

Permutation modules and cohomological singularity

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Dedicated to Henning Krause on the occasion of his 60th birthday.

Abstract. We define a new invariant of finitely generated representations of a finite group, with coefficients in a commutative noetherian ring. This invariant uses group cohomology and takes values in the singularity category of the coefficient ring. It detects which representations are controlled by permutation modules.

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1. Introduction

In the whole paper, G is a finite group and R is a commutative noetherian ring.

Let M be a finitely generated RG -module. In this article we define an invariant which measures how singular the cohomology of M is. It allows us to conclude the theme of [1], where we started exploring how much of the R -linear representation theory of G is controlled by *permutation modules* (equation (2.1)). One motivation is that general RG -modules are typically wild, whereas permutation ones are much simpler. Here we prove a precise version of the following slogan:

The RG -module M is controlled by permutation modules if and only if its cohomology is not singular.

In the remainder of the introduction we describe the invariant and make this statement precise.

It will be convenient to view M as an object of $D_b(RG)$, the bounded derived category of finitely generated RG -modules. We consider the thick triangulated subcategory of $D_b(RG)$ generated by finitely generated permutation modules

$$D_{\text{perm}}(G; R) = \text{thick}\{R(G/H) \mid H \leq G\} \quad (1.1)$$

as the part of $D_b(RG)$ that is ‘controlled by permutation modules’. This interpretation is justified by our result in [1] that the canonical functor which sends a complex of permutation modules to itself viewed in the derived category

$$\Upsilon: K_b(\text{perm}(G; R)) \rightarrow D_b(RG) \quad (1.2)$$

is essentially a localization onto this $D_{\text{perm}}(G; R)$. More precisely, Υ induces, after quotienting-out its kernel and idempotent-completing $(\dots)^\natural$, an equivalence

$$\bar{\Upsilon}: \left(\frac{K_b(\text{perm}(G; R))}{K_{b,\text{ac}}(\text{perm}(G; R))} \right)^\natural \xrightarrow{\sim} D_{\text{perm}}(G; R). \quad (1.3)$$

The announced invariant will be a functor defined on $D_b(RG)$ which vanishes exactly on $D_{\text{perm}}(G; R)$.

To define it, recall the (*small*) *singularity category* [7]

$$D_b^{\text{sing}}(R) = \frac{D_b(R)}{D_{\text{perf}}(R)},$$

which measures how far the ring R is from being regular; see also Stevenson [8]. Since the cohomology of M is typically unbounded, we will also require the ‘*big*’ *singularity category* $D^{\text{sing}}(R)$, following Krause [4]. It is a compactly-generated triangulated category, whose subcategory of compact objects coincides with the idempotent-completion of the above $D_b^{\text{sing}}(R)$. Krause extends the evident quotient functor $D_b(R) \twoheadrightarrow D_b^{\text{sing}}(R)$ to a functor defined on unbounded complexes of arbitrary modules. We call this extension the *singularity functor*

$$\text{sing}: D(R) = D(\text{Mod}(R)) \rightarrow D^{\text{sing}}(R).$$

For each subgroup $H \leq G$, let $(-)^{\text{h}H}: D(RG) \rightarrow D(R)$ be the right-derived functor of the H -fixed-points functor $(-)^H$. We can now state our main result.

Theorem 1.4 (Theorem 4.16). *The subcategory $D_{\text{perm}}(G; R)$ of $D_b(RG)$, given in (1.1), consists of those complexes $X \in D_b(RG)$ such that the invariants*

$$\chi^H(X) := \text{sing}(X^{\text{h}H}) \quad (1.5)$$

vanish in the big singularity category $D^{\text{sing}}(R)$, for every subgroup $H \leq G$.

The functor $(-)^{\text{h}G}: D(RG) \rightarrow D(R)$ is the derived-category version of ordinary group cohomology, that is, the following left-hand square commutes:

$$\begin{array}{ccc} \text{Mod}(RG) & \hookrightarrow & D(RG) \\ \text{H}^*(G, -) \downarrow & & \downarrow (-)^{\text{h}G} \\ \text{Mod}(R) & \xleftarrow{\text{H}^*(=H_*)} & D(R) \end{array} \begin{array}{c} \searrow \chi^G \\ \text{(def)} \end{array} \begin{array}{c} \longrightarrow \\ \text{sing} \end{array} D^{\text{sing}}(R). \quad (1.6)$$

For every subgroup $H \leq G$, we call the invariant appearing in (1.5)

$$\chi^H: D(RG) \xrightarrow{\text{Res}_H^G} D(RH) \xrightarrow{(-)^{hH}} D(R) \xrightarrow{\text{sing}} D^{\text{sing}}(R) \quad (1.7)$$

the *H-cohomological singularity* functor; see Section 3.

To apply Theorem 1.4 to a complex $X \in D_b(RG)$ whose underlying complex of R -modules $\text{Res}_1^G X$ is already perfect it suffices to test $\chi^H(X) = 0$ for the Sylow subgroups H of G , or alternatively for the (maximal) elementary abelian subgroups; see Recollection 2.4. In particular, if G is a p -group, there are two conditions for a complex $X \in D_b(RG)$ to belong to $D_{\text{perm}}(G; R)$: the naïve $\text{Res}_1^G(X) \in D_{\text{perf}}(R)$ and the new $\chi^G(X) = 0$. See Section 4.

To appreciate the strength of Theorem 1.4, observe that for R regular the condition $\chi^H(X) = 0$ is trivially true in $D^{\text{sing}}(R) = 0$. Thereby we recover the non-trivial fact that Υ in (1.2) is surjective-up-to-summands when R is regular.

The article is organized as follows. In Section 2, we explain our conventions and recall the singularity functor. In Section 3, we define the invariant χ^H and prove that the objects X of $D_{\text{perm}}(G; R)$ satisfy $\chi^H(X) = 0$. In Section 4, we prove the converse, namely Theorem 1.4.

2. Recollections and preparations

We recall basic notation and other conventions, mostly following [1]. Then we remind the reader of the singularity category of a ring. Beyond this, Krause's recent monograph [5] provides general background on Grothendieck categories and on representation theory of finite groups.

Conventions. Unless specified, modules are left modules. We denote the category of Λ -modules by $\text{Mod}(\Lambda)$ and its subcategory of finitely generated ones by $\text{mod}(\Lambda)$.

Since fixed points $(-)^H$ and other decorations (duals) appear in the exponent, we use homological notation for complexes

$$\cdots \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots$$

We write $\text{Ch}_?$, $\text{K}_?$, and $\text{D}_?$ for, respectively, the category of chain complexes, its homotopy category and its derived category, with $? \in \{\emptyset, b, +, -\}$ indicating boundedness conditions, as usual. We abbreviate $\text{D}_b(\Lambda)$ for $\text{D}_b(\text{mod}(\Lambda))$ and $\text{D}(\Lambda)$ for $\text{D}(\text{Mod}(\Lambda))$. When we speak of a module as a complex, we mean it is concentrated in degree zero.

All triangulated subcategories are implicitly assumed to be replete (closed under isomorphisms). We abbreviate ‘thick’ for ‘triangulated and thick’ (i.e., closed under direct summands). A triangulated subcategory is called localizing if it is closed

under coproducts. We write $\text{thick}(\mathcal{A})$ (respectively, $\text{Loc}(\mathcal{A})$) for the smallest thick (respectively, localizing) subcategory containing \mathcal{A} .

