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Local-global principle for classical groups over function fields of p-adic curves

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Abstract. Let K be a local field with residue field κ and F the function field of a curve over K. Let G be a connected linear algebraic group over F of classical type. Suppose char(κ) is a good prime for G. Then we prove that projective homogeneous spaces under G over F satisfy a local-global principle for rational points with respect to discrete valuations of F. If G is a semisimple simply connected group over F, then we also prove that principal homogeneous spaces under G over F satisfy a local-global principle for rational points with respect to discrete valuations of F.

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Introduction

Let k be a number field and G a semisimple simply connected linear algebraic group over k. Classical Hasse principle asserts that a principal homogeneous space under G over k has a rational point if it has rational points over all completions of k. This is a theorem due to Kneser (classical groups), Harder (for exceptional groups other than E_8) and Chernousov (for E_8). Harder also proves a Hasse principle for rational points on projective homogeneous spaces under connected linear algebraic groups over k.

Questions related to Hasse principle have been extensively studied over 'semi-global fields', namely function fields of curves over complete discretely valued fields with respect to their discrete valuations. Considerable progress has been made possible due to the patching techniques of Harbater, Hartmann, and Krashen. One could look for analogous Hasse principles for simply connected groups in this context. However, Hasse principle fails for simply connected groups in this generality [10]. If K is a p-adic field and G is a semisimple simply connected quasi split linear algebraic group over the function field of a curve over K with $p \neq 2, 3, 5$, it was proved in [11] that Hasse principle holds for G. This led to the following two conjectures [11].

Let F be the function field of a p-adic curve and Ω_F the set of all discrete valuations of F. For $v \in F$, let F_v be the completion of F at v.

Conjecture 1. *Let Y be a projective homogeneous space under a connected linear algebraic group G over F. Then Y satisfies Hasse principle with respect to Ω_F.*

Conjecture 2. *Let G be a semisimple simply connected linear algebraic group over F and Y a principal homogeneous space under G over F. Then Y satisfies Hasse principle with respect to Ω_F.*

There has been considerable progress towards these conjectures for classical groups in the 'good characteristic case'. Let G be a semisimple simply connected linear algebraic group of classical type over F. We say that the prime p is *good* for G, if $p \neq 2$ for G of type B_n, C_n, D_n (D_4 nontrialitarian) and p does not divide $n+1$ for G of type ${}^1A_n$ and p does not divide $2(n+1)$ for G of type ${}^2A_n$. Let G be any connected linear algebraic group over F. We say that G is of *classical type* if every factor of the simply connected cover $\widetilde{G}$ of the semi-simplification of $G/\operatorname{Rad}(G)$ is of classical type. We say that p is *good* for G if p is good for every factor of $\widetilde{G}$.

Suppose $p \neq 2$. It was proved in [11] that a quadratic form q over F of rank at least 3 is isotropic over F if and only if q is isotropic over F_v for all $v \in \Omega_F$. A local-global principle for generalized Severi–Brauer varieties, under an assumption on the roots of unity in F, is due to Reddy and Suresh [30]. Let A be a central simple algebra over F with an involution σ of either kind. If σ is of the second kind, then assume that $\operatorname{ind}(A) \leq 2$. Let h be a hermitian form over (A,σ). Then Wu [33] proved the validity of Conjecture 1 for the unitary groups of (A,σ). Hence, Conjecture 1 holds for all groups of type B_n, C_n, D_n and for special groups of type ${}^1A_n$ and ${}^2A_n$ in the good characteristic case ([33, Corollary 1.4]).

Conjecture 2 for groups of type B_n, C_n, D_n is due to Hu and Preeti independently [17, 29]. Conjecture 2 for $G = \mathrm{SL}_1(A)$ with index of A square-free is a consequence of the injectivity of the Rost invariant due to Merkurjev–Suslin [25] and a result of Kato [18] on the injectivity of

$$H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v\in\Omega_F} H^3(F_v,\mathbb{Q}/\mathbb{Z}(2)).$$

The case ${}^2A_n$, namely the unitary groups of algebras of index at most 2 with unitary involution is due to Hu and Preeti [17, 29].

The two main open cases concerning Conjectures 1 and 2 for classical groups were types ${}^1A_n$ and ${}^2A_n$. The Conjecture 2 for ${}^1A_n$, namely a local-global principle for reduced norms in the good characteristic case was settled by the authors and Preeti [27].

The aim of this paper is to settle Conjecture 1 and Conjecture 2 in the affirmative in the good characteristic case for all groups of types ${}^1A_n$ and ${}^2A_n$, thereby completing the proof for all classical groups in the good characteristic case. In fact we prove the following more general theorems.

Theorem 0.1 (cf. Theorem 10.1). *Let K be a local field with residue field κ and F the function field of a smooth projective curve over K. Let A be a central simple algebra over F of index coprime to* $\mathrm{char}(\kappa)$. *Then Conjecture 1 holds for* $\mathrm{PGL}(A)$.

Theorem 0.2 (cf. Theorem 11.5). *Let K be a local field with residue field κ and F_0 the function field of a smooth projective curve over K. Let F/F_0 be a quadratic extension and A be central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* $\mathrm{char}(\kappa)$. *Let h be a hermitian form over (A,σ). If $A = F$, then assume that the rank of h is at least 2. Then Conjecture 1 holds for $U(A,\sigma,h)$.*

Theorem 0.3 (cf. Theorem 13.1). *Let K be a local field with residue field κ and F_0 the function field of a smooth projective curve over K. Let F/F_0 be a quadratic extension and A a central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* $\mathrm{char}(\kappa)$. *Then Conjecture 2 holds for* $\mathrm{SU}(A,\sigma)$.

As a consequence we have the following.

Theorem 0.4 (cf. Theorem 14.1). *Let K be a local field with residue field κ and F the function field of a smooth projective curve over K. Let G be a connected linear algebraic group over F of classical type (D_4 nontrialitarian) with* $\mathrm{char}(\kappa)$ *good for G. Then Conjecture 1 holds for G.*

Theorem 0.5 (cf. Theorem 14.2). *Let K be a local field with residue field κ and F the function field of a smooth projective curve over K. Let G be a semisimple simply connected linear algebraic group over F with* $\mathrm{char}(\kappa)$ *good for G. If G is of classical type (D_4 nontrialitarian), then Conjecture 2 holds for G.*

Here is an outline of the structure of the paper. The plan is to reduce the questions on local-global principle with respect to discrete valuations to one for the patching fields in the setting of Harbater, Hartmann and Krashen [13] and then to deal with the question in the patching setting. The reduction to the patching setting requires an understanding of the structure of central simple algebras with involutions of the second kind over the branch fields [13], which are 2-local fields. This leads to the study of cyclic extensions over quadratic extensions of local fields with zero corestriction. Let F_0 be a field, F/F_0 be a quadratic extension and L/F a cyclic extension of degree coprime to $\mathrm{char}(F_0)$. It was proved in [12, Proposition 24] that the corestriction of L/F from F to F_0 is zero if and only if L/F_0 is a dihedral extension. In Section 3 we reprove this statement for the sake of completeness and deduce some consequences for dihedral extensions. In Section 2 we study dihedral extensions over an arbitrary fields. In Section 4 we describe all dihedral extensions over local fields. In Section 6 and Section 8 we describe the structure of central simple algebras with unitary involutions over 2-local fields and 2-dimensional complete fields with finite residue fields. These fields surface in the patching setting. In Section 10, we prove a local-global principle for generalized Severi–Brauer varieties without any assumption on the existence of roots of unity, completing the proof of Conjecture 1 for groups of type ${}^1A_n$. In

Section 11, we prove a local-global principle for isotropy of hermitian forms over division algebras with unitary involutions. The idea is to construct good maximal orders invariant under involution over 2-dimensional complete regular local rings. This is possible due to the complete understanding of the structure of the algebras with unitary involutions studied in Section 6. This settles Conjecture 1 for groups of type ${}^2A_n$ in the good characteristic case. In Section 12, we prove the local-global principle for principal homogeneous spaces under simply connected unitary groups in the patching setting. Finally, in Section 13 we prove the local-global principle for special unitary groups with respect to discrete valuations, thereby completing the validity of Conjecture 2 for groups of type ${}^2A_n$. More generally (cf. Section 14), we prove Conjectures 1 and 2 for groups of classical type over function fields of curves over local fields.

Throughout this paper, a projective homogeneous space Z under a connected linear algebraic group G is a projective variety Z with transitive G-action over the separable closure such that the stabilizer is a parabolic subgroup.

1. Preliminaries

Lemma 1.1. *Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_{q^2}$ the degree two extension of $\mathbb{F}_q$. Suppose q is odd and $\sqrt{-1} \notin \mathbb{F}_q$. Then $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{-1})$. Let d be the maximum integer such that $\mathbb{F}_{q^2}$ contains a primitive 2^d-th root of unity ρ. Then*

$$N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\rho) = -1.$$

Proof. Since $\sqrt{-1} \notin \mathbb{F}_q$ and q is odd, we have $\mathbb{F}_q^*/\mathbb{F}_q^{*2} = \{1, -1\}$. Since there is a unique extension of degree 2 of $\mathbb{F}_q$, we have

$$\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{-1}).$$

Let d be the maximum integer such that $\mathbb{F}_{q^2}$ contains a primitive 2^d-th root of unity ρ. Since there is no 2^{d+1}-th primitive root of unity in $\mathbb{F}_{q^2}$, $\rho \notin \mathbb{F}_{q^2}^{*2}$. Hence,

$$\mathbb{F}_{q^2}^*/\mathbb{F}_{q^2}^{*2} = \{1, \rho\}.$$

Since $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\colon \mathbb{F}_{q^2}^* \to \mathbb{F}_q^*$ is surjective, $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}\colon \mathbb{F}_{q^2}^*/\mathbb{F}_{q^2}^{*2} \to \mathbb{F}_q^*/\mathbb{F}_q^{*2}$ is surjective. Hence,

$$N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\rho) = -\theta^2$$

for some $\theta \in \mathbb{F}_q^{*2}$. Since $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\rho)^{2^d} = 1$ and $\sqrt{-1} \notin \mathbb{F}_q^*$, it follows that $\theta = 1$ and $N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\rho) = -1$. □

Corollary 1.2. *Let K_0 be a local field and K/K_0 the quadratic unramified extension. Suppose that the characteristic of the residue field of K_0 is odd and $\sqrt{-1} \notin K_0$. Then*

$$K = K_0(\sqrt{-1}).$$

Let d be the maximum integer such that K contains a primitive 2^d-th root of unity ρ. Then $N_{K/K_0}(\rho) = -1$.

Lemma 1.3. *Let $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_{q^2}$ the degree two extension of $\mathbb{F}_q$. Let $m \geq 1$. Suppose q is odd and $\sqrt{-1} \notin \mathbb{F}_q$. If $\mathbb{F}_{q^2}$ contains a primitive 2^{m+1}-th root of unity, then $\mathbb{F}_q^* \subset \mathbb{F}_{q^2}^{*2^m}$.*

Proof. Since $\sqrt{-1} \notin \mathbb{F}_q^*$, the only 2^m-th roots of unity in $\mathbb{F}_q$ are ± 1. Hence, we have an exact sequence of groups

$$1 \to \{\pm 1\} \to \mathbb{F}_q^* \to \mathbb{F}_q^{*2^m} \to 1,$$

where the last map is given by $x \to x^{2^m}$. Thus, the order of $\mathbb{F}_q^*/\mathbb{F}_q^{*2^m}$ is 2. Since $-1 \notin \mathbb{F}_q^{*2}$,

$$-1 \notin \mathbb{F}_q^{*2^m} \quad \text{and} \quad \mathbb{F}_q^* = \mathbb{F}_q^{*2^m} \cup (-1)\mathbb{F}_q^{*2^m}.$$

Since $\mathbb{F}_{q^2}^*$ contains a primitive 2^{m+1}-th root of unity, $-1 \in \mathbb{F}_{q^2}^{*2^m}$. Thus, $\mathbb{F}_q^* \subset \mathbb{F}_{q^2}^{*2^m}$. □

Corollary 1.4. *Let K_0 be a local field and K/K_0 the quadratic unramified extension. Suppose that the characteristic of the residue field of K_0 is odd and $\sqrt{-1} \notin K_0$. Let $m \geq 1$. If K contains a primitive 2^{m+1}-th root of unity, then every unit in the valuation ring of K_0 is in K^{*2^m}.*

Lemma 1.5. *Let $\mathbb{F}_q$ be the finite field with q elements. Let $m \geq 1$ be coprime to q. Suppose that $\mathbb{F}_q$ does not contain any nontrivial m-th root of unity. Then $\mathbb{F}_q^* = \mathbb{F}_q^{*m}$.*

Proof. Since $\mathbb{F}_q^*$ does not contain nontrivial m-th roots of unity, the only m-th root of unity in $\mathbb{F}_q$ is 1. Hence, the homomorphism

$$\mathbb{F}_q^* \to \mathbb{F}_q^{*^m}, \quad x \mapsto x^m$$

is an isomorphism. Thus, $\mathbb{F}_q^* = \mathbb{F}_q^{*^m}$. □

Corollary 1.6. *Let K_0 be a local field. Let $m \geq 1$ be coprime to the characteristic of the residue field of K_0. Suppose that K_0 does not contain any nontrivial m-th root of unity. Then every unit in the valuation ring of K_0 is in K_0^m.*

Let F be a discretely valued field with valuation ring R and residue field K. We say that an element $a \in F$ is a *unit in F* if $a \in R$ is a unit. Let $n \geq 1$ be an integer coprime to $\mathrm{char}(K)$. Then we have the residue map

$$\partial\colon H^d(F, \mu_n^{\otimes i}) \to H^{d-1}(K, \mu_n^{\otimes(i-1)}).$$

Let $H^d_{nr}(F, \mu_n^{\otimes i})$ be the kernel of ∂. An element $\alpha \in H^d_{nr}(F, \mu_n^{\otimes i})$ is called an *unramified* element. If F is complete, then we have an isomorphism

$$H^d(K, \mu_n^{\otimes i}) \simeq H^d_{nr}(F, \mu_n^{\otimes i}).$$

We end this section with the following result on reduced norms.

Proposition 1.7. *Let K be a global field with no real places and F a complete discretely valued field with residue field K. Let A be a central simple algebra over F of index n coprime to* $\mathrm{char}(K)$. *Let $(L, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ be the residue of A. Let $\theta \in F^*$ be a unit. If the image of $\theta \in K^*$ is a norm from the extension L/K, then θ is a reduced norm from A.*

Proof. Let E/F be the unramified extension with residue field L and $\tilde{\sigma}$ a generator of $\mathrm{Gal}(E/F)$ lifting σ. Let R be the valuation ring of F and $\pi \in R$ be a parameter. Then $A = A_0 + (E, \tilde{\sigma}, \pi)$ for some central simple algebra A_0 over F representing a class in $H^2_{nr}(F, \mu_n)$ (cf. [27, Lemma 4.1]). Since F is complete and the image of θ in K is a norm from L/K, θ is a norm from E/F. Hence,

$$(E, \tilde{\sigma}, \pi) \cdot (\theta) = 0 \in H^3(F, \mu_n^{\otimes 2}).$$

Since A_0 is unramified on R and θ is a unit, $A_0 \cdot (\theta) \in H^3_{nr}(F, \mu_n^{\otimes 2})$. Since K is a global field with no real places, $\mathrm{cd}(K) = 2$ and $H^3(K, \mu_n^{\otimes 2}) = 0$. Hence,

$$H^3_{nr}(F, \mu_n^{\otimes 2}) = 0 \quad \text{and} \quad A_0 \cdot (\theta) = 0.$$

In particular, $A \cdot (\theta) = 0 \in H^3(F, \mu_n^{\otimes 2})$ and, by [27, Theorem 4.12], θ is a reduced norm from A. □

2. Dihedral extensions

Let G be a dihedral group of order $2m \geq 4$. Let σ and τ be the generators of G with $\sigma^m = 1$, $\tau^2 = 1$, and $\tau\sigma\tau = \sigma^{-1}$. The subgroup generated by σ is the *rotation* subgroup of G and for $0 \leq i \leq m-1$, $\sigma^i\tau$ are the *reflections*.

Let F_0 be a field and E/F_0 a field extension. We say that E/F_0 is a *dihedral extension* if E/F_0 is Galois with Galois group isomorphic to a dihedral group. In this section we prove some basic facts about dihedral extensions.

Lemma 2.1. *Let F_0 be a field and E/F_0 a dihedral extension. Let F be the fixed field of the rotation subgroup of* $\mathrm{Gal}(E/F_0)$. *If M/F is a subextension of E/F with $M \neq F$, then M/F_0 is a dihedral extension.*

Proof. Let $\mathrm{Gal}(E/F_0)$ be generated by σ and τ with $\sigma^m = 1$, $\tau^2 = 1$, and $\tau\sigma\tau = \sigma^{-1}$. Then E/F is cyclic with $\mathrm{Gal}(E/F)$ generated by σ. Let M/F be a subextension of E/F. The extension M/F is cyclic with $\mathrm{Gal}(M/F)$ generated by the restriction of σ to M. Since $M = E^{\sigma^i}$ for some i and $\tau\sigma^i\tau = \sigma^{-i}$, the extension M/F_0 is Galois with the $\mathrm{Gal}(M/F_0)$ generated by the restriction of σ and τ to M. Since $M \neq F$, the restriction of σ to M is nontrivial. Since $F \subset M$, the restriction of τ to M is nontrivial. Hence, M/F_0 is dihedral. □

Lemma 2.2. *Let E/F_0 be a dihedral extension and F the fixed field of the rotation subgroup of* $\mathrm{Gal}(E/F_0)$. *Let $F_0 \subseteq L \subseteq E$ with $F \not\subset L$. If L/F_0 is Galois, then*

$$[L : F_0] \leq 2.$$

Proof. Suppose that $F \not\subset L$ and L/F_0 is Galois. Let $M = FL$. Suppose that $L \neq F_0$. Then $M \neq F$, and hence M/F_0 is dihedral (Lemma 2.1). Since $F \not\subset L$, we have

$$[M : F] = [L : F_0].$$

Since L/F_0 and F/F_0 are Galois extensions, M/F_0 is Galois with $\mathrm{Gal}(M/F_0)$ isomorphic to $\mathrm{Gal}(L/F_0) \times \mathrm{Gal}(F/F_0)$. Since the only dihedral group which is isomorphic to a direct product of two nontrivial subgroups is $\mathbb{Z}/2 \times \mathbb{Z}/2$, we have

$$[L : F_0] = 2. \qquad \square$$

Lemma 2.3. *Let F_0 be a field and E/F_0 a dihedral extension of degree $2m$. Let F be the fixed field of the rotation subgroup of* $\mathrm{Gal}(E/F_0)$. *Then there exist exactly m subfields E' of E containing F_0 with*

$$[E' : F_0] = [E : F] \quad \textit{and} \quad E'F = E.$$

Further, if E' is any such subfield of E and $\ell_1, \ell_2, \cdots, \ell_r$ is any sequence of prime numbers with $[E : F] = \ell_1 \cdots \ell_r$, then there exist subfields

$$F_0 = L_0 \subset L_1 \subset \cdots \subset L_r = E'$$

with $[L_i : L_{i-1}] = \ell_i$.

Proof. Let σ be a generator of the rotation subgroup of $\mathrm{Gal}(E/F_0)$ and τ a reflection. For $0 \leq i \leq m-1$, let $E_i = E^{\tau\sigma^i}$ be the subfield of E fixed by $\tau\sigma^i$. Then

$$[E : E_i] = 2, \quad [E_i : F_0] = m, \quad \text{and} \quad E_iF = E.$$

Since the only elements of order 2 in $\mathrm{Gal}(E/F_0)$ which are not the identity on F are the reflections $\tau\sigma^i$, $0 \leq i \leq m-1$, any E' with the given properties coincides with E_i for some i.

Let $E' = E_i$ for some i. Suppose $m = \ell_1 \cdots \ell_r$ with ℓ_j's primes. Since E/F is a cyclic extension, there exist subfields

$$F = M_0 \subset M_1 \subset \cdots \subset M_r = E$$

such that $[M_j : M_{j-1}] = \ell_i$ for all i. Then $L_j = E' \cap M_j$ have the required property. □

Lemma 2.4. *Let F_0 be a field and F/F_0 a quadratic Galois extension. Let $m \geq 2$ be coprime to* $\operatorname{char}(F_0)$. *Suppose that F contains a primitive m-th root of unity ρ. Let $a \in F_0^*$. Suppose that*

$$[F(\sqrt[m]{a}) : F] = m.$$

Then $F(\sqrt[m]{a})/F_0$ is dihedral if and only if $N_{F/F_0}(\rho) = 1$.

