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# Rationality of even-dimensional intersections of two real quadrics

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**Abstract.** We study rationality constructions for smooth complete intersections of two quadrics over nonclosed fields. Over the real numbers, we establish a criterion for rationality in dimension four.

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## 1. Introduction

Consider a geometrically rational variety $X$, smooth and projective over a field $k$. Is $X$ rational over $k$? A necessary condition is that $X(k) \neq \emptyset$, which is sufficient in dimension one, as well as for quadric hypersurfaces and Brauer–Severi varieties of arbitrary dimension. When the dimension of $X$ is at most two, rationality over $k$ was settled by work of Enriques, Manin, and Iskovksikh [14]. Rationality is encoded in the Galois action on the geometric Néron–Severi group – varieties with rational points that are 'minimal' in the sense of birational geometry need not be rational. In dimension three, recent work [2, 3, 11, 13, 19] has clarified the criteria for rationality: one also needs to take into account principal homogeneous spaces over the intermediate Jacobian, reflecting which curve classes are realized over the ground field. The case of complete intersections of two quadrics was an important first step in understanding the overall structure [12]; rationality in dimension three is equivalent to the existence of a line over $k$ [3, 11, 19].

These developments stimulate investigations in higher dimensions [18]; the examples considered are rational provided there are rational points. In this paper, we focus on the case of four-dimensional complete intersections of two quadrics, especially over the real numbers $\mathbb{R}$. Here we exhibit rational examples without lines and explore further rationality constructions.

**Theorem 1.1.** *A smooth complete intersection of two quadrics $X \subset \mathbb{P}^6$ over $\mathbb{R}$ is rational if and only if $X(\mathbb{R})$ is nonempty and connected.*

In general, a projective variety $X$ that is rational over $\mathbb{R}$ has connected nonempty real locus $X(\mathbb{R})$. The point of Theorem 1.1 is that this necessary condition is also sufficient.

**Corollary 1.2.** *A smooth complete intersection of two quadrics $X \subset \mathbb{P}^6$ is rational over $\mathbb{R}$ if and only if there exists a unirational parametrization $\mathbb{P}^4 \dashrightarrow X$, defined over $\mathbb{R}$, of odd degree.*

Indeed, odd degree rational maps are surjective on real points, which guarantees that $X(\mathbb{R})$ is connected. Smooth complete intersections of two quadrics, of dimension at least two, are unirational provided they have a rational point; see Section 3.1 for references and discussion.

We also characterize rationality in dimension six, with the exception of one isotopy class that remains open (see Section 6.2).

Here is the roadmap of this paper. In Section 2 we recall basic facts about quadrics in even-dimensional projective spaces and their intersections. All interesting cohomology is spanned by the classes of projective subspaces in $X$ of half-dimension, and the Galois group acts on these classes via symmetries of the primitive part of this cohomology, a lattice for the root system $D_{2n+3}$. In Section 3 we present several rationality constructions. The isotopy classification, using Krasnov's invariants, is presented in Section 4; we draw connections with the Weyl group actions. In Section 5 we focus attention on cases where rationality is not obvious, e.g., due to the presence of a line. In Section 6 we prove Theorem 1.1 and discuss the applicability and limitations of our constructions in dimensions four and six. We speculate on possible extensions to more general fields in Section 7.

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## 2. Geometric background

**2.1. Roots and weights.** Let $D_{2n+3}$ be the root lattice of the corresponding Dynkin diagram, expressed in the standard Euclidean lattice

$$\langle L_1, \dots, L_{2n+3} \rangle, \quad L_i \cdot L_j = \delta_{ij}$$

as the lattice generated by simple roots

$$R_1 = L_1 - L_2, \quad R_2 = L_2 - L_3, \quad \dots, \quad R_{2n+1} = L_{2n+1} - L_{2n+2},$$
$$R_{2n+2} = L_{2n+2} - L_{2n+3}, \quad R_{2n+3} = L_{2n+2} + L_{2n+3}.$$

Its discriminant group is cyclic of order four, generated by

$$\frac{1}{4}\big(2(R_1 + 2R_2 + \cdots + (2n+1)R_{2n+1}) + (2n+1)R_{2n+2} + (2n+3)R_{2n+3}\big).$$

Multiplication by $-1$ acts on the discriminant via $\pm 1$. The outer automorphisms of $D_{2n+3}$ also act via automorphisms of $D_{2n+3}$ acting on the discriminant via $\pm 1$, e.g., exchanging $R_{2n+2}$ and $R_{2n+3}$ and keeping the other roots fixed.

The Weyl group $W(D_{2n+3})$ acts in the basis $\{L_i\}$ via signed permutations with an even number of $-1$ entries. The outer automorphisms act via signed permutations with no constraints on the choice of signs, e.g.,

$$L_i \mapsto L_i, \quad i = 1, \ldots, 2n+2, \qquad L_{2n+3} \mapsto -L_{2n+3}.$$

The odd and even half-spin representations have weights indexed by subsets $I \subset \{1, 2, \ldots, 2n+3\}$, with $|I|$ odd or even, written

$$w_I = \frac{1}{2}\Big(\sum_{i \in I} L_i - \sum_{j \in I^c} L_j\Big).$$

The odd and even weights are exchanged by outer automorphisms.

**2.2. Planes.** In this section, we assume that the ground field is algebraically closed of characteristic zero.

Let $X \subset \mathbb{P}^{2n+2}$ be a smooth complete intersection of two quadrics. We will identify subvarieties in $X$ with their classes in the cohomology of $X$ when no confusion may arise.

Let $h$ denote the hyperplane section and consider the primitive cohomology of $X$ under the intersection pairing. Reid [21, Theorem 3.14] shows that

$$(h^n)^{\perp} \simeq (-1)^n D_{2n+3}.$$

In other words, the primitive sublattice of $H^{2n}(X, \mathbb{Z}(n))$ – the Tate twist of singular cohomology for the underlying complex variety – may be identified with the root lattice. This is the target of the cycle class map

$$\mathrm{CH}^n(X) \to H^{2n}(X, \mathbb{Z}(n))$$

so the sign convention is natural.

**Remark 2.1** (Caveat on signs). When $X$ is defined over $\mathbb{R}$, codimension-$n$ subvarieties $Z \subset X$ defined over $\mathbb{R}$ yield classes in $H^{2n}(X, \mathbb{Z}(n))$ that are invariant under complex conjugation. However, the corresponding classes in $H^{2n}(X, \mathbb{Z})$ are multiplied by $(-1)^n$. When we mention invariant classes, it is with respect to the former action.

Given a plane $P \simeq \mathbb{P}^n \subset X$, we have

$$(P \cdot P)_X = c_n,$$

where (see [21, Corollary 3.11])

$$c_0 = 1, \quad c_1 = -1, \quad c_2 = 2, \quad c_3 = -2, \quad \dots, \quad c_n = (-1)^n\Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 1\Big).$$

The projection of $P$ into rational primitive cohomology takes the form

$$P - \frac{1}{4}h^n,$$

which has self-intersection $c_n - 1/4$. The corresponding element $w_P \in D_{2n+3}$ has

$$w_P \cdot w_P = (2n+3)/4.$$

By [21, Corollary 3.9], we obtain bijections

$$\{w_P\}_{P \simeq \mathbb{P}^n \subset X} = \{w_I\}_{|I| \text{ has fixed parity}}.$$

Note that the residual intersections to $\mathbb{P}^n \subset X$

$$X \cap \mathbb{P}^{n+2} = \mathbb{P}^n \cup S$$

give cubic scrolls $S \subset X$; these realize the weights of opposite parity.

