepsilon &= \varepsilon B v^\varepsilon + \varepsilon L_2 u^\varepsilon, \\ u^\varepsilon(0) = u_0&, \quad v^\varepsilon(0) = v_0 \end{aligned} \tag{3.1}$$

on the fast time scale which is given by a semiflow

$$\begin{pmatrix} u^\varepsilon(t) \\ v^\varepsilon(t) \end{pmatrix} = T_\varepsilon(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

For the sake of argument, we assume that the embedding

$$D(A) \times D(B) \to X \times Y$$

is compact so that the intersection of the critical subspace

$$S_0 := \{(u, v) \in D(A) \times D(B) : Au + L_1 v = 0\}$$

with the ball $B(0, R)$ in $D(A) \times D(B)$ around 0 with arbitrary radius $R > 0$ is relatively compact in $X \times Y$. Note that this assumption is frequently satisfied for differential operators on bounded domains. We are thus in a similar situation as in finite dimensions. One would hope that one can apply the theorem given in the introduction of [4] to $S_0 \cap B(0, R)$. However, if one wants to apply this theorem in order to perturb the critical subspace S_0 for (3.1) with $\varepsilon = 0$ to a slow submanifold S_ε for (3.1) with $\varepsilon > 0$, one would – among other assumptions – need that

$$\|T_0(t) - T_\varepsilon(t)\|_{\mathcal{B}(X\times Y)} \to 0 \quad (\varepsilon \to 0) \tag{3.2}$$

for some $t > 0$. In fact, one just needs

$$\|T_0(t) - T_\varepsilon(t)\|_{C^1(N; X\times Y)} \to 0 \quad (\varepsilon \to 0)$$

for a suitable neighborhood N of $S_0 \cap B(0, R)$. But since such a neighborhood already contains a ball in $X \times Y$ around 0 with small radius, this implies (3.2) by linearity. However, (3.2) is not satisfied if B is an unbounded operator. This can be seen as follows: One can use the variation of constants formula together with a standard version of Gronwall's inequality in order to show that there is a constant $C > 0$, such that

$$\sup_{\varepsilon,t\in[0,1]} \big(\|u^\varepsilon(t)\|_X + \|v^\varepsilon(t)\|_Y\big) \leq C\big(\|u_0\|_X + \|v_0\|_Y\big).$$

Therefore, if (3.2) holds then we have that

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \|\operatorname{pr}_Y\big(T_\varepsilon(1) - T_0(1)\big)(u_0, v_0)^T\|_Y \\
&= \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \|v^\varepsilon(1) - v_0\|_Y \\
&= \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \Big\|\big(T_B(\varepsilon) - \mathrm{id}_Y\big)v_0 + \varepsilon\int_0^1 T_B\big(\varepsilon(1-s)\big)L_2u^\varepsilon(s)\,\mathrm{d}s\Big\|_Y \\
&\geq \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \big(\|(T_B(\varepsilon) - \mathrm{id}_Y)v_0\|_Y\big) \\
&\quad - \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \varepsilon\Big\|\int_0^1 T_B\big(\varepsilon(1-s)\big)L_2u^\varepsilon(s)\,\mathrm{d}s\Big\|_Y \\
&= \lim_{\varepsilon\to 0} \sup_{\|(u_0,v_0)^T\|_{X\times Y}=1} \big(\|\big(T_B(\varepsilon) - \mathrm{id}_Y\big)v_0\|_Y\big).
\end{aligned}$$

Hence, we have

$$\|T_B(\varepsilon) - \mathrm{id}_Y\|_{\mathcal{B}(Y)} \to 0 \quad (\varepsilon \to 0),$$

i.e., the semigroup generated by B is norm-continuous at $t = 0$. But this holds if and only if B is a bounded linear operator on Y, see for example [10, Theorem I.3.7]. Therefore, one can not apply [4] directly to fast-slow systems, in which the dynamics of the slow variable are given by a partial differential equation.

3.2. Problems with the notion of normal hyperbolicity. One of the central objects in classical Fenichel theory is the notion of a normally hyperbolic invariant manifold. The important properties of such a manifold M are that it is invariant under the given (semi-)flow $(T^t)_{t\geq 0}$ on the space X and that for each $m \in M$ it admits a splitting

$$X = X_m^c \oplus X_m^s \oplus X_m^u,$$

such that

(i) X_m^c is the tangent space to M at m;

(ii) the splitting is invariant under the linearized flow $\mathrm{D}T^t(m)$;

(iii) $\mathrm{D}T^t(m)|_{X_m^u}$ expands, $\mathrm{D}T^t(m)|_{X_m^s}$ contracts, and both do so to a greater degree than $\mathrm{D}T^t(m)|_{X_m^c}$.

Perturbation results for such normally hyperbolic invariant manifolds in infinite dimensions have been obtained in [4]. Therein, property (iii) includes on a formal level the condition

$$\lambda \min\bigl\{1, \inf\bigl\{\|\mathrm{D}T^t(m)x^c\|_{X_m^c} : x^c \in X_m^c,\ |x^c| = 1\bigr\}\bigr\} > \|\mathrm{D}T^t(m)|_{X_m^s}\|_{\mathcal{B}(X_m^s)} \tag{3.3}$$

for some $\lambda \in (0, 1)$. However, if we consider the uncoupled, linear case of a fast-slow system, i.e., (3.1) with $L_1 = 0$ and $L_2 = 0$, then the center direction X_m^c on the critical manifold will be given by

$$X_m^c = \{(x, y) \in X \times Y : Ax = 0\} \supset \{(x, y) \in X \times Y : x = 0\}.$$

Thus, if B is a standard parabolic operator as the Laplacian Δ on $L_p(\mathbb{R}^d)$ or the Dirichlet Laplacian Δ_D on $L_p(\mathcal{O})$ with $\mathcal{O}$ being a smooth domain, then the left-hand side of (3.3) is equal to 0 so that normal hyperbolicity in the sense of [4, p. 11] can not be satisfied.

3.3. Problems with the splitting in fast and slow time. In infinite dimensions, one has to be careful with the interpretation of the notion "fast-slow system". Many interesting cases can (locally) be written as

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= Au^\varepsilon + f(u^\varepsilon, v^\varepsilon),\\ \partial_t v^\varepsilon &= Bv^\varepsilon + g(u^\varepsilon, v^\varepsilon),\\ u^\varepsilon(0) &= u_0, \quad v^\varepsilon(0) = v_0, \end{aligned} \tag{3.4}$$

where in infinite dimensions the operators A and B are unbounded operators on the Banach spaces X and Y, the Lipschitz continuous nonlinearities f, g have Lipschitz constants which are not too large and u_0, v_0 are certain initial conditions; note that in many examples one may cut off the nonlinearity to make it Lipschitz due to invariant regions [29] or due to global dissipation [28, 31].

Already in finite dimensions, the speed of evolution of the fast variable can only be considered faster than the one of the slow variable if they are related to their norms. Obviously, if $\|v_0\|_Y$ is very large, then $v^\varepsilon(t)$ may change quickly compared to $u^\varepsilon(t)$, even if ε is very small. However, in infinite dimensions $\|\cdot\|_X$ and $\|\cdot\|_Y$ may not be suitable for such a comparison for several reasons. First of all, unlike in finite dimensions, not all norms are equivalent and thus, comparing $\|\cdot\|_X$ and $\|\cdot\|_Y$ might not be very meaningful. But even if ones takes $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y)$, one may run into difficulties. For the sake of argument, we assume for the moment that there is no coupling, i.e., $f = 0$ and $g = 0$. Since B is unbounded in many interesting cases, we may take $u_0 \in D(A)$ with $\|u_0\|_X = 1$ and $v_0 \in Y$ with $\|v_0\|_Y = 1$ such that $\|Bv_0\|_Y > \varepsilon^{-1}\|Au_0\|_X$. Then we have

$$\|\partial_t u^\varepsilon(0)\|_X = \varepsilon^{-1}\|Au_0\|_X < \|Bv_0\|_Y = \|\partial_t v^\varepsilon(0)\|_Y.$$

Therefore, one could argue that $v^\varepsilon(t)$ is faster around $t = 0$ than $u^\varepsilon(t)$, even though it is called "slow variable". Note that this argument breaks down if one takes u_0 and v_0 to have graph norms of the same size, i.e., $\|u_0\|_{D(A)} = \|v_0\|_{D(B)} = 1$. But then we have to problem the other way round: $\|\partial_t v^\varepsilon(0)\|_Y$ might be smaller than $\|\partial_t u^\varepsilon(0)\|_X$ only because $\|v_0\|_Y$ is much smaller than $\|u_0\|_X$. In order to illustrate this, let us consider an example:

Example 3.1. We take $X = L_2(\mathbb{R}^d)$, $Y = H^{-2}(\mathbb{R}^d)$, $A = \Delta - 1$ with domain $H^2(\mathbb{R}^d)$ and $B = \Delta - 1$ with domain $L_2(\mathbb{R}^d)$. Again, we take $f = 0$ and $g = 0$ so that we obtain the system

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= (\Delta - 1)u^\varepsilon, \\ \partial_t v^\varepsilon &= (\Delta - 1)v^\varepsilon, \\ u^\varepsilon(0) = u_0, &\quad v^\varepsilon(0) = v_0. \end{aligned}$$

Now, we take

$$u_0 := \mathscr{F}^{-1}\Big[\xi \mapsto \frac{1}{1+|\xi|^2}\mathbb{1}_{[0,1]^d}(\xi)\Big] \quad \text{and} \quad v_0 := \mathscr{F}^{-1}\big[\xi \mapsto \mathbb{1}_{[0,1]^d}(\xi - \xi_0)\big]$$

for a certain $\xi_0 \in \mathbb{R}^d$. Then we have

$$\begin{aligned} \|u_0\|_{D(A)} &= \|u_0\|_{L_2(\mathbb{R}^d)} + \|(\Delta - 1)u_0\|_{L_2(\mathbb{R}^d)} \\ &\eqsim \|\mathscr{F}^{-1}(1 + |\xi|^2)\mathscr{F}u_0\|_{L_2(\mathbb{R}^d)} = \|\mathbb{1}_{[0,1]^d}\|_{L_2(\mathbb{R}^d)} = 1 \end{aligned}$$

and

$$\begin{aligned} \|v_0\|_{D(B)} &= \|v_0\|_{H^{-2}(\mathbb{R}^d)} + \|(\Delta - 1)v_0\|_{H^{-2}(\mathbb{R}^d)} \\ &\eqsim \|v_0\|_{L_2(\mathbb{R}^d)} = \|\mathbb{1}_{[0,1]^d}(\cdot - \xi_0)\|_{L_2(\mathbb{R}^d)} = 1. \end{aligned}$$

However, it holds that

$$\|u^\varepsilon(t)\|_{L_2(\mathbb{R}^d)} = \|\mathscr{F}^{-1}\mathrm{e}^{-\varepsilon^{-1}(1+|\xi|^2)t}\mathscr{F}u_0\|_{L_2(\mathbb{R}^d)} \geq \mathrm{e}^{-2\varepsilon^{-1}t}\|u_0\|_{L_2(\mathbb{R}^d)}$$

and

$$\|v^\varepsilon(t)\|_{H^{-2}(\mathbb{R}^d)} = \|\mathscr{F}^{-1}\mathrm{e}^{-(1+|\xi|^2)t}\mathscr{F}v_0\|_{H^{-2}(\mathbb{R}^d)} \leq \mathrm{e}^{-|\xi_0|^2 t}\|v_0\|_{H^{-2}(\mathbb{R}^d)}.$$

Hence, $v^\varepsilon(t)$ decays faster in relation to $\|v_0\|_{H^{-2}(\mathbb{R}^d)}$ than $u^\varepsilon(t)$ does in relation to $\|u_0\|_{L_2(\mathbb{R}^d)}$ if $|\xi_0|^2 > 2\varepsilon^{-1}$, even though $\|u_0\|_{D(A)} = \|v_0\|_{D(B)} = 1$.

We also want to point out that norms can be a bad indicator of different time scales in a system. Suppose that B generates a unitary group $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ on a Hilbert space Y and A generates an exponentially stable C_0-semigroup of contractions $(\mathrm{e}^{tA})_{t\geq 0}$ on X. Since $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ is a family of isometric isomorphisms on Y, we obviously have that

$$1 = \|\mathrm{e}^{tB}v_0\|_Y > \|\mathrm{e}^{\varepsilon^{-1}tA}u_0\|_X$$

for all choices of $t > 0$, $v_0 \in Y$ with $\|v_0\|_Y = 1$ and $u_0 \in X$ with $\|u_0\|_X = 1$. But still, the trajectories of $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ can have changes which are much faster than the exponential decay caused by $(\mathrm{e}^{\varepsilon^{-1}tA})_{t\geq 0}$ for certain initial values. Take, for example, $B = \frac{d}{dx}$ on $H^{-1}(\mathbb{R})$ with domain $L_2(\mathbb{R})$. The corresponding group is given by the family of shifts $\mathrm{e}^{tB}v = v(\cdot + t)$. If we take $v_k = \sqrt{k}\mathbb{1}_{[0,k^{-1}]}$, then we have

$$\|v_k\|_{L_2(\mathbb{R})} = 1, \quad \|\mathrm{e}^{k^{-1}B}v_k - v_k\|_{L_2(\mathbb{R})} = \sqrt{2}.$$

Thus, no matter how small $|t|$ is, there will always be an initial value v_0 with $\|v_0\|_{L_2(\mathbb{R})} = 1$ such that $\mathrm{e}^{tB}v_0$ and v_0 have a distance of $\sqrt{2}$.

In principle, the fact that small ε does not provide an intuitive splitting in fast and slow time does not necessarily mean that carrying over the results from the finite to the infinite dimensional setting has to cause problems. However, it shows that both cases are different not only from a technical but also from a conceptual point of view. Looking at the above examples one could even discuss whether using the terminology "fast-slow system" is the most adequate in infinite dimensions as one cannot immediately spot the scale separation from a standard form but we shall nevertheless still use the finite-dimensional terminology as one can then formally refer to the two evolution equations for u^ε and v^ε more easily.

4. General fast-slow systems in infinite dimensions

In Section 3.2 we have seen that the classical notion of normal hyperbolicity is very restrictive in infinite dimensions. Unfortunately, it is not known if or how the Lyapunov–Perron method or Hadamard's graph transform can be carried out without this condition and thus, slow manifolds have not been constructed in a general infinite-dimensional setting so far. The main results of this section, Theorem 4.13 and Corollary 4.15, show that even without the construction of slow manifolds, one can consider the slow flow as a good approximation of the semiflow generated by the fast-slow system. In order to derive these results, we need a weaker version of normal hyperbolicity. The idea behind this condition is that solutions of the fast equation

$$\varepsilon\partial_t u^\varepsilon = Au^\varepsilon + f(u^\varepsilon, v^\varepsilon)$$

should decay unless the contribution of the slow variable v^ε prevents them from doing so. This could be formulated in terms of conditions on the spectrum of $A + \mathrm{D}_x f(x, y)$ or, as we do it later, by the estimate (4.4). For finite-dimensional fast-slow systems, requiring the spectrum of $A + \mathrm{D}_x f(x, y)$ to have an empty intersection with the imaginary axis is equivalent to normal hyperbolicity of the critical manifold. But in infinite dimensions this is clearly not the case, since Section 3.2 shows that classical normal hyperbolicity crucially depends on the operator in the slow variable.

Altogether, one could summarize that in this section we derive weaker results under weaker conditions than classical Fenichel theory. In Section 5 we will then introduce a suitable stronger notion of normal hyperbolicity in infinite dimensions which will suffice to construct slow manifolds. However, this stronger notion will be more restrictive again and there are examples in which we are still forced to rely on the results of Section 4.

4.1. The fast equation. First, we study the equation

$$\begin{aligned} \varepsilon \partial_t u^\varepsilon(t) &= Au^\varepsilon(t) + f(t, u^\varepsilon(t)) \quad (t \in [0,T]), \\ u^\varepsilon(0) &= u_0, \end{aligned} \tag{4.1}$$

under the following assumptions:

(1) $\varepsilon \geq 0$, $T > 0$ are parameters and $u_0 \in X_1 := D(A)$ an initial value which satisfies

$$0 = Au_0 + f(0, u_0) \quad \text{if } \varepsilon = 0.$$

(2) The operator $A\colon X \supset D(A) \to X$ is a closed linear operator on the Banach space X with $D(A)$ being dense in X and with $0 \in \rho(A)$. It generates the C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0} \subset \mathcal{B}(X)$.

(3) We write $(\widetilde{X}_\alpha, A_\alpha)_{\alpha\in[-1,\infty)}$ for the interpolation-extrapolation scale generated by (X, A) and $(X_\alpha)_{\alpha\in[-1,\infty)}$ for a scale of Banach spaces such that the norms $\|\cdot\|_{X_\alpha}$ and $\|\cdot\|_{\widetilde{X}_\alpha}$ are equivalent. Moreover, we take constants $C_A, M_A > 0$, $\omega_A \in \mathbb{R}$ such that

$$\|\mathrm{e}^{tA}\|_{\mathcal{B}(X_1)} \leq M_A \mathrm{e}^{\omega_A t}, \quad \|\mathrm{e}^{tA}\|_{\mathcal{B}(X_\gamma, X_1)} \leq C_A t^{\gamma-1} \mathrm{e}^{\omega_A t} \quad (t > 0),$$

where $\gamma \in (0,1]$ if $(\mathrm{e}^{tA})_{t\geq 0} \subset \mathcal{B}(X)$ is holomorphic and $\gamma = 1$ in the general case.

(4) Take again $\gamma \in (0,1]$ if $(\mathrm{e}^{tA})_{t\geq 0} \subset \mathcal{B}(X)$ is holomorphic and $\gamma = 1$ in the general case. Let $\delta \in [1-\gamma, 1]$. The nonlinearity $f\colon [0,\infty) \times X_\delta \to X$ is continuous and there is an $L_f > 0$ such that

$$\begin{aligned} \|f(t, x_1) - f(t, x_2)\|_{X_\gamma} &\leq L_f \|x_1 - x_2\|_{X_1}, \\ \|f(\cdot, u_1) - f(\cdot, u_2)\|_{C^1([0,t];X_{\delta-1})} &\leq L_f \|u_1 - u_2\|_{C^1([0,t];X_\delta)}, \end{aligned}$$

for all $t \in [0,T]$, $x_1, x_2 \in X_1$ and $u_1, u_2 \in C^1([0,T]; X_\delta)$. Here we assume that

$$\begin{aligned} &f(t,x) \in X_\gamma && \text{for } (t,x) \in [0,T] \times X_1, \\ &f(\cdot, u) \in C^1\big([0,T]; X_{\delta-1}\big) && \text{for } u \in C^1\big([0,T]; X_\delta\big). \end{aligned}$$

(5) We define

$$\omega_f := \omega_A + (2C_A L_f)^{1/\gamma} \Big(\frac{1}{\gamma}\Big)^{(1-\gamma)/\gamma} \quad \text{if } \gamma \in (0,1)$$

and take

$$\omega_f > \omega_A + C_A L_F \quad \text{if } \gamma = 1.$$

According to Remark 2.9 the former definition will not be optimal in most cases, but for the sake of simplicity, we make this choice. However, as the optimal choice for $\gamma = 1$ has a nice representation, we explicitly mention this case. We assume that $\omega_f < 0$, even though it is not necessary for the results in Section 4.1.

We work with these assumptions throughout this subsection.

Remark 4.1. Formally, one has to distinguish the different operators A_α and the corresponding semigroups $(\mathrm{e}^{tA_\alpha})_{t\geq 0}$ for different values of $\alpha \in [-1, \infty)$. However, the difference is not essential for us. So we will in our notation just write A and $(\mathrm{e}^{tA})_{t\geq 0}$ no matter on which X_α we consider them.

Proposition 4.2. (a) *Assume that* $L_f \|A^{-1}\|_{\mathcal{B}(X_{\delta-1}, X_\delta)} < 1$. *Then equation* (4.1) *with* $\varepsilon = 0$ *has a unique solution*

$$u^0 \in C^1\big([0, T]; X_\delta\big).$$

(b) *Equation* (4.1) *with* $\varepsilon > 0$ *has a unique strict solution* u^ε, *i.e., a solution*

$$u^\varepsilon \in C^1\big([0, \infty); X\big) \cap C\big([0, \infty); X_1\big),$$

which satisfies (4.1) *with* $\varepsilon > 0$ *for all* $t \in [0, \infty)$.

Proof. (a) Our assumptions imply that

$$\mathcal{L}\colon C^1\big([0, T]; X_\delta\big) \to C^1\big([0, T]; X_\delta\big), \quad u \mapsto -A^{-1} f(\cdot, u)$$

is a contraction. Since $C^1([0, T]; X_\delta)$ is a Banach space, the assertion follows from Banach's fixed point theorem.

(b) For $\eta \in \mathbb{R}$, let $C_b([0, \infty), \mathrm{e}^{\varepsilon^{-1}\eta t}; X_1)$ be the space of all $u \in C([0, \infty); X_1)$ such that

$$\|u\|_{C_b([0,\infty),\mathrm{e}^{\varepsilon^{-1}\eta t}; X_1)} := \sup_{t\geq 0} \mathrm{e}^{-\varepsilon^{-1}\eta t} \|u(t)\|_{X_1} < \infty.$$

We show that the operator

$$\mathcal{L}(u) := \mathrm{e}^{\varepsilon^{-1} t A} u_0 + \varepsilon^{-1} \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(s, u(s))\, \mathrm{d}s$$

has a unique fixed point in $C_b([0, \infty), \mathrm{e}^{\eta t}; X_1)$ for η large enough. By our assumptions it holds for $\eta > \omega_A$ that

$$\begin{aligned}
&\|\mathcal{L}(u_1) - \mathcal{L}(u_2)\|_{C_b([0,\infty),\mathrm{e}^{\varepsilon^{-1}\eta t}; X_1)} \\
&\quad = \sup_{t\geq 0} \mathrm{e}^{-\varepsilon^{-1}\eta t} \Big\| \varepsilon^{-1} \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} \big(f(s, u_1(s)) - f(s, u_2(s))\big)\, \mathrm{d}s \Big\|_{X_1}
\end{aligned}$$

$$\leq \sup_{t\geq 0} L_f C_A \int_0^t \frac{e^{\varepsilon^{-1}(t-s)(\omega_A-\eta)}}{(t-s)^{1-\gamma}\varepsilon^{\gamma}}\,ds \|u_1-u_2\|_{C_b([0,\infty),e^{\varepsilon^{-1}\eta t};X_1)}$$
$$\leq \frac{L_f C_A \Gamma(\gamma)}{(\eta-\omega_A)^{\gamma}}\|u_1-u_2\|_{C_b([0,\infty),e^{\varepsilon^{-1}\eta t};X_1)},$$

where Γ denotes the gamma function. If even $\eta > (L_f C_A \Gamma(\gamma))^{1/\gamma} + \omega_A$, then $\mathcal{L}$ is a contraction. By Banach's fixed point theorem, it follows that $\mathcal{L}$ has a unique fixed point in $C_b([0,\infty), e^{\varepsilon^{-1}\eta t}; X_1)$. Let u^{ε} be this fixed point. Then we have that

$$u^{\varepsilon}(t) = e^{\varepsilon^{-1}tA}u_0 + \varepsilon^{-1}\int_0^t e^{\varepsilon^{-1}(t-s)A} f(s,u^{\varepsilon}(s))\,ds,$$

and which in turn implies that

$$u^{\varepsilon}(t) = u_0 + \varepsilon^{-1}A\int_0^t u^{\varepsilon}(s)\,ds + \varepsilon^{-1}\int_0^t f(s,u^{\varepsilon}(s))\,ds \quad (t\in[0,\infty)),$$

see for example [23, Proposition 4.1.5]. Hence, it follows that for all $t\geq 0$ we have that

$$\lim_{h\to 0}\frac{u^{\varepsilon}(t+h)-u^{\varepsilon}(t)}{h} = \lim_{h\to 0}\frac{1}{h}\left[\int_t^{t+h}\varepsilon^{-1}Au^{\varepsilon}(s)\,ds + \varepsilon^{-1}\int_t^{t+h} f(s,u^{\varepsilon}(s))\,ds\right]$$
$$= \varepsilon^{-1}Au^{\varepsilon}(t) + \varepsilon^{-1}f(t,u^{\varepsilon}(t)),$$

where to convergence holds in X as $Au^{\varepsilon}, f(\cdot,u^{\varepsilon})\in C([0,\infty);X)$. This shows the assertion. □

Remark 4.3. Note that in the proof of Proposition 4.2 (b) we did not use the estimate

$$\|f(\cdot,u_1)-f(\cdot,u_2)\|_{C^1([0,T];X_{\delta-1})} \leq L_f\|u_1-u_2\|_{C^1([0,T];X_{\delta})} \qquad (u_1,u_2\in C^1([0,T];X_{\delta})),$$

which we assumed to hold for f.

