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Effective drilling and filling of tame hyperbolic 3-manifolds

David Futer, Jessica S. Purcell, and Saul Schleimer

Abstract. We give effective bilipschitz bounds on the change in metric between thick parts of a cusped hyperbolic 3-manifold and its long Dehn fillings. In the thin parts of the manifold, we give effective bounds on the change in complex length of a short closed geodesic. These results quantify the filling theorem of Brock and Bromberg, and extend previous results of the authors from finite volume hyperbolic 3-manifolds to any tame hyperbolic 3-manifold. To prove the main results, we assemble tools from Kleinian group theory into a template for transferring theorems about finite-volume manifolds into theorems about infinite-volume manifolds. We also prove and apply an infinite-volume version of the 6-Theorem.

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1. Introduction

Thurston's celebrated hyperbolic Dehn surgery theorem says that almost all Dehn fillings of a cusped hyperbolic 3-manifold produce closed hyperbolic 3-manifolds. The other direction is also true: drilling a closed geodesic from a hyperbolic 3-manifold produces another hyperbolic 3-manifold, with a cusp [2]. These original results provide the existence of a hyperbolic metric but do not construct it. Hodgson and Kerckhoff's subsequent work [22–24] produces a way to continuously interpolate between the drilled and filled manifolds via a family of manifolds with cone singularities, in a process called *cone deformation*. Their work provides analytic control over quantities such as volume.

Bromberg extended the theory of cone deformations to infinite-volume hyperbolic 3-manifolds [11]. Brock and Bromberg further proved bilipschitz estimates on the change in geometry for such manifolds [9]. Their results are *uniform*, in the sense that the change in geometry is controlled by constants independent of the manifold. However, they are not *effective*, in the sense that the constants are not explicitly given. For instance, Brock and Bromberg's drilling theorem is as follows.

Theorem 1.1 (Drilling theorem, [9]). *Fix $J > 1$ and $\varepsilon > 0$, where ε is smaller than the Margulis constant ε_3. Then there is a number $\ell_0 = \ell_0(\varepsilon, J) > 0$ such that the*

following holds for every geometrically finite hyperbolic 3*-manifold* Y *without rank-*1 *cusps. Suppose that* $\Sigma \subset Y$ *is a geodesic link, whose total length is less than* ℓ_0*. Then* $Y - \Sigma$ *admits a hyperbolic structure* Z *with the same end invariants as those of* Y*. Furthermore, the inclusion*

$$\iota\colon Z \hookrightarrow Y$$

restricts to a J*-bilipschitz diffeomorphism on the complement of* ε*-thin tubes about* Σ*.*

Uniform results such as Theorem 1.1 are very useful for studying convergent sequences of hyperbolic manifolds. Indeed, Brock and Bromberg's application was a special case of the density conjecture (compare Theorem 2.12). On the other hand, ineffective results such as Theorem 1.1 are hard to apply in the study of individual manifolds.

In recent years, there has been a major push to make geometric estimates effective. For instance, in a previous paper, we prove an effective version of Theorem 1.1 in the special case of finite-volume manifolds [20, Theorem 1.2]. Among other applications, effective geometric estimates can be used to control Margulis numbers and rule out cosmetic surgeries.

Our main result in this paper is an effective version of Theorem 1.1 for all tame hyperbolic 3-manifolds. A 3-manifold Y is called *tame* if it is the interior of a compact 3-manifold with boundary. By the tameness theorem, due to Agol [3] and Calegari and Gabai [13], a hyperbolic 3-manifold Y is tame if and only if $\pi_1(Y)$ is finitely generated; see Theorem 2.3. Thus, our results apply to all hyperbolic 3-manifolds with finitely generated fundamental group.

Theorem 1.2 (Effective drilling, tame manifolds). *Let* Y *be a tame hyperbolic* 3*-manifold. Fix any* $0 < \varepsilon \leq \log 3$ *and any* $J > 1$*. Let* Σ *be a geodesic link in* Y *whose total length* ℓ *satisfies*

$$\ell < \frac{1}{4} \min\left\{ \frac{\varepsilon^5}{6771 \cosh^5(0.6\varepsilon + 0.1475)}, \frac{\varepsilon^{5/2} \log(J)}{11.35} \right\}.$$

Then $Y - \Sigma$ *admits a hyperbolic structure* Z *with the same end invariants as* Y*. Moreover, there are* J*-bilipschitz inclusions*

$$\varphi\colon Y^{\geq \varepsilon} \hookrightarrow Z^{\geq \varepsilon/1.2}, \quad \psi\colon Z^{\geq \varepsilon} \hookrightarrow Y^{\geq \varepsilon/1.2}.$$

We remark that existence of the hyperbolic metric Z does not need any numerical hypotheses; see Lemma 6.1. Those hypotheses are only needed for the J-bilipschitz conclusion.

Theorem 1.2 is stronger than Brock and Bromberg's Theorem 1.1 in two respects and weaker in one respect. Most notably, the hypotheses and conclusion of Theorem 1.2 are completely effective. In addition, Theorem 1.2 applies to manifolds with rank-1 cusps and geometrically infinite ends, which are excluded by the hypotheses

of Theorem 1.1. On the other hand, Theorem 1.1 extends its bilipschitz control into the thin parts of Y that do not correspond to components of Σ; this extension is not present in Theorem 1.2. In Theorem 1.5 below, we provide some geometric control in the thin parts of Y by estimating the change in complex length of the core geodesics.

We also prove a version of Theorem 1.2 with hypotheses on the drilled manifold Z rather than the filled manifold Y. That result requires the following definition. If s is a slope on a rank-2 cusp C, the *normalized length* of s is

$$L(s) = \frac{\operatorname{len}(s)}{\sqrt{\operatorname{area}(\partial C)}},$$

where $\operatorname{len}(s)$ denotes the length of a Euclidean geodesic representative of s on ∂C, and $\operatorname{area}(\partial C)$ is the area in the induced Euclidean metric on ∂C. The quantity $L(s)$ is scaling-invariant, hence does not depend on the choice of cusp neighborhood C. Next, suppose we have a tuple of slopes $\mathbf{s} = (s_1, \ldots, s_k)$ on rank-2 cusps $C_1, \ldots, C_k$, respectively. The *total normalized length* $L = L(\mathbf{s})$ is defined by

$$\frac{1}{L(\mathbf{s})^2} = \sum_{j=1}^{k} \frac{1}{L(s_j)^2}. \tag{1.1}$$

In [20, Corollary 9.34], we prove effective bounds on the total normalized length L that guarantee J-bilipschitz inclusions similar to those of Theorem 1.2. We can now generalize that result to all tame hyperbolic 3-manifolds.

Theorem 1.3 (Effective filling, tame manifolds). *Fix any $0 < \varepsilon \leq \log 3$ and any $J > 1$. Let M be a tame 3-manifold and $\Sigma \subset M$ a link, such that $M - \Sigma$ admits a hyperbolic structure Z. Suppose that, in the hyperbolic structure Z on $M - \Sigma$, the total normalized length of the meridians of Σ satisfies*

$$L^2 \geq 4 \max\left\{\frac{2\pi \cdot 6771 \cosh^5(0.6\varepsilon + 0.1475)}{\varepsilon^5} + 11.7,\ \frac{2\pi \cdot 11.35}{\varepsilon^{5/2} \log(J)} + 11.7\right\}.$$

Then M admits a hyperbolic structure Y with the same end invariants as those of Z, in which Σ is a geodesic link. Moreover, there are J-bilipschitz inclusions

$$\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\varepsilon/1.2}, \quad \psi\colon Z^{\geq\varepsilon} \hookrightarrow Y^{\geq\varepsilon/1.2}.$$

The proofs of Theorems 1.2 and 1.3 rely on three major recent results in Kleinian groups, recalled in Section 2. The first of these is the ending lamination theorem, due to Minsky [32] and Brock, Canary, and Minsky [10], with an alternate proof by Bowditch [8] that includes the result for compressible ends. The second is the tameness theorem, due to Agol [3] and Calegari and Gabai [13]. The third and most directly relevant result is the density theorem, which asserts that geometrically finite hyperbolic manifolds are dense in the space of all tame hyperbolic 3-manifolds.

The proof of the density theorem was concluded independently by Ohshika [34] and Namazi and Souto [33], relying on many previous results including the tameness and ending lamination theorems. These major results enable us to transfer bilipschitz control from finite-volume manifolds to infinite-volume manifolds, by expressing the infinite-volume manifolds as geometric limits of finite-volume ones. See Section 1.1 and specifically Theorem 1.6 for a template describing this transfer process.

Because we use finite-volume manifolds to approximate infinite-volume ones, the effective hypotheses on length in Theorems 1.2 and 1.3 are directly derived from the corresponding hypotheses in the finite-volume setting. See Theorems 6.4 and 7.4, and compare Remark 6.6. In turn, the constants appearing in Theorems 6.4 and 7.4 were obtained by bounding a number of analytic quantities appearing in a cone deformation. The given hypotheses on length ensure the existence of a cone deformation with a large embedded tube about the singular locus Σ, and provide control on the infinitesimal change in geometry. See [20, Section 1.3] for a detailed and relatively non-technical summary of the argument. We have no reason to believe the constants in our theorems are sharp. However, they are uniform and effective, so we expect the results to be useful.

If the manifolds Y and Z in Theorems 1.2 and 1.3 are geometrically finite, those theorems can be proved using far less machinery. We only need Brooks' work on circle packings [12], classical results from Ahlfors–Bers theory, and our Theorems 6.4 and 7.4 for finite-volume manifolds. See Remarks 6.3 and 7.3 for details. The geometrically finite case of Theorem 1.2, which is an effective version of Brock and Bromberg's Theorem 1.1, can be used to prove a case of the density theorem; this was its main application in [9]. In the converse direction, the density theorem allows us to extend Theorem 1.1 from geometrically finite manifolds to any tame 3-manifold without rank-1 cusps; see Theorem 6.5.

Our proof of Theorem 1.3 also requires a version of the 6-Theorem for tame manifolds, which is likely of some independent interest, although probably not surprising to experts. The original 6-Theorem, due to Agol [1] and Lackenby [28], states that Dehn filling a finite volume 3-manifold along a slope of length greater than 6 yields a manifold that admits a hyperbolic structure. Their result can be generalized as follows.

Theorem 1.4 (6-Theorem for tame manifolds). *Let Z be a tame hyperbolic 3-manifold, with parabolic locus $P \cup (T_1 \cup \cdots \cup T_k)$, where $T_1, \ldots, T_k$ are a subcollection of the torus ends of Z. Let $H_1 \cup \cdots \cup H_k$ be pairwise disjoint horocusps, with H_i a neighborhood of T_i. Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a tuple of slopes, such that the length of a geodesic representative of each s_i on ∂H_i is strictly greater than 6. Then the manifold $Z(\mathbf{s}) = Z(s_1, \ldots, s_k)$ obtained from Z by Dehn filling along slopes $s_1, \ldots, s_k$ admits a hyperbolic structure Y with end invariants identical to those of Z.*

The proof of Theorem 1.4 for infinite-volume manifolds closely parallels part of the proofs by Agol and Lackenby for finite-volume manifolds [1, 28]. In both arg-

uments, one has to show that the filled manifold $Z(s_1, \dots, s_k)$ does not contain any embedded surfaces that would obstruct hyperbolicity. The proof of the infinite-volume case also uses the above-mentioned major recent results in Kleinian groups, particularly the density theorem. We give the argument in Section 3.

Although we do not have full control on the change in geometry in the thin part of the manifold Y, we do have results that bound the change in length of the short geodesics that lie in at the cores of the thin part. The following result is an effective version of a theorem of Bromberg [11, Theorem 1.4] and an extension of [20, Corollary 7.20] to the infinite-volume case.

Theorem 1.5. *Let Y be a tame hyperbolic 3-manifold. Let Σ be a geodesic link in Y, and γ a closed geodesic disjoint from Σ with complex length*

$$\operatorname{len}_Y(\gamma) + i\,\tau_Y(\gamma),$$

where $\operatorname{len}_Y(\cdot)$ denotes the length in the complete metric on Y.

Suppose that

$$\max(4 \operatorname{len}_Y(\Sigma), \operatorname{len}_Y(\gamma)) < 0.0735.$$

Then $Y - \Sigma$ admits a hyperbolic structure Z with the same end invariants as those of Y. Furthermore, γ is isotopic to a geodesic in Z, whose complex length $\operatorname{len}_Z(\gamma) + i\,\tau_Z(\gamma)$ satisfies

$$1.9793^{-1} \le \frac{\operatorname{len}_Z(\gamma)}{\operatorname{len}_Y(\gamma)} \le 1.9793 \quad \textit{and} \quad |\tau_Z(\gamma) - \tau_Y(\gamma)| \le 0.05417.$$

We will prove Theorem 1.5 in Section 8 as a corollary of Theorem 8.5, which provides explicit bounds on the change in complex length of γ between Y and Z, as a function of the real lengths $\operatorname{len}_Y(\Sigma)$ and $\operatorname{len}_Y(\gamma)$. If we hold $\operatorname{len}_Y(\gamma)$ fixed, we find that the change in complex length of γ tends to 0 as $\operatorname{len}_Y(\Sigma) \to 0$. Thus, when the geodesic link Σ is very short, the geometry of γ barely changes at all under drilling.

In an analogous fashion, we prove a result that bounds the change in complex length of a short geodesic γ under filling, with hypotheses that use the geometry of the drilled manifold Z rather than the filled manifold Y. See Theorem 8.7 for a general result that provides a bound as a function of $\operatorname{len}_Z(\gamma)$ and the normalized length $L(\mathbf{s})$ in Z, with the change in complex length tending to 0 as $L(\mathbf{s}) \to \infty$. See also Corollary 8.9 for a simple statement akin to Theorem 1.5.

1.1. Bootstrapping from finite-volume manifolds to tame manifolds. The proofs of Theorems 1.2 and 1.3 both start with the analogous result for finite-volume manifolds in [20]. Using a strong version of density theorem due to Namazi and Souto [33] (compare Theorem 2.12), we approximate any tame hyperbolic 3-manifold by a sequence of geometrically finite hyperbolic 3-manifolds. These geometrically finite manifolds, in turn, can be perturbed slightly to obtain manifolds that admit circle packings on their ends, by work of Brooks [12]. Finally, manifolds that admit

circle packings on their ends have convex cores embedding isometrically in finite volume manifolds, by a process of "scooping" and "doubling" (see Definition 2.16). At this point, the results for finite-volume manifolds can be applied. By taking better and better finite-volume approximates, we obtain the desired results for any tame manifold.

See Figure 1 for a diagram summarizing the above process. In that figure, $DD(V_n^\circ)$ and $DD(W_n^\circ)$ are finite-volume hyperbolic manifolds obtained by the doubling process, to which we can apply the results of [20]. The construction depicted in Figure 1 can also be summarized as follows:

Theorem 1.6. *Let Y be a tame, infinite-volume hyperbolic 3-manifold. Let $\Sigma \subset Y$ be a geodesic link, such that each component $\sigma \subset \Sigma$ is shorter than $\log 3$. Then $Y - \Sigma$ admits a hyperbolic metric Z with the same end invariants as those of Y. Furthermore, there is a sequence of finite-volume approximating manifolds $DD(V_n^\circ)$ and $DD(W_n^\circ)$ with the following properties:*

(1) *The manifold $DD(V_n^\circ)$ contains a geodesic link $DD(\Sigma_n)$, consisting of four isometric copies of a link Σ_n, such that*

$$DD(W_n^\circ) = DD(V_n^\circ) - DD(\Sigma_n).$$

(2) *For any choice of basepoints $y \in Y$ and $z \in Z$, there are basepoints in the approximating manifolds such that*

$$(DD(V_n^\circ), v_n) \to (Y, y) \quad \text{and} \quad (DD(W_n^\circ), w_n) \to (Z, z)$$

are geometric limits.

(3) *In the geometric limit $(DD(V_n^\circ), v_n) \to (Y, y)$, we have $\Sigma_n \to \Sigma$.*

See Theorem 6.2 for a more detailed statement, of which Theorem 1.6 is a corollary. See also Figure 2 and Theorems 7.1 and 7.2 for a parallel statement about filling rather than drilling.

Our hope is that Theorem 1.6, and the more detailed Theorems 6.2 and 7.2, can serve as user-friendly templates for transferring results about finite-volume manifolds to the infinite-volume setting. While the proofs of those results rely on the full machinery of Kleinian groups that will be described in Section 2, knowledge of this machinery is not needed to *apply* those theorems. We hope that this feature will make these templates useful to other researchers.

1.2. Organization. The paper is organized as follows. In Section 2, we review the tools from Kleinian groups that we need in this paper, particularly results on tame manifolds, their hyperbolic structures, and their limits. In Section 3, we extend the 6-Theorem to tame hyperbolic manifolds. Sections 4 and 5 contain technical results that make the proofs of the main theorems work smoothly. In Section 4, we show that under appropriate hypotheses, geodesics in approximating manifolds converge

to geodesics in the limiting manifold. In Section 5, we prove that if there are two sequences of manifolds converging geometrically, and bilipschitz maps between the approximating manifolds, then one has bilipschitz maps between their limits as well. Then, we combine these technical results with the finite-volume bilipschitz theorems from [20] to establish the effective drilling theorem in Section 6 and the effective filling theorem in Section 7. Finally, the results on short geodesics are proved in Section 8.

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2. Tools from Kleinian groups

This section reviews a number of definitions and results from Kleinian groups that will be needed for our applications. As mentioned above, the proofs of our main theorems use the full trifecta of major results in Kleinian groups from the early 2000s: the tameness theorem, the ending lamination theorem, and the density theorem. We also review the (older) work of Brooks on circle packings, which will similarly prove crucial to our constructions.

Much of our exposition and notation is modeled on that of Namazi and Souto [33]. Another excellent source that surveys these recent results is Canary [15].

The tameness, ending lamination, and density theorems can be seen as results that relate the geometry of a hyperbolic 3-manifold to its topology. While each theorem has a succinct statement, we find it most useful to frame each result in the context, notation, and terminology that will be used for the applications. Setting up this notation and terminology requires a number of definitions. We have endeavored to keep notation to a minimum, and to use consistent letters for parallel notions throughout the paper.

2.1. Topology and geometry.

Throughout the paper, the symbol M denotes a compact 3-manifold with non-empty boundary, which is oriented, irreducible, and atoroidal.

Definition 2.1 (Pared manifolds). Let M be a 3-manifold as above: compact, oriented, irreducible, atoroidal, with $\partial M \neq \emptyset$. We further assume that M is neither a 3-ball nor a solid torus. Let $P \subset \partial M$ be a compact subsurface consisting of incompressible

tori and annuli. The pair (M, P) is called a *pared manifold* if the following additional conditions hold:

- Every π_1-injective map of a torus $T^2 \to M$ is homotopic to a map whose image is contained in P.
- Every π_1-injective map of an annulus $(S^1 \times I, S^1 \times \partial I) \to (M, P)$ is homotopic as a map of pairs to a map whose image is contained in P.

We call P the *parabolic locus* of (M, P). The non-parabolic portion of ∂M is denoted $\partial_0 M = \partial M - P$.

Throughout the paper, variants of the letter M (M, M', etc) always denote a 3-manifold defined only up to topological type. Similarly, variants of the letter P denote the parabolic locus in a pared manifold. We will use variants of the letter N (N', N_n, etc.) to denote a generic a 3-manifold endowed with a hyperbolic metric. In the context of drilling and filling, we will use variants of V, W, Y, Z to denote 3-manifolds with hyperbolic metrics.

A *Kleinian group* Γ is a discrete group of isometries of $\mathbb{H}^3$. For this paper, all Kleinian groups are presumed to be torsion-free and orientation-preserving, ensuring that the quotient $N = \mathbb{H}^3/\Gamma$ is an oriented hyperbolic manifold. All Kleinian groups are also assumed *non-elementary*: this means that Γ has no global fixed points on $\partial\mathbb{H}^3$ and implies that the topological type of N is neither a solid torus nor the product of a torus and an interval. If N is homeomorphic to the interior of M, we say that the Kleinian group Γ, abstractly isomorphic to $\pi_1(M)$, endows M with a *hyperbolic structure*.

A *horocusp* is the quotient $C = H/G$, where $H \subset \mathbb{H}^3$ is an open horoball and G is a discrete group of parabolic isometries of H, isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$. In the first case, C is homeomorphic to $A \times (0, \infty)$ where A is a non-compact annulus, and is called *rank* 1. In the second case, C is homeomorphic to $T \times (0, \infty)$ where T is a torus, and is called *rank* 2. A *horocusp in* N is an isometrically embedded (rank 1 or 2) horocusp in a hyperbolic 3-manifold N. A *tube in* N is a regular neighborhood of a simple closed geodesic, of fixed radius.