By [21, Theorem 3.8], two planes $P_1$ and $P_2$ are disjoint if and only if

$$w_{P_1} \cdot w_{P_2} = (-1)^{n+1}/4.$$

For example, if $n = 1$ and $w_{P_1}$ is identified with $(L_1 - L_2 - L_3 - L_4 - L_5)/2$ then the relevant weights are

$$\begin{aligned} (L_1 + L_2 + L_3 - L_4 - L_5)/2, \quad &\dots, \quad (L_1 - L_2 - L_3 + L_4 + L_5)/2, \\ -L_1 + L_2 + L_3 + L_4 - L_5)/2, \quad &\dots, \quad (-L_1 - L_2 + L_3 + L_4 + L_5)/2, \end{aligned}$$

a total of $10 = \binom{5}{2}$ such lines. When $n = 2$ and $w_{P_1}$ is identified with

$$(L_1 - L_2 - \dots - L_7)/2$$

then the relevant weights are

$$\begin{gathered} (L_1 + L_2 + L_3 + L_4 + L_5 - L_6 - L_7)/2, \quad \dots, \\ (L_1 - L_2 - L_3 + L_4 + L_5 + L_6 + L_7)/2, \\ (-L_1 + L_2 + L_3 + L_4 - L_5 - L_6 - L_7)/2, \quad \dots, \\ (-L_1 - L_2 - L_3 - L_4 + L_5 + L_6 + L_7)/2, \end{gathered}$$

a total of $\binom{6}{4}+\binom{6}{3}=35=\binom{7}{3}$ planes. The planes $P_1$ and $P_2$ meet at a point if and only if

$$w_{P_1}\cdot w_{P_2}=(-1)^n\frac{3}{4}.$$

If they meet along an $r$-plane then an excess intersection computation gives (see [21, Lemma 3.10])

$$P_1\cdot P_2=(-1)^r\Big(\Big\lfloor\frac{r}{2}\Big\rfloor+1\Big),$$
$$w_{P_1}\cdot w_{P_2}=(-1)^{r+n}\Big(\Big\lfloor\frac{r}{2}\Big\rfloor+1\Big)-(-1)^n\frac{1}{4}.$$

In particular, they meet along an $(n-1)$-plane when

$$w_{P_1}\cdot w_{P_2}=-\Big(\Big\lfloor\frac{n-1}{2}\Big\rfloor+1\Big)-(-1)^n\frac{1}{4};$$

for a fixed $w_{P_1}$ there are $2n+3$ planes $P_2$ meeting $P_1$ in this way. For example, if $n=1$ and $w_{P_1}=(L_1-L_2-L_3-L_4-L_5)/2$ then the possibilities for $w_{P_2}$ are

$$(L_1+L_2+L_3+L_4+L_5)/2,\quad (-L_1-L_2+L_3+L_4+L_5)/2,$$
$$(-L_1+L_2-L_3+L_4+L_5)/2,\quad (-L_1+L_2+L_3-L_4+L_5)/2,$$
$$(-L_1+L_2+L_3+L_4-L_5)/2.$$

**2.3. Quadrics.** We retain the notation of Section 2.2.

Our next task is to analyze quadric $n$-folds $Q\subset X$, i.e., $Q$ a degree-two hypersurface in $\mathbb{P}^{n+1}$. Let $\{\mathcal{Q}_t\}$, $t\in\mathbb{P}^1$ denote the pencil of quadric hypersurfaces cutting out $X$. The degeneracy locus

$$D:=\{t\in\mathbb{P}^1:\mathcal{Q}_t \text{ singular}\}$$

consists of $2n+3$ points; since $X$ is smooth, each has rank $2n+2$. The Hilbert scheme of quadric $n$-folds $Q\subset X$ is isomorphic to the relative Fano variety of $(n+1)$-planes

$$\mathcal{F}(\mathcal{Q}/\mathbb{P}^1)=\{\Pi\simeq\mathbb{P}^{n+1}\subset\mathcal{Q}_t \text{ for some } t\in\mathbb{P}^1\},$$

which consists of $2(2n+3)$ copies of the connected isotropic Grassmannian

$$\mathrm{OGr}(n+1,2n+2).$$

Given a quadric $Q$, its projection to rational primitive cohomology

$$Q-\frac{1}{2}h^n$$

corresponds to an element

$$w_Q \in D_{2n+3}, \quad w_Q \in \{\pm L_1, \dots, \pm L_{2n+3}\}.$$

In particular, we have

$$Q \cdot Q = \begin{cases} 2 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

Residuation in a complete intersection of linear forms

$$Q \cup Q' = X \cap h^n$$

reverses signs, i.e., $w_Q = -w_{Q'}$. On the other hand, if $Q_1$ and $Q_2$ are not residual then

$$Q_1 \cdot Q_2 = 1. \tag{2.1}$$

We summarize this as follows:

**Proposition 2.2.** *The signed permutation representation of $W(D_{2n+3})$ is realized via the action on classes $[Q]$, where $Q \subset X$ is a quadric $n$-fold.*

Note that there are

$$2^{2n+1}(2n+3)$$

reducible quadrics – unions of two $n$-planes meeting in an $(n-1)$-plane – with $2^{2n}$ reducible quadrics in each copy of the isotropic Grassmannian.

## 3. Rationality constructions

We now work over an arbitrary field $k$ of characteristic zero.

**3.1. General considerations.** Let $X \subset \mathbb{P}^{d+2}$ denote a smooth complete intersection of two quadrics of dimension at least two. Recall the following:

- If $X(k) \neq \emptyset$ then $X$ is unirational over $k$ and has Zariski dense rational points (see [7, Remark 3.28.3]).
- If there is a line $\ell \subset X$ defined over $k$ then projection induces a birational map $\pi_\ell \colon X \stackrel{\sim}{\dashrightarrow} \mathbb{P}^d$.

For reference, we recall Amer's theorem [20, Theorem 2.2]:

**Theorem 3.1.** *Let $k$ be a field of characteristic not two, $F_1$ and $F_2$ quadrics over $k$, and $\mathcal{Q}_t = \{F_1 + tF_2\}$ the associated pencil of quadrics over $k(t)$. Then*

$$X = \{F_1 = F_2 = 0\}$$

*has an $r$-dimensional isotropic subspace over $k$ if and only if $\mathcal{Q}_t$ has an $r$-dimensional isotropic subspace over $k(t)$.*

We apply this for $k = \mathbb{R}$, where $X \subset \mathbb{P}^{d+2}$ is a smooth complete intersection of two quadrics and $\mathcal{Q} \to \mathbb{P}^1$ is the associated pencil.

Recall Springer's theorem: A quadric hypersurface $\mathcal{Q}$ over a field $L$ has a rational point if it admits a rational point over some odd-degree extension of $L$. Applying this to the pencil $\mathcal{Q} \to \mathbb{P}^1$ associated with a complete intersection of two quadrics, with Amer's theorem, yields the following proposition.

**Proposition 3.2.** *If $d \geq 1$ and $X$ contains a subvariety of odd degree over $k$, then $X(k) \neq \emptyset$.*

We can prove a bit more as follows.