Proposition 4.4. *Consider the situation of Proposition 4.2.*

(a) *Suppose that $L_f\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_{\delta})} < 1$. Let $\varepsilon = 0$ and let u^0 be the solution of* (4.1) *from Proposition 4.2 (a). Then we have the estimate*

$$\|u^0\|_{C^1([0,T];X_{\delta})} \leq \frac{\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_{\delta})}}{1-L_f\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_{\delta})}}\|f(\cdot,0)\|_{C^1([0,T];X_{\delta-1})}.$$

(b) *Let $\varepsilon > 0$ and $\eta > \omega_f$. Then for all $t\geq 0$, we have the estimate*

$$\|u^{\varepsilon}(t)\|_{X_1} \leq 2M_A e^{\varepsilon^{-1}\omega_f t}\|u_0\|_{X_1} + 2C_A\left(\frac{e^{\gamma}}{\gamma^{1-\gamma}} + \Gamma(\gamma)\left|\frac{\eta-\omega_A}{\eta-\omega_f}\right|^{1-\gamma}\right)\frac{\|e^{\varepsilon^{-1}\eta(t-\cdot)}f(\cdot,0)\|_{L_{\infty},([0,t];X_{\gamma})}}{(\eta-\omega_f)^{\gamma}},$$

where u^{ε} denotes the solution of (4.1) *from Proposition 4.2 (b).*

Proof. (a) The assertion follows from

$$\begin{aligned}
&\|u^0\|_{C^1([0,T];X_\delta)} = \|A^{-1} f(\cdot\,, u^0)\|_{C^1([0,T];X_\delta)}\\
&\le \|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)} \|f(\cdot\,, u^0)\|_{C^1([0,T];X_{\delta-1})}\\
&\le \|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)} \big(\|f(\cdot\,, u^0) - f(\cdot\,,0)\|_{C^1([0,T];X_{\delta-1})} + \|f(\cdot\,,0)\|_{C^1([0,T];X_{\delta-1})}\big)\\
&\le \|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)} \big(L_f \|u^0\|_{C^1([0,T];X_\delta)} + \|f(\cdot\,,0)\|_{C^1([0,T];X_{\delta-1})}\big).
\end{aligned}$$

(b) In a first step we assume that $\omega_f < \eta = 0$. For the solution of (4.1) we have the implicit solution formula

$$u^\varepsilon(t) = \mathrm{e}^{\varepsilon^{-1} t A} u_0 + \varepsilon^{-1} \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(s,0)\,\mathrm{d}s + \varepsilon^{-1} \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(s,u^\varepsilon(s)) - f(s,0)\big)\,\mathrm{d}s.$$

Therefore, we obtain

$$\begin{aligned}
\|u^\varepsilon(t)\|_{X_1} &\le \|\mathrm{e}^{\varepsilon^{-1} t A}\|_{\mathcal{B}(X_1)} \|u_0\|_{X_1} + \varepsilon^{-1} \int_0^t \|\mathrm{e}^{\varepsilon^{-1}(t-s)A}\|_{\mathcal{B}(X_\gamma,X_1)} \|f(s,0)\|_{X_\gamma}\,\mathrm{d}s\\
&\quad + L_f \varepsilon^{-1} \int_0^t \|\mathrm{e}^{\varepsilon^{-1}(t-s)A}\|_{\mathcal{B}(X_\gamma,X_1)} \|u^\varepsilon(s)\|_{X_1}\,\mathrm{d}s\\
&\le M_A \mathrm{e}^{\varepsilon^{-1}\omega_A t} \|u_0\|_{X_1} + C_A \int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A (t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\,\mathrm{d}s \|f(\cdot\,,0)\|_{L_\infty([0,t];X_\gamma)}\\
&\quad + C_A L_f \int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A (t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}} \|u^\varepsilon(s)\|_{X_1}\,\mathrm{d}s\\
&= M_A \mathrm{e}^{\varepsilon^{-1}\omega_A t} \|u_0\|_{X_1} + C_A \int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A s}}{\varepsilon^\gamma s^{1-\gamma}}\,\mathrm{d}s \|f(\cdot\,,0)\|_{L_\infty([0,t];X_\gamma)}\\
&\quad + C_A L_f \int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A (t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}} \|u^\varepsilon(s)\|_{X_1}\,\mathrm{d}s.
\end{aligned}$$

Now we choose $t_0 \ge t$ and apply Lemma 2.8 with $p = 2$ together with Corollary 2.5. If $\gamma = 1$, then we apply Lemma 2.8 with p close to 1. Note that

$$t \mapsto \mathrm{e}^{-\varepsilon^{-1}\omega_A t} \int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A s}}{\varepsilon^\gamma s^{1-\gamma}}\,\mathrm{d}s$$

is non-decreasing since $\omega_A < 0$. We get

$$\|u^\varepsilon(t)\|_{X_1} \le 2 M_A \mathrm{e}^{\varepsilon^{-1}\omega_f t} \|u_0\|_{X_1} + 2 C_A \Big(\frac{\mathrm{e}^\gamma}{\gamma^{1-\gamma}} + \Gamma(\gamma)\Big|\frac{\omega_A}{\omega_f}\Big|^{1-\gamma}\Big) \frac{\|f(\cdot\,,0)\|_{L_\infty([0,t_0];X_\gamma)}}{|\omega_f|^\gamma}.$$

Taking $t_0 = t$ yields the assertion for $\omega_f < \eta = 0$. For arbitrary $\omega_f < \eta$, we use the transformation $u^\varepsilon_\eta(t) := \mathrm{e}^{-\varepsilon^{-1}\eta t} u^\varepsilon(t)$. Then u^ε_η satisfies

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon_\eta(t) &= (A-\eta)u^\varepsilon_\eta(t) + \mathrm{e}^{-\varepsilon^{-1}\eta t} f(t, \mathrm{e}^{\varepsilon^{-1}\eta t} u^\varepsilon_\eta(t)) \quad (t \geq 0), \\ u^\varepsilon_\eta(0) &= u_0. \end{aligned}$$

Our previous argument thus implies

$$\begin{aligned} \|u^\varepsilon_\eta(t)\|_{X_1} &\leq 2M_A \mathrm{e}^{\varepsilon^{-1}(\omega_f-\eta)t}\|u_0\|_{X_1} \\ &\quad + 2C_A\Big(\frac{\mathrm{e}^\gamma}{\gamma^{1-\gamma}} + \Gamma(\gamma)\Big|\frac{\eta-\omega_A}{\eta-\omega_f}\Big|^{1-\gamma}\Big)\frac{\|\mathrm{e}^{-\varepsilon^{-1}\eta(\cdot)} f(\,\cdot\,,0)\|_{L_\infty([0,t];X_\gamma)}}{(\eta-\omega_f)^\gamma}. \end{aligned}$$

Multiplying with $\mathrm{e}^{\varepsilon^{-1}\eta t}$ again yields the assertion. □

Proposition 4.5. *Let $\widetilde{f}: X_\delta \to X$ satisfy the same assumptions as f and let $\widetilde{u}^\varepsilon$ be the solution of (4.1) for $\varepsilon > 0$ with f being replaced by $\widetilde{f}$. Let further $\eta > \omega_f$. Then we have the estimate*

$$\begin{aligned} \|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\|_{X_1} &\leq 2C_A\Big(\frac{\mathrm{e}^\gamma}{\gamma^{1-\gamma}} + \Gamma(\gamma)\Big|\frac{\eta-\omega_A}{\eta-\omega_f}\Big|^{1-\gamma}\Big) \\ &\quad \cdot \frac{\sup_{0\leq s\leq t, x\in X_1} \mathrm{e}^{\varepsilon^{-1}\eta(t-s)}\| f(s, \mathrm{e}^{\varepsilon^{-1}\eta s}x) - \widetilde{f}(s, \mathrm{e}^{\varepsilon^{-1}\eta s}x)\|_{X_\gamma}}{(\eta-\omega_f)^\gamma}. \end{aligned}$$

Proof. We only treat the case $\eta = 0$. For the general case, one can use the same transformation as in the proof of Proposition 4.4 (b). Variation of constants yields

$$\begin{aligned} \|u^\varepsilon(t) - \widetilde{u}^\varepsilon(t)\|_{X_1} &\leq \Big\|\varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(s,\widetilde{u}^\varepsilon(s)) - \widetilde{f}(s,\widetilde{u}^\varepsilon(s))\big)\,\mathrm{d}s\Big\|_{X_1} \\ &\quad + \Big\|\varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(s,u^\varepsilon(s)) - f(s,\widetilde{u}^\varepsilon(s))\big)\,\mathrm{d}s\Big\|_{X_1} \\ &\leq C_A\int_0^t \frac{\mathrm{e}^{-\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\,\mathrm{d}s \sup_{0\leq r\leq t_0,\, x\in X_1}\|f(r,x) - \widetilde{f}(r,x)\|_{X_\gamma} \\ &\quad + C_A L_f\int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\|u^\varepsilon(s) - \widetilde{u}^\varepsilon(s)\|_{X_1}\,\mathrm{d}s \\ &\leq C_A\int_0^t \frac{\mathrm{e}^{-\varepsilon^{-1}\omega_A s}}{\varepsilon^\gamma s^{1-\gamma}}\,\mathrm{d}s \sup_{0\leq r\leq t_0,\, x\in X_1}\|f(r,x) - \widetilde{f}(r,x)\|_{X_\gamma} \\ &\quad + C_A L_f\int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\|u^\varepsilon(s) - \widetilde{u}^\varepsilon(s)\|_{X_1}\,\mathrm{d}s, \end{aligned}$$

where $t_0 \geq t$. Applying Lemma 2.8 with $p = 2$ (or p close to 1 if $\gamma = 1$ together with Corollary 2.5 and taking $t_0 = t$ yields the assertion. □

4.2. A modified fast equation. Under the assumptions of Section 4.1, we now consider a modified fast equation

$$\begin{aligned} \varepsilon \partial_t u^{\varepsilon,0}(t) &= Au^{\varepsilon,0}(t) + f(t, u^{\varepsilon,0}(t)) - \varepsilon \partial_t A^{-1} f(t, u^0(t)), \\ u^{\varepsilon,0}(0) &= u_0, \end{aligned} \tag{4.2}$$

where u^0 denotes the solution of (4.1) with $\varepsilon = 0$ from Proposition 4.2 (a). Note that for $\varepsilon = 0$ and u_0 such that $0 = Au_0 + f(0, u_0)$ the equations (4.2) and (4.1) coincide. The reason why we choose $u^{\varepsilon,0}$ as a notation for solutions of (4.2) is that (4.2) inherits properties from (4.1) both with $\varepsilon = 0$ and $\varepsilon > 0$. On the one hand, if u_0 already satisfies $0 = Au_0 + f(0, u_0)$, then u^0 is a solution of (4.2) for all $\varepsilon > 0$. Thus, (4.2) is an ε-dependent extension of (4.1) with $\varepsilon = 0$. On the other hand, Proposition 4.8 shows that solutions of (4.2) and (4.1) with $\varepsilon > 0$ are only an ε-distance away from each other. In this sense, one could say that $u^{\varepsilon,0}$ is just a modified version of u^ε that has been adjusted such that it contains (4.1) with $\varepsilon = 0$ and such that it even approaches the solution u^0. Hence, we choose the notation $u^{\varepsilon,0}$ to emphasize the similarity to both u^0 and u^ε.

Since we work with u^0, we assume that $\|A^{-1}\|_{\mathcal{B}(X_{\delta-1}, X_\delta)} L_f < 1$ in this subsection. Even though it is not necessary for all the results, we will assume $\omega_A < \omega_f < 0$ from now on.

Lemma 4.6. *For all $u_0 \in X_1$ and all $\varepsilon > 0$, there is a unique strict solution of* (4.2)*:*

$$u^{\varepsilon,0} \in C^1\big([0,\infty); X\big) \cap C\big([0,\infty); X_1\big).$$

Proof. Let

$$f_\varepsilon \colon [0,T] \times X_\delta \to X, \quad (t,x) \mapsto f(t,x) - \varepsilon \partial_t A^{-1} f(t, u^0(t)).$$

Since $u^0 \in C^1([0,T]; X_\delta)$ by Proposition 4.2 (a) and since f maps $C^1([0,T]; X_\delta)$ to $C^1([0,T]; X_{\delta-1})$, it follows that $\partial_t A^{-1} f(\cdot, u^0) \in C([0,T]; X_\delta)$ so that f_ε is well-defined. Moreover, we have

$$\|f_\varepsilon(t,x_1) - f_\varepsilon(t,x_2)\|_{X_\gamma} = \|f(t,x_1) - f(t,x_2)\|_{X_\gamma} \le L_f \|x_1 - x_2\|_{X_1}$$

for all $(t,x_1), (t,x_2) \in [0,T] \times X_1$. By Remark 4.3 this suffices to apply Proposition 4.2 with f being replaced by f_ε. □

Proposition 4.7. *Let $u^{\varepsilon,0}$ be the solution of* (4.2) *with $\varepsilon > 0$ and the u^0 solution of* (4.1) *with $\varepsilon = 0$. Then we have the estimate*

$$\|u^{\varepsilon,0}(t) - u^0(t)\|_{X_1} \le 2M_A e^{\varepsilon^{-1}\omega_f t} \|u_0 - u^0(0)\|_{X_1}.$$

Proof. Using variation of constants and integration by parts yields

$$\begin{aligned}
&\|u^{\varepsilon,0}(t) - u^0(t)\|_{X_1} \\
&\leq \Big\| e^{\varepsilon^{-1}tA}u_0 + \int_0^t e^{\varepsilon^{-1}A(t-s)}\big[\varepsilon^{-1}f(s,u^{\varepsilon,0}(s)) - \partial_s A^{-1}f(s,u^0(s))\big]\,ds - u^0(t)\Big\|_{X_1} \\
&= \Big\| e^{\varepsilon^{-1}tA}(u_0 - u^0(0)) + \varepsilon^{-1}\int_0^t e^{\varepsilon^{-1}A(t-s)}\big[f(s,u^{\varepsilon,0}(s)) - f(s,u^0(s))\big]\,ds\Big\|_{X_1} \\
&\leq M_A e^{\varepsilon^{-1}\omega_A t}\|u_0 - u^0(0)\|_{X_1} + C_A L_f \int_0^t \frac{e^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\gamma (t-s)^{1-\gamma}}\|u^{\varepsilon,0}(s) - u^0(s)\|_{X_1}\,ds.
\end{aligned}$$

Now, the assertion follows from Lemma 2.8. □

Proposition 4.8. *Suppose that C_A is chosen such that additionally to the assumptions of Section 4.1 we also have*

$$\|e^{tA}\|_{\mathcal{B}(X_\delta,X_1)} \leq C_A t^{\delta-1}e^{\omega_A t} \quad (t > 0).$$

Let u^ε be the solution of (4.1) and $u^{\varepsilon,0}$ the one of (4.2) for $\varepsilon > 0$ with the same initial data. Then we have the estimate

$$\begin{aligned}
\|u^\varepsilon(t) - u^{\varepsilon,0}(t)\|_{X_1} \leq &\left(\frac{e^\delta}{\delta^{1-\delta}} + \Gamma(\delta)\left|\frac{\omega_A}{\omega_f}\right|^{1-\delta}\right) \\
&\cdot \frac{C_A\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}}{\big(1 - L_f\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}\big)}\frac{\varepsilon}{|\omega_f|^\delta}\|f(t,0)\|_{C_b^1([0,t];X_{\delta-1})}.
\end{aligned}$$

Proof. Using variation of constants and choosing $t_0 \geq t$ yields that

$$\begin{aligned}
&\|u^\varepsilon(t) - u^{\varepsilon,0}(t)\|_{X_1} \\
&\quad= \Big\|\varepsilon^{-1}\int_0^t e^{\varepsilon^{-1}(t-s)A}\big[f(s,u^\varepsilon(s)) - f(s,u^{\varepsilon,0}(s))\big]\,ds \\
&\qquad - \int_0^t e^{\varepsilon^{-1}(t-s)A}\partial_s A^{-1}f(s,u^0(s))\,ds\Big\|_{X_1} \\
&\quad\leq C_A\varepsilon\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}\|\partial_t f(\cdot,u^0)\|_{L_\infty([0,t_0];X_{\delta-1})}\int_0^t \frac{e^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\delta (t-s)^{1-\delta}}\,ds \\
&\qquad + CL_f\int_0^t \frac{e^{-\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^\gamma(t-s)^{1-\gamma}}\|u^\varepsilon(s) - u^{\varepsilon,0}(s)\|_{X_1}\,ds.
\end{aligned}$$

Thus, a combination of Lemma 2.8 and Corollary 2.5 shows that

$$\begin{aligned}
\|u^\varepsilon(t) - u^{\varepsilon,0}(t)\|_{X_1} \leq &\left(\frac{e^\delta}{\delta^{1-\delta}} + \Gamma(\delta)\left|\frac{\omega_A}{\omega_f}\right|^{1-\delta}\right) \\
&\cdot \frac{C_A\varepsilon\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}}{\omega_f^\delta}\|\partial_t f(\cdot,u^0)\|_{L_\infty([0,t_0];X_{\delta-1})}.
\end{aligned}$$

Moreover, it follows from Proposition 4.4 (a) that

$$\begin{aligned}\|\partial_t f(\cdot,u^0)\|_{L_\infty([0,t_0];X_{\delta-1})} &\leq \|f(\cdot,u^0)\|_{C_b^1([0,t_0];X_{\delta-1})}\\ &\leq \|f(\cdot,0)\|_{C_b^1([0,t_0];X_{\delta-1})} + L_f\|u^0\|_{C_b^1([0,t_0];X_\delta)}\\ &\leq \frac{1}{1-\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}L_f}\|f(t,0)\|_{C_b^1([0,t_0];X_{\delta-1})},\end{aligned}$$

so that

$$\begin{aligned}\|u^\varepsilon(t)-u^{\varepsilon,0}(t)\|_{X_1} \leq \Big(\frac{e^\delta}{\delta^{1-\delta}}+\Gamma(\delta)\Big|\frac{\omega}{\omega_f}\Big|^{1-\delta}\Big)\\ \cdot\frac{C_A\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}}{\big(1-L_f\|A^{-1}\|_{\mathcal{B}(X_{\delta-1},X_\delta)}\big)}\frac{\varepsilon}{|\omega_f|^\delta}\|f(t,0)\|_{C_b^1([0,t];X_{\delta-1})}.\end{aligned}$$

This completes the proof. □

4.3. Well-posedness of the full system. Now we consider the nonlinear fast-slow system

$$\begin{aligned}\varepsilon\partial_t u^\varepsilon(t) &= Au^\varepsilon(t)+f(u^\varepsilon(t),v^\varepsilon(t)),\\ \partial_t v^\varepsilon(t) &= Bv^\varepsilon(t)+g(u^\varepsilon(t),v^\varepsilon(t)),\quad (t\in[0,T]),\\ u^\varepsilon(0) &= u_0,\quad v^\varepsilon(0)=v_0.\end{aligned}\tag{4.3}$$

We assume that:

(i) X, Y are Banach spaces, $\varepsilon \geq 0$, $T>0$ are parameters and $u_0 \in X_1 = D(A)$, $v_1 \in Y_1 = D(B)$ are initial values. If $\varepsilon = 0$, then u_0 has to satisfy $0 = Au_0 + f(u_0, v_0)$.

(ii) The closed linear operator $A\colon X \supset D(A) \to X$ generates an exponentially stable C_0-semigroup $(e^{tA})_{t\geq 0} \subset \mathcal{B}(X)$. The closed linear operator $B\colon Y \supset D(B) \to Y$ is the generator of a C_0-semigroup $(e^{tB})_{t\geq 0} \subset \mathcal{B}(Y)$.

(iii) The interpolation-extrapolation scales generated by (X, A) and (Y, B) are, up to equivalence of norms for each fixed $\alpha \in [-1,\infty)$, given by $(X_\alpha)_{\alpha\in[-1,\infty)}$ and $(Y_\alpha)_{\alpha\in[-1,\infty)}$. If $0 \notin \rho(B)$, then $(Y_\alpha)_{\alpha\in[-1,\infty)}$ shall be equivalent to the interpolation-extrapolation scale generated by $B-\lambda$ for some $\lambda \in \rho(B)$.

(iv) Let $\gamma_X \in (0,1]$ if $(e^{tA})_{t\geq 0} \subset \mathcal{B}(X)$ is holomorphic and $\gamma_X = 1$, otherwise. In addition, we choose $\delta_X \in [1-\gamma_X, 1]$. Let further $\delta_Y \in (0,1]$ if $(e^{tB})_{t\geq 0} \subset \mathcal{B}(Y)$ is holomorphic and $\delta_Y = 1$, otherwise. The nonlinearities $f\colon X_{\delta_X} \times Y_{1-\delta_X} \to X$ and $g\colon X_1 \times Y_1 \to Y_{\delta_Y}$ are continuous and there are constants $L_f, L_g > 0$ such that

$$\begin{aligned}\|f(x_1,y_1)-f(x_2,y_2)\|_{\gamma_X} &\leq L_f\big(\|x_1-x_2\|_{X_1}+\|y_1-y_2\|_{Y_1}\big),\\ \|f(u_1,v_1)-f(u_2,v_2)\|_{C^1([0,t];X_{\delta_X-1})} &\leq L_f\big(\|u_1-u_2\|_{C^1([0,t];X_{\delta_X})}\\ &\qquad + \|v_1-v_2\|_{C^1([0,t];Y)}\big),\\ \|g(x_1,y_1)-g(x_2,y_2)\|_{\delta_Y} &\leq L_g\big(\|x_1-x_2\|_{X_1}+\|y_1-y_2\|_{Y_1}\big)\end{aligned}$$

for all $x_1, x_2 \in X_1$, $y_1, y_2 \in Y_1$, $t > 0$, $u_1, u_2 \in C^1([0,t]; X_{\delta_X})$, and all $v_1, v_2 \in C^1([0,t]; Y) \cap C([0,t]; Y_{1-\delta_X})$. Here, we assume that

$$f(x,y) \in X_{\gamma_X},\ g(x,y) \in Y_{\delta_Y} \quad \text{if } (x,y) \in X_1 \times Y_1,$$

as well as

$$f(u,v) \in C^1\big([0,t]; X_{\delta_X - 1}\big) \qquad \text{if } (u,v) \in C^1\big([0,t]; X_{\delta_X} \times Y\big) \text{ and } v \in C\big([0,t]; Y_{1-\delta_X}\big).$$

(v) We assume that $f(0,0) = 0$ and $g(0,0) = 0$.

(vi) We choose constants $M_A, M_B, C_A, C_B > 0$, $\omega_A < 0$ and $\omega_B \in \mathbb{R}$ such that

$$\|e^{tA}\|_{\mathcal{B}(X_1)} \le M_A e^{\omega_A t}, \quad \|e^{tA}\|_{\mathcal{B}(X_{\gamma_X}, X_1)} \le C_A t^{\gamma_X - 1} e^{\omega_A t},$$
$$\|e^{tA}\|_{\mathcal{B}(X_{\delta_X}, X_1)} \le C_A t^{\delta_X - 1} e^{\omega_A t}$$

and

$$\|e^{tB}\|_{\mathcal{B}(Y_1)} \le M_B e^{\omega_B t}, \quad \|e^{tB}\|_{\mathcal{B}(Y_{\delta_Y}, Y_1)} \le C_B t^{\delta_Y - 1} e^{\omega_B t}$$

hold for all $t > 0$.

(vii) Again we define $\omega_f := \omega_A + (2C_A L_f)^{1/\gamma_X} (1/\gamma_X)^{(1-\gamma_X)/\gamma_X}$ if $\gamma_X \in (0,1)$ and take $\omega_f > \omega_A + C_A L_F$ if $\gamma_X = 1$. Even though it is not necessary for all the results, we will assume

$$\begin{aligned} &\omega_f < 0, \\ &L_f \max\big\{\|A^{-1}\|_{\mathcal{B}(X_{\gamma_X}, X_1)}, \|A^{-1}\|_{\mathcal{B}(X_{\delta_X - 1}, X_{\delta_X})}\big\} < 1 \end{aligned} \tag{4.4}$$

in the following. Note that A^{-1} exists as a consequence of the Hille–Yosida theorem, since A generates an exponentially stable C_0-semigroup. Recall that as described at the beginning of Section 4 this is a weak version of normal hyperbolicity, as it ensures that solutions of the fast equation would decay exponentially if there was no influence of the slow variable v^ε in the fast equation.

Note that assumption (v) can in practice very frequently be ensured locally by just moving the point of interest on the critical manifold via a coordinate transformation to the origin and using Taylor expansion, so it is not really a restriction. We work with all the above assumptions for the rest of this paper. Since we also assume global Lipschitz conditions on the nonlinearities, we obtain the following well-posedness results:

Proposition 4.9. (a) *Let $\varepsilon = 0$. Then* (4.3) *has a unique strict solution*

$$(u^0, v^0) \in C^1\big([0,T]; X \times Y\big) \cap C\big([0,T]; X_1 \times Y_1\big).$$

(b) *Let $\varepsilon > 0$. Then* (4.3) *has a unique strict solution*

$$(u^\varepsilon, v^\varepsilon) \in C^1\big([0,T]; X \times Y\big) \cap C\big([0,T]; X_1 \times Y_1\big).$$

Proof. (a) Let $y \in Y_1$. By assumption, it holds that

$$f_y \colon X_{\delta_X} \to X, \quad x \mapsto f_y(x) := f(x, y)$$

is continuous and satisfies

$$\|f_y(x_1) - f_y(x_2)\|_{X_{\gamma_X}} = \|f(x_1, y) - f(x_2, y)\|_{X_{\gamma_X}} \le L_f \|x_1 - x_2\|_{X_1}.$$

Since we assume $\|A^{-1}\|_{\mathcal{B}(X_{\gamma_X}, X_1)} L_f < 1$ it follows from Banach's fixed point theorem that there is a unique solution $x \in X_1$ of

$$0 = Ax + f_y(x).$$

In the following we write $h^0(y)$ for this solution. Given $y_1, y_2 \in Y_1$ it holds that

$$\begin{aligned}\|h^0(y_1) - h^0(y_2)\|_{X_1} &= \|A^{-1} f(h^0(y_1), y_1) - A^{-1} f(h^0(y_2), y_2)\|_{X_1} \\ &\le L_f \|A^{-1}\|_{\mathcal{B}(X_{\gamma_X}, X_1)} \big(\|h^0(y_1) - h^0(y_2)\|_{X_1} + \|y_1 - y_2\|_{Y_1}\big),\end{aligned}$$

and thus

$$\|h^0(y_1) - h^0(y_2)\|_{X_1} \le \frac{1}{1 - L_f \|A^{-1}\|_{\mathcal{B}(X_{\gamma_X}, X_1)}} \|y_1 - y_2\|_{Y_1}.$$

Therefore, the mapping

$$Y_1 \to Y_{\delta_Y}, \quad y \mapsto g(h^0(y), y)$$

is continuous. Moreover, we have the estimate

$$\begin{aligned}&\|g(h^0(y_1), y_1) - g(h^0(y_2), y_2)\|_{Y_{\delta_Y}} \\ &\qquad \le \left(\frac{L_g}{1 - L_f \|A^{-1}\|_{\mathcal{B}(X_{\gamma_X}, X_1)}} + L_g\right) \|y_1 - y_2\|_{Y_1}.\end{aligned}$$

Therefore, it follows from Proposition 4.2 (b) together with Remark 4.3 with $\delta = 1$ and $\gamma = \delta_Y$ that there is a unique strict solution

$$v^0 \in C^1\big([0, T]; Y\big) \cap C\big([0, T]; Y_1\big)$$

of the equation

$$\partial_t v^0(t) = B v^0(t) + g(h^0(v^0(t)), v^0(t)), \quad v^0(0) = v_0.$$

Now we take $u^0(t) := h^0(v^0(t))$, i.e., we have that

$$u^0(t) = A^{-1} f(u^0(t), v^0(t)).$$

Proposition 4.2 (a) shows that $u^0 \in C^1([0,T]; X_{\delta_X}) \subset C^1([0,T]; X)$. Moreover, since $h^0\colon Y_1 \to X_1$ is Lipschitz continuous, it follows that $u^0 \in C([0,T]; X_1)$. Altogether, it follows that

$$(u^0, v^0) = (h^0(v^0), v^0) \in C^1\big([0,T]; X \times Y\big) \cap C\big([0,T]; X_1 \times Y_1\big)$$

is the unique solution of (4.3) with $\varepsilon = 0$.