Given a constant $\varepsilon > 0$, the *thin part of* N is set of points in N with injectivity radius less than $\varepsilon/2$, denoted $N^{<\varepsilon}$. A *Margulis number* for a hyperbolic 3-manifold N is any number $\varepsilon > 0$ such that $N^{<\varepsilon}$ is a disjoint union of tubes and horocusps. The optimal Margulis number of N, denoted $\mu(N)$, is the supremum of its Margulis numbers. The *Margulis constant* ε_3 is the infimum of optimal Margulis numbers over all hyperbolic 3-manifolds. While it is known that $\varepsilon_3 > 0$, the precise value is currently unknown. Since the Weeks manifold W has $\mu(W) \leq 0.776$, it follows that $\varepsilon_3 \leq 0.776$. Meyerhoff [30] showed that $\varepsilon_3 \geq 0.104$.

In the setting of infinite-volume manifolds, we have a stronger estimate. The following result is due to Culler and Shalen [18, Theorem 9.1], combined with the tameness and density theorems. See Shalen [36, Proposition 3.12] for the derivation.

Theorem 2.2 (Margulis numbers). *Let N be a hyperbolic 3-manifold of infinite volume. Then the optimal Margulis number of N satisfies $\mu(N) \geq \log 3$.*

Several results in the paper assume a bound of the form $\varepsilon \leq \log 3$. This can be viewed as ensuring ε is a Margulis number for both N and the manifolds that will be used to approximate N.

2.2. Tameness and compact cores. Let N be a hyperbolic 3-manifold, and suppose that ε is a Margulis number for N. Following Namazi and Souto, we let N^{ε} denote the complement in N of the cusp components of $N^{<\varepsilon}$. Then ∂N^{ε} is a disjoint union of tori and open annuli that satisfies the incompressibility requirements for the parabolic locus of a pared manifold. However, $(N^{\varepsilon}, \partial N^{\varepsilon})$ is not a pared manifold because N^{ε} is typically not compact. The powerful tool that gives us a pared manifold from this data is the tameness theorem, proved independently by Agol [3] and by Calegari and Gabai [13].

Theorem 2.3 (Tameness). *Suppose N is a hyperbolic 3-manifold with finitely generated fundamental group. Then N is homeomorphic to the interior of a compact 3-manifold M. That is, N is tame.*

As a corollary of Theorem 2.3, we obtain:

Corollary 2.4 (Standard compact cores). *Suppose N is a hyperbolic 3-manifold with finitely generated fundamental group and let $0 < \varepsilon \leq \mu(N)$. Then there is a compact 3-manifold M whose boundary ∂M contains a subsurface P, consisting of all toroidal components of ∂M and a possibly empty collection of annuli, such that N^{ε} is homeomorphic to $M - \partial_0 M$.*

The pair (M, P) produced by Corollary 2.4 is a pared manifold, unique up to pared homeomorphism, and independent of the choice of ε. We say that (M, P) is the *pared manifold associated with* N. Note that $M - \partial_0 M$ is homeomorphic to N^{ε}, but is not viewed as a submanifold of N^{ε}. A consequence of the tameness theorem is that there is a compact submanifold $(M', P') \subset (N^{\varepsilon}, \partial N^{\varepsilon})$ that is homeomorphic to (M, P). We call such a submanifold a *standard compact core* of $(N^{\varepsilon}, \partial N^{\varepsilon})$.

The components $F_1, \ldots, F_s$ of $\partial_0 M = \partial M - P$ are called the *free sides* of (M, P). Then the submanifold $N^{\varepsilon} - M'$ consists of s different components called *geometric ends*, each homeomorphic to $F_i \times (0, \infty)$ for some i. We sometimes refer to F_i as an *end* of (M, P). In the main case of interest, where N has infinite volume, ∂M must have some non-torus boundary components, hence (M, P) must contain at least one free side.

2.3. End invariants and the ending lamination theorem. Let $N = \mathbb{H}^3/\Gamma$ be a tame hyperbolic 3-manifold. The *limit set* of Γ, denoted Λ_Γ, is the set of accumulation points of an orbit $\{\Gamma x\}$ in $\partial_\infty \mathbb{H}^3 \cong S^2$. The *convex core* $CC(N) = CH(\Lambda_\Gamma)/\Gamma$ is the quotient by Γ of the convex hull of the limit set.

The *domain of discontinuity* of Γ, denoted Ω_Γ, is $\partial_\infty \mathbb{H}^3 - \Lambda_\Gamma$. The group Γ acts properly discontinuously on $\mathbb{H}^3 \cup \Omega_\Gamma$. The quotient of the action of Γ on Ω_Γ gives a (possibly disconnected) surface with a conformal structure. This surface is the *conformal boundary* of N.

Following Corollary 2.4, let (M, P) be a pared manifold associated to N, and let $\mathcal{E} \cong F \times (0, \infty)$ be a geometric end of N^ϵ associated to a free side $F \subset \partial_0 M$. Then the geometric structure on $\mathcal{E}$ endows $\mathcal{E}$ and F with an *end invariant*, as follows.

A geometric end $\mathcal{E} \subset N^\epsilon$ is called *geometrically finite* if it has a neighborhood whose intersection with the convex core $CC(N)$ is compact. The *end invariant* of a geometrically finite end is the point in the Teichmüller space $\mathcal{T}(F)$ determined by the component of the conformal boundary corresponding to F. If every end of N is geometrically finite, we say N is *geometrically finite*.

If the end $\mathcal{E}$ is not geometrically finite, then it is said to be *degenerate*. In this case, the end comes equipped with a filling geodesic lamination λ on the free side F. This lamination, called the *ending lamination* of $\mathcal{E}$, is the end invariant of $\mathcal{E}$.

The following theorem is due to Minsky [32] and Brock–Canary–Minsky [10]. See also Bowditch [8] for an alternate proof that covers the case of compressible ends.

Theorem 2.5 (Ending lamination). *Let N, N' be tame hyperbolic 3-manifolds. Let (M, P) and (M', P') be standard compact cores of N and N', respectively. Suppose there is a homeomorphism $\phi\colon (M, P) \to (M', P')$ satisfying the following:*

- *If $F \subset \partial M - P$ is a geometrically finite end of N, then $\phi(F)$ is a geometrically finite end of N', and the induced map between conformal boundaries is homotopic to a bi-holomorphic map.*
- *If $F \subset \partial M - P$ is a degenerate end with ending lamination λ, then $\phi(F)$ is a degenerate end of N' with ending lamination $\phi(\lambda)$.*

Then there is an isometry $\Phi\colon N \to N'$, homotopic to ϕ.

2.4. Hyperbolization theorems. We now review several results that guarantee that the topological pared manifold (M, P) admits a hyperbolic structure with specified end invariants.

A hyperbolic structure on a 3-manifold M defines a representation

$$\rho\colon \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C}),$$

and conjugate representations define isometric hyperbolic 3-manifolds. We let $AH(M, P)$ denote the set of conjugacy classes of discrete and faithful representations $\rho\colon \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$ such that those elements whose conjugacy classes are represented by loops on P are mapped to parabolic elements. The space $AH(M, P)$ is endowed with the topology of *algebraic convergence*: a sequence of representations ρ_n converges algebraically to ρ if for all $\gamma \in \pi_1(M)$, the sequence

$\{\rho_n(\gamma)\}$ converges to $\rho(\gamma)$ in $\mathrm{PSL}(2,\mathbb{C})$. Similarly, a sequence $\{[\rho_n]\}$ converges algebraically to $[\rho] \in AH(M)$ if there are representatives $\rho_n \in [\rho_n]$ and $\rho \in [\rho]$ such that $\rho_n \to \rho$. In a slight abuse of notation, we will write "$\rho \in AH(M,P)$" as a shorthand for the correct statement $\rho \in [\rho] \in AH(M,P)$.

A representation $\rho \in AH(M,P)$ is called *minimally parabolic* if it satisfies the following property: $\rho(\gamma) \in \mathrm{PSL}(2,\mathbb{C})$ is parabolic if and only if γ is conjugate into the fundamental group of some component of P. The following result, due to Thurston [37], establishes the existence of at least one such representation.

Theorem 2.6 (Hyperbolization). *Let (M,P) be a pared manifold. Then there is a geometrically finite hyperbolic 3-manifold N such that the pared manifold associated to N is (M,P). Equivalently, there is a geometrically finite, minimally parabolic representation $\rho \in AH(M,P)$.*

The next classical result on hyperbolicity is a parametrization of the set of *all* geometrically finite, minimally parabolic representations in $AH(M,P)$. Two representations $\rho \in AH(M,P)$ and $\rho' \in AH(M,P)$ are called *quasi-conformally conjugated* if there is a quasi-conformal homeomorphism

$$f\colon \partial_\infty \mathbb{H}^3 \to \partial_\infty \mathbb{H}^3$$

with $\rho(\gamma) \circ f = f \circ \rho'(\gamma)$ for all $\gamma \in \pi_1(M)$. Representations that are quasi-conformally conjugated to ρ form an open subset $QH(\rho)$ of $AH(M,P)$. Moreover, if ρ is minimally parabolic and geometrically finite, then so is ρ', and the conformal boundary of $N' = \mathbb{H}^3/\rho'(\pi_1(M))$ gives a point in Teichmüller space.

The statement below combines the work of Ahlfors, Bers, Kra, Marden, Maskit, Mostow, Prasad, Sullivan, and Thurston. Our formulation is drawn from Namazi and Souto [33, Theorem 4.3]. See also Canary [15, Theorem 11.1] and Canary and McCullough [17, Chapter 7].

Theorem 2.7 (Ahlfors–Bers uniformization). *Let (M,P) be a pared manifold. Let $\rho \in AH(M,P)$ be a minimally parabolic, geometrically finite representation, which exists by Theorem 2.6. Then there is a covering map π_{AB} from the Teichmüller space $\mathcal{T}(\partial_0 M)$ to $QH(\rho) \subset AH(M,P)$ with covering group $\mathrm{Mod}_0^+(M,P)$. Furthermore, for all $X \in \mathcal{T}(\partial_0 M)$, the hyperbolic manifold obtained from $\pi_{AB}(X)$ has associated pared manifold (M,P) and conformal boundary bi-holomorphic to X.*

If $\partial_0 M = \emptyset$, then Theorem 2.7 restates the Mostow–Prasad rigidity theorem that $QH(\rho)$ contains a single point. In the main case of interest, if $F_1, \dots, F_s$ are the free sides of (M,P), then a tuple

$$X = (X_1, \dots, X_s) \in \mathcal{T}(\partial_0 M) = \mathcal{T}(F_1) \times \cdots \times \mathcal{T}(F_s)$$

in Techmüller space determines a conjugacy class $\pi_{AB}(X) \in AH(M,P)$. Any representative of this conjugacy class is called the *Ahlfors–Bers representation* corresponding to X. The corresponding hyperbolic manifold has the chosen points $(X_1, \dots, X_s)$ as the full collection of end invariants.

The analogue of Theorem 2.7 in the presence of degenerate ends is Theorem 2.9 below. Stating this result is harder, as it requires definitions involving projectively measured laminations. The following construction follows Thurston [38, pp. 421–422].

Let F be a connected, oriented surface of finite type (for instance, the free side of a pared manifold). Let $\mathcal{C} = \mathcal{C}(F)$ be the set of essential non-peripheral simple closed curves in F, considered up to isotopy. Given $\alpha, \beta \in \mathcal{C}$, we define the *geometric intersection number* $\iota(\alpha, \beta)$ to be the minimal intersection number between isotopy representatives of α and β. For any α we use $\iota_\alpha \colon \mathcal{C} \to \mathbb{R}$ to denote the resulting function $\beta \mapsto \iota(\alpha, \beta)$.

Recall that $\mathbb{R}^{\mathcal{C}}$ is a topological vector space over $\mathbb{R}$, equipped with the product topology. In a minor abuse of notation, we write $\iota \colon \mathcal{C} \to \mathbb{R}^{\mathcal{C}}$ for the resulting injection. We define $\mathcal{MC}(F) = \mathbb{R}_{\geq 0} \cdot \iota(\mathcal{C})$ to be the subset of *measured curves*; that is, functions of the form

$$r\iota_\alpha, \quad \text{where } r \in \mathbb{R}_{\geq 0}, \alpha \in \mathcal{C}.$$

We define $\mathcal{ML}(F)$ to be the space of *measured laminations*; this is the closure of $\mathcal{MC}(F)$ inside of $\mathbb{R}^{\mathcal{C}}$. We define $\mathcal{PML}(F)$ to be the image of $\mathcal{ML}(F)$ in the projectivization $\mathbb{PR}^{\mathcal{C}}$. We end this review by noting the important fact, recorded by Bonahon [6], that ι extends to give a homogeneous, continuous function from

$$\mathcal{ML}(F) \times \mathcal{ML}(F) \to \mathbb{R}.$$

Now, let F be a free side of a pared manifold (M, P). A *meridian* on F is a simple closed curve $\alpha \in \mathcal{C}(F)$ that bounds a disk in M. The *Masur domain* of F consists of all $\lambda \in \mathcal{PML}(F)$ such that $\iota(\lambda, \mu) \neq 0$ for every measured lamination $\mu \in \mathcal{ML}(F)$ that arises as a limit of measured meridians. See [33, Section 6.1].

Definition 2.8 (Filling end invariants). Let (M, P) be a pared manifold with free sides $F_1, \ldots, F_s$. Let $0 \leq r \leq s$. Consider a collection of end invariants

$$(X_1, \ldots, X_r, \lambda_{r+1}, \ldots, \lambda_s),$$

where $X_i \in \mathcal{T}(F_i)$ for $i \leq r$ and λ_i is an ending lamination on F_i for $i \geq r+1$. This collection of end invariants is called *filling* if it satisfies the following conditions:

(∗) If M is an interval bundle over a compact surface S and N has no geometrically finite ends, then the projection of the ending laminations to S has transverse self-intersection.

(∗∗) If a compressible component $F_i \subset \partial_0 M$ corresponds to a degenerate end, then the end invariant λ_i, equipped with some transverse measure, is a Masur domain lamination. Equivalently, λ_i is not contained in the Hausdorff limit of any sequence of meridians.

Canary [14] proved that conditions (∗) and (∗∗) are necessary for the end invariants to be realized by a hyperbolic structure on (M, P). Namazi and Souto proved that these conditions are also sufficient for being realized by a hyperbolic structure [33, Theorem 1.3]:

Theorem 2.9 (Realization). *Let (M, P) be a pared 3-manifold, with a collection of end invariants on the free sides of (M, P). Then there exists a minimally parabolic representation $\rho \in AH(M, P)$ yielding a hyperbolic manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ with the given end invariants if and only if the collection of end invariants is filling.*

2.5. Geometric and strong limits. In addition to the algebraic topology on the space of hyperbolic 3-manifolds, we need to use another, finer topology. Let Γ_n be a sequence of Kleinian groups. We say that Γ_n *converges geometrically* to Γ if the groups converge in the Chabauty topology on closed subsets of $\mathrm{PSL}(2, \mathbb{C})$. Convergence in this topology can be characterized as follows:

- every $\gamma \in \Gamma$ is the limit of some sequence $\{\gamma_n\}$ with $\gamma_n \in \Gamma_n$;
- if $\gamma_n \to \gamma$ is a convergent sequence with $\gamma_n \in \Gamma_n$, then $\gamma \in \Gamma$.

The Chabauty topology on Kleinian groups is metrizable [16, Proposition 3.1.2]. We use the notation d_{Chaub} to denote a conjugation-invariant metric inducing this topology.

We endow $\mathbb{H}^3$ with an origin (denoted 0) and an orthonormal frame at 0. Then each quotient manifold $N_n = \mathbb{H}^3/\Gamma_n$ is endowed with a *baseframe* ω_n, namely the quotient of the fixed orthonormal frame at $0 \in \mathbb{H}^3$. Then Γ_n and the pair (N_n, ω_n) determine one another. Changing Γ_n by conjugation in $\mathrm{PSL}(2, \mathbb{C})$ keeps the quotient manifold the same up to isometry, but changes the baseframe. We emphasize that the Chabauty topology is a topology on Kleinian groups (not conjugacy classes), or equivalently a topology on the set of hyperbolic manifolds endowed with baseframes.

Geometric convergence has the following intrinsic characterization. Let (N_n, ω_n) be a sequence of framed hyperbolic 3-manifolds. Let (N, ω) be another framed hyperbolic 3-manifold, where ω is a baseframe at $x \in N$. For $R > 0$, let $B_R(x) \subset N$ be the metric R-ball in N centered at x, meaning the set of points in N of distance less than R from x. Then (N_n, ω_n) *converges geometrically* to (N, ω) if and only if, for every R, there are embeddings

$$f_{n,R}\colon (B_R(x), \omega) \hookrightarrow (N_n, \omega_n) \tag{2.1}$$

for all n sufficiently large, which converge to isometries in the C^∞ topology as $n \to \infty$. See [16, Theorem 3.2.9] for the equivalence between geometric convergence $(N_n, \omega_n) \to (N, \omega)$ and convergence $\Gamma_n \to \Gamma$ in the Chabauty topology.

In practice, it is often sufficient to keep track of the points where frames are based. If (N_n, x_n) is a sequence of hyperbolic 3-manifolds endowed with basepoints, and (N, x) is another hyperbolic 3-manifold with a basepoint x, we say that

$$(N_n, x_n) \to (N, x)$$

if, for every R, there are embeddings

$$g_{n,R}\colon (B_R(x), x) \hookrightarrow (N_n, x_n) \tag{2.2}$$

for all n sufficiently large, which converge to isometries in the C^∞ topology as $n \to \infty$. Now, suppose we have chosen orthonormal frames ω_n at x_n and ω at x. If there is a geometric limit $(N_n, \omega_n) \to (N, \omega)$, then of course $(N_n, x_n) \to (N, x)$ as well. Conversely, since the set of orthonormal frames at a given point is compact, a pointed limit $(N_n, x_n) \to (N, x)$ implies that there is a subsequence n_i and a frame ν at x such that $(N_{n_i}, \omega_{n_i}) \to (N, \nu)$. See [16, Lemma 3.2.8].

By a mild abuse of notation, we will say that (N_n, x_n) *converges geometrically* to (N, x), meaning that there exists a choice of frames at x_n and x such that $(N_n, \omega_n) \to (N, \omega)$. In most of our limit arguments, basepoints will be important while frames will remain implicit.

Now, consider a sequence of discrete, faithful representations

$$\rho_n\colon \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$$

with image groups Γ_n, and a representation ρ with image group Γ. We say that ρ_n *converges strongly* to ρ if $\rho_n \to \rho$ algebraically and $\Gamma_n \to \Gamma$ in the Chabauty topology.

The question of whether algebraic and geometric limits agree is subtle, and has received extensive attention in the literature. See Marden [29, Chapter 4] for a survey. For our purposes, we will need only the following foundational statement.

Theorem 2.10 (Same topology on $QH(\rho)$). *Let (M, P) be a pared 3-manifold, and let $\rho \in AH(M, P)$ be a geometrically finite, minimally parabolic representation. Then the algebraic and geometric topologies on the open set $QH(\rho) \subset AH(M, P)$ agree. More precisely:*

- *If ρ_n is a sequence in $QH(\rho)$ such that $\rho_n \to \rho$ algebraically, then $\rho_n \to \rho$ strongly.*
- *If ρ_n is a sequence in $QH(\rho)$ such that $\rho_n(\pi_1(M)) \to \rho(\pi_1(M))$ geometrically, then $\rho_n \to \rho$ strongly.*

The first bullet is due to Anderson and Canary [4, Theorem 3.1]. The second bullet is due to Jørgensen and Marden [27, Theorem 4.9]. See also Marden [29, Theorems 4.6.1 and 4.6.2].

2.6. Strong density and approximation theorems. The density theorem, whose proof was concluded independently by Ohshika [34] and Namazi and Souto [33], states that for every $\rho \in AH(M, P)$, there exists a geometrically finite sequence $\rho_n \in AH(M, P)$ converging algebraically to ρ. The results we need, stated below as Theorems 2.12 and 2.13, are stronger. Stating these results requires the notion of a filling sequence.

Definition 2.11 (Filling sequence in $\mathcal{T}(\partial M - P)$). Let (M, P) be a pared manifold with free sides $F_1, \ldots, F_s$. As in Definition 2.8, let $(X_1, \ldots, X_r, \lambda_{r+1}, \ldots, \lambda_s)$ be a filling tuple of end invariants, where $X_1, \ldots, X_r$ are points in Teichmüller space and $\lambda_{r+1}, \ldots, \lambda_s$ are ending laminations. Suppose a sequence

$$(X_1^n, \ldots, X_s^n) \in \mathcal{T}(F_1) \times \cdots \times \mathcal{T}(F_s) = \mathcal{T}(\partial_0 M)$$

satisfies:

(1) for all n and all $i \leq r$, we have $X_i^n = X_i$, and

(2) for all $i \geq r + 1$, there is a sequence of simple closed curves $\gamma_i^n \in \mathcal{C}(F_i)$ which converge to λ_i in $\mathcal{PML}(F_i)$, such that the ratio of lengths $\ell_{X_i^n}(\gamma_i^n)/\ell_{X_i^1}(\gamma_i^n)$ approaches zero.

Then the sequence $(X_1^n, \ldots, X_s^n)$ is said to be *filling*.