**Proposition 3.3.** *If $d \geq 2$ and $X$ has a curve of odd degree defined over $k$, then $X$ is rational over $k$.*

*Proof.* Recall that double projection from a sufficiently general rational point $x \in X(k)$ yields a diagram

$$X \overset{\sim}{\dashrightarrow} Y \to \mathbb{P}^1,$$

where $Y$ is a quadric bundle of relative dimension $d-1$. A curve $C \subset X$ of odd degree yields an multisection of this bundle of odd degree. Thus, $Y \to \mathbb{P}^1$ has a section by Springer's theorem and its generic fiber $Y_t$ is rational over $k(\mathbb{P}^1)$. It follows that $X$ is rational over the ground field. □

**Remark 3.4.** The pencil defining $X$ gives a quadric bundle

$$\mathcal{Q} \to \mathbb{P}^1$$

of relative dimension $d+1$. We apply Witt's Decomposition theorem to $[\mathcal{Q}_t]$ and $[Y_t]$, understood as quadratic forms over $k(\mathbb{P}^1) = k(t)$, to obtain

$$[\mathcal{Q}_t] = [Y_t] \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, a section of $Y \to \mathbb{P}^1$ yields an isotropic line of $\mathcal{Q} \to \mathbb{P}^1$, and Theorem 3.1 implies that $X$ contains a line defined over $k$.

**Corollary 3.5** (see appendix by Colliot-Thélène [12, Theorem A5]). *Let $X \subset \mathbb{P}^{d+2}$ denote a smooth complete intersection of two quadrics of dimension at least two. Suppose there exists an irreducible positive-dimensional subvariety $W \subset X$ of odd degree, defined over $k$. Then $X$ is rational over $k$.*

Given these results, we focus on proving rationality in cases where $X$ does not contain lines or positive-dimensional subvarieties of odd degree.

**3.2. Rationality using half-dimensional subvarieties.** We now turn to even-dimensional intersections of two quadrics

$$X \subset \mathbb{P}^{2n+2}, \quad n \geq 1.$$

Throughout, we assume that $X(k) \neq \emptyset$, and thus $X$ is $k$-unirational and $k$-rational points are Zariski dense.

**Construction I.** Suppose that

- $X$ has a pair of conjugate disjoint $n$-planes $P, \bar{P}$, defined over a quadratic extension $K$ of $k$.

Projecting from a general $x \in X(k)$ gives a birational map

$$X \overset{\sim}{\dashrightarrow} X' \subset \mathbb{P}^{2n+1},$$

where $X'$ is a cubic hypersurface.

Since $X(k) \subset X$ is Zariski dense, we may assume that the images of $P$ and $\bar{P}$ in $X'$ remain disjoint. The 'third point' construction gives a birational map

$$\mathbf{R}_{K/k}(P) \overset{\sim}{\dashrightarrow} X',$$

where the source variety is the restriction of scalars. We conclude that $X$ is rational over $k$. This construction appears in [7, Theorem 2.4].

**Construction II.** Suppose that

- $X$ has a pair of conjugate disjoint quadric $n$-folds $Q, \bar{Q}$, defined over a quadratic extension $K$ of $k$, and meeting transversally at one point.

Projecting from $x \in Q \cap \bar{Q}$, which is a $k$-rational point $X$, gives a birational map

$$X \overset{\sim}{\dashrightarrow} X' \subset \mathbb{P}^{2n+1},$$

where $X'$ is a cubic hypersurface.

The proper transforms $Q', \bar{Q}' \subset X'$ are disjoint unless there exists a line

$$x \in \ell \subset \mathbb{P}^{2n+2}$$

with

$$\{x\} \subsetneq \ell \cap Q, \ell \cap \bar{Q}$$

as schemes. We may assume that $\ell \not\subset X$ as we already know $X$ is rational in this case. Thus, the only possibility is

$$\ell \cap Q = \ell \cap \bar{Q}$$

as length-two subschemes, which is precluded by the intersection assumption.

Repeating the argument for Construction I thus gives rationality.

**Construction III.** Suppose that

- $X$ contains a quadric $Q$ of dimension $n$, defined over $k$.

Projection gives a fibration

$$q: \mathrm{Bl}_Q(X) \to \mathbb{P}^n$$

with fibers quadrics of dimension $n$. Now suppose that $X$ contains a second $n$-fold $T$ with the property that

$$\deg(T) - T \cdot Q$$

is odd, i.e., a multisection for $q$ of odd degree. It follows that the generic fiber of $q$ is rational, and thus $Y$ is rational over $k$.

When $\dim(X) = 4$ a number of $T$ might work, e.g.,

- a plane disjoint from $Q$,
- a second quadric meeting $Q$ in one point,
- a quartic or a sextic del Pezzo surface meeting $Q$ in one or three points,
- a degree 8 K3 surface meeting $Q$ in one, three, five, or seven points.

**Construction IV.** Suppose that

- $\dim(X) = 4$ and $X$ contains a quartic scroll $T$, defined over $k$.

Geometrically, $T$ is the image of the ruled surface

$$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$$

under the linear series of bidegree $(1, 2)$. Over $\mathbb{R}$ we are interested in cases where $T = \mathbb{P}^1 \times C$ with $C$ a nonsplit conic. We do not want to force $X$ to have lines! (Note that quartic scrolls geometrically isomorphic to $\mathbb{F}_2$ contain lines defined over the ground field, and thus are not useful for our purposes.)

On projecting from a point $x \in X(k)$ we get a cubic fourfold

$$X' \subset \mathbb{P}^5,$$

containing a quartic scroll. The Beauville–Donagi construction [1] – concretely, take the image under the linear system of quadrics vanishing along $T$ – shows that $X'$ is birational to a quadric hypersurface thus rational over $k$.

Recall that an $n$-dimensional smooth variety $W \subset \mathbb{P}^{2n+1}$ is said to have *one apparent double point* if a generic point is contained in a unique secant to $W$.

**Construction V.** Suppose that $X$ contains a variety $W$ defined over $k$ of dimension $n \geq 2$ such that

- $W$ spans a $\mathbb{P}^{2n+1}$ and has one apparent double point; or
- $W$ has a rational point $w$ such that projecting from $w$ maps $W$ birationally to a variety with one apparent double point.

Then $X$ is rational over $k$.

As before, one projects from a rational point to a get cubic hypersurface $X' \subset \mathbb{P}^{2n+1}$. Cubic hypersurfaces containing varieties $W$ with one apparent double point are rational [22, Proposition 9]. Indeed, intersecting secant lines of $W$ with $X'$ yields

$$\mathrm{Sym}^2(W) \dashrightarrow X',$$

which is birational when each point lies on a unique secant to $W$.

Quartic scrolls in $\mathbb{P}^5$ have one apparent double point so Construction IV is a special case of Construction V.

## 4. Isotopy classification

**4.1. Krasnov invariants.** We review the classification of smooth complete intersections of two quadrics $X \subset \mathbb{P}^{2n+2}$ over $\mathbb{R}$, following [17].

Express $X = \{F_1 = F_2 = 0\}$ where $F_1$ and $F_2$ are real quadratic forms. We continue to use $D$ for the degeneracy locus of the associated pencil

$$\mathcal{Q}_t = \{t_1 F_1 + t_2 F_2 = 0\}.$$

Let $r = |D(\mathbb{R})|$ which is odd with $r \leq 2n+3$. Consider the signatures $(I^+, I^-)$ of the forms

$$s_1 F_1 + s_2 F_2, \quad (s_1, s_2) \in \mathbb{S}^1 = \{(s_1, s_2) \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}.$$

Record these at the $2r$ points lying over $D$, in order as we trace the circle counterclockwise. We label each of these points with + or − depending on whether the positive part $I^+$ of the signature increases or decreases as we cross the point. Each point of $D(\mathbb{R})$ yields a pair of antipodal points on $\mathbb{S}^1$ labelled with opposite signs.