(b) The proof is similar to the one of Proposition 4.2 (b). This time, for some $\eta \in \mathbb{R}$ we consider the space $C_b([0,\infty), \mathrm{e}^{\eta t}; X_1 \times Y_1)$ of all $(u,v) \in C([0,\infty); X_1 \times Y_1)$ such that

$$\|(u,v)\|_{C_b([0,\infty),\mathrm{e}^{\eta t}; X_1\times Y_1)} := \sup_{t\geq 0} \mathrm{e}^{-\eta t}\big(\|u(t)\|_{X_1} + \|v(t)\|_{Y_1}\big) < \infty.$$

On this space, we define the operator $\mathcal{L}$ by

$$[\mathcal{L}(u,v)](t) := \begin{pmatrix} \mathrm{e}^{\varepsilon^{-1}tA}u_0 + \varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(u(s),v(s))\,\mathrm{d}s \\ \mathrm{e}^{tB}v_0 + \int_0^t \mathrm{e}^{(t-s)B} g(u(s),v(s))\,\mathrm{d}s \end{pmatrix}.$$

We show that this operator is a contraction on $C_b([0,\infty), \mathrm{e}^{\eta t}; X_1 \times Y_1)$ if η is large enough. We have that

$$\begin{aligned}
&\sup_{t\geq 0} \mathrm{e}^{-\eta t}\varepsilon^{-1}\Big\|\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big[f(u_1(s),v_1(s)) - f(u_2(s),v_2(s))\big]\,\mathrm{d}s\Big\| \\
&\quad\leq L_f C_A \sup_{t\geq 0}\int_0^t \frac{\mathrm{e}^{(t-s)(\varepsilon^{-1}\omega_A - \eta)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}}\,\mathrm{d}s\,\|(u_1,v_1)-(u_2,v_2)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)} \\
&\quad\leq \frac{L_f C_A \Gamma(\gamma_X)}{(\varepsilon\eta - \omega_A)^{\gamma_X}}\|(u_1,v_1)-(u_2,v_2)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\sup_{t\geq 0} \mathrm{e}^{-\eta t}\Big\|\int_0^t \mathrm{e}^{(t-s)B}\big[g(u_1(s),v_1(s)) - g(u_2(s),v_2(s))\big]\,\mathrm{d}s\Big\| \\
&\quad\leq L_g C_B \sup_{t\geq 0}\int_0^t \frac{\mathrm{e}^{(t-s)(\omega_B - \eta)}}{(t-s)^{1-\delta_Y}}\,\mathrm{d}s\,\|(u_1,v_1)-(u_2,v_2)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)} \\
&\quad\leq \frac{L_g C_B \Gamma(\delta_Y)}{(\eta - \omega_B)^{\delta_Y}}\|(u_1,v_1)-(u_2,v_2)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\|[\mathcal{L}(u,v)](t)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)} \\
&\quad\leq \left(\frac{L_f C_A \Gamma(\gamma_X)}{(\varepsilon\eta - \omega_A)^{\gamma_X}} + \frac{L_g C_B \Gamma(\delta_Y)}{(\eta - \omega_B)^{\delta_Y}}\right)\|(u_1,v_1)-(u_2,v_2)\|_{C_b([0,\infty),\mathrm{e}^{\eta t};X_1\times Y_1)}.
\end{aligned}$$

In particular, if η is large enough then $\mathcal{L}$ is a contraction. Thus, there is a unique fixed point $(u^\varepsilon, v^\varepsilon) \in C_b([0,\infty), \mathrm{e}^{\eta t}; X_1 \times Y_1)$. By the same line of arguments as in the proof of Proposition 4.2 (b) it now follows that

$$(u^\varepsilon, v^\varepsilon) \in C^1\big([0,T]; X \times Y\big) \cap C\big([0,T]; X_1 \times Y_1\big)$$

and that it solves (4.3) with $\varepsilon > 0$. □

Remark 4.10. (a) In the proof of Proposition 4.9 we introduced the mapping

$$h^0 \colon Y_1 \to X_1, \quad y \mapsto h^0(y),$$

where $h^0(y)$ is the unique solution of

$$0 = Ah^0(y) + f(h^0(y), y).$$

In particular, this mapping describes the critical manifold S_0 over Y_1 by

$$S_0 := \{(h^0(y), y) : y \in Y_1\} \subset X \times Y.$$

Note that Proposition 4.2 (a) shows that if $v^0 \in C^1([0,T]; Y) \cap C([0,T]; Y_1)$, then $h^0(v^0) \in C^1([0,T]; X_{\delta_X})$.

(b) Since (4.3) is autonomous, the solutions (u^0, v^0) and $(u^\varepsilon, v^\varepsilon)$ are given by semiflows, i.e., continuous mappings

$$T_\varepsilon \colon [0,T] \times X_1 \times Y_1 \to X_1 \times Y_1, \quad T_0 \colon [0,T] \times S_0 \to S_0.$$

We write

$$\begin{pmatrix} u^\varepsilon(t) \\ v^\varepsilon(t) \end{pmatrix} = T_\varepsilon(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad \begin{pmatrix} u^0(t) \\ v^0(t) \end{pmatrix} = T_0(t) \begin{pmatrix} h^0(v_0) \\ v_0 \end{pmatrix}.$$

4.4. Extended slow flow. One of our aims is to show that the semiflow of the fast-slow system $(T_\varepsilon(t))_{t\geq 0}$ behaves similarly to the slow flow $(T_0(t))_{t\geq 0}$. However, the slow flow is only defined on the critical manifold S_0, while $(T_\varepsilon(t))_{t\geq 0}$ is defined on $X_1 \times Y_1$. Thus, we will compare $(T_\varepsilon(t))_{t\geq 0}$ to an extension $(T_{\varepsilon,0}(t))_{t\geq 0}$ of the slow flow to $X_1 \times Y_1$. This extension will approach the slow flow at an exponential rate and on the critical manifold it will coincide with the slow flow. This extended flow will be generated by the equation

$$\begin{aligned} \varepsilon\partial_t u^{\varepsilon,0}(t) &= Au^{\varepsilon,0}(t) + f(u^{\varepsilon,0}(t), v^0(t)) - \varepsilon\partial_t A^{-1} f(h^0(v^0(t)), v^0(t)), \\ \partial_t v^0(t) &= Bv^0(t) + g(h^0(v^0(t)), v^0(t)), \\ u^{\varepsilon,0}(0) &= u_0, \quad v^0(0) = v_0. \end{aligned} \tag{4.5}$$

In this equation, the slow variable satisfies the equation of the slow subsystem. The fast variable however satisfies the equation of the fast-slow system with an additional drift in the direction of the slow flow.

Proposition 4.11. *There is a unique solution*

$$(u^{\varepsilon,0}, v^0) \in C^1\big([0,T]; X \times Y\big) \cap C\big([0,T]; X_1 \times Y_1\big)$$

of (4.5) *given by a semiflow* $(T_{\varepsilon,0}(t))_{t\geq 0}$ *on* $X_1 \times Y_1$. *The critical manifold* S_0 *is invariant under* $T_{\varepsilon,0}(t)$ *for all* $t \geq 0$. *Moreover, the restriction of* $(T_{\varepsilon,0}(t))_{t\geq 0}$ *to the critical manifold coincides with the slow flow, i.e.,* $(T_{\varepsilon,0}(t)|_{S_0})_{t\geq 0} = (T_0(t))_{t\geq 0}$.

Proof. In the proof of Proposition 4.9 (a) it was shown that there is a unique solution

$$v^0 \in C^1\big([0,T]; Y\big) \cap C\big([0,T]; Y_1\big)$$

of the equation

$$\partial_t v^0(t) = B v^0(t) + g(h^0(v^0(t)), v^0(t)), \quad v^0(0) = v_0$$

for all $v_0 \in Y_1$. We define

$$f_{\varepsilon,v^0} \colon [0,T] \times X_{\delta_X} \to X, \quad x \mapsto f(x, v^0(t)) - \varepsilon \partial_t A^{-1} f(h(v^0(t)), v^0(t)).$$

Since $v^0 \in C^1([0,T]; Y) \cap C([0,T]; Y_1)$, it follows from Remark 4.10 (a) that

$$[0,T] \times X_{\delta_X} \to X, \quad (t,x) \mapsto A^{-1} \partial_t f(h^0(v^0(t)), v^0(t)),$$

and therefore f_{ε,v^0} is also continuous. Moreover, we have the estimate

$$\begin{aligned} \|f_{\varepsilon,v^0}(t,x_1) - f_{\varepsilon,v^0}(t,x_2)\|_{X_{\gamma_X}} &= \|f(x_1, v^0(t)) - f(x_2, v^0(t))\|_{X_{\gamma_X}} \\ &\leq L_f \|x_1 - x_2\|_{X_1}. \end{aligned}$$

Now Proposition 4.2 (b) together with Remark 4.3 shows that there is a unique solution $u^{\varepsilon,0} \in C^1([0,T]; X) \cap C([0,T]; X_1)$ of

$$\begin{aligned} &\varepsilon \partial_t u^{\varepsilon,0}(t) = A u^{\varepsilon,0}(t) + f(u^{\varepsilon,0}(t), v^0(t)) - \varepsilon \partial_t A^{-1} f(h^0(v^0(t)), v^0(t)), \\ &\quad u^{\varepsilon,0}(0) = u_0. \end{aligned}$$

The desired solution is given by $(u^{\varepsilon,0}, v^0)$. Since (4.5) is autonomous, the solution is given by a semiflow $(T_{\varepsilon,0}(t))_{t\geq 0}$. Note that if $(u_0, v_0) \in S_0$, then the slow flow with initial value v_0 solves (4.5). Therefore, the critical manifold is invariant under $T_{\varepsilon,0}(t)$ for all $t \geq 0$ and $(T_{\varepsilon,0}(t))_{t\geq 0}$ coincides with $(T_0(t))_{t\geq 0}$ on the critical manifold. □

Proposition 4.12. *For all* $t \geq 0$ *it holds that*

$$\left\| T_{\varepsilon,0}(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - T_0(t) \begin{pmatrix} h^0(v_0) \\ v_0 \end{pmatrix} \right\|_{X_1 \times Y_1} \leq 2 M_A \mathrm{e}^{\varepsilon^{-1} \omega_f t} \|u_0 - h^0(v_0)\|_{X_1}.$$

Proof. Since the second components of $T_{\varepsilon,0}(t)(u_0, v_0)^T$ and $T_0(t)(h^0(v_0), v_0)^T$ are equal, we only have to estimate $\|u^{\varepsilon,0}(t) - u^0(t)\|_{X_1}$. But it was shown in Proposition 4.7 that

$$\|u^{\varepsilon,0}(t) - u^0(t)\|_{X_1} \leq 2 M_A \mathrm{e}^{\varepsilon^{-1} \omega_f t} \|u_0 - h^0(v_0)\|_{X_1}.$$

This shows the assertion. □

4.5. Approximation by the slow flow.

Theorem 4.13. *There are constants $C, c > 0$ such that*

$$\left\| T_\varepsilon(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - T_{\varepsilon,0}(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right\|_{X_1\times Y_1} \leq C\mathrm{e}^{(\omega_B+c)t}\big(\varepsilon\|v_0\|_{Y_1} + \varepsilon^{\delta_Y}\|u_0 - h^0(v_0)\|_{X_1}\big)$$

holds for all $(u_0, v_0)^T \in X_1 \times Y_1$, all $t \geq 0$ and all $\varepsilon \in (0, 1]$.

Remark 4.14. Before we turn to the proof we briefly give a rough idea of how large C and c have to be. Actually, we have all the ingredients to explicitly give formulas for these constants and we could also give them by keeping track of the constants in the proof of Theorem 4.13. However, these formulas would be quite involved and probably not sharp. Thus, we refrain from giving precise constants here.

The constant $C > 0$ should not be very large unless δ_Y, γ_X or ω_f are close to 0. If either of these values tends to 0, then C will tend to ∞. C is basically constructed from the constants which were explicitly computed in Proposition 4.4 (b) (with $\varepsilon = 1$ and $\gamma = \delta_Y$), Proposition 4.8 and Proposition 4.7.

For c we are a little bit more precise, even though our rough estimate for c can probably still be improved: The constant c can be taken to be

$$\begin{aligned} c &= 1 + 2(L_g C_B)^{1/\delta_Y}\big[(L+1)^{1/\delta_Y} + (1 + C_1 L_f)^{1/\delta_Y}\big]\Big(\frac{1}{\delta_Y}\Big)^{\frac{1-\delta_Y}{\delta_y}} && \text{if } \delta_Y \in (0,1), \\ c &> 1 + L_g C_B(2 + L + C_1 L_f) && \text{if } \delta_Y = 1, \end{aligned}$$

where C_1 is given by

$$C_1 = 2C_A\left(\frac{\mathrm{e}^{\gamma_X}}{\gamma_X^{1-\gamma_X}} + \Gamma(\gamma_X)\left|\frac{\omega_A}{\omega_f}\right|^{1-\gamma_X}\right)\frac{1}{|\omega_f|^{\gamma_X}},$$

and where L denotes the Lipschitz constant of the critical manifold.

Proof of Theorem 4.13. In this proof, we use the notation

$$\begin{pmatrix} u^\varepsilon(t) \\ v^\varepsilon(t) \end{pmatrix} = T_\varepsilon(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad \begin{pmatrix} u^{\varepsilon,0}(t) \\ v^0(t) \end{pmatrix} = T_{\varepsilon,0}(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Variation of constants shows that

$$\begin{aligned} v^\varepsilon(t) &= \mathrm{e}^{tB}v_0 + \int_0^t \mathrm{e}^{(t-s)B} g(u^\varepsilon(s), v^\varepsilon(s))\,\mathrm{d}s, \\ v^0(t) &= \mathrm{e}^{tB}v_0 + \int_0^t \mathrm{e}^{(t-s)B} g(h^0(v^0(s)), v^0(s))\,\mathrm{d}s. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} &\|v^\varepsilon(t) - v^0(t)\|_{Y_1} \\ &\leq L_g C_B \int_0^t \frac{\mathrm{e}^{(t-s)\omega_B}}{(t-s)^{1-\delta_Y}}\big(\|u^\varepsilon(s) - h^0(v^0(s))\|_{X_1} + \|v^\varepsilon(t) - v^0(t)\|_{Y_1}\big)\,\mathrm{d}s. \quad (4.6) \end{aligned}$$

The aim is to apply Gronwall's inequality. But before we do this, we first estimate the term $\|u^\varepsilon(s) - h^0(v^0(s))\|_{X_1}$. Let $\widetilde{u}^\varepsilon$ be the unique strict solution of

$$\begin{aligned} \varepsilon\partial_t \widetilde{u}^\varepsilon &= A\widetilde{u}^\varepsilon + f(\widetilde{u}^\varepsilon, v^0), \\ \widetilde{u}^\varepsilon(0) &= u_0, \end{aligned}$$

which exists by Proposition 4.2 (b). By the triangle inequality, we have

$$\begin{aligned} \|u^\varepsilon(s) - h^0(v^0(s))\|_{X_1} \leq \|u^\varepsilon(s) - \widetilde{u}^\varepsilon(s)\|_{X_1} + \|\widetilde{u}^\varepsilon(s) - u^{\varepsilon,0}(s)\|_{X_1} \\ + \|u^{\varepsilon,0}(s) - h^0(v^0(s))\|_{X_1}. \end{aligned}$$

Using Proposition 4.5 with $\eta = 0$ we obtain that there is a constant $C_1 > 0$ such that

$$\begin{aligned} \|u^\varepsilon(s) - \widetilde{u}^\varepsilon(s)\|_{X_1} &\leq C_1 \sup_{0\leq r\leq s,\, x\in X_1} \|f(x, v^\varepsilon(r)) - f(x, v^0(r))\|_{X_\gamma} \\ &\leq C_1 L_f \|v^\varepsilon(t) - v^0(t)\|_{Y_1}. \end{aligned}$$

Proposition 4.8 and Proposition 4.4 (b) show that there are constants $C_2, \widetilde{C}_2 \geq 0$ such that

$$\begin{aligned} \|\widetilde{u}^\varepsilon(s) - u^{\varepsilon,0}(s)\|_{X_1} &\leq \widetilde{C}_2 \varepsilon \|f(0, v^0)\|_{C^1([0,s];X_{\delta_X - 1})} \\ &\leq \widetilde{C}_2 L_f \varepsilon \|v^0\|_{C^1([0,s];Y)} \\ &\leq C_2 \varepsilon \mathrm{e}^{\omega_g s} \|v_0\|_{Y_1}, \end{aligned}$$

where

$$\begin{aligned} \omega_g &= \omega_B + (2C_B L_g(L+1))^{1/\delta_Y} \left(\frac{1}{\delta_Y}\right)^{(1-\delta_Y)/\delta_Y} && \text{if } \delta_Y \in (0,1), \\ \omega_g &> \omega_B + C_B L_g(L+1) && \text{if } \delta_Y = 1. \end{aligned} \tag{4.7}$$

Moreover, Proposition 4.7 yields

$$\|u^{\varepsilon,0}(s) - h^0(v^0(s))\|_{X_1} \leq 2M_A \mathrm{e}^{\varepsilon^{-1}\omega_f s} \|u_0 - h^0(v_0)\|_{X_1}.$$

By combining the previous four estimates with (4.6), we obtain that there is a constant $C > 0$ not depending on ω_B, u_0, v_0 and ε such that

$$\begin{aligned} &\|v^\varepsilon(t) - v^0(t)\|_{Y_1} \\ &\quad\leq C \int_0^t \frac{\mathrm{e}^{(t-s)\omega_B}}{(t-s)^{1-\delta_Y}} \big(\varepsilon \mathrm{e}^{\omega_g s} \|v_0\|_{Y_1} + \mathrm{e}^{\varepsilon^{-1}\omega_f s} \|u_0 - h^0(v_0)\|_{X_1}\big)\, \mathrm{d}s \\ &\qquad + L_g C_B (1 + C_1 L_f) \int_0^t \frac{\mathrm{e}^{(t-s)\omega_B}}{(t-s)^{1-\delta_Y}} \|v^\varepsilon(s) - v^0(s)\|_{Y_1}\, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\leq C\mathrm{e}^{\omega_g t}\int_0^t \frac{1}{(t-s)^{1-\delta_Y}}\big(\varepsilon\|v_0\|_{Y_1}+\mathrm{e}^{(\varepsilon^{-1}\omega_f-\omega_g)s}\|u_0-h^0(v_0)\|_{X_1}\big)\,\mathrm{d}s\\
&\qquad + L_g C_B(1+C_1L_f)\int_0^t \frac{\mathrm{e}^{(t-s)\omega_B}}{(t-s)^{1-\delta_Y}}\|v^\varepsilon(s)-v^0(s)\|_{Y_1}\,\mathrm{d}s\\
&\leq C\mathrm{e}^{\omega_g t}\left(\frac{t^{\delta_Y}}{\delta_Y}\varepsilon\|v_0\|_{Y_1}+\frac{e+\delta_Y}{\delta_Y(\varepsilon\omega_g-\omega_f)^{\delta_Y}}\varepsilon^{\delta_Y}\|u_0-h^0(v_0)\|_{X_1}\right)\\
&\qquad + L_g C_B(1+C_1L_f)\int_0^t \frac{\mathrm{e}^{(t-s)\omega_B}}{(t-s)^{1-\delta_Y}}\|v^\varepsilon(s)-v^0(s)\|_{Y_1}\,\mathrm{d}s\\
&\leq C\mathrm{e}^{(\omega_g+1)t}\big(\varepsilon\|v_0\|_{Y_1}+\varepsilon^{\delta_Y}\|u_0-h^0(v_0)\|_{X_1}\big)\\
&\qquad + L_g C_B(1+C_1L_f)\int_0^t \frac{\mathrm{e}^{(t-s)(\omega_g+1)}}{(t-s)^{1-\delta_Y}}\|v^\varepsilon(s)-v^0(s)\|_{Y_1}\,\mathrm{d}s,
\end{aligned}
$$

where we used Lemma 2.6. Thus, Lemma 2.8 shows that there is a constant $C>0$ not depending on ω_B, u_0, v_0 and ε such that

$$
\|v^\varepsilon(t)-v^0(t)\|_{Y_1}\leq C\mathrm{e}^{(\omega_B+c)t}\big(\varepsilon\|v_0\|_{Y_1}+\varepsilon^{\delta_Y}\|u_0-h^0(v_0)\|_{X_1}\big)\quad(t\geq 0),
$$

where

$$
\begin{aligned}
&c=1+2(L_gC_B)^{\frac{1}{\delta_Y}}\big[(L+1)^{1/\delta_Y}+(1+C_1L_f)^{1/\delta_Y}\big]\Big(\frac{1}{\delta_Y}\Big)^{(1-\delta_Y)/\delta_Y} && \text{if } \delta_Y\in(0,1),\\
&c>1+L_gC_B(2+L+C_1L_f) && \text{if } \delta_Y=1.
\end{aligned}
$$

Using this estimate for the slow variable, Proposition 4.5 and Proposition 4.8 we also obtain for the fast variable

$$
\begin{aligned}
\|u^\varepsilon(t)-u^{\varepsilon,0}(t)\|_{X_1}&\leq \|u^\varepsilon(t)-\tilde{u}^\varepsilon(t)\|_{X_1}+\|\tilde{u}^\varepsilon(t)-u^{\varepsilon,0}(t)\|_{X_1}\\
&\leq C\mathrm{e}^{(\omega_B+c)t}\big(\varepsilon\|v_0\|_{Y_1}+\varepsilon^{\delta_Y}\|u_0-h^0(v_0)\|_{X_1}\big).
\end{aligned}
$$

Altogether, we obtain the assertion. □

Corollary 4.15. *There are constants $C, c>0$ such that*

$$
\begin{aligned}
&\left\|T_\varepsilon(t)\begin{pmatrix}u_0\\ v_0\end{pmatrix}-T_0(t)\begin{pmatrix}h^0(v_0)\\ v_0\end{pmatrix}\right\|_{X_1\times Y_1}\\
&\qquad\leq C\Big(\varepsilon\mathrm{e}^{(\omega_B+c)t}\|v_0\|_{Y_1}+\big(\varepsilon^{\delta_Y}\mathrm{e}^{(\omega_B+c)t}+\mathrm{e}^{\varepsilon^{-1}\omega_f t}\big)\|u_0-h^0(v_0)\|_{X_1}\Big)
\end{aligned}
$$

holds for all $(u_0,v_0)^T\in X_1\times Y_1$, all $t\in[0,T]$ and all $\varepsilon\in(0,1]$.

Proof. This is a combination of Proposition 4.12 and Theorem 4.13. □

5. Slow manifolds

Under additional assumptions on the operator B in the equation of the slow variable, we now prove the existence of a family of slow manifolds $S_{\varepsilon,\zeta}$. Unlike in finite dimensions, this family will depend on two parameters. While ε plays the same role as in the finite-dimensional setting, the parameter ζ is new. As explained in Section 3 there might be parts of the slow dynamics which decay faster than other parts in the fast equation evolve. Our idea is to find a certain splitting of the slow variable in a fast and a slow part. The fast part of the slow variable will then be treated together with the fast variable, while the slow manifolds are constructed as graphs over the slow part. The parameter ζ determines which parts of the slow variables are considered as fast and which parts are considered as slow. In the language of normally hyperbolic invariant manifolds one could say that the stable direction will consist of the fast variable and the fast part of the slow variable, and the center direction will consist of the slow part of the slow variable. Since the slow part of the slow variable should not contain parts that evolve faster than the fast variable, we naturally obtain a restriction on how we may choose ζ in relation to ε. This will be reflected in the condition $\varepsilon < c_0(\omega_f/\omega_A)\zeta$ for some fixed $c_0 \in (0,1)$ that we impose below.

The finite dimensional situation will also be recovered as a special case: The family of slow manifolds $S_{\varepsilon,\zeta}$ will then not depend on ζ so that one could omit it in the notation and obtain a family S_ε as usual in finite dimensions. More generally, if B generates a C_0-group, then the family of slow manifolds will not depend on ζ. We will give applications of our techniques to systems of fast-slow partial differential equations in Section 6. In the next subsection, we make our assumptions more precise.

5.1. Our approach on how to resolve the issues of Section 3. For the problems explained in Section 3.2 and Section 3.3, we assume that for small $\zeta > 0$ satisfying $\varepsilon < c_0(\omega_f/\omega_A)\zeta$ with a fixed $c_0 \in (0,1)$ we have a splitting of the slow variable space

$$Y = Y_F^\zeta \oplus Y_S^\zeta$$

in a fast part Y_F^ζ and a slow part Y_S^ζ such that

(i) The spaces Y_F^ζ and Y_S^ζ are closed in Y and the projections $\mathrm{pr}_{Y_F^\zeta}$ and $\mathrm{pr}_{Y_S^\zeta}$ commute with B on Y_1.

(ii) The space $Y_F^\zeta \cap Y_1$ is a closed subspace of Y_1 and will be endowed with the norm $\|\cdot\|_{Y_1}$.

(iii) The space $Y_S^\zeta \cap Y_1$ is a closed subspace of Y_1 and will be endowed with the norm $\|\cdot\|_{Y_1}$. Moreover, the nonlinearity g satisfies

$$\begin{aligned}\|\mathrm{pr}_{Y_S^\zeta}\big[g(x,y_F,y_S) - g(\tilde{x},\tilde{y}_F,\tilde{y}_S)\big]\|_{Y_1}\\ \le L_g\zeta^{\delta_Y-1}\big(\|x-\tilde{x}\|_{X_1} + \|y_F-\tilde{y}_F\|_{Y_1} + \|y_S-\tilde{y}_S\|_{Y_1}\big).\end{aligned}$$

(iv) The realization of B in Y_S^ζ, i.e.,

$$B_{Y_S^\zeta}\colon Y_S^\zeta \supset D(B_{Y_S^\varepsilon}) \to Y_S^\zeta, \quad v \mapsto Bv$$

with

$$D(B_{Y_S^\zeta}) := \{v_0 \in Y_S^\zeta \cap D(B) : Bv_0 \in Y_S^\zeta\}$$

generates a C_0-group $(\mathrm{e}^{tB_{Y_S^\zeta}})_{t\in\mathbb{R}} \subset \mathcal{B}((Y_S^\zeta, \|\cdot\|_Y))$, which satisfies $\mathrm{e}^{tB_{Y_S^\zeta}} = \mathrm{e}^{tB}$ on Y_S^ζ for $t \geq 0$.

(v) The realization of B in Y_F^ζ, i.e.,

$$B_{Y_F^\zeta}\colon Y_F^\zeta \supset D(B_{Y_F^\varepsilon}) \to Y_F^\zeta, \quad v \mapsto Bv$$

with

$$D(B_{Y_F^\zeta}) := \{v_0 \in Y_F^\zeta : Bv_0 \in Y_F^\zeta\}$$

has 0 in its resolvent set.