Proof. Let $E = F(\sqrt[m]{a})$ and $E' = F_0(\sqrt[m]{a})$. Since $a \in F_0^*$, we have $E = E'F$. Since $[E : F_0] = 2m$, we have

$$[E : E'] = 2.$$

Let σ be the automorphism of $F(\sqrt[m]{a})/F$ given by $\sigma(\sqrt[m]{a}) = \rho\sqrt[m]{a}$ and τ the nontrivial automorphism of E/E'. Since τ is nontrivial on F, it follows that $\tau \neq \sigma^i$ for any i. Hence, E/F_0 is Galois and $\operatorname{Gal}(E/F_0)$ is generated by σ and τ. Since the order of σ is m and $\tau^2 = 1$, $\operatorname{Gal}(E/F_0)$ is dihedral if and only if $\tau\sigma\tau = \sigma^{-1}$.

We have

$$\begin{aligned}\tau\sigma\tau(\sqrt[m]{a}) &= \tau\sigma(\sqrt[m]{a}) = \tau(\rho\sqrt[m]{a}) = \tau(\rho)\sqrt[m]{a}\\ &= \tau(\rho)\rho\rho^{-1}\sqrt[m]{a} = \tau(\rho)\rho\sigma^{-1}(\sqrt[m]{a}).\end{aligned}$$

Hence, $\tau\sigma\tau = \sigma^{-1}$ if and only if $N_{F/F_0}(\rho) = \tau(\rho)\rho = 1$. □

We end this section with the following lemma.

Lemma 2.5. *Let F_0 be a field and $n \geq 2$ an integer with $2n$ coprime to* $\operatorname{char}(F_0)$. *Let E/F_0 be a dihedral extension of degree $2n$ and σ and τ generators on* $\operatorname{Gal}(E/F_0)$ *with σ a rotation and τ a reflection. Let $F = E^{\sigma}$ and $E_i = E^{\sigma^i\tau}$ for $1 \leq i \leq n$. Let M/F_0 be a field extension. Suppose $F \otimes_{F_0} M$ is a field and $E \otimes_{F_0} M$ is isomorphic to $\prod_1^n (F \otimes_{F_0} M)$. Then there exists i such that*

$$E_i \otimes_{F_0} M \simeq M \times E_i'$$

for some M-algebra E_i'.

Proof. The proof is by induction on n. Suppose that $n = 2$. Then

$$F = F_0(\sqrt{a}), \quad E_1 = F_0(\sqrt{b}), \quad E_2 = F_0(\sqrt{ab}), \quad E = F(\sqrt{b}).$$

Suppose that $M(\sqrt{a}) = F \otimes_{F_0} M$ is a field and $E \otimes_{F_0} M$ is not a field. Then a is not a square in M and $E \otimes_{F_0} M \simeq M(\sqrt{a}) \times M(\sqrt{a})$. Then either b is a square in M or ab is a square in M. Thus, either

$$E_1 \otimes_{F_0} M \simeq M \times M \quad \text{or} \quad E_2 \otimes_{F_0} M \simeq M \times M.$$

Suppose $n \geq 3$. Suppose that $M(\sqrt{a}) = F \otimes_{F_0} M$ is a field and

$$E \otimes_{F_0} M \simeq \prod_1^n M(\sqrt{a}).$$

Suppose n is odd. Since $E_i \otimes_{F_0} F \simeq E$ and F/F_0 is of degree 2, it follows that

$$E_i \otimes_{F_0} M \simeq \prod_1^r M \times \prod_1^s M(\sqrt{a}).$$

Since $[E_i : F_0] = n$ is odd, $r \geq 1$.

Suppose that n is even. Then, by Lemma 2.3, there exists a quadratic extension F_1/F_0 contained in E and $F_1 \neq F$. Let $F' = FF_1$. Then F'/F_0 is a biquadratic extension. Hence, there is a quadratic extension F_2/F_0 contained in F' with $F \neq F_2$ and $F_1 \neq F_2$. Further, every quadratic extension of F_0 contained in E is either F, F_1, or F_2. Since every E_i contains a quadratic extension of F_0 (Lemma 2.3) and $F \not\subset E_i$, half of E_i contain F_1 and the remaining half of E_i contain F_2. Further, E/F_1 and E/F_2 are dihedral extensions of degree n.

Since $E \otimes_{F_0} M \simeq \prod_1^n M(\sqrt{a})$, we have

$$F' \otimes_{F_0} M \simeq M(\sqrt{a}) \times M(\sqrt{a}).$$

Thus, by the case $n = 2$, either $F_1 \otimes_{F_0} M \simeq M \times M$ or $F_2 \otimes_{F_0} M \simeq M \times M$. Without loss of generality, assume that $F_1 \otimes_{F_0} M \simeq M \times M$. Then F_1 is isomorphic to a subfield of M, and hence M/F_1 is an extension of fields.

Since $F' = F_1(\sqrt{a})$ and a is not a square in M, then $F' \otimes_{F_1} M$ is a field. Since

$$\begin{aligned} E \otimes_{F_0} M &\simeq E \otimes_{F_1} F_1 \otimes_{F_0} M \\ &\simeq E \otimes_{F_1} (M \times M) \simeq E \otimes_{F_1} M \times E \otimes_{F_1} M \end{aligned}$$

and

$$E \otimes_{F_0} M \simeq \prod_1^n M(\sqrt{a}),$$

it follows that

$$E \otimes_{F_1} M \simeq \prod_1^{n/2} M(\sqrt{a}).$$

Since E/F_1 is dihedral and $[E : F_1] < [E : F_0]$, by induction there exists an i such that $E_i \otimes_{F_1} M \simeq M \times E_i''$ for some M-algebra E_i''. We have

$$\begin{aligned} E_i \otimes_{F_0} M &\simeq E_i \otimes_{F_1} F_1 \otimes_{F_0} M \simeq E_i \otimes_{F_1} (M \times M) \\ &\simeq E_i \otimes_{F_1} M \times E_i \otimes_{F_1} M \simeq M \times E_i'' \times E_i \otimes_{F_1} M. \end{aligned}$$

Hence, $E_i \otimes_{F_0} M \simeq M \times E_i'$ for some E_i'. $\square$

3. Corestriction of cyclic extensions over quadratic extensions

In this section we realize cyclic extensions over quadratic extensions with corestriction zero as dihedral extensions.

Let K be a field and A a Galois module over K. For $n \geq 0$, let $H^n(K, A)$ denote the n-th Galois cohomology group with values in A (cf. [26, Ch. VI]). For an extension of fields M/K, let

$$\mathrm{res} = \mathrm{res}_{M/K} \colon H^n(K, A) \to H^n(M, A)$$

be the restriction homomorphism and for a finite extension L/K, and let

$$\mathrm{cores} = \mathrm{cores}_{L/K} \colon H^n(L, A) \to H^n(K, A)$$

be the corestriction homomorphism (cf. [26, p. 47]).

Let F_0 be a field and F/F_0 a Galois extension of degree 2. Let τ_0 be the nontrivial automorphism of F/F_0. Let $\overline{F}$ be an algebraic closure of F. Let $\tilde{\tau} \in \mathrm{Gal}(\overline{F}/F_0)$ be such that $\tilde{\tau}$ restricted to F is τ_0. Since $\tilde{\tau} \notin \mathrm{Gal}(\overline{F}/F)$ and $[F : F_0] = 2$, we have

$$\mathrm{Gal}(\overline{F}/F_0) = \mathrm{Gal}(\overline{F}/F) \cup \mathrm{Gal}(\overline{F}/F)\tilde{\tau} \quad \text{and} \quad \tilde{\tau}^2 \in \mathrm{Gal}(\overline{F}/F).$$

Let $\mathrm{Hom}_c(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/m\mathbb{Z})$ be the group of continuous homomorphisms from $\mathrm{Gal}(\overline{F}/F)$ to $\mathbb{Z}/m\mathbb{Z}$ with profinite topology on $\mathrm{Gal}(\overline{F}/F)$ and discrete topology on $\mathbb{Z}/m\mathbb{Z}$. Since the action of $\mathrm{Gal}(\overline{F}/F)$ on $\mathbb{Z}/m\mathbb{Z}$ is trivial, we have

$$H^1(F, \mathbb{Z}/m\mathbb{Z}) \simeq \mathrm{Hom}_c(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/m\mathbb{Z}).$$

The group $\mathrm{Hom}_c(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/m\mathbb{Z})$ also classifies isomorphism classes of pairs (E, σ) with E/F a cyclic extension of degree dividing m and σ a generator of $\mathrm{Gal}(E/F)$.

Lemma 3.1. *Let* $\phi \in \mathrm{Hom}_c(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/m\mathbb{Z})$. *Then*

$$\mathrm{cores}(\phi) \colon \mathrm{Gal}(\overline{F}/F_0) \to \mathbb{Z}/m\mathbb{Z}$$

is the homomorphism given by

$$\mathrm{cores}(\phi)(\theta) = \phi(\theta) + \phi(\tilde{\tau}\theta\tilde{\tau}^{-1})$$

for all $\theta \in \mathrm{Gal}(\overline{F}/F)$ *and* $\mathrm{cores}(\phi)(\tilde{\tau}) = \phi(\tilde{\tau}^2)$.

Proof. See [26, p. 53]. □

Proposition 3.2 (cf. [12, Proposition 24]). *Let* F_0 *be a field and* F/F_0 *a quadratic Galois field extension. Let* E/F *be a cyclic extension of degree* m *and* σ *a generator of* $\mathrm{Gal}(E/F)$. *Then* $\mathrm{cores}_{F/F_0}(E, \sigma)$ *is zero if and only if* E/F_0 *is a dihedral extension.*

Proof. Since E/F is a cyclic extension with generator σ, we have an isomorphism

$$\phi_0\colon \mathrm{Gal}(E/F) \to \mathbb{Z}/m\mathbb{Z}$$

given by $\phi_0(\sigma^i) \to i \in \mathbb{Z}/m\mathbb{Z}$. Let $\phi\colon \mathrm{Gal}(\bar{F}/F) \to \mathbb{Z}/m\mathbb{Z}$ be the composition

$$\mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(E/F) \xrightarrow{\phi_0} \mathbb{Z}/m\mathbb{Z}.$$

The pair (E,σ) corresponds to the element ϕ in $\mathrm{Hom}_c(\mathrm{Gal}(\bar{F}/F), \mathbb{Z}/m\mathbb{Z})$. Then

$$\mathrm{cores}(\phi)\colon \mathrm{Gal}(\bar{F}/F_0) \to \mathbb{Z}/m\mathbb{Z}$$

is the homomorphism given by $\mathrm{cores}(\phi)(\theta) = \phi(\theta) + \phi(\tilde{\tau}\theta\tilde{\tau}^{-1})$ for all $\theta \in \mathrm{Gal}(\bar{F}/F)$ and $\mathrm{cores}(\phi)(\tilde{\tau}) = \phi(\tilde{\tau}^2)$ (cf. Lemma 3.1).

Suppose $\mathrm{cores}_{F/F_0}(E,\sigma)$ is the zero homomorphism. Then

$$\mathrm{cores}(\phi)\colon \mathrm{Gal}(\bar{F}/F_0) \to \mathbb{Z}/m\mathbb{Z}$$

is the zero homomorphism. Let $\theta \in \mathrm{Gal}(\bar{F}/F)$. Then

$$0 = \mathrm{cores}(\phi)(\theta) = \phi(\theta) + \phi(\tilde{\tau}\theta\tilde{\tau}^{-1}),$$

and hence $\phi(\tilde{\tau}\theta\tilde{\tau}^{-1}) = -\phi(\theta)$.

Suppose $\theta \in \mathrm{Gal}(\bar{F}/E) \subseteq \mathrm{Gal}(\bar{F}/F)$. Since $\mathrm{Gal}(\bar{F}/E)$ is the kernel of ϕ, we have

$$\phi(\tilde{\tau}\theta\tilde{\tau}^{-1}) = -\phi(\theta) = 0,$$

and hence $\tilde{\tau}\theta\tilde{\tau}^{-1} \in \mathrm{Gal}(\bar{F}/E)$. Since $\mathrm{Gal}(\bar{F}/F_0)$ is generated by $\mathrm{Gal}(\bar{F}/F)$ and $\tilde{\tau}$, $\mathrm{Gal}(\bar{F}/E)$ is a normal subgroup of $\mathrm{Gal}(\bar{F}/F_0)$. Hence, E/F_0 is a Galois extension.

Let us denote the restriction of $\tilde{\tau}$ to E by τ. Since $\tau\sigma\tau^{-1}$ is the identity on F and E/F is Galois, $\tau\sigma\tau^{-1} \in \mathrm{Gal}(E/F)$. Let $\tilde{\sigma} \in \mathrm{Gal}(\bar{F}/F)$ with restriction to E equal to σ. Since

$$\phi(\tilde{\tau}\tilde{\sigma}\tilde{\tau}^{-1}) = -\phi(\tilde{\sigma}) = \phi(\tilde{\sigma}^{-1}),$$

it follows that $\phi_0(\tau\sigma\tau^{-1}) = \phi_0(\sigma^{-1})$. Since ϕ_0 is an isomorphism, $\tau\sigma\tau^{-1} = \sigma^{-1}$. Since

$$\phi(\tilde{\tau}^2) = \mathrm{cores}(\phi)(\tilde{\tau}) = 0,$$

it follows that $\phi_0(\tau^2) = 0$. Since ϕ_0 is an isomorphism, τ^2 is the identity on E. Since $\mathrm{Gal}(E/F_0)$ is generated by σ and τ, with $\sigma^m = 1$, $\tau^2 = 1$, and $\tau\sigma\tau^{-1} = \sigma^{-1}$, $\mathrm{Gal}(E/F_0)$ is a dihedral group of order $2m$.

Conversely, suppose $\mathrm{Gal}(E/F_0)$ is a dihedral extension. Since the subgroup of $\mathrm{Gal}(E/F_0)$ generated by σ is of index 2, $\mathrm{Gal}(E/F_0)$ is generated by σ and τ with $\tau^2 = 1$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. Since $\tau \neq \sigma^i$ for all i, τ is not an identity on F. Let $\tilde{\tau}$ be an extension of τ to $\bar{F}$. Then we have

$$\mathrm{cores}(\phi)(\theta) = \phi(\theta) + \phi(\tilde{\tau}\theta\tilde{\tau}^{-1})$$

for all $\theta \in \mathrm{Gal}(\overline{F}/F)$ and $\mathrm{cores}(\phi)(\tilde{\tau}) = \phi(\tilde{\tau}^2)$ (Lemma 3.1). Let $\theta \in \mathrm{Gal}(\overline{F}/F)$. Since $\mathrm{Gal}(E/F)$ is cyclic and generated by σ, θ restricted to E is σ^i for some i. Since $\tau\sigma^i\tau^{-1} = \sigma^{-i}$, we have $\theta\tilde{\tau}\theta\tilde{\tau}^{-1} \in \mathrm{Gal}(\overline{F}/E)$. Since the kernel of ϕ is $\mathrm{Gal}(\overline{F}/E)$, we have

$$\mathrm{cores}(\phi)(\theta) = \phi(\theta) + \phi(\tilde{\tau}\theta\tilde{\tau}^{-1}) = \phi(\theta\tilde{\tau}\theta\tilde{\tau}^{-1}) = 0$$

for all $\theta \in \mathrm{Gal}(\overline{F}/F)$. Since τ^2 is identity on E, we have $\tilde{\tau}^2 \in \mathrm{Gal}(\overline{F}/E)$, and hence

$$\mathrm{cores}(\phi)(\tilde{\tau}) = \phi(\tilde{\tau}^2) = 0.$$

Since $\mathrm{Gal}(\overline{F}/F_0)$ is generated by $\mathrm{Gal}(\overline{F}/F)$ and $\tilde{\tau}$, $\mathrm{cores}(\phi) = 0$. □

Corollary 3.3. *Let F_0 be a field and F/F_0 a quadratic Galois extension. Let $m \geq 2$ be coprime to* $\mathrm{char}(F_0)$. *Suppose that F contains a primitive m-th root of unity ρ. Let $a \in F_0^*$. Suppose that $[F(\sqrt[m]{a}) : F] = m$. Let σ be the automorphism of $F(\sqrt[m]{a})$ given by $\sigma(\sqrt[m]{a}) = \rho\sqrt[m]{a}$. Then* $\mathrm{cores}(F(\sqrt[m]{a}), \sigma)$ *is zero if and only if $N_{F/F_0}(\rho) = 1$.*

Proof. The lemma follows from Proposition 3.2 and Lemma 2.4. □

Lemma 3.4. *Let F_0 be a field of characteristic not 2 and F/F_0 a quadratic extension. Let $n \geq 1$. Let ρ be a 2^n-th root of unity in F. Suppose that $\sqrt{-1} \notin F_0$. Then $N_{F/F_0}(\rho) = \pm 1$.*

Proof. If $n = 1$, then $\rho = -1$, and hence

$$N_{F/F_0}(-1) = (-1)^2 = 1.$$

Suppose $n \geq 2$. Let τ_0 be the nontrivial automorphism of F/F_0. Since ρ is a 2^n-th root of unity, $\tau(\rho)$ is also 2^n-th root of unity, and hence $\rho\tau(\rho)$ is a 2^n-th root of unity in F_0. Since ± 1 are the only 2^n-th roots of unity in F_0, we have

$$N_{F/F_0}(\rho) = \rho\tau(\rho) = \pm 1. \qquad \square$$

Corollary 3.5. *Let F_0 be a field of characteristic not 2 and F/F_0 a quadratic extension. Let $n \geq 2$. Suppose that F contains a primitive 2^n-th root of unity ρ and $\sqrt{-1} \notin F_0$. Let $a \in F_0^*$. Let $1 \leq d \leq n$. Suppose that*

$$[F(\sqrt[2^d]{a}) : F] = 2^d.$$

Let σ_d be the automorphism of $F(\sqrt[2^d]{a})$ given by

$$\sigma_d(\sqrt[2^d]{a}) = \rho^{2^{n-d}} \sqrt[2^d]{a}.$$

If $d < n$, then $\mathrm{cores}_{F/F_0}(F(\sqrt[2^d]{a}), \sigma_d)$ *is zero. Further, $N_{F/F_0}(\rho) = 1$ if and only if* $\mathrm{cores}(F(\sqrt[2^n]{a}), \sigma_n)$ *is zero.*

Proof. Suppose $d < n$. Then

$$N_{F/F_0}(\rho^{2^{n-d}}) = N_{F/F_0}(\rho^{2^{n-d-1}})^2 = 1$$

(see Lemma 3.4), and hence $\mathrm{cores}(F(\sqrt[2^d]{a}), \sigma_d)$ is zero (Corollary 3.3).

Suppose $n = d$. Then, by Corollary 3.3, $\mathrm{cores}(F(\sqrt[2^n]{a}), \sigma_n)$ is zero if and only if $N_{F/F_0}(\rho) = 1$. $\square$

Lemma 3.6. *Let F_0 be a field of characteristic not 2 and F/F_0 a quadratic extension. Let ℓ be a prime not equal to $\mathrm{char}(F_0)$. Let $n \geq 1$. Suppose that F contains a primitive ℓ^n-th root of unity ρ and F_0 does not contain any nontrivial ℓ-th root of unity. Let $a \in F_0^*$. Suppose that*

$$[F(\sqrt[\ell^n]{a}) : F] = \ell^n.$$

Let σ be the automorphism of $F(\sqrt[\ell^n]{a})$ given by

$$\sigma(\sqrt[\ell^n]{a}) = \rho \sqrt[\ell^n]{a}.$$

Then $\mathrm{cores}_{F/F_0}(F(\sqrt[\ell^n]{a}), \sigma)$ is zero.

Proof. Since $\rho^{\ell^n} = 1$, we have $N_{F/F_0}(\rho)^{\ell^n} = 1$. Since F_0 has no nontrivial ℓ-th root of unity, $N_{F/F_0}(\rho) = 1$ and, by Corollary 3.3, $\mathrm{cores}(E, \sigma) = 0$. $\square$

4. Dihedral extensions over local fields

Let F_0 be a complete discrete valued field with residue field κ_0. Let E/F_0 be a dihedral extension of degree $2m$ with $2m$ coprime to $\mathrm{char}(\kappa_0)$. Let $F \subseteq E$ be the fixed field of the rotation subgroup of $\mathrm{Gal}(E/F_0)$. In this section we first determine the degree of E/F_0 if F/F_0 is ramified and then we go on to describe all the dihedral extensions of local fields.

We begin with the following lemma.

Lemma 4.1. *Let F_0 be a complete discrete valued field with residue field κ_0. Let E/F_0 be a dihedral extension of degree $2m$ with $2m$ coprime to $\mathrm{char}(\kappa_0)$. Suppose the subfield F of E fixed by the rotation subgroup of $\mathrm{Gal}(E/F_0)$ is ramified over F_0. Let L/F_0 be an extension contained in E. If $F \not\subseteq L$ and L/F_0 is either unramified or totally ramified, then $[L : F_0] \leq 2$.*

Proof. Let L/F_0 be an extension contained in E with $F \not\subseteq L$. We show that L/F_0 is cyclic.

