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Picard modular forms and the cohomology of local systems on a Picard modular surface

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Abstract. We formulate a detailed conjectural Eichler–Shimura type formula for the cohomology of local systems on a Picard modular surface associated to the group of unitary similitudes $\mathrm{GU}(2,1,\mathbb{Q}(\sqrt{-3}))$. The formula is based on counting points over finite fields on curves of genus three which are cyclic triple covers of the projective line. Assuming the conjecture we are able to calculate traces of Hecke operators on spaces of Picard modular forms. We provide ample evidence for the conjectural formula.

Along the way we prove new results on characteristic polynomials of Frobenius acting on the first cohomology group of cyclic triple covers of any genus, dimension formulas for spaces of Picard modular forms and formulas for the numerical Euler characteristics of the local systems.

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1. Introduction

In his 1963 paper [46], Shimura listed a number of arithmetic ball quotients that are rational and that parametrize Jacobians of finite covers of the projective line. This paper deals with one of these cases and tries to use the link with the moduli of curves to study the modular forms on one of these ball quotients. The case at hand is the 2-dimensional ball quotient studied by Picard in the 1880s ([39–41]) associated to the unitary group in three variables $U(2,1)$ over the field $F=\mathbb{Q}(\sqrt{-3})$. It parametrizes curves of genus 3 that are cyclic covers of degree 3 of the projective line. Around 1979, Shintani considered vector-valued Picard modular forms on such unitary groups in three variables and gave a criterion for such modular forms to be Hecke eigenforms, see [48]. The volume [33] (see also [43]) is devoted to showing that the L-function of a Picard modular surface is the product of automorphic L-functions. But though the literature on automorphic forms on unitary groups is extensive explicit examples are rare. Holzapfel and Feustel studied the rings of scalar-valued modular forms on the group in question, [17, 26], and Finis gave in [18] a list of Hecke eigenforms of weight ≤ 12 and gave a few Hecke eigenvalues.

We decided to use the interpretation of this ball quotient as a Hurwitz space of cyclic triple covers to investigate the traces of Hecke operators using the cohomology of local systems on this moduli space. By counting points on the curves in our family over finite fields we are able to calculate the traces of Frobenius acting on the local systems associated to the cohomology of these curves. We follow the approach initiated in the papers [6, 15] dealing with Siegel modular forms of degree 2 and 3. From the traces of Frobenius on the étale cohomology of our local systems we try to calculate the traces of the Hecke operators on the spaces of vector-valued Picard modular cusp forms.

In the case of modular forms on $\mathrm{SL}(2,\mathbb{Z})$ the basic formula, essentially due to Deligne, expresses the compactly supported cohomology of a local system $\mathbb{V}_k$ on the moduli $\mathcal{A}_1$ of elliptic curves in terms of the motive $S[k+2]$ of modular forms of weight $k+2$ by an Eichler–Shimura type formula

$$e_c(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - 1.$$

The main goal of this paper is to provide a detailed conjectural analogue of this formula for Picard modular forms for F; this formula takes the form

$$e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = \check{S}[n(\lambda)] + e_{\mathrm{extr}}(\lambda),$$

where $\mathbb{W}_\lambda$ is a local system, the term $\check{S}[n(\lambda)]$ is the contribution of genuine Picard modular forms (that have 3-dimensional Galois representations) of weight $n(\lambda)$ and $e_{\mathrm{extr}}(\lambda)$ is a correction term and the analogue of 1 in the earlier formula, but rather complicated due to lifts from smaller groups.

To determine the correction term $e_{\mathrm{extr}}(\lambda)$ we need to subtract the contribution of the boundary, the so-called Eisenstein cohomology, essentially determined by Harder [21]. Since we are interested in the genuine Picard modular eigenforms, the forms that are not lifts from smaller groups and come with 3-dimensional Galois representations, we also need to subtract the so-called endoscopic terms. In the case at hand there is a multitude of endoscopic terms and by analyzing the traces that we computed we were able to make a detailed conjectural description of all the endoscopic contributions. Subtracting the Eisenstein contribution and the conjectured endoscopic terms we find heuristically the traces of the Hecke operators on the spaces of genuine Picard modular forms. These are Picard modular forms on the congruence subgroup of level $\sqrt{-3}$ and thus there is a symmetry group, equal to the symmetric group $\mathfrak{S}_4$, acting. More precisely, we get traces of Hecke operators T_ν on spaces of Picard modular forms of a given weight for primes ν in the ring of integers $\mathcal{O}_F$ with norm $N(\nu) \equiv_3 1$ (that is, $N(\nu) \equiv 1 (\mathrm{mod}\ 3)$) in a equivariant way, that is, taking into account the $\mathfrak{S}_4$-isotypic parts.

The many terms appearing in the correction term $e_{\mathrm{extr}}(\lambda)$ point to the difficulty of getting such detailed results on Picard modular forms using trace formulas, cf. [30].

In order to do this we need to be able to calculate the characteristic polynomial of Frobenius on the étale cohomology $H^1(C_f, \overline{\mathbb{Q}}_\ell)$ of a curve C_f given by $y^3 = f(x)$

over a finite field in an efficient way. More precisely, we need the characteristic polynomial of Frobenius on the part of the cohomology where the cyclic Galois automorphism α of order 3 of C_f acts by a given third root of one. The formula that we give for the characteristic polynomial for arbitrary genus generalizes a theorem of Gauss dealing with the case of genus 1.

The conjectures in this article are based upon counts of curves together with their characteristic polynomials of Frobenius for prime powers $q \equiv_3 1$ with $q \leq 67$. Using this data we can compute traces of Frobenius F_q for *any* local systems $\mathbb{W}_\lambda$ (or it is at least computationally very inexpensive). We settled for those of Deligne weight at most 40. In turn this (assuming our conjectures) gave the traces of $T(\nu)$ for $N(\nu) \leq 67$ on the corresponding spaces of Picard cusp forms.

To make such a computation over a finite field of q elements, we need roughly q operations for each curve (to compute the characteristic polynomial) and there are roughly q^2 points, i.e., curves, in our moduli space (since it is a surface). This tells us that it should be possible to make these computations for significantly larger q.

The evidence that we have for the validity of our conjectures is manifold. In this paper we calculate the dimensions of the spaces of modular forms and we calculate the numerical Euler characteristics of the cohomology of our local systems. To begin with, our procedure for calculating the traces of Hecke operators always yields zero when the dimension of cusp forms is zero. Moreover, since we started this project about ten years ago Fabien Cléry and one of us guided by these heuristic data have constructed explicitly vector-valued modular forms and the results thus obtained in [9] agree with the conjectures. Another striking piece of evidence is provided by congruences of Harder type. Harder predicts congruences modulo primes appearing in the critical values of L-series of Hecke characters and we find quite a number of examples of such congruences.

We now sketch the contents of this paper. After recalling the modular surfaces and modular forms and Hecke operators and local systems, we treat the BGG complex for our Shimura variety and use it to describe the Hodge structure on the cohomology of our local systems. We use it to describe the Eisenstein cohomology. After that we calculate the dimensions of spaces of cusp forms on our groups using Riemann–Roch and the holomorphic Lefschetz formula. We then discuss the moduli of abelian threefolds with multiplication by $\mathcal{O}_F$ and the moduli of curves of genus three with a cyclic Galois automorphism of order three, including degenerations of such curves. We give a theorem describing the characteristic polynomial of Frobenius on the étale cohomology on cyclic triple covers of the projective line. We introduce the Euler characteristics of our local systems and explain how we carried out the counts on our curves over finite fields. We then state the conjectures on the endoscopic terms. We conclude with many examples of our heuristic results on Picard modular forms and explain the evidence for the correctness of these results. In particular we list a number of congruences of Harder type.

We intend to make our results available on a website in the style of [5].

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2. Picard modular surfaces and Picard modular forms

We will first introduce the complex fibres of our spaces using their interpretation as quotients of a complex 2-ball by an arithmetic group, together with the associated modular forms.

2.1. Picard modular groups and surfaces for the Eisenstein integers. Let F be the number field $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\rho)$ with ρ a third root of unity, and with ring of integers $\mathcal{O}_F = \mathbb{Z}[\rho]$. Consider the vector space $V = F^3$ with non-degenerate hermitian form

$$h(z_1, z_2, z_3) = z_1 z_2' + z_1' z_2 + z_3 z_3',$$

where the prime refers to the Galois automorphism of F. Let G be the corresponding algebraic group defined over $\mathbb{Q}$ of similitudes of h

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}(3, F) : \forall z \in V,\ h(gz) = \eta(g)h(z) \text{ with } \eta(g) \in \mathbb{Q}\}.$$

We have that $\eta^3(g) = N_{F/\mathbb{Q}}(\det(g))$ (with $N_{F/\mathbb{Q}}$ the norm) and η defines a homomorphism $G \to \mathbb{G}_m$ called the multiplier or similitude norm. This group is also denoted by $\mathrm{GU}(2, 1, F)$ and it is called the group of unitary similitudes of signature $(2, 1)$. The group $G^0 = \ker(\eta)$, also denoted $\mathrm{U}(2, 1, F)$, is the ordinary unitary group and $G^0 \cap \ker\det$, also denoted $\mathrm{SU}(2, 1, F)$, is the special unitary group. If we do a base change to F our group G becomes isomorphic to $\mathrm{GL}(3, F) \times \mathbb{G}_m$, where the last factor corresponds to the multiplier η.

We identify the Picard modular group $G^0(\mathbb{Z})$ with

$$\{g \in \mathrm{GL}(3, \mathcal{O}_F) : h(gz) = h(z)\}$$

and we use the notation

$$\Gamma := G^0(\mathbb{Z}) \quad \text{and} \quad \Gamma_1 := G^0(\mathbb{Z}) \cap \ker\det .$$

The following congruence subgroups play a central role in this paper:

$$\Gamma[\sqrt{-3}] := \{g \in \Gamma : g \equiv 1 \ (\mathrm{mod}\ \sqrt{-3})\},$$
$$\Gamma_1[\sqrt{-3}] := \{g \in \Gamma_1 : g \equiv 1 \ (\mathrm{mod}\ \sqrt{-3})\}.$$

Note that the center of Γ_1, $\Gamma_1[\sqrt{-3}]$ and $\Gamma[\sqrt{-3}]$ equals μ_3, while the center of Γ is μ_6. There are isomorphisms (see also below)

$$\Gamma/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_6, \quad \Gamma_1/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4,$$

where the symmetric group $\mathfrak{S}_4$ occurs as the special orthogonal group of the $\mathbb{F}_3$-vector space $\mathcal{O}_F^3 \subset V$ modulo $(\sqrt{-3})$.

Choose an embedding $\sigma\colon F \to \mathbb{C}$ and identify $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$. With this identification we get a 3-dimensional complex vector space $Z = V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$, which is a hermitian space of signature $(2,1)$. We let $G(\mathbb{R})$ act on $V_{\mathbb{R}}$ as the standard representation. The set of complex lines in Z on which h is negative definite

$$B := \{U \subset Z : \dim(U) = 1,\ h_{|U} < 0\}$$

gives us a complex 2-ball inside $\mathbb{P}(Z) = \mathbb{P}^2$. If we set $u = z_3/z_2$ and $v = z_1/z_2$ we find an explicit description of this ball,

$$B = \{(u,v) \in \mathbb{C}^2 : v + \overline{v} + u\overline{u} < 0\}.$$

Any element $g = (g_{ij})$ in $G^+(\mathbb{R}) := \{g \in G(\mathbb{R}) : \eta(g) > 0\}$ now acts on B by

$$g \cdot (u,v) := \Big(\frac{g_{31}v + g_{32} + g_{33}u}{g_{21}v + g_{22} + g_{23}u}, \frac{g_{11}v + g_{12} + g_{13}u}{g_{21}v + g_{22} + g_{23}u}\Big).$$

All finite index subgroups Γ_* of Γ act properly discontinuously on B and the complex quotient surface $\Gamma_*\backslash B$, denoted by X_{Γ_*}, is called a Picard modular surface. Such a quotient is not compact, but can be compactified by adding finitely many points, called cusps, which are the orbits of the group action on the set $\partial B \cap \mathbb{P}^2(F)$ of rational points. This is called the Baily–Borel compactification and it will be denoted by $X^*_{\Gamma_*}$.

In the specific cases that we consider, these Picard modular surfaces have been studied in detail by Holzapfel and Feustel and most of the statements in this section can be found in [26, 27] (see also [17]).

The action of Γ on $\partial B \cap \mathbb{P}^2(F)$ has only one orbit since the class number of F is 1, see [50]. The group $\Gamma_1[\sqrt{-3}]$ has four cusps and the isomorphism $\Gamma/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_6$ above is given by $g \mapsto (\sigma(g), \det(g))$, where $\sigma(g)$ is the permutation of the four cusps.

The action on B of $\Gamma_1[\sqrt{-3}]$ modulo its center is not free, but has three orbits of isolated fixed points. These three points become quotient singularities of the form $\mathbb{C}^2/A$ with $A = \langle \mathrm{diag}(\rho, \rho^2)\rangle \subset \mathrm{GL}(2,\mathbb{C})$ on the surface $X_{\Gamma_1[\sqrt{-3}]}$. They can be resolved by a configuration of two non-singular rational curves with self-intersection number -2 intersecting transversally in one point.

The cusps of $X^*_{\Gamma_1[\sqrt{-3}]}$ are singular points and each cusp singularity can be resolved by an elliptic curve $E = \mathbb{C}/\sqrt{-3}\,\mathcal{O}_F$. All these elliptic curves have self-intersection number -3. We number the cusps by $i = 1,2,3,4$ and the resolution curves accordingly by E_i.

The smooth surface resulting from resolving the quotient and cusp singularities of $X^*_{\Gamma_1[\sqrt{-3}]}$ is denoted by $Y_{\Gamma_1[\sqrt{-3}]}$. Both these spaces admit an action of $\mathfrak{S}_4 \times \mu_6$.

We can define a modular curve on $Y_{\Gamma_1[\sqrt{-3}]}$ by considering the embedding of the complex upper half-plane $\mathfrak{H}$ in B by $\tau \mapsto (0, \sqrt{-3}\tau)$ with corresponding embedding of algebraic groups $\mathrm{GL}_2 \to G$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \sqrt{-3}b & 0 \\ c/\sqrt{-3} & d & 0 \\ 0 & 0 & ad-bc \end{pmatrix}.$$

This defines an algebraic curve on $X_{\Gamma_1[\sqrt{-3}]}$ isomorphic to $\Gamma_0(3)\backslash\mathfrak{H}$. Its closure in $X^*_{\Gamma_1[\sqrt{-3}]}$ passes through two cusps. Applying the action of $\mathfrak{S}_4$ we get a curve D_{ij} on $X^*_{\Gamma_1[\sqrt{-3}]}$ passing through the cusps i and j for each $1 \le i < j \le 4$.

After blow-up, we get the following configuration of curves on $Y_{\Gamma_1[\sqrt{-3}]}$.

(i) Four elliptic curves E_i $1 \le i \le 4$ with $E_i^2 = -3$.

(ii) Six rational curves D_{ij} intersecting E_i and E_j transversally.

(iii) Three pairs of rational curves R_{ij}, R_{kl} with $\{i,j,k,l\} = \{1,2,3,4\}$ resolving the quotient singularities, with $D_{ij}R_{ij} = 1$ and $D_{ij}R_{k,l} = 0$ if $\{k,l\} \neq \{i,j\}$.

The surface $X^*_{\Gamma_1[\sqrt{-3}]}$ can be identified with the 3-fold cover of the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ in $\mathbb{P}^3$ given by

$$\zeta^3 = \prod_{1 \le i < j \le 4} (x_i - x_j), \tag{2.1}$$

with the action of $(\sigma, \pm\rho) \in \mathfrak{S}_4 \times \mu_6 \cong \Gamma/\Gamma_1[\sqrt{-3}]$ by $x_i \mapsto \mathrm{sgn}(\sigma)x_{\sigma(i)}$ and $\zeta \mapsto \rho\zeta$. The four cusps correspond to the points with $\zeta = 0$ and

$$(x_1, x_2, x_3, x_4) = (1:1:1:-3),\ (1:1:-3:1),\ (1:-3:1:1),\ (-3:1:1:1),$$

and the remaining three singularities to points with $\zeta = 0$ and

$$(x_1, x_2, x_3, x_4) = (1:1:-1:-1),\ (1:-1:1:-1),\ (1:1:1:-1).$$

The curves $x_i = x_j$ give the images of the curves D_{ij}.

Taking the quotient by

$$\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mu_3$$

gives an identification of $X^*_{\Gamma[\sqrt{-3}]}$ with $\mathbb{P}^2$ as the hyperplane above in $\mathbb{P}^3$ with action by

$$\Gamma/\Gamma[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_2.$$

2.2. Picard modular forms. Fix a point $x_0 \in B$ and let K be the stabilizer of x_0 under the action of $G(\mathbb{R})$. Recall that a factor of automorphy is a mapping

$$J\colon G(\mathbb{R}) \times B \to \mathrm{GL}(W),$$

with W a complex finite-dimensional vector space, fulfilling a cocycle condition and where J restricted to $K \times x_0$ is a representation.

The action of the group $G^+(\mathbb{R})$ on B determines two factors of automorphy for $g = (g_{ij}) \in G^+(\mathbb{R})$ and $(u, v) \in B$ given by

$$j_1(g,u,v) := g_{21}v + g_{22} + g_{23}u,$$
$$j_2(g,u,v) := \det(g)^{-1} \begin{pmatrix} G_{32}u + G_{33} & G_{32}v + G_{31} \\ G_{12}u + G_{13} & G_{12}v + G_{11} \end{pmatrix},$$

with G_{ij} the minor of g_{ij}, see [47]. Note that

$$\det(j_2(g,u,v)) = j_1(g,u,v) \cdot (\det(g))^{-1}$$

and that the transpose of the Jacobian of the action of $G^+(\mathbb{R})$ on B is given by

$$j_1(g,u,v)^{-1} j_2(g,u,v)^{-1}.$$

Let (j,k) be a pair of integers with $j \geq 0$. Define a slash operator on functions $f\colon B \to \mathrm{Sym}^j(\mathbb{C}^2)$ for $g \in G^+(\mathbb{R})$ via

$$(f_{|j,k}g)(u,v) := j_1(g,u,v)^{-k}\,\mathrm{Sym}^j(j_2(g,u,v)^{-1}) f(g \cdot (u,v)).$$

For any finite index subgroup Γ_* in Γ and character χ of finite order on Γ_*, we define the vector space of modular forms of weight (j,k) and character χ on Γ_* by

$$M_{j,k}(\Gamma_*,\chi) := \big\{ f\colon B \to \mathrm{Sym}^j(\mathbb{C}^2) : f \text{ holomorphic, and } \forall g \in \Gamma_* \ f_{|j,k}g = \chi(g) f \big\}.$$

The space of cusp forms of weight (j,k) and character χ on Γ_* is the subspace of modular forms in $M_{j,k}(\Gamma_*,\chi)$ vanishing in the cusps and will be denoted by $S_{j,k}(\Gamma_*,\chi)$. Only the characters that are a power of $\det(g)$ will be considered in this paper. We will just write $M_{j,k}(\Gamma_*)$, or $S_{j,k}(\Gamma_*)$, when χ is trivial. For more details we refer to [9] and the references therein.

Alternatively, we could define another factor of automorphy

$$j_3(g,u,v) := \det(g)$$

and a new slash operator

$$(f_{|j,k,l}g)(u,v) := j_1(g,u,v)^{-k}\,\mathrm{Sym}^j(j_2(g,u,v)^{-1}) j_3(g,u,v)^{-l} f(g \cdot (u,v)),$$

with corresponding spaces of modular forms $M_{j,k,l}(\Gamma_*) = M_{j,k}(\Gamma_*, \det^l)$ and cusp forms $S_{j,k,l}(\Gamma_*) = S_{j,k}(\Gamma_*, \det^l)$.

The group

$$\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mu_3$$

(generated by $\mathrm{diag}(1,1,\rho)$) acts on $M_{j,k}(\Gamma_1[\sqrt{-3}])$. This action is reflected in a decomposition of $M_{j,k}(\Gamma_1[\sqrt{-3}])$ as a sum of spaces of modular forms with character

$$M_{j,k}\big(\Gamma[\sqrt{-3}]\big) \oplus M_{j,k}\big(\Gamma[\sqrt{-3}], \det\big) \oplus M_{j,k}\big(\Gamma[\sqrt{-3}], \det^2\big).$$

Remark 2.1. Note that $M_{j,k}(\Gamma_1[\sqrt{-3}]) = \{0\}$ if $j \not\equiv_3 k$. Moreover, we have

$$M_{j,k}\big(\Gamma[\sqrt{-3}], \det^{\ell}\big) = S_{j,k}\big(\Gamma[\sqrt{-3}], \det^{\ell}\big)$$

if $\ell \not\equiv_3 j$, see [9, Prop. 5.1].

The rings of scalar-valued modular forms on $\Gamma[\sqrt{-3}]$ and $\Gamma_1[\sqrt{-3}]$ were determined by Holzapfel and Feustel ([17, 26]).

Proposition 2.2. *We have*

$$\oplus_{k=0}^{\infty} M_{0,3k}\big(\Gamma[\sqrt{-3}]\big) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 + x_2 + x_3 + x_4),$$

where x_1, x_2, x_3, x_4 are modular forms of weight 3, and the group $\mathfrak{S}_4$ acts on these modular forms by $\sigma\colon x_i \mapsto \mathrm{sgn}(\sigma) x_{\sigma(i)}$. The ring $\oplus_{k=0}^{\infty} M_{0,3k}(\Gamma_1[\sqrt{-3}])$ is the ring extension of $\oplus_{k=0}^{\infty} M_{0,3k}(\Gamma[\sqrt{-3}])$ by the element

$$\zeta \in M_{0,6}\big(\Gamma[\sqrt{-3}], \det\big)$$

satisfying the equation (2.1) and where $\mathfrak{S}_4$ acts on ζ by the sign representation.

Since

$$X^*_{\Gamma[\sqrt{-3}]} \cong \mathrm{Proj}\big(\oplus_{k=0}^{\infty} M_{0,3k}\big(\Gamma[\sqrt{-3}]\big)\big)$$

we retrieve the identification of $X^*_{\Gamma[\sqrt{-3}]}$ with $\mathbb{P}^2$ given at the end of Section 2.1, see [26]. We point out that the x_i have F-integral Fourier–Jacobi expansions, see [9, 18].

Proposition 2.3. *If $k < 0$ and $j \geq 0$ then* $\dim M_{j,k}(\Gamma_1[\sqrt{-3}]) = 0$.

Proof. If one develops a vector-valued modular form $f \in M_{j,k}(\Gamma_1[\sqrt{-3}])$ along the modular curve given in Section 2.1 then the first component of the vector $f(0, \sqrt{-3}\tau)$ is a modular form on $\Gamma_1(3) \subset \mathrm{SL}_2(\mathbb{Z})$ of weight k, see [9, Prop. 8.4], and hence it is zero. The same thing happens for other modular curves. The image of a parametrization

$$\mathfrak{H} \to B, \quad \tau \mapsto (a, \sqrt{-3}\tau)$$

with $a \in F$ yields a modular curve. Restricting modular forms $M_{j,k}(\Gamma_1[\sqrt{-3}])$ leads to modular forms of negative weight on congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ which are zero. Since these curves lie dense, we see that the first component of $f(u,\sqrt{-3}\tau)$ is zero for all u,τ. Applying the invariance of modular forms under the unipotent radical of a parabolic subgroup (see [9, eq. (4)]) we find that all components vanish. $\square$

Remark 2.4. The proof can be easily adapted to show that the proposition holds for any finite index subgroup Γ_* of Γ.

2.3. Hecke Operators. The Hecke rings for the arithmetic group Γ_1 and $\Gamma_1[\sqrt{-3}]$ were studied by Shintani [48] and Finis [18]. Outside the prime 3 these Hecke rings are the same and they are generated by elements $T(\nu)$, $T(\nu,\nu)$ for elements $\nu \in \mathcal{O}_F$ with norm $N(\nu) = p$ for primes $p \equiv_3 1$, and elements $T(p)$ and $T(p,p)$ for primes $p \equiv_3 2$. For Γ_1 we also have elements $T(\sqrt{-3})$ and $T(\sqrt{-3},\sqrt{-3})$. In fact, $T(\nu)$ (respectively, $T(\nu,\nu)$) corresponds to the double coset of $\mathrm{diag}(1,p,\nu)$ (respectively, $\mathrm{diag}(\nu,\nu,\nu)$), while for $p \equiv_3 2$ the operator $T(p)$ (respectively, $T(p,p)$ corresponds to the double coset of $\mathrm{diag}(1,p^2,-p)$ (respectively, $\mathrm{diag}(-p,-p,-p)$). We refer to Finis' paper and to [9] for a description of the Hecke rings and the action on modular forms. Note that for a Hecke eigenform with eigenvalues λ_ν for $T(\nu)$ we have $\lambda_{\overline{\nu}} = \overline{\lambda}_\nu$.

We define for $\nu \in \mathcal{O}_F$ with norm a prime $p \equiv_3 1$ and given weight (j,k) the polynomial

$$Q_\nu^{j,k}(X,\lambda) = 1 - \lambda\, X + \nu^{j+1}\overline{\nu}^{k-2}\overline{\lambda}\, X^2 - \nu^{2j+k}\overline{\nu}^{j+2k-3}\, X^3$$

and for a prime $p \equiv_3 2$ we define $Q_{-p}^{j,k}(X,\lambda)$ by

$$\big(1 - \big(\lambda - (p-1)(-p)^{j+k-3}\big)X + p^{2j+2k-2}X^2\big)\big(1 - (-p)^{j+k-1}X\big).$$

The local factor of the L-function of a Picard modular form f of weight (j,k) that is an eigenform for the Hecke algebra with eigenvalue $\lambda_\nu(f)$ for $T(\nu)$ with

$$N(\nu) = p \equiv_3 1$$

equals the inverse of $Q_\nu^{j,k}(N(\nu)^{-s},\lambda_\nu(f))$, while for a prime $p \equiv_3 2$ the local factor is the inverse of $Q_{-p}^{j,k}(N(p)^{-s},\lambda_{-p}(f))$ with λ_{-p} the eigenvalue for $T(p)$.

2.4. Modular forms as sections of automorphic vector bundles. In this section we will realize our modular forms as sections of some vector bundles. We will use the interpretation of B as the Grassmann variety of negative lines in $Z = V_\mathbb{R}$. This interpretation provides B with two vector bundles T and S fitting in an exact sequence

$$0 \to T \to B \times Z \to S \to 0,$$

where T is the tautological line bundle that associates to a point of B the negative line it represents, and S is the tautological quotient bundle of rank 2. The tangent bundle to B is given by

$$\mathrm{Hom}(T,S)=S\otimes T^{-1},$$

so the cotangent bundle Ω^1_B equals $S^\vee\otimes T$.

Let Γ' be a freely acting finite index subgroup of Γ. Put $X=X_{\Gamma'}=\Gamma'\backslash B$. Let $X^*=X^*_{\Gamma'}$ denote the Baily–Borel compactification of $X_{\Gamma'}$ and $Y=Y_{\Gamma'}$ its minimal resolution. The cotangent bundle Ω^1_X is then equal to the quotient $\Gamma'\backslash(S^\vee\otimes T)$. Let D denote the resolution divisor on Y of the cusps of X^*. Mumford's canonical extension of Ω^1_X extends to $\Omega^1_Y(\log D)$ on Y, see [38, Prop. 3.4]. The bundle on X defined by $\Gamma'\backslash S^\vee$ will be denoted by U and the bundle $\Gamma'\backslash T$ by L. With abuse of notation their canonical extensions to Y will be denoted with the same letters.

If we choose a base point $x_0=(u_0,v_0)\in B$, then B can be identified with $G^0(\mathbb{R})/K^0$, where the maximal compact subgroup $K^0\cong U(2)\times U(1)$ is the stabilizer of x_0. Choosing instead a line in Z for which h is negative definite, then if K is the stabilizer in G of the line spanned by x_0 we find that $K\cong\mathbb{C}^*\cdot K^0$ and that $B\cong G(\mathbb{R})/K$.

For each finite-dimensional complex representation $\lambda\colon K\to\mathrm{GL}(W)$ we get an automorphic vector bundle $\mathcal{W}_\lambda$ by $G(\mathbb{R})\times_K W\to B$, where (g,w) is identified with $(gk,\lambda(k)w)$ for all $g\in G(\mathbb{R})$, $w\in W$ and $k\in K$. The group $G(\mathbb{R})$ acts naturally on the left by $g'.(g,w)=(g'g,v)$. After taking the quotient by Γ' we get an automorphic vector bundle $\mathcal{W}_\lambda$ on X. These vector bundles extend canonically to Y, and will be denoted with the same letters.

Recall from Section 2.2 that a factor of automorphy J, when restricted to $K\times x_0$, is a representation λ, and so J determines a trivialization of $\mathcal{W}_\lambda$ by

$$\Phi_J\colon G(\mathbb{R})\times_K W\to B\times W$$

with $(g,w)\mapsto(gx_0,J(g,x_0)w)$. The global sections of the bundle $\mathcal{W}_\lambda$ on Y can be identified with the modular forms transforming with the factor of automorphy J. It follows directly from our definitions that the bundle L comes with a trivialization given by the factor of automorphy j_1. As mentioned in Section 2.1, the Jacobian of the action of $G(\mathbb{R})$ of B is by $j_1^{-1}j_2^{-1}$. Taking the dual and tensoring with L^{-1} gives that U has a trivialization given by j_2. Finally, we find that the bundle $R:=\det(U)^{-1}\otimes L$ corresponds to the factor of automorphy j_3. In summary, the relations between vector bundles and factors of automorphy are as follows

$$L\leftrightarrow j_1,\quad U\leftrightarrow j_2,\quad R\leftrightarrow j_3,$$

and moreover

$$\det(U)=L\otimes R^{-1},\quad U^\vee\cong U\otimes L^{-1}\otimes R.$$

Definition 2.5. For integers j,k,l with $j\geq 0$, we put

$$\mathcal{W}_{j,k,l}:=\mathrm{Sym}^j(U)\otimes L^k\otimes R^l.$$

We then get the following interpretation of modular forms.