(vi) The space $Y_F^\zeta \cap Y_{\delta_Y}$ contains the parts of Y_{δ_Y} that decay under the semigroup $(\mathrm{e}^{tB})_{t\geq 0}$ almost as fast as the space X_1 under $(\mathrm{e}^{\zeta^{-1}tA})_{t\geq 0}$. The space $Y_S^\zeta \cap Y_1$ contains the parts of Y_1 which do not decay or which only decay slowly under the semigroup $(\mathrm{e}^{tB})_{t\geq 0}$ compared to X_1 under $(\mathrm{e}^{\zeta^{-1}tA})_{t\geq 0}$. More precisely, there are constants $C_B, M_B > 0$ such that for all $\zeta > 0$ small enough there are constants $0 \leq N_F^\zeta < N_S^\zeta < -\zeta^{-1}\omega_A$ such that for all $t \geq 0$, $y_F \in Y_F^\zeta \cap Y_{\delta_Y}$ and $y_S \in Y_S^\zeta \cap Y_1$ we have the estimates

$$\|\mathrm{e}^{tB} y_F\|_{Y_1} \leq C_B \Big(\frac{1}{2} t\zeta\big(N_S^\zeta - N_F^\zeta\big)\Big)^{\delta_Y - 1} \mathrm{e}^{(N_F^\zeta + \zeta^{-1}\omega_A)t} \|y_F\|_{Y_{\delta_Y}}, \tag{5.1}$$

$$\|\mathrm{e}^{-tB} y_S\|_{Y_1} \leq M_B \mathrm{e}^{-(N_S^\zeta + \zeta^{-1}\omega_A)t} \|y_S\|_{Y_1}. \tag{5.2}$$

(vii) We have the estimate

$$\frac{2^{\gamma_X} L_f C_A \Gamma(\gamma_X)}{\big(2(\varepsilon\zeta^{-1} - 1)\omega_A + \varepsilon(N_S^\zeta + N_F^\zeta)\big)^{\gamma_X}} + \frac{2\zeta^{\delta_Y - 1} L_g (C_B \Gamma(\delta_Y) + M_B)}{N_S^\zeta - N_F^\zeta} < 1, \tag{5.3}$$

which will be needed for an application of Banach's fixed point theorem.

These conditions might seem very restrictive at first. However, in many applications it is possible to find such a decomposition. In many cases, it can be obtained by using Riesz projections corresponding to B. This can for example be done if B is a parabolic operator on a bounded domain. If B generates a group, then it will even suffice to take $Y_F^\zeta = \{0\}$ and $Y_S^\zeta = Y$ for small ε. In particular, one can always find such a decomposition if the equation for the slow variable is given by an ordinary differential equation.

Besides the parameters ε and ζ, the quantity $N_S^\zeta - N_F^\zeta$ also plays a certain role. It measures how far one can separate the decay properties of the fast and the slow part in the slow variable. In many situations this number corresponds to size of spectral gaps in the real part of the spectrum of B as one approaches $-\infty$. For example, if B is the Laplace operator Δ on $L_2([0, 2\pi])$ with Dirichlet boundary conditions, then the eigenvalues are of the form $-k^2$. The gaps between two consecutive different eigenvalues will then be given by $2k + 1$, i.e., it will behave almost like the square root of the size of the eigenvalues times a constant. In such a situation, $N_S^\zeta - N_F^\zeta$ will behave like $C\zeta^{-1/2}$ as $\zeta \to 0$. If B generates a group, then it will hold that $N_S^\zeta - N_F^\zeta$ behaves like ζ^{-1}.

We use this splitting to rewrite the fast-slow system (4.3) as

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon(t) &= Au^\varepsilon(t) + f(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t)), \\ \partial_t v_F^\varepsilon(t) &= Bv_F^\varepsilon(t) + \mathrm{pr}_{Y_F^\zeta}\, g(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t)), \\ \partial_t v_S^\varepsilon(t) &= Bv_S^\varepsilon(t) + \mathrm{pr}_{Y_S^\zeta}\, g(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t)), \\ u^\varepsilon(0) = u_0, \quad v_F^\varepsilon(0) &= \mathrm{pr}_{Y_F^\zeta}\, v_0, \quad v_S^\varepsilon(0) = \mathrm{pr}_{Y_S^\zeta}\, v_0, \end{aligned} \qquad (t \in [0, T]) \tag{5.4}$$

with an abuse of notation: Actually, f and g only depend on two variables, but we use the convention

$$f(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t)) := f(u^\varepsilon(t), v_F^\varepsilon(t) + v_S^\varepsilon(t))$$

as well as

$$g(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t)) := g(u^\varepsilon(t), v_F^\varepsilon(t) + v_S^\varepsilon(t)).$$

We should point out that, as already mentioned at the beginning of Section 4, there are also certain situations in which the space of the slow variable does not admit such a splitting. The main example we have in mind is if B is a parabolic operator such as the Laplacian Δ on the whole space $\mathbb{R}^n$. If it is considered on $L_p(\mathbb{R}^n)$, then there are no gaps in the spectrum and it will not be possible to find the constants $0 \leq N_F^\zeta < N_S^\zeta$. In such a situation, we will not be able to construct slow manifolds. If B is a parabolic operator on a bounded domain in dimension $n \geq 3$, then it admits such a splitting, but the spectral gaps will usually not grow as $\zeta \to 0$. This follows for example from Legendre's three-square theorem for a domain such as the n-dimensional torus. In this case, (5.3) will usually not be satisfied, unless the Lipschitz constants of the nonlinearities are small. It should be noted that the case $n = 2$ is different. Here, spectral gaps can indeed become large, but only very slowly, see for example [27]. Nonetheless, even if there are no spectral gaps, we can still use the results of Section 4 to justify that one may reduce the fast-slow system to the slow subsystem.

Remark 5.1. In an earlier version of this work, the estimates in the assumptions were slightly different. Instead of (5.1) it was assumed that

$$\|e^{tB} y_F\|_{Y_1} \leq C_B t^{\delta_Y - 1} e^{(N_F^\zeta + \zeta^{-1}\omega_A)t} \|y_F\|_{Y_{\delta_Y}}.$$

As a consequence of this, Banach's fixed point theorem required

$$\frac{2^{\gamma_X} L_f C_A \Gamma(\gamma_X)}{\left(2(\varepsilon\zeta^{-1}-1)\omega_A + \varepsilon(N_S^\zeta + N_F^\zeta)\right)^{\gamma_X}} + \frac{2^{\delta_Y} L_g C_B \Gamma(\delta_Y)}{(N_S^\zeta - N_F^\zeta)^{\delta_Y}} + \frac{2\zeta^{\delta_Y - 1} L_g M_B}{N_S^\zeta - N_F^\zeta} < 1$$

instead of (5.3). This old set of assumptions was used for example in [11]. It should be noted that both sets of assumptions and their corresponding results are possible and the proofs and results are the same up to modification of some terms according to the modified assumptions. For $\delta_Y = 1$ both settings are even identical. However, it turns out that if $\delta_Y \in (0,1)$, then the new set of assumptions is more realistic for applications, which we will see later in our analysis of the Stommel model.

5.2. Existence of slow manifolds. Now we want to construct a family of slow manifolds $S_{\varepsilon,\zeta}$ which are given as graphs of certain functions

$$h^{\varepsilon,\zeta} \colon (Y_S^\zeta \cap Y_1) \to X_1 \times (Y_F^\zeta \cap Y_1),$$

over the slow part of the slow variable, i.e., we have that

$$S_{\varepsilon,\zeta} := \left\{ (h^{\varepsilon,\zeta}(v_0), v_0) : v_0 \in Y_S^\zeta \cap Y_1 \right\}.$$

In the following, we write $h_{X_1}^{\varepsilon,\zeta}$ for the first and $h_{Y_F^\zeta}^{\varepsilon,\zeta}$ for the second component. We use the Lyapunov–Perron method for the construction of slow manifolds, i.e., we construct fixed points of the operator

$$\mathcal{L}_{v_0,\varepsilon,\zeta} \colon C_\eta \to C_\eta,$$

$$\begin{pmatrix} u \\ v_F \\ v_S \end{pmatrix} \mapsto \left[t \mapsto \begin{pmatrix} \varepsilon^{-1} \int_{-\infty}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(u(s), v_F(s), v_S(s))\, \mathrm{d}s \\ \int_{-\infty}^t \mathrm{e}^{(t-s)B}\, \mathrm{pr}_{Y_F^\zeta}\, g(u(s), v_F(s), v_S(s))\, \mathrm{d}s \\ \mathrm{e}^{tB} v_0 + \int_0^t \mathrm{e}^{(t-s)B}\, \mathrm{pr}_{Y_S^\zeta}\, g(u(s), v_F(s), v_S(s))\, \mathrm{d}s \end{pmatrix} \right],$$

where $v_0 \in Y_S^\zeta$ and $C_\eta := C((-\infty,0], \mathrm{e}^{\eta t}; X_1 \times (Y_F^\zeta \cap Y_1) \times (Y_S^\zeta \cap Y_1))$ for

$$\eta := \zeta^{-1}\omega_A + \frac{N_S^\zeta + N_F^\zeta}{2}$$

is the space of all $(u, v_F, v_S) \in C((-\infty,0]; X_1 \times (Y_F^\zeta \cap Y_1) \times (Y_S^\zeta \cap Y_1))$ such that

$$\|(u, v_F, v_S)\|_{C_\eta} := \sup_{t \le 0} \mathrm{e}^{-\eta t} \left(\|u(t)\|_{X_1} + \|v_F(t)\|_{Y_1} + \|v_S(t)\|_{Y_1} \right) < \infty.$$

Then we obtain the function $h^{\varepsilon,\zeta}$ which describes the family of slow manifolds $S_{\varepsilon,\zeta}$ by

$$h^{\varepsilon,\zeta} \colon (Y_S^\zeta \cap Y_1) \to X_1 \times (Y_F^\zeta \cap Y_1), \quad v_0 \mapsto (u^{v_0}(0), v_F^{v_0}(0))^T,$$

i.e., $h^{\varepsilon,\zeta}$ gives the first two components of the fixed point $(u^{v_0}, v_F^{v_0}, v_S^{v_0})^T$ of $\mathcal{L}_{v_0,\varepsilon,\zeta}$ evaluated at $t = 0$.

Proposition 5.2. *Let $v_0 \in Y_S^\zeta \cap Y_1$. Then $\mathcal{L}_{v_0,\varepsilon,\zeta}$ has a unique fixed point in C_η.*

Proof. We show that $\mathcal{L}_{v_0,\varepsilon,\zeta}$ is a contraction on C_η. Let $(u, v_F, v_S), (\widetilde{u}, \widetilde{v}_F, \widetilde{v}_S) \in C_\eta$. Since showing that $\mathcal{L}_{v_0,\varepsilon,\zeta}$ maps C_η into C_η and showing that $\mathcal{L}_{v_0,\varepsilon,\zeta}$ is a contraction on C_η works in a similar way, we only show the latter. For the first component, we have that

$$\begin{aligned}
&\sup_{t\le 0} \mathrm{e}^{-\eta t} \| \mathrm{pr}_{X_1}\big(\mathcal{L}_{v_0,\varepsilon,\zeta}(u(t), v_F(t), v_S(t))^T - \mathcal{L}_{v_0,\varepsilon,\zeta}(\widetilde{u}(t), \widetilde{v}_F(t), \widetilde{v}_S(t))^T\big)\|_{X_1} \\
&\quad \le L_f C_A \int_{-\infty}^{t} \frac{\mathrm{e}^{(t-s)(\varepsilon^{-1}\omega_A - \eta)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}} \,\mathrm{d}s \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad = \frac{L_f C_A \Gamma(\gamma_X)}{(\varepsilon\eta - \omega_A)^{\gamma_X}} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad = \frac{2^{\gamma_X} L_f C_A \Gamma(\gamma_X)}{\big(2(\varepsilon\zeta^{-1} - 1)\omega_A + \varepsilon(N_S^\zeta + N_F^\zeta)\big)^{\gamma_X}} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta}.
\end{aligned}$$

For the second component, we have that

$$\begin{aligned}
&\sup_{t\le 0} \mathrm{e}^{-\eta t} \| \mathrm{pr}_{Y_F^\zeta}\big(\mathcal{L}_{v_0,\varepsilon,\zeta}(u(t), v_F(t), v_S(t))^T - \mathcal{L}_{v_0,\varepsilon,\zeta}(\widetilde{u}(t), \widetilde{v}_F(t), \widetilde{v}_S(t))^T\big)\|_{Y_1} \\
&\quad \le \frac{\zeta^{\delta_Y - 1} L_g C_B}{2^{\delta_Y - 1}(N_S^\zeta - N_F^\zeta)^{\delta_Y - 1}} \\
&\qquad \cdot \int_{-\infty}^{t} \frac{\mathrm{e}^{(t-s)(\zeta^{-1}\omega_A + N_F^\zeta - \eta)}}{(t-s)^{1-\delta_Y}} \,\mathrm{d}s \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad = \frac{\zeta^{\delta_Y - 1}(N_S^\zeta - N_F^\zeta)^{\delta_Y - 1} L_g C_B \Gamma(\delta_Y)}{2^{\delta_Y - 1}(\eta - \zeta^{-1}\omega_A - N_F^\zeta)^{\delta_Y}} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad = \frac{2\zeta^{\delta_Y - 1} L_g C_B \Gamma(\delta_Y)}{N_S^\zeta - N_F^\zeta} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta}.
\end{aligned}$$

Finally, the third component satisfies

$$\begin{aligned}
&\sup_{t\le 0} \mathrm{e}^{-\eta t} \| \mathrm{pr}_{Y_S^\zeta}\big(\mathcal{L}_{v_0,\varepsilon,\zeta}(u(t), v_F(t), v_S(t))^T - \mathcal{L}_{v_0,\varepsilon,\zeta}(\widetilde{u}(t), \widetilde{v}_F(t), \widetilde{v}_S(t))^T\big)\|_{Y_1} \\
&\quad \le L_g C_B \int_0^t \zeta^{\delta_Y - 1} \mathrm{e}^{(t-s)(\zeta^{-1}\omega_A + N_S^\varepsilon - \eta)} \,\mathrm{d}s \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad \le \frac{\zeta^{\delta_Y - 1} L_g M_B}{\zeta^{-1}\omega_A + N_S^\zeta - \eta} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta} \\
&\quad = \frac{2\zeta^{\delta_Y - 1} L_g M_B}{N_S^\zeta - N_F^\zeta} \|(u-\widetilde{u}, v_F - \widetilde{v}_F, v_S - \widetilde{v}_S)\|_{C_\eta}.
\end{aligned}$$

Thus, if (5.3) is satisfied, then $\mathcal{L}_{v_0,\varepsilon,\zeta}$ is a contraction. Hence, it has a unique fixed point in this case. □

Proposition 5.3. *Consider the situation of Proposition 5.2 and let* $(u^{v_0}, v_F^{v_0}, v_S^{v_0})^T$ *be the unique fixed point of* $\mathcal{L}_{v_0,\varepsilon,\zeta}$. *The mapping*

$$h^{\varepsilon,\zeta} \colon (Y_S^\zeta \cap Y_1) \to X_1 \times (Y_F^\varepsilon \cap Y_1), \quad v_0 \mapsto \big(u^{v_0}(0), v_F^{v_0}(0)\big)^T$$

is Lipschitz continuous.

Proof. Let $v_0, \tilde{v}_0 \in Y_S^\zeta \cap Y_1$ and let $(u, v_F, v_S) \in C_\eta$ and $(\tilde{u}, \tilde{v}_F, \tilde{v}_S) \in C_\eta$ be the fixed points of $\mathcal{L}_{v_0,\varepsilon,\zeta}$ and $\mathcal{L}_{\tilde{v}_0,\varepsilon,\zeta}$, respectively. As in the proof of Proposition 5.2 it follows that

$$\sup_{t\le 0} \mathrm{e}^{-\eta t}\|u(t)-\tilde{u}(t)\|_{X_1} < \frac{2^{\gamma_X} L_f C_A \Gamma(\gamma_X)\|(u-\tilde{u}, v_F-\tilde{v}_F, v_S-\tilde{v}_S)\|_{C_\eta}}{\big(2(\varepsilon\zeta^{-1}-1)\omega_A + \varepsilon(N_S^\zeta + N_F^\zeta)\big)^{\gamma_X}},$$

$$\sup_{t\le 0} \mathrm{e}^{-\eta t}\|v_F(t)-\tilde{v}_F(t)\|_{Y_1} \le \frac{2\zeta^{\delta_Y-1} L_g C_B \Gamma(\delta_Y)\|(u-\tilde{u}, v_F-\tilde{v}_F, v_S-\tilde{v}_S)\|_{C_\eta}}{N_S^\zeta - N_F^\zeta},$$

$$\sup_{t\le 0} \mathrm{e}^{-\eta t}\|v_S(t)-\tilde{v}_S(t)\|_{X_1} \le M_B\|v_0-\tilde{v}_0\|_{Y_1} + \frac{2\zeta^{\delta_Y-1} L_g M_B\|(u-\tilde{u}, v_F-\tilde{v}_F, v_S-\tilde{v}_S)\|_{C_\eta}}{N_S^\zeta - N_F^\zeta}.$$

Thus, if

$$\tilde{L} := \frac{2^{\gamma_X} L_f C_A \Gamma(\gamma_X)}{\big(2(\varepsilon\zeta^{-1}-1)\omega_A + \varepsilon(N_S^\zeta + N_F^\zeta)\big)^{\gamma_X}} + \frac{2\zeta^{\delta_Y-1} L_g (C_B\Gamma(\delta_Y) + M_B)}{N_S^\zeta - N_F^\zeta} < 1$$

then we may sum up the three estimates, subtract $\tilde{L}\|(u-\tilde{u}, v_F-\tilde{v}_F, v_S-\tilde{v}_S)\|_{C_\eta}$ and divide by $1-\tilde{L}$. This gives the Lipschitz continuity. □

Remark 5.4. The proof of Proposition 5.3 even shows that the mapping which maps v_0 to the unique fixed point of $\mathcal{L}_{v_0,\varepsilon,\zeta}$ is Lipschitz continuous from $Y_S^\zeta \cap Y_1$ to C_η.

5.3. Distance to the critical manifold.

Proposition 5.5. *Consider the situation of Proposition 5.2 and choose* $c_0 \in (0,1)$. *There is a constant* $C > 0$ *such that for all* $\varepsilon, \zeta > 0$ *small enough and which satisfy* $\varepsilon < c_0(\omega_f/w_A)\zeta$ *and for all* $v_0 \in Y_S^\zeta \cap Y_1$ *it holds that*

$$\left\| \begin{pmatrix} h_{X_1}^{\varepsilon,\zeta}(v_0) - h^0(v_0) \\ h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0) \end{pmatrix} \right\|_{X_1\times Y_1} \le C\left(\varepsilon + \frac{\zeta^{\delta_Y-1}}{N_S^\zeta - N_F^\zeta}\right)\|v_0\|_{Y_1}.$$

Proof. Let $(\overline{u}, \overline{v}_F, \overline{v}_S) \in C_\eta$ be the unique fixed point of $\mathcal{L}_{v_0,\varepsilon,\zeta}$, i.e.,

$$(\overline{u}, \overline{v}_F, \overline{v}_S) = \big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S), \overline{v}_S\big).$$

Since $(\overline{u}, \overline{v}_F, \overline{v}_S)$ solves (5.4) on $(-\infty, 0]$ we have that $\overline{v}_S \in C^1((-\infty, 0], \mathrm{e}^{\eta t}; Y)$ and

$$\sup_{t\leq 0} \mathrm{e}^{-\eta t}\big(\|\overline{v}_S(t)\|_{Y_1} + \|\partial_t \overline{v}_S(t)\|_Y\big) \leq L\big(1 + \|B\|_{\mathcal{B}(Y_1,Y)} + L_g\big)\|v_0\|_{Y_1},$$

where L denotes the Lipschitz constant of the mapping which maps v_0 to the unique fixed point of $\mathcal{L}_{v_0,\varepsilon,\zeta}$. Moreover, we have that

$$\begin{aligned}
\|h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S(t))\|_{Y_1} &= \Big\| \int_{-\infty}^t \mathrm{e}^{(t-s)B}\, \mathrm{pr}_{Y^\zeta_F}\, g\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S(s)), \overline{v}_S(s)\big)\,\mathrm{d}s \Big\|_{Y_1} \\
&\leq \Big(\frac{1}{2}\zeta\big(N^\zeta_S - N^\zeta_F\big)\Big)^{\delta_Y - 1} L_g C_B \mathrm{e}^{\eta t} \big\|\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S), \overline{v}_S\big)\big\|_{C_\eta} \\
&\qquad \cdot \int_{-\infty}^t \frac{\mathrm{e}^{(t-s)(\zeta^{-1}\omega_A + N^\zeta_F - \eta)}}{(t-s)^{1-\delta_Y}}\,\mathrm{d}s \\
&\leq \frac{L\zeta^{\delta_Y-1}(N^\zeta_S - N^\zeta_F)^{\delta_Y-1} L_g C_B \Gamma(\delta_Y)\mathrm{e}^{\eta t}}{2^{\delta_Y-1}(\eta - \zeta^{-1}\omega_A - N^\zeta_F)^{\delta_Y}}\|v_0\|_{Y_1} \\
&= \frac{2L\zeta^{\delta_Y-1} L_g C_B \Gamma(\delta_Y)\mathrm{e}^{\eta t}}{N^\zeta_S - N^\zeta_F}\|v_0\|_{Y_1}.
\end{aligned} \tag{5.5}$$

Furthermore, integration by parts shows that for $t_0 \leq t \leq 0$ it holds that

$$\begin{aligned}
&h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(t)) - h^0(\overline{v}_S(t)) \\
&\quad = \varepsilon^{-1}\int_{-\infty}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S(s)), \overline{v}_S(s)\big)\,\mathrm{d}s \\
&\qquad + A^{-1} f\big(h^0(\overline{v}_S(t)), 0, \overline{v}_S(t)\big) \\
&\quad = \varepsilon^{-1}\int_{-\infty}^{t_0} \mathrm{e}^{\varepsilon^{-1}(t-s)A} f\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S(s)), \overline{v}_S(s)\big)\,\mathrm{d}s \\
&\qquad + \mathrm{e}^{\varepsilon^{-1}(t-t_0)A} A^{-1} f\big(h^0(\overline{v}_S(t_0)), 0, \overline{v}_S(t_0)\big) \\
&\qquad + \int_{t_0}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} A^{-1}\partial_s f\big(h^0(\overline{v}_S(s)), 0, \overline{v}_S(s)\big)\,\mathrm{d}s \\
&\qquad + \varepsilon^{-1}\int_{t_0}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big[f\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)), h^{\varepsilon,\zeta}_{Y^\zeta_F}(\overline{v}_S(s)), \overline{v}_S(s)\big) \\
&\qquad\qquad - f\big(h^0(\overline{v}_S(s)), 0, \overline{v}_S(s)\big)\big]\,\mathrm{d}s.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\|h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(t)) - h^0(\overline{v}_S(t))\|_{X_1} \\
&\leq L_f C_A \mathrm{e}^{\varepsilon^{-1}\omega_A(t-t_0)+\eta t_0} \big\|\big(h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S), h^{\varepsilon,\zeta}_{Y_F^\zeta}(\overline{v}_S), \overline{v}_S\big)\big\|_{C_\eta} \int_{-\infty}^{t_0} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t_0-s)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}}\,\mathrm{d}s \\
&\quad + L_f M_A \mathrm{e}^{\varepsilon^{-1}\omega_A(t-t_0)+\eta t_0} \|A^{-1}\|_{\mathcal{B}(X_{\gamma_X},X_1)} \big\|\big(h^0(\overline{v}_S), 0, \overline{v}_S\big)\big\|_{C_\eta} \\
&\quad + \varepsilon \mathrm{e}^{\eta t} L_f C_A \|A^{-1}\|_{\mathcal{B}(X_{\delta_X-1},X_{\delta_X})} \\
&\qquad \cdot \int_{t_0}^{t} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t-s)}}{\varepsilon^{\delta_X}(t-s)^{1-\delta_X}}\,\mathrm{d}s \sup_{s\leq 0}\big(\mathrm{e}^{-\eta s}\big(\|\overline{v}_S(s)\|_Y + \|\partial_s \overline{v}_S(s)\|_Y\big)\big) \\
&\quad + \frac{2\zeta^{\delta_Y-1} L L_f C_A L_g C_B \Gamma(\delta_Y)}{N_S^\zeta - N_F^\zeta} \mathrm{e}^{\eta t} \int_{t_0}^{t} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t-s)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}}\,\mathrm{d}s \|v_0\|_{Y_1} \\
&\quad + L_f C_A \int_{t_0}^{t} \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}} \|h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)) - h^0(\overline{v}_S(s))\|_{X_1}\,\mathrm{d}s \\
&\leq C\|v_0\|_{Y_1}\left(\frac{1}{(\varepsilon\eta-\omega_A)^{\gamma_X}} + 1\right)\mathrm{e}^{(\eta-\varepsilon^{-1}\omega_A)t_0}\mathrm{e}^{\varepsilon^{-1}\omega_A t} \\
&\quad + C\|v_0\|_{Y_1}\left(\frac{\varepsilon}{(\varepsilon\eta-\omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y-1}}{(\varepsilon\eta-\omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)}\right)\mathrm{e}^{\eta t} \\
&\quad + L_f C_A \int_{t_0}^{t} \frac{\mathrm{e}^{\varepsilon^{-1}\omega_A(t-s)}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}} \|h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(s)) - h^0(\overline{v}_S(s))\|_{X_1}\,\mathrm{d}s.
\end{aligned}$$

Now, Lemma 2.8 applied to

$$v(r) := \|h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(r+t_0)) - h^0(\overline{v}_S(r+t_0))\|_{X_1} \quad (r \in [0, t-t_0])$$

yields that

$$\begin{aligned}
&\frac{1}{C\|v_0\|_{Y_1}} \|h^{\varepsilon,\zeta}_{X_1}(\overline{v}_S(t)) - h^0(\overline{v}_S(t))\|_{X_1} \\
&\leq \left(\left(\frac{1}{(\varepsilon\eta-\omega_A)^{\gamma_X}} + 1\right) + \frac{\varepsilon}{(\varepsilon\eta-\omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y-1}}{(\varepsilon\eta-\omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)}\right) \\
&\qquad \cdot \mathrm{e}^{\eta t_0 + \varepsilon^{-1}\omega_f(t-t_0)} \\
&\quad + \left(\frac{\varepsilon}{(\varepsilon\eta-\omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y-1}}{(\varepsilon\eta-\omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)}\right) \\
&\qquad \cdot \int_{t_0}^{t} (\eta - \varepsilon^{-1}\omega_A)\mathrm{e}^{\eta s}\mathrm{e}^{\varepsilon^{-1}\omega_f(t-s)}\,\mathrm{d}s
\end{aligned}$$

$$= \left(\left(\frac{1}{(\varepsilon\eta - \omega_A)^{\gamma_X}} + 1 \right) + \frac{\varepsilon}{(\varepsilon\eta - \omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y - 1}}{(\varepsilon\eta - \omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)} \right)$$
$$\cdot \, \mathrm{e}^{\eta t_0 + \varepsilon^{-1}\omega_f (t - t_0)}$$
$$+ \left(\frac{\varepsilon}{(\varepsilon\eta - \omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y - 1}}{(\varepsilon\eta - \omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)} \right)$$
$$\cdot \, \frac{\eta - \varepsilon^{-1}\omega_A}{\eta - \varepsilon^{-1}\omega_f} \left(\mathrm{e}^{t\eta} - \mathrm{e}^{\eta t_0 + \varepsilon^{-1}\omega_f (t - t_0)} \right)$$

Note that since $\eta > \zeta^{-1}\omega_A$, it follows from $\varepsilon < c_0(\omega_f / w_A)\zeta$ that

$$\eta > \zeta^{-1}\omega_A > c_0 \varepsilon^{-1}\omega_f.$$

Hence, choosing $t = 0$ and letting $t_0 \to -\infty$ shows that

$$\|h_{X_1}^{\varepsilon,\zeta}(v_0) - h^0(v_0)\|_{X_1}$$
$$\leq C \left(\frac{\varepsilon}{(\varepsilon\eta - \omega_A)^{\delta_X}} + \frac{\zeta^{\delta_Y - 1}}{(\varepsilon\eta - \omega_A)^{\gamma_X}(N_S^\zeta - N_F^\zeta)} \right) \frac{\eta - \varepsilon^{-1}\omega_A}{\eta - \varepsilon^{-1}\omega_f} \|v_0\|_{Y_1}.$$

Since $\varepsilon\eta - \omega_A$ and $\varepsilon\eta - \omega_f$ are bounded away from 0, it follows that

$$\|h_{X_1}^{\varepsilon,\zeta}(v_0) - h^0(v_0)\|_{X_1} \leq C \left(\varepsilon + \frac{\zeta^{\delta_Y - 1}}{N_S^\zeta - N_F^\zeta} \right) \|v_0\|_{Y_1}$$

for some constant $C > 0$. Moreover, (5.5) turns into

$$\|h_{Y_F^\zeta}^{\varepsilon,\zeta}(\overline{v}_S(t))\|_{Y_1} \leq C \frac{\zeta^{\delta_Y - 1}}{N_S^\zeta - N_F^\zeta} \|v_0\|_{Y_1}.$$

Altogether, we obtain the assertion. □

5.4. Differentiability of the slow manifolds. Now, we suppose that the nonlinearities $f: X_1 \times Y_1 \to X_{\gamma_X}$ and $g: X_1 \times Y_1 \to Y_{\delta_Y}$ are continuously differentiable such that

$$\|\mathrm{D}f(x,y)\|_{\mathcal{B}(X_1 \times Y_1, X_{\gamma_X})} \leq L_f, \quad \|\mathrm{D}g(x,y)\|_{\mathcal{B}(X_1 \times Y_1, Y_{\delta_Y})} \leq L_g. \tag{5.6}$$

The aim is to show that

$$\left(Y_S^\zeta \cap Y_1, \|\cdot\|_{Y_1}\right) \to \left(X_1, \|\cdot\|_{X_1}\right) \times \left(Y_F^\zeta \cap Y_1, \|\cdot\|_{Y_{\delta_Y}}\right),$$
$$v_0 \mapsto \left(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0)\right)$$

is differentiable.