For an additive category \mathcal{A} , we denote by \mathcal{A}^{\natural} its idempotent-completion (also known as Karoubi envelope). Recall from [3] that $\mathbf{K}_b(\mathcal{A}^{\natural}) \cong \mathbf{K}_b(\mathcal{A})^{\natural}$.

Recollection 2.1. If A is a left G -set, we denote by $R(A)$ the free R -module with G -action extended R -linearly from its basis A . An RG -module is called a *permutation module* if it is isomorphic to $R(A)$ for some G -set A . The additive category of permutation modules is denoted by $\text{Perm}(G; R)$ and its subcategory of finitely generated ones by $\text{perm}(G; R)$.

Recollection 2.2. We tensor RG -modules over R and use diagonal G -action:

$$- \otimes_R -: \text{Mod}(RG) \times \text{Mod}(RG) \rightarrow \text{Mod}(RG \otimes_R RG) \rightarrow \text{Mod}(RG).$$

This tensor is right-exact in each argument and can be left-derived as usual:

$$- \otimes_R^{\mathbf{L}} -: D_+(RG) \times D_+(RG) \rightarrow D_+(RG).$$

If either X or $Y \in \text{Ch}_+(RG)$ is degreewise R -flat, we have $X \otimes_R^{\mathbf{L}} Y \cong X \otimes_R Y$. We will also use the scalar-extension functor

$$R' \otimes_R -: \text{Mod}(RG) \rightarrow \text{Mod}(R'G)$$

as well as its derived version

$$R' \otimes_R^{\mathbf{L}} -: D_+(RG) \rightarrow D_+(R'G).$$

It is easy to see that those functors preserve perfect complexes and complexes of permutation modules, and send R -perfect complexes to R' -perfect ones (see Recollection 2.5).

Recollection 2.3. As mentioned in the introduction, the main object of [1] was the canonical tensor-triangulated functors Υ of (1.2) and the induced functor

$$\bar{\Upsilon}: \left(\frac{\mathbf{K}_b(\text{perm}(G; R))}{\mathbf{K}_{b,\text{ac}}(\text{perm}(G; R))} \right)^{\natural} \rightarrow \mathbf{D}_b(RG).$$

We proved [1, Theorem 4.3] that $\bar{\Upsilon}$ is fully faithful, and consequently that its essential image is

$$\mathbf{D}_{\text{perm}}(G; R) := \text{Im}(\bar{\Upsilon}) = \text{thick}(\text{perm}(G; R))$$

in $\mathbf{D}_b(RG)$ as in (1.1).¹ In other words, $\bar{\Upsilon}$ yields the tensor-triangulated equivalence (1.3).

¹In [1], this image $\text{Im} \bar{\Upsilon}$ was denoted by both $\mathcal{P}(G; R)^{\natural}$ and $\mathcal{Q}(G; R)^{\natural}$ and we described its objects as those $X \in \mathbf{D}_b(RG)$ such that $X \oplus \Sigma X$ admits ‘ m -free permutation resolutions’ for all $m \geq 0$. However, in this paper we will not need this description.

Recollection 2.4. It is easy to see that $D_{\text{perm}}(-; R)$ is stable under restriction and induction, and that $X \in D_{\text{perm}}(G; R)$ if and only if $\text{Res}_H^G X \in D_{\text{perm}}(H; R)$ for every Sylow subgroup $H \leq G$. In fact, it suffices to test for $H \leq G$ among the (maximal) elementary abelian subgroups of G . Details can be found in [1, Proposition 2.20, Corollary 2.21 and Remark 4.9]. However, the arguments in the present paper do *not* use reduction to elementary abelian subgroups.

Recollection 2.5. Recall [1, Definition 2.22] that a complex $X \in D_b(RG)$ is *R-perfect* if the underlying complex $\text{Res}_1^G X \in D_b(R)$ is perfect. We denote the thick tensor subcategory of *R-perfect* complexes by $D_{R\text{-perf}}(RG)$. It is obvious that we always have

$$D_{\text{perm}}(G; R) \subseteq D_{R\text{-perf}}(RG).$$

This is an equality if the order $|G|$ of the group is invertible in R ; see [1, Proposition 2.20].

In summary, we have the following inclusions of small ‘derived’ categories

$$D_{\text{perf}}(RG) \subseteq D_{\text{perm}}(G; R) \subseteq D_{R\text{-perf}}(RG) \subseteq D_b(RG).$$

Singularity category. The target of our ‘cohomological singularity’ functor (1.7) is the big singularity category of the coefficient ring R . Let us remind the reader.

Recollection 2.6. As in [4], let \mathcal{A} be a locally noetherian Grothendieck category whose derived category is compactly-generated. For $\mathcal{A} = \text{Mod}(R)$, the subcategory of noetherian objects $\text{noeth } \mathcal{A}$ is $\text{mod}(R)$ and $D(\mathcal{A})$ is generated by $D(\mathcal{A})^c = D_{\text{perf}}(R)$. Similarly for $\mathcal{A} = \text{Mod}(RG)$. The *big singularity category* (or *stable derived category*) of \mathcal{A} is

$$D^{\text{sing}}(\mathcal{A}) = K_{\text{ac}}(\text{Inj } \mathcal{A})$$

the full subcategory of the big homotopy category of injectives $K(\text{Inj } \mathcal{A})$ spanned by acyclic complexes. There is a recollement $Q_\lambda \dashv Q \dashv Q_\rho$ and $I_\lambda \dashv I \dashv I_\rho$

$$\begin{array}{ccc} & D(\mathcal{A}) & \\ & \uparrow \text{ } \downarrow & \\ Q_\lambda \downarrow & Q & \downarrow Q_\rho \\ & K(\text{Inj } \mathcal{A}) & \\ I_\lambda \downarrow & \uparrow \text{ } \downarrow & \\ & I & \\ & \uparrow \text{ } \downarrow & \\ & D^{\text{sing}}(\mathcal{A}) = K_{\text{ac}}(\text{Inj } \mathcal{A}) & \end{array} \quad (2.7)$$

for $I: K_{\text{ac}}(\text{Inj } \mathcal{A}) \hookrightarrow K(\text{Inj } \mathcal{A})$ the inclusion and $Q: K(\text{Inj } \mathcal{A}) \twoheadrightarrow D(\mathcal{A})$ the usual localization $Q^+: K(\mathcal{A}) \twoheadrightarrow D(\mathcal{A})$ restricted to $K(\text{Inj } \mathcal{A})$. The *singularity functor* (Krause’s *stabilization functor*) is defined as the composite

$$\text{sing}_{\mathcal{A}}: D(\mathcal{A}) \xrightarrow{I_\lambda \circ Q_\rho} D^{\text{sing}}(\mathcal{A}).$$

There is a natural transformation $Q_\lambda \rightarrow Q_\rho$ that is invertible on compacts:

$$Q_\lambda(X) \cong Q_\rho(X) \quad \text{if } X \in \mathbf{D}(\mathcal{A})^c \quad (2.8)$$

by [4, Lemma 5.2]. On the larger subcategory $\mathbf{D}_b(\text{noeth } \mathcal{A})$, the right adjoint Q_ρ defines an inverse to the equivalence of [4, Section 2] identifying $\mathbf{K}(\text{Inj } \mathcal{A})^c$

$$Q: \mathbf{K}(\text{Inj } \mathcal{A})^c \xrightarrow{\sim} \mathbf{D}_b(\text{noeth } \mathcal{A}).$$