Proposition 2.6. *For integers j, k, l with $j \geq 0$, we have that*

$$M_{j,k,l}(\Gamma') = H^0(Y_{\Gamma'}, \mathcal{W}_{j,k,l}), \tag{2.2}$$

and

$$S_{j,k,l}(\Gamma') = H^0(Y_{\Gamma'}, \mathcal{W}_{j,k,l} \otimes \mathcal{O}(-D)). \tag{2.3}$$

Furthermore, we have that $\Omega^1_X(\log D) \cong U \otimes L$. The canonical extension of the canonical bundle Ω^2_X equals on the one hand $\Omega^2_Y(D)$ and on the other hand, by a local calculation or since the canonical extension commutes with exterior products (see [16, p. 225]), it equals $\det(\Omega^1_X(\log D))$. We thus find

$$\Omega^1_Y(\log D) \cong U \otimes L, \quad \Omega^2_Y(D) \cong L^3 \otimes R^{-1}. \tag{2.4}$$

The subgroups of Γ that we are mainly considering do not act freely on B. An automorphic vector bundle $\mathcal{W}_\lambda$ on B will become a vector bundle on X precisely if the stabilizer of any point $x \in B$ acts trivially on the fibre $(\mathcal{W}_\lambda)_x$. Let us consider the group $\Gamma_1[\sqrt{-3}]$. The center of $\Gamma_1[\sqrt{-3}]$ is generated by $\rho \cdot 1_3$, and since

$$j_1(\rho \cdot 1_3, u, v) = \rho, \quad j_2(\rho \cdot 1_3, u, v) = \rho^2 \cdot 1_2, \quad j_3(\rho \cdot 1_3, u, v) = 1,$$

a necessary condition for $\mathcal{W}_{j,k,l}$ to be vector bundle on $X_{\Gamma_1[\sqrt{-3}]}$ is that $j \equiv_3 k$. The stabilizer of one the three singular points x in B is generated by a matrix g_x with eigenvalues $1, \rho, \rho^2$ such that

$$j_1(g_x, x) = \rho, \quad j_2(g_x, x) = \mathrm{diag}(\rho, 1), \quad j_3(g_x, x) = 1,$$

so $\mathcal{W}_\lambda$ is only a vector bundle on $X_{\Gamma_1[\sqrt{-3}]}$ if $j = 0$ and $k \equiv_3 0$.

To treat the cases of non-freely acting groups we can replace the group by a freely acting finite index normal subgroup and then take invariants. By the Koecher principle, these forms extend to holomorphic sections of $\mathrm{Sym}^j(U) \otimes L^k \otimes R^l$ over the cusp resolutions. Also the quotient singularities pose no problem. Therefore, the identities of (2.2), (2.3) still hold on $\Gamma[\sqrt{-3}]$ and $\Gamma_1[\sqrt{-3}]$.

Remark 2.7. Proposition 2.3 together with (2.3) shows that

$$H^0\big(Y_{\Gamma[\sqrt{-3}]}, \mathcal{W}_{j,k,l} \otimes \mathcal{O}(-D)\big) = 0$$

for any $j, l \geq 0$ and $k < 0$. This argument is easily generalizable to other Picard modular surfaces and other arithmetic subgroups. Compare with the vanishing results of [36, 37].

3. Cohomology of complex local systems

In this section we introduce the local systems of interest to us and we use the BGG-complex to find information about the cohomology of these local systems.

3.1. Local systems and roots. A vector bundle $\mathcal{W}_\lambda$, as defined in the previous section, is a local system, i.e., locally constant, if the representation λ is a restriction of a representation of $G(\mathbb{R})$.

We have the local system $\mathbb{W}$ on X_{Γ_*} coming from the dual of the standard representation of $G(\mathbb{R})$ acting on $V_{\mathbb{R}}$, as in the beginning of Section 2.4. In terms of a factor of automorphy J, the local systems are the ones for which $J(g,x)$ are independent of x. The bundle R is thus a local system and we find that it is isomorphic to $\wedge^3\mathbb{W}$. It is constant for any Γ_* that is a subgroup of Γ_1 and since $\Gamma/\Gamma_1 \cong \mu_6$ we see that $(\wedge^3\mathbb{W})^6 \cong R^6$ is constant for subgroups of Γ.

The representation theory of G and K over the complex numbers in terms of roots and highest weights will be important for the construction of the BGG-complex which we will use in Section 3.2.

Note first that the base change of G to $\mathbb{C}$ is isomorphic to $\mathrm{GL}(3,\mathbb{C}) \times \mathbb{G}_m$, where the last factor corresponds to the multiplier η.

Let Q be a maximal parabolic subgroup and $Q = M \ltimes \mathcal{U}$ a Levi decomposition with $\mathcal{U}$ the unipotent radical of Q. The complexification of K, namely $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$, is conjugate to that of M.

Let T be the maximal torus of G of diagonal matrices $g = \mathrm{diag}(a_1,a_2,a_3)$ with $a_i \in F^*$ satisfying

$$a_1a_2' = a_1'a_2 = a_3a_3' = \eta(g) \in \mathbb{Q}^*.$$

We have the characters $L_i\colon g \mapsto a_i$. The roots are

$$\pm(L_1 - L_2), \quad \pm(L_1 - L_3), \quad \pm(L_2 - L_3).$$

We can view $\alpha = L_1 - L_2$ and $\beta = L_2 - L_3$ as two simple roots and a system of fundamental weights is $\gamma_1 = L_1$, $\gamma_2 = L_1 + L_2$ and $\gamma_3 = L_1 + L_2 + L_3$.

Then $\Phi_G^+ = \{\alpha, \beta, \alpha+\beta\}$ is a system of positive roots, occurring in the adjoint action on the unipotent radical $\mathcal{U}$ and we can take $\Phi_M^+ = \{\alpha\}$.

The Weyl group W_G of G is generated by the reflections s_α and s_β which act on the fundamental weights by

$$s_\alpha\colon \begin{cases} \gamma_1 \mapsto \gamma_2 - \gamma_1, \\ \gamma_2 \mapsto \gamma_2, \\ \gamma_3 \mapsto \gamma_3, \end{cases} \qquad s_\beta\colon \begin{cases} \gamma_1 \mapsto \gamma_1, \\ \gamma_2 \mapsto \gamma_1 - \gamma_2 + \gamma_3, \\ \gamma_3 \mapsto \gamma_3. \end{cases}$$

Then put $\theta = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ with $\theta(\alpha) = -\beta$ and $\theta(\beta) = -\alpha$, and

$$\theta(n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3) = -n_2\gamma_1 - n_1\gamma_2 + (n_1+n_2+n_3)\gamma_3.$$

The Weyl group W_M equals $\langle s_\alpha \rangle$. Define

$$W^M := \{w \in W_G : \Phi_M^+ \subset w(\Phi_G^+)\}.$$

We find that $W^M = \{1, s_\beta, s_\beta s_\alpha\}$. Put $\delta := \alpha + \beta = \gamma_1 + \gamma_2 - \gamma_3$, which is half the sum of the positive roots. We have an involution on W^M given by $w \mapsto s_\alpha w \theta$ with s_α and θ the elements of longest length in W_M and W_G.

For an element w in the Weyl group and a weight λ we define an action by

$$w * \lambda := w(\lambda + \delta) - \delta.$$

For each weight $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, we get an irreducible finite-dimensional complex representation of G with highest weight λ. The corresponding local system on X_{Γ_*} will be denoted by $\mathbb{W}_\lambda = \mathbb{W}_{n_1,n_2,n_3}$ and it can be found inside

$$\mathrm{Sym}^{n_1}(\mathbb{W}) \otimes \mathrm{Sym}^{n_2}(\wedge^2 \mathbb{W}) \otimes \mathrm{Sym}^{n_3}(\wedge^3 \mathbb{W}).$$

For the same weight λ we get an irreducible finite-dimensional complex representation of K. To identify this representation we consider the factors of automorphy for a diagonal matrix g, and we then find that j_1 corresponds to γ_1, j_3 corresponds to γ_3 and j_2 to $\gamma_2 - \gamma_3$. The vector bundle corresponding to λ is thus $\mathcal{W}_{n_2,n_1,n_2+n_3}$ in the notation of Section 2.4. Note that the cotangent bundle $U \otimes L$ has highest weight $\delta = \alpha + \beta$. A weight λ will be called regular if $n_1 > 0$ and $n_2 > 0$.

We will return to these local systems and vector bundles in terms of the moduli interpretation of our Picard modular surfaces in Section 7.

3.2. The BGG-complex. We will here apply the methods of Faltings and Chai [16, Ch. VI] to our situation, especially the theory exposed on pages 228 to 237. For a given local system one obtains a complex of vector bundles, called the dual BGG complex, with differentials that are differential operators between vector bundles. It is obtained as a direct summand of the de Rham complex of the local system.

Let Γ_* be a finite index subgroup of Γ, and let Γ' be a normal finite index subgroup of Γ_* that acts freely on B. As above, consider the surface $Y_{\Gamma'}$ that is the minimal resolution of the cusp singularities of the Baily–Borel compactification of $X_{\Gamma'} = \Gamma' \backslash B$ with resolution divisor D. Let the inclusion of $X_{\Gamma'}$ in $Y_{\Gamma'}$ be denoted by j.

The BGG-complex that the methods of [16] give for a local system $\mathbb{W}_\lambda$ on $X_{\Gamma'}$ is $K_\lambda^\bullet$ with

$$K_\lambda^q = \oplus_{w \in W^M, \ell(w)=q} \mathcal{W}_{w*\lambda}^\vee.$$

The vector bundles $\mathcal{W}_\xi$ extend canonically over the cusp resolutions and the differential operators do as well. We denote the resulting complex on $Y_{\Gamma'}$ by $\bar{K}_\lambda^\bullet$.

Proposition 3.1. *Let $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$. The dual BGG-complex*

$$\mathrm{Sym}^{n_1}(U) \otimes L^{-n_1-n_2} \otimes R^{n_1+n_2+n_3} \to \mathrm{Sym}^{n_1+n_2+1}(U) \otimes L^{-n_1+1} \otimes R^{n_1+n_2+n_3}$$
$$\to \mathrm{Sym}^{n_2}(U) \otimes L^{n_1+3} \otimes R^{n_2+n_3-1} \to 0$$

is quasi-isomorphic to $Rj_*\mathbb{W}_\lambda$. Similarly, $Rj_!\mathbb{W}_\lambda$ is quasi-isomorphic to the dual BGG-complex tensored with $\mathcal{O}(-D)$.

Proof. This follows as in [16, Prop. 5.4]. Taking the dual corresponds to applying $-\theta$ to λ and then W_M changes to $W_{M'} = \langle s_\beta \rangle$. We thus consider the triples representing $w * (-\theta(\lambda))$ for $w \in W^{M'}$. These are

$$(n_2, n_1, -n_1 - n_2 - n_3), \quad (-n_2 - 2, n_1 + n_2 + 1, -n_1 - n_2 - n_3),$$
$$(n_1 - n_2 - 3, n_2, -n_2 - n_3 + 1).$$

Taking the duals of the resulting $\mathcal{W}_\mu$ gives the result. □

For the case $n_1 = n_2 = n_3 = 0$ we get the usual logarithmic de Rham complex $\Omega^\bullet(\log D)$.

Remark 3.2. The dual of $\mathbb{W}_\lambda$ with $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ corresponds to

$$-\theta(\lambda) = n_2\gamma_1 + n_1\gamma_2 + (-n_1 - n_2 - n_3)\gamma_3.$$

The Serre dual of $\mathcal{W}_\mu = \mathrm{Sym}^a(U) \otimes L^b \otimes R^c$ is

$$\mathrm{Sym}^a(U) \otimes L^{-a-b+3} \otimes R^{a-c-1} \otimes \mathcal{O}(-D).$$

The Serre duals of the terms occurring in the BGG complex for $j_*\mathbb{W}_\lambda$ occur in reverse order in the BGG complex of $j_!\mathbb{W}_{-\theta(\lambda)}$.

Put $|\lambda| := n_1 + 2n_2 + 3n_3$. From [16, Thm. 5.5] it follows that $H^i(X_{\Gamma'}, \mathbb{W}_\lambda)$ has a Hodge structure of weight $\geq i + |\lambda|$ and the compactly supported cohomology $H^i_c(X_{\Gamma'}, \mathbb{W}_\lambda)$ has a Hodge structure of weight $\leq i + |\lambda|$. We also get the following.

Proposition 3.3. *For $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, we have a Hodge filtration on $H^i_c(X_{\Gamma'}, \mathbb{W}_\lambda)$ equal to*

$$F^{n_1+n_2+n_3+2} \subset F^{n_2+n_3+1} \subset F^{n_3}$$

and the graded pieces can be identified with

$$H^{i-2}(Y_{\Gamma'}, \mathrm{Sym}^{n_2}(U) \otimes L^{n_1} \otimes R^{n_2+n_3} \otimes \Omega^2_Y),$$
$$H^{i-1}(Y_{\Gamma'}, \mathrm{Sym}^{n_1+n_2}(U) \otimes L^{-n_1} \otimes R^{n_1+n_2+n_3} \otimes \Omega^1_Y),$$
$$H^{i}(Y_{\Gamma'}, \mathrm{Sym}^{n_1}(U) \otimes L^{-n_1-n_2} \otimes R^{n_1+n_2+n_3} \otimes \mathcal{O}(-D)).$$

For $i = 2$ we see that the first step of the Hodge filtration is isomorphic to the space of cusp forms

$$S_{n_2, n_1+3, n_2+n_3-1}(\Gamma').$$

But note that we want this space of cusp forms to have Hodge weight $n_1 + n_2 + 2$ and the discrepancy is due to a "twisting" that will be described further in Section 7.3.1.

We define the inner cohomology $H^i_!(X_{\Gamma'}, \mathbb{W}_\lambda)$ as the image under the natural map of $H^i_c(X_{\Gamma'}, \mathbb{W}_\lambda)$ in $H^i(X_{\Gamma'}, \mathbb{W}_\lambda)$. It follows that the inner cohomology will have a pure Hodge structure of weight $|\lambda| + i$. By results of Ragunathan, Li–Schwermer and Saper [35, 42, 44] for *regular* λ (that is, $n_1 > 0$, $n_2 > 0$) we know that $H^i_!(X_{\Gamma'}, \mathbb{W}_\lambda) \neq 0$ implies $i = 2$.

Note that by taking invariants under Γ_*/Γ', all results in this section hold also for Γ' replaced by Γ_*. In particular they hold for Γ_* equal to $\Gamma[\sqrt{-3}]$ or $\Gamma_1[\sqrt{-3}]$.

3.3. The neighborhood of a cusp. For a freely acting finite index normal subgroup Γ' of $\Gamma_1[\sqrt{-3}]$, each cusp of $X^*_{\Gamma'}$ is resolved by an elliptic curve and the quotient group Γ/Γ' acts transitively on the set of cusps. Therefore it suffices to look at the cusp at infinity represented by $(1:0:0) \in B$. The isotropy group Γ_∞ of this cusp in Γ consists of elements in Γ of the form

$$\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} 1 & x & -y' \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix} \tag{3.1}$$

satisfying the conditions $t_1 t_2' = t_3 t_3'$ and $x + x' = -yy'$. For the groups Γ and Γ_1 we have $x \in \sqrt{-3}\mathcal{O}_F$ and for Γ (respectively, Γ_1) we have $y \in \mathcal{O}_F$ (respectively, $y \in \sqrt{-3}\,\mathcal{O}_F$. Hence, $t_i \in \mu_6$ and it follows that $t_1 = t_2 = t_3$ and the diagonal matrix lies in the center.

Calculating the factors of automorphy for an element of Γ_∞ as in (3.1) gives

$$j_1 = t_2, \quad j_2 = \begin{pmatrix} 1/t_3 & y'/t_3 \\ 0 & 1/t_1 \end{pmatrix}, \quad j_3 = t_1 t_2 t_3.$$

In particular, for an element of Γ'_∞ as in (3.1) we find $j_1 = 1$, $j_2 = (1, y'; 0, 1)$ and $j_3 = 1$. In this case the restriction of L and R to an elliptic curve E is trivial and U restricted to E is a unipotent bundle of rank 2, cf. [3]. This result also follows from the next proposition for which Thomas Peternell kindly provided us with a proof. We identify a neighborhood of E with the total space of the normal bundle of E in $Y_{\Gamma'}$.

Proposition 3.4. *If E is an elliptic curve and N is a line bundle on E with total space X then the rank 2 bundle $V = \Omega^1_X(\log E)$ restricted to E is an extension of $\mathcal{O}_E$ by $\mathcal{O}_E$ and it is non-trivial if and only if* deg N *is non-trivial.*

Corollary 3.5. *For Γ' as above the restriction of* $\mathrm{Sym}^n(U)$ *to the elliptic curve E is isomorphic to F_{n+1}, the unique indecomposable vector bundle of rank $n+1$ with a 1-dimensional space of sections.*

3.4. Eisenstein cohomology. The Eisenstein cohomology is the contribution to the cohomology coming from the boundary. The study of Eisenstein cohomology was initiated by Harder (see [21, 22]), who used the Borel–Serre compactification and topological methods to determine the Eisenstein cohomology in a closely related case. Here we use, as in [49], coherent cohomology to determine the Eisenstein cohomology.

Let $\Gamma' \subset \Gamma_1[\sqrt{-3}]$ be a normal finite index subgroup that acts freely on B. Let $\mathbb{W}_\lambda$ be a local system on $X_{\Gamma'}$. The full Eisenstein cohomology $e_{\mathrm{Eis,f}}(X_{\Gamma'}, \mathbb{W}_\lambda)$ is defined as

$$e_c(X_{\Gamma'}, \mathbb{W}_\lambda) - e(X_{\Gamma'}, \mathbb{W}_\lambda),$$

where $e_c(X_{\Gamma'}, \mathbb{W}_\lambda)$, respectively, $e(X_{\Gamma'}, \mathbb{W}_\lambda)$, stands for the Euler characteristics

$$\sum_i (-1)^i [H^i_c(X_{\Gamma'}, \mathbb{W}_\lambda)], \quad \text{respectively, } \sum_i (-1)^i [H^i(X_{\Gamma'}, \mathbb{W}_\lambda)],$$

where the square brackets refer to classes in the Grothendieck group of mixed Hodge structures. The compactly supported Eisenstein cohomology $e_{\mathrm{Eis}}(X_{\Gamma'}, \mathbb{W}_\lambda)$ is defined as

$$\sum (-1)^i \big[\ker\big(H^i_c(X_{\Gamma'}, \mathbb{W}_\lambda) \to H^i(X_{\Gamma'}, \mathbb{W}_\lambda)\big)\big].$$

Note that there is a map $e_{\mathrm{Eis}}(X_{\Gamma'}, \mathbb{W}_\lambda) \to e_{\mathrm{Eis,f}}(X_{\Gamma'}, \mathbb{W}_\lambda)$.

Lemma 3.6. *Put $\mathcal{F}_i$, for $i = 0, 1, 2$, equal to*

$$\mathcal{W}_{n_1,-n_1-n_2,n_1+n_2+n_3}, \quad \mathcal{W}_{n_1+n_2+1,-n_1+1,n_1+n_2+n_3}, \quad \mathcal{W}_{n_2,n_1+3,n_2+n_3-1}.$$

(i) *For any $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, we have*

$$e_{\mathrm{Eis,f}}(X_{\Gamma'}, \mathbb{W}_{n_1,n_2,n_3}) = \sum_{i=0}^{2} (-1)^i \big([H^0(D, \mathcal{F}_{i|D})] - [H^1(D, \mathcal{F}_{i|D})]\big).$$

(ii) *For any $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ such that $n_1 + n_2 > 1$, we have*

$$e_{\mathrm{Eis}}(X_{\Gamma'}, \mathbb{W}_{n_1,n_2,n_3}) = -[H^0(D, \mathcal{F}_{0|D})] + [H^1(D, \mathcal{F}_{0|D})] + [H^0(D, \mathcal{F}_{1|D})].$$

Proof. Put $Y = Y_{\Gamma'}$. From [16, Thm. 5.5] it follows (compare the description of the BGG-complexes for $j_* \mathbb{W}_\lambda$ and $j_! \mathbb{W}_\lambda$ in Section 3.2) that

$$e_{\mathrm{Eis,f}}(X_{\Gamma'}, \mathbb{W}_{n_1,n_2,n_3}) = \sum_{i=0}^{2} \sum_{j=0}^{2} (-1)^{i+j} \big([H^i(Y, \mathcal{F}_j(-D))] - [H^i(Y, \mathcal{F}_j)]\big).$$

For any locally free sheaf $\mathcal{F}$ on $Y_{\Gamma'}$ we have an exact sequence

$$0 \to \mathcal{F}(-D) \to \mathcal{F} \to \mathcal{F}_{|D} \to 0.$$

Taking the long exact sequence deduced from this sequence for the three cases $\mathcal{F}_r$, with $r = 0, 1, 2$, statement (i) follows.

For (ii) we again use the BGG-complexes of Section 3.2 to conclude that

$$e_{\mathrm{Eis}}(X_{\Gamma'}, \mathbb{W}_{n_1,n_2,n_3}) = \sum_{i=0}^{2} \sum_{j=0}^{2} (-1)^{i+j} \big[\ker\big(H^i(Y, \mathcal{F}_j(-D)) \to H^i(Y, \mathcal{F}_j)\big)\big].$$

We now consider the three cases $r = 0, 1, 2$ individually. For the case $r = 0$ the vanishing of $H^0(Y, \mathcal{W}_{j,k,l})$ for negative k implies that the sequence reduces to

$$\begin{aligned} 0 &\to H^0(D, \mathcal{F}_{0|D}) \to H^1(Y, \mathcal{F}_0(-D)) \to H^1(Y, \mathcal{F}_0) \\ &\to H^1(D, \mathcal{F}_{0|D}) \to H^2(Y, \mathcal{F}_0(-D)) \to H^2(Y, \mathcal{F}_0) \to 0. \end{aligned}$$

We can identify the last three terms in the sequence with the Serre duals of

$$M_{n_1,n_2+3,-n_2-n_3-1}(\Gamma') \leftarrow S_{n_1,n_2+3,-n_2-n_3-1}(\Gamma') \leftarrow 0.$$

It follows from Corollary 3.5 that $h^0(D, \mathcal{F}_0)$ equals the number c of cusps of $X^*_{\Gamma'}$. Possibly replacing Γ' by a finite index subgroup (and in the end taking invariants) there is, since $n_1 + n_2 > 1$, an Eisenstein series for each cusp, see [9, Prop. 12.1]. Since both U and L have degree 0 when restricted to an elliptic curve in D, Riemann–Roch tells us that $h^1(D, \mathcal{F}_0) = h^0(D, \mathcal{F}_0)$. Finally, since

$$\dim M_{n_1,n_2+3,-n_2-n_3-1}(\Gamma') = c + \dim S_{n_1,n_2+3,-n_2-n_3-1}(\Gamma')$$

it follows that

$$H^1(D, \mathcal{F}_{0|D}) \to H^2(Y, \mathcal{F}_0(-D))$$

is injective.

For $\mathcal{F}_1$, we find that the sequence reduces to

$$0 \to H^0(D, \mathcal{F}_{1|D}) \to H^1(Y, \mathcal{F}_1(-D)) \to H^1(Y, \mathcal{F}_1) \to H^1(D, \mathcal{F}_{1|D}) \to 0,$$

again by the vanishing of $M_{j,k,l}(\Gamma')$ for $k < 0$. Similarly to the case $r = 0$, we can identify $H^0(D, \mathcal{F}_2)$ with a space of Eisenstein series, and hence with the cokernel of the second arrow in the exact sequence

$$\begin{aligned} 0 \to H^0(Y, \mathcal{F}_2(-D)) &\to H^0(Y, \mathcal{F}_2) \to H^0(D, \mathcal{F}_2) \\ \to H^1(Y, \mathcal{F}_2(-D)) &\to H^1(Y, \mathcal{F}_2) \to H^1(D, \mathcal{F}_2) \to 0. \end{aligned}$$

This finishes the proof of (ii). □

Corollary 3.7. *For any regular $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, we have*

$$H^1(Y_{\Gamma'}, \mathcal{F}_0) = 0.$$

Proof. As noted at the end of Section 3.2, since λ is regular, $H^1_!(X, \mathbb{W}_\lambda) = 0$. The result now follows directly from Remark 3.2 together with Proposition 3.3. □

Definition 3.8. We write $\mathbf{s}_\mu$ for the irreducible representation of $\mathfrak{S}_4$ indexed in the usual way by μ a partition of 4. We then define the $\mathfrak{S}_4$-representations,

$$\gamma_i = \begin{cases} \mathbf{s}_4 + \mathbf{s}_{3,1} & i \equiv_6 0, \\ \mathbf{s}_{1^4} + \mathbf{s}_{2,1^2} & i \equiv_6 3, \\ 0 & \text{else.} \end{cases}$$

In the following proposition $\mathbb{L}^{i,j}$ will denote the motive corresponding to a Hecke character, see further in Section 7.1; it is 1-dimensional and has Hodge degree (i, j).

Proposition 3.9. *For regular $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ with $n_1 \equiv_3 n_2$ and $n_1 \equiv_2 n_3$, put $i = n_2 + n_3$. Then $e_{\mathrm{Eis}}(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$ consists of the following three contributions:*

$$\left(-\gamma_i\, \mathbb{L}^{0,0} + \gamma_{i-n_2-1}\mathbb{L}^{n_2+1,0} + \gamma_{i+n_1+1}\mathbb{L}^{0,n_1+1}\right)\mathbb{L}^{n_3,n_3+i}.$$

As another approach to this Eisenstein cohomology is already in the literature [21, 22] we will only provide a sketch of a proof. We will normalize the weights, which corresponds to removing the factor $\mathbb{L}^{n_3,n_3+i}$ in Proposition 3.9, see also further in Section 7.3.1.

For each elliptic curve E occurring in the divisor D each of the contributions in Lemma 3.6 are either zero or 1-dimensional, see Corollary 3.5. We observe that the Hodge weights of the three contributions are 0, 0 and n_2+1. Interchanging n_1 and n_2 amounts to changing the complex structure by its complex conjugate. Hence this limits the Hodge types to $(0,0)$, $(n_2+1,0)$, $(0,n_1+1)$ and (n_2+1,n_1+1). But the last one can occur only in the dual of e_{Eis}. So we find the Hodge types

$$(0,0), \quad (n_2+1,0), \quad (0,n_1+1)$$

for the three contributions.

We now specialize to the situation $\Gamma' = \Gamma_1[\sqrt{-3}]$, where $\Gamma/\Gamma_1[\sqrt{-3}] = \mathfrak{S}_4 \times \mu_6$ acts on $Y_{\Gamma_1[\sqrt{-3}]}$ by permuting the cusps and the group $\mathfrak{S}_4 \times \mu_3$ acts effectively. The group $\langle \rho 1_3 \rangle$ sits in the center of $\Gamma[\sqrt{-3}]$ making the $\mathcal{F}_i$ orbifold bundles. We thus get contributions to the Eisenstein cohomology only if $n_1 \equiv_3 n_2$.

Since for each elliptic curve occurring in the divisor D on $Y_{\Gamma_1[\sqrt{-3}]}$ the cohomology group $H^i(E,\mathcal{F}_j)$ is at most 1-dimensional and the stabilizer of such an elliptic curve is $\mathfrak{S}_3 \times \mu_3$ it contributes either zero or a $\mathfrak{S}_4$-representation of the form

$$\mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_4} \mathbf{s}_3 = \mathbf{s}_4 + \mathbf{s}_{3,1} \quad \text{or} \quad \mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_4} \mathbf{s}_{1^3} = \mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

Furthermore, the element $\gamma = \mathrm{diag}(1,1,\rho)$ acts by multiplication by ρ on each resolution elliptic curve E. The Eisenstein cohomology for $\Gamma[\sqrt{-3}]$ corresponds to the invariant part under the action of γ.

4. The dimension of spaces of Picard modular forms

In this section we are interested in the dimensions of spaces of modular forms and cusp forms on our groups $\Gamma_1[\sqrt{-3}]$ and $\Gamma[\sqrt{-3}]$. We dwell upon this since it provides important checks on the conjectures on the cohomology of local systems.

Recall that the space $M_{j,k}(\Gamma_1[\sqrt{-3}])$ splits as

$$M_{j,k}\big(\Gamma_1[\sqrt{-3}]\big) = M_{j,k}\big(\Gamma[\sqrt{-3}]\big) \oplus M_{j,k}\big(\Gamma[\sqrt{-3}],\det\big) \oplus M_{j,k}\big(\Gamma[\sqrt{-3}],\det^2\big),$$

and similarly for the spaces of cusp forms $S_{j,k}(\Gamma_1[\sqrt{-3}])$. Recall furthermore that the graded ring $\oplus_{k=0}^{\infty} M_{0,3k}(\Gamma_1[\sqrt{-3}])$ is a degree 3 extension of

$$\oplus_{k=0}^{\infty} M_{0,3k}\big(\Gamma[\sqrt{-3}]\big) = \mathbb{C}[x_1,x_2,x_3,x_4]/(x_1+x_2+x_3+x_4)$$

generated by $\zeta \in S_6(\Gamma[\sqrt{-3}], \det)$ satisfying the relation (2.1). Moreover, $\mathfrak{S}_4$ acts by $\sigma\colon x_i \mapsto \mathrm{sgn}(\sigma)x_{\sigma(i)}$ and by the sign on ζ. We find (see [9, Sections 7 and 12] for more details) that

$$\dim M_{0,3k}\big(\Gamma_1[\sqrt{-3}]\big) = (3/2)k(k-1)+4$$

for $k \geq 2$ and $\dim M_{0,3} = 3$; moreover, for $k \geq 2$,

$$\dim S_{0,3k}\big(\Gamma[\sqrt{-3}], \det^{\ell}\big) = \begin{cases} (k^2+3k-6)/2 & \ell = 0, \\ (k^2-k)/2 & \ell = 1, \\ (k^2-5k+6)/2 & \ell = 2. \end{cases}$$

We now deduce a formula for the space of cusp forms on a freely acting finite index subgroup Γ' of $\Gamma_1[\sqrt{-3}]$. The surface $\Gamma'\backslash B$ is smoothly compactified by adding a divisor D consisting of disjoint elliptic curves.

Theorem 4.1. *For $j \geq 0$ and $k > 0$ we have*

$$\dim S_{j,3+k}(\Gamma') = \frac{1}{6}(j+1)(k+1)(j+k+2)\,\mathrm{vol}(\Gamma'\backslash B) + \frac{1}{12}(j+1)D^2$$

with $\mathrm{vol}(\Gamma'\backslash B) = c_2(\Gamma'\backslash B) = 3\,c_1(L)^2$.

Remark 4.2. Note that the formula is symmetric in j and k up to the factor $j\,D^2/12$. For a group Γ' acting on the ball with a compact quotient $\Gamma'\backslash B$ one would get a formula symmetric in j and k.

Before we give the proof we state a simple lemma.