Proposition 5.6. *Under the general assumptions in this section and the differentiability assumptions in this subsection, the slow manifold $S_{\varepsilon,\zeta}$ is differentiable.*

Proof. Given $v_0 \in Y_S^\zeta$ we write $U(\cdot, v_0) := (u(\cdot, v_0), v_F(\cdot, v_0), v_S(\cdot, v_0)) \in C_\eta$ for the fixed point of $\mathcal{L}_{v_0,\varepsilon,\zeta}$. Fix $v_0, \widetilde{v}_0 \in Y_S^\zeta$. Effectively, any classical approach to show smoothness [13, 17, 35] is based around estimates, which show that the derivative exists as the best local linear approximation of the graph of the manifold. We follow this strategy and write

$$U(\cdot\,\widetilde{v}_0) - U(\cdot\,v_0) - T\big[U(\cdot\,\widetilde{v}_0) - U(\cdot\,v_0)\big] = \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{B(\cdot)}(\widetilde{v}_0 - v_0) \end{pmatrix} + I(\widetilde{v}_0, v_0),$$

where

$$T\colon C_\eta \to C_\eta, \quad z \mapsto \left[t \mapsto \begin{pmatrix} \varepsilon^{-1}\int_{-\infty}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\mathrm{D}f(U(s,v_0))z(s)\,\mathrm{d}s \\ \int_{-\infty}^t \mathrm{e}^{(t-s)B}\,\mathrm{pr}_{Y_F^\zeta}\,\mathrm{D}g(U(s,v_0))z(s)\,\mathrm{d}s \\ 0 \end{pmatrix}\right]$$

and $I(\widetilde{v}_0, v_0) = (I_1, I_2, I_3)^T(\widetilde{v}_0, v_0)$, where

$$\begin{aligned} I_1(\widetilde{v}_0, v_0) &= \Big[t \mapsto \varepsilon^{-1}\int_{-\infty}^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(U(s,\widetilde{v}_0)) - f(U(s,v_0)) \\ &\qquad - \mathrm{D}f(U(s,v_0))\big[U(s,\widetilde{v}_0) - U(s,v_0)\big]\big)\,\mathrm{d}s\Big], \\ I_2(\widetilde{v}_0, v_0) &= \Big[t \mapsto \int_{-\infty}^t \mathrm{e}^{(t-s)B}\,\mathrm{pr}_{Y_F^\zeta}\big(g(U(s,\widetilde{v}_0)) - g(U(s,v_0)) \\ &\qquad - \mathrm{D}g(U(s,v_0))[U(s,\widetilde{v}_0) - U(s,v_0)]\big)\,\mathrm{d}s\Big], \\ I_3(\widetilde{v}_0, v_0) &= 0. \end{aligned}$$

The aim is to show that $\|T\|_{\mathcal{B}(C_\eta)} < 1$ and that

$$\|I(\widetilde{v}_0, v_0)\|_{X_1\times(Y_F^\zeta\cap Y_1)\times(Y_S^\zeta\cap Y_1)} = o\big(\|\widetilde{v}_0 - v_0\|_{Y_1}\big) \quad \text{as } \widetilde{v}_0 \to v_0.$$

Then we have

$$U(0,\widetilde{v}_0) - U(0,v_0) = (1-T)^{-1}\begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{B(\cdot)}(\widetilde{v}_0 - v_0) \end{pmatrix} + o\big(\|\widetilde{v}_0 - v_0\|_{Y_1}\big)$$

as $\widetilde{v}_0 \to v_0$, so that

$$U(0,\cdot) = \big(h_X^{\varepsilon,\zeta}, h_{Y_F^\zeta}^{\varepsilon,\zeta}, \mathrm{id}_{Y_S^\zeta}\big)$$

is differentiable. The fact that

$$\|T\|_{\mathcal{B}(X_1\times(Y_F^\xi\cap Y_1)\times(Y_S^\xi\cap Y_1))} < 1$$

follows from the same computation as the one for showing that $\mathcal{L}_{v_0,\varepsilon,\zeta}$ is a contraction in Proposition 5.2. Concerning I one can treat both its components similarly. Hence, we only carry out the usual argument for the first component. By our assumptions on f, for all $\sigma > 0$ there is an $N > 0$ such that

$$\begin{aligned}
&\mathrm{e}^{-\eta t}\Big\|\varepsilon^{-1}\int_{-\infty}^{\min\{-N,t\}} \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(U(s,\tilde{v}_0)) - f(U(s,v_0)) \\
&\qquad - \mathrm{D}f(U(s,v_0))[U(s,\tilde{v}_0) - U(s,v_0)]\big)\,\mathrm{d}s\Big\|_{X_1} \\
&\quad \leq 2L_f C_A\|U(\cdot,\tilde{v}_0) - U(\cdot,v_0)\|_{C_\eta}\int_{-\infty}^{\min\{-N,t\}} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t-s)}}{\varepsilon_X^{\gamma}(t-s)^{1-\gamma_X}}\,\mathrm{d}s \\
&\quad \leq \frac{\sigma}{2}\|\tilde{v}_0 - v_0\|_{Y_1}
\end{aligned}$$

for all $t \leq 0$. Having fixed such an $N > 0$, we obtain that

$$\begin{aligned}
&\mathrm{e}^{-\eta t}\Big\|\varepsilon^{-1}\int_{\min\{-N,t\}}^{t} \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(U(s,\tilde{v}_0)) - f(U(s,v_0)) \\
&\qquad - \mathrm{D}f(U(s,v_0))[U(s,\tilde{v}_0) - U(s,v_0)]\big)\,\mathrm{d}s\Big\|_{X_1} \\
&\quad \leq C_A\|U(\cdot,\tilde{v}_0) - U(\cdot,v_0)\|_{C_\eta}\int_{\min\{-N,t\}}^{t} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t-s)}}{\varepsilon_X^{\gamma}(t-s)^{1-\gamma_X}} \\
&\qquad \cdot \int_0^1 \big\|\mathrm{D}f\big(rU(s,\tilde{v}_0) - (1-r)U(s,v_0)\big) \\
&\qquad\qquad - \mathrm{D}f(U(s,v_0))\big\|_{\mathcal{B}(X_1\times(Y_F^\xi\cap Y_1)\times(Y_S^\xi\cap Y_1),X_{\gamma_X})}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad \leq C\|\tilde{v}_0 - v_0\|_{Y_1}\int_{\min\{-N,t\}}^{t} \frac{\mathrm{e}^{(\varepsilon^{-1}\omega_A-\eta)(t-s)}}{\varepsilon_X^{\gamma}(t-s)^{1-\gamma_X}} \\
&\qquad \cdot \int_0^1 \big\|\mathrm{D}f\big(rU(s,\tilde{v}_0) - (1-r)U(s,v_0)\big) \\
&\qquad\qquad - \mathrm{D}f(U(s,v_0))\big\|_{\mathcal{B}(X_1\times(Y_F^\xi\cap Y_1)\times(Y_S^\xi\cap Y_1),X_{\gamma_X})}\,\mathrm{d}r\,\mathrm{d}s.
\end{aligned}$$

By dominated convergence and the continuity of the integrand, it follows that the integral is smaller than $\sigma/2C$ if $\tilde{v}_0$ is close enough to v_0. Thus, for all $\sigma > 0$ there is a $\widetilde{\sigma} > 0$ such that for all $\widetilde{v}_0 \in Y_S^\zeta$ with $\|\widetilde{v}_0 - v_0\|_{Y_1} < \widetilde{\sigma}$ and all $t \le 0$ it holds that

$$e^{-\eta t}\Big\|\varepsilon^{-1}\int_{-\infty}^{t} \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(f(U(s,\widetilde{v}_0)) - f(U(s,v_0)) \\ - \mathrm{D}f(U(s,v_0))[U(0,\widetilde{v}_0) - U(0,v_0)]\big)\,\mathrm{d}s\Big\|_{X_1} < \sigma\|\widetilde{v}_0 - v_0\|_{Y_1}.$$

A similar computation can be carried out for the second component of I. Thus, we have that

$$\|I(\widetilde{v}_0, v_0)\|_{C_\eta} = o\big(\|\widetilde{v}_0 - v_0\|_{Y_1}\big) \quad \text{as } \widetilde{v}_0 \to v_0,$$

which shows the differentiability of the slow manifolds. □

5.5. Attraction of trajectories. Consider the situation of Proposition 5.6 and let

$$\big(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0\big) \in S_{\varepsilon,\zeta}.$$

Let (u, v_F, v_S) be the solution of (5.4) with initial value $(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0)$ and let $(u^\varepsilon, v_F^\varepsilon, v_S^\varepsilon)$ be the solution of (5.4) with initial value $(u_0, v_{0,F}, v_{0,S})$. Since (u, v_F, v_S) is a strict solution, it holds that

$$\partial_t u(t) = \varepsilon^{-1}Au(t) + \varepsilon^{-1}f(u(t), v_F(t), v_S(t)) \quad (t \ge 0).$$

On the other hand, since $S_{\varepsilon,\zeta}$ is invariant and since it is differentiable, it holds that $u(t) = h_{X_1}^{\varepsilon,\zeta}(v_S(t))$, and therefore

$$\begin{aligned}\partial_t u(t) &= \partial_t h_{X_1}^{\varepsilon,\zeta}(v_S(t)) = \big(\mathrm{D}h_{X_1}^{\varepsilon,\zeta}(v_S(t))\big)\big[\partial_t v_S(t)\big] \\ &= \big(\mathrm{D}h_{X_1}^{\varepsilon,\zeta}(v_S(t))\big)\big[Bv_S(t) + \mathrm{pr}_{Y_S^\zeta}\, g(u(t), v_F(t), v_S(t))\big] \quad (t \ge 0).\end{aligned}$$

Combining both equations for $t = 0$ and using

$$(u(t), v_F(t)) = \big(h_{X_1}^{\varepsilon,\zeta}(v_S(t)), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S(t))\big)$$

yields that

$$h_{X_1}^{\varepsilon,\zeta}(v_0) = \varepsilon A^{-1}\big(\mathrm{D}h_{X_1}^{\varepsilon,\zeta}(v_0)\big)\big[Bv_0 + \mathrm{pr}_{Y_S^\zeta}\, g\big(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0\big)\big] \\ - A^{-1}f\big(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0\big). \quad (5.7)$$

Similarly, it holds that

$$h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0) = B_{Y_F^\zeta}^{-1}\big(\mathrm{D}h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0)\big)\big[Bv_0 + \mathrm{pr}_{Y_S^\zeta}\, g\big(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0\big)\big] \\ - B_{Y_F^\zeta}^{-1}\,\mathrm{pr}_{Y_F^\zeta}\, g\big(h_{X_1}^{\varepsilon,\zeta}(v_0), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_0), v_0\big). \quad (5.8)$$

Note that (5.7) and (5.8) hold for arbitrary $v_0 \in Y_S^\zeta \cap Y_1$. In particular, they also hold for $v_0 = v_S^\varepsilon(t)$. In addition, the differentiability of $h_{X_1}^{\varepsilon,\zeta}$ and $h_{Y_F^\zeta}^{\varepsilon,\zeta}$ shows that

$$\partial_t h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) = \big(\mathrm{D}h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t))\big)\big[Bv_S^\varepsilon(t) + \mathrm{pr}_{Y_S^\zeta}\, g(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t))\big], \tag{5.9}$$

$$\partial_t h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) = \big(\mathrm{D}h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(t))\big)\big[Bv_S^\varepsilon(t) + \mathrm{pr}_{Y_S^\zeta}\, g(u^\varepsilon(t), v_F^\varepsilon(t), v_S^\varepsilon(t))\big]. \tag{5.10}$$

Proposition 5.7. *Consider the situation of Proposition 5.2 together with the assumptions of this subsection. Then there is a constant $K > 0$ such that the following holds: If $\zeta^{\delta_Y - 1}/(N_S^\zeta - N_F^\zeta) < K$ and if ζ and ε are small enough, then there are constants $C, c > 0$ we have the estimate*

$$\left\| \begin{pmatrix} u^\varepsilon(t) - h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) \\ v_F^\varepsilon(t) - h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) \end{pmatrix} \right\|_{X_1\times Y_1} \le C\mathrm{e}^{-ct} \left\| \begin{pmatrix} u_0 - h_{X_1}^{\varepsilon,\zeta}(v_{0,S}) \\ v_{0,F} - h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_{0,S}) \end{pmatrix} \right\|_{X_1\times Y_1},$$

i.e., solutions of (4.3) approach the solutions on the slow manifold at an exponential rate.

Proof. It holds that

$$\begin{aligned} u^\varepsilon(t) - h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) &= \mathrm{e}^{\varepsilon^{-1}tA}\big(u_0 - h_{X_1}^{\varepsilon,\zeta}(v_{0,S})\big) + \mathrm{e}^{\varepsilon^{-1}tA} h_{X_1}^{\varepsilon,\zeta}(v_{0,S}) - h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) \\ &\quad + \varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(u^\varepsilon(s), v_F^\varepsilon(s), v_S^\varepsilon(s))\,\mathrm{d}s \\ &= \mathrm{e}^{\varepsilon^{-1}tA}\big(u_0 - h_{X_1}^{\varepsilon,\zeta}(v_{0,S})\big) - \int_0^t \partial_s\big(\mathrm{e}^{\varepsilon^{-1}(t-s)A} h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s))\big)\,\mathrm{d}s \\ &\quad + \varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(u^\varepsilon(s), v_F^\varepsilon(s), v_S^\varepsilon(s))\,\mathrm{d}s \\ &= \mathrm{e}^{\varepsilon^{-1}tA}\big(u_0 - h_{X_1}^{\varepsilon,\zeta}(v_{0,S})\big) + \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\varepsilon^{-1} A h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s))\,\mathrm{d}s \\ &\quad - \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\partial_s\big[h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s))\big]\,\mathrm{d}s \\ &\quad + \varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A} f(u^\varepsilon(s), v_F^\varepsilon(s), v_S^\varepsilon(s))\,\mathrm{d}s. \end{aligned}$$

Combining this with (5.7) and (5.9) yields

$$\begin{aligned} u^\varepsilon(t) - h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) &= \mathrm{e}^{\varepsilon^{-1}tA}\big(u_0 - h_{X_1}^{\varepsilon,\zeta}(v_{0,S})\big) \\ &\quad + \int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big(\mathrm{D}h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s))\big)\big[\mathrm{pr}_{Y_S^\zeta}\, g\big(u^\varepsilon(s), v_F^\varepsilon(s), v_S^\varepsilon(s)\big) \\ &\qquad - \mathrm{pr}_{Y_S^\zeta}\, g\big(h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s)), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(s)), v_S^\varepsilon(s)\big)\big]\,\mathrm{d}s \end{aligned}$$

$$+\,\varepsilon^{-1}\int_0^t \mathrm{e}^{\varepsilon^{-1}(t-s)A}\big[f\big(u^\varepsilon(s),v_F^\varepsilon(s),v_S^\varepsilon(s)\big) \\ - f\big(h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),v_S^\varepsilon(s)\big)\big]\,\mathrm{d}s.$$

Similarly, it holds that

$$\begin{aligned} v_F^\varepsilon(t)-h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(t)) &= \mathrm{e}^{tB}\big(v_{0,F}-h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_{0,S})\big) \\ &\quad+\int_0^t \mathrm{e}^{(t-s)B}\big(\mathrm{D}h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(s))\big)\big[\mathrm{pr}_{Y_S^\zeta}\, g\big(u^\varepsilon(s),v_F^\varepsilon(s),v_S^\varepsilon(s)\big) \\ &\qquad -\mathrm{pr}_{Y_S^\zeta}\, g\big(h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),v_S^\varepsilon(s)\big)\big]\,\mathrm{d}s \\ &\quad+\int_0^t \mathrm{e}^{(t-s)B}\,\mathrm{pr}_{Y_F^\zeta}\big[g\big(u^\varepsilon(s),v_F^\varepsilon(s),v_S^\varepsilon(s)\big) \\ &\qquad - g\big(h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(s)),v_S^\varepsilon(s)\big)\big]\,\mathrm{d}s. \end{aligned}$$

Thus, if we define

$$\varphi(t):=\|u^\varepsilon(t)-h_{X_1}^{\varepsilon,\zeta}(v_S^\varepsilon(t))\|_{X_1}+\|v_F^\varepsilon(t)-h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_S^\varepsilon(t))\|_{Y_1},$$

then we obtain

$$\begin{aligned} \varphi(t) &\le M_A\mathrm{e}^{\varepsilon^{-1}\omega_A t}\|u_0-h_{X_1}^{\varepsilon,\zeta}(v_{0,S})\|_{X_1}+M_B\mathrm{e}^{(\zeta^{-1}\omega_A+N_F^\zeta)t}\|v_{0,F}-h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_{0,S})\|_{Y_1} \\ &\quad+C_A(\varepsilon L_g L+L_f)\int_0^t \frac{\mathrm{e}^{\varepsilon^{-1}(t-s)\omega_A}}{\varepsilon^{\gamma_X}(t-s)^{1-\gamma_X}}\varphi(s)\,\mathrm{d}s \\ &\quad+\Big(\frac{1}{2}\zeta\big(N_S^\zeta-N_F^\zeta\big)\Big)^{\delta_Y-1}C_B(L_gL+L_g)\int_0^t \frac{\mathrm{e}^{(t-s)(\zeta^{-1}\omega_A+N_F^\zeta)}}{(t-s)^{1-\delta_Y}}\varphi(s)\,\mathrm{d}s. \end{aligned}$$

Here, L denotes the Lipschitz constant of the slow manifold. If now $\zeta^{\delta_Y-1}/(N_S^\zeta-N_F^\zeta) < K$ for a small enough constant $K>0$, then

$$\big(\zeta\big(N_S^\zeta-N_F^\zeta\big)\big)^{\delta_Y-1}<K\big(N_S^\zeta-N_F^\zeta\big)^{\delta_Y}<-K\omega_A\zeta^{-\delta_Y}.$$

Hence, Lemma 2.10 shows that there are constants $C,c>0$ such that

$$\varphi(t)\le C\mathrm{e}^{-ct}\left\|\begin{pmatrix} u_0-h_{X_1}^{\varepsilon,\zeta}(v_{0,S}) \\ v_{0,F}-h_{Y_F^\zeta}^{\varepsilon,\zeta}(v_{0,S})\end{pmatrix}\right\|_{X_1\times Y_1}.$$

This is the assertion. □

5.6. An approximation of the slow flow. In Section 5.3 we measured the distance of the slow manifolds to the subset $S_{0,\zeta}$ of the critical manifold given by

$$S_{0,\zeta} := \{(h^0(v_0), v_0) \in S_0 : \operatorname{pr}_{Y_F^\zeta} v_0 = 0\}.$$

In many cases $S_{0,\zeta}$ will not be invariant under the slow flow. Thus, one might wonder how meaningful the result in Section 5.3 is. However, our aim is not to reduce the fast-slow system (4.3) with $\varepsilon > 0$ to the slow subsystem (4.3) with $\varepsilon = 0$, but to the reduced slow subsystem:

$$\begin{aligned} 0 &= Au_\zeta^0(t) + f(u_\zeta^0(t), v_\zeta^0(t)), \\ 0 &= \operatorname{pr}_{Y_F^\zeta} v_\zeta^0(t), \\ \partial_t v_\zeta^0(t) &= Bv_\zeta^0(t) + \operatorname{pr}_{Y_S^\zeta} g(u_\zeta^0(t), v_\zeta^0(t)), \\ v_\zeta^0(0) &= \operatorname{pr}_{Y_S^\zeta} v_0. \end{aligned} \tag{5.11}$$

Obviously, $S_{0,\zeta}$ is invariant under the reduced slow flow generated by (5.11).

Proposition 5.8. *Let ω_g be defined by (4.7). For all $T > 0$ there is a constant $C > 0$ such that for all $t \in [0, T]$ and all $\zeta > 0$ small enough it holds that*

$$\|v^0(t) - v_\zeta^0(t)\|_{Y_1} \le C\left(\|\operatorname{pr}_{Y_F^\zeta} v_0\|_{Y_1} + \frac{\big(\zeta(N_S^\zeta - N_F^\zeta)\big)^{\delta_Y - 1}}{(\omega_g - \zeta^{-1}\omega_A - N_F^\zeta)^{\delta_Y}}\|v_0\|_{Y_1}\right).$$

Proof. Variation of constants shows that

$$\begin{aligned} &\|v^0(t) - v_\zeta^0(t)\|_{Y_1} \\ &\le M_B e^{(\zeta^{-1}\omega_A + N_F^\zeta)t}\|\operatorname{pr}_{Y_F^\zeta} v_0\|_{Y_1} + \Big(\frac{1}{2}\zeta\big(N_S^\zeta - N_F^\zeta\big)\Big)^{\delta_Y - 1} \\ &\qquad \cdot L_g C_B \int_0^t \frac{e^{(\zeta^{-1}\omega_A + N_F^\zeta)(t-s)}}{(t-s)^{\delta_Y}}\big(\|h^0(v^0(s))\|_{X_1} + \|v^0(s)\|_{Y_1}\big)\,ds \\ &\quad + L_g C_B \int_0^t \frac{e^{\omega_B(t-s)}}{(t-s)^{\delta_Y}}\big(\|h^0(v^0(s)) - h^0(v_\zeta^0(s))\|_{X_1} + \|v^0(s) - v_\zeta^0(s)\|_{Y_1}\big)\,ds \\ &\le C e^{\omega_g t}\left(\|\operatorname{pr}_{Y_F^\zeta} v_0\|_{Y_1} + \frac{\big(\zeta(N_S^\zeta - N_F^\zeta)\big)^{\delta_Y - 1}\|v_0\|_{Y_1}}{(\omega_g - \zeta^{-1}\omega_A - N_F^\zeta)^{\delta_Y}}\right) \\ &\quad + \frac{L_F\|A^{-1}\|_{\mathcal{B}(X_{\delta_X - 1}, X_{\delta_X})} L_g C_B}{1 - L_F\|A^{-1}\|_{\mathcal{B}(X_{\delta_X - 1}, X_{\delta_X})}} \int_0^t \frac{e^{\omega_g(t-s)}}{(t-s)^{\delta_Y}}\|v^0(s) - v_\zeta^0(s)\|_{Y_1}\,ds \end{aligned}$$

Now the assertion follows from Lemma 2.8. □

Corollary 5.9. *Consider the situation of Proposition 5.7. Let ω_g be defined by (4.7). For all $T > 0$ there is a constant $C > 0$ such that for all $t \in [0,T]$ and all $\varepsilon, \zeta > 0$ with $\varepsilon < c_0(\omega_f/\omega_A)\zeta$ small enough it holds that*

$$\left\| \begin{pmatrix} u^\varepsilon(t) - h^0(v^0_\zeta(t)) \\ v^\varepsilon(t) - v^0_\zeta(t) \end{pmatrix} \right\|_{Y_1} \leq C\Bigg(\| \mathrm{pr}_{Y^\zeta_F} v_0\|_{Y_1} + \bigg(\varepsilon + \frac{1}{(\omega_g - \zeta^{-1}\omega_A - N^\zeta_F)^{\delta_Y}} \bigg) \|v_0\|_{Y_1} + \big(\varepsilon^{\delta_Y} + \mathrm{e}^{\varepsilon^{-1}\omega_f t}\big) \|u_0 - h^0(v_0)\|_{X_1} \Bigg).$$

In particular, for initial values on the slow manifold it holds that

$$\left\| \begin{pmatrix} u^\varepsilon(t) - h^0(v^0_\zeta(t)) \\ v^\varepsilon(t) - v^0_\zeta(t) \end{pmatrix} \right\|_{Y_1} \leq C\Bigg(\varepsilon + \frac{\big(\zeta(N^\zeta_S - N^\zeta_F)\big)^{\delta_Y - 1}}{(\omega_g - \zeta^{-1}\omega_A - N^\zeta_F)^{\delta_Y}} + \frac{\zeta^{\delta_Y - 1}}{N^\zeta_S - N^\zeta_F} \Bigg) \|v_0\|_{Y_1}.$$

Proof. The first estimate is a combination of Corollary 4.15 and Proposition 5.8. For the second estimate, we use the first estimate together with Proposition 5.5 and the triangle inequality

$$\| \mathrm{pr}_{Y^\zeta_F} v_0\|_{Y_1} \leq \| \mathrm{pr}_{Y^\zeta_F} v_0 - h^{\varepsilon,\zeta}_{Y^\zeta_F}(v_0)\|_{Y_1} + \|h^{\varepsilon,\zeta}_{Y^\zeta_F}(v_0)\|_{Y_1},$$
$$\|u_0 - h^0(v_0)\|_{X_1} \leq \|u_0 - h^{\varepsilon,\zeta}_{X_1}(v_0)\|_{X_1} + \|h^{\varepsilon,\zeta}_{X_1}(v_0) - h^0(v_0)\|_{X_1}. \qquad \square$$

Remark 5.10. (a) Note that we do not need the existence of slow manifolds for the first estimate in Corollary 5.9.