Let $\rho \in AH(M, P)$ be a discrete, faithful representation corresponding to $(X_1, \ldots, X_r, \lambda_{r+1}, \ldots, \lambda_s)$, with quotient manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1 M)$. Then, for $i > r$, the curves γ_i^n appearing in item (2) define closed geodesics in the end of N_ρ associated with F_i. These curves are said to *exit the end* associated with F_i. This means that all but finitely many lie in the geometric end homeomorphic to $F_i \times (0, \infty)$, and for any compact set $K \subset N_\rho$, only finitely many γ_i^n intersect K.

We can now state a strong form of the density theorem [33, Corollary 12.3].

Theorem 2.12 (Strong density theorem). *Let Γ be a finitely generated Kleinian group. Let (M, P) be the pared manifold associated with $\mathbb{H}^3/\Gamma$, and let $\rho\colon \Gamma \hookrightarrow PSL(2, \mathbb{C})$ be the inclusion map. Then there is a sequence of geometrically finite, minimally parabolic representations $\rho_n \in AH(M, P)$ converging strongly to ρ. Furthermore, the sequence of end invariants corresponding to ρ_n is filling.*

Note that the "furthermore" sentence in our statement of Theorem 2.12 is not stated directly in [33, Corollary 12.3]. However, this assertion is central to the proof of [33, Corollary 12.3]: the approximating manifolds appearing in Namazi and Souto's construction are taken to have ends forming a filling sequence. We also note that the proof of Theorem 2.12 uses both the tameness and the ending lamination theorems.

The following related statement is [33, Corollary 12.5].

Theorem 2.13 (Approximation theorem). *Let (M, P) be a pared 3-manifold with free sides $F_1, \ldots, F_s$. Suppose $(X_1^n, \ldots, X_s^n) \in \mathcal{T}(\partial M - P)$ is a filling sequence converging to the filling end invariants $(X_1, \ldots, X_r, \lambda_{r+1}, \ldots, \lambda_s)$. Let*

$$\rho_n \in \pi_{AB}(X_1^n, \ldots, X_s^n) \in AH(M, P)$$

be the Ahlfors–Bers representation, giving an associated geometrically finite hyperbolic 3-manifold $N_n = \mathbb{H}^3/\rho_n(\pi_1(M))$. Then, up to passing to a subsequence, ρ_n converges strongly to a discrete and faithful representation ρ. If (M', P') is the

pared manifold associated with the hyperbolic manifold $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$*, then there is a homeomorphism* $\phi\colon (M, P) \to (M', P')$ *in the homotopy class determined by* ρ *which maps the filling tuple* $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ *to the end invariants of* N_ρ*.*

2.7. Circle packings. Let R be a Riemann surface of hyperbolic type, meaning that every component $R_i \subset R$ has $\chi(R_i) < 0$. Let Ω be an open subset of the Riemann sphere S^2 that uniformizes R; that is, R is a quotient of Ω by Möbius transformations. A configuration of circles on R (relative to Ω) is a collection of simple closed curves on R that bound disks, such that the interiors of the disks are disjoint, and the lifts of the curves to Ω are round circles on S^2. A configuration of circles is a *circle packing* if the interstitial regions, complementary to the interiors of the disks, consist only of curvilinear triangles.

The following theorem is from Beardon and Stephenson [5, Theorem 6].

Theorem 2.14 (Uniformization theorem for circle packings). *Let* R *be a Riemann surface that admits a circle packing. Then the circle packing uniquely determines a conformal structure on* R*.*

Now, let Γ be a finitely generated Kleinian group, with $\Lambda(\Gamma)$ its limit set and $\Omega(\Gamma)$ its domain of discontinuity. Then $\Omega(\Gamma)/\Gamma$ is a (possibly disconnected) Riemann surface of hyperbolic type.

The following theorem follows from work of Brooks [12].

Theorem 2.15 (Circle packings approximate). *Let* $N = \mathbb{H}^3/\Gamma$ *be a geometrically finite hyperbolic* 3*-manifold with associated pared manifold* (M, P)*, and let* $\rho\colon \pi_1(M) \to \Gamma$ *be the associated representation. Then, for every* $\delta > 0$*, there is a geometrically finite representation* $\rho_\delta \in QH(\rho)$*, representing an* e^δ*-quasiconformal deformation of* ρ*, such that the conformal boundary* $\Omega(\Gamma_\delta)/\Gamma_\delta$ *of the image group* Γ_δ *admits a circle packing.*

Proof sketch. The ideas behind this statement are all contained in Brooks' proof of [12, Theorem 2]. If a component of $R = \Omega(\Gamma)/\Gamma$ is a closed surface S, then we may uniformize S by a component of $\Omega(\Gamma)$. Pack circles into this component S, obtaining interstices that are triangles and quads. Brooks shows that for every $\delta > 0$, there is an e^δ-quasiconformal deformation Γ_δ that eliminates the quads of a sufficiently fine packing, so the deformed conformal structure on S admits a circle packing.

When a component $S \subset R$ has cusps, the packing procedure requires some additional care. The argument is given, for example, in Hoffman and Purcell [25, Lemma 2.3]. Since Γ is geometrically finite, there is a fundamental domain $\mathcal{F}$ for Γ whose sides consist of finitely many geodesic hyperplanes. Where there is a rank-1 cusp, there will be two circular arcs C_1, C_2 bounding a polygonal region of the boundary of $\mathcal{F}$ such that C_1 and C_2 meet tangentially. Begin by adding two circles

meeting orthogonally at that point of tangency, and ensure those circles are small enough that they meet no other sides of $\mathcal{F}$. Then fill in the rest of $\Omega(\Gamma)$ by circles, and use Brooks to perform a quasi-conformal deformation as above to obtain a circle packing. □

We close this section with a definition that will be useful for our constructions.

Definition 2.16 (Scooped manifold, double-double of a circle packed manifold). Let $N = \mathbb{H}^3/\Gamma$ be a tame, geometrically finite hyperbolic manifold with associated pared manifold (M, P), and suppose the conformal boundary associated with each free side of $\partial M - P$ admits a circle packing. Let $\widetilde{C}$ denote the collection of inverse images in $\partial_\infty \mathbb{H}^3$ of the circles in the packing. Scoop out the half spaces in $\mathbb{H}^3$ bounded by Euclidean hemispheres with boundary circles $\widetilde{C}$, and color their boundaries blue. The triangular interstices between circles of $\widetilde{C}$ uniquely determine additional circles dual to the blue ones. Scoop out the half spaces in $\mathbb{H}^3$ bounded by these dual circles as well, and color their boundaries red. The resulting space, denoted $\widetilde{N}^\circ$, is $\mathbb{H}^3$ with the interiors of red and blue hemispheres removed.

The group Γ stabilizes $\widetilde{N}^\circ$. The quotient space $N^\circ = \widetilde{N}^\circ/\Gamma$ is a manifold with corners whose interior is homeomorphic to N, and whose boundary consists of geodesic blue ideal polygons and geodesic red ideal triangles. We call N° the *scooped manifold* associated with N.

Finally, starting with the scooped manifold N° associated with N, double first across the blue polygons, then double again across the red triangles. The result is a finite volume hyperbolic manifold with rank-2 cusps, which we will call the *double-double of* N, and denote by $DD(N^\circ)$.

3. A 6-Theorem for tame manifolds

The 6-Theorem for finite volume 3-manifolds, proved by Agol [1] and Lackenby [28], states that Dehn filling along a slope of length greater than 6 yields a manifold that admits a hyperbolic structure. In this section, we use many of the ideas behind their proof to extend their result to the infinite volume setting.

Theorem 3.1 (6-Theorem for pared manifolds). *Let Z be a tame hyperbolic 3-manifold, with associated pared manifold $(M, P \cup (T_1 \cup \cdots \cup T_k))$, where $T_1, \ldots, T_k$ are torus boundary components of M. Assume that*

$$\partial M - (T_1 \cup \cdots \cup T_k) \neq \emptyset.$$

Let $H_1 \cup \cdots \cup H_k$ be pairwise disjoint horocusps, with H_i a neighborhood of T_i. Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a tuple of slopes, such that the length of a geodesic representative of each s_i on ∂H_i is strictly greater than 6 for each i.

Let $M(\mathbf{s}) = M(s_1, \ldots, s_k)$ denote the 3-manifold obtained by Dehn filling M along the slopes s_i on T_i. Then $(M(\mathbf{s}), P)$ is a pared manifold, such that the free

sides of $\partial_0 M(\mathbf{s})$ are identical to those of $\partial_0 M$. Furthermore, $(M(\mathbf{s}), P)$ admits a hyperbolic structure $Y = Z(\mathbf{s})$ with end invariants identical to those of Z.

Observe that Theorem 1.4 follows immediately from Theorem 3.1 and prior work. If $\partial M(\mathbf{s}) = \emptyset$, then Agol and Lackenby's 6-Theorem [1, 28], combined with Perelman's geometrization theorem, says that $M(\mathbf{s}) = M(s_1, \ldots, s_k)$ admits a hyperbolic structure (with empty end invariants). Otherwise, if $\partial M(\mathbf{s}) \neq \emptyset$, Theorem 3.1 gives the desired conclusion.

There is one case where we re-prove a portion of the original 6-Theorem, by following the same line of argument: the case where $\mathrm{vol}(Z) < \infty$ and $P \neq \emptyset$. In this case, Lemma 3.3 combined with Theorem 2.6 shows that $M(\mathbf{s})$ admits a hyperbolic structure.

Before beginning the proof of Theorem 3.1, we record the following slight generalization Böröczky's theorem on cusp density [7].

Lemma 3.2. *Let S be a hyperbolic surface with finite area, with a positive number of cusps, and with (possibly empty) boundary consisting of geodesics. Let $H \subset S$ be an embedded neighborhood of the cusps, such $H \cap \partial S = \emptyset$ and ∂H is a disjoint union of horocycles. Then*

$$\mathrm{area}(S) \geq \frac{\pi}{3} \mathrm{area}(H).$$

Proof. If $\partial S = \emptyset$, this result is due to Böröczky [7, Theorem 4].

If $\partial S \neq \emptyset$, let DS be the complete hyperbolic surface obtained by doubling S along its geodesic boundary. Since $H \cap \partial S = \emptyset$, the double of H is an embedded cusp neighborhood $DH \subset DS$, with $\partial(DH)$ a disjoint union of horocycles. Böröczky's theorem says that

$$\mathrm{area}(DS) \geq \frac{\pi}{3} \mathrm{area}(DH),$$

hence $\mathrm{area}(S) \geq \frac{\pi}{3} \mathrm{area}(H)$. □

The first step of the proof of Theorem 3.1 is to show that $(M(\mathbf{s}), P)$ is a pared manifold. The main idea of the proof is drawn directly from the arguments of Agol [1] and Lackenby [28].

Lemma 3.3. *With notation and hypotheses as in Theorem 3.1, the pair $(M(\mathbf{s}), P)$ is a pared manifold.*

Proof. Since M is compact and oriented, the filling $M(\mathbf{s})$ is compact and oriented as well. Since $\partial M - (T_1 \cup \cdots \cup T_k) \neq \emptyset$, it follows that $\partial M(\mathbf{s}) \neq \emptyset$. In addition, $M(\mathbf{s})$ cannot be a 3-ball: otherwise, ∂M has a component $\Sigma \cong S^2$ and a separate component $T_1 \cong T^2$, which is impossible because M is irreducible. The remaining obstructions to $(M(\mathbf{s}), P)$ being a pared manifold are as follows.

Claim 3.4. *If $(M(\mathbf{s}), P)$ is not a pared manifold, then there is an essential, embedded surface $(S, \partial S) \to (M(\mathbf{s}), P)$, where S is a sphere, disk, torus, or annulus.*

To prove the claim, we check the remaining conditions of Definition 2.1. If $M(\mathbf{s})$ is reducible or toroidal, then by definition it contains an essential sphere or torus. If $M(\mathbf{s})$ is a solid torus, then $P \subset \partial M$ must be a torus that becomes compressible in $M(\mathbf{s})$, hence $(M(\mathbf{s}), P)$ contains an essential compression disk. If there is a π_1-injective map of an annulus

$$(S^1 \times I, S^1 \times \partial I) \to (M(\mathbf{s}), P),$$

which is not boundary-parallel, then the annulus theorem says that there is also a π_1-injective embedding of an annulus with the same property; see Jaco [26, Theorem VIII.13] or Scott [35]. Finally, if there is a π_1-injective map of a torus $T^2 \to M(\mathbf{s})$, then observe that $M(\mathbf{s})$ is either reducible (hence the claim holds) or Haken. In the latter scenario, the torus theorem [35] says that $M(s_1, \ldots, s_k)$ either contains an essential torus or annulus (hence the claim holds), or is Seifert fibered. But every Seifert fibered 3-manifold with non-empty boundary contains an essential disk or annulus, proving the claim.

Now, let $(S, \partial S) \to (M(\mathbf{s}), P)$ be an essential, embedded surface as in the claim. Assume that S has been moved by isotopy to intersect the filling solid tori a minimal number of times. Then every component of intersection must be a meridian disk of some filling solid torus. Note that the intersection is non-empty, because $(M, P \cup T_1 \cup \cdots \cup T_k)$ is a pared manifold by hypothesis. After removing the meridian disks from S, we obtain an essential surface $S' \subset M$ whose boundary consists of ∂S and a collection of curves on the tori $T_1, \ldots, T_k$, with each boundary on T_i having slope s_i.

Because M is homeomorphic to a standard compact core of $(Z^\varepsilon, \partial Z^\varepsilon)$ for small $\varepsilon > 0$, we obtain an embedding of S' into Z^ε, which we extend by a product into the cusps, obtaining a punctured surface that we continue to call S'.

Using the hyperbolic metric Z, we homotope S' to be a *pleated surface*. This means that after a homotopy, S' becomes an immersed surface consisting of totally geodesic ideal triangles, with bending allowed along the edges of the triangles. A homotopy that moves S' into pleated form exists by [28, Lemma 2.2]. As in Agol's and Lackenby's proof of the original 6-Theorem (see [1, Theorem 5.1], and compare [21, Lemma 2.5]), the horocusp $H = H_1 \cup \cdots \cup H_k$ induces a disjoint union of horocycles in the pleated surface S', such that the length of each horocycle is at least the length of the corresponding slope s_i, hence each has length strictly larger than 6.

Let m be the number of boundary components of S' on components $T_1, \ldots, T_k$. This is the number of horocycles in S' constructed in the previous paragraph. The area of the cusp ends of S' cut off by these horocycles is the sum of the lengths of the horocycles; hence, the total area is strictly larger than $6m$. By Lemma 3.2, the area of S' is at least $\pi/3$ times the area of the cusp neighborhood, hence $\text{area}(S') > 2\pi m$.

On the other hand, by the Gauss–Bonnet theorem, $\text{area}(S') = -2\pi\chi(S')$. Thus, the area of S' satisfies:

$$\text{area}(S') = \begin{cases} 2\pi(m-2) & \text{if } S \text{ is a sphere,} \\ 2\pi(m-1) & \text{if } S \text{ is a disk,} \\ 2\pi(m) & \text{if } S \text{ is a torus or annulus.} \end{cases}$$

In all cases, the area is at most $2\pi m$. This contradiction shows that S cannot exist, hence Claim 3.4 shows that $(M(\mathbf{s}), P)$ is a pared manifold. □

The second step in the proof of Theorem 3.1 is to show that the same end invariants that are realizable in $(M, P \cup (T_1 \cup \dots T_k))$ are also realizable in $(M(\mathbf{s}), P)$.

Proof of Theorem 3.1. By Lemma 3.3, $(M(\mathbf{s}), P)$ is a pared manifold. Notice that $\partial_0 M = \partial_0 M(\mathbf{s})$, hence the free sides of $(M(\mathbf{s}), P)$ are identical to the free sides of

$$(M, P \cup (T_1 \cup \cdots \cup T_k)).$$

Let $F_1, \dots, F_s$ be these free sides. We further assume that the free sides have been ordered so that the end invariants of Z are

$$(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s),$$

where $X_i \in \mathcal{T}(F_i)$ for $i \leq r$ and λ_j is an ending lamination for F_j for $j \geq r+1$.

By the realization theorem, Theorem 2.9, $(M(\mathbf{s}), P)$ admits a hyperbolic structure with these end invariants if and only if the tuple of end invariants $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ is filling. That is, we must check that

$$(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$$

satisfy properties $(*)$ and $(**)$ of Definition 2.8, when viewed as invariants of $(M(\mathbf{s}), P)$.

If $(*)$ is false, then $M(\mathbf{s})$ is an interval bundle over a surface S. Without loss of generality (replacing S by its orientable double cover if needed), we may assume that S is orientable and $M(\mathbf{s}) \cong S \times I$. Since $(*)$ has failed, $M(\mathbf{s})$ has no geometrically finite ends. Furthermore, there is an ending lamination λ_i on a free side $F_i \subset S \times \{0\}$ and an ending lamination λ_j on a free side $F_j \subset S \times \{1\}$ containing parallel, non-isolated leaves. But these leaves are dense in λ_i and λ_j respectively, which means (F_i, λ_i) and (F_j, λ_j) have the same projection to S. Since ∂F_i and ∂F_j are contained in P, and the parabolic locus P cannot admit essential annuli, it follows that $F_i = S \times \{0\}$ and $F_j = S \times \{1\}$ are the only free sides of $(M(\mathbf{s}), P)$. We write λ for the ending lamination on $S \times \{0\}$ and $\overline{\lambda}$ for the ending lamination on $S \times \{1\}$. By the above argument (dense leaves), λ and $\overline{\lambda}$ are parallel.

Let $X^n \in \mathcal{T}(S)$ be a filling sequence converging to λ. Writing $\overline{X}^n \in \mathcal{T}(\overline{S})$ for the same hyperbolic structure with opposite orientation, the sequence $\overline{X}^n$ converges

to $\bar{\lambda}$. Then $(X^n, \bar{X}^n) \in \mathcal{T}(\partial M - P)$ is a filling sequence converging to $(\lambda, \bar{\lambda})$. Theorems 2.6 and 2.7 imply there exists a hyperbolic 3-manifold Z_n with standard compact core homeomorphic to $(M, P \cup (T_1 \cup \cdots \cup T_k))$, and with end invariants $(X^n, \bar{X}^n)$. Because the sequence $(X^n, \bar{X}^n)$ is filling, there exists a sequence of simple closed curves $\gamma^n \subset S$ converging to λ, with $\ell_{X^n}(\gamma^n)/\ell_{X^1}(\gamma^n) \to 0$. In the hyperbolic manifold Z_n, the curve γ^n is homotopic to a geodesic in the lower end whose conformal structure is X^n, and $\bar{\gamma}^n$ is homotopic to a geodesic in the upper end whose conformal structure is $\bar{X}^n$.

The approximation Theorem 2.13 implies that after passing to a subsequence, the manifolds Z_n converge strongly to a manifold homeomorphic to M, with the same end invariants as Z. The ending lamination Theorem 2.5 implies that the limiting manifold is isometric to Z. For each n, let $H^n = H^n_1 \cup \cdots \cup H^n_k$ be a disjoint union of horocusps for $T_1 \cup \cdots \cup T_k$ in Z_n, which converge to the horocusp neighborhood

$$H = H_1 \cup \cdots \cup H_k \subset Z.$$

The strong limit $Z_n \to Z$ implies that for sufficiently large n, we have $\mathrm{len}(s_i) > 6$ in the Euclidean metric on ∂H^n_i.

Consider an annulus

$$A_n = \gamma^n \times I \subset S \times I = M(\mathbf{s}).$$

Let A'_n be the remnant of A_n in $M \subset M(\mathbf{s})$, moved by isotopy to minimize the intersection number with the cores of filling solid tori. Then A'_n has a boundary component along γ^n in the lower end, a boundary component along $\bar{\gamma}^n$ in the upper end, as well as some number of punctures along meridians $s_1, \ldots, s_k$. In the hyperbolic metric Z_n, we may homotope A'_n to be a pleated surface with geodesic boundary along $\gamma^n \cup \bar{\gamma}^n$. For sufficiently large n, the geodesics $\gamma^n \cup \bar{\gamma}^n \subset Z_n$ are disjoint from H^n, because the geodesic realizations of the same curves in Z are exiting the ends of Z as $n \to \infty$. Furthermore, the horocusp neighborhood H^n induces a disjoint union of horocycles in A'_n, where each horocycle has length greater than 6.

Let m_n be the number of punctures in A'_n. As in the proof of Lemma 3.3, the Gauss–Bonnet theorem implies that $\mathrm{area}(A'_n) = 2\pi m_n$. On the other hand, for n large, the cusp area of A'_n is strictly larger than $6m_n$, hence Lemma 3.2 implies that $\mathrm{area}(A'_n) > 2\pi m_n$. This contradiction proves that the end invariants of $(M(\mathbf{s}), P)$ must satisfy $(*)$.