Lemma 4.3. *With $\gamma = c_1(L)$, the Chern character satisfies*

$$\mathrm{ch}(\mathcal{W}_{a,b}) = (a+1)\big(1+(a/2+b)\gamma+(b^2/2+ab/2-a/4)\gamma^2\big).$$

Proof. Since Γ' acts freely we have that $\mathcal{W}_{a,b} = \mathrm{Sym}^a(U) \otimes L^b$ and we know that $\det(U) = L$. Let α_1 and α_2 be the Chern roots of U. Then $\alpha_1+\alpha_2 = \gamma$, and by Hirzebruch–Mumford proportionality ([38]), we have

$$c_1(\Omega^1(\log D))^2 = 3c_2(\Omega^1(\log D)),$$

hence $(3\gamma)^2 = 3(\alpha_1\alpha_2+2\gamma^2)$, so $\alpha_1+\alpha_2 = \gamma$ and $c_2(U) = \alpha_1\alpha_2 = \gamma^2$. This implies $c_1(\mathrm{Sym}^a(U)) = (1/2)a(a+1)\gamma$ and

$$\mathrm{ch}_2(\mathrm{Sym}^a(U)) = \sum_{i=0}^{a}((a-i)\alpha_1+i(\alpha_2))^2/2 = -a(a+1)\gamma^2/4.$$

The result easily follows from this. □

Proof. We work on $Y = Y_{\Gamma'}$. We denote the resolution divisor of the cusps by D. The canonical line bundle K_Y is $L^{\otimes 3} \otimes \mathcal{O}(-D)$. In view of the vanishing of h^1 and h^2 (by Corollary 3.7 and Proposition 2.3) of $\mathcal{W}_{j,k} \otimes \Omega^2$, we have that

$$\dim S_{j,k+3}(\Gamma') = h^0(Y, \mathcal{W}_{j,k} \otimes \Omega_Y^2) = \chi(Y, \mathcal{W}_{j,k} \otimes \Omega_Y^2)$$

with χ denoting the Euler characteristic. By Serre duality we have

$$\chi(Y, \mathcal{W}_{j,k} \otimes \Omega_Y^2) = \chi(Y, \mathcal{W}_{j,-j-k}).$$

We apply the Hirzebruch–Riemann–Roch formula

$$\chi(Y, \mathcal{W}_{a,b}) = (\mathrm{ch}(\mathcal{W}_{a,b}) \cdot \mathrm{td})[Y]$$

with td the Todd class $1 - 3\gamma/2 + D/2 + (c_1^2 + c_2)[Y]/12$ and $c_1(Y) = -3\gamma + D$, where again we write $\gamma = c_1(L)$ for the first Chern class of L.

Using $\chi(Y, \mathcal{O}_Y) = (c_1^2 + c_2)[Y]/12 = \gamma^2 + D^2/12$, we now find

$$\begin{aligned}\chi(Y, \mathcal{W}_{a,b}) &= (a+1)\big((b^2 + ab - 2a - 3b)\gamma^2/2 + \chi(Y, \mathcal{O}_Y)\big) \\ &= (a+1)\big((b^2 + ab - 2a - 3b + 2)\gamma^2/2 + D^2/12\big).\end{aligned}$$

Substituting $a = j$ and $b = -j - k$ gives

$$(j^2k + k^2j + j^2 + 4jk + k^2 + 3j + 3k + 2)\gamma^2/2 + (j+1)D^2/12,$$

which completes the proof. □

Remark 4.4. For our group $\Gamma_1[\sqrt{-3}]$ we have $\mathrm{vol}(\Gamma_1[\sqrt{-3}]\backslash B) = 1$ and for $\Gamma[\sqrt{-3}]$ we have $\mathrm{vol}(\Gamma[\sqrt{-3}]\backslash B) = 1/3$.

For our group $\Gamma_1[\sqrt{-3}]$ we have the following dimension formula.

Proposition 4.5. *For $j \geq 0$ and $k > 0$ with $j \equiv_3 k$, the dimension of $S_{j,k+3}(\Gamma_1[\sqrt{-3}])$ is given by*

$$\frac{1}{6}(j+1)(k+1)(j+k+2) - (j+1) + \begin{cases} 2/3 & j \equiv_3 0, \\ -2/3 & j \equiv_3 1, \\ 0 & j \equiv_3 2. \end{cases}$$

Remark 4.6. We conjecture that the formulas above also hold for $k = 0$ in all cases except when $j \equiv_6 0$ for which the dimension is $(j^2 - 3j + 6)/6$. This is based upon computations of the numerical Euler characteristic in Section 9 and Conjecture 12.12.

Proof. The subgroup $\Gamma_1[3]$ of $\Gamma_1[\sqrt{-3}]$ acts freely and the formula of Theorem 4.1 holds with $\mathrm{vol}(\Gamma_1[3]) = 81$ and D consisting of $4 \cdot 27$ elliptic curves with self-intersection number -9. The quotient group $G' = \Gamma_1[\sqrt{-3}]/\Gamma_1[3]$ is of order 3^5, but its center $\langle \rho\, \mathrm{id} \rangle = \mu_3$ acts trivially. We will apply the holomorphic Lefschetz formula to the action of the group $G = G'/\mu_3$ on $Y_{\Gamma_1[3]}$ and the vector bundle $\mathcal{W}_{j,k} \otimes \Omega^2_{Y_{\Gamma_1[3]}}$. The group G is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$. The action of G on $Y_{\Gamma[3]}$ has $3 \cdot 27$ fixed points lying over the quotient singularities of $X_{\Gamma_1[\sqrt{-3}]}$ and no other fixed points. We know that the quotient singularities are of order 3 and type $(1, 2)$. The holomorphic Lefschetz formula now says that for the action of G on the manifold $Y = Y_{\Gamma_1[3]}$ and vector bundle V, we have

$$\sum_i (-1)^i \dim H^i(Y, V)^G = (1/\#G) \sum_{g \in G} \chi(Y, V, g)$$

with

$$\chi(Y, V, g) = \mathrm{ch}(Y, V, g)\,\mathrm{td}(Y, g)[Y^g],$$

with $\mathrm{td}(Y, g)$ the Todd class associated to Y^g. At a fixed point of a non-trivial element of G the Todd class is $1/(1-\rho^2)(1-\rho) = 1/3$. The action on the fibre of L^3 over a fixed point is trivial, while on the fibre of Ω^1 it acts by (ρ, ρ^2). Hence, on the fibre of $\mathcal{W}_{j,k}$ it acts by eigenvalues $\rho^r \rho^{2(j-r)}$ for $r = 0, \ldots, j$. So one non-trivial element of G yields for one fixed point a contribution

$$c(j) = (1/3) \sum_{r=0}^{j} \rho^{2j-r} = \begin{cases} 1/3 & j \equiv_3 0, \\ -1/3 & j \equiv_3 1, \\ 0 & j \equiv_3 2. \end{cases}$$

In total the 81 fixed points are each fixed by 2 non-trivial elements, hence we get a contribution $(1/81) \cdot 81 \cdot 2 \cdot c(j)$. The identity element contributes

$$(j+1)(k+1)(j+k+2)/6 - (j+1).$$

Together this proves the formula. □

Next we want a formula for the dimension $s_{j,k+3,l}$ of the space of modular forms $S_{j,k+3}(\Gamma[\sqrt{-3}], \det^l)$. Since we have a formula for

$$\sum_{l=0}^{2} s_{j,k+3,l} = \dim S_{j,k+3}\big(\Gamma_1[\sqrt{-3}]\big),$$

it suffices to calculate the trace of a generator $\gamma = \mathrm{diag}(1, 1, \rho)$ of the group

$$\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mathbb{Z}/3\mathbb{Z}$$

on $S_{j,k+3}(\Gamma_1[\sqrt{-3}])$. Indeed, for an operator γ of order 3 acting on a complex vector space of dimension n with eigenspaces of dimension m_1, m_ρ and m_{ρ^2} for the eigenvalues $1, \rho$ and ρ^2 (with $n = m_1 + m_\rho + m_{\rho^2}$) and with $\mathrm{trace}(\gamma) = a + b\rho$ with $a, b \in \mathbb{Z}$, we have

$$m_1 = (n + 2a - b)/3, \quad m_\rho = (n - a + 2b)/3, \quad m_{\rho^2} = (n - a - b)/3.$$

Theorem 4.7. *If $j \not\equiv_3 k$, then* $\dim S_{j,k+3,l}(\Gamma[\sqrt{-3}]) = 0$. *If $j \equiv_3 k$ and $k > 0$, we have*

$$\dim S_{j,k+3,l}\big(\Gamma[\sqrt{-3}]\big) = \frac{1}{18}(j+1)(k+1)(j+k+2) + A_{j,k,l},$$

with $A_{j,k,l}$ depending on the residue class of (j,l) in $(\mathbb{Z}/3\mathbb{Z})^2$ as follows:

	$l \equiv_3 0$	$l \equiv_3 1$	$l \equiv_3 2$
$j \equiv_3 0$	$2k/3 - 10/9$	$-j/3 - 1/9$	$-2j/3 - 2k/3 + 8/9$
$j \equiv_3 1$	$-j/3 - 5/9$	$2k/3 - 14/9$	$-2j/3 - 2k/3 + 4/9$
$j \equiv_3 2$	0	0	$-j - 1$

Remark 4.8. We conjecture that the formulas above also hold for $k = 0$ in all cases except when $l = 0$ and $j \equiv_6 0$, and then $A_{j,0,0} = -1/9$. This is based upon the same evidence as in Remark 4.6.

Proof. We apply the holomorphic Lefschetz formula to the action of a representative $\gamma = \mathrm{diag}(1,1,\rho)$ of $\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mathbb{Z}/3\mathbb{Z}$ on the surface $Y = Y_{\Gamma_1[\sqrt{-3}]}$ and the orbifold vector bundle $\Omega^2_Y \otimes \mathcal{W}_{j,k}$. We assume that $j \equiv_3 k$, otherwise $M_{j,k}(\Gamma_1[\sqrt{-3}])$ is zero. We can write

$$\Omega^2_Y \otimes \mathcal{W}_{j,k} = \Omega^2_Y \otimes \mathrm{Sym}^j(U \otimes L) \otimes L^{k-j},$$

where the center μ_3 acts on all three factors trivially. The fixed point locus of γ on the surface $Y_{\Gamma_1[\sqrt{-3}]}$ consists of the six curves D_{ij} (see Section 2) and the three intersection points of the two resolution curves of the three quotient singularities on $X_{\Gamma_1[\sqrt{-3}]}$. Each of the D_{ij} is a smooth rational curve which is an exceptional curve, the restriction of Ω^1_Y to D_{ij} is $\mathcal{O}(2) \oplus \mathcal{O}(-1)$, with the first factor the cotangent bundle to D_{ij} and the second the conormal bundle. And since the resolution divisor D of the cusps intersects D_{ij} transversally at two points, we find

$$\Omega^1(\log D)_{|D_{ij}} = \mathcal{O} \oplus \mathcal{O}(1).$$

The action of γ preserves the two factors. It acts trivially on the cotangent bundle of D_{ij} and by ρ^2 on the conormal bundle since one D_{ij} is given by $u = 0$ and γ acts by $(u,v) \mapsto (\rho u, v)$. Since $\Omega^1(\log D) \cong U \otimes L$, we get

$$U \otimes L_{|D_{ij}} = \mathcal{O} \oplus \mathcal{O}(1)$$

with γ acting by 1 on the first factor and by ρ^2 on the second one.

We need the Todd class along D_{ij}; if we write the first Chern class of $\mathcal{O}_{D_{ij}}(1)$ as P, the class of a point, then we find

$$\mathrm{td}(D_{ij},\gamma) = \frac{2P}{1-e^{-2P}} \frac{1}{1-\rho^2 e^P},$$

where the first factor comes from the tangent bundle and the second from the normal bundle resulting in

$$\mathrm{td}(D_{ij},\gamma)=\frac{2+\rho}{3}-\frac{1+\rho}{3}P.$$

We have $L^3_{|D_{ij}}=\mathcal{O}(1)$ with γ acting trivially. We need $\mathrm{ch}(\Omega^2_Y\otimes \mathcal{W}_{j,k}|D_{ij})$ in the cohomology of D_{ij} tensored with the representation ring of $\mu_3=\langle\gamma\rangle$. We thus write

$$\mathrm{ch}\big(\mathrm{Sym}^j(U\otimes L)_{|D_{ij}}\big)=r(j)+d(j)P,$$

where P denotes the cohomology class of a point on D_{ij} and we interpret $r(j)$ and $d(j)$ as elements of $\mathbb{Z}[\rho]$. Furthermore, $\mathrm{ch}(\Omega^2_Y|D_{ij})=\rho^2(1-P)$. From the description just given, we obtain

$$\mathrm{ch}(\Omega^2\otimes\mathcal{W}_{j,k}|D_{ij})=(r_j+d_jP)(\rho^2-\rho^2P)\Big(1+\Big(\frac{k-j}{3}\Big)P\Big).$$

We have $r_j=\sum_{a=0}^{j}\rho^{2a}$ and $d_j=\sum_{a=0}^{j}\rho^{2a}a$, and we thus find

$$r(j)+d(j)P=\begin{cases}1+\frac{j}{3}(2+\rho)\,P & j\equiv_3 0,\\ -\rho+\big(\frac{j+2}{3}(-1-2\rho)+\rho\big)\,P & j\equiv_3 1,\\ 0+\frac{j+1}{3}(\rho-1)\,P & j\equiv_3 2.\end{cases}$$

The contribution of the six D_{ij} is the coefficient of P in

$$6\,\rho^2(r(j)+d(j)P)\Big(1+\Big(\frac{k-j-3}{3}\Big)P\Big)\Big(\frac{2+\rho}{3}-\frac{1+\rho}{3}P\Big)$$

and this is

$$\begin{cases}(-2k+2j+6)/3+(-4k-2j+6)\rho/3 & j\equiv_3 0,\\ (-4k-2j+6)/3+(-2k+2j+6)\rho/3 & j\equiv_3 1,\\ 2(j+1)(1+\rho) & j\equiv_3 2.\end{cases}$$

By adding to these three cases $-(j+1)$ (respectively, $-(j+1)\rho$ and $(j+1)\rho^2$) as the contribution of the three isolated fixed points of γ one finds for the trace

$$(2k+j-3)(-1-2\rho)/3,\quad (2k+j-3)(-2-\rho)/3,\quad -(j+1)\rho^2.$$

This gives the desired traces. □

We end this section with a definition.

Definition 4.9. The space $S_{j,k,l}(\Gamma[\sqrt{-3}])$ is a representation of $\mathfrak{S}_4$ and we denote by $S_{j,k,l}(\Gamma[\sqrt{-3}])^{\mu}$ the isotypic component corresponding to the irreducible representation indexed by μ, a partition of 4. We then put

$$\dim_{\mathfrak{S}_4} S_{j,k,l}\big(\Gamma[\sqrt{-3}]\big):=\sum_{\mu\vdash 4}\frac{\dim S_{j,k,l}(\Gamma[\sqrt{-3}])^{\mu}}{\dim \mathbf{s}_{\mu}}\,\mathbf{s}_{\mu}.$$

In Section 10, we define a (conjectural) subset

$$S^{\mathrm{gen}}_{j,k,l}\big(\Gamma[\sqrt{-3}]\big) \subset S_{j,k,l}\big(\Gamma[\sqrt{-3}]\big)$$

of so-called genuine forms and we define $\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{j,k,l}(\Gamma[\sqrt{-3}])$ analogously to the above.

5. Moduli spaces of abelian threefolds with ρ-action

In this part of the article we will study our spaces using their interpretation as moduli spaces of abelian threefolds with an action by ρ. This will give us a Deligne–Mumford stack defined over $\mathbb{Z}[\rho, 1/3]$, which enables us to find cohomological information through its finite fibres. Our goal is the ℓ-adic Euler characteristics of the local systems on our moduli spaces as motives, or more specifically as representations of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.

5.1. Picard modular stacks. For any scheme S defined over $\mathcal{O}_F[1/3]$ consider the groupoid whose objects are tuples (A, λ, ι), where A is an abelian scheme of relative dimension 3 over S,

$$\lambda\colon A \to A^\vee$$

is a principal polarization of A, and

$$\iota\colon \mathcal{O}_F \to \mathrm{End}_S(A)$$

is a homomorphism such that the Rosati involution associated to λ acts by complex conjugation on $\iota(\mathcal{O}_F)$ and that gives $\Omega^1_{A/S}$ a structure of $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{O}_F$-module of signature $(2,1)$. Isomorphisms between (A, λ, ι) and (A', λ', ι') are given by isomorphisms $f\colon A \to A'$ such that

$$\lambda = f^\vee \circ \lambda' \circ f \quad \text{and} \quad f \circ \iota(a) = \iota'(a) \circ f$$

for all $a \in \mathcal{O}_F$. This moduli problem is represented by a Deligne–Mumford stack $\mathcal{X}'_\Gamma$ of relative dimension 2 that is separated, smooth, connected and of finite type over $\mathcal{O}_F[1/3]$, see [32, Cor. 1.4.1.12] and [34]. The Picard modular surface X_Γ is equal to the complex fibre $\mathcal{X}'_\Gamma(\mathbb{C})$.

Notation 5.1. Put $\alpha := \iota(\rho)$.

To our moduli problem we now add the principal level-structure with respect to the endomorphism $1-\alpha$. Consider tuples $(A, \lambda, \iota, \sigma)$, where

$$\sigma\colon (\mathcal{O}_F^3/(1-\alpha)\mathcal{O}_F^3)_S \to A[1-\alpha]$$

is an $\mathcal{O}_F$-equivariant isomorphism, see [32, Def. 1.3.6.1]. This is represented by a Deligne–Mumford stack $\mathcal{X}'_{\Gamma[\sqrt{-3}]}$ with the same properties as above and with $X_{\Gamma[\sqrt{-3}]}$

as complex fibre. Note that $\mathcal{X}'_{\Gamma[\sqrt{-3}]}$ comes with an action of the finite group $\Gamma[\sqrt{-3}]/\Gamma$.

More generally, for any open compact subgroup of $G(\widehat{\mathbb{Z}})$ we get a level-structure that we can add to our moduli problem and get a stack with the same properties as above, see [32]. The open compact subgroup

$$K_\Gamma := \{g \in G(\widehat{\mathbb{Z}}) : g \cdot \mathcal{O}_F^3 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \mathcal{O}_F^3 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}\},$$

corresponds to $\mathcal{X}'_\Gamma$ and if we replace G by $G^0 \cap \ker \det$ in the definition we get $\mathcal{X}'_{\Gamma_1}$. From the subgroup

$$K_{\Gamma[\sqrt{-3}]} := \{g \in G(\widehat{\mathbb{Z}}) : (g-1) \cdot \mathcal{O}_F^3 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subset \sqrt{-3} \cdot \mathcal{O}_F^3 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}\},$$

we get $\mathcal{X}'_{\Gamma[\sqrt{-3}]}$, and replacing G by $G^0 \cap \ker \det$ we get $\mathcal{X}'_{\Gamma_1[\sqrt{-3}]}$, with the corresponding Picard modular surfaces as complex fibres.

5.2. Shimura varieties and complex tori. We briefly revisit our Picard modular surfaces as Shimura varieties and as moduli spaces of complex tori. For any compact open subgroup K in $G(\mathbb{A}_f)$ put

$$S_K(G, B) := G(\mathbb{Q}) \backslash B \times G(\mathbb{A}_f)/K.$$

This is a Shimura variety defined over $\mathbb{C}$. Taking K equal to any of the compact open subgroups of the previous section we get a connected Shimura variety isomorphic to the corresponding Picard modular surface, see [20].

An abelian variety A in $\mathcal{X}'_\Gamma(\mathbb{C})$ is a complex torus V/Γ_* with Γ_* an $\mathcal{O}_F$-module of rank 3. Since the class number is 1, the lattice Γ_* is isomorphic to $\mathcal{O}_F^3$. The polarization of A gives rise to an alternating form E on the underlying real vector space of V, which satisfies $E(Jx, Jy) = E(x, y)$ with J the complex structure on V. This gives that $E(\alpha x, y) = E(x, \alpha' y)$ for all $\alpha \in \mathcal{O}_F$, and where $z \mapsto z'$ denotes the Galois automorphism of $F/\mathbf{Q}$. The corresponding hermitian form h may be normalized (see [46]) so that

$$h(z_1, z_2, z_3) = z_1 z_2' + z_1' z_2 + z_3 z_3'.$$

5.3. Moduli spaces of curves. The Torelli map that sends a smooth curve of genus g to its Jacobian, induces an embedding of coarse moduli spaces. For the corresponding stacks this does not hold in general due to the fact that if G is the automorphism group of a non-hyperelliptic curve C, then the automorphism group of its Jacobian equals $G \times \{-1\}$.

An abelian threefold is either geometrically indecomposable, or a product of the Jacobian of a smooth genus two curve and an elliptic curve, or an unordered product of three elliptic curves. We cut up our spaces $\mathcal{X}'_{\Gamma_*}$, for all Γ_* among the four groups Γ, Γ_1, $\Gamma[\sqrt{-3}]$ and $\Gamma_1[\sqrt{-3}]$, into three pieces $\mathcal{X}'_{1,\Gamma_*}$, $\mathcal{X}'_{2,\Gamma_*}$ and $\mathcal{X}'_{3,\Gamma_*}$ according to this distinction.

We will now shift focus to the corresponding moduli spaces of curves $\mathcal{X}_{\Gamma_*}$, defined over $\mathbb{Z}[\rho, 1/3]$. The stratification on the moduli of abelian varieties induces via Torelli a stratification on the moduli of curves, denoted by $\mathcal{X}_{1,\Gamma_*}$, $\mathcal{X}_{2,\Gamma_*}$ and $\mathcal{X}_{3,\Gamma_*}$.

The space $\mathcal{X}_\Gamma$ will then be the moduli space of curves of compact type together with an automorphism of order three, with action of type $(2, 1)$, that induces an admissible tricyclic cover of the projective line (compare the definitions in [1] or see the next section). The level structure for $\Gamma[\sqrt{-3}]$ will be described in terms of markings of points on the curves. We will not attempt an analogous description for $\Gamma_1[\sqrt{-3}]$.

Define also, in a completely analogous way, the moduli space $\mathcal{X}^{(2)}_{\Gamma_*}$ of abelian surfaces with signature $(1, 1)$ and the moduli space $\mathcal{X}^{(1)}_{\Gamma_*}$ of elliptic curves with signature $(1, 0)$.

The two spaces $\mathcal{X}_{\Gamma_*}$ and $\mathcal{X}'_{\Gamma_*}$ only differ for the open strata of geometrically indecomposable abelian threefolds, due to the difference in automorphism group mentioned above. This is intimately connected to the fact that a geometrically indecomposable abelian threefold is either the Jacobian of a smooth curve of genus three, or the (-1)-twist of the Jacobian of a (non-hyperelliptic) smooth curve of genus three.

Our main interest is the cohomology of local systems of these spaces, and in Remark 7.5 we will relate the cohomology of these two types of spaces. This relation shows that there are no new motives appearing in the cohomology of local systems on $\mathcal{X}_{\Gamma_*}$ other than the ones appearing for $\mathcal{X}'_{\Gamma_*}$. This is in sharp contrast to the situation when comparing the cohomology of local systems on the moduli space of curves $\mathcal{M}_3$ and the moduli space of principally polarized abelian varieties $\mathcal{A}_3$, see [8].

Notation 5.2. Let K denote a field, which is not of characteristic 3, containing a primitive third root of unity $\widetilde{\rho}$ that we fix.

5.4. Smooth curves of genus 3. Let C/K be a smooth curve of genus 3 and let α be an automorphism of C of order 3 of type $(2, 1)$, which means that it will have eigenvalues $(\widetilde{\rho}, \widetilde{\rho}, \widetilde{\rho}^2)$ when acting on the 3-dimensional vector space $H^0(C, \Omega^1_C)$.

Since there are no invariant differentials of α we see that α induces a cyclic triple cover of $\mathbb{P}^1 \cong C/\alpha$. The Riemann–Hurwitz formula tells us that there are five ramification points, which is the same as fixed points of α. Let $c_{\widetilde{\rho}^i}$ be the number of ramification points x such that the action of α on $\Omega^1_{C,x}$ is by multiplication by $\widetilde{\rho}^i$. We then have that $c_{\widetilde{\rho}^2} = 5 - c_{\widetilde{\rho}}$. The Woods–Hole formula or the holomorphic Lefschetz fixed point formula, together with the isomorphism between $H^1(C, \mathcal{O}_C)$ and the dual of $H^0(C, \Omega^1_C)$, tells us that

$$\sum_{i=0}^{1}(-1)^i \operatorname{Tr}\big(\alpha, H^i(C, \mathcal{O}_C)\big) = 1 - (\widetilde{\rho} + 2\widetilde{\rho}^2) = \frac{c_{\widetilde{\rho}}}{1-\widetilde{\rho}} + \frac{c_{\widetilde{\rho}^2}}{1-\widetilde{\rho}^2}$$

giving $c_{\widetilde{\rho}} = 4$.

Elementary Galois theory tells us that a cyclic triple cover of $\mathbb{P}^1$, with coordinate x, can be given by an equation $y^3 = f(x)$ with cover given by $(x, y) \mapsto x$, together with the automorphism $\alpha: (x, y) \mapsto (x, \widetilde{\rho} y)$, and where $f(x)$ does not contain any irreducible factor to the power larger than two. The space $H^0(C, \Omega^1_C)$ of regular differentials of C is generated by

$$\{dx/y, dx/y^2, x dx/y^2\}$$

and the eigenvalues of α are thus of the right form, $(\widetilde{\rho}^2, \widetilde{\rho}, \widetilde{\rho})$ on the given basis. The ramification point for which the action of α is by $\widetilde{\rho}^2$ is necessarily defined over K and using a projective transformation we put it at infinity. The polynomial $f(x)$ should then have four distinct roots over $\bar{K}$ and since the action on the ramification is by $\widetilde{\rho}$ the polynomial should be square-free.

Remark 5.3. Doing the above in more generality, we begin with a covering $y^3 = f_1 f_2^2$, with square-free polynomials f_1, f_2. If the field is not too small we use a projective transformation to make sure that the point over infinity is not ramified, i.e., that 3 divides $\deg(f) = 2(g + 2) - \deg(f_1)$. As above, we then find that the action of α on $H^0(C, \Omega^1_C)$ is of type

$$\big((g - 1 + d)/3, (2g + 1 - d)/3\big).$$

Let $P_d(K) \subset K[x]$ be the subset consisting of polynomials of degree d with non-zero discriminant. To each $f \in P_4(K)$ we associate the cyclic triple cover (C_f, α) given by the equation $y^3 = f(x)$. The isomorphisms between pairs of the form (C_f, α) are given by

$$(x, y) \mapsto (ax + b, cy)$$

with $a, c \in K^*$ and $b \in K$. The groupoid of pairs (C_f, α) with $f \in P_4(K)$ is equivalent to the groupoid $\mathcal{X}_{1,\Gamma}(K)$.

5.4.1. Ramification and $(1 - \alpha)$-torsion. The 3-torsion group of an abelian threefold A in $\mathcal{X}'_{\Gamma[\sqrt{-3}]}(K)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^6$ over $\bar{K}$. We have that

$$3 = (1 - \alpha)(1 - \alpha^2)$$

and the $(1-\alpha)$-torsion group (which equals the $(1-\alpha^2)$-torsion group) is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$ over $\bar{K}$. This is a totally isotropic subspace of the 3-torsion group with respect to the Weil pairing and an isomorphism

$$A[1 - \alpha] \cong (\mathbb{Z}/3\mathbb{Z})^3$$

is acted upon by $\mathrm{O}(h, \mathbb{Z}/3\mathbb{Z})$, the group of orthogonal matrices with coefficients in $\mathbb{Z}/3\mathbb{Z}$ that respect the hermitian form h from Section 2.1. This group is isomorphic to $\Gamma/\Gamma[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_2$, cf. [18, p. 153].

Take a pair (C_f, α) as in the previous section and let p_1, p_2, p_3, p_4 be the ramification points (not necessarily defined over K) where α acts by $\tilde{\rho}$ and let p_0 be the point where it acts by $\tilde{\rho}^2$. The degree 0 divisors $\beta_1 = p_1 - p_0$, $\beta_2 = p_2 - p_0$ and $\beta_3 = p_3 - p_0$ are points on the Jacobian of our curve which are fixed by α, that is, they belong to $J(C_f)[1-\alpha]$.

The morphism

$$\phi\colon (\mathbb{Z}/3\mathbb{Z})^3 \to J(C_f)[1-\alpha]$$

defined by $(c_1, c_2, c_3) \mapsto c_1\beta_1 + c_2\beta_2 + c_3\beta_3$ is an isomorphism. A non-trivial element in the kernel of ϕ can easily be rearranged using

$$\mathrm{div}(y) = \sum_{i=1}^{4}(p_i - p_0)$$

and the relations $3p_i \sim 3p_0$ for all i, to a relation of the form $p_i + p_j \sim p_k + p_l$ for some i, j, k, l. This relation implies that the curve is hyperelliptic which is not possible, see Remark 5.4 below. If we order the points p_1, p_2, p_3, p_4 then we have an action of $\mathfrak{S}_4$ on $J(C_f)[1-\alpha]$. This action corresponds to the group $\mathrm{SO}(h, \mathbb{Z}/3\mathbb{Z})$. And if we add the action $\beta_i \mapsto -\beta_i$ we get the whole $\mathrm{O}(h, \mathbb{Z}/3\mathbb{Z})$.

The groupoid $\mathcal{X}_{1,\Gamma[\sqrt{-3}]}(K)$ is equivalent to the groupoid of pairs (C_f, α) together with an ordering of the four ramification points where α acts by $\tilde{\rho}$.

Remark 5.4. A smooth curve C of genus 3 which is a cyclic triple cover of $\mathbb{P}^1$ of degree 3 is not hyperelliptic. Indeed, if τ denotes the hyperelliptic involution, then τ and α commute, and α permutes the eight fixed points of τ, and vice versa. In particular, since α has an unique fixed point p_0, where it acts by $\tilde{\rho}^2$, we see that τ also has to fix p_0. Since $8 \equiv_3 2$, the action of α must have at least one more fixed point among the fixed points of τ, say p_1. Then $p_0 - p_1$ defines a point in the kernel of both the endomorphism 2 and $(1-\alpha)(1-\alpha^2) = 3$ of the Jacobian of C, hence $p_0 \sim p_1$, a contradiction.

5.5. Smooth curves of genus 2. Arguing as in Section 5.4, a curve C of genus 2 together with an automorphism α of order 3 inducing a cyclic cover of $\mathbb{P}^1$ can be given in the form

$$y^3 = f_1(x) f_2(x)^2$$

with f_1 and f_2 being relatively prime square-free polynomials for which

$$(\deg f_1, \deg f_2) = (2,2),\ (2,1),\ \text{or } (1,2),$$

and where $\alpha\colon (x, y) \mapsto (x, \tilde{\rho} y)$ has eigenvalues $(\tilde{\rho}, \tilde{\rho}^2)$ on $H^0(C, \Omega^1_C)$. Denote the curve corresponding to f_1 and f_2 by C_{f_1,f_2}. The isomorphisms between pairs of the form (C_{f_1,f_2}, α) are generated by $\mathrm{PGL}_2(K)$ acting on x together with $y \mapsto ay$ for any $a \in K^*$.

Let p_1, p_2 be the points above the roots of f_1 and q_1, q_2 be the points above the roots of f_2. They are related by $p_1 + p_2 \sim q_1 + q_2$. The divisors $\beta_1 = p_1 - p_2$ and $\beta_2 = q_1 - q_2$ give a basis of the $(1-\alpha)$-torsion of the Jacobian of C_{f_1,f_2}.

The groupoid $\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}(K)$ is equivalent to the groupoid of pairs (C_{f_1,f_2}, α) together with an ordering of the pair p_1 and p_2 and the pair q_1 and q_2 of ramification points of α. The group $(\mathbb{Z}/2\mathbb{Z})^2$ acts on this moduli space by switching p_1 and p_2, respectively, q_1 and q_2.