We give examples for $n = 0$ and $r = 3$. Consider the sequence of signatures, with alternating singular (underlined) and smooth members,

$$\underline{(0,2)}\,(1,2)\,\underline{(1,1)}\,(2,1)\,\underline{(2,0)}\,(3,0)\,\underline{(2,0)}\,(2,1)\,\underline{(1,1)}\,(1,2)\,\underline{(0,2)}\,(0,3).$$

We encode this with +++−−−. The sequence

$$\underline{(1,1)}\,(2,1)\,\underline{(1,1)}\,(1,2)\,\underline{(1,1)}\,(2,1)\,\underline{(1,1)}\,(1,2)\,\underline{(1,1)}\,(2,1)\,\underline{(1,1)}\,(1,2)$$

yields +−+−+−. These sequences are well-defined up to cyclic permutations and reversals. The signature of a singular fiber may be read off from the signatures of the adjacent smooth fibers; the signatures of the smooth fibers may be recovered, up to cyclic permutation, from the signatures of the singular fibers.

A pencil with anisotropic (definite) smooth members – with signatures $(2n+3, 0)$ or $(0, 2n+3)$ – necessarily has $X(\mathbb{R}) = \emptyset$.

Suppose the sequence of + and − has maximal unbroken chains of +'s of lengths $r_1, r_2, \dots, r_{2s+1}$, where

$$r = r_1 + r_2 + \cdots + r_{2s+1}.$$

The number of terms is odd because antipodal points have opposite signs. In the examples above we have $3 = 3$ and $3 = 1 + 1 + 1$, i.e., (3) and (1, 1, 1). Our invariant is the sequence $(r_1, \dots, r_{2s+1})$ up to cyclic permutations and reversals – a complete isotopy invariant of $X$ over $\mathbb{R}$ (see [17]).

| $r$ | possible invariants |
|---|---|
| 1 | (1) |
| 3 | (3), (1, 1, 1) |
| 5 | (5), (3, 1, 1), (2, 2, 1), (1, 1, 1, 1, 1) |
| 7 | (7), (5, 1, 1), (4, 2, 1), (3, 3, 1), (3, 2, 2), (3, 1, 1, 1, 1), (2, 2, 1, 1, 1), (2, 1, 2, 1, 1), (1, 1, 1, 1, 1, 1, 1) |
| 9 | (9), (7, 1, 1), (6, 2, 1), (5, 3, 1), (5, 2, 2), (4, 4, 1), (4, 3, 2), (3, 3, 3), (5, 1, 1, 1, 1), (4, 2, 1, 1, 1), (4, 1, 2, 1, 1), (3, 3, 1, 1, 1), (3, 1, 3, 1, 1), (3, 2, 2, 1, 1), (3, 2, 1, 2, 1), (3, 2, 1, 1, 2), (3, 1, 2, 2, 1), (2, 2, 2, 2, 1), (3, 1, 1, 1, 1, 1, 1), (2, 2, 1, 1, 1, 1, 1), (2, 1, 2, 1, 1, 1, 1), (2, 1, 1, 2, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1) |

Table 1. Even-dimensional Krasnov invariants for $\dim(X) \le 6$ with $r = |D(\mathbb{R})|$ real singular members.

**4.2. Relation to the Weyl group.** We observed in Section 2 that the primitive cohomology of a complete intersection of two quadrics $X \subset \mathbb{P}^{2n+2}$ has $W(D_{2n+3})$ symmetry. When $X$ is defined over $\mathbb{R}$, complex conjugation yields an involution in this group. Here we relate Krasnov invariants to these involutions. We will use this dictionary in Section 6 to make geometric constructions based on the isotopy class.

Given $(r_1, \dots, r_{2s+1}), r = \sum r_i$, we derive a sequence of $\pm 1$'s of length $r$ as follows: For each point of $D(\mathbb{R})$, record the sign of the discriminant of the associated rank-$(2n+2)$ quadratic form, determined by the parity of $(I^+ - I^-)/2$. In the examples above, we obtain $(-1, +1, -1)$ and $(+1, +1, +1)$. The number of $-1$'s is always even.

The analysis in Section 2.3 shows that complex conjugation acts on $H^{2n}(X, \mathbb{Z}(n))$ in the basis $\{L_1, \dots, L_{2n+3}\}$ as a signed permutation of order two. This is a direct sum of blocks

$$(+1), (-1), \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Actually, we may assume the sign is positive in the third case after conjugating by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Suppose there are $a$ blocks $(+1)$, $2b$ blocks $(-1)$, and $c$ blocks of the third kind, with

$$a + 2b + 2c = 2n + 3.$$

These correspond to the conjugacy classes of involutions $\iota \in W(D_{2n+3})$ [16, Sections 3.2 and 3.3]. We have $r = a + 2b$, reflecting the number of points of $D(\mathbb{R})$ with positive and negative discriminants respectively, and $2c = 2n+3-r$, reflecting the number of complex-conjugate pairs in $D(\mathbb{C}) \setminus D(\mathbb{R})$.

The passage from isotopy classes to conjugacy classes in $W(D_{2n+3})$ results in a loss of information. We give an example for real quartic del Pezzo surfaces $X \subset \mathbb{P}^4$.

**Example 4.1.** The isotopy class (5) has singular members with signatures

$$(0,4)\,(1,3)\,(2,2)\,(3,1)\,(4,0)\,(4,0)\,(3,1)\,(2,2)\,(1,3)\,(0,4)$$

with involution given by the diagonal $5 \times 5$ matrix

$$\mathrm{diag}(1, -1, \mathbf{1}, -1, 1),$$

where the emboldened $\mathbf{1}$ corresponds to a degenerate fiber $\mathcal{Q}_t$ whose rulings sweep out quadric curves (conics) on $X$ defined over $\mathbb{R}$. (There is only one pair of such conics.) Here $X(\mathbb{R}) = \emptyset$ as it is contained in an anisotropic quadric threefold.

The isotopy class (2, 2, 1) has singular members with signatures

$$(1,3)\,(2,2)\,(2,2)\,(2,2)\,(3,1)\,(3,1)\,(2,2)\,(2,2)\,(2,2)\,(1,3)$$

with involution

$$\mathrm{diag}(-1, \mathbf{1}, \mathbf{1}, \mathbf{1}, -1).$$

This has the same Galois action but contains three pairs of conics defined over $\mathbb{R}$, represented by the emboldened $\mathbf{1}$'s.

These are distinguished cohomologically by the arrow

$$\mathbb{Z}^3 \simeq H^0(G, \mathrm{Pic}(X_{\mathbb{C}})) \to H^2(G, \Gamma(\mathcal{O}^*_{X_{\mathbb{C}}})) = \mathrm{Br}(\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}$$

of the Hochschild–Serre spectral sequence.