Next, suppose $(**)$ is false. Then $(M(\mathbf{s}), P)$ has a compressible free side F, and the ending lamination λ is contained in a limit of meridians. It follows that there is a sequence of compression disks D_n for $(M(\mathbf{s}), P)$ such that the boundary curves $\gamma^n = \partial D_n$ converge to a lamination μ containing λ. Since λ is *not* contained in a limit of meridians in (M, P), it follows that the curve γ^n cannot be a meridian in (M, P) for sufficiently large n. In other words, viewing M as a submanifold

of $M(s_1, \dots, s_k)$, it follows that the meridian disks D_n cannot be contained in M for large n. We isotope each D_n to meet the filling solid tori as few times as possible, and set $D'_n = D_n \cap M$.

By strong density, Theorem 2.12, there is a sequence of geometrically finite and minimally parabolic representations $\rho_n \in AH(M, P)$ converging strongly to the representation ρ corresponding to Z. Let Z_n be the associated manifolds. As above, each Z_n is equipped with a disjoint union of horocusps

$$H^n = H^n_1 \cup \cdots \cup H^n_k,$$

where for sufficiently large n we have $\operatorname{len}(s_i) > 6$ in the Euclidean metric on ∂H^n_i. Since the curves γ^n limit to a lamination containing λ, they must exit the end of Z corresponding to F, hence γ^n is disjoint from the horocusp neighborhood H^n for large n. As above, we may pleat the punctured disk D'_n in Z_n, so that it has geodesic boundary along γ^n and some number of punctures corresponding to meridians of $s_1, \dots, s_k$. These punctures are cut off by disjoint horocycles in D'_n, where each horocycle has length greater than 6.

We can now obtain a contradiction as above. Let m_n be the number of punctures in D'_n. The Gauss–Bonnet theorem implies that

$$\operatorname{area}(D'_n) = 2\pi(m_n - 1).$$

On the other hand, for n large, the cusp area of D'_n is strictly larger than $6m_n$, hence Lemma 3.2 implies that $\operatorname{area}(D'_n) > 2\pi m_n$. This contradiction proves that the end invariants $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ must satisfy $(**)$ as well.

Since the tuple of end invariants $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ satisfy both $(*)$ and $(**)$, Theorem 2.9 says that these invariants are realized by a hyperbolic metric Y on $(M(\mathbf{s}), P)$. □

4. Convergence of geodesics

In several theorems in the subsequent sections, we will need to control geodesic links in a convergent sequence of manifolds. The following proposition says that geodesic links behave exactly as expected.

Proposition 4.1. *Let N_∞ be a hyperbolic 3-manifold with standard compact core (M, P) and associated representation $\rho_\infty \in AH(M, P)$. Let $\rho_n \in AH(M, P)$ be a sequence of geometrically finite, minimally parabolic representations converging strongly to ρ_∞. Let N_n be the hyperbolic 3-manifolds associated to ρ_n. Let $\Sigma = \sigma_1 \cup \ldots \cup \sigma_k \subset M$ be a smooth link. Then the following are equivalent:*

(1) *Σ is isotopic to a geodesic link $\Sigma_\infty \subset N_\infty$, where each component has length less than* $\log 3$.

(2) *For all* $n \gg 0$, Σ *is isotopic to a geodesic link* Σ_n *in the hyperbolic structure* N_n, *where each component has length less than* $\log 3 - \delta$ *for some uniform* $\delta > 0$.

Furthermore, assuming either (1) *or* (2) *hold, we have* $\Sigma_n \to \Sigma_\infty$ *in the geometric limit.*

Proof. If $\partial_0 M = \emptyset$, then any convergent sequence in $AH(M, P)$ is eventually constant, hence the result is vacuous. We may now suppose that $\partial_0 M \neq \emptyset$, or equivalently $\mathrm{vol}(N_n) = \mathrm{vol}(N_\infty) = \infty$.

We begin by setting some notation. For each component $\sigma_i \subset \Sigma$, orient σ_i and choose an arc α_i that runs from the basepoint $x \in M$ to σ_i. Then the based loop $\alpha_i \cdot \sigma_i \cdot \overline{\alpha}_i$ represents a homotopy class $\gamma_i \in \pi_1(M, x)$. We also choose basepoints $x_n \in N_n$ and $x_\infty \in N_\infty$ so that

$$(N_n, x_n) \to (N_\infty, x_\infty)$$

is a geometric limit.

We first prove that (1) implies (2). Suppose $\Sigma_\infty \subset N_\infty$ is a geodesic realization of Σ. Since the components $\sigma_1, \ldots, \sigma_k \subset \Sigma$ are isotopic to disjoint, simple geodesics in N_∞, the group elements $\gamma_1, \ldots, \gamma_k$ corresponding to $\sigma_1, \ldots, \sigma_k$ are primitive and pairwise non-conjugate.

Choose a radius R large enough so that $\Sigma_\infty \subset B_R(x_\infty)$. According to the characterization of geometric convergence in (2.2), for large n we have embeddings

$$g_{n,R} \colon (B_R(x_\infty), x_\infty) \hookrightarrow (N_n, x_n)$$

that converge to isometries in the C^∞ topology. These embeddings map the geodesic link Σ_∞ to a link $\Sigma'_n \subset N_n$.

Since each of the k components of Σ_∞ is shorter than $\log 3$, there is a uniform $\delta > 0$ such that each component is shorter than $\log 3 - \delta$.

Let $\sigma_{i,\infty}$ be a component of Σ_∞. Then, for all sufficiently large n, the image

$$\sigma'_{i,n} = g_{n,R}(\sigma_{i,\infty}) \subset \Sigma'_n$$

is shorter than $\log 3 - \delta$. By Theorem 2.2, $\log 3$ is a Margulis number for each N_n. Since the group elements $\gamma_1, \ldots, \gamma_k$ represent distinct, primitive conjugacy classes, the curves $\sigma'_{1,n}, \ldots, \sigma'_{k,n}$ lie in disjoint tube components of $N_n^{<\log 3}$, with each component homotopic to the core of its tube. Furthermore, each $\sigma'_{i,n}$ can be taken to have arbitrarily small geodesic curvature (by choosing n large and applying the definition of a geometric limit), hence σ_n cannot have any local knotting. Thus, $\sigma'_{i,n}$ is *isotopic* to the core of the tube, denoted $\sigma_{i,n}$, where $\mathrm{len}(\sigma_{i,n}) < \log 3 - \delta$. The isotopies in distinct tubes do not interact, hence Σ'_n is isotopic to a geodesic link Σ_n, proving (2).

Next, we check that $\Sigma_n \to \Sigma_\infty$. For each n, let $\rho_n(\gamma_i)$ be the holonomy of γ_i in the hyperbolic structure N_n. For each $n \gg 0$, the group element $\rho_n(\gamma_i)$ stabilizes a geodesic axis $\widetilde{\sigma}_{i,n}$ that covers $\sigma_{i,n}$. The algebraic limit $\rho_n \to \rho_\infty$ implies that

$$\rho_n(\gamma_i) \to \rho_\infty(\gamma_i) \in \mathrm{Isom}(\mathbb{H}^3).$$

Thus, the fixed points of $\rho_n(\gamma_i)$ converge to the fixed points of $\rho_\infty(\gamma_i)$, and the axes $\widetilde{\sigma}_{i,n}$ converge to the axis $\widetilde{\sigma}_{i,\infty}$ that covers $\sigma_{i,\infty}$. Projecting down to the quotient manifold M, we learn that the closed geodesics $\sigma_{i,n} \subset N_n$ converge to $\sigma_{i,\infty}$, as desired.

Now, we prove that (2) implies (1). Suppose that for $n \gg 0$, the hyperbolic structure N_n contains a geodesic link Σ_n isotopic to Σ. Then the component $\sigma_i \subset \Sigma$ is isotopic to a closed geodesic $\sigma_{i,n} \subset N_n$, of length $\mathrm{len}(\sigma_{i,n}) < \log 3 - \delta$ for some uniform δ. By the same argument as in the above paragraph, the algebraic limit $\rho_n \to \rho_\infty$ implies that the geodesic axes $\widetilde{\sigma}_{i,n}$ for $\rho_n(\gamma_i)$ converge to the geodesic axis $\widetilde{\sigma}_{i,\infty}$ for $\rho_\infty(\gamma_i)$. Thus, for each i, the closed geodesics $\sigma_{i,n} \subset N_n$ converge to a closed geodesic $\sigma_{i,\infty} \subset N_\infty$. Since translation lengths converge in the limit, we have

$$\mathrm{len}(\sigma_{i,\infty}) \leq \log 3 - \delta < \log 3$$

for each i. Consequently, the closed geodesics $\sigma_{1,\infty}, \ldots, \sigma_{k,\infty}$ lie at the cores of disjoint tube components of $N_\infty^{<\log 3}$, and $\Sigma_\infty = \sigma_{1,\infty} \cup \ldots \cup \sigma_{k,\infty}$ is a geodesic link.

By construction, the loop $\gamma_i = \alpha_i \cdot \sigma_i \cdot \overline{\alpha}_i$ is freely homotopic to σ_i, hence the closed geodesic $\sigma_{i,\infty}$ is freely homotopic to σ_i. It remains to show that this homotopy can be achieved by isotopy, and that the isotopies for different components of Σ do not interact.

As above, we may choose a radius R large enough that $\Sigma_\infty \subset B_{R/2}(x_\infty)$. Then, for $n \gg 0$, we have embeddings

$$g_{n,R} \colon (B_R(x_\infty), x_\infty) \hookrightarrow (N_n, x_n)$$

that converge to isometries in the C^∞ topology. Since $\Sigma_n \to \Sigma_\infty$, it follows that for $n \gg 0$, we have $\Sigma_n \subset g_{n,R}(B_R(x_\infty))$. Then, for each component σ_i, the preimage $g_{n,R}^{-1}(\sigma_{i,n})$ is an almost-geodesic closed curve in the Margulis tube containing $\sigma_{i,\infty}$. As above, the control on geodesic curvature implies that $g_{n,R}^{-1}(\sigma_{i,n})$ cannot have local knotting, hence is isotopic to $\sigma_{i,\infty}$. Thus, $\sigma_i \subset \Sigma$ is isotopic to $\sigma_{i,\infty}$, with isotopies of different components supported in disjoint Margulis tubes. Thus, $\Sigma \subset M$ is isotopic to Σ_∞, as desired. □

5. Extracting bilipschitz limits

In this section, we describe the construction of a bilipschitz function from a pair of geometric limits. The main result, Theorem 5.1, is a technical statement that will be

used in the proofs of Theorems 1.2 and 1.3. The idea is that if hyperbolic 3-manifolds Y and Z each have a sequence of manifolds limiting to them geometrically, and there are J-bilipschitz maps between the thick parts of the approximating manifolds, then there is also a J-bilipschitz map between the thick parts of Y and Z.

The precise statement that we need involves *convex submanifolds* of a hyperbolic manifold. If Y is a hyperbolic 3-manifold with universal covering map $\pi\colon \mathbb{H}^3 \to Y$, a submanifold $Q \subset Y$ is called *convex* if the full preimage $\pi^{-1}(Q)$ is a convex subset of $\mathbb{H}^3$. The inclusion $Q \hookrightarrow Y$ is necessarily a homotopy equivalence. The convex core $CC(Y)$ is always a convex submanifold; in fact, it is the intersection of all convex submanifolds of Y. Another important example of a convex submanifold is the scooped manifold $N^\circ \subset N$ in Definition 2.16.

One important property of a convex submanifold $Q \subset Y$ is that intrinsic and extrinsic notions of injectivity radius agree. Given a point $x \in Q$, the injectivity radius $\mathrm{injrad}(x) = \varepsilon/2$ is realized by a geodesic loop γ based at x, of length exactly 2ε; compare [20, Lemma 2.11]. By convexity, this loop must be contained in Q. Consequently, $Q^{\geq\varepsilon} = Q \cap Y^{\geq\varepsilon}$.

We have the following theorem.

Theorem 5.1 (Bilipschitz limit). *Fix $\delta > 0$, $\varepsilon > 0$ and $J > 1$. Let $(Y_m, y_m) \to (Y, y)$ and $(Z_m, z_m) \to (Z, z)$ be geometrically converging sequences of based hyperbolic manifolds. For each m, let Y_m° and Z_m° be convex submanifolds of Y_m and Z_m, respectively. Suppose that $y \in Y^{>\varepsilon}$ and $y_m \in (Y_m^\circ)^{\geq\varepsilon}$, while $z_m \in Z_m^\circ$. Suppose $(Y_m^\circ, y_m) \to (Y, y)$ and $(Z_m^\circ, z_m) \to (Z, z)$.*

Suppose that, for each m, there is a J-bilipschitz inclusion

$$\varphi_m\colon (Y_m^\circ)^{\geq\varepsilon} \hookrightarrow (Z_m^\circ)^{\geq\delta},$$

such that $d(\varphi_m(y_m), z_m)$ is uniformly bounded. Then there is also a J-bilipschitz inclusion

$$\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\delta}.$$

The proof of Theorem 5.1 proceeds in two steps. In the first step, carried out in Lemma 5.2, we construct compact sets $K_n \subset Y$ and bilipschitz functions $h_n\colon K_n \to Z$, such that each h_n is *almost* J-bilipschitz and has image *almost* contained in $Z^{\geq\delta}$. These compact sets are nested, with $K_1 \subset K_2 \subset \cdots$, and form an exhaustion of $Y^{>\varepsilon}$. In the second step, carried out in Lemma 5.3, we extract subsequential limits of the locally defined functions h_n to obtain the desired bilipschitz inclusion φ.

Lemma 5.2. *Let the notation and hypotheses be as in Theorem 5.1. For $n \in \mathbb{N}$, define sequences of numbers as follows:*

$$\delta_n = \Big(1 - \frac{1}{n}\Big)\delta, \quad \varepsilon_n = \Big(1 + \frac{1}{n}\Big)\varepsilon, \quad J_n = 2^{1/n} J.$$

Then, for all sufficiently large n, there is a J_n-bilipschitz map

$$h_n\colon \bar{B}_n(y) \cap Y^{\geq\varepsilon_n} \hookrightarrow Z^{\geq\delta_n}.$$

Furthermore, $\{h_n(y)\}$ is a bounded sequence in Z.

Proof. We begin by characterizing what it means for n to be "sufficiently large". Since $y \in Y^{>\varepsilon}$ by hypothesis, and $\varepsilon_n \to \varepsilon$, we choose n large enough to ensure $y \in Y^{\geq \varepsilon_n}$. Next, let D be an upper bound on the distances $d(\varphi(y_m), z_m)$, where $y_m \in (Y_m^\circ)^{\geq \varepsilon}$ is the basepoint of Y_m° and $z_m \in Z_m^\circ$ is the basepoint of Z_m°. We also choose n large enough so that

$$4Jn > 2^{1/n} J \cdot 2n + D.$$

For each sufficiently large n, we will construct a map h_n. To that end, consider the set of points in Y of distance less than $2n$ from y, denoted $B_{2n}(y)$. Since $(Y_m^\circ, y_m) \to (Y, y)$, equation (2.2) says that for large $m \in \mathbb{N}$ there exist embeddings

$$f_{m,2n} \colon (B_{2n}(y), y) \hookrightarrow (Y_m^\circ, y_m)$$

that converge to isometries in the C^∞ topology as $m \to \infty$.

We will collect several desirable properties that hold for large m. First, observe that the closed set $\overline{B}_n(y)$ is compact and the derivatives of $f_{m,2n}$ are converging to the identity. Thus, for all sufficiently large m, we have:

$$\text{The restriction of } f_{m,2n} \text{ to } \overline{B}_n(y) \text{ is } 2^{1/2n}\text{-bilipschitz.} \tag{5.1}$$

Next, a lemma of Canary, Epstein, and Green [16, Lemma 3.2.6] shows that injectivity radii converge in a geometric limit. Since $\varepsilon_n > \varepsilon$, it follows that for all sufficiently large m, we have

$$f_{m,2n}\big(\overline{B}_n(y) \cap Y^{\geq \varepsilon_n}\big) \subset (Y_m^\circ)^{\geq \varepsilon}. \tag{5.2}$$

In particular,

$$f_{m,2n}(y) = y_m \in (Y_m^\circ)^{\geq \varepsilon}.$$

Combining (5.1), (5.2), and the J-bilipschitz property of the map $\varphi_m \colon (Y_m^\circ)^{\geq \varepsilon} \to (Z_m^\circ)^{\geq \delta}$ gives

$$\operatorname{diam}\big(\varphi_m \circ f_{m,2n}\big(\overline{B}_n(y) \cap Y^{\geq \varepsilon_n}\big)\big) \leq J \cdot 2^{1/2n} \cdot 2n.$$

The image set $\varphi_m \circ f_{m,2n}(\overline{B}_n(y) \cap Y^{\geq \varepsilon_n})$ contains $\varphi_m(y_m)$, hence this set is contained in the $(2^{1/2n} J \cdot 2n + D)$-neighborhood of z_m, by the definition of D.

Now, consider the geometric limit $(Z_m^\circ, z_m) \to (Z, z)$. For large m, there exist embeddings

$$g_{m,4Jn} \colon (B_{4Jn}(z), z) \hookrightarrow (Z_m^\circ, z_m)$$

that converge to isometries as $m \to \infty$, with injectivity radii converging as before. Since the image of $B_{4Jn}(z)$ converges to a set of point of distance less than $4Jn$ to z_m, and we have already chosen n so that $4Jn > 2^{1/n} J \cdot 2n + D$, it follows that for large m, we have

$$\varphi_m \circ f_{m,2n}\big(\overline{B}_n(y) \cap Y^{\geq \varepsilon_n}\big) \subset g_{m,4Jn}(B_{4Jn}(z)) \subset (Z_m^\circ)^{\geq \delta}. \tag{5.3}$$

Since the derivatives of $g_{m,4Jn}$ and its inverse converge to the identity as $m \to \infty$, choosing m large ensures that the lipschitz constants on any compact subset are close to 1. Thus, for m large, we have

$$\text{The restriction of } g_{m,4Jn}^{-1} \text{ to } \varphi_m \circ f_{m,2n}\big(\bar{B}_n(y) \cap Y^{\geq \varepsilon_n}\big) \text{ is } 2^{1/2n}\text{-bilipschitz.} \tag{5.4}$$

Finally, since $\varphi_m \circ f_{m,2n}(\bar{B}_n(y) \cap Y^{\geq \varepsilon_n}) \subset (Z_m^\circ)^{\geq \delta}$, and injectivity radii converge in the limit $(Z_m^\circ, z_m) \to (Z, z)$, it follows that choosing m large ensures:

$$g_{m,4Jn}^{-1} \circ \varphi_m \circ f_{m,2n}\big(\bar{B}_n(y) \cap Y^{\geq \varepsilon_n}\big) \subset Z^{\geq \delta_n}. \tag{5.5}$$

We are now ready to define the function h_n and check that it has all the desired properties. For each n, choose a single number $m = m_n$ so that conditions (5.1)–(5.5) hold simultaneously. Then, set

$$h_n = g_{m,4Jn}^{-1} \circ \varphi_m \circ f_{m,2n} \colon \bar{B}_n(y) \cap Y^{\geq \varepsilon_n} \to Z.$$

This function is well-defined by (5.2) and (5.3). It is J_n-bilipschitz for $J_n = 2^{1/n} J$, because $f_{m,2n}$ and $g_{m,4Jn}^{-1}$ are both $2^{1/2n}$-bilipschitz on the relevant domain by (5.1) and (5.4), while φ_m is J-bilipschitz. The image $h_n(\bar{B}_n(y) \cap Y^{\geq \varepsilon_n})$ is contained in $Z^{\geq \delta_n}$ by (5.5). Finally, the points

$$h_n(y) = g_{m,4Jn}^{-1}\big(\varphi_m(y_m)\big) \quad \text{and} \quad z = g_{m,4Jn}^{-1}(z_m)$$

are within distance $2D$, by (5.4) and the definition of D, hence $\{h_n(y)\}$ is a bounded sequence. □

The second step of the proof of Theorem 5.1 is to extract a subsequential limit of the functions h_n that were built in Lemma 5.2. This step does not need any hyperbolic geometry or smoothness, and only needs Y and Z to be metric spaces. So we write down the proof in that generality.

Lemma 5.3. *Let Y and Z be metric spaces. Let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of Y by compact sets. For each n, let $h_n \colon K_n \to Z$ be a continuous function that is a J_n-bilipschitz homeomorphism to its image. Suppose that the sequence of images $\{h_n(y)\}$ is bounded for some basepoint $y \in K_1$, and that $\lim_{n\to\infty} J_n = J$ for some J. Then there is a J-bilipschitz inclusion $h \colon Y \hookrightarrow Z$. Furthermore, for every $x \in Y$, we have $h(x) = \lim h_{n_i}(x)$ for some subsequence n_i.*

Proof. After replacing $\{h_n\}$ by a subsequence, we may assume without loss of generality that $\{J_n\}$ is a monotonic sequence. We will construct h by repeated application of the Arzela–Ascoli theorem.