5.6. Elliptic curves. Let E be an elliptic curve with an automorphism α of order 3 inducing a cyclic cover of $\mathbb{P}^1$ and let the origin be a fixed point of α. Arguing again as in Section 5.4, such a curve can be given in the form

$$y^3 = f(x)^t$$

with $t = 1$ or 2, f a square-free polynomial of degree 2, and $\alpha: (x,y) \mapsto (x, \widetilde{\rho} y)$, and where the origin is placed over infinity. The action of α on $H^0(C, \Omega^1_C)$ has eigenvalue $\widetilde{\rho}^t$. For a fixed t, the isomorphisms between pairs of the form (C_{f^t}, α) are generated by $x \mapsto ax + b$ and $y \mapsto cy$ for any $a, c \in K^*$ and $b \in K$.

If K is algebraically closed then the (coarse) moduli space $\mathcal{X}^{(1)}_{\Gamma}(K)$, which corresponds to $t = 1$, consists of one point. This point can be represented by

$$f(x) = x^2 + x.$$

Let r_1, r_2 be the points above the roots of f. The divisor $r_1 - r_2$ gives a basis for the $(1-\alpha)$-torsion points. The groupoid $\mathcal{X}^{(1)}_{\Gamma[\sqrt{-3}]}(K)$ is equivalent to the groupoid of pairs (C_f, α) together with an ordering of the pair r_1 and r_2. The group $\mathbb{Z}/2\mathbb{Z}$ acts on this moduli space by switching r_1 and r_2. This is also the effect of the involution -1 on E, and so if K is an algebraically closed field then the (coarse) moduli space $\mathcal{X}^{(1)}_{\Gamma[\sqrt{-3}]}(K)$ also consists of one point.

5.7. A smooth genus 2 curve joined with an elliptic curve. A curve C in $\mathcal{X}_{2,\Gamma}(K)$ consists of a curve C_{f_1,f_2} in $\mathcal{X}^{(2)}_{\Gamma}(K)$ and C_f in $\mathcal{X}^{(1)}_{\Gamma}(K)$ joined at a ramification point of each curves. The ramification point of C_{f_1,f_2} should be above a root of f_2 and the ramification point of C_f should be above infinity (isomorphisms are induced by the ones of the individual curves that fixes these points). This leaves us, as we want, with four fixed points of α with eigenvalue $\widetilde{\rho}$ acting on the tangent space, and one with $\widetilde{\rho}^2$. It is then straightforward to see that the groupoid $\mathcal{X}_{2,\Gamma[\sqrt{-3}]}(K)$ is given by adding an ordering of the four fixed points.

5.8. Triples of elliptic curves. A curve in $\mathcal{X}_{3,\Gamma}(K)$ has three components, two curves C_{f_1}, C_{f_2} in $\mathcal{X}^{(1)}_{\Gamma}(K)$ together with a curve of the form $C_{f_3^2}$ corresponding to a curve C_{f_3} in $\mathcal{X}^{(1)}_{\Gamma}(K)$. The two curves C_{f_1}, C_{f_2} are joined to the curve $C_{f_3^2}$ at a ramification point over infinity and at a ramification point over one of the roots of f_3.

This leaves us again with the wanted four fixed points of α with eigenvalue $\widetilde{\rho}$ acting on the tangent space, and one with $\widetilde{\rho}^2$, and the groupoid $\mathcal{X}_{3,\Gamma[\sqrt{-3}]}(K)$ is given by adding an ordering of the four fixed points.

5.9. Stable admissible covers. Let us briefly discuss a compactification, of the moduli space $\mathcal{X}_{\Gamma[\sqrt{-3}]}$ using degenerations of cyclic covers.

Let $\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]}$ be the moduli space defined over $\mathbb{Z}[\rho, 1/3]$ of stable marked admissible $\mathbb{Z}/3$-covers of (stable) curves of genus 0 with action of type $(2,1)$. For the definition of admissible covers, see [25] (and compare with [1]). An element of $\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]}(K)$ is a nodal curve C over K of genus 3 with an action α of an automorphism of order 3 with C/α isomorphic to a curve P of genus 0, stably marked by the ramification points $p_0, p_1, \dots, p_4$ of the cover $C \to P$ and such that α acts by $\widetilde{\rho}$ on the tangent space of the points above $p_1, \dots, p_4$ and by $\widetilde{\rho}^2$ on the point above p_0.

There is a morphism

$$\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]} \to \overline{\mathcal{M}}_{0,1+4},$$

with $\overline{\mathcal{M}}_{0,1+4}$ the moduli space of $(1+4)$-pointed genus 0 curves.

Note that $\overline{\mathcal{M}}_{0,1+4}$ has a stratification with five strata according to the topological type of the genus 0 curve: a 2-dimensional open stratum $\mathcal{M}_{0,1+4}$, two 1-dimensional strata corresponding to a join P_1, P_2 of two $\mathbb{P}^1$'s intersecting in a point with marked points p_0, p_1, p_2 on P_1 and p_3, p_4 on P_2, or p_0, p_1 on P_1 and p_2, p_3, p_4 on P_2 and finally two strata each consisting of one point corresponding to a linear chain of three $\mathbb{P}^1$'s, P_1, P_2, P_3 with marked points p_1, p_2 on P_1, p_0 on P_2 and p_3, p_4 on P_3, or p_0, p_1 on P_1, p_2 on P_2 and p_3, p_4 on P_3 (all described up to the action of $\mathfrak{S}_4$).

This will induce a stratification of $\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]}$. The first three cases above correspond to $\mathcal{X}_{1,\Gamma[\sqrt{-3}]}$, $\mathcal{X}_{2,\Gamma[\sqrt{-3}]}$ and $\mathcal{X}_{3,\Gamma[\sqrt{-3}]}$, respectively.

The fourth strata is 1-dimensional and the curves it parametrizes consist of an elliptic curve C_1 with an order 3 automorphism α_1 with action of type $(1,0)$ and a rational curve $C_0 = \mathbb{P}^1$ with an automorphism α_0 that acts by $x \mapsto \rho x$, joined by identifying the three points of an α_1-orbit of length 3 to the points $1, \widetilde{\rho}, \widetilde{\rho}^2$ on C_0. There are four components depending upon the choice of marking of the ramification point on the rational curve.

The fifth strata is 0-dimensional and the curves it parametrizes consist of an union of an elliptic curve C_1 with an order 3 automorphism α_1 with action of type $(1,0)$ and two $\mathbb{P}^1$'s with automorphism $x \mapsto 1/(1-x)$, say C_0 and C_0', that intersect each other in 0, 1 and ∞ such that C_1 and C_0' are disjoint, while C_1 and C_0 intersect in a fixed point of α_1. This strata consists of twelve points depending upon the choice of marking of the ramification points on the two rational curves.

Let $\mathcal{X}^*_{\Gamma[\sqrt{-3}]}$ denote the Satake–Baily–Borel compactification of $\mathcal{X}'_{\Gamma[\sqrt{-3}]}$ defined over $\mathbb{Z}[\rho, 1/3]$, and so

$$\mathcal{X}^*_{\Gamma[\sqrt{-3}]}(\mathbb{C}) \cong X^*_{\Gamma[\sqrt{-3}]}.$$

Sending a curve to its (generalized) Jacobian gives a morphism

$$\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]} \to \mathcal{X}^*_{\Gamma[\sqrt{-3}]}. \tag{5.1}$$

The Jacobians over the fourth and fifth strata become extensions of an elliptic curve with the multiplicative group $\mathbb{G}_{m,F}$, and these strata will furthermore constitute a $\mathbb{P}^1$-bundle over the four cusps of $\mathcal{X}^*_{\Gamma[\sqrt{-3}]}$.

6. Characteristic polynomials of Frobenius

In this section we find properties of the characteristic polynomial of Frobenius acting on ρ-eigenspaces of the first étale cohomology group of a cyclic triple cover of the projective line.

6.1. Notation for primes, generators and finite fields. Let $k = \mathbb{F}_q$ always denote a finite field with $q = p^r$ elements with $q \equiv_3 1$. For any $n \geq 1$, let $k_n = \mathbb{F}_{q^n}$, a degree n extension of k.

If $p \equiv_3 1$, choose a third root of unity $\widetilde{\rho}$ in $\mathbb{F}_p$. This gives us a third root $\widetilde{\rho}$ in any extension field $k = \mathbb{F}_q$ for $q = p^r$. The choice of $\widetilde{\rho}$ determines a generator $a_{\mathfrak{p}_p} + b_{\mathfrak{p}_p}\rho$ of an ideal $\mathfrak{p}_p$ of norm p, namely let $a_{\mathfrak{p}_p}, b_{\mathfrak{p}_p}$ be the unique pair of integers such that

$$a_{\mathfrak{p}_p}^2 - a_{\mathfrak{p}_p}b_{\mathfrak{p}_p} + b_{\mathfrak{p}_p}^2 = p, \quad a_{\mathfrak{p}_p} \equiv_3 1, \quad b_{\mathfrak{p}_p} \equiv_3 0,$$

and such that an (hence, any) isomorphism between $\mathbb{Z}[\rho]/\mathfrak{p}_p$ and $k = \mathbb{F}_p$, sends ρ to $\widetilde{\rho}$. Define the integers $a_{\mathfrak{p}_q}, b_{\mathfrak{p}_q}$ by the equation $a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho = (a_{\mathfrak{p}_p} + b_{\mathfrak{p}_p}\rho)^r$ and define the ideal $\mathfrak{p}_q = (a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho)$.

For any $p \equiv_3 2$, choose an arbitrary third root of unity $\widetilde{\rho}$ in $\mathbb{F}_{p^2}$. For any even $r \geq 1$ choose an embedding of $\mathbb{F}_{p^2}$ in $k = \mathbb{F}_{p^r}$ and let the chosen third root of unity of $\mathbb{F}_{p^r}$ be the one coming from $\mathbb{F}_{p^2}$. Note that it is the presence of the automorphism $x \mapsto x^p$ of $\mathbb{F}_{p^2}$ that ensures that these choices do not matter for the later results for the moduli spaces, see Proposition 7.3. The ideal $\mathfrak{p}_p = (p)$ is prime in $\mathbb{Z}[\rho]$ and also here we also choose a generator $a_{\mathfrak{p}_p} + b_{\mathfrak{p}_p}\rho$ such that $a_{\mathfrak{p}_p} \equiv_3 1$, $b_{\mathfrak{p}_p} \equiv_3 0$, namely $a_{\mathfrak{p}_p} = -p$ and $b_{\mathfrak{p}_p} = 0$. Define also $a_{\mathfrak{p}_q} = (-p)^r$, $b_{\mathfrak{p}_q} = 0$ and the ideal $\mathfrak{p}_q = (a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho)$.

6.2. The characteristic polynomial. Let χ denote the third power residue symbol, that is, if $a \in k^*$ then $\chi(a) = \rho^i$ where $\widetilde{\rho}^i = a^{(q-1)/3}$, and $\chi(0) = 0$. Let C'_f be a cyclic triple cover of the projective line given by an equation of the form $y^3 = f(x)$, where f is a cube-free polynomial with coefficients in k. With $f(\infty)$ we mean the leading coefficient of f if $\deg f \equiv_3 0$, and 0 otherwise. Let g be the genus of C_f.

Put $C_f := C'_f \otimes_k \bar{k}$ and let F_q denote the geometric Frobenius morphism acting on C_f. The number of points over k of C_f equals

$$|C_f(k)| = |C_f^{F_q}| = \sum_{a \in \mathbb{P}^1(k)} \big(1 + \chi(f(a)) + \overline{\chi(f(a))}\big).$$

Let H_c^i denote compactly supported ℓ-adic étale cohomology. The Lefschetz trace formula (see [13, Th. 3.2]) tells us that

$$|C_f^{F_q}| = \sum_{i=0}^{2} (-1)^i \operatorname{Tr}\big(F_q, H_c^i(C_f, \bar{\mathbb{Q}}_\ell)\big),$$

and so

$$a_1(C_f) := \operatorname{Tr}\big(F_q, H_c^1(C_f, \bar{\mathbb{Q}}_\ell)\big) = -\sum_{a \in \mathbb{P}^1(k)} \big(\chi(f(a)) + \overline{\chi(f(a))}\big).$$

Let α be the automorphism of C_f given by $(y, x) \mapsto (\tilde{\rho} y, x)$, which commutes with Frobenius. We find that

$$|C_f^{F_q \circ \alpha^i}| = \sum_{a \in \mathbb{P}^1(k)} \big(1 + \rho^i\, \chi(f(a)) + \overline{\rho^i\, \chi(f(a))}\big).$$

The automorphism α splits $H_c^j(C_f, \bar{\mathbb{Q}}_\ell)$ into ρ^i-eigenspaces $H_c^j(C_f, \bar{\mathbb{Q}}_\ell)^{\rho^i}$. The projection formula gives

$$\sum_{j=0}^{2} (-1)^j \operatorname{Tr}\big(F_q, H_c^j(C_f, \bar{\mathbb{Q}}_\ell)^{\rho^i}\big) = \frac{1}{3} \sum_{k=0}^{2} \rho^{-ik} |C_f^{F_q \circ \alpha^k}|.$$

The 1-dimensional cohomology groups H_c^0 and H_c^2 are α-invariant. Since the quotient by α has genus 0, H_c^1 has no α-invariant part. These two things can also be deduced using the Lefschetz trace formula for α together with the fact that α has $g + 2$ fixed points (using the Hurwitz formula). It follows that

$$a_{1,\rho^i}(C_f) := \operatorname{Tr}\big(F_q, H_c^1(C_f, \bar{\mathbb{Q}}_\ell)^{\rho^i}\big) = -\sum_{a \in \mathbb{P}^1(k)} \chi(f(a))^i$$

for $i = 1, 2$.

Let $\alpha_1(C_f), \dots, \alpha_g(C_f)$ be the eigenvalues of Frobenius acting on the g-dimensional vector space $H_c^1(C_f, \bar{\mathbb{Q}}_\ell)^{\rho}$ and denote the characteristic polynomial of Frobenius by $\mathfrak{ch}_\rho(C_f)$. Let e_i denote the ith elementary symmetric polynomial in g variables. Note then that

$$e_1\big(\alpha_1(C_f), \dots, \alpha_g(C_f)\big) = a_{1,\rho}(C_f),$$

and that

$$\mathfrak{ch}_\rho(C_f) = \sum_{i=0}^{g} x^{g-i}(-1)^i e_i\big(\alpha_1(C_f), \dots, \alpha_g(C_f)\big).$$

Since $\alpha_i(C_f)\overline{\alpha}_i(C_f) = q$ for $1 \le i \le g$, we immediately get that

$$e_{g-i}q^i = e_g \overline{e}_i \quad \text{for } 1 \le i \le g. \tag{6.1}$$

6.2.1. Characteristic polynomials of elliptic curves. We will first consider the elliptic curves in some detail. Assume that $p \neq 2$. If γ is a generator of k^*, then $y^3 = x^2 + \gamma^i$ for $i = 0, \dots, 5$ are representatives of the six k-isomorphism classes of curves in $\mathcal{X}_\Gamma^{(1)}(k)$.

Let ν denote the second power residue symbol. Using Jacobi sums (see, for instance, [29, Chapter 8]), we have

$$a_{1,\rho}(C_{x^2+D}) = \nu(-4D)\chi(-4D)(a + b\rho) \tag{6.2}$$

for any $D \in k^*$. Note that $-4D$ is the discriminant of the polynomial $x^2 + D$.

Say that σ switches the two marked ramification points of a curve in $\mathcal{X}_{\Gamma[\sqrt{-3}]}^{(1)}(\overline{k})$. The fixed points of Frobenius composed with σ acting on $\mathcal{X}_\Gamma^{(1)}(\overline{k})$ are the curves $y^3 = x^2 + D$ such that $\nu(-4D) = 1$.

Assume now that $p = 2$. If γ is a generator of k^*, then the following are representatives of the six k-isomorphism classes of curves in $\mathcal{X}_\Gamma^{(1)}(k)$, $x^2 + \gamma^i x$ for $i = 0, 1, 2$ and $x^2 + \gamma^i x + \delta_i$ for $i = 0, 1, 2$, where δ_i is any element of k such that the polynomial is irreducible.

For any $a \in k^*$ and $b \in k$, define $\nu_a(b)$ to be 1 if the equation $x^2 + ax + b = 0$ has two solutions in k, and -1 otherwise. Note first that $t^2 + t + 1$ has two roots, say α and β, in k and that $t^2 + at + a^2$ has roots $a\alpha$ and $a\beta$, so in particular $\nu_a(a^2) = 1$. Using that the Jacobi sum $J(\chi, \chi)$ equals $-a - b\rho$ we get

$$a_{1,\rho}(C_{x^2+ax+b}) = -\sum_w \chi(w^2 + aw + b) = \nu_a(b)\chi(a^2)(a + b\rho). \tag{6.3}$$

Note that a^2 is the discriminant of the polynomial $x^2 + ax + b$. The fixed points of Frobenius composed with σ acting on $\mathcal{X}_\Gamma^{(1)}(\overline{k})$ as above are the curves $y^3 = x^2 + ax + b$ such that $\nu_a(b) = 1$.

Theorem 6.5 below is a generalization of the formulas (6.2) and (6.3), which go back to Gauss.

6.2.2. The characteristic polynomial modulo $1-\rho$. Let $p_1, \dots, p_{g+2}$ be the roots of $f = f_1 f_2^2$. The elements $(v_1, \dots, v_g)$ with $v_i = p_i - p_{i+1}$ form a basis of the g-dimensional $\mathbb{Z}/3\mathbb{Z}$-vector space $J(C_f)[1-\alpha]$. Using the Tate module of $J(C_f)$ we see that the action of Frobenius on $H_c^1(C_f, \overline{\mathbb{Q}}_\ell)^\rho$ modulo $(1-\rho)$ is equal to the action of Frobenius on $J(C_f)[1-\alpha]$. Let $\mathfrak{ch}_\rho(C_f)_{\rho=1} \in (\mathbb{Z}/3\mathbb{Z})[x]$ denote the polynomial $\mathfrak{ch}_\rho(C_f)$ modulo $(1-\rho)$. This polynomial is then equal to the characteristic polynomial of Frobenius acting on $J(C_f)[1-\alpha]$.

Say that the Frobenius F_q induces a permutation $\sigma \in S_{g+2}$, that has $c_i(\sigma)$ cycles of length i, on the set of points $\{p_1,\dots,p_{g+2}\}$. We will now describe the $g\times g$-matrix A_σ induced by F_q acting on the basis $(v_1,\dots,v_g)$ with $v_i = p_i - p_{i+1}$. Put $d_1 = \deg f_1$, $d_2 = \deg f_2$ and $\mathfrak{ch}(A_\sigma) = \det(xI - A)$.

Let us first handle the cases for which $c_1(\sigma)\geq 1$. In this case we can look at any σ without it having to correspond to an actual curve. By reordering, we can assume that $g+2$ is fixed by σ. If $\sigma = (1,\dots,g+1)(g+2)$, then

$$F_q(v_i) = v_{i+1}$$

for $i\leq g-1$, and

$$F_q(v_g) = -v_1 - v_2 - \cdots - v_g.$$

We find that $\mathfrak{ch}(A_\sigma) = (x^{g+1}-1)/(x-1)$. Say now that we have computed A_τ for some τ with $c_1(\tau)\geq 1$. If σ consists of a cycle $(1,\dots,h)$ followed by τ reordered such that j is replaced by $j+h$, then

$$F_q(v_i) = v_{i+1}$$

for $i\leq h-2$, and

$$F_q(v_{h-1}) = -v_1 - v_2 - \cdots - v_h \quad\text{and}\quad F_q(v_h) = v_1 + v_2 + \cdots + v_{h+1}.$$

If we define B to be the $h\times h$-matrix given by $w_i \mapsto w_{i+1}$ for $i\leq h-2$, and

$$w_{h-1} \mapsto -w_1 - w_2 - \cdots - w_h \quad\text{and}\quad v_h \mapsto w_1 + w_2 + \cdots + w_h$$

on a basis $(w_1,\dots,w_h)$, then $\mathfrak{ch}(A_\sigma) = \mathfrak{ch}(A_\tau)\mathfrak{ch}(B) = \mathfrak{ch}(A_\tau)(x^h-1)$. This describes, by induction, the structure of A_σ for any σ with $c_1(\sigma)\geq 1$.

Example 6.1. For $\sigma = (1)(23)(4567)(8)$, we have

$$A_\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}.$$

Let us now handle the cases for which $c_1(\sigma)=0$. Reorder the points $p_1,\dots,p_{g+2}$ such that the roots of f_1 come before the roots of f_2 and put $v_{g+1} = p_{g+1} - p_{g+2}$.

Note first that $3p_i \sim 3p_j$ for any i,j, and then that on the one hand

$$\operatorname{div}_0(y) = \sum_{i=1}^{d_1} p_i + \sum_{i=1+d_1}^{d_1+d_2} 2p_i,$$

and on the other

$$\mathrm{div}_\infty(y) \sim (d_1 + 2d_2)p_i$$

for any i. If $d_1 \equiv_3 0$, then this can be used to give the relation

$$\sum_{i=1}^{d_1} p_i + \sum_{i=1+d_1}^{d_1+d_2} 2p_i \sim \sum_{i=1}^{d_1/3} 3p_{3i-1} + \sum_{i=1+d_1/3}^{d_2/3} 6p_{3i-1}$$

from which it follows that

$$\sum_{i=1}^{d_1/3}(v_{3i-2} - v_{3i-1}) + \sum_{i=1+d_1/3}^{(d_1+d_2)/3} (-v_{3i-2} + v_{3i-1}) = 0.$$

Similarly, if $d_1 \equiv_3 1$, then

$$\sum_{i=1}^{(d_1-1)/3} (v_{3i-2} - v_{3i-1}) + v_{d_1} + \sum_{i=1+(d_1-1)/3}^{(d_1+d_2-2)/3} (-v_{3i} + v_{3i+1}) = 0,$$

and if $d_1 \equiv_3 2$, then

$$\sum_{i=1}^{(d_1-2)/3} (v_{3i-2} - v_{3i-1}) + v_{d_1-1} - v_{d_1} - v_{d_1+1} + \sum_{i=2+(d_1-2)/3}^{(d_1+d_2-1)/3} (-v_{3i-1} + v_{3i}) = 0.$$

We will now use the same reasoning as above. The difference is that if σ contains the cycle $(s, \dots, s+t-1)(s+t, s+t+1, \dots, g+2)$, then

$$\begin{aligned} F_q(v_g) &= v_{g+1} && \text{if } s+t \le g, \\ F_q(v_g) &= v_s + \cdots + v_{g+1} && \text{if } s+t = g+1. \end{aligned}$$

We can express v_{g+1} in terms of $v_1, \dots, v_g$ using the formulas above, but we find that only the coefficients of $v_{t-1}, \dots, v_g$ will affect $\mathfrak{ch}(A_\sigma)$. Using that σ necessarily permutes the roots of f_1 and f_2, respectively, we find that the contribution of the cycles $(s, \dots, s+t-1)(t, t+1, \dots, g+2)$ to $\mathfrak{ch}(A_\sigma)$ equals $(x^t-1)(x^{g+2-s-t}-1)/(x-1)^2$.

Example 6.2. For $\sigma_1 = (123)(456)$ and $\sigma_2 = (12)(34)(56)$, we have

$$A_{\sigma_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad A_{\sigma_2} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Summing up, we find the following.

Theorem 6.3. *Let C_f be a tricyclic cover of genus g. If Frobenius induces a permutation σ on the set of $g+2$ ramification points with $c_i(\sigma)$ cycles of length i, then we have*

$$\mathfrak{ch}_\rho(C_f)_{\rho=1} = \mathfrak{ch}(A_\sigma) = \frac{1}{(x-1)^2}\prod_{i=1}^{g+2}(x^i-1)^{c_i(\sigma)}$$

as polynomials in $(\mathbb{Z}/3\mathbb{Z})[x]$.

Remark 6.4. Computing

$$a_{n,\rho}(C_f) := -\sum_{a\in\mathbb{P}^1(k_n)} \chi(f(a)) = \sum_{i=1}^{g} \alpha_i(C_f)^n \in \mathbb{Z}[\rho]$$

modulo $1-\rho$ is more straightforward since

$$-\sum_{a\in\mathbb{P}^1(k_n)} \chi(f(a)) = -(q^n+1-r_i) = 1+r_n \bmod (1-\rho),$$

where r_n is the number of roots of f defined over k_n.

6.2.3. The determinant of Frobenius. Say that α has s eigenvalues equal to $\widetilde{\rho}$ when acting on $H^0(C,\Omega^1_C)$. It then follows from [19, Th. 1] that e_g generates the ideal $\mathfrak{p}_q^s\cdot(\overline{\mathfrak{p}}_q)^{g-s}$, and hence

$$e_g\big(\alpha_1(C_f),\dots,\alpha_g(C_f)\big) = (-1)^{j_1}\rho^{j_2}(a_{\mathfrak{p}_p}+b_{\mathfrak{p}_p}\rho)^{rs}(a_{\mathfrak{p}_p}+b_{\mathfrak{p}_p}\rho^2)^{r(g-s)} \quad (6.4)$$

for some integers j_1 and j_2.

Theorem 6.5. *For any polynomial h, let $D(h)$ denote the discriminant of h, and $\varepsilon(h)$ the number of irreducible factors (over k) of h. If we assume that $f = f_1f_2^2$ and $3\mid \deg f$, then*

$$\frac{e_g(\alpha_1(C_f),\dots,\alpha_g(C_f))}{(a_{\mathfrak{p}_p}+b_{\mathfrak{p}_p}\rho)^{rs}(a_{\mathfrak{p}_p}+b_{\mathfrak{p}_p}\rho^2)^{r(g-s)}} = (-1)^{g+\varepsilon(f_1)+\varepsilon(f_2)}\chi(D(f_1))\overline{\chi(D(f_2))}. \quad (6.5)$$

Remark 6.6. Note that if $p\neq 2$, then by Stickelberger's theorem (see [10, Thm. 1.3] or [11])

$$(-1)^{g+\varepsilon(f_1)+\varepsilon(f_2)} = \nu(D(f_1)D(f_2)) = \nu(D(f_1))\overline{\nu(D(f_2))},$$

where ν denotes the second power residue symbol.

Proof. Since, $e_g(\alpha_1(C_f),\dots,\alpha_g(C_f)) = (-1)^{j_1}$ modulo $(1-\rho)$, Theorem 6.3 tells us immediately that $j_1 = g+\varepsilon(f_1)+\varepsilon(f_2)$.

The action of Frobenius on $H^1_c(C_f, \overline{\mathbb{Q}}_\ell)^\rho$ modulo $1-\rho$ is equal to the action on $J(C_f)[1-\alpha]$. To determine j_2 it suffices to calculate the expression (6.4) modulo 3 since it equals $(-1)^{j_1}(1-\theta)^{j_2}$ in $\mathcal{O}_F/(3) \cong \mathbb{F}_3[\theta]$ with $\theta = 1-\rho \pmod 3$. That means that it suffices to calculate the determinant of Frobenius on the $\mathbb{F}_3[\theta]$-module $J(C_f)[3]$. In view of the exact sequence

$$0 \to J(C_f)[1-\alpha] \to J(C_f)[3] \xrightarrow{1-\alpha} J(C_f)[1-\alpha] \to 0$$

and the fact that $J(C_f)[1-\alpha]$ is isotropic for the Weil pairing, as kernel of an endomorphism, we see that the action of α on C_f induces a cyclic μ_3-action on the three possibilities for this determinant. (If we lift our abelian variety together with α to $\mathbb{C}$ then this action corresponds to the action of $\mathrm{diag}(1,1,\rho) \in \Gamma[\sqrt{-3}]$.) The determinant of Frobenius on $J(C_f)[3]$ is determined up to a third root of 1 by the level structure

$$J(C_f)[1-\alpha] \sim (\mathcal{O}_F/(1-\alpha))^3.$$

The moduli stack of triples (J, l, d) with J a Jacobian of a cyclic triple cover $C_f \to \mathbb{P}^1$ of signature $(s, g-s)$ with a level $(1-\alpha)$-structure l on $J(C_f)[1-\alpha]$ and the determinant d of the cohomology modulo 3, is a threefold étale cover of the moduli stack of tuples (J, l). It is étale since the ramification points of the cover $C_f \to \mathbb{P}^1$ determine the level $(1-\alpha)$-structure and α then induces the μ_3-action. This degree three cover extends to the appropriate moduli stacks (Picard modular stacks, see Section 5.1) of principally polarized abelian varieties with level structure.

In the case of $g = 3$ and the covers considered in Section 5.4 this étale cover is given by the μ_3-cover

$$\mathcal{X}_{\Gamma_1[\sqrt{-3}]} \to \mathcal{X}_{\Gamma[\sqrt{-3}]},$$

which is étale outside the locus where the discriminant of f vanishes. Indeed, we know that this cover $\mathcal{X}_{\Gamma_1[\sqrt{-3}]} \to \mathcal{X}_{\Gamma[\sqrt{-3}]}$ is the cover defined over $\mathcal{O}_F$ by equation (2.1), see also Proposition 2.2. Therefore, the action of Frobenius on the fibres of $\mathcal{X}_{\Gamma_1[\sqrt{-3}]} \to \mathcal{X}_{\Gamma[\sqrt{-3}]}$ is determined by the cubic character of the discriminant of f, hence j_2 is. The normalization of the cubic character then follows by checking that it satisfies the formula of the theorem in examples for the case of genus 3 or by checking it for abelian threefolds that are a product of elliptic curves.

In the general case the threefold cover of stacks is ramified along the codimension 1 locus where the discriminant of f vanishes. Therefore, j_2 is determined by the cubic character of the discriminant of f. Then we can specialize to the case where the Jacobian $J(C_f)$ splits as a direct sum of Jacobians of curves of lower genus to check the formula inductively starting from the cases of $g \leq 3$. $\square$

7. Euler characteristics of ℓ-adic local systems

In this section we will introduce the motivic Euler characteristics of local systems on our moduli spaces, stating basic results, showing how the Lefschetz trace formula can be used to find cohomological information and presenting a formula for the integer valued Euler characteristic of any local system.

7.1. Hecke characters. Recall the notation of Section 6.1. Define the Hecke character ψ of conductor (3) for any $\mathfrak{p}_p$ by putting

$$\psi(\mathfrak{p}_p) = a_{\mathfrak{p}} + b_{\mathfrak{p}}\rho.$$

This gives a 1-dimensional "motive" that we will denote $\mathbb{L}^{1,0}$, pure of weight 1 and Hodge type $(1,0)$ and as an ℓ-adic $\mathrm{Gal}(\bar{F}/F)$-representation, the trace of a Frobenius element $F_{\mathfrak{p}_p}$, corresponding to a prime ideal $\mathfrak{p}_p$, is given by $\psi(\mathfrak{p}_p)$.