**Proposition 4.2.** *Fix a conjugacy class* $[\iota = \iota_{abc}]$ *of involutions in* $W(D_{2n+3})$. *Consider the isotopy classes of* $X(\mathbb{R}) \subset \mathbb{P}^{2n+2}$ *such that complex conjugation acts by* $\iota$. *The possible isotopy classes correspond to shuffles of*

$$(\underbrace{1,\dots,1}_{a \text{ times}}, \underbrace{-1,\dots,-1}_{2b \text{ times}})$$

*up to cyclic permutations and reversals.*

*Proof.* Observe first that points in $D(\mathbb{C}) \setminus D(\mathbb{R})$ are irrelevant to the Krasnov invariant, assuming the dimension is given. So it makes sense they are not relevant in the enumeration.

We have already seen that sequences of the prescribed form arise from each isotopy class; we present the reverse construction.

Choose such a sequence, e.g.,

$$(1, 1, 1, -1, 1, -1, -1, 1, -1).$$

The key observation is that local maxima and minima of $I^+ - I^-$ – which necessarily occur at smooth points – arise precisely between points of the degeneracy locus where signs do *not* change. We indicate smooth fibers achieving maxima/minima with $|$, e.g.,

$$1 \mid 1 \mid 1, -1, 1, -1 \mid -1, 1, -1,$$

or equivalently

$$\mid 1 \mid 1, -1, 1, -1 \mid -1, 1, -1, 1 \mid$$

after cyclic permutation.

Lifting to the double cover $\mathbb{S}^1$ entails concatenating two such expressions:

$$\mid 1 \mid 1, -1, 1, -1 \mid -1, 1, -1, 1 \mid 1 \mid 1, -1, 1, -1 \mid -1, 1, -1, 1 \mid .$$

From this, we read off the points of $D(\mathbb{R})$ on which $I^+$ increases and decreases

$$+----++++-++++----,$$

which determines the Krasnov invariant – $(1, 4, 4)$ in this example. $\square$

## 5. Applying quadratic forms

**5.1. Quadric fibrations over real curves.** Let $C$ be a smooth projective geometrically connected curve over $\mathbb{R}$ with function field $K = \mathbb{R}(C)$. Let $Q \subset \mathbb{P}^{d+1}$ be a smooth (rank $d + 2$) quadric hypersurface over $K$ and $F_i(Q)$ the variety parametrizing $i$-dimensional isotropic subspaces, so that $F_0(Q) = Q$ and $F_m(Q)$

is empty when $2m > d$. If $d = 2m$ then $F_m(Q)$ has two geometrically connected components; otherwise it is connected.

Suppose that $\pi: \mathcal{Q} \to C$ is a regular projective model of $Q$, such that the fibers are all quadric hypersurfaces of rank at least $d + 1$. The locus $D \subset C$ corresponding to fibers of rank $d + 1$ is called the *degeneracy locus*.

Fundamental results of Witt – see [5, Section 2] and [23] for a modern formulation in terms of local-global principles – assert that:

- if $d > 0$, then $Q(K) \neq \emptyset$ if $\mathcal{Q}_c = \pi^{-1}(c)$ has a smooth real point for each $c \in C(\mathbb{R})$;
- if $d > 2$, then $F_1(Q)(K) \neq \emptyset$ if $F_1(\mathcal{Q}_c)$ has a smooth real point for each $c \in C(\mathbb{R})$.

We can translate these into conditions on the signatures of the smooth fibers

- if $d > 0$, then $Q(K) \neq \emptyset$ if $\mathcal{Q}_c$ is not definite for any $c \in (C \setminus D)(\mathbb{R})$;
- if $d > 2$, then $F_1(Q)(K) \neq \emptyset$ if $\mathcal{Q}_c$ does not have signatures $(d+2, 0)$, $(d+1, 1)$, $(1, d+1)$, or $(0, d+2)$ for any $c \in (C \setminus D)(\mathbb{R})$.

In other words, we have points and lines over $K$ if the fibers permit them.

This reflects a general principle: Suppose $\mathcal{X}$ is regular and has a flat proper morphism $\varpi: \mathcal{X} \to C$ to a curve $C$, all defined over $\mathbb{R}$. The local-global and reciprocity obstructions to sections of $\varpi$ are reflected in the absence of *continuous* sections $C(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ for the induced map of the underlying real manifolds (see [8]).

**5.2. Implications of Amer's theorem.** Let $X \subset \mathbb{P}^{d+2}$ be a smooth complete intersection of two quadrics over $\mathbb{R}$ and $\mathcal{Q} \to \mathbb{P}^1$ the associated pencil of quadrics.

The results of Section 5.1 imply that $\mathcal{Q} \to \mathbb{P}^1$ has a section unless the Krasnov invariant is $(d + 3)$; the variety of lines $F_1(\mathcal{Q}/\mathbb{P}^1) \to \mathbb{P}^1$ has a section unless the Krasnov invariant is

$$(d + 3),\ (d + 2 - e, e, 1) \quad \text{with } 1 \leq e \leq \frac{d+2}{2},\ (d + 1).$$

Thus, the Krasnov invariant determines which dimensional linear subspaces and quadrics appear on $X$:

**Proposition 5.1.** *Let $X \subset \mathbb{P}^{d+2}$ be a smooth complete intersection of two quadrics defined over $\mathbb{R}$. The only isotopy classes of $X$ that fail to contain a line are:*

- $(d + 3)$*;*
- $(d + 2 - e, e, 1)$ *with* $1 \leq e \leq (d + 2)/2$*;*
- $(d + 1)$.

As observed in Section 4.1, the pencil in case $(d + 3)$ has anisotropic members so $X(\mathbb{R}) = \emptyset$. The case $(d + 1, 1, 1)$ has disconnected real locus $X(\mathbb{R})$ [17, p. 117]; thus, $X$ cannot be rational over $\mathbb{R}$. The cases $(d+2-e, e, 1)$ with $2 \leq e \leq (d+2)/2$ are connected.

**5.3. Quadric $n$-folds over $\mathbb{R}$.** Assume now that $X$ has even dimension $d = 2n$. We may read off from the invariant $(r_1, \dots, r_{2s+1})$ which classes of quadric $n$-folds $Q \subset X_{\mathbb{C}}$ are realized by quadrics defined over $\mathbb{R}$.

Fix a smooth real quadric hypersurface

$$\mathbf{Q} = \{F = 0\} \subset \mathbb{P}^{2n+1}.$$

Recall that $F$ is classified up to real changes in coordinates by the signatures $I^+(F)$ and $I^-(F)$. The following conditions are equivalent [10, Section 85]:

- the geometric components of the variety of maximal isotropic subspaces $\mathrm{OGr}(\mathbf{Q})$ are defined over $\mathbb{R}$;
- the discriminant of $F$ is positive;
- $I^+(F) - I^-(F)$ is divisible by four.

This means that complex conjugation fixes the *class* of a maximal isotropic subspace. Witt's Decomposition theorem [10, Section 8] gives an equivalence between:

- there is a maximal isotropic subspace $\mathbb{P}^n \subset \mathbf{Q}$ defined over $\mathbb{R}$;
- $I^+(F) = I^-(F) = n + 1$.

Thus, quadric $n$-folds

$$Q \subset X \subset \mathbb{P}^{2n+2}$$

defined over $\mathbb{R}$ correspond to rulings of degenerate fibers $\mathcal{Q}_t, t \in D(\mathbb{R})$, where $\mathcal{Q}_t$ has signature $(n + 1, n + 1)$. As in Example 4.1, the corresponding $(+1)$-blocks in the complex conjugation involution $\iota \in W(D_{2n+3})$ will be designated $\mathbf{1}$, in boldface.