Focusing attention on a single compact set K_m, the functions h_n are defined on K_m for all $n \geq m$. The family $\{h_n : n \geq m\}$ is equicontinuous on K_m because each h_n is $\bar{J}$-lipschitz, where $\bar{J} = \sup J_n$. Furthermore, the set of images $\{h_n(y)\}$ is

bounded by hypothesis, and $d(h_n(y), h_n(x))$ is also uniformly bounded for $x \in K_m$ because the functions h_n are uniformly lipschitz. Thus, $\{h_n : n \geq m\}$ is uniformly bounded on K_m. By Arzela–Ascoli, some subsequence converges uniformly on K_m.

Now, consider K_1. By the above paragraph, there is a subsequence $\{h_n^1\}$ of $\{h_n\}$ that converges uniformly on K_1. Define $h(x) = \lim_{n\to\infty} h_n^1(x)$ for $x \in K_1$.

Next, consider $K_2 \supset K_1$. As above, there is a subsequence $\{h_n^2\}$ of $\{h_n^1\}$ that converges uniformly on K_2. Define $h(x) = \lim_{n\to\infty} h_n^2(x)$ for $x \in K_2$. This agrees with the previous definition of h on $K_1 \subset K_2$ because we have taken a subsequence of a sequence that already converges on K_1.

Continuing inductively in this manner, we have a subsequence $\{h_n^m\}$ of $\{h_n^{m-1}\}$ that converges uniformly on K_m. We then have $h(x) = \lim_{n\to\infty} h_n^m(x)$ for $x \in K_m$. Since the K_m provide an exhaustion of Y, and the definition is consistent as m grows, this defines h on all of Y.

It remains to show that h is J-bilipschitz, where $J = \lim J_n$. Consider a pair of points $x, x' \in Y$, and let K_m be such that $x, x' \in K_m$. Recall our assumption at the beginning of the proof that $\{J_n\}$ is monotonic. If J_n is monotonically increasing with n, then every h_n^m is already J-bilipschitz. In particular, we have

$$J^{-1} \cdot d(x, x') \leq d\left(h_n^m(x), h_n^m(x')\right) \leq J \cdot d(x, x').$$

As $n \to \infty$, the middle term converges to $d(h(x), h(x'))$, hence h is J-bilipschitz.

If J_n is monotonically decreasing with n, then J_n^m is also monotonically decreasing. Fix an integer $k \gg 0$. Then, for $n \geq k$, every h_n^m is J_k^m-bilipschitz. Thus, for all $n \geq k$, we have:

$$(J_k^m)^{-1} \cdot d(x, x') \leq d\left(h_n^m(x), h_n^m(x')\right) \leq J_k^m \cdot d(x, x').$$

As $n \to \infty$ (holding k fixed), the middle term converges to $d(h(x), h(x'))$. Then we take a limit as $k \to \infty$, and J_k^m converges to J. We obtain

$$J^{-1} \cdot d(x, x') \leq d\left(h(x), h(x')\right) \leq J \cdot d(x, x'),$$

hence h is J-bilipschitz. □

Proof of Theorem 5.1. As in the statement of the theorem, we have hyperbolic manifolds Y and Z, with a basepoint $y \in Y^{>\varepsilon}$. Let $\delta_n \to \delta$, $\varepsilon_n \to \varepsilon$, and $J_n \to J$ be the convergent sequences of Lemma 5.2, and let $K_n = \bar{B}_n(y) \cap Y^{\geq \varepsilon_n}$ be a compact set. Lemma 5.2 says that for all sufficiently large n, there is a J_n-bilipschitz map $h_n \colon K_n \hookrightarrow Z^{\geq \delta_n}$, such that the images of the basepoint $\{h_n(y)\}$ are bounded in Z. We reindex the sequence so that it starts at $n = 1$.

Observe that $K_1 \subset K_2 \subset \cdots$ is an exhaustion of $Y^{>\varepsilon}$ by compact sets. Now, Lemma 5.3 constructs a J-bilipschitz function $h \colon Y^{>\varepsilon} \hookrightarrow Z$. For every $x \in Y$, there is a subsequence h_{n_i} (depending on the compact set containing x) such that

$$h(x) = \lim h_{n_i}(x).$$

Since $h_{n_i}(x) \in Z^{\geq \delta_{n_i}}$, and $\delta_{n_i} \to \delta$, it follows that in fact $h(x) \in Z^{\geq \delta}$. Thus, we have a J-bilipschitz map

$$h\colon Y^{>\varepsilon} \hookrightarrow Z^{\geq \delta}.$$

Since h is J-bilipschitz, it has a continuous and J-bilipschitz extension to $\partial Y^{>\varepsilon}$. This achieves our goal: a J-bilipschitz inclusion $\varphi = h\colon Y^{\geq \varepsilon} \hookrightarrow Z^{\geq \delta}$. □

Remark 5.4. In the proof of Theorem 6.5, we will rely on the following straightforward extension of Theorem 5.1. Suppose that ε and δ are Margulis numbers for Y and Z, respectively. Let $Y_\Sigma^{<\varepsilon}$ be the disjoint union of certain components of $Y^{<\varepsilon}$, and let $Z_\Sigma^{<\delta}$ be the disjoint union of certain components of $Z^{<\delta}$. Let $(Y_m)_\Sigma^{<\varepsilon}$ and $(Z_m)_\Sigma^{<\delta}$ be the corresponding subsets of the approximating manifolds in the geometric limits $(Y_m, y_m) \to (Y, y)$ and $(Z_m, z_m) \to (Z, z)$. Suppose that, for each m, there is a J-bilipschitz inclusion

$$\varphi_m\colon \big(Y_m - (Y_m)_\Sigma^{<\varepsilon}\big) \hookrightarrow \big(Z_m - (Z_m)_\Sigma^{<\delta}\big),$$

such that $d(\varphi_m(y_m), z_m)$ is uniformly bounded. Then there is also a J-bilipschitz inclusion $\varphi\colon (Y - Y_\Sigma^{<\varepsilon}) \hookrightarrow (Z - Z_\Sigma^{<\delta})$.

The main change from the statement of Theorem 5.1 is that $Y^{\geq \varepsilon}$ has been replaced by $(Y - Y_\Sigma^{<\varepsilon})$, and similarly for Z, so that the J-bilipschitz inclusion φ now extends into some of the thin parts of Y. The only change needed in the proof is that in Lemma 5.2 the compact set $\bar{B}_n(y) \cap Y^{\geq \varepsilon}$ should be replaced by the compact set

$$\bar{B}_n(y) \cap (Y - Y_\Sigma^{<\varepsilon}).$$

This again gives a compact exhaustion of the desired submanifold of Y; the rest of the argument works verbatim.

6. The effective drilling theorem

In this section, we prove Theorem 1.2. The proof proceeds in two main steps. In the first step, accomplished in Theorem 6.2, we use a number of results from Kleinian groups (Section 2) to approximate both Y and the drilled manifold $Z \cong Y - \Sigma$ with geometrically finite manifolds that admit a circle packing. We also double-double the scooped versions of the geometrically finite manifolds to obtain finite-volume hyperbolic manifolds converging to Y and Z, respectively. In the second step, we use the geometric limits constructed in Theorem 6.2, combined with the finite-volume Theorem 6.4 and the bilipschitz limit Theorem 5.1, to build bilipschitz maps between the thick parts of Y and Z. See Figure 1 for a visual outline.

We also employ the same outline to prove Theorem 6.5, which extends Brock and Bromberg's Theorem 1.1 to all tame hyperbolic 3-manifolds without rank-1 cusps.

Before beginning the proofs of those results, we need to verify that for any geodesic link Σ, the complement $Y - \Sigma$ admits a hyperbolic structure Z with end invariants identical to those of Y.

Lemma 6.1. *Let Y be a tame hyperbolic 3-manifold with standard compact core (M, P). Let $\Sigma \subset Y$ be a geodesic link with a regular neighborhood $\mathcal{N}(\Sigma) \subset M$. Then $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a pared manifold that admits a hyperbolic metric Z with the same end invariants as those of Y.*

Proof. We begin by checking that $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a pared manifold. Since M is compact, oriented, and not a 3-ball or solid torus, the same is true of $M - \mathcal{N}(\Sigma)$. For the other properties, recall a theorem of Kerckhoff that $M - \Sigma$ admits a complete metric of variable negative curvature. (See Agol [2, pages 908–909] for a proof.) It follows that $M - \mathcal{N}(\Sigma)$ is irreducible and algebraically atoroidal, meaning that every π_1-injective map of a torus $T^2 \to M - \mathcal{N}(\Sigma)$ is homotopic into some boundary torus belonging to either P or to $\partial\mathcal{N}(\Sigma)$.

Now, consider a π_1-injective map of an annulus

$$f\colon (A, \partial A) \to (M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma)).$$

Since Σ is a disjoint union of geodesics in Y, an essential curve on $\partial\mathcal{N}(\Sigma)$ cannot be homotopic to P through A, and two distinct essential curves on $\partial\mathcal{N}(\Sigma)$ cannot be homotopic to each other through A. Thus, both components of ∂A must be mapped to P, which means that we in fact have

$$f\colon (A, \partial A) \to (M - \mathcal{N}(\Sigma), P).$$

Since (M, P) is a pared manifold, it follows that f is homotopic into P through M. Since we have already checked that $M - \mathcal{N}(\Sigma)$ is irreducible and atoroidal, and the geodesic components of Σ cannot be homotopic into P, the homotopy of f can be taken to avoid $\mathcal{N}(\Sigma)$. This proves that $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a pared manifold.

To prove that the end invariants of Y are realized by a hyperbolic structure on $M - \mathcal{N}(\Sigma)$, we need to check that the end invariants of Y are still filling when viewed as end invariants for $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$. That is, we need to check conditions $(*)$ and $(**)$ of Definition 2.8. Condition $(*)$ holds automatically: the only way $M - \mathcal{N}(\Sigma)$ can be an interval bundle over a surface is if M is a solid torus, which we have already ruled out. For condition $(**)$, let F_i be a free side of $\partial_0 M$ and λ_i be the ending lamination on F_i. If λ_i is contained in the Hausdorff limit of a sequence of meridians in $M - \mathcal{N}(\Sigma)$, then the same meridians are also meridians in M, a contradiction. Thus, the end invariants of $M - \mathcal{N}(\Sigma)$ are filling, and Theorem 2.9 says that these end invariants are realized by a hyperbolic structure Z. □

6.1. Scooped manifolds approaching Y and Z. The following theorem encapsulates the limiting construction that will be used in the proof of Theorem 1.2. We will also use this result in the proof of Theorem 8.5.

Theorem 6.2. *Let Y be a tame, infinite-volume hyperbolic 3-manifold with associated pared manifold (M, P) and associated representation $\rho \in AH(M, P)$. Let $\Sigma \subset Y$ be a geodesic link, such that each component of Σ is shorter than $\log 3$. Then $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a pared manifold admitting a hyperbolic metric Z with the same end invariants as those of Y. Furthermore, there exist approximating sequences such that the following properties hold for all $n \gg 0$:*

(1) *There is a geometrically finite, minimally parabolic representation $\rho_n \in AH(M, P)$. The conformal boundary of $V_n = \mathbb{H}^3/\rho_n(\pi_1 M)$ admits a circle packing C_n. Furthermore, $\rho_n \to \rho$ is a strong limit.*

(2) *For every $y \in Y$, there is a choice of basepoints $v_n \in V_n^\circ$, such that (V_n°, v_n) converges geometrically to (Y, y). There is a geodesic link $\Sigma_n \subset V_n^\circ$, carried to Σ by a homeomorphism $V_n \to Y$, such that $\Sigma_n \to \Sigma$ in the geometric limit.*

(3) *There is a geometrically finite, minimally parabolic representation*

$$\xi_n \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$$

such that the associated hyperbolic manifold W_n is homeomorphic to $Y - \Sigma$ and $V_n - \Sigma_n$ and has end invariants that are identical to those of V_n. In particular, the conformal boundary of W_n admits the same circle packing C_n.

(4) *For every $z \in Z$, there is a choice of basepoints $w_n \in W_n^\circ$, such that (W_n°, w_n) converges geometrically to (Z, z). Furthermore, there is a strong limit $\xi_n \to \xi$, where $\xi \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a representation associated to Z.*

(5) *The operations of drilling and double-doubling commute. That is:*

$$DD(V_n^\circ) - DD(\Sigma_n) \cong DD(W_n^\circ).$$

(6) *The manifolds Y and Z are geometric limits of finite-volume hyperbolic manifolds, as follows: $(DD(V_n^\circ), v_n) \to (Y, y)$ and $(DD(W_n^\circ), w_n) \to (Z, z)$.*

Observe that Theorem 1.6, stated in the Introduction, is an immediate corollary of the above theorem.

See Figure 1 for a commutative diagram encapsulating the main objects in the statement of Theorem 6.2, as well as in its proof. We will begin with the top-right of the diagram, with the strong limit $Y_n \to Y$, and then construct the approximating manifolds V_n and V_n° by proceeding right to left. We will then drill out an appropriate copy of Σ from each of these manifolds and construct the limiting sequences in the bottom row of the diagram.

Proof of Theorem 6.2. Let $\Gamma = \rho(\pi_1(M))$ be the Kleinian group associated to Y. Let $\{O_n : n \in \mathbb{N}\}$ be an open neighborhood system about $[\rho] \in AH(M, P)$. By strong density, Theorem 2.12, there exists a strongly convergent sequence $\sigma_n \to \rho$, such that $[\sigma_n] \in O_n$. Let $\Gamma_n = \sigma_n(\pi_1(M))$, and let $Y_n = \mathbb{H}^3/\Gamma_n$ be the associated

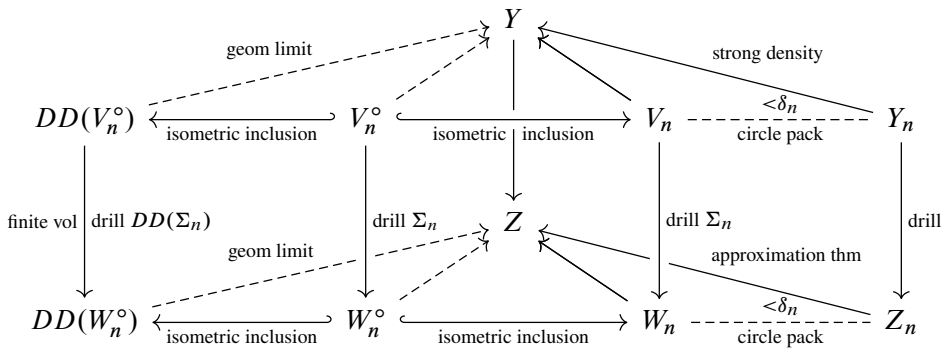

Figure 1. The manifolds appearing in the statement and proof of Theorem 6.2. Hooked horizontal arrows represent isometric inclusions. Dashed horizontal lines represent a small quasiconformal deformation that produces a circle-packed manifold. Vertical arrows represent drilling out an embedded link. Solid diagonal arrows represent strong limits. Dashed diagonal arrows represent geometric limits only.

geometrically finite hyperbolic 3-manifolds. After passing to a subsequence, we can ensure that $d_{\text{Chaub}}(\Gamma, \Gamma_n) < 2^{-n}$. By Theorem 2.12, the end invariants of Y_n form a filling sequence, converging to the end invariants of Y. Fix an arbitrary basepoint $y \in Y$.

For each n, pick a constant $\delta_n > 0$ such that $\lim \delta_n = 0$. (In subsequent paragraphs, we will impose additional constraints, all of which hold when δ_n is sufficiently small.) For each n, Brooks' Theorem 2.15 says that there is a geometrically finite Kleinian group Γ_{n,δ_n} representing an e^{δ_n}-quasiconformal deformation of Γ_n, such that the conformal boundary of Γ_{n,δ_n} admits a circle packing. Let $\rho_n \colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be the associated representation. The bound of e^{δ_n} on the quasiconformal deformation means that the distance in $\mathcal{T}(\partial_0 M)$ between the conformal end invariants of Γ_n and Γ_{n,δ_n} is at most δ_n. Here, we are measuring distances in the Teichmüller metric on the Teichmüller space of the (possibly disconnected) surface $\partial_0 M = \partial M - P$. We choose δ_n small enough that

$$d_{\text{Chaub}}(\Gamma_{n,\delta_n}, \Gamma_n) < 2^{-n}. \tag{6.1}$$

We also choose δ_n small enough to ensure $[\rho_n] \in O_n$; this is possible by Theorem 2.10. The restrictions on δ_n ensure that $\rho_n \to \rho$ is a strong limit. Let $V_n = \mathbb{H}^3/\Gamma_{n,\delta_n}$ be the quotient manifold.

Next, we construct three sets of hyperbolic manifolds with pared manifold $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$:

- A hyperbolic manifold Z whose end invariants agree with those of Y.
- A hyperbolic manifold Z_n whose end invariants agree with those of Y_n.
- A hyperbolic manifold W_n whose end invariants agree with those of V_n.

The pared manifold $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ and the hyperbolic structure Z are both guaranteed to exist by Lemma 6.1. Let $\xi\colon \pi_1(M - \mathcal{N}(\Sigma)) \to PSL(2,\mathbb{C})$ be the discrete faithful representation corresponding to Z, with image

$$\Delta = \xi(\pi_1(M - \mathcal{N}(\Sigma))).$$

Let $\xi' \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ be a geometrically finite, minimally parabolic representation; this exists by Theorem 2.6. Thus, by Theorem 2.7, there exist representations $\tau_n \in QH(\xi')$ with image group Δ_n and quotient manifold Z_n, as well as representations $\xi_n \in QH(\xi')$ with image group Δ_{n,δ_n} and quotient manifold W_n.

Let $Q_0 = AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$. Let $\{Q_n : n \in \mathbb{N}\}$ be a nested system of open neighborhoods about $[\xi] \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$. Now, define a sequence $m(n)$ as follows. If $[\tau_n] = [\xi]$, let $m(n) = n$; otherwise, let $m(n)$ be the largest integer m such that $[\tau_n] \in Q_m$. In either case, we have $[\tau_n] \in Q_{m(n)}$. We will check below that $[\tau_n] \to [\xi]$, which implies $m(n) \to \infty$.

Observe that $\partial_0(M - \mathcal{N}(\Sigma)) = \partial_0 M$, because $\partial\mathcal{N}(\Sigma)$ is part of the parabolic locus. Since the (geometrically finite) end invariants of Z_n and W_n agree with those of Y_n and V_n, respectively, the distance in $\mathcal{T}(\partial_0(M - \mathcal{N}(\Sigma)))$ between the conformal end invariants of Z_n and W_n is at most δ_n. Recall that by the Ahfors–Bers Uniformization Theorem 2.7, $\mathcal{T}(\partial_0(M - \mathcal{N}(\Sigma)))$ provides a local parametrization of the interior

$$QH(\xi') \subset AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma)),$$

and by Theorem 2.10 the algebraic topology on $QH(\xi')$ agrees with the Chabauty topology. Thus, a sufficiently small choice of δ_n ensures that ξ_n has the following properties. We choose δ_n so that

$$d_{\text{Chaub}}(\Delta_{n,\delta_n}, \Delta_n) < 2^{-n}. \tag{6.2}$$

and so that $[\xi_n] \in Q_{m(n)}$ for each n. This completes the list of requirements that δ_n needs to satisfy.

Now, we proceed to verify that the sequences V_n and W_n have all of the required properties. Conclusion (1) holds by construction, since we have chosen Γ_{n,δ_n} so that the conformal boundary of $V_n = \mathbb{H}^n/\Gamma_{n,\delta_n}$ admits a circle packing. We call this circle packing C_n.

Equation (6.1) implies that $\Gamma_{n,\delta_n} \to \Gamma$ in the Chabauty topology. Thus, there is a choice of basepoints $v_n \in V_n$ giving a geometric limit $(V_n, v_n) \to (Y, y)$. In particular, for any fixed R and large n, there is an almost-isometric embedding

$$f_n\colon (B_R(y), y) \hookrightarrow (V_n, v_n).$$

Since R is fixed, a sufficiently large choice of n ensures that the image $f_n(B_R(y))$ will be contained in the scooped manifold V_n°. Thus, we also have a geometric limit $(V_n^\circ, v_n) \to (Y, y)$. Since $\rho_n \to \rho$ is a strong limit, Proposition 4.1 says that the

homeomorphic image of Σ is a link in V_n, isotopic to a geodesic link Σ_n when $n \gg 0$. Observe that $\Sigma_n \subset CC(V_n) \subset V_n^\circ$, hence (2) holds.

Conclusion (3) holds by construction, because W_n has pared manifold $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ and conformal boundary identical to that of V_n. Thus, C_n is also a circle packing on the conformal boundary of W_n. Observe that $\xi_n \in QH(\xi')$ is geometrically finite and minimally parabolic by construction. The homeomorphism

$$W_n \cong Y - \Sigma \cong V_n - \Sigma_n$$

holds by (2).