In summary, we have a finite localization sequence

$$\mathbf{D}(\mathcal{A}) \xrightarrow{Q_\lambda} \mathbf{K}(\text{Inj } \mathcal{A}) \xrightarrow{J_\lambda} \mathbf{D}^{\text{sing}}(\mathcal{A})$$

as in (2.7) and the triangulated category $\mathbf{D}^{\text{sing}}(\mathcal{A})$ is compactly-generated with compact part the (usual) *small singularity category*, idempotent-completed:

$$\mathbf{D}^{\text{sing}}(\mathcal{A})^c \cong \left(\frac{\mathbf{K}(\text{Inj } \mathcal{A})^c}{(Q_\lambda \mathbf{D}(\mathcal{A}))^c} \right)^\natural \cong \left(\frac{\mathbf{D}_b(\text{noeth } \mathcal{A})}{\mathbf{D}(\mathcal{A})^c} \right)^\natural = \mathbf{D}_b^{\text{sing}}(\mathcal{A})^\natural.$$

Finally, we note that since $\mathbf{K}(\text{Inj } \mathcal{A})$ is compactly-generated and since the inclusion $\mathbf{K}(\text{Inj } \mathcal{A}) \hookrightarrow \mathbf{K}(\mathcal{A})$, that we denote J , preserves products and coproducts (because \mathcal{A} is locally noetherian), there is another useful triple of adjoints $J_\lambda \dashv J \dashv J_\rho$:

$$\begin{array}{ccc} & \mathbf{K}(\mathcal{A}) & \\ J_\lambda \downarrow & \uparrow J & \downarrow J_\rho \\ & \mathbf{K}(\text{Inj } \mathcal{A}) & \end{array} \quad (2.9)$$

Lemma 2.10. *Keep the above notation, e.g., for $\mathcal{A} = \text{Mod}(RG)$. Let X be in $\mathbf{K}(\mathcal{A})$.*

- (a) *The object $Q_\rho(X) \in \mathbf{K}(\text{Inj } \mathcal{A})$ in (2.7) is a K -injective resolution of X . In particular, if $X \in \mathbf{K}_-(\mathcal{A})$ is left-bounded then $Q_\rho(X)$ belongs to $\mathbf{K}_-(\text{Inj } \mathcal{A})$.*
- (b) *If $X \in \mathbf{K}_-(\mathcal{A})$ is left-bounded, the object $J_\lambda(X) \in \mathbf{K}(\text{Inj } \mathcal{A})$ in (2.9) is an injective resolution of X . Hence, if $X \in \mathbf{K}_{-, \text{ac}}(\mathcal{A})$ is also acyclic, then $J_\lambda(X) = 0$.*
- (c) *The restrictions of the two functors Q_ρ and J_λ to $\mathbf{K}_-(\mathcal{A})$ are isomorphic.*

Proof. By [4, Remark 3.7], we have $Q J_\lambda \cong Q^+$, where $Q^+: \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ is the (Bousfield) localization defining $\mathbf{D}(\mathcal{A})$. It follows that $J Q_\rho$ is right adjoint to Q^+ . So, if we let $i := J Q_\rho \circ Q^+$, every $X \in \mathbf{K}(\mathcal{A})$ fits in an exact triangle

$$a(X) \rightarrow X \xrightarrow{\eta} i(X) \rightarrow \Sigma a(X) \quad (2.11)$$

in $\mathbf{K}(\mathcal{A})$, where $a(X) \in \text{Ker}(Q^+) = \mathbf{K}_{\text{ac}}(\mathcal{A})$ and $i(X)$ belongs to $\text{Ker}(Q^+)^\perp = \mathbf{K}_{\text{ac}}(\mathcal{A})^\perp$, that is, $i(X)$ is K -injective by definition. In other words, (2.11) is the essentially unique triangle providing the K -injective resolution of X (see [4,

Corollary 3.9] if necessary). Suppressing the functors J and Q^+ that are just the identity on objects, we have $i(X) = Q_\rho(X)$, which gives (a).

Let now $A \in \mathbf{K}_{-,ac}(\mathcal{A})$. The unit $\eta': A \rightarrow JJ_\lambda(A)$ is a map from a left-bounded acyclic to a complex of injectives, hence $\eta' = 0$ in $\mathbf{K}(\mathcal{A})$. But,

$$J_\lambda(\eta'): J_\lambda \xrightarrow{\sim} J_\lambda JJ_\lambda$$

is invertible (J being fully faithful). Thus, $J_\lambda(A) = 0$, as in the second claim of (b).

Take now $X \in \mathbf{K}_-(\mathcal{A})$ arbitrary and an injective resolution $i(X) \in \mathbf{K}_-(\text{Inj } \mathcal{A})$. There is a triangle (2.11) with $a(X)$ acyclic and left-bounded since X and $i(X)$ are. By the above for $A = a(X)$, we already know that $J_\lambda(a(X)) = 0$. Applying J_λ to the triangle (2.11) in question, we get

$$J_\lambda(X) \cong J_\lambda(i(X)) \cong i(X),$$

since $i(X) \in \mathbf{K}(\text{Inj } \mathcal{A})$. Hence, (b).

Part (c) is now immediate from the uniqueness of \mathbf{K} -injective resolutions. \square

Let us now specialize to $\mathcal{A} = \text{Mod}(R)$.

Lemma 2.12. *Let X be an object of $\mathbf{D}(R)$. The following are equivalent:*

- (i) *The image of X in $\mathbf{D}^{\text{sing}}(R)$ is zero: $\text{sing}(X) = 0$.*
- (ii) *$Q_\rho(X)$ belongs to the localizing subcategory of $\mathbf{K}(\text{Inj}(R))$ generated by $Q_\rho(R)$.*
- (iii) *For every $Y \in \mathbf{D}_b(R)$, every map $Y \rightarrow X$ in $\mathbf{D}(R)$ factors via a perfect complex.*

Proof. We have $\text{sing} = I_\lambda \circ Q_\rho$ by definition. So we have $\text{sing}(X) = 0$ if and only if $Q_\rho(X) \in \text{Ker}(I_\lambda)$, and by (2.7) that kernel is

$$\text{Ker}(I_\lambda) = Q_\lambda(\mathbf{D}(R)) = Q_\lambda(\text{Loc}(R)) = \text{Loc}(Q_\lambda(R)),$$

where the last equality holds since Q_λ is coproduct-preserving and fully faithful. We get the formulation (ii) from $Q_\lambda(R) \cong Q_\rho(R)$ since $R \in \mathbf{D}(R)^c$; see (2.8). To reformulate this as (iii), recall from [6, Section 2] that in a compactly-generated triangulated category, the localizing subcategory generated by a thick subcategory \mathcal{J} of compacts consists of those X such that every map from the generators to X factors via an object of \mathcal{J} . If we apply this to $\mathbf{K}(\text{Inj}(R))$ and the object $Q_\rho(X)$, we see that

$$Q_\rho(X) \in \text{Loc}(Q_\lambda(\mathbf{D}_{\text{perf}}(R))) = \text{Loc}(Q_\rho(\mathbf{D}_{\text{perf}}(R)))$$

if and only if for every $Y \in \mathbf{D}_b(R)$ every map $Q_\rho(Y) \rightarrow Q_\rho(X)$ factors via $Q_\rho(P)$ for some $P \in \mathbf{D}_{\text{perf}}(R)$. This is equivalent to (iii), since Q_ρ is fully faithful. \square

Remark 2.13. In fact, $\text{sing}(X) = 0$ is also equivalent to $Q_\lambda(X) \xrightarrow{\sim} Q_\rho(X)$, but we will not need this in the sequel.