Suppose that L/F_0 is unramified. Let κ be the residue field of F and κ' the residue field of L. Then

$$\kappa = \kappa_0 \quad \text{and} \quad [\kappa' : \kappa_0] = [L : F_0].$$

Since F/F_0 is totally ramified, LF/F is an extension of degree $[L : F_0]$ and the residue field of LF is also κ'. Since $LF \subset E$ and E/F is cyclic, LF/F is cyclic. In particular, κ'/κ_0 is cyclic. Since L/F_0 is unramified and F_0 is complete, L/F_0 is cyclic and by Lemma 2.2, $[L : F_0] \leq 2$.

Suppose that L/F_0 is totally ramified of degree d. Since d is coprime to $\operatorname{char}(\kappa_0)$, $L = F_0(\sqrt[d]{\pi})$ for some parameter $\pi \in F_0$ (cf. [27, Lemma 2.4]). Since $F \not\subset L$, we have

$$[LF : F] = [L : F_0].$$

Since E/F is cyclic, LF/F is cyclic. Since

$$LF = F(\sqrt[d]{\pi}) \quad \text{and} \quad [LF : F] = [L : F_0] = d,$$

then F contains a primitive d-th root of unity. Since F/F_0 is totally ramified, F_0 contains a primitive d-th root of unity. In particular, L/F_0 is cyclic and by Lemma 2.2, $[L : F_0] \leq 2$. $\square$

Proposition 4.2. *Let F_0 be a complete discrete valued field with residue field κ_0. Let E/F_0 be a dihedral extension of degree $2m$ with $2m$ coprime to* $\operatorname{char}(\kappa_0)$. *If the subfield of E fixed by the rotation subgroup of* $\operatorname{Gal}(E/F_0)$ *is ramified over F_0, then* $[E : F_0] \leq 4$.

Proof. Let F be the subfield of E fixed the rotation subgroup of $\operatorname{Gal}(E/F_0)$. Then

$$[F : F_0] = 2.$$

Suppose that F/F_0 is ramified. Then F/F_0 is totally ramified.

Suppose that $[E : F_0] = 2m \geq 5$. Suppose there is an odd prime ℓ dividing m. Then there exists an extension L/F_0 of degree ℓ such that $L \subset E$ and $F \not\subset L$ (Lemma 2.3). Since ℓ is a prime, L/F_0 is either unramified or totally ramified. Then, by Lemma 4.1, we have

$$[L : F_0] = \ell \leq 2,$$

leading to a contradiction.

Suppose there is no odd prime dividing m. Then 4 divides m. Thus, there exists an extension L/F_0 of degree 4 such that $L \subset E$ and $F \not\subset L$ (Lemma 2.3). Since $[L : F_0] = 4$, by Lemma 4.1, L/F_0 is neither totally ramified nor unramified. Also since $[L : F_0] = 4$, we have

$$L = F_0(\sqrt{u})(\sqrt{\pi})$$

for some $u \in F_0$ a unit and π a parameter in $F_0(\sqrt{u})$. Since F/F_0 is ramified,

$$F = F_0(\sqrt{\pi_1})$$

for some parameter π_1 in F_0. Since $F_0(\sqrt{u})/F_0$ is unramified, π_1 is a parameter in $F_0(\sqrt{u})$, and hence $\pi = v\pi_1$ for some unit $v \in F_0(\sqrt{u})$. Let $L' = F_0(\sqrt{u})(\sqrt{v})$. Since $[LF : F_0] = 8$ and $LF = F_0(\sqrt{u})(\sqrt{v}, \sqrt{\pi_1})$, we have

$$[L' : F_0] = 4.$$

Since L'/F_0 is unramified, by Lemma 4.1, we have

$$[L' : F_0] \leq 2,$$

leading to a contradiction. □

Corollary 4.3. *Let F_0 be a complete discrete valued field with residue field κ_0 of characteristic not 2. Let F/F_0 be a ramified quadratic field extension. Let E/F be a cyclic extension of degree coprime to* $\mathrm{char}(\kappa_0)$ *and σ a generator of* $\mathrm{Gal}(E/F)$*. If* $\mathrm{cores}_{F/F_0}(E,\sigma)$ *is zero, then $[E : F] \leq 2$.*

Proof. Suppose $\mathrm{cores}_{F/F_0}(E,\sigma)$ is zero. Then E/F_0 is Galois with $\mathrm{Gal}(E/F_0)$ dihedral (Proposition 3.2). Since F/F_0 is ramified, by Proposition 4.2, $[E : F] \leq 2$. □

Proposition 4.4. *Let K_0 be a local field and L/K_0 be a dihedral extension of degree $2m$. Let K be the subfield of L fixed by the rotation subgroup of* $\mathrm{Gal}(L/K_0)$*. If K/K_0 is unramified, then L/K is totally ramified.*

Proof. Let L^{nr} be the maximal unramified subextension of L/K_0. Suppose K/K_0 is unramified. Then $K \subseteq L^{nr}$. Suppose that $K \neq L^{nr}$. Then, by Lemma 2.1, L^{nr}/K_0 is dihedral. Since K_0 is a local field and L^{nr}/K_0 is unramified, L^{nr}/K_0 is cyclic. Since a dihedral group can not be cyclic, $L^{nr} = K$. □

Remark 4.5. Let K_0 be a local field with characteristic of the residue field not 2. Let $\pi \in K_0$ be a parameter and $u \in K_0$ a unit which is not a square. Since $K_0^*/K_0^{*2} = \{1, \pi, u, u\pi\}$ (cf. [32, Theorem 4.1, p. 217]), $L = K_0(\sqrt{u}, \sqrt{\pi})$ is the unique degree four extension with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the dihedral group of order 4, L/K_0 is the unique dihedral extension of degree 4.

Theorem 4.6. *Let K_0 be a local field with characteristic of the residue field not 2 and $\pi \in K_0$ be a parameter. Let d be the maximum integer such that $K_0(\sqrt{-1})$ contains a primitive 2^d-th root of unity. Then there exists a dihedral extension of K_0 of degree 2^{n+1} with $n \geq 2$ if and only if $\sqrt{-1} \notin K_0$ and $n < d$. In this case, $K_0(\sqrt{-1}, \sqrt[2^n]{\pi})$ is the unique dihedral extension of degree 2^{n+1}.*

Proof. Suppose that $\sqrt{-1} \notin K_0$ and $2 \leq n < d$. Let $\pi \in K_0$ be a parameter and

$$L_n = K_0(\sqrt{-1}, \sqrt[2^n]{\pi}).$$

Let $\rho \in K_0(\sqrt{-1})$ be a primitive 2^d-th root of unity. Then $\rho' = \rho^{2^{d-n}}$ is a primitive 2^n-th root of unity and

$$N_{K_0(\sqrt{-1})/K_0}(\rho') = (-1)^{2^{d-n}} = 1$$

(cf. Corollary 1.2). Hence, by Lemma 2.4, L_n/K_0 is a dihedral extension.

Suppose, conversely, there exists a dihedral extension L/K_0 of degree 2^{n+1} with $n \geq 2$. Let K be the subfield of L fixed by the rotation subgroup of $\mathrm{Gal}(L/K_0)$. Since $n \geq 2$, by Proposition 4.2, K/K_0 is unramified. By Proposition 4.4, L/K is totally ramified. By Proposition 2.3, there exists a subfield L' of L with $[L' : K_0] = [L : K]$ and $L'K = L$. Since K/K_0 is unramified and L/K is totally ramified, L'/K_0 is totally ramified. Since the characteristic of the residue field of K_0 is not 2 and $[L' : K_0] = [L : K] = 2^n$, we have that

$$L' = K_0(\sqrt[2^n]{\pi_0})$$

for some parameter $\pi_0 \in K_0$. Hence, $L = K(\sqrt[2^n]{\pi_0})$.

Suppose that $\sqrt{-1} \in K_0$. Let $L_1 = K_0(\sqrt[4]{\pi_0}) \subset L'$. Then L_1/K_0 is cyclic of degree 4, leading to a contradiction (Lemma 2.2). Hence, $\sqrt{-1} \notin K_0$.

Since $L = K(\sqrt[2^n]{\pi_0})$ is a cyclic extension of K of degree 2^n, K contains a primitive 2^n-th root of unity ρ. Since $n \geq 2$, we have $\sqrt{-1} \in K$, and hence

$$K = K_0(\sqrt{-1}).$$

Thus, by the maximality of d, $n \leq d$. Suppose $n = d$. Since K_0 is a local field, by Corollary 1.2, we have

$$N_{K/K_0}(\rho) = -1.$$

Hence, by Lemma 2.4, L/K_0 is not dihedral, a contradiction. Therefore, $n < d$. Let R be the valuation ring of K_0. Since $\sqrt{-1} \notin K_0^*$, we have

$$R^* = R^{*2} \cup (-R^{*2}),$$

and hence $R^* = R^{*2^n} \cup (-R^{*2^n})$. Since -1 is a 2^n-th power in K, $R^* \subset K^{*2^n}$. Since $\pi = u\pi_0$ for some $u \in R^*$, we have $L \simeq L_n$, proving the uniqueness of dihedral extensions of degree 2^{n+1} over K_0. □

Theorem 4.7. *Let K_0 be a local field with characteristic of the residue field not 2 and $\pi \in K_0$ be a parameter. Let ℓ be an odd prime not equal to the characteristic of the residue field of K_0. Let ρ be a primitive ℓ-th root of unity and d be the maximum integer such that $K_0(\rho)$ contains a primitive ℓ^d-th root of unity. Then there exists a dihedral extension of K_0 of degree $2\ell^n$ with $n \geq 1$ if and only if $[K_0(\rho) : K_0] = 2$ and $1 \leq n \leq d$. In this case $K_0(\rho, \sqrt[\ell^n]{\pi})$ is the unique dihedral extension of degree $2\ell^n$.*

Proof. Suppose $[K_0(\rho) : K_0] = 2$ and $1 \le n \le d$. Let $\pi \in K_0$ be a parameter. Let $L_n = K_0(\rho, \sqrt[\ell^n]{\pi})$. Let $\rho_n \in K_0(\rho)$ be a primitive ℓ^n-th root of unity. Since $N_{K_0(\rho)/K_0}(\rho_n)$ is an ℓ^n-th root of unity in K_0 and the only ℓ^n root of unity in K_0 is 1, we have

$$N_{K_0(\rho)/K_0}(\rho_n) = 1.$$

By Lemma 2.4, L_n/K_0 is a dihedral extension.

Suppose, conversely there exists a dihedral extension L/K_0 of degree $2\ell^n$. Let K be the subfield of L fixed by the rotation subgroup of $\mathrm{Gal}(L/K_0)$. Since $[L : K] = \ell^n \ge 3$, by Proposition 4.2, K/K_0 is unramified. Then, by Proposition 4.4, L/K is totally ramified. Let L' be a subfield of L with $[L' : K_0] = [L : K]$ and $L'K = L$ (Proposition 2.3). Since K/K_0 is unramified and L/K is totally ramified, L'/K_0 is totally ramified. Since the characteristic of the residue field of K_0 is not ℓ and $[L : K_0] = [L : K] = \ell^n$, we have

$$L' = K_0(\sqrt[\ell^n]{\pi_0})$$

for some parameter $\pi_0 \in K_0$. Hence, $L = K(\sqrt[\ell^n]{\pi_0})$. Since L/K is cyclic, K contains a primitive ℓ^n-th root of unity. Thus, $n \le d$. Since $n \ge 1$, $\rho \in K^*$.

Suppose $[K_0(\rho) : K_0] \neq 2$. Since $K_0(\rho) \subseteq K$ and $[K : K_0] = 2$, we have $\rho \in K_0$. Let $L_1 = K_0(\sqrt[\ell]{\pi_0})$, then L_1/K_0 is cyclic. Since $K \not\subseteq L_1$, by Lemma 2.2, we have

$$[L_1 : K_0] = \ell \le 2,$$

leading to a contradiction. Hence,

$$[K_0(\rho) : K_0] = 2.$$

Since $\rho \in K$ and $[K : K_0] = 2$, we have $K = K_0(\rho)$. Since $\rho \notin K_0$ (as in the proof of Theorem 4.6), every unit in the valuation ring of K_0 is an ℓ^n-th power in K. In particular, $L \simeq L_n$, proving the uniqueness of dihedral extensions of degree $2\ell^n$ over K_0. □

Corollary 4.8. *Let K_0 be a local field with the residue field κ_0 and $m \ge 3$ with $2m$ coprime to $\mathrm{char}(\kappa_0)$. Let L/K_0 be an extension of degree $2m$ and $\pi \in K_0$ be a parameter. Then L/K_0 is a dihedral extension if and only if there exists a primitive m-th root of unity $\rho \in L$ with $[K_0(\rho) : K_0] = 2$, $N_{K_0(\rho)/K_0}(\rho) = 1$, and $L = K_0(\rho, \sqrt[m]{\pi})$.*

Proof. Suppose L/K_0 is a dihedral extension of degree $2m$. Suppose $m = 2^n$. Let d be the maximum integer such that $K_0(\sqrt{-1})$ contains a primitive 2^d-th root of unity. Then, by Theorem 4.6, we have $\sqrt{-1} \notin K_0$, $n < d$, and $L = K_0(\sqrt{-1}, \sqrt[2^n]{\pi})$. Let $\rho \in K_0(\sqrt{-1})$ be a primitive 2^n-th root of unity. Since $K_0(\rho)/K_0$ is unramified and $K_0(\sqrt{-1})$ is the maximal unramified extension of L/K_0, we have

$$K_0(\sqrt{-1}) = K_0(\rho).$$

In particular,

$$[K_0(\rho) : K_0] = 2 \quad \text{and} \quad L = K_0(\rho, \sqrt[2^n]{\pi}).$$

Since L/K_0 is dihedral, by Lemma 2.4, we have

$$N_{K_0(\sqrt{-1})/K_0}(\rho) = 1.$$

Assume that there is an odd prime dividing m. Let $m = \ell_0^{n_0}\ell_1^{n_1}\cdots\ell_r^{n_r}$ with $\ell_0 = 2$, for $i \geq 1$, ℓ_i are distinct odd primes, $n_0 \geq 0$ and $n_i \geq 1$ for all $i \geq 1$. Let σ be a generator of the rotation subgroup of $\mathrm{Gal}(L/K_0)$. For $0 \leq i \leq r$, let

$$M_i = L^{\sigma^{\ell_i^{n_i}}},$$

then $[M_i : K_0] = 2\ell_i^{n_i}$.

Let $1 \leq i \leq r$. Then M_i/K_0 is a dihedral extension of degree $2\ell_i^{n_i}$ with ℓ_i odd. By Theorem 4.7, there exists a primitive $\ell_i^{n_i}$-th root of unity $\rho_i \in M_i$, $[K_0(\rho_i) : K_0] = 2$ and

$$M_i = K_0(\rho_i, \sqrt[\ell_i^{n_i}]{\pi}).$$

Let $m_0 = \ell_1^{n_1}\cdots\ell_r^{n_r}$. Since ℓ_i are distinct primes and $M_i \subseteq L$, $\sqrt[m_0]{\pi} \in L$ and $\rho' = \rho_1\cdots\rho_r \in L$ is a primitive m_0-th root of unity. If $n_0 = 0$, then $m_0 = m$. Since $\rho_i \notin K_0$ for all $i \geq 1$ and ℓ_i's are distinct primes, it follows that $\rho' \notin K_0$. Since $K_0(\rho')/K_0$ is an unramified extension and $K_0(\sqrt[m]{\pi})/K_0$ is a totally ramified extension of degree m, it follows that

$$[K_0(\rho') : K_0] = 2 \quad \text{and} \quad L = K_0(\rho', \sqrt[m]{\pi}).$$

By Lemma 2.4, $N_{K_0(\rho')/K_0}(\rho') = 1$.

Suppose $n_0 = 1$. Then M_0/K_0 is the unique bi-quadratic extension, and hence

$$M_0 = K_0(\sqrt{u}, \sqrt{\pi})$$

(cf. Remark 4.5). Suppose $n_0 \geq 2$. Then, as in the first case, M_0 contains a primitive 2^{n_0}-th root of unity ρ_0,

$$[K(\rho_0) : K_0] = 2 \quad \text{and} \quad M_0 = K_0(\rho_0, \sqrt[2^{n_0}]{\pi}).$$

Hence, in either case, the maximal unramified extension of M_0/K_0 is of degree 2 over K_0.

Since $M_0 \subseteq L$, we have $\sqrt[2^{n_0}]{\pi} \in L$. Since $m = 2^{n_0}m_0$, we have $\sqrt[m]{\pi} \in L$. Since $K_0(\sqrt[m]{\pi})/K_0$ is a totally ramified extension of degree m, the degree of the maximal unramified extension of L/K_0 is 2. Since L contains a primitive 2^{n_0}-th root of unity and m_0-th root of unit, L contains a primitive m-th root of unity ρ. Since $m \geq 3$, either $n_0 \geq 2$ or $m_0 \geq 2$. Hence,

$$[K_0(\rho) : K_0] = 2 \quad \text{and} \quad L = K_0(\rho, \sqrt[m]{\pi}).$$

By Lemma 2.4, $N_{K_0(\rho)/K_0}(\rho) = 1$.

Conversely, suppose there exists a primitive m-th root of unity $\rho \in L$, with

$$[K_0(\rho) : K_0] = 2, \quad N_{K/K_0}(\rho) = 1, \quad \text{and} \quad L = K_0(\rho, \sqrt[m]{\pi}).$$

Then, by Lemma 2.4, L/K_0 is a dihedral extension. □

We conclude this section with the following result on norms from dihedral extensions over local fields.

Proposition 4.9. *Let K_0 be a local field and $m \geq 2$ with $2m$ coprime to the characteristic the residue field of K_0. Let L/K_0 be a dihedral extension of degree $2m$. Let K be the subfield of L fixed by the rotation subgroup of* $\mathrm{Gal}(L/K_0)$. *Let $L_0, \dots, L_{m-1}$ be the subfields of L with $L_iK = L$ and $[L : L_i] = 2$ (see Proposition 2.3). Let $\theta_0 \in K_0^*$. Then for every $0 \leq i \leq m-1$, there exists $\mu_i \in L_i$, such that*

$$\prod_{i=0}^{m-1} N_{L_i/K_0}(\mu_i) = \theta_0.$$

Proof. Suppose $m = 2$. Then L/K_0 is a biquadratic extension, L_0 and L_1 are non isomorphic quadratic extensions of K_0. Then, by local class field theory (cf. [8, Proposition 3, p. 142]), $N_{L_0/K_0}(L_0^*)$ and $N_{L_1/K_0}(L_1^*)$ are two distinct subgroups of K_0^* of index 2. Let $b \in N_{L_0/K_0}(L_0^*)$, which is not in $N_{L_1/K_0}(L_1^*)$. Let $a \in K_0^*$. Suppose $a \notin N_{L_1/K_0}(L_1^*)$, then $a \in bN_{L_1/K_0}(L_1^*)$. Hence,

$$a = bc$$

for some $c \in N_{L_1/K_0}(L_1^*)$. In particular, $a \in N_{L_0/K_0}(L_0^*)N_{L_1/K_0}(L_1^*)$.

Suppose $m \geq 3$. Let ρ be a primitive m-th root of unity. Then, for any parameter $\pi \in K_0$, by Corollary 4.8,

$$L = K_0(\rho, \sqrt[m]{\pi}).$$

Let $\pi \in K_0$ be a parameter. Since $L = K_0(\rho, \sqrt[m]{\pi})[L : K_0(\sqrt[m]{\pi})] = 2$ and $K = K_0(\rho)$, we have

$$K_0(\sqrt[m]{\pi}) = L_r$$

for some r. In particular, $(-1)^{m-1}\pi$ is a norm from the extension L_r/K_0. Let $u \in K_0$ be a unit. Since $u\pi$ is a parameter in K_0, we have

$$\sqrt[m]{u\pi} \in L \quad \text{and} \quad K_0(\sqrt[m]{u\pi}) = L_s$$

for some s. Hence, $(-1)^{m-1}u\pi$ is a norm from the extension L_s/K_0. In particular, u is a product of norms from the extensions L_r/K_0 and L_s/K_0. Since every element in K_0 is $u\pi^r$ for some $u \in K_0$ a unit, it follows that every element in K_0 is a product of norms from the extensions L_i/K_0. □

5. Approximation of norms from dihedral extensions over global fields

Proposition 5.1. *Let K_0 be a global field and $n \geq 2$ an integer with $2n$ coprime to* $\operatorname{char}(K_0)$. *Let E/K_0 be a dihedral extension of degree $2n$ and σ and τ generators of* $\operatorname{Gal}(E/K_0)$ *with σ a rotation and τ a reflection. Let $K = E^{\sigma}$ and $E_i = E^{\sigma^i \tau}$ for $1 \leq i \leq n$. Let ν be a place of K_0 and $\lambda_\nu \in K_{0\nu}$. Suppose that the characteristic of the residue field at ν is coprime to $2n$. If λ_ν is a norm from the extension*

$$E \otimes_{K_0} K_{0\nu}/K \otimes_{K_0} K_{0\nu},$$

then λ_ν is a product of norms from the extensions

$$E_i \otimes K_{0\nu}/K_{0\nu}.$$

Proof. Suppose λ_ν is a norm from the extension $E \otimes_{K_0} K_{0\nu}/K \otimes_{K_0} K_{0\nu}$.