Next, we turn to conclusion (4). Recall that the end invariants of σ_n form a filling sequence, converging to the end invariants $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ associated to ρ and to Y. Thus, the end invariants of τ_n (which are the same as those of σ_n) also form a filling sequence, converging to the end invariants $(X_1, \dots, X_r, \lambda_{r+1}, \dots, \lambda_s)$ of ξ. Now, the approximation Theorem 2.13 combined with the ending lamination Theorem 2.5 says that (after passing to a subsequence) we have a strong limit $\tau_n \to \xi$. Having passed to this subsequence, we have $[\tau_n] \in Q_{m(n)}$ where $m(n) \to \infty$, as well as a Chabauty limit $\Delta_n \to \Delta$.

In (6.2), we chose δ_n so that

$$d_{\text{Chaub}}(\Delta_{n,\delta_n}, \Delta_n) < 2^{-n}.$$

Thus, $\Delta_{n,\delta_n} \to \Delta$. Similarly, δ_n was chosen so that $[\xi_n] \in Q_{m(n)}$ for the same sequence $m(n) \to \infty$. Since $Q_{m(n)}$ is a nested system of open neighborhoods of $[\xi]$, it follows that $[\xi_n] \to [\xi]$ in the algebraic topology. Thus, $\xi_n \to \xi$ is a strong limit.

Since the geometric topology is the Chabauty topology, for every $z \in Z$ there is a choice of basepoints w_n such that $(W_n, w_n) \to (Z, z)$. Furthermore, as above, the almost-isometric image of $B_R(z)$ will be contained in the scooped manifold W_n° for large n, hence we also have a geometric limit $(W_n^\circ, w_n) \to (Z, z)$, hence (4) holds.

After double-doubling the scooped manifold V_n°, as in Definition 2.16, we obtain a finite-volume hyperbolic manifold $DD(V_n^\circ)$. This finite-volume manifold contains a link $DD(\Sigma_n)$, consisting of four isometric copies of Σ_n. Recall that by (3), W_n is homeomorphic to $V_n - \Sigma_n$, and has identical conformal boundary admitting the same circle packing C_n. Applying Definition 2.16 again, we may double W_n° twice, first in the blue faces and then the red, to obtain a finite-volume hyperbolic manifold homeomorphic to $DD(V_n^\circ - \Sigma_n)$. Thus, by Mostow–Prasad rigidity (or by the rigidity of circle packings, Theorem 2.14), we have isometries

$$DD(V_n^\circ) - DD(\Sigma_n) = DD(V_n^\circ - \Sigma_n) = DD(W_n^\circ),$$

establishing conclusion (5).

Finally, conclusion (6) is a corollary of (2) and (4), because for any $R > 0$, a metric R-ball about $v_n \in DD(V_n^\circ)$ will in fact be contained in the original copy of V_n° for $n \gg 0$. A similar statement holds in $DD(W_n^\circ)$. □

Remark 6.3. If the hyperbolic manifold Y is geometrically finite, the preceding proof becomes considerably more lightweight. In this special case, one can take constant sequences $Y_n = Y$ and $Z_n = Z$. Consequently, the strong density Theorem 2.12 and the approximation Theorem 2.13 become unnecessary, as does the ending lamination theorem. Thurston's hyperbolization, Theorem 2.6, becomes unnecessary because the representation ξ corresponding to Z is already geometrically finite and minimally parabolic. Finally, the realization Theorem 2.9, which is used inside Lemma 6.1 to establish a hyperbolic structure on $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ with the correct end invariants, can be replaced with the Ahlfors–Bers Theorem 2.7. Thus, in the geometrically finite case, the only tools required in the proof are Theorem 2.7 and Brooks' Theorem 2.15.

The final tool that we need to prove Theorem 1.2 is a finite-volume analogue of the same result. The following is a restatement of [20, Theorem 1.2].

Theorem 6.4 (Effective drilling in finite volume, [20]). *Fix* $0 < \varepsilon \leq \log 3$ *and* $J > 1$. *Let* V *be a finite-volume hyperbolic* 3*-manifold and* Σ *a geodesic link in* V *whose total length* ℓ *satisfies*

$$\ell \leq \min\left\{\frac{\varepsilon^5}{6771\cosh^5(0.6\varepsilon + 0.1475)}, \frac{\varepsilon^{5/2}\log(J)}{11.35}\right\}.$$

Then $V - \Sigma$ *admits a complete hyperbolic metric* W. *There are canonical* J*-bilipschitz inclusions*

$$\varphi\colon V^{\geq\varepsilon} \hookrightarrow W^{\geq\varepsilon/1.2}, \quad \psi\colon W^{\geq\varepsilon} \hookrightarrow V^{\geq\varepsilon/1.2}.$$

The maps φ *and* ψ *are equivariant with respect to the symmetry group of the pair* (V, Σ).

We now have all the necessary tools to bootstrap from Theorem 6.4 to Theorem 1.2. The proof involves chasing the left half of the diagram in Figure 1. Starting from Y, we will consider a circle-packed approximating manifold V_n, the finite-volume manifold $DD(V_n^\circ)$, its drilling $DD(W_n^\circ)$, and the scooped submanifold W_n° that approximates Z, the hyperbolic structure on $Y - \Sigma$.

Proof of Theorem 1.2. If $\mathrm{vol}(Y) < \infty$, the desired statement already appears in Theorem 6.4, substituting $V = Y$ and $W = Z$. For the rest of the proof, we assume $\mathrm{vol}(Y) = \infty$.

Let V_n and W_n be the sequences of geometrically finite manifolds constructed in Theorem 6.2. In particular, every V_n is homeomorphic to Y and every W_n is homeomorphic to Z. Furthermore, the conformal boundaries of each V_n and each W_n admit the same circle packing C_n. By Theorem 6.2, we have strong limits $\rho_n \to \rho$ (corresponding to $V_n \to Y$ and $\xi_n \to \xi$ (corresponding to $W_n \to Z$).

Recall that by Theorem 2.2, $\varepsilon < \log 3$ is a Margulis number for every infinite-volume hyperbolic 3-manifold, hence $Y^{<\varepsilon}$ is a disjoint union of tubes and horocusps.

We let $T^{<\varepsilon}$ denote a component of $\partial Y^{<\varepsilon}$; this is either a horotorus about a cusp or an equidistant torus about a short geodesic. We can choose a number $\eta \in (\varepsilon, 2\varepsilon)$ such that η is still a Margulis number for Y.

Fix a basepoint $y \in Y$ so that $2\text{injrad}(y) = \eta \in (\varepsilon, 2\varepsilon)$, and furthermore y lies on an embedded, η-thick equidistant torus $T^{\eta}(\sigma)$ about the first component $\sigma \subset \Sigma$. Such a choice of $y \in Y^{>\varepsilon}$ is possible because η is a Margulis number for Y, hence the thick part $Y^{>\eta}$ is non-empty. Similarly, fix a basepoint $z \in Z$ so that $2\text{injrad}(z) = \varepsilon$, and furthermore z lies on an embedded ε-thick horotorus $T^{\varepsilon}(\sigma)$ that bounds an embedded horocusp about the same component $\sigma \subset \Sigma$. Again, such a choice is possible because ε is a Margulis number for Z.

Now, Theorem 6.2 says that for $n \gg 0$, there exist choices of basepoints $v_n \in V_n^\circ$ and $w_n \in W_n^\circ$, such that $(V_n^\circ, v_n) \to (Y, y)$ and $(W_n^\circ, w_n) \to (Z, z)$. Furthermore, the homeomorphic image of Σ is a link in V_n, isotopic to a geodesic link Σ_n when $n \gg 0$. Note that $\Sigma_n \subset CC(V_n) \subset V_n^\circ$.

After doubling V_n° twice to obtain the finite-volume manifold $DD(V_n^\circ)$, as in Definition 2.16, we also obtain a geodesic link $DD(\Sigma_n) \subset DD(V_n^\circ)$ consisting of four isometric copies of Σ_n. By Proposition 4.1, the strong limit $\rho_n \to \rho$ means that $\Sigma_n \to \Sigma$. Thus, for large n, the length of Σ_n is arbitrarily close to $\text{len}(\Sigma) = \ell$. In particular, we have

$$\begin{aligned} \text{len}(DD(\Sigma_n)) &= 4\,\text{len}(\Sigma_n) \\ &\leq \min\left\{ \frac{\varepsilon^5}{6771\cosh^5(0.6\varepsilon + 0.1475)}, \frac{\varepsilon^{5/2}\log(J)}{11.35} \right\}. \end{aligned} \tag{6.3}$$

Thus, we may apply the finite-volume effective drilling result, Theorem 6.4, to $DD(V_n^\circ)$ and $DD(\Sigma_n)$. For the unique hyperbolic metric on $DD(V_n^\circ) - DD(\Sigma_n)$, Theorem 6.4 gives J-bilipschitz inclusions

$$\begin{aligned} \varphi_n &\colon DD(V_n^\circ)^{\geq\varepsilon} \to \big(DD(V_n^\circ) - DD(\Sigma_n)\big)^{\geq\varepsilon/1.2}, \\ \psi_n &\colon \big(DD(V_n^\circ) - DD(\Sigma_n)\big)^{\geq\varepsilon} \to DD(V_n^\circ)^{\geq\varepsilon/1.2}. \end{aligned}$$

Furthermore, φ_n and ψ_n respect the symmetries of the pair $(DD(V_n^\circ), DD(\Sigma_n))$.

The pair $(DD(V_n^\circ), DD(\Sigma_n))$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of symmetries, where the generator of the first $\mathbb{Z}_2$ acts by reflection in the blue faces of V_n° and the generator of the second $\mathbb{Z}_2$ acts by reflection in the red faces. This action restricts to a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of symmetries of

$$DD(V_n^\circ) - DD(\Sigma_n) = DD(V_n^\circ - \Sigma_n),$$

with a fundamental domain of the form $V_n^\circ - \Sigma_n$. Since φ_n and ψ_n respect these symmetries, we obtain J-bilipschitz inclusions

$$\varphi_n \colon (V_n^\circ)^{\geq\varepsilon} \to (V_n^\circ - \Sigma_n)^{\geq\varepsilon/1.2}, \quad \psi_n \colon (V_n^\circ - \Sigma_n)^{\geq\varepsilon} \to (V_n^\circ)^{\geq\varepsilon/1.2},$$

isotopic to the topological drilling of Σ_n. By Theorem 6.2, we have $V_n^\circ - \Sigma_n \cong W_n^\circ$.

We can now construct the J-bilipschitz inclusions $\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\varepsilon/1.2}$ and $\psi\colon Z^{\geq\varepsilon} \hookrightarrow Y^{\geq\varepsilon/1.2}$ using Theorem 5.1. Most of the hypotheses of that theorem have already been verified. We have geometrically convergent sequences $(V_n^\circ, v_n) \to (Y, y)$ and $(W_n^\circ, w_n) \to (Z, z)$ We have $y \in Y^{>\varepsilon}$ as required. Since injectivity radii converge in a geometric limit [16, Lemma 3.2.6], it follows that $v_n \in (V_n^\circ)^{\geq\varepsilon}$ for large n. For large n, we have a J-bilipschitz inclusion

$$\varphi_n\colon (V_n^\circ)^{\geq\varepsilon} \to (W_n^\circ)^{\geq\varepsilon/1.2}.$$

To apply Theorem 5.1, it remains to check that $d(\varphi_n(v_n), w_n)$ is uniformly bounded.

This can be checked as follows. By construction, the basepoint $y \in Y$ lies on an equidistant torus $T^\eta(\sigma)$ about $\sigma \subset \Sigma$, where

$$\varepsilon < 2\mathrm{injrad}(y) < 2\varepsilon.$$

By [16, Lemma 3.2.6], the same two-sided bound holds for $\mathrm{injrad}(v_n)$ for large n. By [20, Theorem 9.30], of which Theorem 6.4 is a corollary, the injectivity radius of $v_n \in (V_n^\circ)^{\geq\varepsilon}$ changes by a factor of at most 1.2 under φ_n. Thus,

$$\varepsilon/1.2 < 2\mathrm{injrad}(\varphi_n(v_n)) = \mu_n < 2.4\varepsilon,$$

and furthermore $\varphi_n(v_n)$ still lies on a μ_n-thin horospherical torus about the same component $\sigma \subset \Sigma$. Meanwhile, observe that $2\mathrm{injrad}(w_n)$ is nearly equal to ε, and w_n also lies on a horospherical torus about $\sigma \subset \Sigma$. The diameter of each of those tori is uniformly bounded, by the geometric limit $(W_n, w_n) \to (Z, z)$, while the distance between the μ_n-thin and ε-thin tori is uniformly bounded by [19, Proposition 1.4]. Thus, $d(\varphi_n(v_n), w_n)$ is uniformly bounded. Hence Theorem 5.1 gives a J-bilipschitz inclusion $\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\varepsilon/1.2}$.

The reverse inclusion $\psi\colon Z^{\geq\varepsilon} \hookrightarrow Y^{\geq\varepsilon/1.2}$ is constructed in exactly the same way, tracing points backwards to check the hypotheses of Theorem 5.1. The points v_n and $\psi_n(w_n)$ lie on equidistant tori in a tube about $\sigma \subset \Sigma$, hence $d(v_n, \psi_n(w_n))$ is uniformly bounded by [19, Proposition 5.7]. □

6.2. Extended and strengthened results. The above proof method can be used to extend Brock and Bromberg's Theorem 1.1 to all tame hyperbolic manifolds without rank-1 cusps.

Theorem 6.5. *Fix $J > 1$ and $\varepsilon > 0$, where ε is smaller than the Margulis constant ε_3. Then there is a number $\ell_0 = \ell_0(\varepsilon, J) > 0$ such that the following holds for every tame hyperbolic 3-manifold Y without rank-1 cusps. Suppose that $\Sigma \subset Y$ is a geodesic link, whose total length is less than ℓ_0. Then $Y - \Sigma$ admits a hyperbolic structure Z with the same end invariants as those of Y. Furthermore, the inclusion*

$$\iota\colon Z \hookrightarrow Y$$

restricts to a J-bilipschitz diffeomorphism on the complement of ε-thin tubes about Σ.

In contrast with Theorem 1.2, this result is not effective because the function $\ell_0(\varepsilon, J)$ is not explicitly given. On the other hand, this result has the advantage that J-bilipschitz control extends into the thin parts of Y and Z that do not correspond to components of Σ.

For the following proof, it is helpful to consult the commutative square in Figure 1 with corners at Y, V_n, W_n, and Z.

Proof of Theorem 6.5. Fix $\varepsilon < \varepsilon_3$, and recall from Section 2.1 that $\varepsilon_3 < 1$. One of the first constraints in Brock and Bromberg's proof of Theorem 1.1 is that $\ell_0 < \varepsilon$.

Let Y be a tame hyperbolic 3-manifold without rank-1 cusps. We may assume that Y is geometrically infinite, as otherwise Theorem 1.1 already gives the desired conclusion. Let $\Sigma \subset Y$ be a geodesic link with $\operatorname{len}(\Sigma) < \ell_0 < \varepsilon$. Then Lemma 6.1 implies that $Y - \Sigma$ admits a hyperbolic structure Z with the same invariants. Theorem 6.2 constructs (type-preserving) geometric limits $(V_n, v_n) \to (Y, y)$ and $(W_n, w_n) \to (Z, z)$, where W_n is obtained by drilling a geodesic link Σ_n from V_n, and where $\Sigma_n \to \Sigma$ in the geometric limit.

We introduce the following notation. Let $Y_\Sigma^{<\varepsilon}$ be the union of the ε-thin Margulis tubes about the components of Σ. This is a disjoint union, because $\operatorname{len}(\Sigma) < \varepsilon < \varepsilon_3$. Let $Z_\Sigma^{<\varepsilon}$ be the union of ε-thin cusps in Z corresponding to components of Σ. Define $(V_n)_\Sigma^{<\varepsilon}$ and $(W_n)_\Sigma^{<\varepsilon}$ similarly.

Since V_n is geometrically finite, and $\operatorname{len}(\Sigma_n) < \ell_0 = \ell_0(\varepsilon, J)$ for all sufficiently large n, Theorem 1.1 says that the homeomorphism $V_n - \Sigma_n \to W_n$ induces a J-bilipschitz diffeomorphism

$$\varphi_n \colon V_n - (V_n)_\Sigma^{<\varepsilon} \to W_n - (W_n)_\Sigma^{<\varepsilon}$$

As in the proof of Theorem 1.2, the distance estimate [19, Proposition 1.4] ensures that basepoints on equidistant tori about a component $\sigma \subset \Sigma$ do not escape under φ_n. Then Theorem 5.1 and Remark 5.4 imply that the J-bilipschitz maps φ_n converge (after passing to a subsequence) to the desired J-bilipschitz map

$$\varphi \colon Y - Y_\Sigma^{<\varepsilon} \to Z - Z_\Sigma^{<\varepsilon}. \qquad \square$$

We also discuss a quantitative strengthening of Theorem 1.2.

Remark 6.6. An astute reader will notice that the quantitative hypothesis on $\ell = \operatorname{len}(\Sigma)$ in Theorem 1.2 differs from the hypothesis in Theorem 6.4 by a factor of 4. This discrepancy occurs because we need to double-double the scooped manifold V_n° to obtain a finite-volume manifold $DD(V_n^\circ)$. See equation (6.3), where the transition from $\operatorname{len}(\Sigma)$ to $\operatorname{len}(DD(\Sigma_n))$ happens.

One may ask whether paying a factor of 4 is a necessary price in adapting finite-volume results in the infinite-volume setting. We suspect that the factor of 4 can probably be eliminated, at the cost of a different price: reexamining some of the technical analytic estimates that were used to prove Theorem 6.4. Here are the two most salient points.

In the cone deformation theory of Hodgson and Kerckhoff [23, 24], the length of the geodesic link Σ is used to control the radius R of an embedded tube about Σ; in turn, this tube radius is used to control almost all other quantities. (See the discussion around Definition 8.1 for a more quantitative summary.) Hodgson and Kerckhoff proved the original radius bound [24, Theorem 5.6], and we adapted their proof in [20, Theorem 4.21]. The proof is essentially a packing argument. In the context of the symmetric manifold $DD(V_n^\circ)$, one should be able to adapt the argument to use $\operatorname{len}(\Sigma_n)$ rather than $\operatorname{len}(DD(\Sigma_n))$, provided that the tubes in different copies of V_n° do not meet during the cone deformation. This can probably be ensured, because the tubes stay in the core portion of V_n°, whereas the circle-packing and scooping happen deep into the ends.

Since the length of Σ and its cone angle α both change throughout the cone deformation, the change in length must itself be controlled. A key differential inequality, proved in [24, Proposition 5.5 and p. 1079] and restated in [20, Lemma 6.7], bounds the change in the ratio $\alpha/\operatorname{len}(\Sigma)$ in terms of functions of the tube radius R. This estimate does not scale correctly when we pass from V_n° to $DD(V_n^\circ)$, because the length gets quadrupled but the radius stays the same. Thus, removing the factor of 4 would also require adapting the proof of [24, Proposition 5.5] to work directly in V_n°, thought of as a hyperbolic orbifold with mirrored boundary. This can be done, because the proof of [24, Proposition 5.5] is essentially an application of the Cauchy–Schwartz inequality.

Provided the above technical points are addressed, the rest of the estimates from [20] should go through unchanged, with $\operatorname{len}(\Sigma)$ in place of $\operatorname{len}(DD(\Sigma_n))$.

7. The effective filling theorem

In this section, we prove Theorem 1.3, giving effective bilipschitz bounds on Dehn fillings of tame hyperbolic 3-manifolds. The proof follows the same two-step process as in the previous section. The first step, Theorem 7.2, is an analogue of Theorem 6.2. It uses a number of results from Kleinian groups to approximate both the drilled and filled manifolds with sequences of geometrically finite manifolds whose conformal boundaries admit a circle packing. See Figure 2 for a visual preview. The second step uses the sequences constructed in Theorem 7.2, together with a finite-volume bilipschitz theorem (Theorem 7.4, proved in [20]), to complete the proof of Theorem 1.3.

The first step can be summarized by the following theorem, which is analogous to Theorem 1.6, but in the filling rather than drilling case.

Theorem 7.1. *Let Z be a tame, infinite-volume hyperbolic 3-manifold with a fixed collection of rank-2 cusps, and with fixed slopes on those cusps of total normalized length at least $L^2 \geq 230.1$. Then the Dehn filling of Z along those slopes is a tame manifold that admits a hyperbolic metric Y with the same end invariants as*

those of Z, and a geodesic link $\Sigma \subset Y$ such that Z is homeomorphic to $Y - \Sigma$. Furthermore, there is a sequence of finite-volume approximating manifolds $DD(W_n^\circ)$ and $DD(V_n^\circ)$ with the following properties:

(1) *The manifold $DD(V_n^\circ)$ contains a geodesic link $DD(\Sigma_n)$, consisting of four isometric copies of a link Σ_n, such that*

$$DD(W_n^\circ) = DD(V_n^\circ) - DD(\Sigma_n).$$

(2) *For any choice of basepoints $y \in Y$ and $z \in Z$, there are basepoints in the approximating manifolds such that*

$$(DD(V_n^\circ), v_n) \to (Y, y) \quad \text{and} \quad (DD(W_n^\circ), w_n) \to (Z, z)$$

are geometric limits.