3. Cohomological singularity

In this section, we define the announced cohomological singularity functor (1.5).

Recollection 3.1. The functor that equips every R -module with trivial G -action

$$\mathrm{Infl}_1^G \cong \mathrm{hom}_R(R, -) \cong R \otimes_R -: \mathrm{Mod}(R) \rightarrow \mathrm{Mod}(RG)$$

has adjoints the usual G -orbits $(-)_G$ and G -fixed points $(-)^G$

$$R \otimes_{RG} - = (-)_G \left(\begin{array}{c} \mathrm{Mod}(RG) \\ \uparrow \mathrm{Infl}_1^G \\ \mathrm{Mod}(R) \end{array} \right) \mathrm{hom}_{RG}(R, -) = (-)^G \quad (3.2)$$

This triple of adjoints passes to homotopy categories of complexes on the nose. For derived categories, we left-derive the left adjoint and right-derive the right one:

$$R \otimes_{RG}^L - =: (-)_{hG} \left(\begin{array}{c} \mathrm{D}(RG) \\ \uparrow \mathrm{Infl}_1^G \\ \mathrm{D}(R) \end{array} \right) \mathrm{Rhom}_{RG}(R, -) =: (-)^{hG} \quad (3.3)$$

So $(-)^{hG}$ provides a complex whose homology groups are G -cohomology as in (1.6).

Definition 3.4. Let $H \leq G$ be a subgroup. The H -cohomological singularity functor $\chi^H = \mathrm{sing}_R \circ (-)^{hH}$ is the following composite (see Recollection 2.6 for sing):

$$\chi^H: \mathrm{D}(RG) \xrightarrow{\mathrm{Res}_H^G} \mathrm{D}(RH) \xrightarrow{(-)^{hH}} \mathrm{D}(R) \xrightarrow{\mathrm{sing}_R} \mathrm{D}^{\mathrm{sing}}(R).$$

We say that a complex $X \in \mathrm{D}(RG)$ is H -cohomologically perfect if $\chi^H(X) = 0$. We say that X is cohomologically perfect, if it is H -cohomologically perfect for all subgroups $H \leq G$, that is, if $\bigoplus_H \chi^H(X) = 0$ in $\mathrm{D}^{\mathrm{sing}}(R)$.

Example 3.5. For the trivial subgroup $H = 1$, a complex $X \in \mathrm{D}_b(RG)$ is 1-cohomologically perfect if and only if its underlying complex $\mathrm{Res}_1^G X \in \mathrm{D}_b(R)$ is perfect, that is, if and only if X is R -perfect in the sense of Recollection 2.5.

Remark 3.6. We remind the reader that although $\mathrm{Ker}(\mathrm{sing}) \cap \mathrm{D}_b(R) = \mathrm{D}_{\mathrm{perf}}(R)$, the kernel of $\mathrm{sing}: \mathrm{D}(R) \rightarrow \mathrm{D}^{\mathrm{sing}}(R)$ on the big derived category is larger than $\mathrm{D}_{\mathrm{perf}}(R)$. For instance, we will see in Lemma 3.13 that R^{hG} belongs to that kernel.

Even when $H = 1$, a big object $X \in \mathrm{D}(RG)$ being 1-cohomologically perfect is more flexible than being R -perfect although the two notions coincide when $X \in \mathrm{D}_b(RG)$ is bounded, i.e., when $\mathrm{Res}_1^G(X) \in \mathrm{D}_b(R)$, as we saw in Example 3.5.

For more general subgroups $H \leq G$, even a bounded complex $X \in \mathrm{D}_b(RG)$ can be H -cohomologically perfect without X^{hH} being perfect; see Example 3.10.

We provide a further justification of the terminology in Remark 4.20.

Remark 3.7. The functor $(-)^G \cong \text{hom}_{RG}(R, -)$ is a special value of the bifunctor

$$\text{Mod}(RG)^{\text{op}} \times \text{Mod}(RG) \xrightarrow{\text{hom}_{RG}(-, -)} \text{Mod}(R).$$

It follows that for any $X \in \text{D}(RG)$ the object $X^{\text{h}G}$ is represented by both

$$\text{hom}_{RG}(P_R, X) \quad \text{and} \quad \text{hom}_{RG}(R, i(X)), \quad (3.8)$$

where $P_R \rightarrow R$ is a projective resolution of R as an RG -module, and $X \rightarrow i(X)$ is a K -injective resolution of X , for both are quasi-isomorphic to $\text{hom}_{RG}(P_R, i(X))$.

Remark 3.9. If the order $|G|$ is invertible in R , then the trivial RG -module R is projective by Maschke. In that case, $(-)^G$ is exact and coincides with $(-)^{\text{h}G}$.

We can use this to see that being G -cohomologically perfect does not imply being H -cohomologically perfect for each subgroup $H \leq G$, even for $H = 1$.

Example 3.10. Let $R = \mathbb{Z}/9$ and $G = C_2 = \langle x \mid x^2 = 1 \rangle$. Consider the R -module $M = \mathbb{Z}/3$ with the action of x by -1 . We have $M^G = 0$, and hence $M^{\text{h}G} = 0$ by Remark 3.9. In particular, M is G -cohomologically perfect, but it is not H -cohomologically perfect for the subgroup $H = 1$, since $\mathbb{Z}/3$ is not perfect over $\mathbb{Z}/9$.

Let us establish some generalities about the cohomological singularity functor.

Proposition 3.11. *Let $H \leq G$ be a subgroup. There are canonical isomorphisms*

$$(-)^{\text{h}G} \circ \text{Ind}_H^G \cong (-)^{\text{h}H} \quad \text{and} \quad \chi^G \circ \text{Ind}_H^G \cong \chi^H.$$

Proof. The first isomorphism follows from the relation $\text{Res}_H^G \circ \text{Infl}_1^G = \text{Infl}_1^H$, by taking right adjoints and right-deriving. We use here that induction is also right-adjoint to restriction (because $[G : H] < \infty$) and is exact; see [5], if necessary. The second isomorphism follows by post-composing with sing_R . \square

Corollary 3.12. *Let $H \leq G$. Then induction $\text{Ind}_H^G : \text{D}_b(RH) \rightarrow \text{D}_b(RG)$ and restriction $\text{Res}_H^G : \text{D}_b(RG) \rightarrow \text{D}_b(RH)$ preserve cohomologically perfect complexes.*

Proof. Restriction is built into Definition 3.4. For induction, it follows immediately from the Mackey formula and Proposition 3.11. \square

Here is a key computation of our invariant χ^G of Definition 3.4.