Suppose that $K \otimes_{K_0} K_{0\nu}$ is not a field. Then

$$K \otimes_{K_0} K_{0\nu} \simeq K_{0\nu} \times K_{0\nu}.$$

Since $KE_i = E$, we have

$$E \otimes_{K_0} K_{0\nu} \simeq E_i \otimes_{K_0} K \otimes_{K_0} K_{0\nu} \simeq E_i \otimes_{K_0} K_{0\nu} \times E_i \otimes_{K_0} K_{0\nu}.$$

Since λ_ν is a norm from the extension $E \otimes_{K_0} K_{0\nu}/K \otimes_{K_0} K_{0\nu}$, it follows that λ_ν is a norm from $E_i \otimes_{K_0} K_{0\nu}/K_{0\nu}$.

Suppose $K \otimes_{K_0} K_{0\nu}$ is a field. Suppose $E \otimes_{K_0} K_{0\nu} \simeq \prod_1^n K \otimes_{K_0} K_{0\nu}$. Then, by Lemma 2.5, there exists an i such that

$$E_i \otimes_{K_0} K_{0\nu} \simeq K_{0\nu} \times E'_{i\nu}$$

for some $K_{0\nu}$-algebra $E'_{i\nu}$. In particular, λ_ν is norm from $E_i \otimes_{K_0} K_{0\nu}/K_{0\nu}$.

Suppose $E \otimes_{K_0} K_{0\nu}$ is not isomorphic to $\prod_1^n K \otimes_{K_0} K_{0\nu}$. Since E/K_0 is Galois, we have

$$E \otimes_{K_0} K_{0\nu} \simeq \prod E_\nu$$

for some field extension $E_\nu/K_{0\nu}$ and $K \otimes_{K_0} K_{0\nu}$ is a proper subfield of E_ν. Hence, $E_\nu/K_{0\nu}$ is a dihedral extension. Since the characteristic of the residue field at ν is coprime to $2n$, by Proposition 4.9, λ_ν is a product of norms from $E_i \otimes_{K_0} K_{0\nu}/K_{0\nu}$. □

Corollary 5.2. *Let K_0, E and K be as in Proposition 5.1. Let S be a finite set of places of K_0 with $2n$ coprime to the characteristic of the residue field at places in S. For $\nu \in S$, let $\lambda_\nu \in K_{0\nu}$ be a norm from the extension*

$$E \otimes_{K_0} K_{0\nu}/K \otimes_{K_0} K_{0\nu}.$$

Then there exists $\lambda \in K_0$ such that λ is a norm from the extension E/K and

$$\lambda\lambda_\nu^{-1} \in (K \otimes_{K_0} K_{0\nu})^{*n}$$

for all $\nu \in S$.

Proof. Let $\sigma, \tau \in \mathrm{Gal}(E/K_0)$ be as in Proposition 5.1. Let $E_i = E^{\sigma^i \tau}$ for $1 \leq i \leq n$. Let $\nu \in S$. Then, by Proposition 5.1, for $1 \leq i \leq n$, there exists $z_{i\nu} \in E_i \otimes_{K_0} K_{0\nu}$ such that

$$\lambda_\nu = \prod N_{E_i \otimes_{K_0} K_{0\nu}/K_{0\nu}}(z_{i\nu}).$$

For $1 \leq i \leq n$, let $z_i \in E_i$ be close to $z_{i\nu}$ for all $\nu \in S$. Let $\lambda = \prod N_{E_i/K_0}(z_i)$. Since z_i is close to $z_{i\nu}$ for all $\nu \in S$, λ is close to λ_ν for all $\nu \in S$. In particular,

$$\lambda\lambda_\nu^{-1} \in (K \otimes_{K_0} K_{0\nu})^{*n}.$$

Since $KE_i = E$, λ is a norm from the extension E/K. □

6. Central simple algebras with involutions of second kind over 2-local fields

In this section we give a description of central simples algebras having involutions of second kind over complete discretely valued fields with residue fields local fields (such fields are called 2-local fields).

We begin with the following lemma.

Lemma 6.1. *Let F be a complete discretely valued field and $\pi \in F$ a parameter. Let E/F be a cyclic unramified extension and σ a generator of* $\mathrm{Gal}(E/F)$. *Then the cyclic algebra (E, σ, π) is unramified if and only if $E = F$.*

Proof. Let $m = [E : F]$. Since E/F is unramified, the order of the class of π in $F^*/N_{E/F}(E^*)$ is m, and hence $D = (E, \sigma, \pi)$ is a division algebra of degree m (see [3, Theorem 6, p. 95]). Let ν be the discrete valuation on F and $\widetilde{\nu}$ be the extension of ν to D (see [31, Theorem 12.10, p. 138]). Let e be the ramification index of D. Since there exists $y \in D$ with $y^m = \pi$, we have $\widetilde{\nu}(\pi) \geq m$, and hence $e = m$ (see [31, Theorem 13.7, p. 142]). Suppose D is unramified. Then $e = 1$, and hence $m = 1$. In particular, $E = F$. □

Lemma 6.2. *Let F_0 be a complete discrete valued field with residue field of characteristic not 2. Let $\pi \in F_0$ be a parameter and $F = F_0(\sqrt{\pi})$. Let E/F be an unramified cyclic extension and σ a generator of* $\mathrm{Gal}(E/F)$. *If* $\mathrm{cores}_{F/F_0}(E, \sigma, \sqrt{\pi})$ *is unramified, then $(E, \sigma, \sqrt{\pi})$ is zero.*

Proof. Let E_0 be the maximal unramified extension of E/F_0. Since E/F is unramified and F/F_0 is ramified extension of degree 2, E/E_0 is of degree 2 and $E = E_0F$.

Since F/F_0 is ramified, E_0/F_0 is unramified. Since E/F is cyclic, E_0/F_0 is cyclic (cf. proof of Lemma 4.1). Let σ_0 be the restriction of σ to E_0. Then

$$(E_0, \sigma_0) \otimes F = (E, \sigma).$$

Hence,

$$\operatorname{cores}_{F/F_0}(E, \sigma, \sqrt{\pi}) = (E_0, \sigma_0, N_{F/F_0}(\sqrt{\pi})) = (E_0, \sigma_0, -\pi)$$

(see [26, Proposition 1.5.3]).

Suppose that $\operatorname{cores}_{F/F_0}(E, \sigma, \sqrt{\pi}) = (E_0, \sigma_0, -\pi)$ is unramified. Since π is a parameter in F_0 and E_0/F_0 is unramified, by Lemma 6.1, $E_0 = F_0$. In particular, $E = F$ and $(E, \sigma, \sqrt{\pi})$ is zero. □

Lemma 6.3. *Let F_0 be a complete discrete valued field with residue field K_0 and F/F_0 a ramified quadratic field extension. Let $m \geq 1$ with $2m$ coprime to* $\operatorname{char}(K_0)$ *and $\alpha \in H^2(F, \mu_m)$. If* $\operatorname{cores}_{F/F_0}(\alpha)$ *is zero, then $\alpha = \alpha_0 \otimes F$ for some $\alpha_0 \in H^2_{nr}(F_0, \mu_m)$. In particular,* $\operatorname{per}(\alpha) \leq 2$.

Proof. Since F/F_0 is a ramified quadratic extension and $\operatorname{char}(K_0) \neq 2$, $F = F_0(\sqrt{\pi})$ for some $\pi \in F_0$ a parameter. Since m is coprime to $\operatorname{char}(K_0)$, we have

$$\alpha = \alpha' + (E, \sigma, \sqrt{\pi})$$

for some $\alpha' \in H^2_{nr}(F, \mu_m)$ and E/F an unramified cyclic field extension of F (cf. [27, Lemma 4.1]). Since $\operatorname{cores}_{F/F_0}(\alpha) = 0$, we have

$$\operatorname{cores}_{F/F_0}(-\alpha') = \operatorname{cores}_{F/F_0}(E, \sigma, \sqrt{\pi}).$$

Since α' is unramified, $\operatorname{cores}_{F/F_0}(-\alpha')$ is also unramified (cf. [9, p. 48]), and hence $\operatorname{cores}_{F/F_0}(E, \sigma, \sqrt{\pi})$ is unramified. Thus, by Lemma 6.2, $(E, \sigma, \sqrt{\pi})$ is zero, and hence $\alpha = \alpha'$. Since the residue field of F and F_0 are equal and both F and F_0 are complete, it follows that $\alpha = \alpha' = \alpha_0 \otimes F$ for some $\alpha_0 \in H^2_{nr}(F_0, \mu_m)$. □

Lemma 6.4. *Let F_0 be a complete discrete valued field and F/F_0 an unramified quadratic extension. Let $\pi \in F_0$ be a parameter and $m \geq 1$. Suppose $2m$ is coprime to the characteristic of the residue field of F_0. Let*

$$\alpha = \alpha' + (E, \sigma, \pi) \in H^2(F, \mu_m)$$

for some $\alpha' \in H^2_{nr}(F, \mu_m)$ and let E/F be an unramified cyclic field extension. If $\operatorname{cores}_{F/F_0}(\alpha)$ *is zero, then* $\operatorname{cores}_{F/F_0}(\alpha')$ *and* $\operatorname{cores}_{F/F_0}(E, \sigma, \pi)$ *are zero.*

Proof. Since F/F_0 is unramified extension, π is a parameter in F. Since $\pi \in F_0$, we have

$$\operatorname{cores}_{F/F_0}(E, \sigma, \pi) = \operatorname{cores}_{F/F_0}(E, \sigma) \cdot (\pi).$$

Since $\mathrm{cores}_{F/F_0}(\alpha) = \mathrm{cores}_{F/F_0}(\alpha') + \mathrm{cores}_{F/F_0}(E,\sigma,\pi)$ is zero, we have

$$\mathrm{cores}_{F/F_0}(E,\sigma)\cdot(\pi) = -\mathrm{cores}_{F/F_0}(\alpha').$$

Since α' is unramified, then $\mathrm{cores}_{F/F_0}(\alpha')$ is unramified. Since E/F is unramified, then $\mathrm{cores}_{F/F_0}(E,\sigma)$ is unramified, and hence

$$\partial(E,\sigma)\cdot(\pi) = (\overline{E},\overline{\sigma}),$$

where $\overline{E}$ is the residue homomorphism and $\overline{\sigma}$ is the induced automorphism. Since $\mathrm{cores}_{F/F_0}(E,\sigma)\cdot(\pi)$ is unramified and F_0 is complete, $\mathrm{cores}_{F/F_0}(E,\sigma)$ is zero (Lemma 6.1). Hence,

$$\mathrm{cores}_{F/F_0}(E,\sigma,\pi) = \mathrm{cores}_{F/F_0}(E,\sigma)\cdot(\pi)$$

is zero and, in particular, $\mathrm{cores}_{F/F_0}(\alpha')$ is zero. □

Lemma 6.5. *Let F_0 be a complete discrete valued field with residue field K_0 a local field, F/F_0 a quadratic field extension and $\pi \in F_0$ a parameter. Let $m \geq 1$ with $2m$ coprime to* $\mathrm{char}(K_0)$. *Let $\alpha \in H^2(F,\mu_m)$ with* $\mathrm{cores}_{F/F_0}(\alpha) = 0$. *If* $\mathrm{ind}(\alpha) \geq 3$, *then F/F_0 is unramified and $\alpha = (E,\sigma,\pi)$ for some unramified cyclic extension E/F.*

Proof. Suppose $\mathrm{cores}_{F/F_0}(\alpha) = 0$ and $\mathrm{ind}(\alpha) \geq 3$. Suppose also that F/F_0 is ramified. Then, by Lemma 6.3, α is unramified and $\mathrm{per}(\alpha) \leq 2$. Let K be the residue field of F and $\beta \in H^2(K,\mu_m)$ be the image of α. Since $\mathrm{per}(\alpha) \leq 2$, we have that $\mathrm{per}(\beta) \leq 2$. Since K is a local field, we have

$$\mathrm{ind}(\beta) = \mathrm{per}(\beta).$$

Since F is complete, we have

$$\mathrm{ind}(\alpha) = \mathrm{ind}(\beta)$$

(cf. [6, Proof of Corollary 6.2]). Hence,

$$\mathrm{ind}(\alpha) \leq 2,$$

leading to a contradiction. Hence, F/F_0 is unramified and π is a parameter in F. Since m is coprime to $\mathrm{char}(K_0)$, we have

$$\alpha = \alpha' + (E,\sigma,\pi)$$

for some $\alpha' \in H^2_{nr}(F,\mu_m)$ and E/F is an unramified cyclic extension (cf. [27, Lemma 4.1]). Then, by Lemma 6.4, $\mathrm{cores}_{F/F_0}(\alpha')$ and $\mathrm{cores}_{F/F_0}(E,\sigma,\pi)$ are zero. Let $\beta' \in H^2(K,\mu_m)$ be the image of α'. Since $\mathrm{cores}_{F/F_0}(\alpha') = 0$, we have

$$\mathrm{cores}_{K/K_0}(\beta') = 0.$$

Since K/K_0 is a quadratic field extension of local fields, $\beta' = 0$ (cf. [21, Theorem 10, p. 237]). Since F is complete, $\alpha' = 0$, and hence $\alpha = (E,\sigma,\pi)$. □

Let F be a field and $m \geq 1$ coprime to $\mathrm{char}(F)$. Suppose F contains a primitive m-th root of unity ρ. For $a, b \in F^*$, let $(a,b)_m$ be the cyclic algebra generated by x and y with relations $x^m = a$, $y^m = b$, and $yx = \rho xy$.

Proposition 6.6. *Let F_0 be a complete discrete valued field with residue field K_0 a local field. Let $m \geq 3$ with $2m$ coprime to the characteristic of the residue field of K_0. Let $\pi \in F_0$ be a parameter and $\delta \in F_0$ a unit such that the image of δ in K_0 is a parameter. Let F/F_0 be a quadratic field extension and $\alpha \in H^2(F, \mu_m)$. If*

$$\mathrm{cores}_{F/F_0}(\alpha) = 0 \quad \textit{and} \quad \mathrm{ind}(\alpha) = m,$$

then F/F_0 is unramified, F contains a primitive m-th root of unity ρ, $N_{F/F_0}(\rho) = 1$ and $\alpha = (\delta, \pi)_m$.

Proof. Suppose $\mathrm{cores}_{F/F_0}(\alpha) = 0$ and $\mathrm{ind}(\alpha) = m$. Since $m \geq 3$, by Lemma 6.5, F/F_0 is unramified and $\alpha = (E, \sigma, \pi)$ for some E/F an unramified cyclic extension.

Let K be the residue field of F and L the residue field of E. Since F/F_0 and E/F are unramified, K/K_0 is an extension of degree 2 and L/K is a cyclic extension of degree $[E : F]$. We denote the image of ρ in K by ρ again. Let σ_0 denote the automorphism of L/K induced by σ. Since $\mathrm{cores}_{F/F_0}(E, \sigma) = 0$, we have

$$\mathrm{cores}_{K/K_0}(L, \sigma_0) = 0.$$

Hence, by Proposition 3.2, L/K_0 is a dihedral extension. Let $\overline{\delta} \in K_0$ be the image of δ. Then, by the assumption, $\overline{\delta}$ is a parameter in K_0. Since K_0 is a local field and $[L : K] = m \geq 3$, by Corollary 4.8, then $K = K_0(\rho)$ for a primitive m-th root of unity,

$$N_{K/K_0}(\rho) = 1 \quad \text{and} \quad L = K_0(\rho, \sqrt[m]{\overline{\delta}}).$$

Since F_0 is complete, $F = F_0(\rho)$ and $E = F(\sqrt[m]{\delta})$. Since $N_{K/K_0}(\rho) = 1$, we have $N_{F/F_0}(\rho) = 1$. Since F contains a primitive m-th root of unity, we have

$$\alpha = (E, \sigma, \pi) = (\delta, \pi)_m. \qquad \square$$

Proposition 6.7. *Let F be a complete discrete valued field with residue field K, valuation ring R, $\pi \in R$ a parameter and $u \in R$ a unit. Let $n \geq 2$ which is coprime to* $\mathrm{char}(K)$. *Suppose that F contains a primitive n-th root of unity and the cyclic algebra $D = (\pi, u)_n$ is a division algebra. Let $x, y \in D$ be with $x^n = \pi$, $y^n = u$, and $xy = \rho yx$. Then*

$$R[x] + R[x]y + \cdots + R[x]y^{n-1} = R[y] + R[y]x + \cdots + R[y]x^{n-1} \subset D$$

is the maximal order of D.

Proof. Since D is a division algebra and F is complete, $\widetilde{\nu}\colon D^* \to \mathbb{Z}$ given by $\widetilde{\nu}(z) = \nu(\mathrm{Nrd}(z))$ is a discrete valuation on D and $\Lambda = \{z \in D^* \mid \widetilde{\nu}(z) \geq 0\} \cup \{0\}$

is the unique maximal order of D (see [31, Theorem 12.8]). Since every element in $R[y]x^i$ has reduced norm in R, we have

$$R[y] + R[y]x + \cdots + R[y]x^{n-1} \subseteq \Lambda.$$

Since $\mathrm{Nrd}(x) = (-1)^{n-1}\pi$, we have $\widetilde{\nu}(x) = 1$. Since $y^n = u$ is a unit in R and n is coprime to the characteristic of K, the extension $F(y)/F$ is unramified and $R[y]$ is the integral closure of R in $F(y)$. Since $\deg(D) = n$, we have $[F(y) : F] = n$. Hence, for any $a \in R[y]$, we have

$$\mathrm{Nrd}(a) = N_{F(y)/F}(a).$$

Since $F(y)/F$ is unramified, $\widetilde{\nu}(a)$ is divisible by n for all $a \in R[y]$. Let $z \in \Lambda$. Then

$$z = \frac{1}{b}(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})$$

for some $b \in R$, $b \neq 0$ and $a_i \in R[y]$. Since $\widetilde{\nu}a_i$ is divisible by n and $\widetilde{\nu}(x) = 1$, we have that

$$\widetilde{\nu}(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})$$

is equal to the minimum of $\widetilde{\nu}(a_i x^i)$ for $0 \leq i \leq n-1$. Since $\widetilde{\nu}(z) \geq 0$, we have

$$\widetilde{\nu}(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) \geq \widetilde{\nu}(b).$$

In particular, $\widetilde{\nu}(a_i x^i) \geq \widetilde{\nu}(b)$. Since $\widetilde{\nu}(a_i x^i) = \widetilde{\nu}(a_i) + i$ and $\widetilde{\nu}(a_i)$, we have that $\widetilde{\nu}(b)$ is divisible by n, and it follows that $\widetilde{\nu}(a_i) \geq \widetilde{\nu}(b)$ for all $0 \leq i \leq n-1$. Hence,

$$\frac{a_i}{b} \in R[y] \quad \text{and} \quad z \in R[y] + R[y]x + \cdots + R[y]x^{n-1}.$$

Hence, $\Lambda = R[y] + R[y]x + \cdots + R[y]x^{n-1}$ is a maximal R-order of D. □

We end this section with the following proposition.

Proposition 6.8. *Let F_0 be a field and F/F_0 a quadratic extension. Let $m \geq 2$ with $2m$ coprime to* $\mathrm{char}(F_0)$. *Suppose that F contains a primitive m-th root of unity ρ. Let $a, b \in F_0^*$. Suppose that $[F(\sqrt[m]{a}) : F] = m$. Let $A = (a, b)_m$ be the cyclic algebra generated by x and y with relations $x^m = a$, $y^m = b$ and $yx = \rho xy$. Then there exists an F/F_0-involution σ on A with $\sigma(x) = x$ and $\sigma(y) = y$ if and only if $N_{F/F_0}(\rho) = 1$.*

Proof. Let τ_0 be the nontrivial automorphism of F/F_0. Then $N_{F/F_0}(\rho) = \tau_0(\rho)\rho$.

Suppose there exists an F/F_0-involution σ on A with $\sigma(x) = x$ and $\sigma(y) = y$. Since $yx = \rho xy$, we have

$$xy = \sigma(yx) = \sigma(\rho xy) = \tau_0(\rho) yx = \tau_0(\rho)\rho xy.$$

Hence, $N_{F/F_0}(\rho) = \tau_0(\rho)\rho = 1$.