(b) If the initial values are not on the slow manifold, then it looks like the term $\| \mathrm{pr}_{Y^\zeta_F} v_0\|_{Y_1}$ might prevent the trajectories of semiflow generated by the fast-slow system from converging to the ones of the reduced slow flow as $\varepsilon, \zeta \to 0$. However, sometimes it holds that $\| \mathrm{pr}_{Y^\zeta_F} v_0\|_{Y_1} \to 0$ as $\zeta \to 0$ uniformly in v_0 running through certain sets. For example, if one takes v_0 from a bounded set in Y_2, then this will hold in many situations.

6. Three examples

In most applications nonlinearities are not Lipschitz continuous as we assume in our abstract theory, but only locally Lipschitz continuous. This means that we have to use cutoff techniques and our theory can only be applied locally. So before we start with the examples, let us briefly explain how this can be done for polynomial nonlinearities in toroidal Bessel potential spaces $H_2^s(\mathbb{T}^n)$. The space $H_2^s(\mathbb{T}^n)$ with $s \geq 0$ is defined as the space of all $f \in L_2(\mathbb{T}^n)$ such that

$$\|f\|_{H_2^s(\mathbb{T}^n)} := \Big\| x \mapsto \sum_{k\in\mathbb{Z}^n} \big(1+|k|^2\big)^{s/2} \widehat{f}(k) \mathrm{e}^{ikx} \Big\|_{L_2(\mathbb{T}^n)} < \infty,$$

where $\widehat{f}(k)$ denotes the k-th Fourier coefficient.

The main idea is to use Young's inequality for convolutions together with Plancherel's theorem. First, we observe that for $u, v \in H_2^s(\mathbb{T}^n)$ it holds that

$$\begin{aligned}
\big(1+|k|^2\big)^{s/2}\widehat{uv}(k) &= \big(1+|k|^2\big)^{s/2} \sum_{l\in\mathbb{Z}^n} \hat{u}(k-l)\hat{v}(l) \\
&\leq 2^s \sum_{l\in\mathbb{Z}^n} \big(1+|k-l|^2\big)^{s/2}\hat{u}(k-l)\hat{v}(l) + 2^s \sum_{l\in\mathbb{Z}^n} \big(1+|l|^2\big)^{s/2}\hat{u}(k-l)\hat{v}(l) \\
&= 2^s \big((1+|k|^2)^{s/2}\hat{u}(k)\big)_{k\in\mathbb{Z}^n} * \big(\hat{v}(k)\big)_{k\in\mathbb{Z}^n} \\
&\quad + 2^s \big(\hat{u}(k)\big)_{k\in\mathbb{Z}^n} * \big((1+|k|^2)^{s/2}\hat{v}(k)\big)_{k\in\mathbb{Z}^n}.
\end{aligned}$$

Now it follows from Young's inequality for convolutions together with Plancherel's theorem that

$$\begin{aligned}
\|uv\|_{H_2^s(\mathbb{T}^n)} &= \|\big(\big(1+|k|^2\big)^{s/2}\widehat{uv}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)} \\
&\leq 2^s \|\big(\big(1+|k|^2\big)^{s/2}\hat{u}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)} \|\big(\hat{v}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \\
&\qquad + 2^s \|\big(\hat{u}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \|\big(\big(1+|k|^2\big)^{s/2}\hat{v}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \\
&= 2^s \|\big(\hat{v}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \|u\|_{H_2^s(\mathbb{T}^n)} + 2^s \|\big(\hat{u}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \|v\|_{H_2^s(\mathbb{T}^n)}.
\end{aligned} \tag{6.1}$$

Therefore, a first idea to make polynomial nonlinearities Lipschitz continuous via a cutoff would be to use a smooth cutoff function on $\ell_1(\mathbb{Z}^n)$ and cut off functions for which the $\ell_1(\mathbb{Z}^n)$-norm of the Fourier series is large. However, it is well known that nontrivial smooth functions with bounded support do not exist on $\ell_1(\mathbb{Z}^n)$, see for example [7, Theorem 2]. Fortunately, the situation is better for other integrability parameters. More generally, it was observed in [26] that a separable Banach space X admits an equivalent continuously differentiable norm (i.e., a norm that is continuously Fréchet differentiable on $X \setminus \{0\}$) if and only if its topological dual is also separable. This also includes our Bessel potential spaces with integrability parameter 2. Note that (6.1) together with

$$\begin{aligned}
&\|\big(\hat{u}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_1(\mathbb{Z}^n)} \\
&\quad \leq \|\big(\big(1+|k|^2\big)^{-s/2}\big)_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)} \|\big(\big(1+|k|^2\big)^{s/2}\hat{u}(k)\big)_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)} \\
&\quad = c_s \|u\|_{H_2^s(\mathbb{T}^n)}
\end{aligned}$$

for $s > n/2$ and $c_s := \|((1+|k|^2)^{-s/2})_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)}$ yields the well-known fact that the Bessel potential spaces with regularity $s > n/2$ and integrability parameter 2 are an algebra. More precisely, we obtain

$$\|uv\|_{H_2^s(\mathbb{T}^n)} \leq 2^s c_s \|u\|_{H_2^s(\mathbb{T}^n)} \|v\|_{H_2^s(\mathbb{T}^n)}. \tag{6.2}$$

On Banach spaces X with continuously differentiable norm, e.g., $H_2^s(\mathbb{T}^n)$, one can now define a cutoff function as follows: One may take a function $\chi \in C^\infty(\mathbb{R};[0,1])$ with support in $B(0, C_2)$ for some $C_2 > 0$ and with $\chi(x) = 1$ for $|x| \leq C_1$ for some $C_1 \in (0, C_2)$ and compose it with the differentiable norm $\|\cdot\|_X$. Then $\chi_X := \chi(\|\cdot\|_X)$ is a continuously differentiable and Lipschitz continuous cutoff function from X to $[0,1]$. The function χ_X may even have an arbitrarily small Lipschitz constant, if χ is chosen accordingly.

The observations on smooth cutoff functions can now be combined with the estimate (6.1) in order to obtain the following result:

Proposition 6.1. *Let $p\colon \mathbb{R}^2 \to \mathbb{C}$ be a polynomial of degree $d \in \mathbb{N}_0$ given by*

$$p(u,v) = \sum_{\alpha\in\mathbb{N}_0^2,\, |\alpha|_1\leq d} a_\alpha u^{\alpha_1} v^{\alpha_2}$$

for some $a_\alpha \in \mathbb{C}$. We fix $s > n/2$ and define

$$c_s := \|\big((1+|k|^2)^{-s/2}\big)_{k\in\mathbb{Z}^n}\|_{\ell_2(\mathbb{Z}^n)}$$

as above. Let χ be a continuously differentiable cutoff function on $H^s(\mathbb{T}^n)$ with Lipschitz constant $L_\chi > 0$ which is supported in $B(0,R) \subset H^s(\mathbb{T}^n)$ for some $R > 0$ as described before. Then the mapping

$$p_\chi\colon H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n), \quad (u,v) \to p(\chi(u)u, \chi(v)v)$$

is globally Lipschitz continuous with Lipschitz constant not larger than

$$\sum_{\alpha\in\mathbb{Z}^2,\, |\alpha|_1\leq d} |a_\alpha||\alpha|_1 (2^s c_s R)^{|\alpha|_1 - 1}(1 + 2^{s+1} c_s R L_\chi).$$

Proof. (i) In a first step, we show that

$$H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n), \quad u \mapsto \chi(u)u$$

is Lipschitz continuous. So let $u, u_0 \in H^s(\mathbb{T}^n)$. If $\|u\|_{H^s(\mathbb{T}^n)}, \|u_0\|_{H^s(\mathbb{T}^n)} > R$, then it trivially holds that

$$\|\chi(u)u - \chi(u_0)u_0\|_{H^s(\mathbb{T}^n)} = 0 \leq \|u - u_0\|_{H^s(\mathbb{T}^n)}.$$

If $\|u\|_{H^s(\mathbb{T}^n)}, \|u_0\|_{H^s(\mathbb{T}^n)} \leq 2R$, then it follows that

$$\begin{aligned}
&\|\chi(u)u - \chi(u_0)u_0\|_{H^s(\mathbb{T}^n)} \\
&\qquad \leq \|\chi(u)u - \chi(u)u_0\|_{H^s(\mathbb{T}^n)} + \|(\chi(u) - \chi(u_0))u_0\|_{H^s(\mathbb{T}^n)} \\
&\qquad \leq \|u - u_0\|_{H^s(\mathbb{T}^n)} + 2^{s+1} c_s R L_\chi \|u - u_0\|_{H^s(\mathbb{T}^n)} \\
&\qquad = (1 + 2^{s+1} c_s R L_\chi)\|u - u_0\|_{H^s(\mathbb{T}^n)}
\end{aligned}$$

If $\|u\|_{H^s(\mathbb{T}^n)} \leq R$ and $\|u_0\|_{H^s(\mathbb{T}^n)} > 2R$, then $\|u - u_0\|_{H^s(\mathbb{T}^n)} \geq R$ so that

$$\|\chi(u)u - \chi(u_0)u_0\|_{H^s(\mathbb{T}^n)} = \|\chi(u)u\|_{H^s(\mathbb{T}^n)} \leq R \leq \|u - u_0\|_{H^s(\mathbb{T}^n)}.$$

The same holds if $\|u\|_{H^s(\mathbb{T}^n)} > 2R$ and $\|u_0\|_{H^s(\mathbb{T}^n)} \leq R$. Altogether, we have shown that

$$\|\chi(u)u - \chi(u_0)u_0\|_{H^s(\mathbb{T}^n)} \leq (1 + 2^{s+1} c_s R L_\chi)\|u - u_0\|_{H^s(\mathbb{T}^n)}$$

for all $u, u_0 \in H^s(\mathbb{T}^n)$.

(ii) In the second step, we show that

$$H^s(\mathbb{T}^n) \to H^s(\mathbb{T}^n), \quad u \mapsto (\chi(u)u)^k$$

is Lipschitz continuous for any $k \in \mathbb{N}$. Let again $u, u_0 \in H^s(\mathbb{T}^n)$. Then it holds that

$$\begin{aligned}
&\|(\chi(u)u)^k - (\chi(u_0)u_0)^k\|_{H^s(\mathbb{T}^n)} \\
&\qquad = \Big\| \sum_{j=0}^{k-1} (\chi(u)u)^j (\chi(u_0)u_0)^{k-1-j} (\chi(u)u - \chi(u_0)u_0) \Big\|_{H^s(\mathbb{T}^n)} \\
&\qquad \leq 2^s c_s \sum_{j=0}^{k-1} \|(\chi(u)u)^j (\chi(u_0)u_0)^{k-1-j}\|_{H^s(\mathbb{T}^n)} \|\chi(u)u - \chi(u_0)u_0\|_{H^s(\mathbb{T}^n)} \\
&\qquad \leq (2^s c_s)^{k-1} k R^{k-1} (1 + 2^{s+1} c_s R L_\chi)\|u - u_0\|_{H^s(\mathbb{T}^n)}.
\end{aligned}$$

(iii) Finally, we turn to the assertion. So let $u, u_0, v, v_0 \in H^s(\mathbb{T}^n)$. Then it holds that

$$\begin{aligned}
&\Big\| \sum_{\substack{\alpha \in \mathbb{Z}^2 \\ |\alpha|_1 \leq d}} a_\alpha (\chi(u)u)^{\alpha_1} (\chi(v)v)^{\alpha_2} - \sum_{\substack{\alpha \in \mathbb{Z}^2 \\ |\alpha|_1 \leq d}} a_\alpha (\chi(u_0)u_0)^{\alpha_1} (\chi(v_0)v_0)^{\alpha_2} \Big\|_{H^s(\mathbb{T}^n)} \\
&\qquad \leq 2^s c_s \sum_{\substack{\alpha \in \mathbb{Z}^2 \\ |\alpha|_1 \leq d}} |a_\alpha| \big(\|(\chi(u)u)^{\alpha_1} - (\chi(u_0)u_0)^{\alpha_1}\|_{H^s(\mathbb{T}^n)} \|(\chi(v)v)^{\alpha_2}\|_{H^s(\mathbb{T}^n)} \\
&\qquad\qquad + \|(\chi(v)v)^{\alpha_2} - (\chi(v_0)v_0)^{\alpha_2}\|_{H^s(\mathbb{T}^n)} \|(\chi(u_0)u_0)^{\alpha_1}\|_{H^s(\mathbb{T}^n)} \big)
\end{aligned}$$

$$\leq 2^s c_s \sum_{\substack{\alpha\in\mathbb{Z}^2\\|\alpha|_1\leq d}} |a_\alpha|\big((2^s c_s)^{\alpha_1-1}\alpha_1 R^{\alpha_1-1}(1+2^{s+1}c_s RL_\chi)(2^s c_s)^{\alpha_2-1} \cdot R^{\alpha_2}\|u-u_0\|_{H^s(\mathbb{T}^n)}$$
$$+ (2^s c_s)^{\alpha_2-1}\alpha_2 R^{\alpha_2-1}(1+2^{s+1}c_s RL_\chi)(2^s c_s)^{\alpha_1-1}R^{\alpha_1}\|u-u_0\|_{H^s(\mathbb{T}^n)}\big)$$
$$= \Bigg(\sum_{\substack{\alpha\in\mathbb{Z}^2\\|\alpha|_1\leq d}} |a_\alpha||\alpha|_1(2^s c_s R)^{|\alpha|_1-1}(1+2^{s+1}c_s RL_\chi)\Bigg)\|u-u_0\|_{H^s(\mathbb{T}^n)}. \qquad \square$$

6.1. The spatial Stommel model. Now we apply our methods to a version of Stommel's box model for oceanic circulation in the North Atlantic (see [30] and [6, (6.2.4)]) in which we add diffusion in both variables. These equations are then given by

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= \Delta u^\varepsilon - u^\varepsilon + 1 - \varepsilon u^\varepsilon\big[1+\eta^2((u^\varepsilon)^2-(w^\varepsilon)^2)\big],\\ \partial_t w^\varepsilon &= \Delta w^\varepsilon + \mu - w^\varepsilon\big[1+\eta^2((u^\varepsilon)^2-(w^\varepsilon)^2)\big],\\ u^\varepsilon(0) &= u_0, \quad w^\varepsilon(0)=v_0, \end{aligned} \tag{6.3}$$

where $\mu,\eta>0$ are certain parameters. We study this system with periodic boundary conditions, i.e., on the torus $\mathbb{T}$, and partly also on $\mathbb{T}^n$.

Theorem 6.2. *Let $s\geq 0$, $n\in\mathbb{N}$ and $\delta_Y\in(\frac{1}{2},1)$ such that $2s+4(1-\delta_Y)>n$. Let further $T>0$ be fixed. We write $(u^\varepsilon,w^\varepsilon)$ for the strict solution of (6.3) with $\varepsilon>0$ and (u^0,w^0) for corresponding slow flow. Then for all $R>0$ there are constants $\varepsilon_0>0$ and $C,c>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$,*

$$\begin{aligned} u_0 &\in H_2^{s+2}(\mathbb{T}^n) && \text{with } \|u_0\|_{H_2^{s+2}(\mathbb{T}^n)}\leq R,\\ v_0 &\in H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) && \text{with } \|u_0\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)}\leq R \end{aligned}$$

it holds that

$$\sup_{0\leq t\leq T(R)}\big(\|u^\varepsilon(t)-u^0(t)\|_{H_2^{s+2}(\mathbb{T}^n)} + \|w^\varepsilon(t)-v^0(t)\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)}\big) \leq C(\varepsilon^{\delta_Y}+\mathrm{e}^{-c\varepsilon^{-1}t}),$$

where $T(R)$ is defined by

$$T(R) := \inf\big\{t\in[0,T] : \max\big\{\|u^0(t)\|_{H_2^{s+2}(\mathbb{T}^n)}, \|w^0(t)\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)}, \|u^\varepsilon(t)\|_{H_2^{s+2}(\mathbb{T}^n)}, \|w^\varepsilon(t)\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)}\big\} > R\big\}.$$

Theorem 6.3. *Let $n=1$, $s\geq 0$ and $\delta_Y\in(\frac{1}{2},1)$ such that $2s+4(1-\delta_Y)>1$ and let $T>0$ be fixed. Then for all $R>0$ there are $\zeta_0>0$ and a family of finite-dimensional slow manifolds $S_{\varepsilon,\zeta}\subset H_2^{s+2}(\mathbb{T})\times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})$ with $0<\zeta\leq\zeta_0$ and $0<\varepsilon\leq c(\omega_f/\omega_A)\zeta$ for constants ω_f,ω_A which we define later and some $c\in(0,1)$ such that the following assertions hold:*

(a) *For each* $\zeta \in (0, \zeta_0]$ *there is a splitting*

$$H_2^{s+2(1-\delta_Y)}(\mathbb{T}) = Y_F^\zeta \oplus Y_S^\zeta,$$

where Y_S^ζ *is the projection of* $H_2^{s+2(1-\delta_Y)}(\mathbb{T})$ *to the* k*-th Fourier modes with* $|k|$ *being smaller than a certain number* $k(\zeta)$ *depending on* ζ*. We also have that* Y_F^ζ *is the projection to the remaining Fourier modes.*

(b) *Let* $B_{Y_S^\zeta}(0, R)$ *be defined as*

$$B_{Y_S^\zeta}(0, R) := \big\{ f \in Y_S^\zeta : \|f\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})} < R \big\}.$$

Then $S_{\varepsilon,\zeta}$ *is given as the graph of a differentiable mapping*

$$\begin{aligned} h^{\varepsilon,\zeta} \colon \big(B_{Y_S^\zeta}(0, R), \|\cdot\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})}\big) \\ \to H_2^{s+2}(\mathbb{T}) \times \big(Y_F^\zeta \cap H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}), \|\cdot\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})}\big). \end{aligned}$$

(c) $S_{\varepsilon,\zeta}$ *is locally invariant under the semiflow generated by* (6.3)*, i.e., the semiflow can only leave* $S_{\varepsilon,\zeta}$ *through its boundary.*

(d) *Let*

$$S_{0,\zeta} := \big\{ (u, w) \in S_0 : w \in B_{Y_S^\zeta}(0, R) \big\}$$

be the submanifold of the critical manifold which consists of all points whose slow components are elements of $B_{Y_S^\zeta}(0, R)$*. Then there is a constant* $C > 0$ *depending on* R *such that*

$$\operatorname{dist}(S_{\varepsilon,\zeta}, S_{0,\zeta}) \leq C\big(\varepsilon + \zeta^{\delta_Y - 1/2}\big) \leq C\zeta^{\delta_Y - 1/2}.$$

(e) *Suppose that*

$$\|u_0\|_{H_2^{s+2}(\mathbb{T})} \leq R, \quad \|v_0\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})} \leq R, \quad \|h^0(v_0)\|_{H_2^{s+2}(\mathbb{T})} \leq R$$

and let (u_ζ^0, w_ζ^0) *be the solution of the truncated slow subsystem of the diffusive Stommel model given by*

$$\begin{aligned} 0 &= \Delta u_\zeta^0 - u_\zeta^0 + 1, \\ \partial_t w_\zeta^0 &= \operatorname{pr}_{Y_S^\zeta}\big[\Delta w_\zeta^0 + \mu - w^\varepsilon[1 + \eta^2((u_\zeta^0)^2 - (w_\zeta^0)^2)]\big], \\ u_\zeta^0 &= h^0(\operatorname{pr}_{Y_S^\zeta} v_0), \quad w_\zeta^0(0) = \operatorname{pr}_{Y_S^\zeta} v_0. \end{aligned} \tag{6.4}$$

Assume that $(u_0, v_0) \in S_{\varepsilon,\zeta}$*. Then for each* $T > 0$ *there is a constant* $C > 0$ *such that*

$$\sup_{0 \leq t \leq T(R)} \big(\|u^\varepsilon(t) - u_\zeta^0(t)\|_{H_p^{s+2}(\mathbb{T})} + \|w^\varepsilon(t) - w_\zeta^0(t)\|_{H_p^{s+2+2(1-\delta_Y)}(\mathbb{T})}\big) \leq C\zeta^{\delta_Y - 1/2},$$

where $T(R)$ is defined by

$$T(R) := \inf\big\{t \in [0,T] : \max\big\{\|u_\zeta^0(t)\|_{H_p^{s+2}(\mathbb{T})}, \|w_\zeta^0(t)\|_{H_p^{s+2+2(1-\delta_Y)}(\mathbb{T})}, \\ \|u^\varepsilon(t)\|_{H_p^{s+2}(\mathbb{T})}, \|w^\varepsilon(t)\|_{H_p^{s+2+2(1-\delta_Y)}(\mathbb{T})}\big\} > R\big\}.$$

Remark 6.4. (a) In both theorems the condition $2s+4(1-\delta_Y) > n$ is not essential, but cutoff techniques would get more tedious without this assumption. This condition has the advantage that the nonlinearities are already well-defined and locally Lipschitz continuous in the spaces we work with later on without having to cut them off. Cutoff techniques are then only required to turn local Lipschitz continuity into global Lipschitz continuity.

(b) In Theorem 6.2 we may allow $\mathbb{T}^n$ as underlying domain, as its proof only uses the results of Section 4, which do not require large spectral gaps in the slow variable.

(c) We could also work in H_p^s with $p \neq 2$. But then the proofs would be more complicated since we would have to use Fourier multiplier theorems instead of just Plancherel's theorem.

Now we show how our general theory can be applied to derive Theorem 6.2 and Theorem 6.3.

In order to remove constants in the nonlinear terms, we introduce the dummy variable $\widetilde{w}$ which takes values in $\mathbb{R}^3$ and satisfies

$$\partial_t \widetilde{w}^\varepsilon = 0, \quad \widetilde{w}^\varepsilon(0) = (\sqrt{\varepsilon}, M, \mu)$$

for some $M > 0$ that we choose later. We make the following choices:

- The fast variable is given by u^ε. The slow variable is given by $v^\varepsilon = (w^\varepsilon, \widetilde{w}_1^\varepsilon, \widetilde{w}_2^\varepsilon, \widetilde{w}_3^\varepsilon)$. As underlying spaces we choose

$$X = H_2^s(\mathbb{T}^n) \quad \text{and} \quad Y = H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3$$

such that $2s + 4(1-\delta_Y)) > n$.

- The linear operator in the fast variable is given by

$$A\colon H_2^s(\mathbb{T}^n) \supset H_2^{s+2}(\mathbb{T}^n) \to H_2^s(\mathbb{T}^n), \quad u \mapsto \Delta u - u.$$

The linear operator in the slow variable is given by

$$B\colon H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 \supset H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 \to H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3, \\ (v, z_1, z_2, z_3)^T \mapsto (\Delta v_1, -z_1, -z_2, -z_3)^T$$

for some $\delta_Y \in (\frac{1}{2}, 1)$; we compensate the terms $z_j \mapsto -z_j$ from the linear part by inserting maps $z_j \mapsto z_j$ in the nonlinear part defined below.