Recall that

$$f_\psi(z) = \sum_\alpha \psi(\alpha) q^{N(\alpha)} \in \mathbb{Z}[q],$$

where $q = e^{2\pi i z}$ and the sum is over all integral ideals α prime to (3) and $N(\alpha)$ is the norm of α, is a cusp form of weight 2 and level $\Gamma_0(27)$. Moreover, we have that

$$f_\psi(z) = \eta(3z)^2\eta(9z)^2 = q - 2q^2 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} - 5q^{25} + \cdots.$$

Similarly, we define $\mathbb{L}^{0,1}$ of Hodge type $(0,1)$ by using the Hecke character

$$\bar{\psi}(\mathfrak{p}_p) := a_{\mathfrak{p}_p} + b_{\mathfrak{p}_p}\rho^2.$$

Finally, for any pair of integers n, m we define $\mathbb{L}^{n,m}$ by taking tensor products of the "motives" above. Note that $\mathbb{L}^{1,1}$ becomes the usual Lefschetz motive, also denoted $\mathbb{L}^1$.

7.2. Euler characteristics of ℓ-adic local systems. For any of our moduli spaces $\mathcal{X}$ (which are stacks) introduced in Section 5.1 and Section 5.3 we have a universal family $\pi\colon \mathcal{C} \to \mathcal{X}$ and we consider the ℓ-adic local system $\mathbb{V} := R^1\pi_*\bar{\mathbb{Q}}_\ell$. This local system has rank 6, where the fiber of a geometric point represented by an abelian variety A equals the ℓ-adic étale cohomology group $H^1(A, \bar{\mathbb{Q}}_\ell)$. It is provided with a non-degenerate alternating pairing $\mathbb{V} \times \mathbb{V} \to \bar{\mathbb{Q}}_\ell(-1)$.

The action of α gives rise to a decomposition of the base change to F of $\mathbb{V}$, as a direct sum of two local systems of rank 3 over $F \otimes \bar{\mathbb{Q}}_\ell$:

$$\mathbb{V} \otimes F = \mathbb{W} \oplus \mathbb{W}'$$

with $\mathbb{W}$ (respectively, $\mathbb{W}'$) the ρ-eigenspace (respectively, the ρ^2-eigenspace) of α.

Note that we can also take $\mathcal{X}(\mathbb{C})$ and define the (Betti) local system $\mathbb{V} := R^1\pi_*\mathbb{Q}$, and then the ρ-eigenspace $\mathbb{W}$ is the same local system as the one defined in Section 3.1.

We define local systems $\mathbb{W}_\lambda$ using the representations of $\mathrm{GL}(3,\mathbb{C}) \times \mathbb{G}_m$ as in Section 3.1. The multiplier defines the constant local system $F(-1)$.

Let $\mathbb{W}^\vee$ denote the F-linear dual. Then note that

$$(\mathbb{W}_{n_1,n_2,n_3})^\vee \cong \mathbb{W}_{n_2,n_1,-n_1-n_2-n_3}.$$

The non-degenerate pairing implies that the conjugate takes the form

$$\mathbb{W}' \cong \mathbb{W}^\vee \otimes F(-1),$$

and so

$$\mathbb{W}'_{n_1,n_2,n_3} \cong \mathbb{W}_{n_2,n_1,-n_1-n_2-n_3} \otimes F(-n_1 - 2n_2 - 3n_3). \tag{7.1}$$

Let H_c^* denote compactly supported ℓ-adic étale cohomology. The action of the symmetric group $\mathfrak{S}_4 \cong \Gamma[\sqrt{-3}]/\Gamma$ on $\mathcal{X}_{\Gamma[\sqrt{-3}]}$ induces an action on its cohomology groups. We define the Euler characteristic of the local system $\mathbb{W}_\lambda$ on $\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \overline{F}$ in $K_0(\mathrm{Gal}_F^{\mathfrak{S}_4})$, the Grothendieck group of ℓ-adic $\mathrm{Gal}(\overline{F}/F)$-representations equipped with an action of $\mathfrak{S}_4$ by

$$e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) := \sum_{i=0}^{4} (-1)^i \big[H_c^i(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{F}, \mathbb{W}_\lambda)\big].$$

Similarly, consider compactly supported Betti cohomology and define by (abuse of) the same notation

$$e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) := \sum_{i=0}^{4} (-1)^i \big[H_c^i(\mathcal{X}_{\Gamma[\sqrt{-3}]}(\mathbb{C}), \mathbb{W}_\lambda)\big]$$

in the Grothendieck group of Hodge modules equipped with an action of $\mathfrak{S}_4$. Let $e_{c,\mu}(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$ correspond to a μ-isotypic component of the Euler characteristics in the sense that

$$e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) = \sum_{\mu \vdash 4} e_{c,\mu}(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)\mathbf{s}_\mu.$$

The statements in Section 12 will be called motivic, and by this we will mean that these are statements about the Euler characteristic in both these Galois groups.

Proposition 7.1. *For all λ and all i we have the following:*

(1) $H_c^i(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) = 0$ *if* $n_1 \not\equiv_3 n_2$*;*

(2) $H_c^i(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda \otimes F(-k)) = H_c^i(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)\, \mathbb{L}^k$.

Proof. We prove (1), the proof of (2) is standard. The automorphism α acts on the fibre $(\mathbb{W}_\lambda)_A$ by $\rho^{n_1+2n_2}$ for any closed point A of $\mathcal{X}_{\Gamma[\sqrt{-3}]}$. So if $n_1 \not\equiv_3 n_2$, then this action has no invariants and hence the cohomology has to vanish. $\square$

7.3. Traces of Frobenius. Recall the notation of Section 6.1 and Section 6.2. Compare the following section to the article [6] and the references therein. We define the (geometric) Frobenius $F_q \in \mathrm{Gal}(\bar{k}/k)$ to be the inverse of $x \mapsto x^q$. We choose a Frobenius element $F_{\mathfrak{p}_q} \in \mathrm{Gal}(\bar{F}/F)$, using an element of the Galois group of the p-adic completion of F that is mapped to the Frobenius element $F_q \in \mathrm{Gal}(\bar{k}/k)$. These Frobenii satisfy

$$\mathrm{Tr}\big(F_{\mathfrak{p}_q}, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{F}, \mathbb{W}_\lambda\big)\big) = \mathrm{Tr}\big(F_q, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big), \tag{7.2}$$

and these traces are elements of $\mathbb{Z}[\rho]$. The traces of $F_{\mathfrak{p}_p}$ for (almost) all unramified primes $\mathfrak{p}_p$ will (using a Chebotarov density argument) determine

$$e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{F}, \mathbb{W}_\lambda\big)$$

as an element of $K_0(\mathrm{Gal}_F)$, cf. [12, Prop. 2.6]. For any element of $V \in K_0(\mathrm{Gal}_F)$ we can define a virtual representation $\bar{V}$ by the property $\mathrm{Tr}(F_{\mathfrak{p}_p}, \bar{V}) = \overline{\mathrm{Tr}(F_{\mathfrak{p}_p}, V)}$.

Proposition 7.2. *For any μ and $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, we have*

$$\overline{e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_{n_1,n_2,n_3}\big)} = e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_{n_2,n_1,-n_1-n_2-n_3}\big)\mathbb{L}^{-n_1-2n_2-3n_3},$$

as elements of $K_0(\mathrm{Gal}_F)$.

Proof. This follows directly from equation (7.1). □

Proposition 7.3. *For any $p \equiv_3 2$, even $r \geq 1$, μ and λ, we have*

$$\mathrm{Tr}\big(F_q, e_{c,\mu}(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda)\big) \in \mathbb{Z}.$$

Proof. If $p \equiv_3 2$, then the automorphism $x \mapsto x^p$ sends $\tilde{\rho}$ to $\tilde{\rho}^2$ in $k = \mathbb{F}_{p^r}$ for any even $r \geq 1$. This immediately shows that

$$\mathrm{Tr}\big(F_q, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big) = \overline{\mathrm{Tr}\big(F_q, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big)}. \qquad \square$$

Let e_i denote the ith elementary symmetric polynomial and p_i the ith power sum polynomial. The number of variables should in the future be clear from context. Partitions $\nu \vdash n$ will be written on the form $\nu = (1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n})$, where

$$n = \sum_{i=1}^{n} i\,\nu_i.$$

Now, for any partition $\nu \vdash n$, let s_ν denote the Schur polynomial associated to ν and put

$$e_\nu := \prod_{i=1}^{n} e_i^{\nu_i} \quad \text{and} \quad p_\nu := \prod_{i=1}^{n} p_i^{\nu_i}.$$

Recall that s_ν is a polynomial with integer coefficients in the elementary symmetric polynomials $e_1, e_2, \dots, e_n$, and with rational coefficients in the power sum polynomials and we define $c_{\nu,\xi}$ by

$$s_\nu = \sum_{\xi \vdash n} c_{\nu,\xi} \frac{p_\xi}{z_\xi}, \quad \text{where } z_\xi := \prod_{i=1}^{n} \xi_i ! i^{\xi_i}.$$

In the representation ring of $\mathfrak{S}_n$ (tensored with $\mathbb{Q}$) we have the corresponding equality

$$\mathbf{s}_\nu = \sum_{\xi \vdash n} c_{\nu,\xi} \frac{\mathbf{p}_\xi}{z_\xi}.$$

Recall the notation in Section 6.2. For any $C \in \mathcal{X}_{\Gamma[\sqrt{-3}]}(k)$, using Poincaré duality in étale cohomology between $H^1(C, \overline{\mathbb{Q}}_\ell)$ and $H^1_c(C, \overline{\mathbb{Q}}_\ell)$, we have that

$$\mathrm{Tr}\big(F_q, (\mathbb{W})_{C \otimes \bar{k}}\big) = \overline{\alpha_1(C)} + \overline{\alpha_2(C)} + \overline{\alpha_3(C)},$$

and so for any partition λ, we have

$$\mathrm{Tr}\big(F_q, (\mathbb{W}_\lambda)_{C \otimes \bar{k}}\big) = s_\lambda\big(\overline{\alpha_1(C)}, \overline{\alpha_2(C)}, \overline{\alpha_3(C)}\big).$$

For any partition ν of 4, let $\mathcal{X}_\nu(k)$ inside $\mathcal{X}_{\Gamma[\sqrt{-3}]}(k)$ consist of the curves whose ramification points, where α acts by ρ, are defined over k_{ν_i} but not over a subfield of k_{ν_i} for $1 \le i \le 4$. From the Lefschetz trace formula (see [13, Th. 3.2]), it follows that if $\sigma_\nu \in \mathfrak{S}_4$ has cycle type ν, then

$$\mathrm{Tr}\big(F_q \circ \sigma_\nu, e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda)\big) = \sum_{C \in \mathcal{X}_\nu(k)/\cong_k} \frac{s_\lambda\big(\overline{\alpha_1(C)}, \overline{\alpha_2(C)}, \overline{\alpha_3(C)}\big)}{|\mathrm{Aut}_k(C)|}. \quad (7.3)$$

By the projection formula, we then have

$$\mathrm{Tr}\big(F_q, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big) = \sum_{\nu \vdash 4} c_{\mu,\nu} \, \mathrm{Tr}\big(F_q \circ \sigma_\nu, e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big),$$

giving the equality

$$\mathrm{Tr}\big(F_q, e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big)\mathbf{s}_\mu = \sum_{\nu \vdash 4} c_{\mu,\nu} \, \mathrm{Tr}\big(F_q \circ \sigma_\nu, e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big) \frac{\mathbf{p}_\nu}{z_\nu}.$$

Proposition 7.4. *For any λ, we have*

$$e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda \otimes \det(\mathbb{W})^3\big) = e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big)\mathbb{L}^{6,3}\mathbf{s}_{1^4},$$

as elements of $K_0^{\mathfrak{S}_4}(\mathrm{Gal}_F)$.

Proof. Let C be an element of $\mathcal{X}_\mu(k)$. We see from Theorem 6.5 that

$$\begin{aligned}\mathrm{Tr}\big(F_q, \big(\mathbb{W}_\lambda \otimes (\det \mathbb{W})^3\big)_{C\otimes\bar{k}}\big) &= \mathrm{Tr}\big(F_q, (\mathbb{W}_\lambda)_{C\otimes\bar{k}}\big) e_3\big(\alpha_1(C), \alpha_2(C), \alpha_3(C)\big)^3 \\ &= \mathrm{Tr}\big(F_q, (\mathbb{W}_\lambda)_{C\otimes\bar{k}}\big)(-1)^{\mathrm{sign}(\nu)}(a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho)^6(a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho^2)^3.\end{aligned}$$

From this, the result follows. □

Remark 7.5. If the weight $n_1+2n_2+3n_3$ of the local system $\lambda = n_1\gamma_1+n_2\gamma_2+n_3\gamma_3$ is even then

$$e_c\big(\mathcal{X}'_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big),$$

but if the weight is odd then

$$e_c\big(\mathcal{X}'_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 0$$

due to the presence of the automorphism -1 of the abelian varieties that $\mathcal{X}'_{\Gamma[\sqrt{-3}]}$ parametrizes. But we see from Proposition 7.4 that there are no new motives appearing for a local system $\mathbb{W}_\lambda$ on $\mathcal{X}_{\Gamma[\sqrt{-3}]}$ of odd weight, since these motives will, after being tensored with the "trivial factor" $\mathbb{L}^{6,3} \otimes \mathbf{s}_{1^4}$, appear in

$$e_c\big(\mathcal{X}'_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda \otimes \det(\mathbb{W})^3\big).$$

7.3.1. Normalization of the Euler characteristic. In the proof of Proposition 7.4 we also see, for any μ and λ, that

$$e_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda \otimes \det(\mathbb{W})^i\big) = \mathbb{L}^{2i,i} V_{\lambda,\mu}$$

for some element $V_{\lambda,\mu}$ of $K_0(\mathrm{Gal}_F^{\mathfrak{S}_4})$.

Definition 7.6. If $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ and $\lambda' = n_1\gamma_1 + n_2\gamma_2$, then $\mathbb{W}_\lambda = \mathbb{W}_{\lambda'\otimes\det(\mathbb{W})^{n_3}}$, and by taking away the factor $\mathbb{L}^{2n_3,n_3}$ we define the *normalized* Euler characteristic to be

$$e^{\mathrm{norm}}_{c,\mu}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = V_{\lambda',\mu}.$$

7.3.2. The appearance of Picard modular cusp forms. Using Propositions 7.1, 7.2 and 7.4, we can restrict ourselves to determining the Euler characteristics of local systems

$$\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$$

for which $n_2 \le n_1$, $n_1 \equiv_3 n_2$, $n_1 \equiv_2 n_3$ and $0 \le n_3 \le 5$.

Proposition 3.3 suggests that one finds a motive in the Euler characteristic $e_c^{\mathrm{norm}}(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$ that will correspond to the space of Picard modular cusp forms S_{n_2,n_1+3,n_2+n_3-1}, see further in Section 10.

8. Counts over finite fields

8.1. Information needed. In this section we will see what information is needed to compute

$$\mathrm{Tr}\big(F_q, e_{c,\mu}(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda)\big).$$

The results will be used in Section 10.1 and are the basis of our conjectures in Sections 11 and 12.

First, define the contributions of the strata to the trace

$$\mathrm{Tr}_{i,\nu,\lambda,q} := \mathrm{Tr}\big(F_q \circ \sigma_\nu, e_c\big(\mathcal{X}_{i,\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda\big)\big).$$

8.1.1. Counts of smooth curves of genus 3. Let $P_{1,\mu}(k)$ denote the set of square-free polynomials f with coefficients in k of degree four such that f has μ_i roots defined over k_i but not over any proper subfield of k_i. From equation (7.3), together with the results of Section 5.4, we find that

$$\mathrm{Tr}_{1,\nu,q} = \frac{1}{q(q-1)^2} \sum_{f \in P_{1,\mu}(k)} s_\lambda\big(\overline{\alpha_1(C)}, \overline{\alpha_2(C)}, \overline{\alpha_3(C)}\big).$$

If we have computed

$$e_1\big(\alpha_1(C_f), \alpha_2(C_f), \alpha_3(C_f)\big) = a_{1,\rho}(C_f) = -\sum_{a \in \mathbb{P}^1(k)} \chi(f(a)),$$

then we can use equation (6.1) together with Theorem 6.5 to easily compute

$$e_i\big(\alpha_1(C_f), \alpha_2(C_f), \alpha_3(C_f)\big)$$

for $i = 2, 3$. With this information we can compute

$$s_\lambda\big(\alpha_1(C_f), \alpha_2(C_f), \alpha_3(C_f)\big)$$

for any λ.

One can then simplify the computation of $\mathrm{Tr}_{1,\nu,q}$ by using the group of isomorphisms to find normal forms. Fix a generator γ of k^*. If p is odd and $p \neq 3$, then a curve of the form

$$y^3 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{8.1}$$

with a_2, a_3 and a_4 non-zero, is isomorphic (over k) to a curve of the form

$$y^3 = \gamma^i(a_4'x^4 + x^2 + x + a_0'), \tag{8.2}$$

for some a_4', a_0' and $0 \leq i \leq 2$. The curves of the latter form are all non-isomorphic and have an automorphism group of order 3 generated by $y \mapsto \tilde{\rho}y$. If $D_1 \subset P_{1,\mu}(k)$

is the subset of polynomials of the form (8.1) and $D_2 \subset P_{1,\mu}(k)$ of the form (8.2) then

$$\frac{1}{q(q-1)^2}\sum_{f\in D_1} s_\lambda\big(\overline{\alpha_1(C)},\overline{\alpha_2(C)},\overline{\alpha_3(C)}\big)$$
$$=\frac{1}{3}\sum_{i=1}^{3}\sum_{f=a_4'x^4+x^2+x+a_0'\in D_2} s_\lambda\big(\tilde{\rho}^i\overline{\alpha_1(C_f)},\tilde{\rho}^i\overline{\alpha_2(C_f)},\tilde{\rho}^i\overline{\alpha_3(C_f)}\big).$$

In a similar way, one can construct other normal forms if a_2 or a_3 is zero.

If $p=2$, then a curve of the form

$$y^3 = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

with a_1, a_0 and a_4 non-zero, will be isomorphic to a curve of the form

$$y^3 = \gamma^i(x^4 + x^3 + a_1'x + a_0')$$

for some a_1', a_0' and $0 \le i \le 2$.

In this manner, we can reduce the number of free parameters in the polynomials in the sum $\mathrm{Tr}_{1,\nu,q}$ from 5 to 2 (which is optimal since we are considering a surface).

8.1.2. Counts of smooth genus 2 curves joined with elliptic curves. We will denote by $P_{2,\mu}(k)$ the set of triples of polynomials (f_1, f_2, f) with coefficients in k, as in Section 5.7, but where we assume that f_2 is of degree 1 by putting the point q_2 in infinity using a linear transformation in x, and such that f and f_1 together have μ_i roots defined over k_i but not over any subfield of k_i. We find that

$$\mathrm{Tr}_{2,\nu,q} = \frac{1}{(q^2(q-1)^4}\sum_{(f_1,f_2,f)\in P_{2,\mu}(k)} s_\lambda\big(\overline{\alpha_1(C_{f_1,f_2})},\overline{\alpha_2(C_{f_1,f_2})},\overline{\alpha_1(C_f)}\big).$$

If we have computed

$$e_1\big(\alpha_1(C_{f_1,f_2}),\alpha_2(C_{f_1,f_2})\big) = -\sum_{a\in\mathbb{P}^1(k)} \chi(f_1(a))\chi(f_2(a))^2,$$

we can use Theorem 6.5 to compute $e_2(\alpha_1(C_{f_1,f_2}),\alpha_2(C_{f_1,f_2}))$. Using the equations in Section 6.2.1 we can compute $e_1(\alpha_1(C_f))$. From this we can determine

$$e_i\big(\alpha_1(C_{f_1,f_2}),\alpha_2(C_{f_1,f_2}),\alpha_1(C_f)\big)$$

for $i=1,2,3$.

As in Section 8.1.1, we can use the group of isomorphisms to find normal forms which will simplify the computation of $\mathrm{Tr}_{2,\nu,q}$. Fix γ, a generator of k^*. If p is odd, and

$$(b_2x^2 + b_1x + b_0, c_1x + c_0, d_2x^2 + d_1x + d_0)$$

is in $P_{2,\mu}(k)$ with $c_0 \neq 0$, then we can find an isomorphism to a curve given by (f_1', f_2', f_3') in $P_{2,\mu}(k)$, where

$$f_3' = x^2 + \gamma^i, \quad f_2' = x + 1, \quad f_1' = \gamma^j(x^2 + b_1)$$

for some $b_1 \in k$, $0 \leq i \leq 5$ and $0 \leq j \leq 2$. The curves of the latter form are all non-isomorphic and have an automorphism group of order 3 generated by $y \mapsto \tilde{\rho} y$. Similar normal forms can be found if $c_0 = 0$ and if p is even.

In this manner, we can reduce the free parameters in the polynomials in the sum $\mathrm{Tr}_{2,\nu,q}$ from 8 to 1 (which is optimal since we are here considering a curve).

8.1.3. Counts of triples of elliptic curves. Let $P_{3,\mu}(k)$ denote the set of triples of polynomials (f_1, f_2, f_3^2) with coefficients in k, as in Section 5.8, such that f_1 and f_2 together have μ_i roots defined over k_i but not over any subfield of k_i. Define $P'_{3,\mu}(k)$ in the same way, but where f_1, f_2 are defined over k_2 and where Frobenius sends f_1 to f_2. Note that we have more isomorphisms between the curves corresponding to elements of these sets, namely by switching the two "wings" of these curves, that is between C_{f_1,f_2,f_3^2} and C_{f_2,f_1,f_3^2}. We have that

$$\mathrm{Tr}_{3,\nu,q} = \frac{1}{2q^3(q-1)^3} \sum_{(f_1,f_2,f_3^2)\in P_{3,\mu}(k)} s_\lambda\big(\overline{\alpha_1(C_{f_1})}, \overline{\alpha_1(C_{f_2})}, \overline{\alpha_1(C_{f_3^2})}\big)$$
$$+ \frac{1}{2q^3(q-1)(q^2-1)} \sum_{(f_1,f_2,f_3^2)\in P'_{3,\mu}(k)} s_\lambda\big(\overline{\alpha_1(C_{f_1})}, \overline{\alpha_2(C_{f_2})}, \overline{\alpha_1(C_{f_3^2})}\big).$$

If (f_1, f_2, f_3^2) are in $P_{3,\mu}(k)$ then using the equations in Section 6.2.1, we can compute $e_1(\alpha_1(C_{f_i}))$ for $i = 1, 2, 3$. For (f_1, f_2, f_3^2) in $P'_{3,\mu}(k)$, we can determine $e_1(\alpha_1(C_{f_3^2}))$ in the same way. Moreover, $e_1(\alpha_1(C_{f_1}), \alpha_2(C_{f_2})) = 0$ and

$$p_2\big(\alpha_1(C_{f_1}), \alpha_2(C_{f_2})\big) = -\sum_{a\in\mathbb{P}^1(k_2)} \chi(f_1(a)) - \sum_{a\in\mathbb{P}^1(k_2)} \chi(f_2(a)),$$

where χ is the third power residue symbol for k_2. In both cases, this gives enough information to determine $e_i(\alpha_1(C_{f_1}), \alpha_1(C_{f_2}), \alpha_1(C_{f_3^2}))$ for $i = 1, 2, 3$.

In Section 6.2.1, a representative of each k-isomorphism class of $\mathcal{X}_\Gamma^{(1)}(k)$ is given. With this information $\mathrm{Tr}_{3,\nu,q}$ is easily computed for any ν and q.

8.2. Counts with constant coefficients. Let us consider the Euler characteristic when $\mathbb{W}_\lambda = \overline{\mathbb{Q}}_\ell$.

We will repeatedly use the trick below that summing over all elements defined over k, of one of the groupoids at hand, and then dividing by the number of k-isomorphisms between these elements is the same as summing elements weighted by the reciprocal of their number of k-automorphisms, compare for instance [4, Sec. 5].

The elements of $\mathcal{X}_{1,\Gamma}(k)$ together with their isomorphisms are described in Section 5.4. For $\Gamma[\sqrt{-3}]$ we divide into the different choices of four branch points on $\mathbb{P}^1$ giving

$$\begin{aligned}\mathrm{Tr}\big(F_q, e_c(\mathcal{X}_{1,\Gamma[\sqrt{-3}]}\otimes \bar{k}, \bar{\mathbb{Q}}_\ell)\big) &= \Big(q(q-1)(q-2)(q-3)\frac{\mathbf{p}_1^4}{24}\\ &+(q^2-q)q(q-1)\frac{\mathbf{p}_1^2\mathbf{p}_2}{4}+(q^3-q)q\frac{\mathbf{p}_1\mathbf{p}_3}{3}+(q^2-q)(q^2-q-2)\frac{\mathbf{p}_2^2}{8}\\ &+(q^4-q^2)\frac{\mathbf{p}_4}{4}\Big)/\big(q(q-1)\big) = q^2\mathbf{s}_4+(1-q)\mathbf{s}_{3,1}-q\mathbf{s}_{2,2}+\mathbf{s}_{2,1,1}.\end{aligned}$$

Similarly for $\mathcal{X}_{2,\Gamma[\sqrt{-3}]}$ we use Sections 5.5, 5.6 and 5.7, and we find that

$$\begin{aligned}&\mathrm{Tr}\big(F_q, e_c(\mathcal{X}_{2,\Gamma[\sqrt{-3}]}\otimes \bar{k}, \bar{\mathbb{Q}}_\ell)\big)\\ &= \Big(\Big(\frac{\mathbf{p}_1^2}{2}+\frac{\mathbf{p}_2}{2}\Big)\Big(q(q-1)(q-2)\frac{\mathbf{p}_1^2}{2}+q(q^2-q)\frac{\mathbf{p}_2\mathbf{p}_2}{2}\Big)\Big)/\big(q(q-1)\big)\\ &= (q-1)\mathbf{s}_4+(q-2)\mathbf{s}_{3,1}+(q-1)\mathbf{s}_{2,2}-\mathbf{s}_{2,1,1}.\end{aligned}$$

For $\mathcal{X}_{3,\Gamma[\sqrt{-3}]}$, we use Sections 5.6 and 5.8, and we recall the plethysm $\circ$ to deal with the symmetry of the two elliptic curves that form the "wings". We find that

$$\begin{aligned}\mathrm{Tr}\big(F_q, e_c\big(\mathcal{X}_{3,\Gamma[\sqrt{-3}]}\otimes \bar{k}, \bar{\mathbb{Q}}_\ell\big)\big) &= \Big(\frac{\mathbf{p}_1^2}{2}+\frac{\mathbf{p}_2}{2}\Big)\circ\Big(\frac{\mathbf{p}_1^2}{2}+\frac{\mathbf{p}_2}{2}\Big)\\ &= \frac{1}{2}\Big(\frac{\mathbf{p}_1^2}{2}+\frac{\mathbf{p}_2}{2}\Big)^2+\frac{1}{2}\Big(\frac{\mathbf{p}_2^2}{2}+\frac{\mathbf{p}_4}{2}\Big) = \mathbf{s}_4+\mathbf{s}_{2,2}.\end{aligned}$$

The trace of Frobenius on $e_c(\mathcal{X}_{3,\Gamma[\sqrt{-3}]}\otimes \bar{k})$ for all $q\equiv_3 1$ determines

$$e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}\otimes \bar{k}, \bar{\mathbb{Q}}_\ell\big)$$

as an element in $K_0(\mathrm{Gal}_F^{\mathfrak{S}_4})$, see Section 7.3. Summing the three cases above we then get the following.

Proposition 8.1. *We have an equality of elements in* $K_0(\mathrm{Gal}_F^{\mathfrak{S}_4})$*:*

$$e_c\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}\otimes \bar{k}, \bar{\mathbb{Q}}_\ell\big) = (\mathbb{L}^2+\mathbb{L})\mathbf{s}_4-\mathbf{s}_{3,1}.$$

We continue with case (iv) of Section 5.9. On the elliptic curve C_1, there is a choice of a point not equal to any of the ramification points. This gives a contribution $q+1-a_1(C_1)-r_1(C_1)$, where $r_1(C_1)$ is the number of ramification points of C_1 defined over k. A computation similar to the one in Section 6.2.1 shows, due to the symmetry, that the contribution from $a_1(C_1)$ vanishes. The genus 0 curve contributes a p_1, and so the trace of Frobenius on the Euler characteristic of the strata for case (iv) equals

$$\begin{aligned}p_1\Big((q+1-3)\frac{\mathbf{p}_1^3}{6}+(q+1-1)\frac{\mathbf{p}_1\mathbf{p}_2}{2}+(q+1)\frac{\mathbf{p}_3}{3}\Big)&\\ = q\mathbf{s}_4+(q-1)\mathbf{s}_{3,1}-\mathbf{s}_{2,2}-\mathbf{s}_{2,1,1}&.\end{aligned}$$

Case (v) is straightforward and the trace of Frobenius on its Euler characteristic equals

$$\mathbf{p}_1^2\Big(\frac{\mathbf{p}_1^2}{2}+\frac{\mathbf{p}_2}{2}\Big)=\mathbf{s}_4+2\mathbf{s}_{3,1}+\mathbf{s}_{2,2}+\mathbf{s}_{2,1,1}.$$

Summing cases (i) to (v) and using the purity of a smooth and proper (Deligne–Mumford) stack we find the following.

Proposition 8.2. *We have an equality of elements in* $\mathrm{Gal}_F^{\mathfrak{S}_4}$*:*

$$H_c^i\big(\widetilde{\mathcal{X}}_{\Gamma[\sqrt{-3}]}\otimes\bar{k},\bar{\mathbb{Q}}_\ell\big)=\begin{cases}\mathbb{L}^2\mathbf{s}_4 & \text{if } i=4,\\ \mathbb{L}(2\mathbf{s}_4+\mathbf{s}_{3,1}) & \text{if } i=2,\\ \mathbb{L}^0\mathbf{s}_4 & \text{if } i=0,\\ 0 & \text{if } i \text{ odd}.\end{cases}$$

Together, cases (iv) and (v) contribute $(q+1)(\mathbf{s}_4+\mathbf{s}_{3,1})$ and they form a $\mathbb{P}^1$-bundle under the morphism (5.1). Hence, their image (the four cusps) contribute $\mathbf{s}_4+\mathbf{s}_{3,1}$ and we get a trace of Frobenius equal to

$$\mathrm{Tr}\big(F_q,e_c\big(\mathcal{X}^*_{\Gamma[\sqrt{-3}]}\otimes\bar{k},\bar{\mathbb{Q}}_\ell\big)\big)=(q^2+q+1)\mathbf{s}_4,$$

which echoes the fact that $X^*_{\Gamma[\sqrt{-3}]}\cong\mathbb{P}^2$.