Suppose $N_{F/F_0}(\rho) = 1$. Let $c \in F_0^*$ with $F = F_0(\sqrt{c})$ and $E = F(x)$. Then

$$A = E \oplus Ey \oplus \cdots \oplus Ey^{m-1}.$$

Since $\{x^i y^j, \sqrt{c}x^i y^j \mid 0 \le i,\ j \le m-1\}$ is an F_0-basis of A, we have an F_0-vector space isomorphism $\sigma\colon A \to A$ given by

$$\sigma(x^i y^j) = y^j x^i \quad \text{and} \quad \sigma(\sqrt{c}x^i y^j) = -\sqrt{c}y^j x^i.$$

Then $\sigma(z) = \tau_0(z)$ for all $z \in F$. Since $a, b \in F_0$, we have

$$\sigma(x^m) = \sigma(a) = a = x^m = \sigma(x)^m.$$

Similarly, $\sigma(y^m) = b = \sigma(y)^m$. Since $\rho = c_1 + c_2\sqrt{c}$ for some $c_i \in F_0$ and $\tau_0(\sqrt{c}) = -\sqrt{c}$, we have

$$\begin{aligned}\sigma(\rho xy) &= \sigma(c_1 xy + c_2\sqrt{c}xy) = c_1 yx - c_2\sqrt{c}yx\\ &= (c_1 - c_2\sqrt{c})yx = \tau_0(\rho)yx.\end{aligned}$$

Since $yx = \rho xy$, we have

$$\sigma(yx) = \sigma(\rho xy) = \tau_0(\rho)yx = \tau_0(\rho)\rho xy = xy = \sigma(x)\sigma(y).$$

Hence, σ is an F/F_0-involution. □

7. Reduced norms of central simple algebras over two dimensional complete fields

Let R be a complete regular local ring of dimension 2 with residue field κ and field of fractions F. For a prime $\theta \in R$, let F_θ be the completion of F at the discrete valuation given by the prime ideal (θ) of R and $\kappa(\theta)$ the residue field at θ. Let A be a central simple algebra over F of index coprime to char(κ). Let $m = (\pi, \delta)$ be the maximal ideal of R. Suppose that A is unramified on R except possibly at π and δ. Let $\lambda = v\pi^s\delta^t \in F^*$ for some unit $v \in R$ and $r, s \in \mathbb{Z}$. In this section we show that if κ is a finite field and $\lambda \in \mathrm{Nrd}(A \otimes F_\pi)$, then $\lambda \in \mathrm{Nrd}(A)$.

Remark 7.1. Let $\mu \in F_\pi^*$ and $n \ge 1$ coprime to char(κ). Then $\mu = u\pi^r$ for some $u \in F_\pi$ which is a unit at π. Let $\overline{u}$ be the image of u in $\kappa(\pi)$. Since $\kappa(\pi)$ is the field of fractions of $R/(\pi)$ and R is complete, $\kappa(\pi)$ is a complete discrete valued field with residue field κ and $\overline{\delta} \in R/(\pi)$ is a parameter. Hence, $\overline{u} = \overline{v}\delta^s$ for some $v \in R$ a unit. Then $\mu(v\pi^r\delta^s)^{-1}$ is a unit at π and maps to 1 in $\kappa(\pi)$. Since n is coprime to n, we have $\mu = v\pi^r\delta^s c^n$ for some $c \in F_\pi^*$.

We begin by extracting the following from [27].

Proposition 7.2. *Let R be a complete regular local ring of dimension 2 with residue field κ and field of fractions F. Let A be a central simple algebra over F of index n coprime to* $\mathrm{char}(\kappa)$ *and $\alpha \in H^2(F, \mu_n)$ be the class of A. Let $m = (\pi, \delta)$ be the maximal ideal of R. Suppose that κ is a finite field and A is unramified on R except possibly at π and δ. Let $\lambda = v\pi^s\delta^t \in F^*$ for some unit $v \in R$ and $s, t \in \mathbb{Z}$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then $\lambda \in \mathrm{Nrd}(A)$.*

Proof. As in [27, Theorem 4.12], we assume that $\mathrm{ind}(A) = \ell^d$ with ℓ a prime and F contains a primitive ℓ-th root of unity. Since $\mathrm{ind}(A)$ is coprime to $\mathrm{char}(\kappa)$, we have $\ell \neq \mathrm{char}(\kappa)$. We prove the result by induction on d. If $d = 0$, then A is a matrix algebra, and hence every element is a reduced norm from A. Suppose that $d \geq 1$.

Suppose $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$. Suppose s is coprime to ℓ. Then, by [27, Lemma 6.1], $A = (E, \sigma, (-1)^s\lambda)$ for some cyclic extension E/F with σ a generator of $\mathrm{Gal}(E/F)$. In particular,

$$(-1)^{\ell^d+1}(-1)^s\lambda \in \mathrm{Nrd}(A).$$

Suppose ℓ is odd, then $-1 \in \mathrm{Nrd}(A)$, and hence $\lambda \in \mathrm{Nrd}(A)$. Suppose $\ell = 2$. Since s is odd, we have

$$\lambda = (-1)^{\ell^d+1}(-1)^s\lambda \in \mathrm{Nrd}(A).$$

Similarly, if t is coprime to ℓ, then $\lambda \in \mathrm{Nrd}(A)$.

Suppose that s and t are divisible by ℓ. Then, by [27, Lemma 4.10], there exists an unramified cyclic field extension L_π/F_π of degree ℓ and $\mu_\pi \in L_\pi$ such that

$$N_{L_\pi/F_\pi}(\mu) = \lambda, \quad \mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(A \otimes F_\pi), \quad \alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_{\ell^d}^{\otimes 2}).$$

Since L_π/F_π is an unramified cyclic extension of degree ℓ and F contains a primitive ℓ-th root of unity, we have $L_\pi = F_\pi(\sqrt[\ell]{a})$ for some $a \in F_\pi$, which is a unit at π. Since $\mathrm{char}(\kappa) \neq \ell$ and the residue field $\kappa(\pi)$ of F_π is the field of fractions of $R/(\pi)$, we have

$$a = w\delta^\varepsilon \in F_\pi^*/F_\pi^{*\ell}$$

for some unit $w \in R$ and $0 \leq \varepsilon \leq \ell - 1$ (cf. Remark 7.1). Suppose $\varepsilon \geq 1$. Let $1 \leq \varepsilon' \leq \ell - 1$ with $\varepsilon\varepsilon' = 1$ modulo ℓ. Since

$$F_\pi(\sqrt[\ell]{w\delta^\varepsilon}) = F_\pi(\sqrt[\ell]{w^{\varepsilon'}\delta}),$$

replacing w by $w^{\varepsilon'}$, we assume that

$$L_\pi = F_\pi(\sqrt[\ell]{w\delta^\varepsilon})$$

with $0 \leq \varepsilon \leq 1$. Let $L = F(\sqrt[\ell]{w\delta^\varepsilon})$. Then L/F is a cyclic extension of degree ℓ and $L \otimes F_\pi \simeq L_\pi$. Let S be the integral closure of R in L. Then S is a regular local ring with maximal ideal (π, δ_1), where $\delta_1 = \delta$ or $\sqrt[\ell]{w\delta}$ depending on whether $\varepsilon = 0$ or 1 (see [28, Lemmas 3.1 and 3.2]). Since $\operatorname{ind}(\alpha \otimes L_\pi) < \operatorname{ind}(\alpha)$, by [27, Proposition 5.8], we have

$$\operatorname{ind}(\alpha \otimes L) < \operatorname{ind}(\alpha).$$

Since S is a regular local ring with maximal ideal (π, δ_1) and L the field of fractions of S, there exists $u \in S$ a unit such that

$$\mu_\pi = u\pi^i \delta_1^j \mu_1^{\ell^{d+1}}$$

for some $i, j \in \mathbb{Z}$ and $\mu_1 \in L_\pi$ (cf. Remark 7.1). Let $\mu' = u\pi^i \delta_1^j \in L$ and let $\lambda' = N_{L/F}(\mu')$. Then $\lambda' = v'\pi^{\ell i}\delta^{j'}$ for some unit $v' \in R$. Since

$$\lambda = N_{L_\pi/F_\pi}(\mu_\pi) = N_{L_\pi/F_\pi}(\mu' \mu_1^{\ell^{d+1}}) = \lambda' N_{L_\pi/F_\pi}(\mu_1)^{\ell^{d+1}},$$

we have $\lambda'\lambda^{-1} \in F_\pi^{\ell^{d+1}}$. Hence, by [27, Corollary 5.5], $\lambda = \lambda'\theta^{\ell^{d+1}}$ for some $\theta \in F$. Let $\mu = \mu'\theta^{\ell^d} \in L$. Then $N_{L/F}(\mu) = \lambda$. Since

$$\mu = \mu'\theta^{\ell^d} = \mu_\pi \mu_1^{-\ell^{d+1}} \theta^{\ell^d} \quad \text{and} \quad \alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_{\ell^d}^{\otimes 2}),$$

we have $\alpha \cdot (\mu) = 0 \in H^3(L_\pi, \mu_{\ell^d}^{\otimes 2})$. Hence, by [27, Corollary 5.5], we have

$$\alpha \cdot (\mu) = 0 \in H^3(L, \mu_{\ell^d}^{\otimes 2}).$$

Since $\operatorname{ind}(\alpha \otimes L) < \operatorname{ind}(\alpha)$, we have $\mu \in \operatorname{Nrd}(A \otimes L)$ by induction. Finally. since $N_{L/F}(\mu) = \lambda$, we have $\lambda \in \operatorname{Nrd}(A)$. $\square$

Corollary 7.3. *Assuming the notation and hypothesis of Proposition 7.2, if we have $\lambda \in \operatorname{Nrd}(A \otimes F_\pi)$, then $\lambda \in \operatorname{Nrd}(A)$.*

Proof. Let $\alpha \in H^2(F, \mu_n)$ be the class of A. Since $\lambda \in \operatorname{Nrd}(A \otimes F_\pi)$, we have

$$\alpha \cdot (\lambda) = 0 \in H^3(F_\pi, \mu_n^{\otimes 2}).$$

Since α is unramified on R except possibly at π, δ, and $\lambda = c\pi^s\delta^t$, we have that $\alpha \cdot (\lambda)$ is unramified on R except possibly at π and δ. Since $\alpha \cdot (\lambda) = 0 \in H^3(F_\pi, \mu_n^{\otimes 2})$, by [27, Corollary 5.5], we have

$$\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2}).$$

Hence, by Proposition 7.2, $\lambda \in \operatorname{Nrd}(A)$. $\square$

8. Central simple algebras with involutions of second kind over two dimensional complete fields

Let R_0 be a complete regular local ring of dimension two with residue field κ_0 a finite field of characteristic not 2 and F_0 the field of fractions of R_0. Let $m = (\pi, \delta)$ be the maximal ideal of R_0. Let F/F_0 be a quadratic field extension with $F = F_0(\sqrt{u\pi^\varepsilon})$ for some $u \in R_0$ a unit and $\varepsilon = \{0, 1\}$. Let R be the integral closure of R_0 in F. Then R is a regular local ring with maximal ideal (π_1, δ), where $\pi_1 = \pi$ if $\varepsilon = 0$ and $\pi_1 = \sqrt{u\pi}$ if $\varepsilon = 1$ (see [28, Lemmas 3.1 and 3.2]). Let κ be the residue field of R. Then $[\kappa : \kappa_0] \leq 2$.

Let A be a central division algebra over F which is unramified on R except possibly at π_1 and δ. Suppose that $n = \mathrm{ind}(A)$ is coprime to $\mathrm{char}(\kappa_0)$. In this section we show that if there is an involution τ on A of second kind and A is division, then there exists a maximal R-order in A invariant under τ with some additional structure. We then prove a local-global principle for certain classes of hermitian forms over (A, τ).

We begin with the following lemma.

Lemma 8.1. *Let $v \in R$ be a unit and $\mu = v\pi_1^r\delta^s \in F^*$ for some $r, s \in \mathbb{Z}$. Suppose $\theta_\pi \in F_{0\pi}^*$ is such that $\mu\theta_\pi \in \mathrm{Nrd}(A \otimes_F F_{\pi_1})$. Then there exists $\theta \in F_0$ such that $\mu\theta \in \mathrm{Nrd}(A)$ and $\theta_\pi\theta^{-1} \in F_{0\pi}^n$.*

Proof. Since F_0 is the field of fractions of R_0 and (π, δ) is the maximal ideal of R_0, we have

$$\theta_\pi = w\pi^{r_1}\delta^{s_1}c^n$$

for some $w \in R_0$ a unit, $c \in F_{0\pi}$ (cf. Remark 7.1). Let

$$\theta = w\pi^{r_1}\delta^{s_1} \in F_0^*.$$

Since $\mathrm{ind}(A) = n$, we have $c^n \in \mathrm{Nrd}(A \otimes_F F_{\pi_1})$, and hence $\mu\theta \in \mathrm{Nrd}(A \otimes_F F_{\pi_1})$. Since A is unramified on R except possibly at π_1 and δ and the support of $\mu\theta$ is at most π_1 and δ, by Corollary 7.3, $\mu\theta \in \mathrm{Nrd}(A)$. □

Proposition 8.2. *Let $\alpha \in H^2(F, \mu_n)$ be the class of A. Suppose $\mathrm{ind}(\alpha) = n \geq 3$. If $\mathrm{cores}_{F/F_0}(\alpha) = 0$, then F/F_0 is unramified on R_0, F contains a primitive n-th root of unity ρ, $N_{F/F_0}(\rho) = 1$ and $\alpha = (\delta, \pi)_n$.*

Proof. Since $F = F_0(\sqrt{u\pi^\varepsilon})$, we have that R_0 is complete and $\mathrm{char}(\kappa) \neq 2$, and it follows that $F \otimes F_{0\pi} = F_{\pi_1}$ is a field. Since α is unramified on R except possibly at π_1 and δ, we have

$$\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha \otimes F_{\pi_1})$$

(see [27, Proposition 5.8]).

The residue field of $F_{0\pi}$ is a local field with residue field κ_0. Suppose that $\mathrm{cores}_{F/F_0}(\alpha) = 0$. We then have, $\mathrm{cores}_{F_{\pi_1}/F_{0\pi}}(\alpha) = 0$. Since $\mathrm{ind}(\alpha \otimes F_{\pi_1}) = n \geq 3$, by Proposition 6.6, $F_{\pi_1}/F_{0\pi}$ is unramified, F_{π_1} contains a primitive n-th root of unity ρ, $N_{F_{\pi_1}/F_{0\pi}}(\rho) = 1$ and $\alpha \otimes F_{\pi_1} = (\delta, \pi)_n$. Since the residue field $\kappa(\pi_1)$ of F_{π_1} is a complete discretely valued field with residue field κ, κ contains a primitive n-th root of unity. Hence, F contains a primitive n-th root of unity. By [27, Corollary 5.5], we have $\alpha = (\delta, \pi)_n$. Since F/F_0 is unramified except possibly at π and $F_{\pi_1}/F_{0\pi}$ is unramified, we then have that F/F_0 is unramified on R_0. Since

$$N_{F_{\pi_1}/F_{0\pi}}(\rho) = 1,$$

we have $N_{F/F_0}(\rho) = 1$. □

Let $\alpha \in H^2(F, \mu_n)$ be the class of A. We suppose that $\mathrm{ind}(\alpha) = n \geq 3$ and $\mathrm{cores}_{F/F_0}(\alpha) = 0$. Since (π, δ) is a maximal ideal of R, $(\delta, \pi)_n$ is a division algebra. Let $D = (\pi, \delta)_n$. Then, by Proposition 8.2, α is the class of D. Thus, there exist $x, y \in D$ such that

$$x^n = \delta, \quad y^n = \pi \quad \text{and} \quad yx = \rho xy.$$

Since $D \otimes F_\pi$ and $D \otimes F_\delta$ are division algebras (see [27, Proposition 5.8]), the valuation ν_π and ν_δ given by π and δ on F extend to valuations w_π and w_δ on $D \otimes F_\pi$ and $D \otimes F_\delta$, respectively (see [31, Theorem 12.6]). We have

$$e_\pi := [w_\pi(D^*) : \nu_\pi(F^*)] = n \quad \text{and} \quad e_\delta := [w_\delta(D^*) : \nu_\delta(F^*)] = n.$$

Let $S = R[\sqrt[n]{\delta}] = R[x]$. Then S is the integral closure of R in $F(\sqrt[n]{\delta})$ and S is a regular local ring of dimension 2 with maximal ideal $(\sqrt[n]{\delta}, \pi)$ (see [28, Lemma 3.2]). Since $D \simeq (\delta, \pi)_n$ and $N_{F/F_0}(\rho) = 1$, by Proposition 6.8, there exists an F/F_0-involution σ on D with $\sigma(x) = x$ and $\sigma(y) = y$.

Lemma 8.3 (cf. [33, Lemma 3.7]). *Let $\Lambda = S + Sy + \cdots + Sy^{n-1} \subset D$. Then Λ is a maximal R-order in D.*

Proof. Since S is a free R-module, Λ is a free R-module. Let $P \subset R$ be a height one prime ideal. Suppose $P \neq (\pi)$ and $P \neq (\delta)$. Since π and δ are units at P, $\Lambda_P = \Lambda \otimes R_P$ is an Azumaya algebra, and hence a maximal R_P-order in D. Suppose $P = (\pi)$ or (δ). Then, by Proposition 6.7 and [31, Theorem 11.5], Λ_P is a maximal R_P-order in D. Since R is noetherian, integrally closed and Λ is a reflexive R-module, by [5, Theorem 1.5], Λ is a maximal R-order of D. □

Lemma 8.4 (cf. [33, Lemma 3.1]). *Let σ and Λ be as above. Let $a \in \Lambda$ with $\sigma(a) = a$. If $\mathrm{Nrd}(a) = u\pi^r\delta^s$ for some unit $u \in R_0$ and $r, s \in \mathbb{Z}$, then there exist a unit $\theta \in \Lambda$, $r', s' \in \{0, 1\}$ with $r \equiv r'$ and $s \equiv s'$ modulo 2 such that $\langle a \rangle \simeq \langle \theta x^{r'} y^{s'} \rangle$ as hermitian forms over (D, σ).*

Proof. Let $r = 2r_1 + r'$ and $s = 2s_1 + s'$ with $r', s' \in \{0, 1\}$. Let $z = x^{s_1} y^{r_1} \in \Lambda$. Then

$$\mathrm{Nrd}(z) = \mathrm{Nrd}(\sigma(z)) = \delta^{s_1}\pi^{r_1}.$$

Let $\theta = \sigma(z)^{-1} a z^{-1} (x^{s'} y^{r'})^{-1}$. Then

$$a = \sigma(z)\theta x^{s'} y^{r'} z,$$

and hence $\langle a\rangle \simeq \langle \theta x^{s'} y^{r'}\rangle$. Since $\mathrm{Nrd}(\theta) = u \in R_0$ is a unit, it follows that $\theta \in \Lambda$ and is a unit in Λ, as in the proof of [33, Lemma 3.1]. $\square$

Corollary 8.5 (cf. [33, Corollary 3.2]). *Let σ and Λ be as in Lemma 8.3. Let $h = \langle a_1, \ldots, a_r\rangle$ be a hermitian form over (A, σ) with $a_i \in \Lambda$, $\sigma(a_i) = a_i$ and* $\mathrm{Nrd}(a_i)$ *is a product of a unit in R, a power of π and a power of δ. Then*

$$h \simeq \langle u_1, \ldots, u_{m_0}\rangle \perp \langle v_1, \ldots, v_{n_1}\rangle x \perp \langle w_1, \ldots, w_{n_2}\rangle y \perp \langle \theta_1, \ldots, \theta_{n_3}\rangle xy$$

for some $u_i, v_i, w_i, \theta_i \in \Lambda$ units.

We have the following (cf. [33, Corollary 3.3]).

Corollary 8.6. *Let σ and Λ as above. Let $a_i \in \Lambda$ be as in Corollary 8.5 and $h = \langle a_1, \ldots, a_r\rangle$. If $h \otimes F_{0\pi}$ is isotropic, then h is isotropic over F_0.*

Proof. Since $\sigma(xy) = yx = \rho xy$ and $\rho\sigma(\rho) = N_{F/F_0}(\rho) = 1$, it follows that $\mathrm{Int}(xy) \circ \sigma$ is an involution on D. Following the proof of [33, Corollary 3.3], it follows that if h is isotropic over $F_{0\pi}$, then h is isotropic over F_0. $\square$

9. An application of refinement of patching to local-global principle

Let T be a complete discrete valuation ring and K its field of fractions. We recall a few basic definitions from [15, 16]. Let F be a function field of a curve over K. Let $\mathcal{Y} \to \mathrm{Spec}(T)$ be a proper normal model of F and Y the special fibre. For a point x of Y, let F_x be the field of fractions of the completion $\widehat{R}_x$ of the local ring at x. Let U be a nonempty proper subset of an irreducible component of Y not containing the singular points of Y. Let R_U be the subset of F containing all those elements of F which are regular at every closed point of U. Let $t \in T$ be a parameter, $\widehat{R}_U$ be the (t)-adic completion of R_U and F_U the field of fractions of $\widehat{R}_U$. Let $P \in Y$ be a closed point. A height one prime ideal $\mathfrak{p}$ of $\widehat{R}_P$ containing t is called a *branch* at P. For a branch $\mathfrak{p}$, let $F_{\mathfrak{p}}$ be the completion of F_P at the discrete valuation given by $\mathfrak{p}$.