(3) *In the geometric limit $(DD(V_n^\circ), v_n) \to (Y, y)$, we have $\Sigma_n \to \Sigma$.*

Indeed, Theorem 7.1 is an immediate consequence of the following more detailed result.

Theorem 7.2. *Let M be a compact 3-manifold, and $P \subset \partial M$ a collection of annuli and tori. Let $\Sigma \subset M$ be a smooth link with regular neighborhood $\mathcal{N}(\Sigma)$. Suppose that $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$ is a pared manifold that admits an infinite-volume hyperbolic structure Z, uniformized by a representation ξ, where the total normalized length of the meridians of Σ satisfies $L^2 \geq 230.1$.*

Then (M, P) is a pared manifold that admits a hyperbolic structure Y, uniformized by a representation $\rho \in AH(M, P)$, with the same end invariants as those of Z. Furthermore, there exist approximating sequences such that the following hold for all $n \gg 0$:

(1) *There is a geometrically finite, minimally parabolic representation $\xi_n \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$, such that the 3-manifold*

$$W_n = \mathbb{H}^3/\xi_n(\pi_1(M - \Sigma))$$

has conformal boundary admitting a circle packing C_n. Furthermore, $\xi_n \to \xi$ is a strong limit.

(2) *For every $z \in Z$, there is a choice of basepoints $w_n \in W_n^\circ$, such that (W_n°, w_n) converges geometrically to (Z, z).*

(3) *There is a geometrically finite, minimally parabolic representation $\rho_n \in AH(M, P)$ such that the associated hyperbolic 3-manifold V_n has end invariants that are identical to those of W_n. In particular, the conformal boundary of V_n admits the same circle packing C_n. Furthermore, $\Sigma \subset M$ is isotopic to a geodesic link $\Sigma_n \subset V_n^\circ$.*

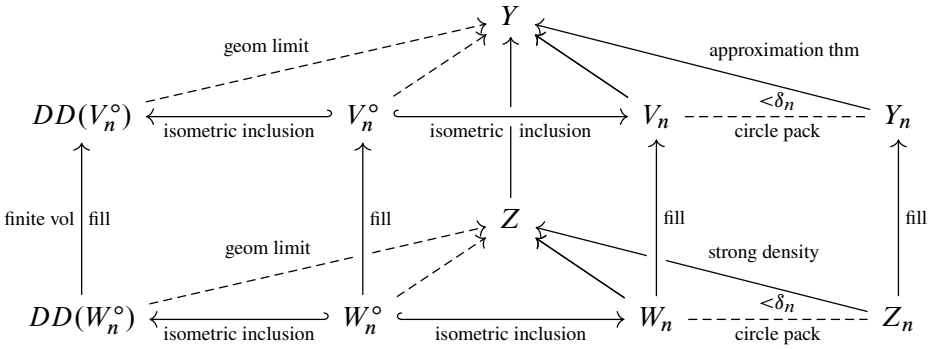

Figure 2. The manifolds appearing in the statement and proof of Theorem 7.2. Hooked horizontal arrows represent isometric inclusions. Dashed horizontal lines represent a small quasiconformal deformation that produces a circle-packed manifold. Vertical arrows represent Dehn filling. Solid diagonal arrows represent strong limits. Dashed diagonal arrows represent geometric limits only.

(4) *For every $y \in Y$, there is a choice of basepoints $v_n \in V_n^\circ$, such that (V_n°, v_n) converges geometrically to (Y, y). In the geometric limit, we have $\Sigma_n \to \Sigma_\infty$, where $\Sigma_\infty \subset Y$ is a geodesic link isotopic to Σ. Furthermore, there is a strong limit $\rho_n \to \rho$, where $\rho \in AH(M, P)$ is a representation associated to Y.*

(5) *The operations of filling and double-doubling commute. That is:*

$$DD(V_n^\circ) - DD(\Sigma_n) \cong DD(W_n^\circ).$$

(6) *The manifolds Y and Z are geometric limits of finite-volume hyperbolic manifolds, as follows:*

$$(DD(V_n^\circ), v_n) \to (Y, y) \quad \text{and} \quad (DD(W_n^\circ), w_n) \to (Z, z).$$

See Figure 2 for a visual summary of the theorem.

Proof. The proof is very similar to the proof of Theorem 6.2, with the notable difference that drilling is replaced by filling. This change of direction means that some more work is required to ensure that the filled manifolds are hyperbolic and contain geodesic links representing Σ.

We begin by verifying the existence of a hyperbolic manifold Y with pared manifold (M, P) and the same end invariants as those of Z. Let $H_1, \dots, H_k$ be disjoint horocusps in Z about the components $\sigma_1, \dots, \sigma_k$ of Σ. By a theorem of Meyerhoff [30, Section 5], the H_i can be chosen so that $\text{area}(\partial H_i) \geq \sqrt{3}/2$ for every i. Let s_i be a Euclidean geodesic on ∂H_i representing the meridian of σ_i. Let $L_i = \text{len}(s_i)/\sqrt{\text{area}(\partial H_i)}$ be the normalized length of s_i, and $L = L(\mathbf{s})$ the

total normalized length of $\mathbf{s} = (s_1, \ldots, s_k)$. Thus, our hypothesis $L(\mathbf{s})^2 \geq 230.1$ combined with (1.1) implies

$$\frac{1}{230.1} \geq \frac{1}{L(\mathbf{s})^2} \geq \frac{1}{L(s_i)^2} = \frac{\text{area}(\partial H_i)}{\text{len}(s_i)^2} \geq \frac{\sqrt{3}/2}{\text{len}(s_i)^2}\,, \tag{7.1}$$

hence $\text{len}(s_i) > 14$ for each i. Consequently, Theorem 3.1 tells us that (M, P) is a pared manifold that admits a hyperbolic structure Y with end invariants matching those of Z.

Starting from Z, we apply strong density, Theorem 2.12, to find a sequence of geometrically finite, minimally parabolic representations

$$\tau_n \in AH(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma)),$$

such that $\tau_n \to \xi$ is a strong limit. The end invariants of the associated hyperbolic manifolds Z_n form a filling sequence, converging to the end invariants of Z. The Kleinian groups Δ_n associated to Z_n converge (both geometrically and algebraically) to the Kleinian group Δ associated to Z. In particular, we have $\Delta_n \to \Delta$ in the Chabauty topology. We also choose an open neighborhood system Q_n about $[\xi]$, such that $[\tau_n] \in Q_n$.

For each n, we choose a positive constant δ_n such that $\lim \delta_n = 0$ and several more constraints (specified below) are all satisfied. For each n, Brooks' Theorem 2.15 says that there is a geometrically finite Kleinian group Δ_{n,δ_n} representing an e^{δ_n}-quasiconformal deformation of Δ_n, and such that the conformal boundary of Δ_{n,δ_n} admits a circle packing. By picking δ_n small enough, we ensure that

$$d_{\text{Chaub}}(\Delta_{n,\delta_n}, \Delta_n) < 2^{-n}.$$

Picking δ_n small enough also ensures that the associated representation ξ_n satisfies $[\xi_n] \in Q_n$. These choices imply that $\xi_n \to \xi$ is a strong limit. We let $W_n = \mathbb{H}^3/\Delta_{n,\delta_n}$.

Next, we fill in the meridians of Σ to recover the pared manifold (M, P). We obtain three sets of hyperbolic structures on (M, P):

- A hyperbolic structure Y, whose end invariants agree with those of Z.
- A hyperbolic structure Y_n, whose end invariants agree with to those of Z_n.
- A hyperbolic structure V_n, whose end invariants agree with those of W_n.

Since (M, P) is a pared manifold, Theorem 2.6 says there is also a geometrically finite, minimally parabolic representation $\rho' \in AH(M, P)$. The hyperbolic structures Y_n and V_n, represented by $\sigma_n \in QH(\rho')$ and $\rho_n \in QH(\rho')$, respectively, exist by Theorem 2.7.

By construction, the conformal boundary of V_n is an e^{δ_n}-quasiconformal deformation of the conformal boundary of Y_n. Thus, the distance in $\mathcal{T}(\partial_0 M)$ between the conformal boundaries of those manifolds is at most δ_n. By choosing δ_n

small enough, we ensure that Y_n and V_n are represented by Kleinian groups Γ_n and Γ_{n,δ_n}, respectively, such that

$$d_{\text{Chaub}}(\Gamma_{n,\delta_n}, \Gamma_n) < 2^{-n}.$$

We also ensure that, for a neighborhood system $\{O_n\}$ about $[\rho]$, we have $\rho_n \in O_{m(n)}$ whenever $\rho \neq \sigma_n \in O_{m(n)}$, and $\rho_n \in O_n$ whenever $\sigma_n = \rho$. This completes the list of conditions that δ_n needs to satisfy.

Now, we check the conclusions of the proposition. Conclusion (1) holds by construction, because we have chosen Δ_{n,δ_n} so that the conformal boundary of $W_n = \mathbb{H}^3/\Delta_{n,\delta_n}$ admits a circle packing C_n, and because we have chosen δ_n small enough to ensure $\xi_n \to \xi$ is a strong limit.

For conclusion (2), recall that we have already checked that $\Delta_{n,\delta_n} \to \Delta$ in the Chabauty topology. Thus, for every basepoint $z \in Z$, there exist basepoints $w_n \in W_n$ such that $(W_n, w_n) \to (Z, z)$. For any fixed radius, the ball $B_R(w_n)$ will lie in the scooped manifold W_n° when $n \gg 0$, hence the sequence of scooped manifolds (W_n°, w_n) also converges geometrically to (Z, z).

We will prove conclusions (3) and (5) together. Observe that since the conformal boundary of V_n agrees with that of W_n, it also admits the same circle packing C_n. Thus, we may double-double the scooped manifolds V_n° and W_n° to obtain finite-volume manifolds $DD(V_n^\circ)$ and $DD(W_n^\circ)$, respectively. The scooped manifold W_n° contains a tuple of slopes $\mathbf{s}^n = (s_1^n, \ldots, s_k^n)$ corresponding to the meridians of Σ, with the property that $L(s_i^n) \to L(s_i)$ in the geometric limit $(W_n^\circ, w_n) \to (Z, z)$. Thus, for $n \gg 0$, we have $L(\mathbf{s}^n)^2 > 230.08$. Thus, the finite-volume manifold $DD(W_n^\circ)$ contains a tuple of slopes $DD(\mathbf{s}^n)$, containing four copies of each meridian of Σ. The definition of normalized length in (1.1) implies that for all $n \gg 0$, we have

$$\frac{1}{L(DD(\mathbf{s}^n))^2} = \frac{4}{L(\mathbf{s}^n)^2} < \frac{4}{230.08} = \frac{1}{57.52},$$

and hence

$$L(DD(\mathbf{s}^n)) > \sqrt{57.52} > 7.584. \tag{7.2}$$

Consequently, a theorem of Hodgson and Kerckhoff [24, Theorem 1.2], implies that Dehn filling $DD(W_n^\circ)$ along the tuple of slopes $DD(\mathbf{s}^n)$ produces a hyperbolic manifold U_n, where the union of cores of the Dehn filling solid tori is a geodesic link Υ_n. By [24, Corollary 5.13], the total length of Υ_n is less than 0.16.

Now, recall that each V_n is homeomorphic to M, and contains a smoothly embedded copy of Σ, such that W_n is homeomorphic to $V_n - \Sigma$. After an isotopy, we may assume that the embedded copy of Σ lies in V_n°, hence $DD(V_n^\circ)$ contains the double-doubled link $DD(\Sigma)$. Since ∂V_n° and ∂W_n° contain exactly the same pattern of red and blue faces, the double-doubling construction shows that $DD(W_n^\circ)$ is homeomorphic to $DD(V_n^\circ) - DD(\Sigma)$ by a homeomorphism that carries $DD(\mathbf{s}^n)$

to the tuple of meridians of $DD(\Sigma)$. The above homeomorphism extends to a homeomorphism of pairs

$$(U_n, \Upsilon_n) \to (DD(V_n^\circ), DD(\Sigma)).$$

Since $DD(V_n^\circ)$ is Haken, Waldhausen's topological rigidity theorem [39, Theorem 7.1] implies that $DD(\Sigma)$ is isotopic to the geodesic link Υ_n. The components of Υ_n that lie in the original copy of V_n° form a geodesic link Σ_n, with the property that

$$DD(V_n^\circ) - DD(\Sigma_n) \cong DD(V_n^\circ - \Sigma_n) \cong DD(W_n^\circ),$$

proving (3) and (5). We note that $\operatorname{len}(\Sigma_n) = \operatorname{len}(\Upsilon_n)/4 < 0.04$.

Conclusion (4) is established exactly as in Theorem 6.2. By construction, the end invariants of Y_n agree with those of Z_n. Thus, those end invariants form a filling sequence that limits to the end invariants of Y. Thus, by the approximation Theorem 2.13, the representations σ_n associated to Y_n converge strongly (after a subsequence) to a representation of a hyperbolic manifold homeomorphic to Y and having the same end invariants as Y. By the ending lamination Theorem 2.5, we have a strong limit $\sigma_n \to \rho$ for a representation ρ associated to Y. In particular, $\Gamma_n \to \Gamma$ in the Chabauty topology and $[\sigma_n] \in O_{m(n)}$ for a sequence $m(n) \to \infty$.

By our choice of δ_n, we have a Chabauty limit $\Gamma_{n,\delta_n} \to \Gamma$ and $[\rho_n] \in O_{m(n)}$ for the same sequence $m(n)$, hence $\rho_n \to \rho$ is a strong limit. As above, we can fix a basepoint $y \in Y$ and then find basepoints $v_n \in V_n$, which lie in V_n° for $n \gg 0$, such that (V_n°, v_n) converges geometrically to (Y, y). Since $\rho_n \to \rho$ is a strong limit, and $\operatorname{len}(\Sigma_n) < 0.04$ for all $n \gg 0$, Proposition 4.1 says that $\Sigma_n \to \Sigma_\infty \subset Y$, a geodesic link isotopic to Σ.

Finally, conclusion (6) is a corollary of (2) and (4). □

Remark 7.3. If the end invariants of Y (equivalently, the end invariants of Z) are geometrically finite, the preceding proof becomes considerably more lightweight. In this case, one can take constant sequences $Y_n = N$ and $Z_n = Z$. Thus, as in Remark 6.3, the hyperbolization, strong density, ending lamination, and approximation theorems become unnecessary. Thus, in the geometrically finite case, the only tools required in the proof are Theorem 2.7 and Brooks' Theorem 2.15.

Next, we record a finite-volume version of Theorem 1.3, with some additional information. The following result is [20, Corollary 9.34]. As in Remark 6.6, the hypotheses of Theorem 7.4 differ from those of Theorem 1.3 by a factor of 4.

Theorem 7.4 (Effective filling in finite volume, [20]). *Fix any $0 < \varepsilon \leq \log 3$ and any $J > 1$. Let M be a 3-manifold with empty or toroidal boundary, and Σ a link in M. Suppose that $M - \Sigma$ admits a complete, finite-volume hyperbolic metric W, in which the total normalized length of the meridians of Σ satisfies*

$$L^2 \geq \max\left\{ \frac{2\pi \cdot 6771 \cosh^5(0.6\varepsilon + 0.1475)}{\varepsilon^5} + 11.7,\ \frac{2\pi \cdot 11.35}{\varepsilon^{5/2} \log(J)} + 11.7 \right\}.$$

Then M has a hyperbolic metric V, in which Σ a geodesic link. Furthermore, there are canonical J-bilipschitz inclusions

$$\varphi\colon V^{\geq\varepsilon} \hookrightarrow W^{\geq\varepsilon/1.2}, \quad \psi\colon W^{\geq\varepsilon} \hookrightarrow V^{\geq\varepsilon/1.2},$$

which are equivariant with respect to the symmetry group of the pair (M, Σ).

We can now use Theorems 7.2 and 7.4 to prove Theorem 1.3. The proof involves chasing the left half of the diagram in Figure 2. Starting from the hyperbolic structure Z on the complement of Σ, we will consider a circle-packed approximating manifold W_n, the finite-volume manifold $DD(W_n^\circ)$, its filling $DD(V_n^\circ)$, and the scooped submanifold V_n° that approximates Y.

Proof of Theorem 1.3. Let M be a tame 3-manifold and $\Sigma \subset M$ a link such that $M - \Sigma$ admits a hyperbolic structure Z. If $\mathrm{vol}(Z) < \infty$, then the desired conclusion is covered by Theorem 7.4, substituting $V = Y$ and $W = Z$. For the rest of the proof, we assume $\mathrm{vol}(Z) = \infty$.

Let $\mathbf{s} = (s_1, \ldots, s_k)$ be a tuple of slopes on the cusps of Z, with one slope for the meridian of each component of Σ. We may think of each s_i as a slope on a torus of $\mathcal{N}(\Sigma)$. Recall our hypothesis on the total normalized length of the meridians:

$$L(\mathbf{s})^2 > 4\max\left\{\frac{2\pi \cdot 6771\cosh^5(0.6\varepsilon + 0.1475)}{\varepsilon^5} + 11.7,\ \frac{2\pi \cdot 11.35}{\varepsilon^{5/2}\log(J)} + 11.7\right\}. \tag{7.3}$$

Since $\varepsilon \leq \log 3$, and since $\cosh^5(x) \geq 1$ for any x, hypothesis (7.3) is considerably stronger than the normalized length hypothesis of Theorem 7.2. Thus, $Y = Z(\mathbf{s})$ is a hyperbolic structure on M with the same end invariants as those of Z.

Fix a basepoint $z \in Z$ so that $\varepsilon = 2\mathrm{injrad}(z)$, and furthermore z lies on an embedded, ε-thick horotorus $T^\eta(\sigma)$ about the first component $\sigma \subset \Sigma$. Such a choice of $z \in Z^{\geq\varepsilon}$ is possible because ε is a Margulis number for Z by Theorem 2.2; hence the thick part $Z^{>\varepsilon}$ is non-empty.

Let V_n and W_n be the sequences of geometrically finite manifolds constructed in Theorem 7.2. In particular, the conformal boundaries of each V_n and each W_n admit the same circle packing C_n. Let $V_n^\circ \subset V_n$ and $W_n^\circ \subset W_n$ be the scooped submanifolds defined by C_n, as in Definition 2.16. We also have strong limits $\rho_n \to \rho$ and $\xi_n \to \xi$, as described in Theorem 7.2.

By Theorem 7.2, each approximating manifold W_n has pared manifold $(M - \mathcal{N}(\Sigma), P \cup \partial\mathcal{N}(\Sigma))$. Thus, each W_n has a rank-2 horocusp H_i^n corresponding to each component $\sigma_i \subset \Sigma$, with meridian slope s_i^n. We may assume $H_i^n \subset W_n^\circ$, after shrinking the horocusp as needed. For each i, the normalized length $L(s_i^n)$ converges to the normalized length $L(s_i)$ measured in Z. Write $\mathbf{s}^n = (s_1^n, \ldots, s_k^n)$ for the tuple of slopes in W_n representing the meridians of Σ. Since $L(s_i^n) \to L(s_i)$ as $n \to \infty$, it follows that for $n \gg 0$, the total normalized length $L(\mathbf{s}^n)$ must also satisfy the lower bound (7.3).

After doubling W_n° twice, as in Definition 2.16, we obtain a finite-volume hyperbolic manifold $DD(W_n^\circ)$. This manifold contains the disjoint union of four isometric copies of H_i^n for each i. Then the tuple of slopes $\mathbf{s}^n$ is also double-doubled to become a tuple of slopes $DD(\mathbf{s}^n)$ on the cusps of $DD(W_n^\circ)$. Because each slope s_i^n appears four times in $DD(\mathbf{s}^n)$, the total normalized length of the meridians of $DD(H^n)$ satisfies

$$L^2 > \max\left\{\frac{2\pi \cdot 6771 \cosh^5(0.6\varepsilon + 0.1475)}{\varepsilon^5} + 11.7,\ \frac{2\pi \cdot 11.35}{\varepsilon^{5/2}\log(J)} + 11.7\right\}.$$

Thus, for $n \gg 0$, Theorem 7.4 enables us to fill $DD(W_n^\circ)$ along the tuple of meridians of $\mathbf{s}^n$ and obtain a hyperbolic 3-manifold U_n, in which the union of cores of the filled solid tori is a geodesic link Υ_n. By Theorem 7.2, we have $U_n = DD(V_n^\circ)$ and $\Upsilon_n = DD(\Sigma_n)$ for a geodesic link $\Sigma_n \subset V_n^\circ$.