**5.4. Analysis of the remaining even-dimensional isotopy classes.** We continue to assume that $X$ has even dimension $d = 2n$, focusing on the isotopy classes without lines.

The cases

$$(2n + 2 - e, e, 1) = (e, 1, 2n + 2 - e), \quad 2 \le e \le n + 1$$

have degeneracy consisting of $2n+3$ real points. The signatures of nonsingular members are

$$\begin{aligned}&(1, 2n + 2) \dots (e + 1, 2n + 2 - e)\,(e, 2n + 3 - e)\,(e + 1, 2n + 2 - e) \dots\\ &(2n + 1, 2)\,(2n + 2, 1) \dots (2n + 2 - e, e + 1)\,(2n + 3 - e, e)\\ &(2n + 2 - e, e + 1) \dots (2, 2n + 1).\end{aligned}$$

For $(n + 1, n + 1, 1)$ the resulting signed permutation matrix is the diagonal matrix

$$\mathrm{diag}(\underbrace{(-1)^n, \dots, -1, \mathbf{1}}_{n+1 \text{ terms}}, \mathbf{1}, \underbrace{\mathbf{1}, -1, \dots, (-1)^n}_{n+1 \text{ terms}}), \tag{5.1}$$

with the emboldened $\mathbf{1}$'s corresponding to singular fibers with signature $(n+1, n+1)$.

The number of $+1$'s

$$a = \begin{cases} n+2 & \text{if } n \text{ odd,} \\ n+3 & \text{if } n \text{ even.} \end{cases}$$

For $e \neq n+1$ we have

$$\operatorname{diag}(\underbrace{(-1)^n, \dots, (-1)^{n+e-1}}_{e \text{ terms}}, (-1)^{n+e-1}, \underbrace{(-1)^{n+(2n+2-e)-1}, \dots, (-1)^n}_{2n+2-e \text{ terms}}).$$

Note that $(-1)^{n+e-1} = (-1)^{n+(2n+2-e)-1}$, so the three middle terms are equal. There is exactly one **1** corresponding to the singular fiber with signature $(n+1, n+1)$. The number of $+1$'s is given

$$a = \begin{cases} n & \text{if } n, e \text{ odd,} \\ n+2 & \text{if } n \text{ odd and } e \text{ even,} \\ n+3 & \text{if } n \text{ even and } e \text{ odd,} \\ n+1 & \text{if } n, e \text{ even.} \end{cases} \tag{5.2}$$

For case $(2n+1)$ the signatures of nonsingular members are

$$(2, 2n+1) \ \dots \ (2n, 3)\ (2n+1, 2)\ (2n+2, 3) \ \dots \ (2, 2n+1).$$

The signed permutation matrix has one factor

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and diagonal entries

$$((-1)^{n-1}, \dots, -1, 1, -1, \dots, (-1)^{n-1})$$

The number of positive terms is

$$a = \begin{cases} n & \text{if } n \text{ odd,} \\ n+1 & \text{if } n \text{ even.} \end{cases}$$

## 6. Application of the constructions

**6.1. Proof of Theorem 1.1.** Rationality is evident for isotopy classes of varieties that contain a line defined over $\mathbb{R}$. Proposition 5.1 enumerates the remaining cases

$$(5),\ (1, 3, 3),\ (1, 2, 4).$$

These are covered by the following propositions.

**Proposition 6.1.** *Let $X \subset \mathbb{P}^{2n+2}$ be a smooth complete intersection of two quadrics over $\mathbb{R}$ with invariant $(2n+1)$. Then $X$ is rational.*

*Proof.* The analysis in Section 5.4 indicates that complex conjugation exchanges two classes of quadric $n$-folds associated to complex conjugate points in $D(\mathbb{C}) \setminus D(\mathbb{R})$. Denote these by $[Q]$ and $[\bar{Q}]$ – recall from (2.1) that

$$[Q] \cdot [\bar{Q}] = 1.$$

Choosing a suitably general complex representation $Q \subset X_{\mathbb{C}}$, the intersection $Q \cap \bar{Q}$ is proper. Then $Q \cap \bar{Q}$ consists of a single rational point of $X$ with multiplicity one. In particular, the hypotheses of Construction II are satisfied. □

**Proposition 6.2.** *Let $X \subset \mathbb{P}^{2n+2}$ be a smooth complete intersection of two quadrics over $\mathbb{R}$ with invariant*

$$(2n+2-e, e, 1) = (e, 1, 2n+2-e), \quad 2 \leq e \leq n+1.$$

*Assume that either $e$ is even or $e = n+1$. Then $X$ is rational.*

*Proof.* Assume first that $e = n+1$. It follows from (5.1) in Section 5.4 that $X$ admits *three* classes $Q_1$, $Q_2$, and $Q_3$ (see Table 2) with each $Q_i$ realized by a quadric $n$-fold defined over $\mathbb{R}$. Construction III gives rationality in this case.

| | $h^n$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $h^n$ | 4 | 2 | 2 | 2 |
| $Q_1$ | 2 | $1+(-1)^n$ | 1 | 1 |
| $Q_2$ | 2 | 1 | $1+(-1)^n$ | 1 |
| $Q_3$ | 2 | 1 | 1 | $1+(-1)^n$ |

Table 2.

Now assume that $e$ is even. The formula (5.2) shows that the numbers of $+1$'s and $-1$'s appearing in the $\iota \in W(D_{2n+3})$ associated with complex conjugation are as close as possible. If $n$ is even then we have $n+1$ of the former and $n+2$ of the latter; when $n$ is odd we have $n+2$ of the former and $n+1$ of the letter. Given an $n$-plane $P \subset X_{\mathbb{C}}$, the formulas in Section 2.2 yield

$$w_P \cdot w_{\bar{P}} = (-1)^{n+1}/4,$$

whence $P$ and $\bar{P}$ are disjoint in $X_{\mathbb{C}}$. Thus, we may apply Construction I to conclude rationality. □

**Remark 6.3.** Remark 3.4 implies that $X$ does not admit curves (or surfaces!) of odd degree defined over $\mathbb{R}$. These would force the existence of lines defined over $\mathbb{R}$, which do not exist in this isotopy class.

**6.2. Remaining six-dimensional case.** To settle the rationality of six-dimensional complete intersections of two quadrics $X \subset \mathbb{P}^8$, there is one remaining case in the Krasnov classification: $(1, 3, 5)$.

The sequence of signatures of nonsingular elements of $\{\mathcal{Q}_t\}$ is:

$$(1,8)\,(2,7)\,(3,6)\,(4,5)\,(5,4)\,(6,3)\,(5,4)\,(6,3)\,(7,2)$$
$$(8,1)\,(7,2)\,(6,3)\,(5,4)\,(4,5)\,(3,6)\,(4,5)\,(3,6)\,(2,7).$$

The signed permutation is the diagonal matrix

$$\mathrm{diag}(-1, 1, -1, \mathbf{1}, -1, -1, -1, 1, -1)$$

and the invariant cycles are as in Table 3.

| | $h^3$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $h^3$ | 4 | 2 | 2 | 2 |
| $Q_1$ | 2 | 0 | 1 | 1 |
| $Q_2$ | 2 | 1 | 0 | 1 |
| $Q_3$ | 2 | 1 | 1 | 0 |

Table 3.

Here, $Q_1$ corresponds to the singular fiber of signature $(4, 4)$ and $Q_2$ and $Q_3$ correspond to the singular fibers of signatures $(2, 6)$ and $(6, 2)$. If $P \subset X$ is a three-plane then $w_P \cdot w_{\overline{P}} = -3/4$ and $P$ and $\overline{P}$ meet in a single point.