Lemma 3.13. *The object $\chi^G(R) = \text{sing}(R^{\text{h}G})$ is zero in $\text{D}^{\text{sing}}(R)$.*

Proof. Recall from Remark 3.7 that $R^{\text{h}G} = \text{hom}_{RG}(P_R, R)$, where P_R is any projective resolution of R over RG . Let $\Omega R = \text{Ker}(RG \twoheadrightarrow R)$ be the kernel of augmentation. By additivity, it suffices to prove that

$$\text{sing}(X) = 0,$$

where $X = \text{hom}_{RG}(P, R)$, for any RG -projective resolution P of $R \oplus \Omega R$.

By [1, Corollary 5.3], there exists a sequence of quasi-isomorphisms of bounded complexes in $\text{Ch}_{\geq 0}(RG)$:

$$\cdots \rightarrow Q(n+1) \rightarrow Q(n) \rightarrow \cdots \rightarrow Q(1) \rightarrow R \oplus \Omega R$$

such that $Q(n)$ consists of finitely generated \natural -permutation RG -modules (i.e., direct summands of finitely generated permutation modules), and in the range $0 \leq d < n$, the RG -module $Q(n)_d$ is projective and the map $Q(n+1)_d \rightarrow Q(n)_d$ is the identity. In particular, the above sequence of complexes

$$\cdots \rightarrow Q(n) \rightarrow \cdots \rightarrow Q(1)$$

is eventually stationary in each degree and the limit $P = \lim_{n \rightarrow \infty} Q(n)$, computed degreewise, is a projective resolution of $R \oplus \Omega R$.

Let us write for simplicity $(-)^{\dagger}$ for $\text{hom}_{RG}(-, R)$. This additive functor induces degreewise the functor

$$(-)^{\dagger} = \text{hom}_{RG}(-, R): \mathbf{K}(RG)^{\text{op}} \rightarrow \mathbf{K}(R).$$

Our goal is to show that $\text{sing}(X) = 0$ for $X = P^{\dagger}$. Note that since $P = \lim_{n \rightarrow \infty} Q(n)$ in a degreewise stationary way, we also have $P^{\dagger} = \text{colim}_{n \rightarrow \infty} Q(n)^{\dagger}$ in a degreewise stationary way, say, in $\text{Ch}(R)$. The maps

$$Q(n)^{\dagger} \rightarrow Q(n+1)^{\dagger} \rightarrow \cdots \rightarrow P^{\dagger}$$

are the identity in degree $d > -n$. Note also that $P^{\dagger} \in \mathbf{K}_-(R)$ is left-bounded.

The key remark is that for every \natural -permutation RG -module Q , the R -module Q^{\dagger} is projective. Indeed, for Q permutation, Q^{\dagger} is R -free. In our case, the complexes $Q(n)^{\dagger}$ are therefore perfect over R .

We prove $\text{sing}(P^{\dagger}) = 0$ via criterion (iii) in Lemma 2.12. Let $Y \in \text{D}_b(R)$. A morphism $Y \rightarrow X = P^{\dagger}$ in $\text{D}(R)$ is given by a fraction $Y \leftarrow L \rightarrow P^{\dagger}$ in $\mathbf{K}(R)$, where $L \rightarrow Y$ is a projective resolution of Y , in particular $L \in \mathbf{K}_+(R)$ is right-bounded. It is then easy to see that any morphism

$$L \rightarrow P^{\dagger} = \text{colim}_{n \rightarrow \infty} Q(n)^{\dagger}$$

in $\mathbf{K}(R)$ must factor via $Q(n)^{\dagger} \rightarrow P^{\dagger}$ for $n \gg 0$. Since $Q(n)^{\dagger}$ is perfect, we have established condition (iii) of Lemma 2.12 for our X , giving us $\text{sing}(X) = 0$, as wanted. \square

Recall from (1.1) the thick subcategory $\text{D}_{\text{perm}}(G; R)$ of $\text{D}_b(RG)$, generated by finitely generated permutation modules.

Proposition 3.14. *Every object of $\text{D}_{\text{perm}}(G; R)$ is cohomologically perfect.*

Proof. As cohomologically perfect complexes form a thick subcategory of $D_b(RG)$, it suffices to show that $R(G/H)$ is cohomologically perfect for all subgroups $H \leq G$. The latter follows easily from Lemma 3.13 and Corollary 3.12. \square

We can now apply Proposition 3.14 to show that being R -perfect (Recollection 2.5) is not sufficient to belong to $D_{\text{perm}}(G; R)$.

Example 3.15. Let $k = \mathbb{F}_2$ and consider the ring $R = k[x]/\langle x^2 - 1 \rangle$. Take $G = C_2 = \langle y \mid y^2 = 1 \rangle$ to be cyclic of order 2. Let $M = R_x$ denote the ring R viewed as an RG -module with the non-trivial action of y via x . This $M \in D_b(RC_2)$ is R -perfect, but we claim that $\chi^{C_2}(M) \neq 0$. As RC_2 is self-injective, the following resolution:

$$0 \rightarrow R_x \xrightarrow{y-x} RC_2 \xrightarrow{y-x} RC_2 \rightarrow \dots$$

is an injective resolution of R_x . Computing $(R_x)^{hG}$ in $D(R)$ as in (3.8) with the above

$$i(M) = \dots 0 \rightarrow RC_2 \xrightarrow{y-x} RC_2 \xrightarrow{y-x} RC_2 \rightarrow \dots,$$

we get that $(R_x)^{hG}$ is

$$\dots 0 \rightarrow R \xrightarrow{1-x} R \xrightarrow{1-x} R \rightarrow \dots,$$

and we deduce that $(R_x)^{hG} \simeq k$ in $D(R)$. But $k \in D_b(R)$ is not perfect, hence $\chi^G(M) \simeq \text{sing}(k) \neq 0$. In other words, M is not cohomologically perfect. Using Proposition 3.14, this means $M \notin D_{\text{perm}}(G; R)$.

4. Main result

We saw in Proposition 3.14 that $D_{\text{perm}}(G; R)$ is contained in the subcategory of cohomologically perfect complexes (Definition 3.4). Our goal in this section is to prove the reverse inclusion. Two ideas will be key: the ‘‘cohomology’’ comonad $\text{Infl}_1^G \circ (-)^{hG}$ on cohomologically perfect objects, and compactness arguments. To make both work at the same time, we lift that comonad to the homotopy category of injectives, $K(\text{Inj}(RG))$, whose compact part is the bounded derived category. The proof of our main result being somewhat long, we prove several shorter lemmas. Let us first set the notation.