8.2.1. The genus 2 case. There are two strata in $\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}$, one consisting of smooth genus 2 curves and one consisting of pairs of genus one curves, one with action of type $(1,0)$ and one of type $(0,1)$, joined at a ramification point on each curve. There is an action of $\mathfrak{S}_2\times\mathfrak{S}_2$ on the two pairs of ramification points, one pair where the action of α is by ρ and one by ρ^2. Let us use the notation $\mathbf{p}_i$ and $\widetilde{\mathbf{p}}_i$, and $\mathbf{s}_\mu$ and $\widetilde{\mathbf{s}}_\mu$, for the basis of representations of the two components of $\mathfrak{S}_2\times\mathfrak{S}_2$. A consideration analogous to the ones above shows that

$$\begin{aligned}\mathrm{Tr}\big(F_q,e_c\big(\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}\otimes\bar{k},\bar{\mathbb{Q}}_\ell\big)\big)&=\Big((q+1)q(q-1)(q-2)\frac{\mathbf{p}_1^2\widetilde{\mathbf{p}}_1^2}{4}\\&+(q^2-q)(q+1)q\,\frac{\mathbf{p}_2\widetilde{\mathbf{p}}_1^2+\mathbf{p}_1^2\widetilde{\mathbf{p}}_2}{4}\\&+(q^2-q)(q^2-q-2)\frac{\mathbf{p}_2\widetilde{\mathbf{p}}_2}{4}\Big)/\big((q+1)q(q-1)\big)\\&+\frac{\mathbf{p}_1^2\widetilde{\mathbf{p}}_1^2}{4}+\frac{\mathbf{p}_1^2\widetilde{\mathbf{p}}_2}{4}+\frac{\mathbf{p}_2\widetilde{\mathbf{p}}_1^2}{4}+\frac{\mathbf{p}_2\widetilde{\mathbf{p}}_2}{4}=q\mathbf{s}_2\widetilde{\mathbf{s}}_2-\mathbf{s}_{1,1}\widetilde{\mathbf{s}}_{1,1},\end{aligned}$$

and we can conclude the following.

Proposition 8.3. *We have an equality of elements in* $K_0(\mathrm{Gal}_F^{\mathfrak{S}_4})$*:*

$$e_c\big(\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}\otimes\bar{k},\bar{\mathbb{Q}}_\ell\big)=\mathbb{L}\mathbf{s}_2\widetilde{\mathbf{s}}_2-\mathbf{s}_{1,1}\widetilde{\mathbf{s}}_{1,1}.$$

8.3. Euler characteristics of local systems for elliptic curves. Using the results of Section 6.2.1 and Section 7.3, we find that

$$\mathrm{Tr}\big(F_q, e'_c\big(\mathcal{X}^{(1)}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_k\big)\big) = \sum_{i=0}^{2} \rho^{ki} (a_{\mathfrak{p}_q} + b_{\mathfrak{p}_q}\rho^2)^k \big(\mathbf{p}_1^2 + (-1)^k \mathbf{p}_2\big)$$

for all q such that $q \equiv_3 1$. This equality, together with the fact that an element in $K_0(\mathrm{Gal}_F)$ is determined by all traces of Frobenius, shows the following.

Proposition 8.4. *For any $k \geq 0$, we have the equality in $K_0^{\mathfrak{S}_4}(\mathrm{Gal}_F)$:*

$$e_c\big(\mathcal{X}^{(1)}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_k\big) = \begin{cases} \mathbb{L}^{0,k}\mathbf{s}_2 & \text{if } k \equiv_6 0, \\ \mathbb{L}^{k,0}\mathbf{s}_{1^2} & \text{if } k \equiv_6 3, \\ 0 & \text{if } k \not\equiv_3 0. \end{cases}$$

9. Numeric Euler characteristics of local systems

In this section the ground field will be $\mathbb{C}$, we will consider the compactly supported Betti cohomology, and we will find a formula for the integer-valued Euler characteristic,

$$E_{c,\mu}\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) := \sum_{i=0}^{4} (-1)^i \dim_{\mathbb{C}} H^i_{c,\mu}\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) \in \mathbb{Z}$$

for any λ and μ, where $H^i_{c,\mu}$ is the μ-isotypic component of H^i_c. Examples of computations using this formula will be found in Section 13.2.

Similarly to the above, we will write

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) := \sum_{\mu \vdash 4} \frac{E_{c,\mu}\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big)}{\dim \mathbf{s}_\mu} \mathbf{s}_\mu \in \mathbb{Z}[\mathfrak{S}_4].$$

The reader should compare this section to the article [7] and the references therein. Note that by comparison theorems this numerical Euler characteristic will be the same if Betti cohomology is replaced by ℓ-adic étale cohomology as described in Section 7. So, $E_c(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$ equals $\dim e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$ for any λ of even weight.

We stratify our moduli space X_Γ, first into $X_{1,\Gamma}$, $X_{2,\Gamma}$ and $X_{3,\Gamma}$ as in Section 5.3. We then stratify further into strata $\Sigma_i(G)$ for $i = 1, 2, 3$ and G a finite group, consisting of the curves corresponding to points of $X_{i,\Gamma}$ whose automorphism group equals G. As usual, let $H^1(C, \mathbb{C})^\rho$ denote the ρ-eigenspace of $H^1(C, \mathbb{C})$ when acting by α. Say that $g \in G$ has eigenvalues $\xi_1(g)$, $\xi_2(g)$ and $\xi_3(g)$ when acting on $H^1(C, \mathbb{C})^\rho$ of a curve $C \in \Sigma(G)$. Say furthermore that the induced action of $g \in G$

on the four ramification points of $C \in \Sigma_i(G)$ where α acts by ρ has ν_i cycles of length i. Note that this data will be constant on the strata, i.e., independent of the choice of $C \in \Sigma_i(G)$. If $e_c\big(\Sigma_i(G)\big)$ denotes the usual compactly supported Euler characteristic of $\Sigma_i(G)$, then

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = \sum_{i=1}^{3} \sum_{G} \frac{E_c\big(\Sigma_i(G)\big)}{|G|} \sum_{g \in G} s_\lambda\big(\xi_1(g), \xi_2(g), \xi_3(g)\big)\mathbf{p}_\nu. \quad (9.1)$$

In the sections below, we will find the necessary information to compute this formula for any given λ.

In all cases below, the automorphism groups that appear are cyclic, so it is enough to give the three eigenvalues of a generator, which we will denote by ϕ, together with its cycle type as a permutation of the four ramification points.

9.1. Numerical Euler characteristics for smooth curves of genus 3. One easily finds that there are four different cyclic automorphism groups in this case, namely the generic case $\mathbf{C}_3$ and then $\mathbf{C}_6$, $\mathbf{C}_9$ and $\mathbf{C}_{12}$. The strata for $\mathbf{C}_9$ and $\mathbf{C}_{12}$ consist of a single point. For $\mathbf{C}_6$, the stratum is 1-dimensional. Each isomorphism class can be represented by a curve of the form

$$f = x^4 + ax^2 + 1$$

with $a \in \mathbb{C}$. To make this a smooth curve we need $a^2 \neq 0, 1$. Moreover, two curves of this form are isomorphic precisely if their coefficients a differ by a sign. This shows that the Euler characteristic of this stratum equals -1. The whole moduli space $X_{1,\Gamma}$ is described in the end of Section 2 and we find that it has Euler characteristic 1 (compare with the point count in Section 8.2). From this it follows that the Euler characteristic of the generic (open dense) stratum must be 0.

We have an isomorphism

$$H^1(C, \mathbb{C}) \cong H^0(C, \Omega) \oplus \overline{H^0(C, \Omega)}$$

and the subspace $H^1(C, \mathbb{C})^\rho$ has a basis consisting of dx/y^2, xdx/y^2 and the dual of dx/y. The eigenvalues of the action of ϕ on $H^1(C, \mathbb{C})^\rho$ can thus be found through its action on this basis.

We exemplify such a computation in the case of $\mathbf{C}_9$. The other cases are completely analogous. We have that ϕ applied to dx/y^2, xdx/y^2, dx/y equals

$$\rho dx/(\varepsilon^2 y^2) = \varepsilon dx/y^2, \quad \rho^2 dx/(\varepsilon^2 y^2) = \rho\varepsilon dx/y^2, \quad \rho dx/(\varepsilon y) = \varepsilon^2 dx/y,$$

respectively. The action should be on the dual of $\rho dx/y$ and hence this eigenvalue becomes $\varepsilon^{-2} = \rho^2\varepsilon$. We see that the action of ϕ cyclically permutes three of the ramification points and fixes the fourth.

The data to compute the contribution from the strata $X_{1,\Gamma[\sqrt{-3}]}$ to equation (9.1), is found in Table 1.

G	f	$\Sigma_1(G)$	$\phi(x,y)$	$\xi_1(\phi),\xi_2(\phi),\xi_3(\phi)$	$\mathbf{p}_\nu$
$\mathbf{C}_3$		0	$(x,\rho y)$	ρ,ρ,ρ	$\mathbf{p}_1^4$
$\mathbf{C}_6$	x^4+ax^2+1	-1	$(-x,\rho y)$	$\rho,-\rho,-\rho$	$\mathbf{p}_2^2$
$\mathbf{C}_9$	$x(x^3-1)$	1	$(\rho x,\varepsilon y)$	$\varepsilon,\rho\varepsilon,\rho^2\varepsilon$	$\mathbf{p}_1\mathbf{p}_3$
$\mathbf{C}_{12}$	x^4-1	1	$(ix,\rho y)$	$-\rho,i\rho,-i\rho$	$\mathbf{p}_4$

Table 1.

9.2. Numerical Euler characteristics for smooth genus 2 curves joined with elliptic curves. Let us first consider smooth genus 2 curves C_{f_1,f_2} together with a marked root of f_2, which we place in infinity, see Section 5.7. Note that the hyperelliptic involution does not fix the marked point. There are two strata. The generic strata, with automorphism group $\mathbf{C}_3$, has a representative

$$f_1 = x^2+ax+1, \quad f_2 = x$$

for each $a \neq 0 \in \mathbb{C}$. This gives an Euler characteristic equal to -1. The strata with automorphism group $\mathbf{C}_6$ consists of a point, given by $a=0$. In this case, the involution switches the two ramification points where α acts by ρ. In both cases, the subspace $H^1(C_{f_1,f_2},\mathbb{C})^\rho$ has a basis consisting of $x dx/y^2$ and the dual of dx/y. Computations as in the previous section give Table 2.

G	$f_1f_2^2$	$\Sigma_1(G)$	$\phi(x,y)$	$\xi_1(\phi),\xi_2(\phi),\xi_3(\phi)$	$\mathbf{p}_\nu$
$\mathbf{C}_3$	$(x^2+ax+1)x^2$	-1	$(x,\rho y)$	ρ,ρ	$\mathbf{p}_1^2$
$\mathbf{C}_6$	$(x^2+1)x^2$	1	$(-x,\rho y)$	$\rho,-\rho$	$\mathbf{p}_2$

Table 2.

The elliptic curves come with a marked ramification point at infinity and there is only one stratum consisting of the curve with equation $y^3=x^2+1$ and automorphism group $\mathbb{Z}/6\mathbb{Z}$ generated by the element $\phi\colon (x,y)\mapsto(-x,\rho y)$. The single eigenvalue of ϕ acting on $H^1_c(C,\mathbb{C})^\rho$ is $-\rho$. Furthermore ϕ^i permutes the ramification points if i is odd and fixes them if i is even.

The possible automorphism groups of curves $C_{f_1,f_2,f_3}\in X_{2,\Gamma[\sqrt{-3}]}$ are just products of the automorphism groups for C_{f_1,f_2} and C_{f_3}. Moreover, we have that

$$H^1(C_{f_1,f_2,f_3},\mathbb{C})^\rho \cong H^1(C_{f_1,f_2},\mathbb{C})^\rho \oplus H^1(C_{f_3},\mathbb{C})^\rho.$$

So, piecing together the information above enables us to compute the contribution from $X_{2,\Gamma[\sqrt{-3}]}$ to equation (9.1).

9.3. Numerical Euler characteristics for triples of elliptic curves. Triples of elliptic curves C_{f_1,f_2,f_3^2} are described in Section 5.8. The “backbone”, corresponding to f_3^2, will only have an automorphism group generated by α, because two of its ramification points are fixed. Then there is an additional automorphism σ by switching the two "wings" corresponding to f_1 and f_2. This gives rise to an automorphism group of the form $G = \mathbf{C}_3 \times (\mathbf{C}_6 \wr \mathfrak{S}_2)$, where $\wr$ denotes the wreath product. Say that α_i is an automorphism of C_{f_i} for $i = 1, \dots, 3$ with eigenvalues τ_i acting on $H^1(C_{f_i}, \mathbb{C})^\rho$, then these will also be the eigenvalues of the induced action on

$$H^1(C_{f_1,f_2,f_3^2}, \mathbb{C})^\rho \cong H^1(C_{f_1}, \mathbb{C})^\rho \oplus H^1(C_{f_2}, \mathbb{C})^\rho \oplus H^1(C_{f_3^2}, \mathbb{C})^\rho.$$

If the previous automorphism is composed with the involution in $\mathfrak{S}_2$, the eigenvalues will be $(\tau_1\tau_2)^{1/2}$, $-(\tau_1\tau_2)^{1/2}$, τ_3. From this information one can compute the contribution from $X_{3,\Gamma[\sqrt{-3}]}$ to equation (9.1).

10. Our approach

Here we will explain the approach that led us to the conjectures on Picard modular forms in Section 11 and 12.

10.1. Computer counts over finite fields. Using the results of Section 8 we computed

$$\mathrm{Tr}\big(F_q, e_{c,\mu}^{\mathrm{norm}}(\mathcal{X}_{\Gamma[\sqrt{-3}]} \otimes \bar{k}, \mathbb{W}_\lambda)\big)$$

for all prime powers $q \equiv_3 1$ such that $q \leq 67$, and all partitions λ such that $n_1 + n_2 + 2 \leq 40$. These traces always turned out to be in $\mathbb{Z}[\rho]$ as they should be, see Section 7.3.

The conjectures of this section are based upon these computer counts (using the equality (7.2)).

10.2. Preview. We are interested in calculating the trace of Hecke operators on the $\mathfrak{S}_4$-isotypic components of the space $S_{j,k,l}(\Gamma[\sqrt{-3}])$ of cusp forms of given weight. In the analogous case of the space S_k of cusp forms of weight k on $\mathrm{SL}(2,\mathbb{Z})$ one can use for even $k > 0$ the formula

$$\mathrm{Tr}\big(T(p), S_{k+2}\big) = \mathrm{Tr}\big(F_p, S[k+2]\big)$$

with $S[k+2]$ the Chow motive of dimension $2 \dim S_{k+2}$ associated by Scholl ([45]) to the space S_{k+2} and F_p denotes Frobenius. By Deligne's result the motive $S[k+2]$ can be found inside the cohomology of a local system $\mathbb{V}_k$ on the moduli space $\mathcal{A}_1$ of elliptic curves

$$e_c(\mathcal{A}_1, \mathbb{V}_k) = -S[k+2] - 1, \tag{10.1}$$

and we thus can use counts of points over finite fields to calculate the trace of Frobenius on this cohomology and thus the traces of the Hecke operators. Note that the -1 in the last formula comes from the Eisenstein cohomology. By replacing e_c by the inner cohomology $e_!$ we get rid of it. We remark that equation (10.1) still holds for $k = 0$ if we put $S[2] = -\mathbb{L} - 1$.

We want the analogue of this for our Picard modular case. Ideally, in our case one would hope for the existence of a motive $S[j,k,l]$ of dimension $3 \dim S_{j,k,l}(\Gamma[\sqrt{-3}])$ defined over F such that

$$\mathrm{Tr}\big(T(\nu), S_{j,k,l}(\Gamma[\sqrt{-3}])\big) = \mathrm{Tr}\big(F_\nu, S[j,k,l]\big)$$

with $T(\nu)$ the Hecke operator for any $\nu \in \mathbb{Z}[\rho]$ such that $\nu \equiv_3 1$ and $\nu\bar{\nu} = p$ a prime and F_ν is the Frobenius as in Section 7.3 (see also Section 6.1). We refer to the paper by Blasius and Rogawski [33, p. 421] and Harder's book [22]. Moreover, $S[j,k,l]$ should appear $\mathfrak{S}_4$-equivariantly in the second inner cohomology group of the corresponding local system on our moduli space, see Proposition 3.3.

However, one must expect deviations from this due to the fact that there will be liftings from U(1) and GL(2). In Section 11 we make precise conjectures on all such lifts. To any of these lifts f that is a Hecke eigenform we can (conjecturally) associate a reducible 3-dimensional Galois representation M_f defined over F such that

$$\mathrm{Tr}(T(\nu), f) = \mathrm{Tr}(F_\nu, M_f).$$

But for most of these we only see a contribution from a 1-dimensional or 2-dimensional part of M_f in the étale cohomology of our local systems. After removing these cusp forms we are left with a (conjectural) Hecke-invariant subspace of what we call *genuine* Picard modular forms and that we denote by $S^{\mathrm{gen}}_{j,k,l}(\Gamma[\sqrt{-3}])$. So, to each Hecke eigenform in this space there should be a 3-dimensional Galois representation appearing in the cohomology and its (normalized) Hodge degrees in Betti cohomology should be $(j+k-1, 0)$, $(j+1, k-2)$ and $(0, j+k-1)$.

For any $n_1 \equiv_3 n_2$ and $n_1 \equiv_2 n_3$, our goal is to have a formula analogous to equation (10.1), namely,

$$e_c^{\mathrm{norm}}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = \check{S}[n(\lambda)] + e_{\mathrm{extr}}(\lambda) \tag{10.2}$$

equivariant for the action of $\mathfrak{S}_4$ and with $e_{\mathrm{extr}}(\lambda)$ coming from endoscopic groups such that (except for the case $n_1 = n_2 = n_3 = 0$)

$$\mathrm{Tr}\big(T(\nu), S^{\mathrm{gen}}_{n(\lambda)}(\Gamma[\sqrt{-3}])\big) = \mathrm{Tr}\big(F_\nu, \check{S}[n(\lambda)]\big). \tag{10.3}$$

for any $\nu \equiv_3 1$ with norm p a prime and with $n(\lambda) = (n_2, n_1+3, n_2+n_3-1)$. Ideally equation (10.2) should be an equality of motives, but we can also treat it as an equality of bookkeeping devices for calculating traces as in equation (10.3). To make equation (10.2) hold also in the case $n_1 = n_2 = n_3 = 0$ we put

$$\check{S}[0,3,2] = (\mathbb{L}^2 + \mathbb{L} + 1)\mathbf{s}_4,$$

see Proposition 8.1.

Analogously, for the numerical Euler characteristic we should have

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{n(\lambda)}(\Gamma[\sqrt{-3}]) = \frac{1}{3}\big(E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) - E_{\mathrm{extr}}(\lambda)\big).$$

This is formulated in Main conjecture 12.9 and Conjecture 12.12. All of the conjectures of the following sections were found by analyzing the data described in Section 10.1 together with all of the knowledge acquired in the previous sections.

10.3. Notation for local systems and Hecke operators. In the following sections we will use a slightly different notation for our local systems $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ writing

$$\lambda = (a+i, i, -b+i)$$

with $n_1 = a$, $n_2 = b$, $n_3 = -b+i$. This reflects that instead of taking $\mathbb{W}$, $\wedge^2\mathbb{W}$ and $\det(\mathbb{W})$ as building blocks we take $\mathbb{W}$, $\mathbb{W}'$ and $\det(\mathbb{W})$, see Section 7.2. The reason for this switch of notation is that it makes the formulas of Section 12.1 less cumbersome.

Assume from now on that $a \equiv_3 b$ and put

$$n(\lambda) = (b, a+3, i+2).$$

This is the corresponding weight of the modular forms appearing for λ. So, if $i \equiv_2 a+b$ then we expect a Galois representation (of dimension 1, 2 or 3 times the dimension of an irreducible representation of $\mathfrak{S}_4$) corresponding to each eigenform in $S_{n(\lambda)}(\Gamma[\sqrt{-3}])$ appearing (with positive coefficient) in $e_c(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda)$. If $i \not\equiv_2 a+b$ we expect the same contribution but with the $\mathfrak{S}_4$-action twisted by $\mathbf{s}_{1^4}$, compare Proposition 7.4.

By ν we will always mean an element of $\mathbb{Z}[\rho]$ such that $\nu \equiv_3 1$ and $\nu\bar{\nu} = p$ with p a prime. To such an element there is a corresponding Hecke operator acting on $S_{n(\lambda)}(\Gamma[\sqrt{-3}])$ that we denote by $T(\nu)$, see Section 2.3.

Finally, μ will denote a partition of 4.

11. Conjectured lifts

By analyzing the data described in Section 10.1 we see Galois representations in the cohomology of our local systems that seem to be associated to lifts of modular forms from U(1) or GL(2). The conjectures of this section are based upon these examples. The authors have not been able to connect these with the liftings described in Rogawski's book [43], but a reader that is more well-acquainted with representation theory might be more successful.

Each such lifted eigenform then contributes a piece of dimension 1, 2 or 3 to the cohomology. It is important for us to identify the 1-dimensional and 2-dimensional pieces in order to be left with the “genuine” Picard modular forms with associated irreducible 3-dimensional Galois representations.

We will use the notation from Section 10.3.

11.1. Notation for elliptic modular forms. We write $\Gamma_0(N)$ and $\Gamma_1(N)$ for the usual subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and we write $S_k(\Gamma_0(N))$ for the space of cusp forms of weight k on $\Gamma_0(N)$ and $S_k^{\mathrm{new}}(\Gamma_0(N))$ for the subspace of new forms. The dimensions of these spaces will be denoted by $s_k(\Gamma_0(N))$ and $s_k^{\mathrm{new}}(\Gamma_0(N))$. In our case we will have level $N = 3$ or $N = 9$.

We note for level 3 and even $k > 2$ the dimension formula

$$s_k^{\mathrm{new}}(\Gamma_0(3)) = \begin{cases} \lfloor \frac{k}{6} \rfloor + 1 & \text{if } k \equiv_{12} \pm 2, \\ \lfloor \frac{k}{6} \rfloor - 1 & \text{if } k \equiv_{12} 0, \\ \lfloor \frac{k}{6} \rfloor & \text{else.} \end{cases}$$

For odd k, we split the space $S_k(\Gamma_1(3))$ as

$$S_k(\Gamma_1(3)) = S_k^-(\Gamma_1(3)) \oplus S_k^+(\Gamma_1(3))$$

into the $\pm$-eigenspace for the Fricke operator W_3. For odd $k \geq 3$ the dimension of $S_k^{\pm}(\Gamma_1(3))$ is given by

$$s_k^-(\Gamma_1(3)) = \Big\lfloor \frac{k-3}{6} \Big\rfloor, \quad s_k^+(\Gamma_1(3)) = \begin{cases} \lfloor \frac{k-3}{6} \rfloor + 1 & \text{if } k \equiv_6 1, \\ \lfloor \frac{k-3}{6} \rfloor & \text{else.} \end{cases}$$

Inside the space $S_k^{\mathrm{new}}(\Gamma_0(9))$, for even k, we consider the eigenforms f for which the twist f_χ, with χ the non-trivial character modulo 3, is also an eigenform in $S_k^{\mathrm{new}}(\Gamma_0(9))$. These generate a subspace $\Sigma_k \subset S_k^{\mathrm{new}}(\Gamma_0(9))$, and twisting $f \mapsto f_\chi$ defines an involution on this space. It may happen that $f = f_\chi$ and then f will have Hecke eigenvalues $a(p) = 0$ for $p \equiv_3 2$ and f has CM by $\mathbb{Q}(\sqrt{-3})$. This happens for $k \equiv_3 1$, and thus $k \equiv_6 4$. We decompose Σ_k for even k

$$\Sigma_k = S_k^-(\Gamma_0(9)) \oplus S_k^+(\Gamma_0(9)),$$

where $S_k^{\pm}(\Gamma_0(9))$ is the $\pm$-eigenspace for the twisting $f \mapsto f_\chi$ on Σ_k. We found experimentally the dimension formulas for $\Sigma_k^{\pm}(\Gamma_0(9))$

$$s_k^-(\Gamma_0(9)) = \begin{cases} \lfloor \frac{k+4}{12} \rfloor - 1 & k \equiv_{12} 10, \\ \lfloor \frac{k+4}{12} \rfloor & \text{else,} \end{cases}$$

$$s_k^+(\Gamma_0(9)) = \begin{cases} \lfloor \frac{k+4}{12} \rfloor + 1 & k \equiv_{12} 4, \\ \lfloor \frac{k+4}{12} \rfloor & \text{else.} \end{cases}$$

For odd k, and with χ the quadratic character modulo 3, we consider the subspace $S_k^{\chi}(\Gamma_0(9))$ of $S_k^{\mathrm{new}}(\Gamma_0(9),\chi)$ generated by eigenforms such that both f and its twist f_χ belong to $S_k^{\mathrm{new}}(\Gamma_0(9),\chi)$ and are distinct. This space has dimension

$$s_k^{\chi}(\Gamma_0(9)) = \left\lfloor \frac{k+1}{6} \right\rfloor.$$

11.2. One-dimensional lifts. Here we present the conjectured lifts of modular forms from $\mathrm{U}(1)$. For each of these lifts we see a 1-dimensional piece in the cohomology of the corresponding local system $\mathbb{W}_\lambda$ with trace of the Frobenius F_ν equal to ν^{a+b+2}, compare Definition 12.4.

Case 1. For $a \equiv_6 3$ there is a theta series $\zeta_{a+3} \in S_{0,a+3,1}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_{1^4}$. This eigenform is constructed in [18, Prop. 2]. For $a = 3$ we find the form ζ. The Hecke eigenvalue of $T(\nu)$ is given by

$$\nu^{a+2} + (p+1)\overline{\nu}^{a+1},$$

see [18, Prop. 9].

Case 2. For $(a,b) \equiv_6 (5,2)$ we find an eigenform in $S_{b,a+3,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_4$. It will have a Hecke eigenvalue of $T(\nu)$ given by

$$\nu^{a+b+2} + \nu^{b+1}\overline{\nu}^{a+1} + \overline{\nu}^{a+b+2}.$$

The first example is found in $S_{2,8,2}(\Gamma[\sqrt{-3}])$ and it is described in [9, Example 16.7].

Case 3. We conjecture that there is a lift

$$S_{b+2}^{-}(\Gamma_0(9)) \to S_{b,a+3,1-a}\big(\Gamma[\sqrt{-3}]\big)$$

with representation $\mathbf{s}_4$, and the lift of an eigenform f will have Hecke eigenvalue of $T(\nu)$ given by

$$a_p(f) + \nu^{a+b+2}.$$

11.3. Two-dimensional lifts. The Hecke eigenvalue of $T(p)$ for an elliptic eigenform f will be denoted by $a_p(f)$. For each lift of an elliptic eigenform f, i.e., a lift from $\mathrm{GL}(2)$, described in this section we see a 2-dimensional piece (times the dimension of the accompanying $\mathfrak{S}_4$-representation) in the cohomology of the corresponding local system $\mathbb{W}_\lambda$ with trace of Frobenius F_p equal to $a_p(f)\nu^{b+1}$ in all cases but the first, where we just see $a_p(f)$, compare Definition 12.5.

Case 1. We conjecture that there is a lift

$$S_{a+b+3}^{-}(\Gamma_0(9)) \to S_{b,a+3,2}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_4$ and the lift of an eigenform f will have Hecke eigenvalue of $T(\nu)$ given by

$$a_p(f) + \nu^{b+1}\overline{\nu}^{a+1}.$$

An example is given by the lift from $S_8^-(\Gamma_0(9))$ to $S_{1,7,2}(\Gamma[\sqrt{-3}])$; the generating lift is Ψ_1 described in [9, p. 44]. Another example is the lift from $S_{12}^-(\Gamma_0(9))$ to $S_{0,12,2}(\Gamma[\sqrt{-3}]) = \mathbb{C}\zeta^2$, already considered by Finis in [18, p. 178].

In the following five cases we will have a lift of an elliptic eigenform f to $S_{b,a+3,l}(\Gamma[\sqrt{-3}])$ and it will have a Hecke eigenvalue of $T(\nu)$ given by

$$a_p(f)\nu^{b+1} + \overline{\nu}^{a+b+2}.$$

Case 2a. We conjecture that there is a lift

$$S_{a+2}(\Gamma_0(1)) \to S_{b,a+3,b}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}$.

The first example is the lift of $\Delta \in S_{12}(\Gamma_0(1))$ to a form in $S_{1,13,1}(\Gamma[\sqrt{-3}])$ given in the table on page 43 of [9]. Lifts of this type were constructed by Kudla in [31, Thm. 5.3].

Case 2b. We conjecture that there is a lift

$$S_{a+2}^-(\Gamma_1(3)) \to S_{b,a+3,b}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_4 + \mathbf{s}_{3,1}$.

The first example is the lift from $S_9^-(\Gamma_1(3))$ to $S_{1,10,1}(\Gamma[\sqrt{-3}])$ which appears in the table on page 43 of [9]. Lifts of this type were constructed by Kudla as in Case 2a.

Case 3. We conjecture that there is a lift

$$S_{a+2}^{\mathrm{new}}(\Gamma_0(3)) \to S_{b,a+3,b}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_{2,1^2}$.

The lift of $(\eta(3\tau)\eta(\tau))^6 \in S_6(\Gamma_0(3))$ to an element of $S_{1,7,1}(\Gamma[\sqrt{-3}])$ is an example.

Case 4. We conjecture that there is lift

$$S_{a+2}^-(\Gamma_0(9)) \to S_{b,a+3,b}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_{3,1}$.

An example is the form $F_{9,2}$ of Finis [18, p. 151] found in $S_{0,6,0}(\Gamma[\sqrt{-3}])$.

Case 5. We conjecture that there is a lift

$$S_{a+2}^{\chi}(\Gamma_0(9)) \to S_{b,a+3,b}\big(\Gamma[\sqrt{-3}]\big)$$

with $\mathfrak{S}_4$-representation $\mathbf{s}_{2,2}$.

The first example is the lift from $S_5^{\chi}(\Gamma_0(9))$ to $S_{3,6,0}(\Gamma[\sqrt{-3}])$ described in [9, p. 50].

11.4. Genuine Picard modular forms. We conjecture that the space $S_{n(\lambda)}(\Gamma[\sqrt{-3}])$ decomposes into a Hecke-invariant direct sum of a subspace generated by all the lifts described in Sections 11.2 and 11.3 and a subspace $S^{\mathrm{gen}}_{n(\lambda)}(\Gamma[\sqrt{-3}])$ which we call the space of genuine Picard modular forms.

11.5. Three-dimensional lifts. As above, the Hecke eigenvalue of $T(p)$ for an elliptic eigenform f of weight k will be denoted by $a_p(f)$. In the cohomology of our local systems we see examples of 3-dimensional pieces of the form $\mathrm{Sym}^2(M_f)$, with M_f the motive associated to the elliptic eigenform, and with Hecke eigenvalue for $T(\nu)$ equal to $a_p(f)^2 - p^{k-1}$.

We list these (conjectural) examples of lifts, which should be eigenforms in $S^{\mathrm{gen}}_{n(\lambda)}(\Gamma[\sqrt{-3}])$, with $n(\lambda)$ defined in Section 10.3, without formulating more general conjectures.

- The eigenform $f \in S_5^{\chi}(\Gamma_0(9))$ lifts to an eigenform in $S_{3,6,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}$.
- The eigenform $f \in S_6(\Gamma_0(3))$ lifts to an eigenform in $S_{4,7,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_{3,1}$.
- The eigenform $f \in S_7^{\chi}(\Gamma_0(9))$ lifts to an eigenform in $S_{5,8,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_{1^4}$.
- The eigenform $f \in S_8^{-}(\Gamma_0(9))$ lifts to an eigenform in $S_{6,9,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_4$.
- The eigenform $f \in S_9^{-}(\Gamma_1(3))$ lifts to an eigenform in $S_{7,10,2}(\Gamma[\sqrt{-3}])$ with $\mathfrak{S}_4$-representation $\mathbf{s}_4$.