Let $\mathcal{P}$ be a finite set of closed points of Y containing all singular points of Y and at least one point from each irreducible component of Y. Let $\mathcal{U}$ be the set of irreducible components of $Y \setminus \mathcal{P}$ and $\mathcal{B}$ the set of branches at points in $\mathcal{P}$. Let G be a linear algebraic group over F. We say that *factorization* holds for G with respect to $(\mathcal{P},\mathcal{U})$ if given $(g_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in\mathcal{B}} G(F_{\mathfrak{p}})$, there exists

$$(g_Q) \in \prod_{Q\in\mathcal{P}} G(F_Q) \quad \text{and} \quad (g_U) \in \prod_{U\in\mathcal{U}} G(F_U)$$

such that if $\mathfrak{p}$ is a branch at P along U, then $g_{\mathfrak{p}} = g_Q g_U$. If the factorization holds for G with respect to all possible pairs $(\mathcal{P},\mathcal{U})$, then we say that *factorization* holds for G over F with respect to $\mathcal{Y}$. Let Z be a variety over F with a G-action. We say that G *acts transitively* on points of Z if $G(E)$ acts transitively on $Z(E)$ for all extensions E/F with $Z(E) \neq \emptyset$.

Let $\mathcal{X} \to \mathcal{Y}$ be a sequence of blow ups and X the special fibre of $\mathcal{X}$. Let $P \in \mathcal{Y}$ be a closed point and V the fibre over P. Suppose that $\dim(V) = 1$. Let $\mathcal{P}'$ be a finite set of closed points of V containing all the singular points of V and at least one point from each irreducible component of V. Let $\mathcal{U}'$ be the set of connected components of $V \setminus \mathcal{P}'$. Let $\mathcal{B}'$ be the set of branches at the points of $\mathcal{P}'$. We say that *factorization* holds for G with respect to $(\mathcal{P}',\mathcal{U}')$ if given $(g_{\mathfrak{p}}) \in \prod_{\mathfrak{p}\in\mathcal{B}'} G(F_{\mathfrak{p}})$, there exists

$$(g_Q) \in \prod_{Q\in\mathcal{P}'} G(F_Q) \quad \text{and} \quad (g_U) \in \prod_{U\in\mathcal{U}'} G(F_U)$$

such that if $\mathfrak{p}$ is a branch at P along U, then $g_{\mathfrak{p}} = g_Q g_U$.

Let $\mathcal{P}_X$ be a finite set of closed points of X containing $\mathcal{P}'$, all singular points of X and at least one closed point from each irreducible component of X. Let $\mathcal{U}_X$ be the set of irreducible components of $X \setminus \mathcal{P}$ and $\mathcal{B}_X$ the set of branches at points in $\mathcal{P}$.

The following results are immediate consequences of results of Harbater, Hartmann and Krashen [16].

Theorem 9.1. *Let F, P, $\mathcal{P}_X$, $\mathcal{P}'$, $\mathcal{U}_X$ and $\mathcal{U}'$ be as above. Let G be a connected linear algebraic group over F. If the factorization holds for G with respect to $(\mathcal{P}_X,\mathcal{U}_X)$, then the kernel of natural map*

$$H^1(F_P,G) \to \prod_{U'\in\mathcal{U}'} H^1(F_{U'},G) \times \prod_{Q\in\mathcal{P}'} H^1(F_Q,G)$$

is trivial.

Proof. Suppose the factorization holds for G with respect to $(\mathcal{P},\mathcal{U})$. Then, by [16, Proposition 3.14], factorization holds for G with respect to $(\mathcal{P}',\mathcal{U}')$. By [16, Proposition 3.10], patching holds for the injective diamond

$$F_{P_\bullet} = \Big(F_P \le \prod_{Q\in\mathcal{P}'} F_Q, \prod_{U'\in\mathcal{U}'} F_{U'} \le \prod_{b'\in\mathcal{B}'} F_{b'}\Big).$$

Hence, by [16, Theorem 2.13], the map

$$H^1(F_P, G) \to \prod_{U' \in \mathcal{U}'} H^1(F_{U'}, G) \times \prod_{Q \in \mathcal{P}'} H^1(F_Q, G)$$

is injective. □

Corollary 9.2. *Let F, $\mathcal{Y}$, P and $\mathcal{X}$ be as above. Let G be a connected linear algebraic group over F. If the factorization holds for G over F with respect to $\mathcal{Y}$, then the kernel of natural map*

$$H^1(F_P, G) \to \prod_{x \in V} H^1(F_x, G)$$

is trivial.

Proof. Let ξ be in the kernel of the map $H^1(F_P, G) \to \prod_{x \in V} H^1(F_x, G)$. Then, as in [15, Corollary 5.9], there exists a finite set $\mathcal{P}'$ of closed points of V containing all the singular points of V and at least one closed point from each irreducible component of V such that if $\mathcal{U}'$ is the set of irreducible components of $V \setminus \mathcal{P}'$, then ξ is in the kernel of

$$H^1(F_P, G) \to \prod_{U' \in \mathcal{U}} H^1(F_U, G) \times \prod_{Q \in \mathcal{P}'} H^1(F_Q, G).$$

Hence, by Theorem 9.1, ξ is trivial. □

Theorem 9.3. *Let F, P, $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{U}$, $\mathcal{U}'$, F_P and F_U be as above. Let G be a connected linear algebraic group over F. Suppose the factorization holds for G with respect to $(\mathcal{P}, \mathcal{U})$. Let Z be a F-variety with G acting transitively on points Z. If $Z(F_{U'}) \neq \emptyset$ and $Z(F_Q) \neq \emptyset$ for all $U' \in \mathcal{U}'$ and $Q \in \mathcal{P}'$, then $Z(F_P) \neq \emptyset$.*

Proof. The result follows from Theorem 9.1 and [15, Corollary 2.8]. □

Corollary 9.4 (cf. [15, Theorem 9.1]). *Let F, $\mathcal{Y}$, P, $\mathcal{X}$ and V be as above. Let G be a connected linear algebraic group over F. Suppose the factorization holds for G over F. Let Z be a F-variety with G acting transitively on points of Z. If $Z(F_x) \neq \emptyset$ for all $x \in V$, then $Z(F_P) \neq \emptyset$.*

Proof. Suppose $Z(F_x) \neq \emptyset$ for all $x \in V$. Let X_i be an irreducible component of V and $\eta_i \in V$ the generic point of X_i. Since $Z(F_{\eta_i}) \neq \emptyset$, by [15, Proposition 5.8], there exists a nonempty affine open subset U_i of X_i such that $Z(F_{U_i}) \neq \emptyset$. Let $\mathcal{P}'$ be the complement of the union of U_i's in V. Let $Q \in \mathcal{P}'$. Then, by the assumption on Z, we have $Z(F_Q) \neq \emptyset$. Hence, by Theorem 9.3, $Z(F_P) \neq \emptyset$. □

10. Local-global principle for projective homogeneous spaces under general linear groups

Let K be a complete discretely valued field with residue field κ and F the function field of a smooth projective curve over K. Let A be a central simple algebra over F of index n coprime to $\mathrm{char}(k)$ and $G = \mathrm{PGL}(A)$. Let Z be a projective homogeneous space under G over F. If F contains a primitive n-th root of unity, then from the results in [30], it follows that $Z(F) \neq \emptyset$ if and only if $Z(F_\nu) \neq \emptyset$ for all divisorial discrete valuations of F. In this section, we dispense with the condition on the roots of unity if K is a local field.

Let $\mathcal{X}$ be a normal proper model of F over the valuation ring of K. Let P be a closed point of $\mathcal{X}$. A discrete valuation ν of F (respectively, F_P) is called a *divisorial* discrete valuation if it is given by a codimension one point of a model of F (respectively, with center P).

Let M be a field and A a central simple algebra over M of degree n. For a sequence of integers $0 < n_1 < n_2 < \cdots < n_k < n$, let

$$X(n_1, \ldots, n_k) = \{(I_1, \ldots, I_k) \mid I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq A, \text{sequence of right ideals of } A \textit{ with } \dim_F(I_j) = n \cdot n_j, j = 1, \ldots, k\}.$$

Theorem 10.1. *Let K be a local field with residue field κ and F the function field of a smooth projective curve over K. Let A be a central simple algebra over F of index coprime to* $\mathrm{char}(\kappa)$. *Let $\mathcal{X}$ be a normal proper model of F over the valuation ring of K and $P \in \mathcal{X}$ be a closed point. Let L be the field F or F_P. Let Z be a projective homogeneous space under* $\mathrm{PGL}(A)$ *over L. If $Z(L_\nu) \neq \emptyset$ for all divisorial discrete valuation ν of L, then $Z(L) \neq \emptyset$.*

Proof. Let $f : \mathcal{X}' \to \mathcal{X}$ be a sequence of blow ups such that $\mathcal{X}'$ is regular, the ramification locus of A on $\mathcal{X}'$ and the special fibre of $\mathcal{X}'$ is a union of regular curves with normal crossings. By blowing up P, we assume that the dimension of the fibre over P is 1. Let V be either the special fibre of $\mathcal{X}'$ or the fibre of f over P, depending on $L = F$ or $L = F_P$.

Let $n = \deg(A)$. Then Z is isomorphic to $X(n_1, \ldots, n_r)$ for some sequence of integers $0 < n_1 < \cdots < n_r < n$ and $\mathrm{PGL}(A)$ acts transitively on points of Z (cf. [23, Section 5]). Let d be the lcm of $n_1, \ldots, n_r, n$. Then, for any field extension M/F, $Z(M) \neq \emptyset$ if and only if $\mathrm{ind}(A \otimes_F M)$ divides d (cf. [23, p. 561, 5.3]).

Suppose that $Z(L_\nu) \neq \emptyset$ for all divisorial discrete valuations ν of L. Since $\mathrm{PGL}(A)$ is rational, factorization holds for $\mathrm{PGL}(A)$ over F ([13, Theorem 3.6.]). Thus, by [15, Theorem 5.10 and Theorem 9.1] for the case $L = F$ and Corollary 9.4 for the case $L = F_P$, it is enough to show that $Z(F_x) \neq \emptyset$ for all $x \in V$.

Let $x \in V$. Suppose x is a generic point of V. Then x defines a divisorial discrete valuation ν_x of L and $L_x = L_{\nu_x}$. Hence, $Z(L_x) \neq \emptyset$.

Suppose $x = Q$ is a closed point of V. Then, by the choice of $\mathcal{X}'$, the local ring R_Q at Q on $\mathcal{X}'$ is generated by (π, δ) such that A is unramified on R_Q except possibly at (π) and (δ).

Let ν_π be the discrete valuation given by π on L and L_{ν_π} the completion of L at ν_π. Since ν_π is a divisorial discrete valuation of L, we have $Z(L_{\nu_\pi}) \neq \emptyset$. Hence, $\operatorname{ind}(A \otimes_F L_{\nu_\pi})$ divides d. Since $L_{\nu_\pi} \subseteq L_{Q,\pi}$, we have $\operatorname{ind}(A \otimes_F L_{Q,\pi})$ divides d. Since, by [27, Corollary 5.6],

$$\operatorname{ind}(A \otimes_F L_Q) = \operatorname{ind}(A \otimes_F L_{Q,\pi}),$$

$\operatorname{ind}(A \otimes_F L_Q)$ divides d. Hence, $Z(L_Q) \neq \emptyset$. □

Corollary 10.2. *Let L be the field F or F_P, and A as in Theorem 10.1. Then*

$$\operatorname{ind}(A \otimes_F L) = \operatorname{lcm}\{\operatorname{ind}(A \otimes_F L_\nu) \mid \nu \text{ a divisorial discrete valuation of } L\}.$$

Proof. Let $d = \operatorname{lcm}\{\operatorname{ind}(A \otimes_F L_\nu) \mid \nu$ a divisorial discrete valuation of $L\}$. Then clearly d divides $\operatorname{ind}(A \otimes_F L)$. Thus, it is enough to show that $\operatorname{ind}(A \otimes_F L)$ divides d.

Let $Z = X(d)$. Since for every divisorial discrete valuation ν of L, $\operatorname{ind}(A \otimes_F L_\nu)$ divides d, $Z(F_\nu) \neq \emptyset$ (cf. [23, p. 561 5.3]). Hence, by Theorem 10.1, $Z(L) \neq \emptyset$. Thus, $\operatorname{ind}(A \otimes_F L)$ divides d (cf. [23, p. 561 5.3]). □

Corollary 10.3. *Let L be the field F or F_P and A as in Theorem 10.1. Let $G = \mathrm{GL}(A)$ and Z be a projective homogeneous space under G over L. If $Z(L_\nu) \neq \emptyset$ for all divisorial discrete valuation of L, then $Z(L) \neq \emptyset$.*

Proof. Since the projective homogeneous spaces under $\mathrm{GL}(A)$ are in bijection with the projective homogeneous spaces under $\mathrm{PGL}(A)$ [7, Theorem 2.20 (i)], the corollary follows from Theorem 10.1. □

11. Local-global principle for homogeneous spaces under unitary groups

Let K be a local field with residue field κ of characteristic not 2 and F_0 the function field of a smooth projective curve over K. Let F/F_0 be a quadratic field extension. Let A be a central simple algebra over F with an involution σ of second kind and $F^\sigma = F_0$. Let (V, h) be a hermitian form over (A, σ) and $G = U(A, \sigma, h)$. If $\operatorname{ind}(A) = 1$, then the validity of Conjecture 1 for G is a consequence of results proved in [11]. If $\operatorname{ind}(A) = 2$, Wu [33] proved the validity of Conjecture 1 for G. In this section we dispense with the condition $\operatorname{ind}(A) \leq 2$ for the good characteristic case.

We begin by recalling the structure of projective homogeneous spaces under a unitary group over any field. Let F_0 be a field and F/F_0 a separable quadratic extension. Let A be a central simple algebra over F of degree n with an involution σ of second kind and $F^\sigma = F_0$. Let (V, h) be a hermitian form over (A, σ) and $G = U(A, \sigma, h)$.

Let W be a finitely generated module over A. The *reduced dimension* $\mathrm{rdim}_A(W)$ of W over A is defined as $\dim_F(W)/n$ [19, Definition 1.9]. For a sequence of integers $0 < n_1 < \cdots < n_r \leq n/2$ and for any field extension L/F, let

$$X(n_1, \ldots, n_r) = \{(W_1, \ldots, W_r) \mid \{0\} \subsetneq W_1 \subsetneq \cdots \subsetneq W_r, W_i \text{ a totally isotropic subspace of } V \text{ with } \mathrm{rdim}_F W_i = n_i\}.$$

We recall the following from [23, 24], cf. [33, Section 2].

Theorem 11.1. *Let F_0 be a field and F/F_0 a separable quadratic extension. Let A be a central simple algebra over F of degree n with an involution σ of second kind and $F^\sigma = F_0$. Let (V, h) be a hermitian form over (A, σ) and $G = U(A, \sigma, h)$. Then*

(i) *A projective variety X over F_0 is a projective homogeneous space under G over F_0 if and only if $X \simeq X(n_1, \ldots, n_r)$ for some increasing sequence of integers $0 < n_1 < \cdots < n_r \leq n/2$.*

(ii) *For any field extension L/F_0, we have $X(n_1, \ldots, n_r)(L) \neq \emptyset$ if and only if $X(n_r)(L) \neq \emptyset$ and $\mathrm{ind}(A_L)$ divides n_i for all i.*

(iii) *If $A = M_r(D)$ for some central simple algebra over F and $G_0 = U(D, \sigma_0)$ for some unitary involution σ_0 on D, then there is a bijection assigning projective homogeneous spaces X under G and to projective homogeneous spaces X_0 under G_0. Further, for any field extension L/F_0, we have $X(L) \neq \emptyset$ if and only if $X_0(L) \neq \emptyset$.*

The proof of the following theorem is parallel to Abhyankar's proof in the case of algebraic surfaces [1].

Theorem 11.2. *Let $\mathcal{X}_0$ be a normal integral excellent two dimensional scheme with function field F_0 and F/F_0 a quadratic field extension. Suppose that 2 is invertible on $\mathcal{X}_0$. Let D be a divisor on $\mathcal{X}_0$. Then there exists a sequence of blowups $\mathcal{X}' \to \mathcal{X}_0$ with $\mathcal{X}'$ regular such that the integral closure $\mathcal{X}$ of $\mathcal{X}'$ in F is regular and support of the pull back of D on $\mathcal{X}$ is a union of regular curves with normal crossings.*

Proof. Let $d \in F_0^*$ be such that $F = F_0(\sqrt{d})$. Since $\mathcal{X}_0$ is excellent and dimension two, there exists a sequence of blowups $\mathcal{X}' \to \mathcal{X}_0$ such that the union E of $\mathrm{supp}_{\mathcal{X}'}(d)$ and the support of the pullback of D on $\mathcal{X}'$ is a union of regular curves with normal crossings [20].

Then for any closed point P of $\mathcal{X}'$, the maximal ideal at P is generated by (π_P, δ_P) such that $d = u_P \pi_P^{n_P} \delta_P^{m_P}$ for some unit u_P at P and $n_P, m_P \in \mathbf{Z}$. Further, replacing $\mathcal{X}'$ by a sequence of blowups, we assume that for each closed point P of $\mathcal{X}'$, either n_P is even or m_P is even (see, for instance, [33]). We now show that $\mathcal{X}'$ has the required properties.

Let $P \in \mathcal{X}'$ be a closed point. Then, by the choice of $\mathcal{X}'$, we have $d = u_P d_1^2$ or $u_P \pi_P d_1^2$ or $u_P \delta_P d_1^2$ for some unit u_P at P and $d_1 \in F_0^*$. Thus, the integral closure $\mathcal{X}$ of $\mathcal{X}'$ in F is regular (see, for instance, [28, Lemma 3.3]).

Let $\widetilde{D}$ be the pull back of D to $\mathcal{X}$. Let $Q \in \mathcal{X}$ be a closed point which is on the support of $\widetilde{D}$. Let $P \in \mathcal{X}'$ be the image of Q. Let A_P be the local ring at P and B_P be the integral closure of A_P in F. Then B_P is a regular local ring.

Suppose d is not a square in the field of fractions of the completion of A_P. Then Q is the only point in $\mathcal{X}$ which maps to P. Let C be an irreducible curve which is in the support of D. Then C is regular at P on $\mathcal{X}'$, and hence there is a unique irreducible curve in the support of $\widetilde{D}$ mapping to C. Since the support of D is a union of regular curves with normal crossings, it follows that the support of $\widetilde{D}$ at Q is a union of regular curves with normal crossings.

Suppose d is a square in the field of fractions of the completion of A_P. Then by the choice of $\mathcal{X}'$, we have that $d = u_P d_1^2$ for some unit $u_P \in A_P$ and $d_1 \in F_0^*$. Thus, $F = F_0(\sqrt{u_P})$, and hence B_P/A_P is étale. Since the support of D is a union of regular curves with normal crossings, it follows that the support of $\widetilde{D}$ at Q is a union of regular curves with normal crossings. □

Corollary 11.3. *Let $\mathcal{X}_0$ be a normal integral excellent two dimensional scheme with function field F_0 and F/F_0 a quadratic field extension. Suppose that 2 is invertible on $\mathcal{X}_0$. Let D' be a divisor on the integral closure $\mathcal{X}_0'$ of $\mathcal{X}_0$ in F. Then there exists a sequence of blowups $\mathcal{X}' \to \mathcal{X}_0$ with $\mathcal{X}'$ regular such that the integral closure $\mathcal{X}$ of $\mathcal{X}'$ in F is regular and support of the pull back of D' on $\mathcal{X}'$ is a union of regular curves with normal crossings.*

Proof. Let D be a divisor on $\mathcal{X}_0$ containing the image of all irreducible curves in the support of D'. Then applying Theorem 11.2 to D we get the required $\mathcal{X}'$. □

Theorem 11.4. *Let K be a local field with residue field κ. Let F_0 be the function field of a curve over K. Let F/F_0 be a quadratic extension and A be a central simple algebra over F with an F/F_0- involution σ. Suppose that* $2\,\mathrm{ind}(A)$ *is coprime to* $\mathrm{char}(\kappa)$. *Let h be a hermitian form over (A,σ) and $G = U(A,\sigma,h)$. If $A = F$, then assume that rank of h is at least 2. Let Z be a projective homogeneous space under G over F. Let $\mathcal{X}_0$ be a normal proper model of F_0 and $P \in \mathcal{X}_0$ be a closed point with $F \otimes_{F_0} F_{0P}$ a field. If $Z(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then $Z(F_{0P}) \neq \emptyset$.*

Proof. Let n be the degree of A. Since Z is a projective homogeneous space under G, by Theorem 11.1,

$$Z \simeq X(n_1, \ldots, n_r)$$

for some sequence of integers $0 < n_1 < \cdots < n_r \leq n/2$. Suppose that $Z(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0. Then, by Theorem 11.1, $\operatorname{ind}(A \otimes_{F_0} F_{0\nu})$ divides n_i for all i. Since K is a local field, $\operatorname{ind}(A)$ is the lcm of $\operatorname{ind}(A \otimes_{F_0} F_{0\nu})$ as ν varies over all divisorial discrete valuations of F (cf. Corollary 10.2). Hence, $\operatorname{ind}(A)$ divides n_i for all i. By Theorem 11.1, $X(n_r)(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F. To prove the theorem, by Theorem 11.1, it suffices to show that

$$X(n_r)(F_{0P}) \neq \emptyset.$$

Thus, we assume that $Z = X(m)$ with $m = n_r$.