Construction 4.1. We will assemble the following diagram via [4, Section 6]:

$$\begin{array}{ccc}
 & & K(\text{Inj}(RG)) \\
 & \begin{array}{c} \xrightarrow{Q_\lambda} \\ \xrightarrow{Q} \\ \xrightarrow{Q_\rho} \end{array} & \uparrow \hat{c}_* \quad \downarrow \hat{c}! \\
 D(RG) & & K(\text{Inj}(R)) \\
 \begin{array}{c} \uparrow c_* \\ \downarrow c! \end{array} & \begin{array}{c} \xrightarrow{Q_\lambda} \\ \xrightarrow{Q} \\ \xrightarrow{Q_\rho} \end{array} &
 \end{array} \tag{4.2}$$

We already encountered the slanted arrows $Q_\lambda \dashv Q \dashv Q_\rho$ in the recollement (2.7). The left-hand vertical arrows $\mathbf{c}_* \dashv \mathbf{c}^\dagger$ are simply a shorthand for (3.2):

$$\mathbf{c}_* := \mathrm{Infl}_1^G \quad \text{and} \quad \mathbf{c}^\dagger := (-)^{hG}.$$

There are several reasons for this notation. First, it is lighter in formulas involving iterated compositions. Second, it evokes the algebro-geometric notation $\mathbf{c}^* \dashv \mathbf{c}_* \dashv \mathbf{c}^\dagger$ for an imaginary closed immersion $\mathbf{c}: \mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(RG)$, which actually makes sense if G is abelian. (And we do have a left adjoint \mathbf{c}^* too, namely the left-derived functor of G -orbits $(-)^{hG}$.) Finally, it allows for a simple notation at the level of $\mathbf{K}(\mathrm{Inj})$, namely the yet-to-be-explained $\widehat{\mathbf{c}}_* \dashv \widehat{\mathbf{c}}^\dagger$ on the right-hand side of (4.2).

For this, we apply [4, Section 6] to the *exact* functor (denoted F in loc. cit.)

$$\mathrm{Infl}_1^G: \mathrm{Mod}(R) \rightarrow \mathrm{Mod}(RG).$$

Its right adjoint

$$(-)^G: \mathrm{Mod}(RG) \rightarrow \mathrm{Mod}(R)$$

preserves injectives and our $\widehat{\mathbf{c}}^\dagger: \mathbf{K}(\mathrm{Inj}(RG)) \rightarrow \mathbf{K}(\mathrm{Inj}(R))$ is simply $(-)^G$ degree-wise. Its left adjoint

$$\widehat{\mathbf{c}}_*: \mathbf{K}(\mathrm{Inj}(R)) \rightarrow \mathbf{K}(\mathrm{Inj}(RG))$$

is more subtle than just inflation. It is Krause's construction, namely $\widehat{\mathbf{c}}_*$ is defined as the composite

$$\widehat{\mathbf{c}}_*: \mathbf{K}(\mathrm{Inj}(R)) \xrightarrow{J} \mathbf{K}(\mathrm{Mod}(R)) \xrightarrow{\mathrm{Infl}_1^G} \mathbf{K}(\mathrm{Mod}(RG)) \xrightarrow{J_\lambda} \mathbf{K}(\mathrm{Inj}(RG)),$$

where $J: \mathbf{K}(\mathrm{Inj}) \hookrightarrow \mathbf{K}(\mathrm{Mod})$ is the inclusion and $J_\lambda: \mathbf{K}(\mathrm{Mod}) \twoheadrightarrow \mathbf{K}(\mathrm{Inj})$ its left adjoint, as in (2.9). Using that J is fully faithful, it is easy to see that $\widehat{\mathbf{c}}_* \dashv \widehat{\mathbf{c}}^\dagger$. (Although we had a derived left adjoint $\mathbf{c}^* \dashv \mathbf{c}_*$ there is no $\widehat{\mathbf{c}}^* \dashv \widehat{\mathbf{c}}_*$ on $\mathbf{K}(\mathrm{Inj})$.)

By [4, Lemma 6.3], since inflation is exact, we have

$$Q \circ \widehat{\mathbf{c}}_* \cong \mathbf{c}_* \circ Q: \mathbf{K}(\mathrm{Inj}(R)) \rightarrow \mathrm{D}(RG). \quad (4.3)$$

From this we deduce, by taking right adjoints, that

$$\widehat{\mathbf{c}}^\dagger \circ Q_\rho \cong Q_\rho \circ \mathbf{c}^\dagger. \quad (4.4)$$

Note that since the functor $(-)^G: \mathrm{Mod}(RG) \rightarrow \mathrm{Mod}(R)$ preserves coproducts, so does the induced $\widehat{\mathbf{c}}^\dagger$ on $\mathbf{K}(\mathrm{Inj})$. Thus, its left adjoint preserves compacta:

$$\widehat{\mathbf{c}}_*(\mathbf{K}(\mathrm{Inj}(R))^c) \subseteq \mathbf{K}(\mathrm{Inj}(RG))^c. \quad (4.5)$$

Remark 4.6. On every $Y \in \mathbf{K}(\mathrm{Inj}(RG))$ the comonad $\widehat{\mathbf{c}}_*\widehat{\mathbf{c}}^\dagger$ equals by construction

$$\widehat{\mathbf{c}}_*\widehat{\mathbf{c}}^\dagger(Y) = J_\lambda \mathrm{Infl}_1^G J \widehat{\mathbf{c}}^\dagger(Y) = J_\lambda \mathrm{Infl}_1^G (J(Y)^G) \cong J_\lambda \mathrm{hom}_{RG}(R, Y),$$

where $\mathrm{hom}_{RG}(R, Y)$ has trivial G -action. This leads us to bimodule actions:

Lemma 4.7. *There is an action of the bounded derived category of $R(G \times G^{\text{op}})$ -modules on $\mathbf{K}_-(\text{Inj}(RG))$, in the form of a well-defined bi-exact functor*

$$[-, -]: \mathbf{D}_b(R(G \times G^{\text{op}}))^{\text{op}} \times \mathbf{K}_-(\text{Inj}(RG)) \rightarrow \mathbf{K}_-(\text{Inj}(RG))$$

given by the formula $[L, Y] = J_\lambda(\text{hom}_{RG}(L, Y))$.

Proof. At the level of module categories, there is an action

$$\text{hom}_{RG}(-, -): \text{Mod}(R(G \times G^{\text{op}}))^{\text{op}} \times \text{Mod}(RG) \rightarrow \text{Mod}(RG),$$

which takes (L, Y) to the abelian group $\text{Hom}_{RG}(L, Y)$ built by viewing L as a left RG -module via its left G -action, and then making the output $\text{Hom}_{RG}(L, Y)$ into a left G -module $\text{hom}_{RG}(L, Y)$ by using the ‘remaining’ right G -action on L . Being additive in both variables, this passes to a bi-exact functor on homotopy categories

$$\text{hom}_{RG}(-, -): \mathbf{K}(R(G \times G^{\text{op}}))^{\text{op}} \times \mathbf{K}(RG) \rightarrow \mathbf{K}(RG) \quad (4.8)$$

(by totalizing via \prod , which is irrelevant in our bounded case). This yields

$$[-, -]: \mathbf{K}_b(R(G \times G^{\text{op}}))^{\text{op}} \times \mathbf{K}_-(\text{Inj}(RG)) \xrightarrow{(4.8)} \mathbf{K}_-(RG) \xrightarrow{J_\lambda} \mathbf{K}_-(\text{Inj}(RG)).$$

The preservation of left-boundedness by J_λ is Lemma 2.10 (b). To show that this descends to the derived category in the first variable, let $L \in \mathbf{K}_b(R(G \times G^{\text{op}}))$ be acyclic and let $Y \in \mathbf{K}_-(\text{Inj}(RG))$, and let us show that $J_\lambda(\text{hom}_{RG}(L, Y)) = 0$. By Lemma 2.10 (b) again, it suffices to show that $\text{hom}_{RG}(L, Y)$ is acyclic. But

$$H_n \text{hom}_{RG}(L, Y) = \text{Hom}_{\mathbf{K}(RG)}(L[n], Y)$$

vanishes since Y is a left-bounded complex of injectives and L is acyclic (as complex of RG -modules as well). \square

Remark 4.9. Each object L in $\mathbf{D}_b(R(G \times G^{\text{op}}))$ thus defines an exact endofunctor

$$[L, -]: \mathbf{K}_-(\text{Inj}(RG)) \rightarrow \mathbf{K}_-(\text{Inj}(RG)).$$

For instance, $[RG, -] \cong \text{Id}$, whereas $[R, -] \cong \widehat{\mathfrak{C}}_* \widehat{\mathfrak{C}}^!$ is our comonad, by Remark 4.6. We use this to show that some $Y \in \mathbf{K}_-(\text{Inj}(RG))$ can be recovered from $\widehat{\mathfrak{C}}_* \widehat{\mathfrak{C}}^!(Y)$.