- The Banach scales are given by

$$X_\alpha = H_2^{s+2\alpha}(\mathbb{T}^n) \quad \text{and} \quad Y_\alpha = H_2^{s+2(1-\delta_Y)+2\alpha}(\mathbb{T}^n) \times \mathbb{R}^3.$$

- We have already chosen $\delta_Y \in (\frac{1}{2}, 1)$. Moreover, we take $\gamma_X = 1-\delta_Y$ and $\delta_X = 1$. Thus, we have to define continuous nonlinearities

$$f\colon X_1 \times Y \to X, \quad g\colon X_1 \times Y_1 \to Y_{\delta_Y}$$

satisfying the Lipschitz conditions

$$\begin{aligned} \|f(x_1,y_1) - f(x_2,y_2)\|_{X_{1-\delta_Y}} &\le L_f\big(\|x_1-x_2\|_{X_1} + \|y_1-y_2\|_{Y_1}\big),\\ \|f(u_1,v_1) - f(u_2,v_2)\|_{C^1([0,t];X)} &\le L_f\big(\|u_1-u_2\|_{C^1([0,t];X_1)}\\ &\qquad + \|v_1-v_2\|_{C^1([0,t];Y)}\big),\\ \|g(x_1,y_1) - g(x_2,y_2)\|_{Y_{\delta_Y}} &\le L_g\big(\|x_1-x_2\|_{X_1} + \|y_1-y_2\|_{Y_1}\big). \end{aligned}$$

With our choices of spaces this translates into

$$\begin{aligned} f\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s}(\mathbb{T}^n),\\ g\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2}(\mathbb{T}^n) \times \mathbb{R}^3 \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} &\|f(x_1,y_1) - f(x_2,y_2)\|_{H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n)}\\ &\quad \le L_f\big(\|x_1-x_2\|_{H_2^{s+2}(\mathbb{T}^n)} + \|y_1-y_2\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)\times\mathbb{R}^3}\big),\\ &\|f(u_1,v_1) - f(u_2,v_2)\|_{C^1([0,t];H_2^{s}(\mathbb{T}^n))}\\ &\quad \le L_f\big(\|u_1-u_2\|_{C^1([0,t];H_2^{s+2}(\mathbb{T}^n))} + \|v_1-v_2\|_{C^1([0,t];H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n)\times\mathbb{R}^3)}\big),\\ &\|g(x_1,y_1) - g(x_2,y_2)\|_{H_2^{s+2}(\mathbb{T}^n)\times\mathbb{R}^3}\\ &\quad \le L_g\big(\|x_1-x_2\|_{H_2^{s+2}(\mathbb{T}^n)} + \|y_1-y_2\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)\times\mathbb{R}^3}\big). \end{aligned} \tag{6.6}$$

Note that if

$$\begin{aligned} f\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n),\\ g\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2}(\mathbb{T}^n) \times \mathbb{R}^3 \end{aligned} \tag{6.7}$$

are differentiable with

$$\begin{aligned} \|\mathrm{D}f(x,y)\|_{\mathcal{B}(H_2^{s+2}(\mathbb{T}^n)\times H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n)\times\mathbb{R}^3,\, H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n))} &\le L_f,\\ \|\mathrm{D}g(x,y)\|_{\mathcal{B}(H_2^{s+2}(\mathbb{T}^n)\times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)\times\mathbb{R}^3,\, H_2^{s+2}(\mathbb{T}^n)\times\mathbb{R}^3)} &\le L_g, \end{aligned} \tag{6.8}$$

then both (6.5) and (6.6) as well as (5.6) are satisfied. Since $H_2^s(\mathbb{T}^n)$ is a multiplication algebra whenever $2s > n$, it follows that the nonlinearities

$$\begin{aligned}\tilde{f}\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n),\\ (x, y, z_1, z_2, z_3) &\mapsto \frac{1}{M} z_2 + z_1^2 x\big[1 + \eta^2(x^2 - y^2)\big],\\ \tilde{g}\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2}(\mathbb{T}^n) \times \mathbb{R}^3,\\ (x, y, z_1, z_2, z_3) &\mapsto \big(z_3 - y\big[1 + \eta^2(x^2 - y^2)\big], z_1, z_2, z_3\big),\end{aligned}$$

are well-defined and satisfy (6.8) locally. In order to obtain these properties globally, we use cutoff techniques as described in Proposition 6.1. Let $R > 0$ be arbitrary and choose C^1-functions

$$\chi_1\colon H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \to [0,1], \quad \chi_2\colon H_2^{s+2}(\mathbb{T}^n) \to [0,1],$$
$$\chi_3\colon H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \to [0,1],$$

which equal 1 on the ball $B(0, R)$ around 0 with radius R in their respective topologies and which equal to 0 in the complement of $B(0, 2R)$. For $\sigma > 0$ we further choose $\psi_\sigma \in C^\infty(\mathbb{R})$ taking values in $[0, 1]$ such that

$$\psi_\sigma(z) = 1 \text{ if } |z| \le \sigma, \quad \psi_\sigma(z) = 0 \text{ if } |z| \ge 2\sigma, \quad \text{and} \quad |\psi_\sigma'(z)| \le \frac{2}{\sigma} \text{ for } z \in \mathbb{R}.$$

Now, the nonlinearities

$$\begin{aligned}f\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n),\\ (x, y, z_1, z_2, z_3) &\mapsto \frac{1}{M} z_2 + \psi_\sigma(z_1) z_1^2 \chi_2(x) x\big[1 + \eta^2\big((\chi_2(x)x)^2 - (\chi_1(y)y)^2\big)\big],\\ g\colon H_2^{s+2}(\mathbb{T}^n) \times H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3 &\to H_2^{s+2}(\mathbb{T}^n) \times \mathbb{R}^3,\\ (x, y, z_1, z_2, z_3) &\mapsto \big(z_3 - \chi_3(y) y\big[1 + \eta^2\big((\chi_2(x)x)^2 - (\chi_3(y)y)^2\big)\big], z_1, z_2, z_3\big),\end{aligned}$$

satisfy (6.8) globally. Moreover, if we choose $\sigma > 0$ small enough, then we have

$$L_f < \frac{2}{M}.$$

With these choices, we may rewrite (6.3) as

$$\begin{aligned}\varepsilon \partial_t u^\varepsilon &= A u^\varepsilon + f(u^\varepsilon, v_1^\varepsilon, \sqrt{\varepsilon}, M, \mu),\\ \partial_t v^\varepsilon &= B v^\varepsilon + g(u^\varepsilon, v_1^\varepsilon, \sqrt{\varepsilon}, M, \mu),\\ u^\varepsilon(0) &= u_0, \quad v_1^\varepsilon(0) = v_0.\end{aligned} \tag{6.9}$$

If $\|u^\varepsilon\|_{H_2^{s+2}(\mathbb{T}^n)}, \|v_1^\varepsilon\|_{H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}^n)} \le R$ and $\varepsilon \le \sigma^2$, then (6.3) and (6.9) coincide. This is why we have to introduce $T(R)$ in the statements of Theorem 6.2 and Theorem 6.3. For the proof Theorem 6.2 we just have to check whether the assumptions of Section 4.3 are satisfied:

(i) It is well known that $X = H_2^s(\mathbb{T}^n)$ and $Y = H_2^{s+2(1-\delta_Y)}(\mathbb{T}^n) \times \mathbb{R}^3$ are Banach spaces.

(ii) The Laplacian generates a bounded holomorphic C_0-semigroup $(\mathrm{e}^{t\Delta})_{t\geq 0}$ on any of the spaces $H_2^{s+\alpha}(\mathbb{T}^n)$, $\alpha \in \mathbb{R}$, which is given by

$$\mathrm{e}^{t\Delta} f(x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-|k|^2 t} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

where $\hat{f}(k)$ denotes the k-th Fourier coefficient. Accordingly, A generates an exponentially decaying holomorphic C_0-semigroup and B generates a holomorphic C_0-semigroup.

(iii) It follows from complex interpolation that the spaces $X_\alpha = H_2^{s+2\alpha}(\mathbb{T}^n)$ and $Y_\alpha = H_2^{s+2(1-\delta_Y)+2\alpha}(\mathbb{T}^n) \times \mathbb{R}^3$ are valid choices for our Banach scales.

(iv) We used cutoff techniques in order to ensure that f and g satisfy the continuity assumptions of Section 4.3.

(v) We introduced the dummy variable $\tilde{w}^\varepsilon$ the ensure that $f(0,0) = 0$ and $g(0,0) = 0$.

(vi) Theorem 2.1 ensures that there are constants M_A, C_A and C_B such that

$$\|\mathrm{e}^{tA}\|_{\mathcal{B}(X_1)} \leq M_A \mathrm{e}^{\omega_A t}, \quad \|\mathrm{e}^{tA}\|_{\mathcal{B}(X_{\gamma_X}, X_1)} \leq C_A t^{\gamma_X - 1}\mathrm{e}^{\omega_A t},$$
$$\|\mathrm{e}^{tA}\|_{\mathcal{B}(X_{\delta_X}, X_1)} \leq C_A t^{\delta_X - 1}\mathrm{e}^{\omega_A t},$$

and

$$\|\mathrm{e}^{tB}\|_{\mathcal{B}(Y_1)} \leq M_B \mathrm{e}^{\omega_B t}, \quad \|\mathrm{e}^{tB}\|_{\mathcal{B}(Y_{\delta_Y}, Y_1)} \leq C_B t^{\delta_Y - 1}\mathrm{e}^{\omega_B t}$$

hold for all $t > 0$. Since $(\mathrm{e}^{t\Delta})_{t\geq 0}$ is a bounded holomorphic semigroup on any of the spaces $H_p^{s+\alpha}(\mathbb{T}^n)$, $\alpha \in \mathbb{R}$, we may take ω_A to be an arbitrary number larger than -1.

(vii) We chose $\sigma > 0$ such that $L_f < 2/M$. If we take $M > 32C_A$ and ω_A close to -1, then we have

$$\omega_A + (2C_A L_f)^{1/\gamma_X}\left(\frac{1}{\gamma_X}\right)^{(1-\gamma_X)/\gamma_X} < \omega_A + \frac{1}{\sqrt{2}} < 0,$$

so that we may take ω_f such that

$$\omega_A + (2C_A L_f)^{1/\gamma_X}\left(\frac{1}{\gamma_X}\right)^{(1-\gamma_X)/\gamma_X} < \omega_f < 0.$$

Altogether, all the assumptions of Section 4.3 are satisfied and we obtain Theorem 6.2 Let us turn to Theorem 6.3. Our task now is to find the splitting

$$Y = Y_F^\zeta \oplus Y_S^\zeta$$

for any $\zeta > 0$ small enough. For the Stommel model we may simply take the truncation to certain Fourier modes. If $-(|k_0|+1)^2 < \zeta^{-1}\omega_A \le -|k_0|^2$ for some $k_0 \in \mathbb{N}$, then we take

$$\begin{aligned}\widetilde{Y}_S^\zeta &:= \operatorname{span}\big\{[x \mapsto e^{ikx}] : k \in \mathbb{Z},\ |k| \le |k_0| - 1\big\},\\ \widetilde{Y}_F^\zeta &:= \operatorname{cl}_{H_p^{s+2(1-\delta_Y)}(\mathbb{T})}\big(\operatorname{span}\big\{[x \mapsto e^{ikx}] : k \in \mathbb{Z},\ |k| \ge |k_0|\big\}\big),\end{aligned}$$

where $\operatorname{cl}_{H_p^{s+2(1-\delta_Y)}(\mathbb{T})} A$ means that we take the closure of a set

$$A \subset H_p^{s+2(1-\delta_Y)}(\mathbb{T}) \quad \text{in } H_p^{s+2(1-\delta_Y)}(\mathbb{T}).$$

Now we choose

$$Y_S^\zeta := \widetilde{Y}_S^\zeta \times \mathbb{R}^3, \quad Y_F^\zeta := \widetilde{Y}_F^\zeta \times \{0_{\mathbb{R}^3}\}.$$

These definitions indeed yield a splitting

$$Y = Y_F^\zeta \oplus Y_S^\zeta.$$

Let us check the conditions of Section 5.1.

(i) Since Y_S^ζ is finite-dimensional and since Y_F^ζ is defined as a closure, both spaces are closed. Moreover, in the Fourier image it is easy to see that the their projections commute with B.

(ii) By our construction $\widetilde{Y}_F^\zeta$ consists of all $f \in H_2^{s+2(1-\delta_Y)}(\mathbb{T})$ such that $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$ such that $|k| \le |k_0| - 1$. Therefore,

$$\widetilde{Y}_F^\zeta \cap H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})$$

consists of all $f \in H_2^{s+2+2(1-\delta_Y)}(\mathbb{T})$ such that $\widehat{f}(k) = 0$ for all $k \in \mathbb{Z}$ such that $|k| \le |k_0| - 1$. This makes Y_F^ζ a closed subspace of $Y_1 = H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}) \times \mathbb{R}^3$.

(iii) Obviously, Y_S^ζ is a closed subspace of Y_1 and thus the same holds trivially for $Y_S^\zeta \cap Y_1$. In addition, we know that

$$g \colon X_1 \times Y_1 \to Y_{\delta_Y}$$

is Lipschitz continuous and Plancherel's theorem yields

$$\| \operatorname{pr}_{Y_S^\zeta} \|_{\mathcal{B}(Y_{\delta_Y}, Y_1)} \le \zeta^{\delta_Y - 1}.$$

Hence, we obtain that

$$\operatorname{pr}_{Y_S^\zeta} g \colon X_1 \times Y_1 \to Y_1$$

is Lipschitz continuous with Lipschitz constant $L_g \zeta^{\delta_Y - 1}$.

(iv) Y_S^ζ is a finite-dimensional space. Therefore, the realization of B in Y_S^ζ is bounded and thus generates a C_0-group $(\mathrm{e}^{tB_{Y_S^\zeta}})_{t\in\mathbb{R}}$. It is obvious that is group coincides with $(\mathrm{e}^{tB}|_{Y_S^\zeta})$ for $t \geq 0$.

(v) We show that the realization of B in Y_F^ζ has 0 in its resolvent set by simply giving a formula for the inverse. It is given by

$$B_{Y_F^\zeta}^{-1}\colon H_2^{s+2+2(1-\delta_Y)}(\mathbb{T}) \times \{0_{\mathbb{R}^3}\} \to H_2^{s+2(1-\delta_Y)}(\mathbb{T}) \times \{0_{\mathbb{R}^3}\},$$

$$\Big(\sum_{\substack{k\in\mathbb{Z},\\ |k|\geq|k_0|}} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}, 0, 0, 0\Big) \mapsto \Big(\sum_{\substack{k\in\mathbb{Z},\\ |k|\geq|k_0|}} \frac{\hat{f}(k)\mathrm{e}^{\mathrm{i}kx}}{|k|^2}, 0, 0, 0\Big).$$

This is well-defined if ζ is small as $k = 0$ does not appear in the sum.

(vi) We have already observed that $(\mathrm{e}^{tB})_{t\geq 0}$ is given by

$$\mathrm{e}^{tB} f = \Big[x \mapsto \sum_{k\in\mathbb{Z}} \mathrm{e}^{-|k|^2 t} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}\Big].$$

Thus, Plancherel's theorem shows that for $y_S \in Y_S^\zeta$ and $t \geq 0$ it holds that

$$\|\mathrm{e}^{-tB} y_S\|_{H_2^{s+2(1-\delta_Y)}(\mathbb{T})} \leq \mathrm{e}^{(|k_0|-1)^2 t},$$

so that we may take

$$N_S^\zeta := -\zeta^{-1}\omega_A - \big(|k_0| - 1\big)^2.$$

Since $-(|k_0| + 1)^2 < \zeta^{-1}\omega_A \leq -|k_0|^2$ it holds that $N_S^\zeta > 0$. Similarly, we can take

$$N_F^\zeta = -\zeta^{-1}\omega_A - |k_0|^2 + |k_0| + 1,$$

so that $N_S^\zeta - N_F^\zeta = |k_0| \geq \sqrt{-\zeta^{-1}\omega_A}$. Therefore, we have $N_S^\zeta - N_F^\zeta > \frac{1}{2}\zeta^{-1/2}$ if ζ is small and if ω_A is close to -1. For (5.1) we now have to verify that there is a constant $C > 0$ which is independent of $y_F \in H^{s+2+2(1-\delta_Y)}$, $|k| \geq |k_0|$, $t \geq 0$ and $\zeta > 0$ small enough such that

$$\Big(\sum_{|k|^2\geq|k_0|^2} \big(1 + |k|^2\big)^{s+2+2(1-\delta_Y)} \hat{y}_{F,k}^2 e^{-2|k|^2 t}\Big)^{1/2}$$
$$\leq C\big(t\zeta\big(N_S^\zeta - N_F^\zeta\big)\big)^{\delta_Y-1} e^{(N_F^\zeta+\zeta^{-1}\omega_A)t}\Big(\sum_{|k|^2\geq|k_0|^2} \big(1 + |k|^2\big)^{s+2} \hat{y}_{F,k}^2\Big)^{1/2}.$$

We do this buy showing that there is a constant $C > 0$ independent of $|k| \geq |k_0|$, $t \geq 0$ and $\zeta > 0$ such that

$$\big(1 + |k|^2\big)^{1-\delta_Y} e^{-|k|^2 t} \leq C\big(t\zeta\big(N_S^\zeta - N_F^\zeta\big)\big)^{\delta_Y-1} e^{(N_F^\zeta+\zeta^{-1}\omega_A)t}.$$

Using the definition of N_F^ζ and the fact that $N_S^\zeta - N_F^\zeta \sim \zeta^{-1/2} \sim |k_0|$ this is the case if and only if there is a constant $C > 0$ such that

$$|k|^{2(1-\delta_Y)}e^{-|k|^2 t} \sim \left(1+|k|^2\right)^{1-\delta_Y} e^{-|k|^2 t} \leq C\left(\frac{t}{|k_0|}\right)^{\delta_Y - 1} e^{(-|k_0|^2+|k_0|+1)t}.$$

Thus, it suffices to show that

$$(t,k,k_0) \mapsto \left(\frac{t|k|^2}{|k_0|}\right)^{1-\delta_Y} e^{(|k_0|^2-|k|^2-|k_0|-1)t}$$

for $t \geq 0$ and $|k| \geq |k_0|$ is bounded. For fixed k, k_0, one can compute the critical points in t. This yields that the maximum is attained at

$$t = \frac{1-\delta_Y}{|k|^2+|k_0|+1-|k_0|^2}$$

and is given by

$$\left(\frac{(1-\delta_Y)|k|^2}{\left(|k|^2+|k_0|+1-|k_0|^2\right)|k_0|}\right)^{1-\delta_Y} e^{\delta_Y - 1}.$$

But for $|k| \geq |k_0|$ this expression is decreasing in $|k|$ so that its maximum is attained at $|k_0| = |k|$ and is given by

$$\left(\frac{(1-\delta_Y)|k_0|^2}{\left(|k_0|+|k_0|^2\right)}\right)^{1-\delta_Y} e^{\delta_Y - 1},$$

which is bounded by $(1-\delta_Y)^{1-\delta_Y} e^{\delta_Y - 1}$. This shows that (5.1) is satisfied.

(vii) If we take $\zeta > 0$ small enough and $\varepsilon < c(\omega_f/\omega_A)\zeta$ for some constant $c \in (0,1)$, then (5.3) is satisfied for $\delta_Y > \frac{1}{2}$.

Altogether, all the assumptions we need to apply our theory are satisfied. The application of our abstract results to the diffusive Stommel model to obtain Theorem 6.3 is straightforward. We should point out though that for the proof of Theorem 6.3 (e) one formally has different initial conditions for $(u^\varepsilon, w^\varepsilon)$ and (u_0^ζ, w_0^ζ) due to our dummy variables: For (6.3) we have $z_2 = \sqrt{\varepsilon}$ and for (6.4) we have $z_2 = 0$. However, the well-posedness (6.3) ensures that the difference of the solutions of (6.3) with $z_2 = \sqrt{\varepsilon}$ and $z_2 = 0$ are of the order $O(\sqrt{\varepsilon})$ on bounded time intervals. Thus, for the derivation of Theorem 6.3 (e) we can just use Corollary 5.9 together with an application of the triangle inequality.

6.2. The doubly-diffusive FitzHugh–Nagumo equation. The techniques we used for the Stommel model can also be applied to the doubly-diffusive FitzHugh–Nagumo equation, which has recently been of interest in pattern formation [8]. It is a modification of the classical FitzHugh–Nagumo equation and given by

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= \Delta u^\varepsilon + u^\varepsilon(1-u^\varepsilon)(u^\varepsilon - a) - w^\varepsilon, \\ \partial_t w^\varepsilon &= \Delta w^\varepsilon + u^\varepsilon - \gamma w^\varepsilon, \\ u^\varepsilon(0) &= u_0, \quad w^\varepsilon(0) = v_0, \end{aligned} \tag{6.10}$$

where $\gamma > 0$ and $a \in (0, \frac{1}{2})$. Of course, it is well known from many works (see [21] and references therein) that at the two fold points of nonlinearity, there is loss of normal hyperbolicity even without the Laplacian terms. Our methods can be applied to describe the dynamics in the stable parts of the phase space away from the set of bifurcation points. In order to study the full dynamics, additional techniques are needed to treat the behavior close to the bifurcation points. First steps in this direction for a transcritical bifurcation have been taken in [12], but much more work has to be done before this can be applied to describe pattern forming phenomena. Hence, we just illustrate our methods locally at a point on an attracting branch of the critical manifold. We simply select this point as the origin but other points could be treated similarly upon translation of the coordinates locally. Furthermore, compared to the Stommel model we have the additional difficulty that the nonlinearity in the fast variable does not get small as $\varepsilon \to 0$. However, we have the advantage that we do not have to introduce dummy variables and that all terms are actually linear in the slow variable. The latter property will help us to derive better convergence results, since we can avoid certain cutoffs that would cause problems with different topologies. This way, we obtain:

Theorem 6.5. *Let $\mathbb{E} \in \{\mathbb{T}, \mathbb{R}\}$, i.e., let $\mathbb{E}$ either be the torus or the real line. We write $(u^\varepsilon, w^\varepsilon)$ for the strict solution of (6.10) with $\varepsilon > 0$ and (u^0, w^0) for corresponding slow flow. Then there is a neighborhood $U \subset H_2^2(\mathbb{E}^n)$ of 0 which only depends on a, and constants $\varepsilon_0 > 0$ and $C, c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, $u_0 \in U$ and $v_0 \in H_2^2(\mathbb{E}^n)$ it holds that*

$$\sup_{0 \le t \le T(R,U)} \big(\|u^\varepsilon(t) - u^0(t)\|_{H_2^2(\mathbb{E}^n)} + \|w^\varepsilon(t) - v^0(t)\|_{H_2^2(\mathbb{E}^n)} \big) \le C(\varepsilon + \mathrm{e}^{-c\varepsilon^{-1}t}),$$

where $T(R, U)$ is defined by

$$T(R,U) := \inf\big\{t \in [0,T] : u^0 \notin U \text{ or } u^\varepsilon \notin U\big\}.$$

Theorem 6.6. *There is a neighborhood $U \subset H_2^2(\mathbb{T})$ of 0 which only depends on a, a constant $\zeta_0 > 0$, and a family of finite-dimensional manifolds $S_{\varepsilon,\zeta} \subset H_2^2(\mathbb{T}) \times H_2^2(\mathbb{T})$ with $0 < \zeta \le \zeta_0$ and $0 < \varepsilon \le C(\omega_f/\omega_A)\zeta$ for some $C \in (0, 1)$ such that the following assertions hold:*

(a) *For each $\zeta \in (0, \zeta_0]$ there is a splitting*

$$L_2(\mathbb{T}) = Y_F^\zeta \oplus Y_S^\zeta,$$

where Y_S^ζ is the projection of $L_2(\mathbb{T})$ to the k-th Fourier modes with $|k|$ being smaller than a certain number $k(\zeta)$ depending on ζ. We also have that Y_F^ζ is the projection to the remaining Fourier modes.

(b) *The manifolds $S_{\varepsilon,\zeta}$ are given as the graph of a differentiable mapping*

$$h^{\varepsilon,\zeta}\colon \big(Y_S^\zeta, \|\cdot\|_{H_2^{s+2}(\mathbb{T})}\big) \to H_2^2(\mathbb{T}) \times \big(Y_F^\zeta \cap H_2^2(\mathbb{T}), \|\cdot\|_{H_2^2(\mathbb{T})}\big).$$

(c) *The intersection of $S_{\varepsilon,\zeta}$ with $U \times Y$ is a slow manifold which is locally invariant under the semiflow generated by* (6.10)*, i.e., the semiflow can only leave $S_{\varepsilon,\zeta} \cap U \times Y$ through its boundary.*

(d) *Let*

$$S_{0,\zeta,U} := \big\{(u,w) \in S_0 : w \in Y_S^\zeta\big\} \cap U \times Y$$

be the intersection of $U \times Y$ with the submanifold of the critical manifold which consists of all points whose slow components are elements of Y_S^ζ. Then constant $C > 0$ such that

$$\operatorname{dist}(S_{\varepsilon,\zeta}, S_{0,\zeta}) \le C(\varepsilon + \zeta^{1/2}) \le C\zeta^{1/2}.$$

(e) *Suppose that $u_0 \in U$ and let (u_ζ^0, w_ζ^0) be the solution of the truncated slow subsystem of* (6.10) *given by*

$$\begin{aligned} 0 &= \Delta u_\zeta^0 - u_\zeta^0(1-u_\zeta^0)(u_\zeta^0 - a) - v_\zeta^0, \\ \partial_t w_\zeta^0 &= \operatorname{pr}_{Y_S^\zeta}\big[\Delta w_\zeta^0 + u_\zeta^0 - \gamma v_\zeta^0\big], \\ u^\varepsilon(0) &= h^0(\operatorname{pr}_{Y_S^\zeta} v_0), \quad w^\varepsilon(0) = \operatorname{pr}_{Y_S^\zeta} v_0. \end{aligned} \tag{6.11}$$

Assume that $(u_0, v_0) \in S_{\varepsilon,\zeta} \cap U \times Y$. Then for each $T > 0$ there is a constant $C > 0$ such that

$$\sup_{0 \le t \le T(U)} \big(\|u^\varepsilon(t) - u_\zeta^0(t)\|_{H_2^2(\mathbb{T})} + \|w_\zeta^0(t) - v^0(t)\|_{H_2^2(\mathbb{T})}\big) \le C\zeta^{1/2},$$

where $T(R,U)$ is defined by

$$T(U) := \inf\big\{t \in [0,T] : u_\zeta^0 \notin U \text{ or } u^\varepsilon \notin U\big\}.$$

Remark 6.7. One might wonder why we have to introduce the neighborhood U in Theorem 6.5 and Theorem 6.6. The reason is that we have only treated the attracting case in our general theory. In order to ensure that we stay in this attracting case, we use cutoff techniques to modify the nonlinearity in the fast variable where it would be positive. However, this means that our results are only related to the system (6.10) as long as the fast variable stays in the region where we did not modify the nonlinearity.