## 7. Extensions and more general fields

We work over a field $k$ of characteristic zero. In this section, we give further examples of rationality constructions for $2n$-dimensional intersections of two quadrics over $k$, relying on special subvarieties of dimension $n$.

**7.1. Dimension four: Intersection computations.** Given $X \subset \mathbb{P}^6$, a smooth complete intersection of two quadrics, we have

$$\begin{aligned} c_t(\mathcal{T}_X) &\equiv (1 + 7ht + 21h^2t^2)/(1 + 2ht)^2 \pmod{t^3} \\ &\equiv (1 + 7ht + 21h^2t^2)(1 - 2ht + 4h^2t^2)^2 \pmod{t^3} \\ &\equiv 1 + 3ht + 5h^2t^2 \pmod{t^3}. \end{aligned}$$

If $T \subset X$ is a smooth projective geometrically connected surface then

$$\begin{aligned} c_t(\mathcal{N}_{T/X}) &= (1 + 3ht + 5h^2t^2)/(1 - K_T t + \chi(T)t^2) \\ &= 1 + (3h + K_T)t + (5h^2 + 3hK_T + K_T^2 - \chi(T))t^2, \end{aligned}$$

where $\chi(T)$ is the topological Euler characteristic. The expected dimension of the deformation space of $T$ in $X$ is

$$\begin{aligned}\chi(\mathcal{N}_{T/X}) &= 2 - \frac{1}{2}(3h + K_T)K_T + \frac{1}{2}(3h + K_T)^2 \\ &\quad - (5h^2 + 3hK_T + K_T^2 - \chi(T)) \\ &= 2\chi(\mathcal{O}_T) + \frac{1}{2}(-h^2 - 3hK_T) - K_T^2 + \chi(T)).\end{aligned}$$

For example,

- if $T = \mathbb{P}^2$ is embedded as a plane then $(T \cdot T)_X = 2$ and $T$ is rigid;
- if $T$ is a quadric then $(T \cdot T)_X = 2$ and $T$ moves in a three-parameter family;
- if $T$ is a quartic scroll then $(T \cdot T)_X = 6$ and moves in a five-parameter family;
- if $T$ is a sextic del Pezzo surface then $(T \cdot T)_X = 12$ and $T$ moves in an eight-parameter family.

**7.2. Dimension four: Surfaces with one apparent double point.** Recall that Construction V gives the rationality of fourfolds admitting a surface with one apparent double point. A classical result asserted by Severi – see [4, Theorem 4.10] for a modern proof – characterizes smooth surfaces $T \subset \mathbb{P}^5$ with one apparent double point, i.e., surfaces that acquire one singularity on generic projection into $\mathbb{P}^4$:

- $\deg(T) = 4$: $T$ is a quartic scroll

$$T \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)^2), \quad \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3));$$

- $\deg(T) = 5$: $T$ is a quintic del Pezzo surface.

Thus, Construction V says that a smooth complete intersection of two quadrics $X \subset \mathbb{P}^6$ is rational if it contains a quartic scroll, a quintic del Pezzo surface, or a sextic del Pezzo surface with a rational point. Rationality always holds when there are positive-dimensional subvarieties of odd degree (see Section 3.1), so we focus attention to the first case.

We seek criteria for the existence of a quartic scroll $T \subset X$, defined over $k$. Clearly, the class $[T]$ must be Galois-invariant; however, Galois-invariant classes need not be represented by cycles over $k$.

The intersection computations above imply that

$$[T] = [Q_2] + [Q_3],$$

where $Q_2$ and $Q_3$ represent quadric surfaces in $X$, defined over the algebraic closure. Assume that the class $[Q_2] + [Q_3]$ is Galois invariant and represents algebraic cycles defined over the ground field. We look for quartic scrolls $T \subset X$ with class $[T] = [Q_2] + [Q_3]$.

**Remark 7.1.** Over $k = \mathbb{R}$, case $(1, 2, 4)$ has signed permutation

$$\mathrm{diag}(1, -1, -1, -1, \mathbf{1}, -1, 1).$$

The intersection form on the invariant part of $H^4(X_{\mathbb{C}}, \mathbb{Z})$ is shown in Table 4.

| | $h^2$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $h^2$ | 4 | 2 | 2 | 2 |
| $Q_1$ | 2 | 2 | 1 | 1 |
| $Q_2$ | 2 | 1 | 2 | 1 |
| $Q_3$ | 2 | 1 | 1 | 2 |

Table 4.

Here, we assume $Q_1$ (and $h^2 - Q_1$) are associated with the element of $\{\mathcal{Q}_t\}$ with signature $(3, 3)$ and $Q_2$ and $Q_3$ are contributed by the elements with signatures $(1, 5)$ and $(5, 1)$. While $Q_1$ is definable over $\mathbb{R}$, $Q_2$ and $Q_3$ are not definable over $\mathbb{R}$ (see Section 5.3).

The requisite real cycles exist in $[T] = [Q_2] + [Q_3]$. This follows from the exact sequence in [6, Section 4], using the rationality of $X$ over $\mathbb{R}$ to ensure the vanishing of the unramified cohomology. It would be interesting to deduce this directly using cohomological machinery [15]. However, we do not know whether such $X$ contain quartic scrolls, in general.

Let $M$ denote the moduli space of quartic scrolls in a fixed cohomology class on $X$. There is a morphism

$$\begin{aligned} M &\to (\mathbb{P}^6)^\vee, \\ T &\mapsto \mathrm{span}(T), \end{aligned}$$

which assigns to each scroll the hyperplane it spans.

Hyperplane sections $Y = H \cap X$ containing such scrolls are singular by the Lefschetz hyperplane theorem. Computations in `Macaulay2` indicate that a generic such $Y$ has four ordinary singularities. If a complete intersection of two quadrics $Y \subset \mathbb{P}^5$ contains a quartic scroll, it contains two families of such scrolls, each parametrized by $\mathbb{P}^3$: These arise from residual intersections in quadrics in

$$I_T(2)/I_Y(2).$$

Thus, the residual family has class

$$2h^2 - [T] = \big(h^2 - [Q_2]\big) + \big(h^2 - [Q_3]\big).$$

The hyperplane sections of $X$ with four singularities should be parametrized by a reducible surface with distinguished component $\Sigma \subset (\mathbb{P}^6)^\vee$.

We speculate that $\Sigma$ is a quartic del Pezzo surface, constructed as follows: Consider the pencil of quadrics $\mathcal{Q}_t$ defining $X$ and fix the pair of rank-six quadrics

$$\mathcal{Q}_{t_2}, \mathcal{Q}_{t_3},$$

whose maximal isotropic subspaces sweep out $Q_2$ and $Q_3$. Let $v_i \in \mathcal{Q}_{t_i}$ denote the vertices and $\ell$ the line they span, which is defined over $k$ even when $t_2$ and $t_3$ are conjugate over $k$. Projecting from $\ell$ gives a degree-four cover

$$X \to \mathbb{P}^4.$$

Geometrically, the covering group is the Klein four-group and the branch locus consists of two quadric hypersurfaces $Y_2, Y_3$ intersecting in a degree-four del Pezzo surface $S_{23}$. Is $\Sigma \simeq S_{23}$ over $k$?