Lemma 4.10. *Let G be a finite p -group and $Y \in \mathbf{K}_-(\text{Inj}(RG))$ such that $p^n \cdot \text{id}_Y = 0$ for $n \gg 0$. Then Y belongs to $\text{thick}(\widehat{\mathfrak{C}}_* \widehat{\mathfrak{C}}^!(Y))$ in $\mathbf{K}_-(\text{Inj}(RG))$.*

Proof. As explained in Remark 4.9, we need to show that in $\mathbf{K}_-(\text{Inj}(RG))$

$$[RG, Y] \in \text{thick}([R, Y]).$$

Since $p^n \cdot Y = 0$, we also have $p^n \cdot [RG, Y] = 0$ and the octahedron axiom gives (see [1, Remark 2.27])

$$[RG, Y] \in \text{thick}(\text{cone}([RG, Y] \xrightarrow{p} [RG, Y])) = \text{thick}([\text{cone}(RG \xrightarrow{p} RG), Y])$$

using exactness of $[-, Y]$. Hence, it suffices to prove in $D_b(R(G \times G^{\text{op}}))$ that

$$\text{cone}(RG \xrightarrow{p} RG) \in \text{thick}(R).$$

This last statement is independent of Y . By scalar extension along $\mathbb{Z} \rightarrow R$ (Recollection 2.2), it suffices to prove that in $D_b(\mathbb{Z}(G \times G^{\text{op}}))$, we have

$$\text{cone}(\mathbb{Z}G \xrightarrow{p} \mathbb{Z}G) \in \text{thick}(\mathbb{Z}).$$

Consider the exact functor

$$i_*: D_b(\mathbb{F}_p(G \times G^{\text{op}})) \rightarrow D_b(\mathbb{Z}(G \times G^{\text{op}}))$$

obtained by restriction-of-scalars. Then the above $\text{cone}(\mathbb{Z}G \xrightarrow{p} \mathbb{Z}G)$ is nothing but $i_*(\mathbb{F}_p G)$, and

$$i_*(\mathbb{F}_p) \cong \text{cone}(\mathbb{Z} \xrightarrow{p} \mathbb{Z})$$

belongs to $\text{thick}(\mathbb{Z})$. So we are reduced to showing that

$$\mathbb{F}_p G \in \text{thick}(\mathbb{F}_p)$$

in $D_b(\mathbb{F}_p(G \times G^{\text{op}}))$, which is clear since $G \times G^{\text{op}}$ is also a p -group. \square

Lemma 4.11. *Let G be a p -group. Let $X \in D_b(RG)$ be p -torsion (i.e., $p^n \cdot X = 0$ for $n \gg 0$) and G -cohomologically perfect. Then X belongs to $\text{thick}(R)$.*

Proof. By Lemma 2.12, the assumption $0 = \chi^G(X) = I_\lambda Q_\rho c^!(X)$ implies that

$$Q_\rho c^!(X) \in \text{Loc}(Q_\rho(R)) \tag{4.12}$$

in $K(\text{Inj}(R))$. Applying to this relation the (coproduct-preserving) left adjoint

$$\widehat{c}_*: K(\text{Inj}(R)) \rightarrow K(\text{Inj}(RG))$$

of (4.2), we obtain in $K(\text{Inj}(RG))$ that

$$\widehat{c}_* \widehat{c}^! Q_\rho(X) \stackrel{(4.4)}{\cong} \widehat{c}_* Q_\rho c^!(X) \stackrel{(4.12)}{\in} \text{Loc}(\widehat{c}_* Q_\rho(R)).$$

Hence, by Lemma 4.10 with $Y = Q_\rho(X)$, which is p -torsion since X is, we have

$$Q_\rho(X) \in \text{thick}(\widehat{c}_* \widehat{c}^! Q_\rho(X)) \subseteq \text{Loc}(\widehat{c}_* Q_\rho(R)) \tag{4.13}$$

in $\mathbf{K}(\text{Inj}(RG))$. Now $Q_\rho(X)$ is compact in $\mathbf{K}(\text{Inj}(RG))$ by Recollection 2.6, and $\widehat{\mathbf{c}}_* Q_\rho(R)$ is compact because $Q_\rho(R)$ is and because $\widehat{\mathbf{c}}_*$ preserves compacts (4.5). So, by [6, Lemma 2.2], we can replace ‘Loc’ by ‘thick’ in (4.13), giving us the relation

$$Q_\rho(X) \in \text{thick}(\widehat{\mathbf{c}}_* Q_\rho(R))$$

in $\mathbf{K}(\text{Inj}(RG))$. Applying $Q: \mathbf{K}(\text{Inj}(RG)) \rightarrow \mathbf{D}(RG)$ and $Q Q_\rho \cong \text{Id}$, we get

$$X \in \text{thick}(Q \widehat{\mathbf{c}}_* Q_\rho(R)) \stackrel{(4.3)}{=} \text{thick}(\mathbf{c}_* Q Q_\rho(R)) = \text{thick}(\mathbf{c}_*(R)).$$

Of course, $\mathbf{c}_*(R)$ is just R with trivial G -action, viewed in $\mathbf{D}_b(RG)$. \square

Lemma 4.14. *Let G be a p -group and $X \in \mathbf{D}_b(RG)$. The following are equivalent:*

- (i) $X \in \mathbf{D}_{\text{perm}}(G; R)$; see (1.1).
- (ii) X is cohomologically perfect (Definition 3.4).
- (iii) X is G -cohomologically perfect and R -perfect (Recollection 2.5).

Proof. The implication (i) \Rightarrow (ii) is Proposition 3.14, and the implication (ii) \Rightarrow (iii) is trivial by Definition 3.4. For the implication (iii) \Rightarrow (i), suppose that $\chi^G(X) = 0$ and X is R -perfect. By [1, Corollary 2.26], there exists an exact triangle in $\mathbf{D}_b(RG)$

$$P \rightarrow X \oplus \Sigma X \rightarrow T \rightarrow \Sigma P,$$

where P is a bounded complex of permutation modules (and therefore belongs to $\mathbf{D}_{\text{perm}}(G; R)$) and where $T \in \mathbf{D}_b(RG)$ is p -torsion. Since P and X are G -cohomologically perfect so is T . Then Lemma 4.11 tells us that $T \in \text{thick}(R)$ in $\mathbf{D}_b(RG)$. As $R \in \mathbf{D}_{\text{perm}}(G; R)$, we get $X \in \mathbf{D}_{\text{perm}}(G; R)$ as well. \square

Remark 4.15. For G a p -group the equivalence (ii) \Leftrightarrow (iii) in Lemma 4.14 shows that G -cohomological perfection together with R -perfection does imply H -cohomological perfection for all $H \leq G$. This is sharp by Examples 3.10 and 3.15.