Let us give a sketch on how these results can be obtained. Again, we only treat the case $\mathbb{E} = \mathbb{T}$. First, we rescale the slow variable and define $v^\varepsilon = \frac{2}{a} w^\varepsilon$ so that (6.10) turns into

$$\begin{aligned} \varepsilon \partial_t u^\varepsilon &= \Delta u^\varepsilon + u^\varepsilon(1-u^\varepsilon)(u^\varepsilon - a) - \frac{a}{2} v^\varepsilon, \\ \partial_t v^\varepsilon &= \Delta v^\varepsilon + \frac{2}{a} u^\varepsilon - \gamma v^\varepsilon, \\ u^\varepsilon(0) &= u_0, \quad v^\varepsilon(0) = \frac{2}{a} v_0. \end{aligned} \tag{6.12}$$

Now we make the following choices:

- As underlying spaces we choose $X = L_2(\mathbb{T}^n)$ and $Y = L_2(\mathbb{T}^n)$.
- The linear operator in the fast variable is given by

$$A\colon L_2(\mathbb{T}^n) \supset H_2^2(\mathbb{T}^n) \to L_2(\mathbb{T}^n), \quad u \mapsto \Delta u - au.$$

The linear operator in the slow variable is given by

$$B\colon L_2(\mathbb{T}^n) \supset H_2^2(\mathbb{T}^n) \to L_2(\mathbb{T}^n), \quad u \mapsto \Delta u - \gamma u.$$

- The Banach scales are given by $X_\alpha = H_2^{2\alpha}(\mathbb{T}^n)$ and $Y_\alpha = H_2^{2\alpha}(\mathbb{T}^n)$.
- We choose $\gamma_X = \delta_X = \delta_Y = 1$. This is the main difference to the Stommel model and will lead to better convergence rates. With these parameters, it suffices to choose a differentiable mapping $f\colon X_1 \times Y \to X$ which is also differentiable as a mapping from $X_1 \times Y_1$ to X_1 such that

$$\begin{aligned} \|Df(x,y)\|_{\mathcal{B}(X_1 \times Y, X)} &\le L_f < a, \\ \|Df(x,y)\|_{\mathcal{B}(X_1 \times Y_1, X_1)} &\le L_f < a. \end{aligned}$$

Moreover, for the nonlinearity in the slow variable we may choose a continuous mapping $g\colon X \times Y \to Y$ which is differentiable as a mapping $g\colon X_1 \times Y_1 \to Y_1$ with bounded derivative. With our choices of spaces this translates into

$$\begin{aligned} f\colon H_2^2(\mathbb{T}^n) \times L_2(\mathbb{T}^n) &\to L_2(\mathbb{T}^n), \\ g\colon L_2(\mathbb{T}^n) \times L_2(\mathbb{T}^n) &\to L_2(\mathbb{T}^n), \end{aligned}$$

and

$$\begin{aligned} \|\mathrm{D} f(x,y)\|_{\mathcal{B}(H_2^2(\mathbb{T}^n) \times L_2(\mathbb{T}^n), L_2(\mathbb{T}^n))} &\le L_f < a, \\ \|\mathrm{D} f(x,y)\|_{\mathcal{B}(H_2^2(\mathbb{T}^n) \times H_2^2(\mathbb{T}^n), H_2^2(\mathbb{T}^n))} &\le L_f < a, \\ \|\mathrm{D} g(x,y)\|_{\mathcal{B}(H_2^2(\mathbb{T}^n) \times H_2^2(\mathbb{T}^n), H_2^2(\mathbb{T}^n)))} &\le L_g. \end{aligned}$$

For the definition of f, we choose a small number $1 > \sigma > 0$ and a C^1-function

$$\chi\colon H_2^2(\mathbb{T}^n) \to [0,1]$$

such that

$$\begin{aligned} \chi(u) &= 1 \quad \text{if } \|u\|_{H_2^2(\mathbb{T}^n)} \leq \sigma^2, \\ \chi(u) &= 0 \quad \text{if } \|u\|_{H_2^2(\mathbb{T}^n)} \geq 2\sigma, \end{aligned}$$

and $\|D\chi\|_{\mathcal{B}(H_2^2(\mathbb{T}^n);\mathbb{R})} \leq \sigma$. Then we define

$$\begin{aligned} f\colon H_2^2(\mathbb{T}^n) \times L_2(\mathbb{T}^n) &\to L_2(\mathbb{T}^n), \\ (u,v) &\mapsto -(\chi(u)u)^3 + (1+a)(\chi(u)u)^2 - \frac{a}{2}v, \\ g\colon L_2(\mathbb{T}^n) \times L_2(\mathbb{T}^n) &\to L_2(\mathbb{T}^n), \\ (u,v) &\mapsto \frac{2}{a}u. \end{aligned}$$

If σ is small enough, then it will hold that $L_f < a$.

With these choices, the equation

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= Au^\varepsilon + f(u^\varepsilon, v^\varepsilon), \\ \partial_t v^\varepsilon &= Bv^\varepsilon + g(u^\varepsilon, v^\varepsilon) \end{aligned}$$

is equivalent to (6.12) as long as $\|u^\varepsilon\|_{H_2^2(\mathbb{T}^n)} \leq \sigma^2$. Concerning the splitting $Y = Y_F^\zeta \oplus Y_S^\zeta$ we make analogous choices as for the Stommel model. Now, as for the Stommel model one can verify that our theory can be applied.

6.3. The Maxwell–Bloch equations. We consider the Maxwell–Bloch equations in the slow time scale

$$\begin{aligned} \varepsilon\partial_t u_1^\varepsilon &= \mu w^\varepsilon u_2^\varepsilon - (1+i\delta)u_1^\varepsilon, \\ \varepsilon\partial_t u_2^\varepsilon &= \gamma_\|(\lambda + 1 - u_2^\varepsilon) - \frac{\mu}{2}\big(\overline{w^\varepsilon}u_1^\varepsilon + w^\varepsilon\overline{u_1^\varepsilon}\big), \\ \partial_t w^\varepsilon &= -\partial_x w^\varepsilon + \kappa\Big(\frac{1}{\mu}u_1^\varepsilon - w^\varepsilon\Big), \\ u_1^\varepsilon(0) &= u_{0,1}, \quad u_2^\varepsilon(0) = u_{0,2}, \quad w^\varepsilon(0) = v_0, \end{aligned} \tag{6.13}$$

on the one-dimensional torus $\mathbb{T}$. Here, $\gamma_\|, \kappa, \delta, \lambda > 0$ are certain parameters and $\mu = \sqrt{\lambda\gamma_\|}$. The existence of slow manifolds for this system which are given as graphs over a certain subset of the slow variable space has been shown in [25] by a direct approach. We want to illustrate that these equations are a special case accessible through our more general methods. In order to be consistent with [25], we work with the scale generated by $(C(\mathbb{T}), -(\partial_x + \kappa))$ and write $C^{-1}(\mathbb{T})$ for the extrapolation space of $(C(\mathbb{T}), -(\partial_x + \kappa))$. Concerning the cutoff of nonlinearities, this scale is more difficult and easier at the same time. It is more difficult because smooth functions with bounded support do not exist on these spaces; see [7]. But on the other hand, cutoff techniques are easier because we may use superposition operators instead. Although the set of globally Lipschitz continuous superposition operators on $C^k(\mathbb{T})$ for $k \in \mathbb{N}$ is quite small (see, for example, [3, Theorem 8.4]), they have nice properties in $C(\mathbb{T})$; see [3, Section 6]. In particular, if we choose a globally Lipschitz continuous C^∞-function $\chi\colon \mathbb{R}^n \to [0,1]$ which is equal to 1 in a ball of a given radius and which is 0 outside of a ball of a given radius, then the superposition operator

$$p_\chi\colon C(\mathbb{T}) \times C(\mathbb{T}) \to C(\mathbb{T}), \quad (u,v) \mapsto p(\chi(u)u, \chi(v)v)$$

for a polynomial $p\colon \mathbb{R}^2 \to \mathbb{K}$ is smooth and globally Lipschitz continuous.

Theorem 6.8. *Let $R > 0$ be large enough, $T > 0$ and $w_0 \in C(\mathbb{T}, \mathbb{C})$ be fixed. Let further $(u^\varepsilon, w^\varepsilon)$ be the strict solution of (6.13) with $\varepsilon > 0$ and let (u^0, w^0) be the corresponding slow flow. Then there are a neighborhood $U \subset C(\mathbb{T}, \mathbb{C})$ of w_0 and constants $\varepsilon_0, C, c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, $u_0 \in C(\mathbb{T}; \mathbb{C}) \times C(\mathbb{T}; \mathbb{R})$ with $\|u_{0,1}\|_{C(\mathbb{T};\mathbb{C})} + \|u_{0,2}\|_{C(\mathbb{T};\mathbb{R})} \le R$ and $v_0 \in U$ it holds that*

$$\sup_{0 \le t \le T(R,U)} \big(\|u^\varepsilon(t) - u^0(t)\|_{C(\mathbb{T};\mathbb{C}) \times C(\mathbb{T};\mathbb{R})} + \|w^\varepsilon(t) - w^0(t)\|_{C(\mathbb{T};\mathbb{C})}\big) \le C(\varepsilon + e^{-c\varepsilon^{-1}t}),$$

where $T(R,U)$ is defined by

$$T(R,U) := \inf\big\{t \in [0,T] : \max\big\{\|u^0(t)\|_{C(\mathbb{T};\mathbb{C}) \times C(\mathbb{T};\mathbb{R})}, \|u^\varepsilon(t)\|_{C(\mathbb{T};\mathbb{C}) \times C(\mathbb{T};\mathbb{R})}\big\} > R \text{ or } w^0(t) \notin U \text{ or } w^\varepsilon(t) \notin U\big\}. \tag{6.14}$$

Theorem 6.9. *Let $R > 0$ be large enough and let $w_0 \in C(\mathbb{T}, \mathbb{C})$ be fixed. Then there are $\varepsilon_0 > 0$, a neighborhood $U \subset C(\mathbb{T}, \mathbb{C})$ of w_0 and a family of infinite-dimensional slow manifolds $S_\varepsilon \subset C(\mathbb{T}, \mathbb{C}) \times C(\mathbb{T}, \mathbb{R}) \times C(\mathbb{T}, \mathbb{C})$ with $0 < \varepsilon \le \varepsilon_0$ such that the following assertions hold:*

(a) *The slow manifold S_ε is given as the graph of a differentiable mapping*

$$h^\varepsilon\colon \big(U, \|\cdot\|_{C(\mathbb{T},\mathbb{C})}\big) \to C(\mathbb{T}, \mathbb{C}) \times C(\mathbb{T}, \mathbb{R}).$$

(b) *S_ε is locally invariant under the semiflow generated by (6.13), i.e., the semiflow can only leave S_ε through its boundary.*

(c) *Let*

$$S_{0,U} := \{(u, w) \in S_0 : w \in U\}$$

be the submanifold of the critical manifold which consists of all points whose slow components are elements of U*. Then there is a constant depending on* R *such that*

$$\mathrm{dist}(S_\varepsilon, S_{0,U}) \leq C\varepsilon.$$

(d) *Suppose that* $\|u_0\|_{C^1(\mathbb{T};\mathbb{C})\times C(\mathbb{T};\mathbb{R})} \leq R$*,* $v_0 \in U$*. Assume that* $(u_0, v_0) \in S_\varepsilon$*. Then for each* $T > 0$ *there is a constant* $C > 0$ *such that*

$$\sup_{0\leq t\leq T(R,U)} \left(\|u^\varepsilon(t) - u^0(t)\|_{C(\mathbb{T};\mathbb{C})\times C(\mathbb{T};\mathbb{R})} + \|w^\varepsilon(t) - w^0(t)\|_{C(\mathbb{T};\mathbb{C})}\right) \leq C\varepsilon,$$

where $T(R, U)$ *is again defined by* (6.14).

First, we rescale (6.13) so that the constants in front of the nonlinearities in the fast variable can be chosen small. We define $\widetilde{v}^\varepsilon := \sigma^{-1}w^\varepsilon$ for some $\sigma > 0$ and obtain

$$\begin{aligned}
\varepsilon\partial_t u_1^\varepsilon &= \sigma\mu\widetilde{v}^\varepsilon u_2^\varepsilon - (1 + i\delta)u_1^\varepsilon,\\
\varepsilon\partial_t u_2^\varepsilon &= -\gamma_\| u_2^\varepsilon + \gamma_\|(1 + \lambda) - \frac{\sigma\mu}{2}\left(\overline{\widetilde{v}^\varepsilon}u_1^\varepsilon + \widetilde{v}^\varepsilon\overline{u_1^\varepsilon}\right),\\
\partial_t\widetilde{v}^\varepsilon &= -\partial_x\widetilde{v}^\varepsilon + \kappa\left(\frac{1}{\sigma\mu}u_1^\varepsilon - \widetilde{v}^\varepsilon\right),\\
u_1^\varepsilon(0) &= u_{0,1}, \quad u_2^\varepsilon(0) = u_{0,2}, \quad \widetilde{v}^\varepsilon(0) = \frac{v_0}{\sigma}.
\end{aligned} \tag{6.15}$$

A straightforward calculation shows that the critical manifold to this rescaled equation is given as the graph of

$$h_\sigma^0\left(\frac{v_0}{\sigma}\right) = \begin{pmatrix} \mu(1 - i\delta)\frac{(\lambda+1)\sigma v_0}{1+\delta^2+\sigma^2\lambda|v_0|^2} \\ \frac{(1+\delta^2)(\lambda+1)}{1+\delta^2+\sigma^2\lambda|v_0|^2} \end{pmatrix}. \tag{6.16}$$

In particular, h_σ^0 will be bounded in the spaces we choose later with a bound that can be chosen independently of σ. This fact will be useful for the cutoff procedure of the nonlinearities.

As for the Stommel model, we introduce the dummy variable $\widetilde{w}^\varepsilon$ to ensure that the nonlinearities vanish at 0. This way, we may rewrite (6.15) as

$$\begin{aligned}
\varepsilon\partial_t u_1^\varepsilon &= \sigma\mu\left(\widetilde{v}^\varepsilon - \frac{v_0}{\sigma}\right)u_2^\varepsilon - (1 + i\delta)u_1^\varepsilon + \mu v_0 u_2^\varepsilon,\\
\varepsilon\partial_t u_2^\varepsilon &= -\frac{\mu}{2}\left(\overline{v_0}u_1^\varepsilon + v_0\overline{u_1^\varepsilon}\right) - \gamma_\| u_2^\varepsilon + \sigma\widetilde{w}^\varepsilon - \frac{\sigma\mu}{2}\left(\overline{\left(\widetilde{v}^\varepsilon - \frac{v_0}{\sigma}\right)}u_1^\varepsilon + \left(\widetilde{v}^\varepsilon - \frac{v_0}{\sigma}\right)\overline{u_1^\varepsilon}\right),\\
\partial_t\widetilde{v}^\varepsilon &= -\partial_x\widetilde{v}^\varepsilon + \kappa\left(\frac{1}{\sigma\mu}u_1^\varepsilon - \widetilde{v}^\varepsilon\right),\\
\partial_t\widetilde{w}^\varepsilon &= 0,\\
u_1^\varepsilon(0) &= u_{0,1}, \quad u_2^\varepsilon(0) = u_{0,2}, \quad \widetilde{v}^\varepsilon(0) = \frac{v_0}{\sigma}, \quad \widetilde{w}^\varepsilon(0) = \frac{(\lambda + 1)\gamma_\|}{\sigma}.
\end{aligned} \tag{6.17}$$

Now we make the following choices:

- As base spaces we take

$$X := C(\mathbb{T};\mathbb{C}) \times C(\mathbb{T};\mathbb{R}) \quad \text{and} \quad Y := C^{-1}(\mathbb{T};\mathbb{C}) \times \mathbb{C}.$$

Here, we identify $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ and treat it as a real vector space. This way complex conjugation is a differentiable mapping.

- The fast variable is given by $u^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon)$ and the slow variable is given by $v^\varepsilon := (\widetilde{v}^\varepsilon, \widetilde{w}^\varepsilon)$.

- The linear operator A of the fast variable is even a bounded operator

$$A\colon X \to X, \quad \begin{pmatrix} \mathrm{Re}(u_1) \\ \mathrm{Im}(u_1) \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} -\mathrm{Re}(u_1) + \delta \mathrm{Im}(u_1) + \mu \mathrm{Re}(v_0) u_2 \\ -\delta \mathrm{Re}(u_1) - \mathrm{Im}(u_1) + \mu \mathrm{Im}(v_0) u_2 \\ -\mu \mathrm{Re}(v_0)\mathrm{Re}(u_1) - \mu \mathrm{Im}(v_0)\mathrm{Im}(u_1) - \gamma_\| \end{pmatrix},$$

i.e., it is given by the multiplication with matrix

$$\begin{pmatrix} -1 & \delta & \mu\mathrm{Re}(v_0) \\ -\delta & -1 & \mu\mathrm{Im}(v_0) \\ -\mu\mathrm{Re}(v_0) & \mu\mathrm{Im}(v_0) & -\gamma_\| \end{pmatrix}.$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of this matrix have a negative real part. Let

$$K := |\max\{\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3)\}|.$$

The linear operator B of the slow variable is given by

$$B\colon Y \supset D(B) \to Y, \quad (v_1, v_2) \mapsto (-\partial_x v_1 - \kappa v_1, 0),$$

where the domain is given by

$$D(B) = C(\mathbb{T};\mathbb{C}) \times \mathbb{C}.$$

- We choose the parameters $\gamma_X = \delta_Y = 1$ and $\delta_X = 0$. Thus, we only need the Banach scales for $\alpha \in \{0, 1\}$. Since A is a bounded operator, the Banach scale in the fast variable is just given by $X = X_1$. For the fast variable we have $Y_1 = C(\mathbb{T};\mathbb{C}) \times \mathbb{C}$ endowed with the norm

$$\|(v_1, v_2)\|_{\mathbb{Y}_1} = \|v_1\|_{C(\mathbb{T};\mathbb{C})} + |v_2|.$$

- The nonlinearities $\widetilde{f}, \widetilde{g}$ are given by

$$\widetilde{f}\colon X \times Y_1 \to X, \quad \begin{pmatrix} (x_1, x_2)^T \\ (y_1, y_2)^T \end{pmatrix} \mapsto \begin{pmatrix} \sigma\mu\big(y_1 - \frac{v_0}{\sigma}\big)x_2 \\ \sigma y_2 - \frac{\sigma\mu}{2}\Big(\overline{\big(y_1 - \frac{v_0}{\sigma}\big)}x_1 - \big(y_1 - \frac{v_0}{\sigma}\big)\overline{x_1}\Big) \end{pmatrix},$$

$$g\colon X \times Y \to Y, \quad \begin{pmatrix} (x_1, x_2)^T \\ (y_1, y_2)^T \end{pmatrix} \mapsto \begin{pmatrix} \frac{\kappa}{\sigma\mu} x_1 \\ 0 \end{pmatrix}.$$

In order to make $\widetilde{f}$ globally Lipschitz continuous, we use cutoff functions again, this time in form of superposition operators. Suppose that the critical manifold is bounded by

$$M := \sup_{v \in C^1(\mathbb{T};\mathbb{C}), 0<\sigma<1} \|h^0_\sigma(v)\|_{C^1(\mathbb{T};\mathbb{C}) \times C^1(\mathbb{T};\mathbb{R})}.$$

Let further $R \geq 2M$ and $\chi_1\colon \mathbb{R} \to [0,1]$ be a C^∞-function such that

$$\begin{aligned} \chi_1(x) &= 1 \quad \text{for } |x| \leq 2R, \\ \chi_1(x) &= 0 \quad \text{for } |x| \geq 2R + 2. \end{aligned}$$

Moreover, let $\widetilde{K} > 0$ large enough and $\chi_2\colon \mathbb{R} \to [0,1]$ be a C^∞-function such that

$$\begin{aligned} \chi_2(x) &= 1 && \text{for } |x| \leq K/2\widetilde{K}\mu\sigma, \\ \chi_2(x) &= 0 && \text{for } |x| \geq K/\widetilde{K}\mu\sigma, \\ |\chi'(x)| &\leq 3\widetilde{K}\mu\sigma/K && \text{for all } x \in \mathbb{R}. \end{aligned}$$

Now we define

$$f\colon X \times Y \to X,$$

$$\begin{pmatrix} (x_1, x_2)^T \\ (y_1, y_2)^T \end{pmatrix} \mapsto \begin{pmatrix} \sigma\mu\big(y_1 - \frac{v_0}{\sigma}\big)x_2\chi_1(x_2)\chi_2\big(y_1 - \frac{v_0}{\sigma}\big) \\ \sigma y_2 - \frac{\sigma\mu}{2}\Big(\big(\overline{y_1 - \frac{v_0}{\sigma}}\big)x_1 - \big(y_1 - \frac{v_0}{\sigma}\big)\overline{x_1}\Big)\chi_1(x_1)\chi_2\big(y_1 - \frac{v_0}{\sigma}\big) \end{pmatrix}.$$

With these choices it holds that (6.13) is given by

$$\begin{aligned} \varepsilon\partial_t u^\varepsilon &= Au^\varepsilon + f\left(u^\varepsilon, v_1^\varepsilon, \frac{\gamma_\parallel(1+\lambda)}{\sigma}\right), \\ \partial_t v^\varepsilon &= Bv^\varepsilon g(u^\varepsilon, v^\varepsilon), \\ u_1^\varepsilon(0) &= u_{0,1}, \quad u_2^\varepsilon(0) = u_{0,2}, \quad \widetilde{v}^\varepsilon(0) = \frac{v_0}{\sigma}, \end{aligned} \tag{6.18}$$

as long as $\|u_1^\varepsilon\|_{C(\mathbb{T};\mathbb{C})} \leq R$, $\|u_2^\varepsilon\|_{C(\mathbb{T};\mathbb{R})} \leq R$ and $\|\sigma v_1^\varepsilon - v_0\|_{C(\mathbb{T};\mathbb{C})} \leq K/10\widetilde{K}\mu$. Let us now check the conditions of Section 4.3 for this example.

(i) It is well known that $X = C(\mathbb{T};\mathbb{C}) \times C(\mathbb{T};\mathbb{R})$ and $Y = C^{-1}(\mathbb{T};\mathbb{C}) \times \mathbb{C}$ are Banach spaces.

(ii) Since all eigenvalues of A have a negative real part and since A is bounded, it follows that it generates an exponentially stable analytic semigroup. Moreover, it is well known and straightforward to verify that

$$\partial_x\colon C^{-1}(\mathbb{T};\mathbb{C}) \supset C(\mathbb{T};\mathbb{C}) \to C^{-1}(\mathbb{T};\mathbb{C}), \quad v \mapsto v$$

generates the translation group $(T(t))_{t\in\mathbb{R}}$ given

$$T(t)v(x) = v(t + x).$$

Therefore, also B generates a C_0-group which even is exponentially decaying.

(iii) Since $\gamma_X = \delta_Y = 1$ and $\delta_X = 0$, we only need the spaces X, Y, X_1, Y_1 which we already defined. If we wanted, we could complete the scales by adding Hölder spaces, but this is not necessary for our considerations.

(iv) The differentiability of $f : X \times Y_1 \to X$ and $g : X \times Y \to Y$ in the real sense is obvious. It is also clear that $g : X_1 \times Y_1 \to Y_1$ is Lipschitz continuous. We also have that $f : X \times Y_1 \to X$ is globally Lipschitz continuous due to the cutoff. We need the Lipschitz constant of f to be smaller than the decay rate of e^{tA}, i.e., smaller than K. But if $\sigma \to 0$ and $\widetilde{K} \to \infty$, then we have that

$$\|\mathrm{D}f(x, y)\|_{\mathcal{B}(X\times Y_1,X)} \to 0.$$

This shows that both Lipschitz conditions on f hold true with small Lipschitz constant L_f.

(v) We introduced the dummy variable $\widetilde{w}^\varepsilon$ so that $f(0, 0) = 0$ and $g(0, 0) = 0$.

(vi) Let $\omega_A \in (-K, 0)$ be close to $-K$. Since we have $\gamma_X = \delta_Y = 1$ and $\delta_X = 0$, the estimates

$$\|\mathrm{e}^{tA}\|_{\mathcal{B}(X_1)} \le M_A \mathrm{e}^{\omega_A t}, \quad \|\mathrm{e}^{tA}\|_{\mathcal{B}(X_{\gamma_X},X_1)} \le C_A t^{\gamma_X-1}\mathrm{e}^{\omega_A t},$$
$$\|\mathrm{e}^{tA}\|_{\mathcal{B}(X_{\delta_X},X_1)} \le C_A t^{\delta_X-1}\mathrm{e}^{\omega_A t},$$
$$\|\mathrm{e}^{tB}\|_{\mathcal{B}(Y_1)} \le M_B \mathrm{e}^{\omega_B t}, \quad \|\mathrm{e}^{tB}\|_{\mathcal{B}(Y_{\delta_Y},Y_1)} \le C_B t^{\delta_Y-1}\mathrm{e}^{\omega_B t}$$

hold trivially.

(vii) Since we can make L_f arbitrarily small by choosing σ small and $\widetilde{K}$ large enough, we can choose an ω_f satisfying $\omega_A + C_A L_f < \omega_f < 0$.

Now, the proof of Theorem 6.8 is a direct application of Corollary 4.15. Concerning Theorem 6.9 we are in the easy situation that B already generated a C_0-group. Thus, we may choose the trivial splitting

$$Y = Y_F^\zeta \oplus Y_S^\zeta := \{0\} \oplus Y$$

for all $\zeta > 0$. Therefore, we may take $\zeta = C\varepsilon$ for some $C \in (0, 1)$, $N_F^\zeta = 0$ and $N_S^\zeta = -\omega_A\zeta^{-1} - \kappa$. If $\varepsilon > 0$ is small enough, then all the conditions of Section 5.1 can easily be verified and Theorem 6.9 follows from the results in Section 5.

7. Outlook

We have provided a quite general theory to use time scale separation in infinite-dimensional evolution equations with a focus on slow manifolds. Evidently, there are always further generalizations one could pursue. Examples are trying to weaken the conditions on the linear operators A and B, trying to lift the theory into a completely non-standard form setting [34], or extending it to quasilinear problems [2]. In addition, the case of loss of invertibility/hyperbolicity of the fast dynamics has been a key focus in many finite-dimensional problems [21], i.e., in this scenario one has to track invariant slow manifolds through special regions. Therefore, combining our slow manifold theory here with the recent development of the blow-up method for fast-slow PDEs [12] is a natural challenge for future work. Furthermore, it would be interesting to connect our results to the normally elliptic (instead of the normally hyperbolic) case occurring in fast-slow Hamiltonian systems [20, 24].

From the viewpoint of applications, several directions are likely to be important. First, one may want to compute the invariant slow manifolds numerically, and we refer to [21, Ch. 11] for a survey of methods available for computing slow manifolds for finite-dimensional fast-slow systems. In fact, our analytically intermediate approximation (4.5) provides a hint, how to prove rigorous error estimates for computational methods based upon the invariance equation and/or iterated asymptotics for infinite-dimensional fast-slow dynamics. Second, working out concrete examples from pattern formation problems will be relevant as this can provide additional insights, which aspects of the theory need extensions, while others are immediately applicable. Third, trying to make many results, which have been obtained only via formal asymptotic matching methods for PDEs, rigorous is likely to be possible since a similar strategy using Fenichel theory has worked already in finite dimensions [19, 21].

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F. Hummel, Department of Mathematics, Technical University of Munich,
Boltzmannstraße 3, 85748 Garching bei München, Germany

E-mail: hummel@ma.tum.de

C. Kuehn, Department of Mathematics, Technical University of Munich,
Boltzmannstraße 3, 85748 Garching bei München, Germany

E-mail: ckuehn@ma.tum.de