Let T be the valuation ring of K. Then there exists a sequence of blow ups $\mathcal{X}_0' \to \mathcal{X}_0$ such that the normalization $\mathcal{X}$ of $\mathcal{X}_0'$ in F is regular and the ramification locus of A on $\mathcal{X}$ and the special fibre of $\mathcal{X}$ is a union of regular curves with normal crossings Corollary 11.3. If necessary, by blowing up P, we assume that the fibre V over P is of dimension 1. Then, by Corollary 9.4, it is enough to show that

$$Z(F_{0x}) \neq \emptyset$$

for all $x \in V$.

Let $x \in V$ be a generic point. Then x gives a divisorial discrete valuation ν_x on F_0 such that $F_{0x} = F_{\nu_x}$. Hence, $Z(F_{0x}) \neq \emptyset$.

Let $Q \in V$ be a closed point. We show that $Z(F_{0Q}) \neq \emptyset$ by induction on $\operatorname{ind}(A \otimes_{F_0} F_{0Q})$. Suppose

$$\operatorname{ind}(A \otimes_{F_0} F_{0Q}) = 1.$$

Then the hermitian form h corresponds to a quadratic form over q_h over F_{0Q} such that h is isotropic over any field extension M of F_Q if and only if q_h is isotropic over M (see [32, Theorem 1.1, p. 348]). Since $Z = X(m)$, for every divisorial discrete valuation ν of F_0, there is a totally isotropic subspace of $V \otimes_{F_0} F_{0\nu}$ of dimension m. Thus, to prove the theorem, it is enough to show that there is a totally isotropic subspace of $V \otimes_{F_0} F_{0Q}$ of dimension m. By induction on $\dim(q_h)$, it is enough to show that q_h is isotropic over F_{0Q}. By the assumption on the rank of h, rank of q_h is at least 4. Since for every (divisorial) discrete valuation ν of F_0 centered on Q, q_h is isotropic over F_{0Q_ν}, by [16, Corollary 4.7], q_h is isotropic over F_{0Q}.

Suppose $\operatorname{ind}(A \otimes_{F_0} F_{0Q}) \geq 2$. Then by the choice of the model $\mathcal{X}_0'$, we have the following:

(i) the local ring R_Q at Q on $\mathcal{X}_0'$ is regular with (π, δ) as the maximal ideal;

(ii) $F \otimes F_{0Q} = F_{0Q}(\sqrt{u\pi^\varepsilon})$ for some unit $u \in R_Q$ and $\varepsilon = 0, 1$;

(iii) A is unramified on the integral closure of R_Q in $F_{0Q}(\sqrt{u\pi^\varepsilon})$, except possibly at π_1 and δ, where $\pi_1 = \pi$ or $\sqrt{u\pi}$ depending on $\varepsilon = 0$ or 1.

Let D_Q be the central division algebra over $F \otimes_{F_0} F_{0Q}$ which is Brauer equivalent to $A \otimes_{F_0} F_{0Q}$. Then there is a unitary involution σ_Q on D_Q and the hermitian form (V, h) corresponds to a hermitian form (V_Q, h_Q) over D_Q. By Theorem 11.1,

$$Z_{F_{0Q}} = X(m)_{F_{0Q}}$$

corresponds to a projective homogeneous space $Z_Q = X(m')$ under $U(D_Q, \sigma_Q)$ for some suitable m' which is divisible by $\mathrm{ind}(D_Q)$. Further, to show that

$$Z(F_{0Q}) \neq \emptyset,$$

it is enough to show that $Z_Q(F_{0Q}) \neq \emptyset$.

Since $\mathrm{ind}(A \otimes_{F_0} F_{0Q}) \geq 2$, we have $\deg(D_Q) \geq 2$. If $\deg(D_Q) = 2$, let Λ be the maximal R_Q-order of D_Q as in [33, Lemma 3.7]. If $\deg(D_Q) \geq 3$, let Λ be the maximal R_Q-order of D_Q as in Lemma 8.3. Since D_Q is a division algebra, we have

$$h_Q \simeq \langle a_1, \ldots, a_n \rangle$$

for some $a_i \in \Lambda_P$. Once again there exists a sequence of blow ups $\mathcal{X}_0'' \to \mathcal{X}_0'$ such that support of $\mathrm{Nrd}(a_i)$ for all i is a union of regular curves with normal crossings (see [2, 20], cf. [33, Section 4]). Further, by blowing up, we also assume that $\mathcal{X}_0''$ satisfies (i), (ii) and (iii). Let V'' be the fibre over Q. Once again we assume that $\dim(V'') = 1$. Thus, to show that

$$Z_Q(F_{0Q}) \neq \emptyset,$$

by Corollary 9.4, it is enough to show that $Z_Q(F_{0x}) \neq \emptyset$ for all $x \in V''$.

Let $x' \in V''$ be a generic point then, as above, $Z_Q(F_{0x'}) \neq \emptyset$.

Let $Q' \in V''$ be a closed point. Suppose that $D_Q \otimes_{F_0Q} F_{0Q'}$ is division. By Theorem 11.1, it is enough to show that there is a totally isotropic subspace of $V_Q \otimes F_{0Q'}$ of dimension m'. By induction on the reduced dimension of V_Q, it is enough to show that h_Q is isotropic over $F_{0Q'}$. Since $\mathcal{X}_0''$ satisfies (i), (ii) and (iii), the maximal ideal at Q' on $\mathcal{X}_0''$ is generated by (π, δ), $h_Q = \langle a_1, \ldots, a_n \rangle$ for some $a_i \in \Lambda_P$ with $\mathrm{Nrd}(a_i)$ is a supported along only π, δ, and $h_Q \otimes F_{0Q'\pi}$ is isotropic. Hence, by Corollary 8.6, h_Q is isotropic over $F_{0Q'}$.

Suppose that $D_Q \otimes_{F_0Q} F_{0Q'}$ is not division. Then

$$\mathrm{ind}(A \otimes_{F_0} F_{0Q'}) < \mathrm{ind}(A \otimes_{F_0} F_{0Q}).$$

Hence, by induction, $Z_Q(F_{0Q'}) \neq \emptyset$. □

Theorem 11.5. *Let K be a local field with residue field κ. Let F_0 be a function field of a curve over K. Let F/F_0 be a quadratic extension and A a central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* $\mathrm{char}(\kappa)$. *Let h be a hermitian form over (A, σ). If $A = F$, then assume that the rank of h is at least 2. Let Z be a projective homogeneous space under $U(A, \sigma, h)$ over F. If $Z(F_{0v}) \neq \emptyset$ for all (divisorial) discrete valuations v of F_0, then $Z(F_0) \neq \emptyset$.*

Proof. Suppose $Z(F_{0\nu}) \neq \emptyset$ for all (divisorial) discrete valuations ν of F_0. Let $\mathcal{X}_0$ be a normal proper model of F_0 over the valuation ring of K and X_0 the special fibre. Let $x \in X_0$ be a codimension 0 point in X_0 and ν_x the discrete valuation of F_0 given by x. Then $F_{0x} = F_{0\nu_x}$ and $Z(F_{0x}) \neq \emptyset$.

Let $P \in X_0$ be a closed point. We have

$$A \otimes_{F_0} F \simeq A_1 \times A_1^{\mathrm{op}}$$

for some central simple algebra A_1/F (see [19, Proposition 2.14]) and

$$U(A, \sigma, h) \otimes F \simeq \mathrm{GL}(A_1)$$

(see [19, p. 346]). Suppose $F \otimes_{F_0} F_{0P}$ is not a field. Then $F \subset F_{0P}$ and

$$A \otimes_{F_0} F_{0P} \simeq A_1 \otimes F_{0P} \times A_1^{\mathrm{op}} \otimes F_0.$$

Let $\mathcal{X}$ be the normal closure of $\mathcal{X}_0$ in F. Since $F \otimes_{F_0} F_{0P}$ is not a field, there exists a closed point Q of $\mathcal{X}$ such that $F_Q \simeq F_{0P}$. Hence, by Corollary 10.3, $Z(F \otimes_{F_0} F_{0P}) \neq \emptyset$.

Suppose $F \otimes_{F_0} F_{0P}$ is a field. Then, by Theorem 11.4, $Z(F_{0P}) \neq \emptyset$. Since $U(A, \sigma, h)$ is rational and connected (see [22, Lemma 1, p. 195]), by [15, Corollary 6.5 and Theorem 9.1], $Z(F_0) \neq \emptyset$. □

Theorem 11.6. *Let K be a local field with residue field κ. Let F_0 be a function field of a curve over K. Let F/F_0 be a quadratic extension and A a central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* $\mathrm{char}(\kappa)$. *Let h be a hermitian form over (A, σ). Then the canonical map*

$$H^1(F_0, U(A, \sigma, h)) \to \prod_{\nu \in \Omega_{F_0}} H^1(F_{0\nu}, U(A, \sigma, h))$$

has trivial kernel.

Proof. Let $\xi \in H^1(F_0, U(A, \sigma, h))$. Then ξ corresponds to a hermitian space h' over (A, σ) of reduced rank equal to the reduced rank of h. Let $h_0 = h \perp -h'$ and m the reduced rank of h. Let $G = U(A, \sigma, h_0)$ and $Z = X(m)$. Then Z is a projective homogeneous variety under G over F_0. Suppose ξ maps to the trivial element in $H^1(F_{0\nu}, U(A, \sigma, h))$ for all (divisorial) discrete valuations ν of F_0. Then

$$h' \otimes F_{0\nu} \simeq h \otimes F_{0\nu},$$

and hence h_0 is hyperbolic. Thus, $Z(F_{0\nu}) \neq \emptyset$ for all (divisorial) discrete valuations ν of F_0. Hence, by Theorem 11.5, $Z(F_0) \neq \emptyset$. In particular, h_0 is hyperbolic. Since the reduced ranks of h and h' are equal, $h \simeq h'$ and ξ is the trivial element in $H^1(F_0, U(A, \sigma, h))$. □

12. Local-global principle for special unitary groups: Patching setup

Let K be a local field with residue field κ and F_0 the function field of a smooth projective curve over K. Let F/F_0 be a separable quadratic extension. Let A be a central simple algebra over F of degree n with an involution σ of second kind and $F^\sigma = F_0$. Suppose that $2n$ is coprime to $\mathrm{char}(\kappa)$. In this section we show that there is a local-global principle for principal homogeneous spaces under $\mathrm{SU}(A,\sigma)$ over F_0 in the patching setup (cf. Theorem 12.5).

Let $\mu \in F^*$. Let $F = F_0(\sqrt{d})$, $d \in F_0^*$. Let T be the valuation ring of K. Then there exists a regular proper model $\mathcal{X}_0 \to \mathrm{Spec}(T)$ of F_0 with the normalization $\mathcal{X}$ of $\mathcal{X}_0$ in F regular and with the property that the special fibre X of $\mathcal{X}$, the ramification locus of F/F_0 on $\mathcal{X}$, the ramification locus of A on $\mathcal{X}$ and the support of μ on $\mathcal{X}$ are a union of regular curves with normal crossings [2, 20]. Let X_0 be the reduced special fibre of $\mathcal{X}_0$ and $\{\eta_1, \dots, \eta_m\}$ be the generic points of X_0.

Let $\mathcal{P}_0$ be a finite set of closed points of X_0 containing all the singular points of X_0 and at least one closed point from each irreducible component of X_0. Let $\mathcal{U}_0$ be the set of irreducible components of $X_0 \setminus \mathcal{P}_0$. We fix the data $\mu \in F^*$, $\mathcal{X}_0, \mathcal{P}_0$ and $\mathcal{U}_0$ for until Theorem 12.5. Let $\mathcal{B}_0$ be the set of branches at $\mathcal{P}_0$. Since X_0 is a union of regular curves with normal crossings, $\mathcal{B}_0$ is in bijection with pairs (P, U) with $P \in \mathcal{P}_0$, $U \in \mathcal{U}_0$ and P is in the closure of U.

Let $\eta \in X_0$ be a generic point and $P \in \overline{\{\eta\}}$ a closed point. Then η defines a discrete valuation ν_η on F_{0P}. Then the completion of F_0 at the restriction of ν_η to F_0 is denoted by $F_{0\eta}$ and the completion of F_{0P} at ν_η denoted by $F_{0P,\eta}$. The closed point P induces a discrete valuation ν_P on the residue field $\kappa(\eta)$ of $F_{0\eta}$ such that the completion $\kappa(\eta)_P$ of $\kappa(\eta)$ at ν_P is the residue field of $F_{0P,\eta}$.

Let $P \in X_0$ be a closed point and A_P the local ring at P on $\mathcal{X}_0$. Since the normalization of $\mathcal{X}_0$ in F is regular, $d = \pi u$ or $d = u$ for some $\pi \in A_P$ a regular parameter and $u \in A_P$ a unit. Hence, $B_P = A_P[\sqrt{d}]$ is the integral closure of A_P in F. Let $\delta \in A_P$ be such that $m_P = (\pi, \delta)$ is the maximal ideal of A_P. If $d = \pi u$, then B_P is local and $(\sqrt{\pi u}, \delta)$ is the maximal ideal of B_P. Suppose $d = u$ a unit in A_P. If u is not a square in the residue field $\kappa(P)$, then B_P is local and the maximal ideal of B_P is generated by π and δ.

We begin with the following lemma.

Lemma 12.1. *Let η be a generic point of X_0 and S be a finite set of closed points of $\overline{\{\eta\}}$. For every $P \in S$, let $\theta_{\eta,P} \in F^*_{0P,\eta}$ be a unit at η which is a reduced norm from $A \otimes F_{0P,\eta}$. Then there exists $\theta_\eta \in F_{0\eta}$, which is a reduced norm from $A \otimes F_{0\eta}$ such that $\theta_\eta \theta_{\eta,P}^{-1} \in F^{*n}_{0P,\eta}$ for all $P \in S$.*

Proof. Suppose $F_\eta = F \otimes F_{0\eta}/F_{0\eta}$ is a ramified field extension. Then, by Lemma 6.3, there exists an unramified algebra A_0 over $F_{0\eta}$ such that

$$A \otimes_{F_0} F_{0\eta} \simeq A_0 \otimes_{F_{0\eta}} F_\eta.$$

For $P \in S$, let $\overline{\theta}_{\eta,P} \in \kappa(\eta)_P^*$ be the image of $\theta_{\eta,P} \in F_{0P,\eta}^*$. We choose $\overline{\theta}_\eta \in \kappa(\eta)^*$ be close to $\overline{\theta}_{\eta,P}$ for all $P \in S$. Since A_0 is unramified over $F_{0\eta}$, its specialization B_0 is a central simple algebra over $\kappa(\eta)$. Since $\kappa(\eta)$ is a global field of positive characteristic, by the Hasse–Maass–Schilling theorem, $\overline{\theta}_\eta$ is a reduced norm from B_0. Let $\theta_\eta \in F_{0\eta}$ be a lift of $\overline{\theta}_\eta$. Since $F_{0\eta}$ is complete, θ_η is a reduced norm from $A_0 \otimes_{F_0} F_{0\eta}$, and hence a reduced norm from $A \otimes_F F_\eta$. Since $\overline{\theta}_\eta$ is close to $\overline{\theta}_{\eta,P}$ for all $P \in S$ and n is coprime to char(κ), we have $\overline{\theta}_\eta \overline{\theta}_{\eta,P}^{-1} \in \kappa(\eta)_P^{*n}$ for all $P \in S$. Since $F_{0P,\eta}$ is complete with residue field $\kappa(\eta)_P$, we have $\theta_\eta \theta_{\eta,P}^{-1} \in F_{0P,\eta}^{*n}$ for all $P \in S$.

Suppose that $F_\eta / F_{0\eta}$ is an unramified field extension. Then the residue field $\widetilde{\kappa}(\eta)$ of F_η is a quadratic extension of $\kappa(\eta)$. Let (L_η, σ_η) be the residue of A at η. Since the residue commutes with the corestriction,

$$\mathrm{cores}_{\widetilde{\kappa}(\eta)/\kappa(\eta)}(L_\eta, \sigma_\eta) = 0.$$

Thus, by Proposition 3.2, $L_\eta / \kappa(\eta)$ is a dihedral extension. Since $\theta_{\eta,P}$ is a reduced norm from $A \otimes F_{0P,\eta}$, we have

$$A \cdot (\theta_{\eta,P}) = 0 \in H^3(F \otimes F_{0P,\eta}, \mu_n^{\otimes 2}).$$

Let $\overline{\theta}_{\eta,P}$ be the image of $\theta_{\eta,P}$ in the residue field $\kappa(\eta)_P$ of $F_{P,\eta}$. By taking the residue of $A \cdot (\theta_{\eta,P})$, we get that $(L_\eta, \sigma_\eta, \overline{\theta}_{\eta,P}) = 0$ (cf. [27, Proof of Lemma 4.7]). Hence, $\overline{\theta}_{\eta,P}$ is a norm from the extension

$$L_\eta \otimes_{\kappa(\eta)} \kappa(\eta)_P / \widetilde{\kappa}(\eta) \otimes_{\kappa(\eta)} \kappa(\eta)_P.$$

Since $\kappa(\eta)$ is a global field, by Corollary 5.2, there exists $\overline{\theta}_\eta \in \kappa(\eta)^*$ with $\overline{\theta}_\eta$ a norm from $L_\eta / \widetilde{\kappa}(\eta)$ and $\overline{\theta}_\eta \overline{\theta}_{\eta,P}^{-1} \in \kappa(\eta)_P^{*n}$.

Let $\theta_\eta \in F_{0\eta}$ be a lift of $\overline{\theta}_\eta \in \kappa(\eta)$. Then $\theta_\eta \theta_{\eta,P}^{-1} \in F_{0P,\eta}^{*n}$. Since $\overline{\theta}_\eta$ is a norm from $L_\eta / \widetilde{\kappa}(\eta)$, by Proposition 1.7, θ_η is a reduced norm from $A \otimes F_{0\eta}$.

Suppose $F_\eta = F \otimes_{F_0} F_{0\eta}$ is not a field. Then

$$F_\eta \simeq F_{0\eta} \times F_{0\eta} \quad \text{and} \quad A \otimes_{F_0} F_{0\eta} \simeq A_1 \times A_1^{\mathrm{op}},$$

where A_1^{op} is the opposite algebra. Since $\theta_{\eta,P} \in F_{0P,\eta}$ is a reduced norm from $A \otimes F_{0P,\eta}$, we have $\theta_{\eta,P}$ is a reduced norm from $A_1 \otimes F_{0P,\eta}$. Then, as above, we can find $\theta_\eta \in F_{0\eta}$ such that $\theta_\eta \theta_{\eta,P}^{-1} \in F_{0P,\eta}^{*n}$ and θ_η is a reduced norm from A_1. Then θ_η is a reduced norm from $A \otimes_{F_0} F_{0\eta}$. □

Lemma 12.2. *Suppose that for every generic point η of X_0 there exists $c_\eta \in F_{0\eta}^*$ such that μc_η is a reduced norm from $A \otimes F_{0,\eta}$. Then for every generic point η of X_0, there exists $a_\eta \in F_{0\eta}^*$ such that μa_η is a reduced norm from $A \otimes F_{0\eta}$ with the following property: if η_1 and η_2 are two generic points of X_0 and $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$ with $F \otimes F_{0P}$ a field, then there exists $a_P \in F_{0P}^*$ such that μa_P is a reduced norm from $A \otimes F_{0P}$ and $a_{\eta_i} a_P^{-1} \in F_{0P,\eta_i}^{*n}$ for $= 1, 2$.*

Proof. Since the special fibre is a union of regular curves with normal crossings, for a generic point η of X_0, there exists $\pi_\eta \in F_0$ a parameter at η such that for every closed point $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$ for any two distinct generic points η_1 and η_2 of X_0, the maximal ideal at P is $(\pi_{\eta_1}, \pi_{\eta_2})$.

Suppose that for every generic point η of X_0 there exists $c_\eta \in F_{0\eta}$ such that μc_η is a reduced norm from $A \otimes F_{0,\eta}$. For every generic point η of X_0, let $r_\eta = \nu_\eta(c_\eta)$. For every closed point $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$ with $F \otimes F_{0P}$ is a field, let $a_P = \pi_{\eta_1}^{r_{\eta_1}} \pi_{\eta_2}^{r_{\eta_2}} \in F_{0P}^*$.