Here is the main result. The category $\mathbf{D}_{\text{perm}}(G; R) = \text{Im}(\bar{\Upsilon})$ can be found in (1.1) and in Recollection 2.3. The invariant χ^H is in Definition 3.4.

Theorem 4.16. *Let G be a finite group and R a commutative noetherian ring. Let $X \in \mathbf{D}_b(RG)$ be a bounded complex. The following properties of X are equivalent:*

- (i) *The complex X belongs to $\mathbf{D}_{\text{perm}}(G; R)$.*
- (ii) *It is cohomologically perfect: $\chi^H(X) = 0$ in $\mathbf{D}^{\text{sing}}(R)$ for all subgroups $H \leq G$.*
- (iii) *It is R -perfect, i.e., the underlying complex $\text{Res}_1^G(X) \in \mathbf{D}_b(R)$ is perfect, and it is H -cohomologically perfect, $\chi^H(X) = 0$, for every Sylow subgroup $H \leq G$.*

Proof. Implication (i) \Rightarrow (ii) is Proposition 3.14. The implication (ii) \Rightarrow (iii) is trivial (Definition 3.4). If we assume (iii), then Lemma 4.14 implies that $\text{Res}_H^G(X) \in \text{D}_{\text{perm}}(H; R)$ for every Sylow subgroup $H \leq G$. Since the indices of all Sylow subgroups are coprime, it is easy to deduce that $X \in \text{D}_{\text{perm}}(G; R)$; see [1, Corollary 2.21]. So the three conditions (i) \Leftrightarrow (ii) \Leftrightarrow (iii) are equivalent. \square

Remark 4.17. As in Recollection 2.5, it suffices to test (iii) for the p -Sylow subgroups H corresponding to primes p that are non-invertible on X (and in R).

One can also replace (iii) by only asking X to be E -cohomologically perfect for every elementary abelian p -subgroup $E \leq G$; see Recollection 2.4.

Remark 4.18. Theorem 1.4 of the introduction follows from Theorem 4.16.

Corollary 4.19. *With $X \in \text{D}_b(RG)$ as in Theorem 4.16, the conditions (i), (ii), (iii) are also equivalent to:*

(iv) *In $\text{D}(R)$, we have $X^{\text{h}H} \in \text{thick}(\{R^{\text{h}K} \mid K \leq G\})$ for every subgroup $H \leq G$.*

Proof. Let us denote by $\mathcal{J} := \text{thick}(\{R^{\text{h}K} \mid K \leq G\})$ the thick subcategory of $\text{D}(R)$ appearing in (iv). For (iv) \Rightarrow (ii), note that

$$\text{sing}(R^{\text{h}K}) = \chi^K(R) = 0,$$

by Lemma 3.13. So $\mathcal{J} \subseteq \text{Ker}(\text{sing})$, and therefore $X^{\text{h}H} \in \mathcal{J}$ implies

$$\chi^H(X) = \text{sing}(X^{\text{h}H}) = 0.$$

For (i) \Rightarrow (iv), it is sufficient to prove that for all subgroups $H, L \leq G$, we have $(R(G/L))^{\text{h}H} \in \mathcal{J}$. This follows from the Mackey formula and Proposition 3.11. \square

Remark 4.20. The inflation functor $c_*: \text{D}(R) \rightarrow \text{D}(RG)$ is monoidal and its right adjoint $c^! = (-)^{\text{h}G}: \text{D}(RG) \rightarrow \text{D}(R)$ is therefore lax monoidal. In particular, $c^!c_*(\mathbb{1}) = R^{\text{h}G}$ is a ring object, namely the ‘cohomology ring’ of G with coefficients in R , and every object $X \in \text{D}(RG)$ gives rise to a module $X^{\text{h}G}$ over this ring.

With this in mind, and the fact that for every ring Λ we have $\text{D}_{\text{perf}}(\Lambda) = \text{thick}(\Lambda)$, the terminology ‘cohomologically perfect’ of Definition 3.4 is somewhat justified by the equivalent formulation given in part (iv) of Corollary 4.19.

Remark 4.21. Neeman’s Localization theorem [6, Theorem 2.1] suggests that the equivalence (1.3) is the compact tip of an iceberg. To describe this iceberg, we define the (*big*) *derived category of permutation modules*

$$\text{DPerm}(G; R) := \text{Loc}(\text{perm}(G; R)) = \text{Loc}(\{R(G/H) \mid H \leq G\})$$

as the localizing subcategory of $\text{K}(\text{Mod}(RG))$ generated by permutation modules. Its compact part is precisely $\text{K}_b(\text{perm}(G; R)^{\text{h}})$. It follows from Lemma 2.10(c) that

$$J_\lambda(R(G/H)) \cong Q_\rho(R(G/H)),$$

so that we obtain a coproduct-preserving and compact-preserving exact functor

$$\Upsilon^+ := (J_\lambda)_{|\text{DPerm}(G;R)} : \text{DPerm}(G; R) \rightarrow \text{K Inj}_{\text{perm}}(G; R), \quad (4.22)$$

where the latter is the localizing subcategory of $\text{K}(\text{Inj}(RG))$ generated by the $Q_\rho(R(G/H))$, $H \leq G$. This Υ^+ is a finite localization which extends beyond compacts the canonical functor Υ of (1.2). In particular, it induces an equivalence

$$\bar{\Upsilon}^+ : \frac{\text{DPerm}(G; R)}{\text{Loc}(\text{K}_{\text{b,ac}}(\text{perm}(G; R)))} \xrightarrow{\sim} \text{K Inj}_{\text{perm}}(G; R).$$

On compacts, this equivalence identifies with the equivalence $\bar{\Upsilon}$ of (1.3). Note that if R is regular, (4.22) exhibits Krause's homotopy category of injectives $\text{K}(\text{Inj}(RG))$ as a finite localization of $\text{DPerm}(G; R)$.²

Unfortunately, for general R , we are unable to characterize the subcategory

$$\text{K Inj}_{\text{perm}}(G; R) \subseteq \text{K}(\text{Inj}(RG))$$

along the lines of Theorem 4.16. An immediate extension of that result is blocked because the cohomological singularity functor is not coproduct-preserving.

Remark 4.23. The above definition of $\text{DPerm}(G; R)$ does not do justice to the big derived category of permutation modules. We refer to the expository note [2] for a more conceptual approach. There we also explain that $\text{DPerm}(G; R)$ is equivalent to the derived category of cohomological R -linear Mackey functors on G and, after suitably extending to profinite groups, to the triangulated category of Artin motives over a field with absolute Galois group G and with coefficients in R , in the sense of Voevodsky.

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²We mention as a curiosity that for $R = k$ a field (or just self-injective) we have the inclusion $\text{K}(\text{Inj}(kG)) \subseteq \text{DPerm}(G; k)$ as localizing subcategories of $\text{K}(\text{Mod}(RG))$, and the finite localization Υ^+ is simply the left adjoint to that inclusion.

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