12. Conjectures on the cohomology of local systems

Recall the notation from Section 10.3. We will consider the normalized motivic (in the sense of Definition 7.6) Euler characteristic with compact support

$$e_c(\lambda) := e_c^{\mathrm{norm}}\big(\mathcal{X}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big)$$

and the inner variant of this $e_!(\lambda)$. Define also $e_{c,\mu}(\lambda)$ as in Section 7.2. Put also

$$E_c(\lambda) := E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big),$$

see Section 9.

Put $\lambda' = (b-i, -i, -a-i)$, so $(\lambda')' = \lambda$, and note by Proposition 7.2 that we have a duality

$$e_c(\lambda') = \overline{e_c(\lambda)}.$$

Note that $\lambda = \lambda'$ precisely when $a = b$ and $i = 0$.

12.1. The main conjecture. In this section we formulate the main conjectures of the article. First we present a series of definitions of different contributions to the cohomology.

The contributions making up the *extraneous* contribution $e_{\mathrm{extr}}(\lambda)$, see Definition 12.8, will consist of Hecke characters $\mathbb{L}^{i,j}$, see Section 7.1, and motives from elliptic modular forms. Namely, to all the spaces of elliptic modular forms

$$S_k(\Gamma_0(1)),\quad S_k^{\mathrm{new}}(\Gamma_0(3)),\quad S_k^{\pm}(\Gamma_1(3)),\quad S_k^{\pm}(\Gamma_0(9)),\quad S_k^{\chi}(\Gamma_0(9))$$

introduced in Section 11.1, we have corresponding motives (of twice their dimension)

$$S[\Gamma_0(1),k],\quad S^{\mathrm{new}}[\Gamma_0(3),k],\quad S_k^{\pm}[\Gamma_1(3),k],\quad S^{\pm}[\Gamma_0(9),k],\quad S^{\chi}[\Gamma_0(9),k]$$

with the property that for any prime p the trace of the Hecke operator $T(p)$ on the space of modular forms equals the trace of Frobenius F_v on the corresponding motive.

Definition 12.1. We define the $\mathfrak{S}_4$-representations

$$\alpha_j = \begin{cases} \mathbf{s}_4 & j \equiv_6 0, \\ \mathbf{s}_{1^4} & j \equiv_6 3, \\ 0 & \text{else}, \end{cases} \qquad \beta_j = \begin{cases} \mathbf{s}_{3,1} & j \equiv_6 0, \\ \mathbf{s}_{2,1^2} & j \equiv_6 3, \\ 0 & \text{else}, \end{cases}$$

and we define δ_j to be 1 if $j \equiv_6 0$ and 0 else.

Proposition 3.9 gives a formula for the (normalized) Eisenstein cohomology $e_{\mathrm{Eis}}(\lambda) = e_c(\lambda) - e_!(\lambda)$ for all regular λ. Proposition 3.9 is only formulated in Betti cohomology, but Harder's result is actually motivic in the sense of Section 7.2. We generalize the formula in the following definition.

Definition 12.2. We define $e'_{\mathrm{Eis}}(\lambda)$ as

$$-(\alpha_i+\beta_i)\mathbb{L}^{0,0} + (\alpha_{i-b-1}+\beta_{i-b-1})\mathbb{L}^{b+1,0} + (\alpha_{i+a+1}+\beta_{i+a+1})\,\mathbb{L}^{0,a+1} - \begin{cases} \alpha_{i-1}(\mathbb{L}+1)\mathbb{L}^{b+1,0} & \text{if } a=0 \text{ and } b\equiv_2 0, \\ \alpha_{i+1}(\mathbb{L}+1)\mathbb{L}^{0,a+1} & \text{if } b=0 \text{ and } a\equiv_2 0. \end{cases}$$

All the following contributions should be found in $e_!(\lambda)$. Note that the Hodge degrees of all these contributions, for a regular local system, are either $(a+b+2,0)$, $(b+1,a+1)$, or $(0,a+b+2)$, as they need to be, see Section 3.2. Note also that the sign is always positive, which it should be for regular local systems since in that case only the second inner cohomology group can be non-zero, see Section 3.2.

Before we formulate the contributions from lifts we recall that expressions $s_k(\Gamma_0(N))$ denote dimensions of elliptic modular cusp forms and an expression $S[k,\Gamma_0(N)]$ denotes a motive corresponding to a space of elliptic modular cusp forms.

Definition 12.3. We define the central endoscopic term $e_{\mathrm{ce}}(\lambda)$ as

$$\begin{aligned}\alpha_i\big(s_{a+b+3}(\Gamma_0(1)) + s^-_{a+b+3}(\Gamma_1(3))\big)\mathbb{L}^{b+1,a+1}\\ +\beta_i\big(s_{a+b+3}(\Gamma_0(1)) + s^{\mathrm{new}}_{a+b+3}(\Gamma_0(3)) + s^+_{a+b+3}(\Gamma_1(3))\big)\mathbb{L}^{b+1,a+1}\\ +\beta_{i+3}\, s^+_{a+b+3}(\Gamma_0(9))\,\mathbb{L}^{b+1,a+1} + (\delta_i + \delta_{i+3})\,\mathbf{s}_{2^2}\, s^\chi_{a+b+3}(\Gamma_0(9))\mathbb{L}^{b+1,a+1}\\ +\alpha_i\mathbb{L}^{b+1,a+1} \quad \text{if } (a,b) \equiv_6 (5,5).\end{aligned}$$

The following two contributions are connected to the lifted (holomorphic) forms in $S_{b,a+3,i+2}(\Gamma[\sqrt{-3}])$ described in the conjectures of Section 11.

Definition 12.4 (Holomorphic 1-dimensional lifts). We define $e_{1\ell}(\lambda)$ as

$$\alpha_{i+a+4}s^-_{b+2}(\Gamma_0(9))\,\mathbb{L}^{a+b+2,0} + \begin{cases}\alpha_{i+3}\mathbb{L}^{a+b+2,0} & \text{if } (a,b) \equiv_6 (5,2),\\ \alpha_{i+4}\mathbb{L}^{a+2,0} & \text{if } a \equiv_6 3 \text{ and } b = 0.\end{cases}$$

Definition 12.5 (Holomorphic 2-dimensional lifts). We define $e_{2\ell}(\lambda)$ as

$$\begin{aligned}\alpha_{i+3}S^-[\Gamma_0(9), a+b+3]\\ +\alpha_{i-b-1}\big(S[\Gamma_0(1), a+2] + S^-[\Gamma_1(3), a+2]\big)\mathbb{L}^{b+1,0}\\ +\beta_{i-b-1}\big(S[\Gamma_0(1), a+2] + S^{\mathrm{new}}[\Gamma_0(3), a+2] + S^-[\Gamma_1(3), a+2]\big)\mathbb{L}^{b+1,0}\\ +\beta_{i-b+2}S^-[\Gamma_0(9), a+2]\mathbb{L}^{b+1,0}\\ +(\delta_{i-b-1} + \delta_{i-b+2})\,\mathbf{s}_{2^2}\, S^\chi[\Gamma_0(9), a+2]\mathbb{L}^{b+1,0}.\end{aligned}$$

Let $\overline{S}_{j,k,l}(\Gamma[\sqrt{-3}])$ be isomorphic as a vector space to $\overline{S}_{j,k,l}(\Gamma[\sqrt{-3}])$ but such that the Hecke operator $T(\nu)$ acts on $\overline{S}_{j,k,l}(\Gamma[\sqrt{-3}])$ as $T(\overline{\nu})$ acts on $S_{j,k,l}(\Gamma[\sqrt{-3}])$. The following two contributions (i.e., Definitions 12.6 and 12.7) are connected to the lifted anti-holomorphic forms. In other words, they areconnected to the lifted (holomorphic) forms in $\overline{S}_{a,b+3,-i-1}(\Gamma[\sqrt{-3}])$ except for the contribution of the form $\alpha_{i+3}S^-[\Gamma_0(9), a+b+3]$. Compare this with Proposition 7.2.

Definition 12.6 (Anti-holomorphic 1-dimensional lifts). We define $e_{\overline{1\ell}}(\lambda)$ as

$$\alpha_{i-b+2}s^-_{a+2}(\Gamma_0(9))\mathbb{L}^{0,a+b+2} + \begin{cases}\alpha_{i+3}\mathbb{L}^{0,a+b+2} & \text{if } (a,b) \equiv_6 (2,5),\\ \alpha_{i-2}\mathbb{L}^{0,b+2} & \text{if } a = 0 \text{ and } b \equiv_6 3.\end{cases}$$

Definition 12.7 (Anti-holomorphic 2-dimensional lifts). We define $e_{\overline{2\ell}}(\lambda)$ as

$$\begin{aligned}\alpha_{i+a+1}\big(S[\Gamma_0(1), b+2] + S^-[\Gamma_1(3), b+2]\big)\mathbb{L}^{0,a+1}\\ +\beta_{i+a+1}\big(s[\Gamma_0(1), b+2] + S^{\mathrm{new}}[\Gamma_0(3), b+2] + S^-[\Gamma_1(3), b+2]\big)\mathbb{L}^{0,a+1}\\ +\beta_{i+a+4}S^-[\Gamma_0(9), b+2]\mathbb{L}^{0,a+1}\\ +(\delta_{i+a+1} + \delta_{i+a+4})\,\mathbf{s}_{2^2}\, S^\chi[\Gamma_1(9), b+2]\mathbb{L}^{0,a+1}.\end{aligned}$$

Definition 12.8. We put

$$e_{\text{extr}}(\lambda) = e'_{\text{Eis}}(\lambda) + e_{\text{ce}}(\lambda) + e_{1\ell}(\lambda) + e_{2\ell}(\lambda) + e_{\overline{1\ell}}(\lambda) + e_{\overline{2\ell}}(\lambda)$$

and we define $e_{\text{extr},\mu}(\lambda)$ as in Section 7.2.

Main conjecture 12.9. *We conjecture that for any $\lambda \neq (0,0,0)$, ν and μ, if $i \equiv_2 a+b$, then*

$$\operatorname{Tr}\big(T(\nu), S^{\text{gen}}_{n(\lambda)}\big(\Gamma[\sqrt{-3}]\big)^{\mu}\big) = \operatorname{Tr}\big(F_\nu, e_{c,\mu}(\lambda) - e_{\text{extr},\mu}(\lambda)\big).$$

Assuming that the conjecture is true, this gives a possibility to compute the trace of the Hecke operators $T(\nu)$ by counts of points over $\mathbb{F}_p$ as described in Section 8.

Remark 12.10. Note that $e_{\text{extr}}(\lambda') = \overline{e_{\text{extr}}(\lambda)}$.

The main conjecture then implies that there is a Hecke-invariant isomorphism between $S^{\text{gen}}_{j,k+3,l}(\Gamma[\sqrt{-3}])^{\mu}$ and $\overline{S}^{\text{gen}}_{k,j+3,1-l}(\Gamma[\sqrt{-3}])^{\mu}$. Note that in general

$$\dim S_{j,k+3,l}\big(\Gamma[\sqrt{-3}]\big) \neq \dim S_{k,j+3,1-l}\big(\Gamma[\sqrt{-3}]\big).$$

According to Theorem 4.7, the difference of dimensions is equal to

	$l \equiv_3 0$	$l \equiv_3 1$	$l \equiv_3 2$
$j \equiv_3 0$	$k-1$	$1-j$	0
$j \equiv_3 1$	$1-j$	$k-1$	0
$j \equiv_3 2$	0	0	$k-j$

which is a consequence of the presence lifts.

Definition 12.11. We define $E_{\text{extr}}(\lambda)$ as $e_{\text{extr}}(\lambda)$ but replacing

- $\mathbb{L}^{i,j}$ by 1 for any i, j;
- $S[\Gamma_0(1),k]$, $S^{\text{new}}[\Gamma_0(3),k]$, $S^{\pm}[\Gamma_1(3),k]$, $S^{\pm}[\Gamma_0(9),k]$, $S^{\chi}[\Gamma_0(9),k]$ by

$$2\dim S_k(\Gamma_0(1)),\ 2\dim S_k^{\text{new}}(\Gamma_0(3)),\ 2\dim S_k^{\pm}(\Gamma_1(3)),$$
$$2\dim S_k^{\pm}(\Gamma_0(9)),\ 2\dim S_k^{\chi}(\Gamma_0(9)),$$

  respectively, for any k,

in the formulas of Definitions 12.2–12.7. This gives an element in the Galois group of representations of $\mathfrak{S}_4$.

Conjecture 12.12. *We conjecture that, for any λ,*

$$\dim_{\mathfrak{S}_4} S^{\text{gen}}_{n(\lambda)}\big(\Gamma[\sqrt{-3}]\big) = \frac{1}{3}\big(E_c(\lambda) - E_{\text{extr}}(\lambda)\big).$$

12.2. A congruence modulo 9. Our experimental data lead us to conjecture a congruence for the eigenvalues of Hecke operators.

Conjecture 12.13. *For any $j, k, l \geq 0$, ν and $\mu \vdash 4$, we conjecture that*

$$\mathrm{Tr}\big(T(\nu), S^{\mathrm{gen}}_{j,k+3,l}(\Gamma[\sqrt{-3}])^{\mu}\big)/\dim \mathbf{s}_{\mu} \equiv_9 3 \dim S^{\mathrm{gen}}_{j,k+3,l}(\Gamma[\sqrt{-3}])^{\mu}/\dim \mathbf{s}_{\mu}.$$

Remark 12.14. If one uses the evidence of the results in [9] one might also conjecture that for a prime $p \equiv_3 2$ the trace on the space of genuine forms is divisible by 9.

For the lifted forms described in Section 11.5 this means that for a prime $p \equiv_3 1$ the congruence

$$a_p(f)^2 - p^{k-1} \equiv_9 3$$

should hold, and similarly a congruence $a_p(f)^2 - p^{k-1} \equiv_9 0$ for primes $p \equiv_3 2$.

12.3. Evidence. The conjectures of this section and the previous were based upon the computations described in Section 10.1. Here we list a series of regularities in this data that lends credence to the conjectures.

The following holds for all λ such that $a + b + 2 \leq 40$:

- The integer
$$E_c(\lambda) - E_{\mathrm{extr}}(\lambda)$$
is divisible by 3.
- We find, using Theorem 4.7, that $\dim S_{n(\lambda)}(\Gamma[\sqrt{-3}])$ equals
$$\frac{1}{3}\big(E_c(\lambda) - E_{\mathrm{extr}}(\lambda)\big),$$
when replacing $\mathbf{s}_{\mu}$ with $\dim \mathbf{s}_{\mu}$, and adding the dimension of the lifts described in Section 11.
- If
$$E_c(\lambda) - E_{\mathrm{extr}}(\lambda) = 0,$$
then
$$\mathrm{Tr}\big(F_q, e_c(\lambda) - e_{\mathrm{extr}}(\lambda)\big) = 0$$
for $q \leq 67$ and $q \equiv_3 1$.
- We have that (compare with Conjecture 12.13)
$$\mathrm{Tr}\big(F_q, e_c(\lambda) - e_{\mathrm{extr}}(\lambda)\big) \equiv_9 E_c(\lambda) - E_{\mathrm{extr}}(\lambda)$$
for $q \leq 67$ and $q \equiv_3 1$.
- All traces computed (for $j = 0$) in [18] match with the ones computed using Conjecture 12.9 for $p \leq 67$ and $p \equiv_3 1$.

- The ring of scalar valued modular forms, i.e., when $j = 0$, is given in Proposition 2.2. This gives a formula for $\dim S^{\mathrm{gen}}_{0,k+3,l}(\Gamma[\sqrt{-3}])^{\mu}$ for any k, l, μ, which matches the one given by Conjecture 12.12 (for $a + b + 2 \le 40$). For instance, we have that

$$S_{0,6k+3,2}\big(\Gamma[\sqrt{-3}]\big) = M_{0,6k-9,0}\big(\Gamma[\sqrt{-3}]\big)\zeta^2,$$
$$\dim_{\mathfrak{S}_4} S_{0,6k+3,2}\big(\Gamma[\sqrt{-3}]\big) = \mathrm{Sym}^{2k-3}(\mathbf{s}_{2,1^2})$$

  for $k \ge 2$.

- All traces computed in [9] match with the ones computed using Conjecture 12.9 for $p \le 67$ and $p \equiv_3 1$.

Note that the information

$$\mathrm{Tr}\big(F_{q^r}, e_c(\lambda) - e_{\mathrm{extr}}(\lambda)\big)$$

for $r = 1, \ldots, (E_c(\lambda) - E_{\mathrm{extr}}(\lambda))$ gives a way to compute the characteristic polynomial of F_q acting on $e_c(\lambda) - e_{\mathrm{extr}}(\lambda)$, assuming that it is effective of dimension $E_c(\lambda) - E_{\mathrm{extr}}(\lambda)$.

For all λ such that $a + b + 2 \le 40$ and $E_c(\lambda) - E_{\mathrm{extr}}(\lambda) = 3$ (see further in Section 13.1) the characteristic polynomial for $q = 4$ and the partial information for $q = 7$ has the expected structure (namely the one derived from the results of Section 2.3).

See also Section 14 for evidence coming from congruences studied by Harder.

12.4. Modules of vector-valued forms. Define

$$\mathcal{M}_j = \mathcal{M}_j^0 \oplus \mathcal{M}_j^1 \oplus \mathcal{M}_j^2$$

with

$$\mathcal{M}_j^{\ell} = \oplus_k M_{j,k,l}\big(\Gamma[\sqrt{-3}]\big).$$

Then $\mathcal{M}_j$ is a module over $\mathcal{M}_0$; for $\mathcal{M}_0$ see Proposition 2.2. Guided by the heuristics of our conjectures the structure of some modules $\mathcal{M}_j$ was determined in [9], e.g., for $j = 1, 2, 3$. For example, the module $\mathcal{M}_1^0$ is generated over $\mathcal{M}_0^0$ by three forms $\Phi_0, \Phi_1, \Phi_2 \in S_{1,7,0}(\Gamma[\sqrt{-3}])$ satisfying a relation

$$\varphi_0\Phi_0 + \varphi_1\Phi_1 + \varphi_2\Phi_2 = 0$$

with $\varphi_0, \varphi_1, \varphi_2$ generators of the ring $\mathcal{M}_0^0$. For a table of Hecke eigenvalues of the Φ_i, we refer to [9, Table 7].

12.5. Conjecture for the moduli space of genus 2 curves. In this section we will be brief and give a similar conjecture to the one above but in the case of genus 2.

Define the normalized compactly supported Euler characteristic e_c^{norm} analogously to Definition 7.6. Recall the notation from Section 8.2.1 and define the representation

$$\alpha_k = \begin{cases} \mathbf{s}_2\widetilde{\mathbf{s}}_2 & k \equiv_2 0, \\ \mathbf{s}_{1,1}\widetilde{\mathbf{s}}_2 & k \equiv_2 1, \end{cases} \qquad \beta_k = \begin{cases} \mathbf{s}_{1,1}\widetilde{\mathbf{s}}_{1,1} & k \equiv_2 0, \\ \mathbf{s}_2\widetilde{\mathbf{s}}_{1,1} & k \equiv_2 1.x \end{cases}$$

Let W_3 denote the Fricke operator and for any prime p let $T(p)$ be the Hecke operator. For the proofs of the following properties of W_3, see [2].

If $k > 0$ is even and $f \in S_{k+2}(\Gamma_1(3))$ is an eigenform with $T(p)f = a_p f$, then $a_p = \overline{a}_p$ for all primes p. Moreover, $a_3 = \pm 3^{k/2}$ and

$$W_3(f) = -\mathrm{sgn}(a_3) f.$$

The $\pm$-spaces of W_3 are clearly Hecke invariant and we denote the $\pm$-eigenspaces of $S_{k+2}(\Gamma_1(3))$ by $S^{\pm}_{k+2}(\Gamma_1(3))$. Define $S^{\pm}[\Gamma_1(3), k+2]$ analogously.

If k is odd, then

$$S_{k+2}(\Gamma_1(3)) = S_{k+2}(\Gamma_0(3), \chi),$$

where χ is the Dirichlet character of order 2. If $f \in S_{k+2}(\Gamma_0(3), \chi)$ is an eigenform with $T(p)f = a_p f$, then $\overline{a}_p = \chi(p) a_p$ for all $p \nmid 3$ and

$$W_3(f) = c\overline{f}$$

for some $c \in \mathbb{C}$ with $|c| = 1$, and

$$W_3^2(f) = -f.$$

If $f \neq \overline{f}$, then $\pm i\overline{c} f + \overline{f}$ is an eigenvector for W_3 with eigenvalue $\mp i$. If $p \equiv_3 1$, then both these are also eigenvectors of $T(p)$ with eigenvalue a_p. For $k \equiv_3 2$, then there is an eigenvector $f \in S_{k+2}(\Gamma_0(3), \chi)$ such that $a_3 = (-3)^{(k+1)/2}$,

$$W_3(f) = (-1)^{(k+1)/2} i f$$

and

$$a_p(f) = \mathrm{Tr}\big(F_p, \mathbb{L}^{k+1,0} + \mathbb{L}^{0,k+1}\big).$$

We denote the $\pm i$-eigenspaces of $S_{k+2}(\Gamma_1(3))$ by $S^{\pm}_{k+2}(\Gamma_1(3))$, and we define $S^{\pm}[\Gamma_1(3), k+2]$ analogously.

Conjecture 12.15. *For any $k > 0, l \geq 0$ such that $k \equiv_6 l$, we have*

$$\begin{aligned} e_c^{\mathrm{norm}}\big(\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_{k,l}\big) &= -\alpha_k - \beta_k - S^+[\Gamma_1(3), k+2]\alpha_k - S^-[\Gamma_1(3), k+2]\,\beta_k \\ &\quad + \delta_{k+1}\big(\mathbb{L}^{k+1,0}\alpha_k + \mathbb{L}^{0,k+1}\beta_k\big) + \delta_{k+7}\big(\mathbb{L}^{0,k+1}\alpha_k + \mathbb{L}^{k+1,0}\beta_k\big) \end{aligned}$$

as elements of $K_0^{\mathfrak{S}_2\times\mathfrak{S}_2}(\mathrm{Gal}_F)$, and where $\delta_i = 1$ if $i \equiv_{12} 0$ and 0 otherwise.

For $k = 0$, put

$$S^+[\Gamma_1(3), 2] = -\mathbb{L} - 1 \quad \text{and} \quad S^-[\Gamma_1(3), 2] = 0$$

to make the formula correct, by Proposition 8.3. Looking at how $\mathfrak{S}_2 \times \mathfrak{S}_2$ acts on the two cusps of the Bailey–Borel compactification, we find that the Eisenstein contribution to $e_c^{\text{norm}}(\mathcal{X}^{(2)}_{\Gamma[\sqrt{-3}]}, \mathbb{W}_{k,l})$ for $k \equiv_6 l$ is equal to $-\alpha_k - \beta_k$ (compare with the proof of Proposition 3.9).

13. Examples

We now illustrate our conjectures and calculations by examples. First, we recall the dimensions

$$\dim \mathbf{s}_4 = \dim \mathbf{s}_{1^4} = 1, \quad \dim \mathbf{s}_{3,1} = \dim \mathbf{s}_{2,1^2} = 3, \quad \dim \mathbf{s}_{2^2} = 2.$$

Furthermore, recall that for $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ we now write

$$\lambda = (a+i, i, -b+i) = (n_1+n_2+n_3, n_2+n_3, n_3), \quad n(\lambda) = (b, a+3, i+2).$$

13.1. One-dimensional spaces of genuine forms. The cases for which

$$\dim S^{\text{gen}}_{j,k,l}\big(\Gamma[\sqrt{-3}]\big)^{\mu} = \dim \mathbf{s}_{\mu}$$

are of special importance to us since in these cases counts of points over finite fields gives (using Conjecture 12.9) Hecke eigenvalues rather than just traces. Namely, when this holds, there is a Hecke eigenform F in $S^{\text{gen}}_{j,k,l}(\Gamma[\sqrt{-3}])^{\mu}$ such that the linear span of its orbit under $\mathfrak{S}_4$ equals the whole of $S^{\text{gen}}_{j,k,l}(\Gamma[\sqrt{-3}])^{\mu}$. So, if $\lambda_\nu(F)$ denote its Hecke eigenvalue, then

$$\operatorname{Tr}\big(T(\nu), S^{\text{gen}}_{j,k,l}\big(\Gamma[\sqrt{-3}]\big)^{\mu}\big) = \lambda_\nu(F) \cdot \dim \mathbf{s}_{\mu}.$$

For $\mu = (4)$ we found 78 such cases using Conjecture 12.12. We list all such (j, k, l) below, but because of Remark 12.10 we only list them up to duality:

(0, 15, 1)	(0, 21, 1)	(0, 24, 1)	(0, 27, 0)	(0, 30, 1)	(0, 30, 2)	(0, 33, 0)
(0, 36, 0)	(0, 36, 2)	(0, 39, 2)	(0, 42, 0)	(0, 45, 2)	(1, 16, 0)	(1, 19, 0)
(1, 19, 1)	(1, 19, 2)	(1, 22, 1)	(1, 25, 1)	(1, 28, 2)	(2, 11, 0)	(2, 11, 1)
(2, 14, 0)	(2, 14, 1)	(2, 20, 2)	(2, 23, 2)	(3, 9, 1)	(3, 12, 0)	(3, 12, 1)
(3, 15, 0)	(3, 18, 2)	(4, 7, 0)	(4, 10, 0)	(4, 10, 2)	(4, 13, 2)	(5, 8, 0)
(5, 11, 2)	(5, 14, 2)	(6, 9, 0)	(6, 9, 2)	(7, 10, 2)		

For $\mu = (3,1), (2^2), (2,1^2), (1^4)$ we found 35, 44, 35, 76 cases, respectively.

13.2. Explicit examples. Let us study a series of local systems in some detail.

13.2.1. $\lambda = (10, 4, 4)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{0,9,0}(\Gamma[\sqrt{-3}])$.

From the formula described in Section 7 we get the numerical Euler characteristic,

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = \mathbf{s}_4 + 2\mathbf{s}_{3,1} + 3\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

Following Section 12.1, the only non-zero contributions we have to $e_{\text{extr}}(\lambda)$ are

$$e'_{\text{Eis}}(\lambda) = \mathbb{L}^{1,0}(\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}), \quad e_{\overline{1\ell}}(\lambda) = \mathbb{L}^{0,8}\mathbf{s}_4,$$
$$e_{2\ell}(\lambda) = S^-[\Gamma_0(9), 8]\,\mathbb{L}^{1,0}\mathbf{s}_{3,1} + S^{\text{new}}[\Gamma_0(3), 8]\,\mathbb{L}^{1,0}\mathbf{s}_{2,1^2}.$$

In $e_{2\ell}(\lambda)$, the $\mathbf{s}_{3,1}$-term comes from the Kudla lift denoted $F_{9,1} = \varphi_0\varphi_1(\varphi_0 - \varphi_1)$ and the $\mathbf{s}_{2,1^2}$-term comes from the Kudla lift of weight 9 denoted $F_{9,2}$ by Finis [18, Tables, pp. 151, 177].

Removing the extraneous contributions should leave us with contributions from the genuine Picard modular cusp forms. Starting with the numerical Euler characteristic we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) - E_{\text{extr}}\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 0,$$

so there should be no genuine forms. And indeed we find that

$$\text{Tr}\big(F_\nu, e_c(\lambda)\big) - \text{Tr}\big(F_\nu, e_{\text{extr}}(\lambda)\big),$$

which conjecturally equals

$$\text{Tr}\big(T(\nu), S^{\text{gen}}_{0,9,0}(\Gamma[\sqrt{-3}])\big)$$

is 0 for all $p \le 67$.

This also fits (adding the lifts and recalling Definition 4.9) with the formula

$$\dim_{\mathfrak{S}_4} S_{0,9,0}\big(\Gamma[\sqrt{-3}]\big) = \mathbf{s}_{3,1} + \mathbf{s}_{2,1^2}$$

that we find by Proposition 2.2.

13.2.2. $\lambda = (16, 4, 4)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{0,15,0}(\Gamma[\sqrt{-3}])$.

Again, from the formula in Section 7, we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = \mathbf{s}_4 + 5\mathbf{s}_{3,1} + 3\mathbf{s}_{2^2} + 7\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

Similarly to the previous case, following Section 12.1, we find that $e_{\text{extr}}(\lambda)$ consists of

$$e'_{\text{Eis}}(\lambda) = \mathbb{L}^{1,0}(\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}), \quad e_{\overline{1\ell}}(\lambda) = \mathbb{L}^{0,14}\mathbf{s}_4,$$
$$e_{2\ell}(\lambda) = S^-[\Gamma_0(9), 14]\mathbb{L}^{1,0}\mathbf{s}_{3,1} + S^{\text{new}}[\Gamma_0(3), 14]\mathbb{L}^{1,0}\mathbf{s}_{2,1^2}.$$

p	$S^{\mathrm{gen}}_{0,15,0}(\Gamma[\sqrt{-3}])^{(2^2)}$	$S^{\mathrm{gen}}_{0,15,0}(\Gamma[\sqrt{-3}])^{(3,1)}$
7	$-388107\rho - 1608891$	$-524625 - 205857\rho$
13	$-60967989\rho - 9061701$	$-36504663 + 20888505\rho$
19	$-578216997\rho - 665720736$	$-398615136 - 1035916731\rho$
31	$-690422256\rho - 8829510909$	$32032766937 + 14052080592\rho$
37	$-111679368147\rho - 59483009571$	$30023590017 - 12661429743\rho$
43	$-98981609184\rho + 131622854187$	$298590045213 + 634311769248\rho$

Table 3.

Removing the extraneous contributions should leave us with contributions from the genuine Picard modular cusp forms. Starting with the numerical Euler characteristic we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) - E_{\mathrm{extr}}\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 3\mathbf{s}_{3,1} + 3\,\mathbf{s}_{2^2}.$$

Dividing this expression by 3 gives the conjectural result

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{0,15,0}\big(\Gamma[\sqrt{-3}]\big) = \mathbf{s}_{3,1} + \mathbf{s}_{2^2}.$$

Together with the lifts, we get

$$\dim_{\mathfrak{S}_4} S_{0,15,0}\big(\Gamma[\sqrt{-3}]\big) = 2\mathbf{s}_{3,1} + \mathbf{s}_{2^2} + 3\mathbf{s}_{2,1^2},$$

which fits with the formula we find by Proposition 2.2.