Let η be a generic point of X_0. Let $P \in \overline{\{\eta\}} \cap \overline{\{\eta'\}}$ for some generic point $\eta' \neq \eta$. Suppose that $F \otimes F_{0P}$ is a field. By the choice of $\mathscr{X}_0$, F/F_0 is unramified at P except possibly at π_η and $\pi_{\eta'}$. Since the maximal ideal at P is $(\pi_\eta, \pi_{\eta'})$, by [27, Corollary 5.5], $F \otimes F_{0P,\eta}$ is a field. Since n is coprime to $\operatorname{char}(\kappa(P))$, we have

$$c_\eta = u_P \pi_\eta^{r_\eta} \pi_{\eta'}^{s_P} (b_P)^n$$

for some $s_P \in \mathbb{Z}$, a unit $u_P \in \widehat{A}_P$ and $b_P \in F_{0P,\eta}^*$ (cf. Remark 7.1). Let

$$\theta_{\eta,P} = u_P^{-1} \pi_{\eta'}^{r_{\eta'} - s_P}.$$

Let $\alpha \in H^2(F, \mu_m)$ be the class of A. Since A admits an F/F_0-involution, we have

$$\operatorname{cores}_{F/F_0}(\alpha) = 0.$$

Since $\theta_{\eta,P} \in F_{0P,\eta}^*$, we have

$$\operatorname{cores}_{F \otimes_{F_0} F_{0P,\eta}/F_{0P,\eta}}(\alpha \cdot (\theta_{\eta,P})) = \operatorname{cores}_{F \otimes_{F_0} F_{0P,\eta}/F_{0P,\eta}}(\alpha) \cdot (\theta_{\eta,P}) = 0.$$

Since $F \otimes_{F_0} F_{0P,\eta}$ is a field and K_0 is a local field, we have

$$\text{cores}\colon H^3(F, \mu_n^{\otimes 2}) \to H^3(F_0, \mu_n^{\otimes 2})$$

is injective (see [27, Proposition 4.6]), and hence $\alpha \cdot (\theta_{\eta,P}) = 0$. By [27, Theorem 4.12], $\theta_{\eta,P}$ is a reduced norm from A.

Since $\theta_{\eta,P}$ is a unit at η, by Lemma 12.1, there exists $\theta_\eta \in F_{0\eta}$ which is a reduced norm from $A \otimes F_{0\eta}$ such that $\theta_\eta \theta_{\eta,P}^{-1} \in F_{0P,\eta}^{*n}$.

Let $a_\eta = c_\eta \theta_\eta$. Since μc_η and θ_η are reduced norms from $A \otimes F_{0\eta}$, μa_η is a reduced norm from $A \otimes F_{0\eta}$. Let $P \in \overline{\{\eta\}} \cap \overline{\{\eta'\}}$ for some generic point $\eta' \neq \eta$ with $F \otimes F_{0P}$ is a field. Then, by the choice of a_η, we have

$$a_\eta = \pi_\eta^{r_\eta} \pi_{\eta'}^{r_{\eta'}} \text{ modulo } F_{0P,\eta}^{*n}.$$

Hence, $a_\eta a_P^{-1} \in F_{0P,\eta}^{*n}$. □

Lemma 12.3. *Let η_1 and η_2 be two distinct generic points of X_0. Suppose that $P \in \overline{\{\eta_1\}} \cap \overline{\{\eta_2\}}$ is a closed point with $F \otimes F_{0P}$ not a field. Suppose $a_{\eta_i} \in F_{\eta_i}^*$ is such that $\mu a_{\eta_i} \in \operatorname{Nrd}(A \otimes F_{0\eta_i})$. Then there exists $a_P \in F_{0P}^*$ such that μa_P is a reduced norm from $A \otimes F_{0P}$ and $a_{\eta_i} a_P^{-1} \in F_{0P,\eta_i}^{*n}$ for $i = 1, 2$.*

Proof. Since $F \otimes F_{0P}$ is not a field and $\mathrm{cores}_{F/F_0}(A) = 0$, we have

$$F \otimes F_{0P} \simeq F_{0P} \times F_{0P} \quad \text{and} \quad A \otimes F_{0P} \simeq A_1 \times A_1^{\mathrm{op}}$$

for some central simple algebra A_1 over F_{0P}. Write $\mu = (\mu_1, \mu_2)$. Since $a_{\eta_i}\mu$ is a reduced norm from $A \otimes F_{\eta_i}$ and $a_{\eta_i} \in F^*_{0\eta_i}$, we have $a_{\eta_i}\mu_1$ and $\mu_1\mu_2^{-1}$ are reduced norms from $A_1 \otimes F_{0P,\eta_i}$. Since by the choice of $\mathcal{X}_0$, the union of the support of μ on $\mathcal{X}$ and the ramification locus of A on $\mathcal{X}$ is a union of regular curves with normal crossings, by Corollary 7.3, $\mu_1\mu_2^{-1}$ is a reduced norm from $A_1 \otimes F_{0P}$.

The generic points η_1 and η_2 give discrete valuations ν_1 and ν_2 on F_{0P} with completions $F_{0\eta_1,P}$ and $F_{0\eta_2,P}$. Let $z_i \in A_1 \otimes F_{0P,\eta_i}$ with reduced norm $a_{\eta_i}\mu_1$. Let $z \in A_1 \otimes F_{0P}$ be close to z_i for $i = 1, 2$. Let $a_P = \mu_1^{-1}\,\mathrm{Nrd}(z) \in F_{0P}$. Then $\mu_1 a_P$ is a reduced norm from $A_1 \otimes F_{0P}$. Since z is close to z_i and $\mathrm{Nrd}(z_i) = a_{\eta_i}\mu_1$, we have $\mathrm{Nrd}(z)$ is close to $a_{\eta_i}\mu_1$. Hence, a_P is close to a_{η_i}. Therefore, $a_{\eta_i}a_P^{-1} \in F^{*n}_{0P,\eta_i}$. Since $\mu_1\mu_2^{-1}$ is a reduced norm and $a_P\mu_1$ is a reduced norm, $a_P\mu_2$ is a reduced norm. In particular, $a_P\mu$ is a reduced norm. □

Lemma 12.4. *Let η be a generic point of X_0 and $P \in \overline{\{\eta\}}$ a closed point. Suppose $a_\eta \in F_\eta$ is such that $\mu a_\eta \in \mathrm{Nrd}(A \otimes F_{0\eta})$. Then there exists $a_P \in F^*_{0P}$ such that μa_P is a reduced norm from $A \otimes F_{0P}$ and $a_\eta a_P^{-1} \in F^{*n}_{0P,\eta}$.*

Proof. Suppose that $F \otimes F_{0P}$ is a field. Then, by the choice of $\mathcal{X}_0$ and by Lemma 8.1, there exists $a_P \in F_{0P}$ such that μa_P is a reduced norm from $A \otimes F_{0P}$ and $a_\eta a_P^{-1} \in F^n_{0U,P}$.

Suppose $F \otimes F_{0P}$ is a not field. Then, we get the required a_P as in the proof of Lemma 12.3. □

We have an exact sequence of algebraic groups

$$1 \to \mathrm{SU}(A,\sigma) \to U(A,\sigma) \to R^1_{F/F_0}(\mathbf{G}_{\mathrm{m}}) \to 1.$$

For any field extension L/F_0, we have an induced exact sequence

$$U(A,\sigma)(L) \to (L \otimes_{F_0} F)^{*1} \to H^1(L, \mathrm{SU}(A,\sigma)) \to H^1(L, U(A,\sigma)), \qquad (\star)$$

where $(L \otimes_{F_0} F)^{*1} = R^1_{F/F_0}(\mathbf{G}_{\mathrm{m}})(L)$ is the subgroup of $(L \otimes_{F_0} F)^*$ consisting of norm one elements and the map $U(A,\sigma)(L) \to (L \otimes_{F_0} F)^{*1}$ is given by the reduced norm. Further, the image of $U(A,\sigma)(L) \to (L \otimes_{F_0} F)^{*1}$ is equal to

$$\{\theta^{-1}\sigma(\theta) \mid \theta \in \mathrm{Nrd}(A \otimes_{F_0} L^*)\}$$

(see [19, p. 202]).

Theorem 12.5. *Let K be a local field with the residue field κ and valuation ring T. Let F_0 be the function field of a smooth projective curve over K and F/F_0 a separable quadratic extension. Let A be a central simple algebra over F of degree n with an* *involution σ of second kind and $F^\sigma = F_0$. Suppose that $2n$ is coprime to* char(κ). *Let $\mathcal{X}_0 \to$* Spec(T) *be a proper normal model of F_0 with special fibre X_0. Let $\mathcal{P}_0$ be a finite set of closed points of X_0 containing all the singular points of X_0 and $\mathcal{U}_0$ the set of irreducible components of $X_0 \setminus \mathcal{P}_0$. Then the canonical map*

$$H^1(F_0, \mathrm{SU}(A,\sigma)) \to \prod_{U \in \mathcal{U}_0} H^1(F_{0U}, \mathrm{SU}(A,\sigma)) \times \prod_{P \in \mathcal{P}_0} H^1(F_{0P}, \mathrm{SU}(A,\sigma))$$

has trivial kernel.

Proof. Let $\xi \in H^1(F_0, \mathrm{SU}(A,\sigma))$. Suppose that ξ maps to 0 in $H^1(F_{0x}, \mathrm{SU}(A,\sigma))$ for all $x \in \mathcal{U}_0 \cup \mathcal{P}_0$. Since $U(A,\sigma)$ is rational and connected (see [22, Lemma 1, p. 195]), by [13, Theorem 3.7], ξ maps to 0 in $H^1(F_0, U(A,\sigma))$. Hence, from the exact sequence ($\star$), there exists $\lambda \in F^{*1}$ such that λ maps to ξ in $H^1(F_0, \mathrm{SU}(A,\sigma)$. Let $\mu \in F^*$ be such that $\lambda = \mu^{-1}\sigma(\mu)$. Since ξ maps to 0 in $H^1(F_{0U}, \mathrm{SU}(A,\sigma))$, there exists $c_U \in F_{0U}$ such that $c_U\mu$ is a reduced norm from $A \otimes_{F_0} F_{0U}$ (cf. [19, p. 202]).

Then, there exists a sequence of blow-ups $\mathcal{X}'_0 \to \mathcal{X}_0$ such that $\mathcal{X}'_0$ is regular, the integral closure $\mathcal{X}'$ of $\mathcal{X}'_0$ in F is regular and the union of the special fibre of $\mathcal{X}'$, the ramification locus of A on $\mathcal{X}'$ and the support of μ on $\mathcal{X}'$ is a union of regular curves with normal crossings Corollary 11.3. Let $\mathcal{P}'_0$ be a finite set of closed points of $\mathcal{X}'_0$ containing all the singular points of the special fibre X'_0 of $\mathcal{X}'_0$ and at least one closed point lying over points of $\mathcal{P}_0$. Let $\mathcal{U}'_0$ be the set of components of $X'_0 \setminus \mathcal{P}'_0$. Then ξ maps to 0 in $H^1(F_{0x'}, \mathrm{SU}(A,\sigma))$ for all $x' \in \mathcal{P}'_0 \cup \mathcal{U}'_0$ (see [15, Section 5]). Thus, replacing $\mathcal{X}_0$ by $\mathcal{X}'_0$, we assume that the integral closure $\mathcal{X}$ of $\mathcal{X}_0$ in F is regular and the union of the special fibre of $\mathcal{X}$, the ramification locus of A on $\mathcal{X}$ and the support of μ on $\mathcal{X}$ is a union of regular curves with normal crossings.

Let η be a generic point of X_0. Then $\eta \in U_\eta$ for some $U_\eta \in \mathcal{U}$. Let $c_\eta = c_{U_\eta}$. Since $F_{0U_\eta} \subset F_{0\eta}$, we have $c_\eta \in F^*_{0\eta}$ and $c_\eta\mu$ is a reduced norm from $A \otimes_{F_0} F_{0\eta}$. Let $a_\eta \in F_{0\eta}$ be as in Lemma 12.2. Then, by Artin's approximation [4, Theorem 1.10], as in the proof of [27, Lemma 7.2], there exists a nonempty open subset V_η of U_η such that $a_\eta \in F_{0V_\eta}$ (see [14, Lemma 3.2.1]) and $a_\eta\mu$ is a reduced norm from $A \otimes_{F_0} F_{0V_\eta}$. Let $a_{V_\eta} = a_\eta \in F_{0V_\eta}$. Let $\mathcal{U}'$ be the set of these V_η's. Let $\mathcal{P}'_0$ be the complement of the union of V_η's in X_0. Then $\mathcal{U}'$ is the set of components of $X_0 \setminus \mathcal{P}'_0$.

Let $P \in \mathcal{P}'_0$. Suppose that $P \in \overline{\{\eta\}} \cap \overline{\{\eta'\}}$ for two distinct generic points η and η' of X_0. Then $P \in \mathcal{P}_0$. If $F \otimes F_{0P}$ is a field, then let $a_P \in F_{0P}$ be as in Lemma 12.2. If $F \otimes F_{0P}$ is not a field, let $a_P \in F_{0P}$ be as in Lemma 12.3. Suppose $P \in \overline{\{\eta\}}$ for some generic point η of X_0 and $P \notin \overline{\{\eta'\}}$ for all generic points η' of X_0 not equal to η. Let $a_P \in F_{0P}$ be as in Lemma 12.4.

Let (V, P) be a branch. Then $P \in \overline{V}$. By the choice of a_P and a_V, we have

$$a_V a_P^{-1} = b^n_{V,P}$$

for some $b_{V,P} \in F^*_{0V,P}$. By [15, Corollary 3.4], for every $x \in \mathcal{U}'_0 \cup \mathcal{P}'_0$, there exists $b_x \in F^*_{0x}$ such that $b_{V,P} = b_V b_P$ for all branches (V, P).

For $V \in \mathcal{U}_0$, let $a'_V = a_V b_V^{-n}$ and for $P \in \mathcal{P}'_0$, let $a'_P = a_P b_P^n$. Then, we have

$$a'_V = a'_P \in F_{0V,P}$$

for all branches (V, P). Hence, there exists $a' \in F_0$ such that $a' = a'_x \in F_{0x}$ for all $x \in \mathcal{U}'_0 \cup \mathcal{P}'_0$ (see [15, Section 3]). Since μa_x is a reduced norm from $A \otimes F_{0x}$ for all $x \in \mathcal{U}'_0 \cup \mathcal{P}'_0$ and n is the degree of A, $\mu a'_x$ is a reduced norm from $A \otimes F_{0x}$ for all $x \in \mathcal{U}'_0 \cup \mathcal{P}'_0$. In particular, by [15, Proposition 8.2], $\mu a'$ is a reduced norm from $A \otimes_{F_0} F_{0\nu}$ for all discrete valuations ν of F_0. Thus, by [27, Corollary 11.2], $\mu a'$ is a reduced norm from A. Since $\lambda = (\mu a')^{-1}\sigma(\mu a')$, we have λ is in the image of $U(A,\sigma)(F_0) \to F^{*1}$, and hence ξ is trivial. □

The following is immediate from Theorem 12.5 and [15, Corollary 5.9].

Corollary 12.6. *Let K be a local field with residue field κ. Let F_0 be a function field of a curve over K. Let F/F_0 be a quadratic extension and A a central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* char(κ). *Then the canonical map*

$$H^1(F_0, \mathrm{SU}(A,\sigma)) \to \prod_{x \in X_0} H^1(F_{0x}, \mathrm{SU}(A,\sigma))$$

has trivial kernel.

13. Local-global principle for special unitary groups: Discrete valuations

Theorem 13.1. *Let K be a local field with residue field κ. Let F_0 be a function field of a curve over K. Let F/F_0 be a quadratic extension and A a central simple algebra over F of index n with an F/F_0- involution σ. Suppose that $2n$ is coprime to* char(κ). *Then the canonical map*

$$H^1(F_0, \mathrm{SU}(A,\sigma)) \to \prod_{\nu \in \Omega_{F_0}} H^1(F_{0\nu}, \mathrm{SU}(A,\sigma))$$

has trivial kernel.

Proof. Let $\xi \in H^1(F_0, \mathrm{SU}(A,\sigma))$. Suppose that ξ maps to 0 in $H^1(F_{0\nu}, \mathrm{SU}(A,\sigma))$ for all $\nu \in \Omega_{F_0}$. By Theorem 11.6, the image of ξ in $H^1(F_0, U(A,\sigma))$ is zero. Hence, from the exact sequence ($\star$) of Section 12, there exists $\lambda \in F^{*1}$ such that λ maps to ξ in $H^1(F_0, \mathrm{SU}(A,\sigma))$. Write $\lambda = \mu^{-1}\sigma(\mu)$ for some $\mu \in F^*$.

Let $d \in F_0^*$ be such that $F = F_0(\sqrt{d})$. There exists a regular proper model $\mathcal{X}_0$ of F_0 such that the special fibre and the support of d is a union of regular curves with normal crossings. Further, the integral closure $\mathcal{X}$ of $\mathcal{X}_0$ in F has the following property: $\mathcal{X}$ is regular, the special fibre of $\mathcal{X}$, the ramification locus of (A,σ), the support of d, μ and λ is a union of regular curves with normal crossings Corollary 11.3. Let X_0 be the special fibre of $\mathcal{X}_0$.

Let $x \in X_0$ be a codimension zero point. Then x gives a discrete valuation ν_x on F_0 and $F_{0x} = F_{0\nu_x}$. Hence, ξ maps to zero in $H^1(F_{0x}, \mathrm{SU}(A,\sigma))$.

Let $P \in X_0$ be a closed point. Let A_P be the local ring at P and B_P the integral closure of A_P in F. Since B_P is regular, there is at most one irreducible curve of $\mathcal{X}$ in the support of d which passes through P. Further, there are at most two curves passing through P which are in the union of special fibre of $\mathcal{X}'$, the support of μ and ramification locus of A. Let x be one such curve and ν_x the discrete valuation of F_0 given by x. Then $F_{0\nu_x} \subset F_{0P,\nu_x}$, and hence ξ maps to 0 in $H^1(F_{0P,\nu_x}, \mathrm{SU}(A,\sigma))$.

Since λ maps to ξ, there exists $\theta_x \in F_{0P,\nu_x}$ such that $\mu\theta_x \in \mathrm{Nrd}(A \otimes F_{0P,\nu_x})$. Hence, by Lemma 8.1, there exists $\theta_P \in F_{0P}$ such that $\mu\theta_P \in \mathrm{Nrd}(A \otimes F_{0P})$. In particular, $\xi \otimes F_{0P}$ is trivial. Hence, by Corollary 12.6, ξ is trivial. $\square$

14. Conjectures 1 and 2 for classical groups

In this section, we prove the validity of Conjecture 1 and Conjecture 2 for all groups of classical type in the good characteristic case. In fact we prove local-global principles for function fields of curves over any local field.

Theorem 14.1. *Let K be a local field with residue field κ and F the function field of a curve over K. Let G be a connected linear algebraic group over F of classical type (D_4 nontrialitarian) with $\mathrm{char}(\kappa)$ good for G. Let Z be a projective homogeneous space under G over F. If $Z(F_\nu) \neq \emptyset$ for all divisorial discrete valuations of F, then $Z(F) \neq \emptyset$. Thus, Conjecture 1 holds for G.*

Proof. Let G^{ss} be the semisimplification of $G/\mathrm{rad}(G)$. Since G is of classical type, there exists a central isogeny $G_1 \times \cdots \times G_r \to G^{\mathrm{ss}}$ with each G_i an almost simple simply connected group of the classical type (D_4 nontrialitarian) with $\mathrm{char}(\kappa)$ good. It is well known that using the results of [7, Theorem 2.20] and [24, Proposition 6.10], one reduces to the case $r = 1$ (cf. proof of [33, Corollary 4.6]).

Let G be a connected linear algebraic group with an isogeny $G' \to G^{\mathrm{ss}}$ for some almost simple simply connected group G' of classical type (D_4 nontrialitarian). If G' is of type ${}^1A_n$, then the result follows from Theorem 10.1. If G' is of type ${}^2A_n$, then the result follows from Theorem 11.5. If G' is of type B_n, C_n or D_n, then the result follows from [33]. $\square$

Theorem 14.2. *Let K be a local field with residue field κ and F the function field of a curve over K. Let G be a semisimple simply connected linear algebraic group over F with $\mathrm{char}(\kappa)$ is good for G. Suppose G is of the classical type (D_4 nontrialitarian). Let Z be a principal homogeneous space under G over F. If $Z(F_\nu) \neq \emptyset$ for all divisorial discrete valuations of F, then $Z(F) \neq \emptyset$. Thus, Conjecture 2 holds for G.*

Proof. For G of type B_n, C_n or D_n (D_4 nontrialitarian), this result is proved in [17, 29].

Suppose G is of type ${}^1A_n$. Then $G \simeq \mathrm{SL}(A)$ for some central simple algebra A over F and the principal homogeneous spaces under G are classified by

$$H^1(F,G) \simeq F^*/\operatorname{Nrd}(A).$$

Since $\operatorname{char}(\kappa)$ is good for G, the degree of A is coprime to $\operatorname{char}(\kappa)$. Hence, the result follows from [27, Corollary 11.2].

Suppose G is of type ${}^2A_n$. Then there exists a separable quadratic extension F/F_0 and central simple algebra over F with an F/F_0-involution σ such that $G \simeq \mathrm{SU}(A,\sigma)$. Since $\operatorname{char}(\kappa)$ is good for G, $2(\deg(A))$ is coprime to $\operatorname{char}(\kappa)$. Hence, the result follows from Theorem 13.1. □

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