**7.3. Dimension six: Threefolds with one apparent double point.** Construction V indicates that the existence of a threefold $W \subset X$ with one apparent double point yields rationality. The following classification [4] builds on constructions of Edge [9]:

- $\deg(W) = 5$: a scroll in planes associated with two lines and twisted cubic, or one line and two conics;
- $\deg(W) = 6$: an Edge variety constructed as a residual intersection

  $$Q \cap (\mathbb{P}^1 \times \mathbb{P}^3) = \Pi_1 \cup \Pi_2 \cup W,$$

  where $Q$ is a quadric hypersurface and the $\Pi_i \simeq \mathbb{P}^3$ are fibers of the Segre variety under the first projection;
- $\deg(W) = 7$: an Edge variety constructed as a residual intersection

  $$Q \cap (\mathbb{P}^1 \times \mathbb{P}^3) = \Pi \cup W$$

  with the notation as above;
- $\deg(W) = 8$: a scroll in lines over $\mathbb{P}^2$ of the form $\mathbb{P}(\mathcal{E})$ (one-dimensional quotients of $\mathcal{E}$), where $\mathcal{E}$ is a rank-two vector bundle given as an extension

  $$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_{p_1,\dots,p_8}(4) \to 0$$

  for eight points $p_1, \dots, p_8 \in \mathbb{P}^2$, no four collinear or seven on a conic.

Given that rationality follows when there are positive-dimensional subvarieties of odd degree (see Section 3.1) we focus on the varieties of even degree.

**7.4. Dimension six: Degree six Edge variety.** Let $W \subset \mathbb{P}^7$ denote an Edge variety arising as follows: Consider the Segre fourfold

$$\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$$

and take the residual intersection to two copies of $\mathbb{P}^3$

$$\{0, \infty\} \times \mathbb{P}^3 \subset \mathbb{P}^1 \times \mathbb{P}^3$$

in a quadric hypersurface. The resulting threefold

$$W \simeq \mathbb{P}^1 \times \Sigma,$$

where $\Sigma \subset \mathbb{P}^3$ is a quadric hypersurface. Note that the ideal of $W \subset \mathbb{P}^7$ is generated by nine quadratic forms. Complete intersections of two quadrics

$$W \subset Y \subset \mathbb{P}^7$$

depend on five parameters. A `Magma` computation shows that a generic such $Y$ has eight ordinary singularities.

Suppose we have an embedding $W \hookrightarrow X$, where $X \subset \mathbb{P}^8$ is a smooth complete intersection of two quadrics. For fixed $X$, the Hilbert scheme of such threefolds has dimension eight. Realizing

$$W \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

we have

$$\begin{aligned} c_1(\mathcal{N}_{W/X}) &= 3(h_1 + h_2 + h_3), \\ c_2(\mathcal{N}_{W/X}) &= 8(h_1h_2 + h_1h_3 + h_2h_3), \\ c_3(\mathcal{N}_{W/X}) &= 4h_1h_2h_3. \end{aligned}$$

The Riemann–Roch formula gives $\chi(N_{W/X}) = 8$. Since $(W \cdot W)_X = 4$, the primitive class is

$$\Big([W] - \frac{3}{2}h^3\Big)^2 = 4 - 18 + 9 = -5.$$

**Remark 7.2.** Suppose that $X$ is defined over $\mathbb{R}$, and corresponds to the $(1, 2, 6)$ case, with signed permutation matrix (see Section 5.4)

$$\mathrm{diag}(-1, 1, 1, 1, -1, \mathbf{1}, -1, 1, -1).$$

The invariant cycles are as in Table 5. Here, $Q_1$ corresponds to the singular fiber of signature $(4, 4)$ and the classes $Q_2, \dots, Q_5$ correspond to the singular fibers of signatures $(2, 6)$ and $(6, 2)$.

The class

$$[Q_2] + [Q_3] + [Q_4] + [Q_5] - [Q_1]$$

has degree six and self-intersection four. Is it represented by cycles defined over $\mathbb{R}$? Does it admit an Edge variety of degree six over $\mathbb{R}$? Over more general $k$ where the requisite cycles exist?

| | $h^3$ | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ | $Q_5$ |
|---|---|---|---|---|---|---|
| $h^3$ | 4 | 2 | 2 | 2 | 2 | 2 |
| $Q_1$ | 2 | 0 | 1 | 1 | 1 | 1 |
| $Q_2$ | 2 | 1 | 0 | 1 | 1 | 1 |
| $Q_3$ | 2 | 1 | 1 | 0 | 1 | 1 |
| $Q_4$ | 2 | 1 | 1 | 1 | 0 | 1 |
| $Q_5$ | 2 | 1 | 1 | 1 | 1 | 0 |

Table 5.

**7.5. Dimension six: Degree eight variety.** Let $\mathcal{E}$ be a stable rank-two vector bundle on $\mathbb{P}^2$ with invariants $c_1(\mathcal{E}) = 4L$ and $c_2(\mathcal{E}) = 8L^2$. Note that $\Gamma(\mathcal{E})$ has dimension eight, giving an inclusion

$$V := \mathbb{P}(\mathcal{E}^\vee) \subset \mathbb{P}^7.$$

We have a tautological exact sequence

$$0 \to \mathcal{O}_V(-\xi) \to \mathcal{E}_V^\vee \to Q \to 0,$$

whence

$$0 \to Q(\xi) \to T_V \to T_{\mathbb{P}^2} \to 0.$$

Thus, we have the following

$$\begin{aligned}
\xi^2 - 4L\xi + 8L^2 &= 0, \\
c(Q(\xi)) &= 1 + (2\xi - 4L) + (\xi^2 - 4L\xi + 8L^2), \\
c(\mathcal{T}_V) &= (1 + 3L + 3L^2)(c(Q(\xi)) \\
&= 1 + (2\xi - L) + (\xi^2 - 4L\xi + 8L^2 + 6\xi L - 12L^2 + 3L^2),
\end{aligned}$$

and we find

$$\begin{aligned}
c_1(\mathcal{N}_{V/X}) &= 3\xi + L, \\
c_2(\mathcal{N}_{V/X}) &= 5\xi^2 - \xi L + 2L^2, \\
c_3(\mathcal{N}_{V/X}) &= 3\xi^3 - 3\xi^2 L + 2L^2\xi - 9L^3.
\end{aligned}$$

Note that

$$\deg(L^3) = 0, \quad \deg(L^2\xi) = 1, \quad \deg(L\xi^2) = 4, \text{ and } \deg(\xi^3) = 8,$$

so we conclude that

$$(V \cdot V)_X = 14.$$

The primitive class $[V] - 2h^3$ has self-intersection $14 - 4 \cdot 8 + 16 = -2$, which means that

$$[V] = h^3 + [Q_2] + [Q_3], \quad (Q_2 \cdot Q_3) = 1,$$

where $Q_2$ and $Q_3$ are classes of quadric threefolds in $X$, defined over the algebraic closure. (Up to the action of the Weyl group $W(D_9)$ this is the only possibility.)

Returning to the only remaining case in dimension six where rationality over $\mathbb{R}$ remains open (see Section 6.2):

**Question 7.3.** *Let $X \subset \mathbb{P}^8$ be a smooth complete intersection of two quadrics over $\mathbb{R}$ in isotopy class $(1, 3, 5)$. Which classes of codimension-three cycles $X$ are realized over $\mathbb{R}$? Are there varieties with one apparent double point, defined over $\mathbb{R}$, representing these classes?*

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