We can then compute

$$\mathrm{Tr}\big(F_\nu, e_c(\lambda)\big) - \mathrm{Tr}\big(F_\nu, e_{\mathrm{extr}}(\lambda)\big),$$

which conjecturally equals

$$\mathrm{Tr}\big(T(\nu), S^{\mathrm{gen}}_{0,15,0}(\Gamma[\sqrt{-3}])\big).$$

For the two $1 \cdot \dim \mathbf{s}_\mu$-dimensional isotypic components of the space of genuine forms we then (conjecturally) get Hecke eigenvalues as described in Section 13.1, and a few of them are given in Table 3. Note that the analogue of the Ramanujan conjecture for this situation holds; $N(\xi_\nu) \leq 3N(\nu)^{a+b+2}$ for eigenvalues ξ_ν. Similar observations can be made for the other tables appearing in this section.

13.2.3. $\boldsymbol{\lambda = (32, 2, 2)}$. In the cohomology of this local system we expect to find contributions from the forms in $S_{0,33,1}(\Gamma[\sqrt{-3}])$.

Again, from the formula in Section 7, we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 9\mathbf{s}_4 + 27\mathbf{s}_{3,1} + 9\mathbf{s}_{2^2} + 19\mathbf{s}_{2,1^2} + 2\mathbf{s}_{1^4}.$$

p	$S^{\mathrm{gen}}_{0,33,1}(\Gamma[\sqrt{-3}])^{(1^4)}$
7	$-17187741337239\rho - 27371045932368$
13	$619757358250752891\rho - 73897512261622296$
19	$3239797517682438611\rho + 1611097296842413037555$
31	$-450614269323285049766016\rho + 463109207345192219515905$
37	$4464950074069806168802623\rho - 6793659375871694908403 76$
43	$-62575475768038597846807512\rho - 83275045472246397000970011$

Table 4.

Following Section 12.1, we find that $e_{\mathrm{extr}}(\lambda)$ only consists of

$$e'_{\mathrm{Eis}}(\lambda) = \mathbb{L}^{0,31}\mathbf{s}_{2,1^2} - \mathbb{L}^{1,32}\mathbf{s}_{1^4}.$$

Removing this contribution from the Euler characteristic and dividing by 3, as in the previous example, we get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{0,33,1}(\Gamma[\sqrt{-3}]) = 3\mathbf{s}_4 + 9\mathbf{s}_{3,1} + 3\mathbf{s}_{2^2} + 6\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

Since there are no lifts, this is the same as the dimensions of all cusp forms and it equals the formula $\mathbf{s}_{1^4}\mathrm{Sym}^9(\mathbf{s}_{2,1^2})$, found using Proposition 2.2.

Some (conjectural) Hecke eigenvalues for the 1-dimensional isotypic component of the space of genuine forms corresponding to $\mathbf{s}_{1^4}$ are given in Table 4.

13.2.4. $\lambda = (7, 1, -2)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{3,9,0}(\Gamma[\sqrt{-3}])$. Note that the modular forms occurring here are described in [9, Prop. 15.2].

From the formula described in Section 7, we get

$$E_c(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) = \mathbf{s}_4 + 5\mathbf{s}_{3,1} + 6\mathbf{s}_{2^2} + 9\mathbf{s}_{2,1^2} + 4\mathbf{s}_{1^4}.$$

From Proposition 3.9 and Section 12.1, we get

$$e'_{\mathrm{Eis}}(\lambda) = e_{\mathrm{Eis}}(\lambda) = \mathbb{L}^{4,0}(\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}).$$

The only other non-zero contributions we have to $e_{\mathrm{extr}}(\lambda)$ are

$$e_{\overline{1\ell}}(\lambda) = \mathbb{L}^{0,11}\mathbf{s}_4,$$
$$e_{2\ell}(\lambda) = S^{-}[\Gamma_0(9), 8]\mathbb{L}^{4,0}\mathbf{s}_{3,1} + S^{\mathrm{new}}[\Gamma_0(3), 8]\mathbb{L}^{4,0}\mathbf{s}_{2,1^2}.$$

Again, removing the extraneous contributions from the Euler characteristic and dividing by 3, we get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{3,9,0}(\Gamma[\sqrt{-3}]) = \mathbf{s}_{3,1} + 2\,\mathbf{s}_{2^2} + 2\,\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

p	$S^{\text{gen}}_{3,9,0}\big(\Gamma[\sqrt{-3}]\big)^{(3,1)}$	$S^{\text{gen}}_{3,9,0}\big(\Gamma[\sqrt{-3}]\big)^{(1^4)}$
7	$-2661 - 3735\rho$	$-39273 - 37755\rho$
13	$697611 - 853785\rho$	$-616209 - 1939509\rho$
19	$-4019046 - 4493727\rho$	$2924922 + 16469397\rho$
31	$236296587 + 26549946\rho$	$-13532361 - 40067046\rho$
37	$381974925 - 151949367\rho$	$-294789795 - 270210663\rho$
43	$685398387 + 28100862\rho$	$1093524015 + 1099688094\rho$

Table 5.

Together with the lifts we get the conjecture

$$\dim_{\mathfrak{S}_4} S_{3,9,0}\big(\Gamma[\sqrt{-3}]\big) = 2\mathbf{s}_{3,1} + 2\,\mathbf{s}_{2^2} + 3\,\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

This conjectural expression fits with what Theorem 4.7 tells us, namely,

$$\dim S_{3,9,0}\big(\Gamma[\sqrt{-3}]\big) = 20.$$

For the two $1 \cdot \dim \mathbf{s}_\mu$-dimensional isotypic components of the space of genuine forms we then (conjecturally) get Hecke eigenvalues as described in Section 13.1, and a few of them are given in Table 5.

13.2.5. $\boldsymbol{\lambda = (11, 5, 2)}$. In the cohomology of this local system we expect to find contributions from the forms in $S_{3,9,1}(\Gamma[\sqrt{-3}])$.

Again, from the formula in Section 7, we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 4\mathbf{s}_4 + 7\mathbf{s}_{3,1} + 5\mathbf{s}_{2^2} + 6\mathbf{s}_{2,1^2} + 3\mathbf{s}_{1^4}.$$

From Proposition 3.9 and Section 12.1, we get that the only non-zero contributions we have to $e_{\text{extr}}(\lambda)$ are

$$e'_{\text{Eis}}(\lambda) = e_{\text{Eis}}(\lambda) = \mathbb{L}^{0,7}(\mathbf{s}_4 + \mathbf{s}_{3,1}),$$
$$e_{\overline{2\ell}}(\lambda) = \mathbb{L}^{0,7} S^{\chi}[\Gamma_0(9), 5]\,\mathbf{s}_{2,2}.$$

Removing this contribution from the Euler characteristic and dividing by 3, as in the previous example, we get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\text{gen}}_{3,9,1}\big(\Gamma[\sqrt{-3}]\big) = \mathbf{s}_4 + 2\mathbf{s}_{3,1} + \mathbf{s}_{2^2} + 2\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

This is the same as the dimensions of all cusp forms since there are no lifts which fits with the result

$$\dim S_{3,9,1}\big(\Gamma[\sqrt{-3}]\big) = 16,$$

following from Theorem 4.7.

p	$S^{\mathrm{gen}}_{3,9,1}(\Gamma[\sqrt{-3}])^{(2^2)}$	$S^{\mathrm{gen}}_{3,9,1}(\Gamma[\sqrt{-3}])^{(1^4)}$
7	$-39753\rho - 15702$	$-3303\rho - 20562$
13	$-2259729\rho - 462012$	$39537\rho - 662244$
19	$-813897\rho - 7616175$	$-12094443\rho - 15482085$
31	$62423118\rho + 189603705$	$13979610\rho - 2791545$
37	$154008855\rho - 213937620$	$-132007005\rho - 420798660$
43	$-1091048814\rho - 311480763$	$-1442196450\rho - 484155105$

Table 6.

Using the same method as above, we compute some (conjectural) Hecke eigenvalues as described in Section 13.1 for two of the $1 \cdot \dim \mathbf{s}_\mu$-dimensional spaces in Table 6. Note that the eigenform in $S^{\mathrm{gen}}_{3,9,1}(\Gamma[\sqrt{-3}])^{(1^4)}$ equals $(E_0+E_2+E_2-E_3)\zeta$, with E_i the Eisenstein series given in [9, Lemma 15.1].

13.2.6. $\lambda = (9,3,0)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{3,9,2}(\Gamma[\sqrt{-3}])$.

Once again, using the formula in Section 7, we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 2\mathbf{s}_4 + 7\mathbf{s}_{3,1} + 3\mathbf{s}_{2^2} + 4\mathbf{s}_{2,1^2}.$$

From Proposition 3.9 and Section 12.1, we find the non-zero contributions to $e_{\mathrm{extr}}(\lambda)$:

$$e'_{\mathrm{Eis}}(\lambda) = e_{\mathrm{Eis}}(\lambda) = -\mathbb{L}^{0,0}(\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}),$$
$$e_{\mathrm{ce}}(\lambda) = \mathbb{L}^{4,7}(\mathbf{s}_{3,1} + 2\,\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}), \quad e_{2\ell}(\lambda) = S^{-}[\Gamma_0(9), 12]\mathbf{s}_4.$$

As in the previous examples, we use the numerical Euler characteristic to get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{3,9,2}\big(\Gamma[\sqrt{-3}]\big) = 2\,\mathbf{s}_{3,1} + \mathbf{s}_{2^2} + \mathbf{s}_{2,1^2}.$$

Together with the 1-dimensional space of lifts above, we get

$$\dim S_{3,9,2}\big(\Gamma[\sqrt{-3}]\big) = 12,$$

which fits with Theorem 4.7.

Some (conjectural) Hecke eigenvalues for the $1 \cdot \dim \mathbf{s}_\mu$-dimensional spaces are found in Table 7.

13.2.7. $\lambda = (5,0,-5)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{5,8,2}(\Gamma[\sqrt{-3}])$.

p	$S^{\mathrm{gen}}_{3,9,2}\big(\Gamma[\sqrt{-3}]\big)^{(2^2)}$	$S^{\mathrm{gen}}_{3,9,2}\big(\Gamma[\sqrt{-3}]\big)^{(2,1^2)}$
7	$-522\rho - 771$	$-42405 - 73422\rho$
13	$64731 - 1053828\rho$	$2150805 + 1144836\rho$
19	$9397530\rho + 10858953$	$1117083 - 2630970\rho$
31	$-199487250\rho - 223012887$	$17764311 - 9145350\rho$
37	$-283226796\rho - 170478933$	$144010695 + 424906308\rho$
43	$456864210\rho - 855993435$	$-365663985 - 1035862434\rho$

Table 7.

Note that in this case $\lambda = \lambda'$ and hence for any μ, we have

$$\mathrm{Tr}\big(F_q, e_{c,\mu}(\lambda)\big) \in \mathbb{Z}$$

by Proposition 7.2. From Section 12.1, it also follows that

$$\mathrm{Tr}\big(F_q, e_{c,\mu}(\lambda) - e_{\mathrm{extr},\mu}(\lambda)\big) \in \mathbb{Z}.$$

Using the formula in Section 7, we get

$$E_c\big(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda\big) = 3\mathbf{s}_4 + 9\mathbf{s}_{3,1} + 6\mathbf{s}_{2^2} + 9\mathbf{s}_{2,1^2} + 3\mathbf{s}_{1^4}.$$

From Proposition 3.9 and Section 12.1, we find the non-zero contributions to $e_{\mathrm{extr}}(\lambda)$:

$$e'_{\mathrm{Eis}}(\lambda) = e_{\mathrm{Eis}}(\lambda) = (-\mathbb{L}^{0,0} + \mathbb{L}^{6,0} + \mathbb{L}^{0,6})(\mathbf{s}_4 + \mathbf{s}_{3,1}),$$
$$e_{\mathrm{ce}}(\lambda) = 2\mathbb{L}^{6,6}(\mathbf{s}_4 + \mathbf{s}_{3,1} + \mathbf{s}_{2,2}),$$
$$e_{\overline{2\ell}}(\lambda) = \mathbb{L}^{0,6} S^{\chi}[\Gamma_1(9), 7]\mathbf{s}_{2^2}, \quad e_{2\ell}(\lambda) = \mathbb{L}^{6,0} S^{\chi}[\Gamma_1(9), 7]\mathbf{s}_{2^2}.$$

As in the previous examples, we use the numerical Euler characteristic to get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{5,8,2}\big(\Gamma[\sqrt{-3}]\big) = 2\mathbf{s}_{3,1} + 3\mathbf{s}_{2,1^2} + \mathbf{s}_{1^4}.$$

Together with the 2-dimensional space of lifts above, we get

$$\dim S_{5,8,2}\big(\Gamma[\sqrt{-3}]\big) = 18,$$

which fits with Theorem 4.7.

Some (conjectural) Hecke eigenvalues for the 1-dimensional space of genuine cusp forms are found in Table 8.

p	$S^{\mathrm{gen}}_{5,8,2}\big(\Gamma[\sqrt{-3}]\big)^{(1^4)}$
7	156927
13	−4708473
19	−41663481
31	−775905585
37	−1985783685
43	1950903255

Table 8.

13.2.8. $\lambda = (11, 0, -5)$. In the cohomology of this local system we expect to find contributions from the forms in $S_{5,14,2}(\Gamma[\sqrt{-3}])$.

Using the formula in Section 7, we get

$$E_c(X_{\Gamma[\sqrt{-3}]}, \mathbb{W}_\lambda) = 9\mathbf{s}_4 + 27\mathbf{s}_{3,1} + 18\mathbf{s}_{2^2} + 27\mathbf{s}_{2,1^2} + 9\mathbf{s}_{1^4}.$$

From Proposition 3.9 and Section 12.1, we find the non-zero contributions to $e_{\mathrm{extr}}(\lambda)$:

$$e'_{\mathrm{Eis}}(\lambda) = e_{\mathrm{Eis}}(\lambda) = (-\mathbb{L}^{0,0} + \mathbb{L}^{6,0} + \mathbb{L}^{0,12})(\mathbf{s}_4 + \mathbf{s}_{3,1}),$$
$$e_{\mathrm{ce}}(\lambda) = 3\mathbb{L}^{6,12}(\mathbf{s}_4 + \mathbf{s}_{3,1} + \mathbf{s}_{2,2}), \quad e_{\overline{2\ell}}(\lambda) = \mathbb{L}^{0,12} S^{\chi}[\Gamma_1(9), 7]\mathbf{s}_{2^2},$$
$$e_{2\ell}(\lambda) = \mathbb{L}^{6,0} S^{-}[\Gamma_1(3), 13](\mathbf{s}_4 + \mathbf{s}_{3,1}) + \mathbb{L}^{6,0} S^{\chi}[\Gamma_1(9), 13]\mathbf{s}_{2^2}.$$

As in the previous examples, we use the numerical Euler characteristic to get the following conjecture:

$$\dim_{\mathfrak{S}_4} S^{\mathrm{gen}}_{5,14,2}\big(\Gamma[\sqrt{-3}]\big) = \mathbf{s}_4 + 7\mathbf{s}_{3,1} + 3\mathbf{s}_{2^2} + 9\mathbf{s}_{2,1^2} + 3\mathbf{s}_{1^4}.$$

Together with the 8-dimensional space of lifts above, we get

$$\dim S_{5,14,2}\big(\Gamma[\sqrt{-3}]\big) = 66,$$

which fits with Theorem 4.7.

Some (conjectural) Hecke eigenvalues for the 1-dimensional space of genuine cusp forms are found in Table 9.

14. Congruences of Harder type

According to Harder a prime appearing in the denominator of a certain ratio of critical values of an L-function sometimes leads to a congruence between modular forms, see [23]. The shape of these congruences was discussed by Harder after we found instances of congruences, see [24] and see also Dummigan's discussion in [14].

p	$S^{\text{gen}}_{5,14,2}(\Gamma[\sqrt{-3}])^{(4)}$
7	$-38516760\rho - 13589673$
13	$-6017408280\rho - 7487727117$
19	$546522935760\rho + 368972351247$
31	$20336092789320\rho + 55796255768703$
37	$-147394045113480\rho + 55302806453187$
43	$134094712536720\rho - 23648747132697$

Table 9.

In the case at hand we look at the standard L-function associated to an algebraic Hecke character ψ_m with the following Euler factors. For a prime $p \equiv_3 1$ with $p = \nu_p \overline{\nu}_p$ and $\nu_p \equiv \overline{\nu}_p \equiv_3 1$, we have

$$L_p(\psi_m, s) = 1/(1 - \nu_p^m p^{-s})(1 - \overline{\nu}_p^m p^{-s})$$

and for a prime $p \equiv_3 2$, we have

$$L_p(\psi_m, s) = 1/\bigl(1 - (-p)^m p^{-2s}\bigr),$$

while for $p = 3$, we have $L_3(\psi_m, s) = 1$ unless $m \equiv_6 0$ and then

$$L_3(\psi_m, s) = 1/\bigl(1 - (\sqrt{-3})^m 3^{-s}\bigr).$$

The completed L-function

$$\Lambda(\psi_m, s) = \frac{\Gamma(s)}{(2\pi)^s} \prod L_p(\psi_m, s)$$

extends to a holomorphic function of s and satisfies a functional equation relating s with $m + 1 - s$. According to (an analogue of) a result of Hurwitz (see [28]) we get rational quotients of critical values

$$Q(m,n) := \frac{\Lambda(\psi_m, n-1)}{\Lambda(\psi_m, n)} \in \mathbb{Q} \quad \text{for } n = m, m-1, \ldots, \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Conjecture 14.1 (Harder's conjecture). *If a prime $\ell > m$ divides the denominator of $Q(m,n)$, there exists a Picard modular cusp form of weight*

$$(b, a+3) = (m-n, 2n-m+1),$$

which is an eigenform of the Hecke algebra, such that its Hecke eigenvalues λ_{ν_p} for $p \equiv_3 1$ satisfy the congruence

$$\lambda_{\nu_p} \equiv_\ell \overline{\nu}_p^{a+b+2} + (p^{a+1} + 1)\nu_p^{b+1}.$$

j,k,l	μ	(m,n)	ℓ	$Q(m,n)$
$2,11,2$	(1^4)	$(15,13)$	53	$2^4/3\cdot 53$
$1,13,1$	(1^4)	$(15,14)$	19	$53/2\cdot 5\cdot 19$
$1,13,1$	$(2,1^2)$	$(15,14)$	19	$53/2\cdot 5\cdot 19$
$6,9,0$	(1^4)	$(21,15)$	271	$233/2\cdot 5\cdot 271$
$3,12,0$	(4)	$(18,15)$	29	$3\cdot 5/2\cdot 29$
$5,11,2$	(1^4)	$(21,16)$	17	$271/2\cdot 3\cdot 11\cdot 17$
$2,20,2$	(4)	$(24,22)$	97	$11\cdot 457/2^2\cdot 3\cdot 5^2\cdot 97$
$1,22,1$	(4)	$(24,23)$	41	$23\cdot 97/2\cdot 3\cdot 5\cdot 11\cdot 41$
$0,27,0$	(1^4)	$(27,27)$	449	$3^2\cdot 179\cdot 223/2^2\cdot 11\cdot 17\cdot 23\cdot 449$
$0,33,0$	(1^4)	$(33,33)$	17093	$19\cdot 84802789/2^2\cdot 5\cdot 11\cdot 17\cdot 23\cdot 29\cdot 17093$

Table 10.

p	$S^{\mathrm{gen}}_{2,11,2}(\Gamma[\sqrt{-3}])^{(1^4)}$	$S^{\mathrm{gen}}_{6,9,0}(\Gamma[\sqrt{-3}])^{(1^4)}$
7	$113760\rho+180273$	$-742581\rho-967245$
13	$6574680\rho+4136763$	$-11444355\rho+37295661$
19	$-3105720\rho+22527309$	$-1411116471\rho-1183781976$
31	$1128613680\rho-206255175$	$2162847960\rho+20439895125$
37	$-1059546600\rho-631344705$	$113910723225\rho+29288724825$
43	$-3998935080\rho-6398875995$	$-55912815000\rho-92116884255$

Table 11.

We find the following cases where the data available to us are in accordance with this conjecture (see Table 10). If the index of the local system $\mathbb{W}_\lambda$ is $\lambda=(a+i,i,-b+i)$, we list the weight $(j,k,l)=(b,a+3,i+2)$ of the modular forms, the representation of $\mathfrak{S}_4$, the index $(m,n)=(a+2b+3,a+b+3)$, the congruence prime ℓ and the value of $Q(m,n)$. These values were calculated by a Mathematica program provided to us by Harder.

The Hecke eigenvalues in two of these cases are found in Table 11.

In fact, we searched for congruences in our heuristic data (for the cases where the space $S^{\mathrm{gen}}_{j,k,l}(\Gamma[\sqrt{-3}])^{\mu}$ has dimension $\dim \mathbf{s}_\mu$) and then checked the value of the corresponding quotient of critical L-values. In all cases except one the congruence prime showed up in $Q(m,n)$. The one extra congruence not explained by the above conjecture occurs for the local system $\mathbb{W}_\lambda$ with $\lambda=(16,1,1)$ and $\mu=(3,1)$. We found a congruence modulo $\ell=37$. But the corresponding

$$Q(18,18)=3\cdot 7\cdot 19/2\cdot 5\cdot 11\cdot 17$$

p	$S_{0,18,0}(\Gamma[\sqrt{-3}])^{(3,1)}$
7	$-37133403 - 19436265\rho$
13	$-114953793 - 826184565\rho$
19	$82348187646 + 48917648907\rho$
31	$2339550247917 - 489600934794\rho$
37	$6061060465185 + 27008238932829\rho$
43	$-13426382809671 - 41363330321286\rho$

Table 12.

does not show 37. Harder thinks that this congruence might be due to the second factor $c(\phi, 0)$ in [21, p. 590]. Indeed, in the case at hand $c(\phi, 0) = \zeta(-30)/\zeta(-31)$ and 37 divides (the numerator of) $\zeta(-31)$. In Table 12, we list some eigenvalues for the case $\lambda = (16, 1, 1)$.

References

[1] J. D. Achter and R. Pries, The integral monodromy of hyperelliptic and trielliptic curves. *Math. Ann.* **338** (2007), no. 1, 187–206 Zbl 1129.11027 MR 2295509

[2] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution. *J. Math. Soc. Japan* **28** (1976), no. 1, 48–61 Zbl 0313.10026 MR 427235

[3] M. F. Atiyah, Vector bundles over an elliptic curve. *Proc. London Math. Soc. (3)* **7** (1957), 414–452 Zbl 0084.17305 MR 131423

[4] J. Bergström, Cohomology of moduli spaces of curves of genus three via point counts. *J. Reine Angew. Math.* **622** (2008), 155–187 Zbl 1158.14025 MR 2433615

[5] J. Bergström, F. Cléry, C. Faber, and G. van der Geer, Siegel modular forms of degree 2 and 3. 2017, `http://smf.compositio.nl`

[6] J. Bergström, C. Faber, and G. van der Geer, Siegel modular forms of degree three and the cohomology of local systems. *Selecta Math. (N.S.)* **20** (2014), no. 1, 83–124 Zbl 1343.11051 MR 3147414

[7] J. Bergström and G. van der Geer, The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three. *J. Topol.* **1** (2008), no. 3, 651–662 Zbl 1156.14032 MR 2417447

[8] F. Cléry, C. Faber, and G. van der Geer, Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three. *Selecta Math. (N.S.)* **26** (2020), no. 4, Paper No. 55 Zbl 1465.11119 MR 4125987

[9] F. Cléry and G. van der Geer, Generators for modules of vector-valued Picard modular forms. *Nagoya Math. J.* **212** (2013), 19–57 Zbl 1285.11080 MR 3290679

[10] D. Cox, Stickelberger and the eigenvalue theorem. 2020, arXiv:2007.12573v1

[11] K. Dalen, On a theorem of Stickelberger. *Math. Scand.* **3** (1955), 124–126 Zbl 0068.03102 MR 71460

[12] H. Darmon, F. Diamond, and R. Taylor, Fermat's last theorem. In *Current developments in mathematics (Cambridge, MA, 1995)*, pp. 1–154, Int. Press, Cambridge, MA, 1995 Zbl 0877.11035 MR 1474977

[13] P. Deligne, Rapport sur la formule des traces. In *Cohomologie étale*, pp. 76–109, Lecture Notes in Math. 569, Springer, Berlin, 1977 Zbl 0349.14009 MR 3727434

[14] N. Dummigan, Eisenstein congruences for unitary groups. 2015, https://neil-dummigan.sites.sheffield.ac.uk/publications

[15] C. Faber and G. van der Geer, Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes. II. *C. R. Math. Acad. Sci. Paris* **338** (2004), no. 6, 467–470 Zbl 1055.14026 MR 2057727

[16] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*. Ergeb. Math. Grenzgeb. (3) 22, Springer, Berlin, 1990 Zbl 0744.14031 MR 1083353

[17] J. M. Feustel, Ringe automorpher Formen auf der komplexen Einheitskugel und ihre Erzeugung durch Theta-konstanten. Preprint, Ser. Akad. Wiss. DDR, P-Math-13, 1986

[18] T. Finis, Some computational results on Hecke eigenvalues of modular forms on a unitary group. *Manuscripta Math.* **96** (1998), no. 2, 149–180 Zbl 0990.11031 MR 1624517

[19] J. Giraud, Remarque sur une formule de Shimura–Taniyama. *Invent. Math.* **5** (1968), 231–236 Zbl 0165.54801 MR 227172

[20] B. B. Gordon, Canonical models of Picard modular surfaces. In *The zeta functions of Picard modular surfaces*, pp. 1–29, Centre de Recherches Mathématiques, Univ. de Montréal, Montreal, QC, 1992 Zbl 0756.14011 MR 1155224

[21] G. Harder, Eisensteinkohomologie für Gruppen vom Typ $GU(2,1)$. *Math. Ann.* **278** (1987), no. 1–4, 563–592 Zbl 0633.10030 MR 909239

[22] G. Harder, *Eisensteinkohomologie und die Konstruktion gemischter Motive*. Lecture Notes in Math. 1562, Springer, Berlin, 1993 Zbl 0795.11024 MR 1285354

[23] G. Harder, A congruence between a Siegel and an elliptic modular form. In *The 1-2-3 of modular forms*, pp. 247–262, Universitext, Springer, Berlin, 2008 Zbl 1259.11049 MR 2409680

[24] G. Harder, The shape of the congruences for $U(2,1)$. Unpublished manuscript, 4 November 2013

[25] J. Harris and I. Morrison, *Moduli of curves*. Grad. Texts in Math. 187, Springer, New York, 1998 Zbl 0913.14005 MR 1631825

[26] R.-P. Holzapfel, Around Euler partial differential equations. VEB Deutscher Verlag der Wissenschaften, Berlin, 1986

[27] R.-P. Holzapfel, *The ball and some Hilbert problems*. Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 1995 Zbl 0905.14013 MR 1350073

[28] A. Hurwitz, Ueber die Entwickelungscoefficienten der lemniskatischen Functionen. *Math. Ann.* **51** (1899), no. 2, 196–226 Zbl 29.0385.02 MR 1511022

[29] K. Ireland and M. Rosen, *A classical introduction to modern number theory*. Second edn., Grad. Texts in Math. 84, Springer, New York, 1990 Zbl 0712.11001 MR 1070716

[30] J. Keranen, Compact support cohomology of Picard modular surfaces. PhD thesis, UCLA, 2015, https://escholarship.org/uc/item/6dd2h1zb

[31] S. S. Kudla, On certain arithmetic automorphic forms for $\mathrm{SU}(1,q)$. *Invent. Math.* **52** (1979), no. 1, 1–25 Zbl 0408.10016 MR 532744

[32] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*. London Math. Soc. Monogr. Ser. 36, Princeton Univ. Press, Princeton, NJ, 2013 Zbl 1284.14004 MR 3186092

[33] R. P. Langlands and D. Ramakrishnan (eds.), *The zeta functions of Picard modular surfaces*. Centre de Recherches Mathématiques, Univ. de Montréal, Montreal, QC, 1992 Zbl 0752.00024 MR 1155223

[34] M. J. Larsen, Arithmetic compactification of some Shimura surfaces. In *The zeta functions of Picard modular surfaces*, pp. 31–45, Centre de Recherches Mathématiques, Univ. de Montréal, Montreal, QC, 1992 MR 1155225

[35] J.-S. Li and J. Schwermer, On the Eisenstein cohomology of arithmetic groups. *Duke Math. J.* **123** (2004), no. 1, 141–169 Zbl 1057.11031 MR 2060025

[36] Y. Miyaoka, Stable Higgs bundles with trivial Chern classes. Several examples. *Tr. Mat. Inst. Steklova* **264** (2009), 129–136 Zbl 1312.14100 MR 2590843

[37] S. Müller-Stach, X. Ye, and K. Zuo, Mixed Hodge complexes and L^2-cohomology for local systems on ball quotients. *Doc. Math.* **17** (2012), 517–543 Zbl 1254.14027 MR 2992957

[38] D. Mumford, Hirzebruch's proportionality theorem in the noncompact case. *Invent. Math.* **42** (1977), 239–272 Zbl 0365.14012 MR 471627

[39] E. Picard, Sur une classe de groupes discontinus de substitutions linéaires et sur les fonctions de deux variables indépendantes restant invariables par ces substitutions. *Acta Math.* **1** (1882), no. 1, 297–320 Zbl 15.0356.02 MR 1554586

[40] E. Picard, Sur des fonctions de deux variables indépendantes analogues aux fonctions modulaires. *Acta Math.* **2** (1883), no. 1, 114–135 Zbl 15.0432.01 MR 1554595

[41] E. Picard, Sur les formes quadratiques ternaires indéfinies á indéterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes. *Acta Math.* **5** (1884), no. 1, 121–182 Zbl 16.0385.01 MR 1554651

[42] M. S. Raghunathan, On the first cohomology of discrete subgroups of semisimple Lie groups. *Amer. J. Math.* **87** (1965), 103–139 Zbl 0132.02102 MR 173730

[43] J. D. Rogawski, *Automorphic representations of unitary groups in three variables*. Ann. of Math. Stud. 123, Princeton Univ. Press, Princeton, NJ, 1990 Zbl 0724.11031 MR 1081540

[44] L. Saper, $\mathcal{L}$-modules and the conjecture of Rapoport and Goresky–MacPherson. Automorphic forms. I. *Astérisque* **298** (2005), 319–334 Zbl 1083.11033 MR 2141706

[45] A. J. Scholl, Motives for modular forms. *Invent. Math.* **100** (1990), no. 2, 419–430 Zbl 0760.14002 MR 1047142

[46] G. Shimura, On purely transcendental fields automorphic functions of several variable. *Osaka Math. J.* **1** (1964), no. 1, 1–14 Zbl 0149.04302 MR 176113

[47] G. Shimura, The arithmetic of automorphic forms with respect to a unitary group. *Ann. of Math. (2)* **107** (1978), no. 3, 569–605 Zbl 0409.10016 MR 563087

[48] T. Shintani, On automorphic forms on unitary groups of order 3. Unpublished manuscript

[49] G. van der Geer, Rank one Eisenstein cohomology of local systems on the moduli space of abelian varieties. *Sci. China Math.* **54** (2011), no. 8, 1621–1634 Zbl 1242.14034 MR 2824963

[50] T. Zink, Über die Anzahl der Spitzen einiger arithmetischer Untergruppen unitärer Gruppen. *Math. Nachr.* **89** (1979), 315–320 Zbl 0424.10020 MR 546890

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