Now, Theorem 7.4 says that there are J-bilipschitz inclusions

$$\varphi_n \colon U_n^{\geq\varepsilon} \to DD(W_n^\circ)^{\geq\varepsilon/1.2}, \quad \psi_n \colon DD(W_n^\circ)^{\geq\varepsilon} \to U_n^{\geq\varepsilon/1.2},$$

which are equivariant with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of symmetries of the pair (U_n, Υ_n). Since (V_n°, Σ_n) is a fundamental domain for this group action, φ_n and ψ_n restrict to J-bilipschitz inclusions

$$\varphi_n \colon (V_n^\circ)^{\geq\varepsilon} \to (W_n^\circ)^{\geq\varepsilon/1.2}, \quad \psi_n \colon DD(W_n^\circ)^{\geq\varepsilon} \to (V_n^\circ)^{\geq\varepsilon/1.2}.$$

By Theorem 7.2, the geodesic links $\Sigma_n \subset V_n^\circ$ converge to a geodesic link $\Sigma_\infty \subset Y$, isotopic to Σ. After performing this isotopy, we may suppose that $\Sigma \subset Y$ is a geodesic link.

In preparation for Theorem 5.1, we choose appropriate basepoints for our geometric limits. Recall that we have picked a component $\sigma \subset \Sigma$, and chosen a basepoint $z \in Z$ so that

$$2\mathrm{injrad}(z) = \varepsilon,$$

and furthermore z lies on an embedded ε-thick horotorus $T^\varepsilon(\sigma)$. In a similar fashion, we choose a basepoint $y \in Y$ so that

$$2\mathrm{injrad}(y) = \eta \in (\varepsilon, 2\varepsilon),$$

and furthermore y lies on an embedded, η-thick equidistant torus $T^\eta(\sigma)$ about the same component $\sigma \subset \Sigma$. Such a choice of $y \in Y^{>\varepsilon}$ is possible because ε is a Margulis number for N; hence the thick part $N^{>\varepsilon}$ is non-empty.

Theorem 7.2 says that for $n \gg 0$, there exist choices of basepoints $v_n \in V_n^\circ$ and $w_n \in W_n^\circ$, such that $(V_n^\circ, v_n) \to (Y, y)$ and $(W_n^\circ, w_n) \to (Z, z)$. The convergence of injectivity radii in a geometric limit implies that $2\mathrm{injrad}(v_n) \in (\varepsilon, 2\varepsilon)$ for large n. Similarly, $2\mathrm{injrad}(w_n) \to \varepsilon$ as $n \to \infty$.

We are now ready to construct the J-bilipschitz inclusion $\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\varepsilon/1.2}$, using Theorem 5.1. We have geometrically convergent sequences $(V_n^\circ, v_n) \to (Y, y)$ and $(W_n, w_n) \to (Z, z)$. We have $y \in Y^{>\varepsilon}$ and $v_n \in (V_n^\circ)^{\geq\varepsilon}$ for large n, as required. For large n, we have a J-bilipschitz inclusion

$$\varphi_n\colon (V_n^\circ)^{\geq\varepsilon} \to (W_n^\circ)^{\geq\varepsilon/1.2}.$$

Furthermore, $d(\varphi_n(v_n), w_n)$ is uniformly bounded, by exactly the same argument as in the end of the proof of Theorem 1.2. (Essentially, this follows because injectivity radii are well-behaved under both geometric limits and bilipschitz maps.) Thus, Theorem 5.1 gives a J-bilipschitz inclusion $\varphi\colon Y^{\geq\varepsilon} \hookrightarrow Z^{\geq\varepsilon/1.2}$.

The reverse inclusion $\psi\colon Z^{\geq\varepsilon} \hookrightarrow Y^{\geq\varepsilon/1.2}$ is constructed in exactly the same way, tracing points backwards to check the hypotheses of Theorem 5.1. □

8. Short geodesics in infinite-volume manifolds

The main results of this section are Theorem 8.5 and Theorem 8.7, which bound the change in complex length of a short geodesic under drilling and filling, respectively. Corollaries of those results include Theorem 1.5 and Corollary 8.9, where the functions that estimate the change in complex length are replaced by constants.

As in Section 6, the proof of Theorem 8.5 combines an approximation result (Theorem 6.2) with a previously proved theorem that works in finite volume (Theorem 8.6). Similarly, the proof of Theorem 8.7 combines an approximation result (Theorem 7.2) with a finite-volume theorem (Theorem 8.8).

To set up our results, we need to define the functions that will estimate the change in length.

Definition 8.1. Let $z_0 = \sqrt{\sqrt{5} - 2} = 0.5306\ldots$. For $z \in [z_0, 1]$, define a function

$$\operatorname{haze}(z) = 3.3957\,\frac{z(1 - z^2)}{1 + z^2}.$$

By a derivative computation, the function $\operatorname{haze}(z)$ is decreasing and invertible in this domain. Using Cardano's formula, the inverse function haze^{-1} can be expressed as follows:

$$\operatorname{haze}^{-1}(3.3957x) = \frac{2\sqrt{x^2 + 3}}{3}\cos\left(\frac{\pi}{3} + \frac{1}{3}\tan^{-1}\left(\frac{-3\sqrt{-3x^4 - 33x^2 + 3}}{x^3 + 18x}\right)\right) - \frac{x}{3}.$$

Note that haze^{-1} is defined and monotonically decreasing on $[0, 1.0196]$. Compare [20, Remark 4.23].

Here is the geometric meaning of haze and haze^{-1}. If (N, Σ, g_t) is a hyperbolic cone manifold whose singular locus Σ has angle $\alpha \in [0, 2\pi]$, the *visual area of* Σ is defined to be

$$\mathcal{A}(\Sigma) = \alpha \operatorname{len}(\Sigma).$$

Hodgson and Kerckhoff showed that under appropriate hypotheses, there is an embedded tube about Σ of radius

$$R \geq \operatorname{arctanh}(\operatorname{haze}^{-1}(\mathcal{A}(\Sigma))).$$

See [23, Theorem 5.6] and [20, Corollary 4.25] for details. In turn, the radius of this tube is used to control a number of geometric quantities through the cone deformation [20, Sections 5–7]. One of those quantities is the complex length of a non-singular closed geodesic, which we seek to control here.

Definition 8.2. Let γ be a closed geodesic in a hyperbolic 3-manifold N. Then γ corresponds to a loxodromic isometry $\varphi = \varphi(\gamma) \in \operatorname{Isom}^+(\mathbb{H}^3)$. This loxodromic isometry φ has an invariant axis in $\mathbb{H}^3$, which it translates by distance $\operatorname{len}(\gamma)$ and rotates by angle $\tau(\gamma)$. We define the complex length

$$\mathcal{L}_N(\gamma) = \mathcal{L}(\gamma) = \operatorname{len}(\gamma) + i\,\tau(\gamma).$$

Observe that $i\,\mathcal{L}(\gamma)$ lies in the upper half-plane of $\mathbb{C}$, which we identify with the hyperbolic plane $\mathbb{H}^2$.

Given two complex lengths $\mathcal{L}_Y(\gamma)$, $\mathcal{L}_Z(\delta)$, we define the *hyperbolic distance* between them to be

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\delta)) = d_{\mathbb{H}^2}(i\,\mathcal{L}_Y(\gamma),\, i\,\mathcal{L}_Z(\delta)).$$

This is closely related to distance in the Teichmüller space of the torus, which is isometric to $\mathbb{H}^2$. See Minsky [31, Section 6.2] for details.

The hyperbolic distance between lengths can be translated into a bound on the real and imaginary parts of length. The following elementary lemma is [20, Lemma 7.14].

Lemma 8.3. *Let $\mathcal{L}_Y(\gamma)$ and $\mathcal{L}_Z(\delta)$ be complex lengths. Suppose that*

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\delta)) \leq K$$

for some $K > 0$. Then the real and imaginary parts of $\mathcal{L}_Y(\gamma)$ and $\mathcal{L}_Z(\delta)$ are bounded as follows:

$$e^{-K} \leq \frac{\operatorname{len}_Z(\delta)}{\operatorname{len}_Y(\gamma)} \leq e^K, \quad |\tau_Z(\delta) - \tau_Y(\gamma)| \leq \sinh(K) \cdot \min\{\operatorname{len}_Y(\gamma),\, \operatorname{len}_z(\delta)\}.$$

Finally, given a filled manifold Y and a drilled manifold Z, we control the hyperbolic distance $d_{\text{hyp}}(\mathcal{L}_Y(\gamma),\, \mathcal{L}_Z(\gamma))$ via the following function.

Definition 8.4. For $z \in [z_0, 1]$ and $\ell \in (0, 0.5085]$, define a function

$$F(z, \ell) = \frac{(1 + z^2)}{z^3(3 - z^2)} \cdot \frac{\ell}{10.667 - 20.977\ell}.$$

Note that F is positive everywhere on its domain, decreasing in z, and increasing in ℓ. Compare [20, Definition 7.2].

8.1. Short geodesics under drilling. The first main result of this section controls the complex length of a short geodesic γ under the drilling of a geodesic link Σ. In the next theorem, $\ell = \mathrm{len}_Y(\Sigma)$ is the length of the geodesic link that we wish to drill, $m = \mathrm{len}_Y(\gamma)$ is the length of the geodesic that we wish to control, and

$$z = z_{\min} = \tanh R_{\min},$$

where $R_{\min}$ is the minimum radius of an embedded tube about $\Sigma^+ = \Sigma \cup \gamma$. Then the function $F(z, \ell)$ controls the change in the complex length $\mathcal{L}(\gamma)$. We will not compute $R_{\min}$ or $z_{\min}$ directly; we will merely estimate $z_{\min}$ as a function of ℓ and m.

Theorem 8.5. *Let Y be a tame hyperbolic 3-manifold. Let Σ be a geodesic link in Y, and γ a closed geodesic disjoint from Σ. Let $\ell = \mathrm{len}_Y(\Sigma)$ and $m = \mathrm{len}_Y(\gamma)$ be the lengths of Σ and γ in the complete metric on Y. Suppose $\ell < 0.018375$ and $m < 0.0996 - 1.408 \cdot \ell$. Let*

$$z_{\min} = \mathrm{haze}^{-1}(2\pi(4\ell + m + 10^{-5})).$$

Then $Y - \Sigma$ also admits a complete hyperbolic metric Z, with the same end invariants as those of Y. The closed curve γ is isotopic to a geodesic in this metric. Furthermore, the complex lengths of γ in Y and Z are related as follows:

$$d_{\mathrm{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\gamma)) \leq 4\pi^2 F(z_{\min}, 4\ell).$$

The proof of Theorem 8.5 relies on the following finite-volume analogue [20, Theorem 7.19].

Theorem 8.6 (Short geodesics under drilling, [20]). *Let V be a complete, finite volume hyperbolic 3-manifold. Let Σ be a geodesic link in V, and γ a closed geodesic disjoint from Σ. Let $\ell = \mathrm{len}_V(\Sigma)$ and $m = \mathrm{len}_V(\gamma)$ be the lengths of Σ and γ in the complete metric on V. Suppose $\ell \leq 0.0735$ and $m \leq 0.0996 - 0.352 \cdot \ell$. Let*

$$z'_{\min} = \mathrm{haze}^{-1}(2\pi(\ell + m + 10^{-5})) > 0.6288.$$

Then $V - \Sigma$ also admits a complete hyperbolic metric W, in which γ is again isotopic to a geodesic. Furthermore, the complex lengths of γ in V and W are related as follows:

$$d_{\mathrm{hyp}}(\mathcal{L}_V(\gamma), \mathcal{L}_W(\gamma)) \leq 4\pi^2 F(z'_{\min}, \ell).$$

Proof of Theorem 8.5. If $\mathrm{vol}(Y) < \infty$, the desired result already follows from Theorem 8.6. (Although the definition of $z'_{\min}$ in Theorem 8.6 differs from the definition of $z_{\min}$ in Theorem 8.5, the monotonicity of haze^{-1} implies that $z'_{\min} > z_{\min}$. Then, the monotonicity of F ensures that the conclusion of Theorem 8.6 still applies with $z_{\min}$ in place of $z'_{\min}$ and 4ℓ in place of ℓ.) For the rest of the proof, we assume that $\mathrm{vol}(Y) = \infty$.

We will apply Theorem 6.2. Let V_n and W_n be the sequences of geometrically finite manifolds constructed in that theorem. Let Z be the hyperbolic manifold homeomorphic to $Y - \Sigma$, with the same end invariants as those of Y. Then, by Theorem 6.2, the conformal boundaries of each V_n and each W_n admit the same circle packing C_n. That theorem also guarantees a strong limit $\rho_n \to \rho$ (where ρ_n is the representation corresponding to W_n and ρ corresponds to Y) and a strong limit $\xi_i \to \xi$ (where ξ_n is the representation corresponding to W_n and ξ corresponds to Z).

Now, let $\gamma \subset Y$ be a closed geodesic satisfying the length bound of the theorem. By Meyerhoff's theorem [30, Section 7], we have

$$\gamma \cap \Sigma = \emptyset,$$

hence $\Sigma^+ = \Sigma \cup \gamma$ is a geodesic link where each component is shorter than 0.1. Then Proposition 4.1 implies that for $n \gg 0$, the approximating manifold V_n contains a geodesic link $\Sigma_n^+ = \Sigma_n \cup \gamma_n$, where the sequence $\{\Sigma_n^+\}$ converges to Σ^+ as $n \to \infty$. In particular, setting

$$\ell_n = \mathrm{len}_{V_n}(\Sigma_n) \quad \text{and} \quad m_n = \mathrm{len}_{V_n}(\gamma_n),$$

we have $\ell_n \to \ell$ and $m_n \to m$. Consequently, for all $n \gg 0$, we have

$$\ell_n \leq 0.018375 \quad \text{and} \quad m_n \leq 0.0996 - 1.408 \cdot \ell_n.$$

For every n where Σ_n^+ is defined, we have $\Sigma_n^+ \subset CC(V_n) \subset V_n^\circ$. Thus, for all $n \gg 0$, the double-double $DD(V_n^\circ)$ contains the double-double $DD(\Sigma_n)$, a geodesic link of total length $4\ell_n \leq 0.0735$. Furthermore, by construction, we have

$$m_n \leq 0.0996 - 1.408 \cdot \ell_n = 0.0996 - 0.352 \cdot 4\ell_n.$$

Thus, $DD(V_n^\circ)$ satisfies the hypotheses of Theorem 8.6. Combining that result with Theorem 6.2, we may drill the link $DD(\Sigma_n)$ and obtain a cusped hyperbolic 3-manifold $DD(W_n^\circ) = DD(V_n^\circ) - DD(\Sigma_n)$ containing a closed geodesic isotopic to γ_n. Furthermore, Theorem 8.6 gives

$$d_{\mathrm{hyp}}(\mathcal{L}_{DD(V_n^\circ)}(\gamma_n), \mathcal{L}_{DD(W_n^\circ)}(\gamma_n)) \leq 4\pi^2 F(z^n_{\min}, 4\ell_n),$$

where

$$z^n_{\min} = \mathrm{haze}^{-1}(2\pi(4\ell_n + m_n + 10^{-5})).$$

Observe that the isotopy class of γ_n in $V_n^\circ - \Sigma_n$ contains a representative disjoint from the scooped boundary (the red and blue faces). Thus, the closed geodesic γ_n in the hyperbolic metric on $DD(W_n^\circ) = DD(V_n^\circ) - DD(\Sigma_n)$ must be disjoint from the red and blue totally geodesic surfaces that partition the four copies of the fundamental domains W_n°. In short, we may take γ_n to be a closed geodesic in W_n°. Thus, by the above displayed equation, we also have

$$d_{\mathrm{hyp}}(\mathcal{L}_{V_n}(\gamma_n), \mathcal{L}_{W_n}(\gamma_n)) \leq 4\pi^2 F(z^n_{\min}, 4\ell_n). \tag{8.1}$$

Observe that

$$\lim_{n\to\infty} z^n_{\min} = \lim_{n\to\infty} \operatorname{haze}^{-1}(2\pi(4\ell_n + m_n + 10^{-5})) = \operatorname{haze}^{-1}(2\pi(4\ell + m + 10^{-5})) = z_{\min}.$$

Since $z^n_{\min} \geq 0.6288$ by Theorem 8.6, we can substitute $F(z^n_{\min}, 4\ell_n) \leq 0.0174$ in (8.1), hence Lemma 8.3 implies $\operatorname{len}_{W_n}(\gamma_n) < 2m_n < 0.2$.

Now, recall the strong limit $\xi_n \to \xi$. The closed geodesics $\gamma_n \subset W_n$ have length universally bounded by a constant less than $\log 3$, hence Proposition 4.1, says that $\gamma_n \subset W_n$ converge to a closed geodesic $\gamma \subset Z$ in the geometric limit. In particular, $\mathcal{L}_{W_n}(\gamma_n) \to \mathcal{L}_Z(\gamma)$. Taking limits of the bound in (8.1) as $n \to \infty$ gives

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma),\ \mathcal{L}_Z(\gamma)) \leq 4\pi^2\, F(z_{\min}, 4\ell),$$

as desired. □

We can now derive Theorem 1.5, which was stated in the introduction.

Proof of Theorem 1.5. Let $\ell = \operatorname{len}_Y(\Sigma)$ and $m = \operatorname{len}_Y(\gamma)$, and assume $\max\{4\ell, m\} < 0.0735$. This hypothesis implies $\ell < 0.018375$, and thus $m + 1.408\ell < 0.0996$, hence the hypotheses of Theorem 8.5 are satisfied. By Lemma 6.1, $Y - \Sigma$ admits a hyperbolic metric Z, with the same end invariants. In addition, the above hypothesis on ℓ and m, combined with the monotonicity of haze^{-1}, implies

$$z_{\min} = \operatorname{haze}^{-1}(2\pi(4\ell + m + 10^{-5})) \geq 0.6299.$$

Now, Theorem 8.5 gives

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma),\ \mathcal{L}_Z(\gamma)) \leq 4\pi^2 F(z_{\min}, 4\ell) \leq 4\pi^2 F(0.6299, 0.0735) \leq 0.6827,$$

where the second inequality uses the monotonicity of F and the third inequality comes from evaluating Definition 8.4. Finally, Lemma 8.3 gives

$$1.9793^{-1} \leq \frac{\operatorname{len}_Z(\gamma)}{\operatorname{len}_Y(\gamma)} \leq 1.9793 \quad \text{and} \quad |\tau_Z(\gamma) - \tau_Y(\gamma)| \leq 0.05417.$$ □

8.2. Short geodesics under filling. Next, we turn our attention to bounding the length of a short geodesic under filling rather than drilling. The following result is the filling analogue of Theorem 8.5.

Theorem 8.7. *Let V be a tame, hyperbolic 3-manifold and Σ a geodesic link in V. Suppose that $V - \Sigma$ admits a hyperbolic structure W with the same end invariants as those of V, such that the total normalized length of the meridians of Σ in W satisfies $L^2 > 512$. Let $\gamma \subset W$ be a closed geodesic of length $m = \operatorname{len}_W(\gamma) < 0.056$. Define*

$$z_{\min} = \operatorname{haze}^{-1}\left(\frac{(2\pi)^2}{(L/2)^2 - 14.7} + 2\pi \cdot 1.656\, m\right).$$

Then γ is isotopic to a closed geodesic in V. Furthermore, the complex lengths of γ in V and W are related as follows:

$$d_{\mathrm{hyp}}(\mathcal{L}_V(\gamma), \mathcal{L}_W(\gamma)) \leq 4\pi^2 F\left(z_{\min}, \frac{2\pi}{(L/2)^2 - 14.7}\right).$$

Just as with the drilling argument, the proof of Theorem 8.7 relies on the following finite-volume analogue [20, Theorem 7.21].

Theorem 8.8 (Short geodesics under filling, [20]). *Let Y be a complete, finite-volume hyperbolic 3-manifold and Σ a geodesic link in Y. Suppose that $Y - \Sigma$ admits a hyperbolic structure Z, such that the total normalized length of the meridians of Σ in Z satisfies $L^2 \geq 128$. Let $\gamma \subset Z$ be a closed geodesic of length $m = \mathrm{len}_Z(\gamma) \leq 0.056$. Define*

$$z'_{\min} = \mathrm{haze}^{-1}\left(\frac{(2\pi)^2}{L^2 - 14.7} + 2\pi \cdot 1.656\, m\right) > 0.624.$$

Then γ is isotopic to a closed geodesic in Y. Furthermore, the complex lengths of γ in Y and Z are related as follows:

$$d_{\mathrm{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\gamma)) \leq 4\pi^2 F\left(z'_{\min}, \frac{2\pi}{L^2 - 14.7}\right).$$

Proof of Theorem 8.7. If $\mathrm{vol}(Y) < \infty$, the desired result already follows from Theorem 8.8. (Although the definition of $z'_{\min}$ in Theorem 8.8 differs from the definition of $z_{\min}$ in Theorem 8.7, the monotonicity of haze^{-1} and F ensures that the conclusion of Theorem 8.6 still applies with $(L/2)^2$ in place of L^2.) For the rest of the proof, we assume that $\mathrm{vol}(Y) = \infty$.

Let V_n and W_n be the sequences of geometrically finite manifolds constructed in Theorem 6.2. By that theorem, the conformal boundaries of each V_n and each W_n admit the same circle packing C_n. Furthermore, there is a strong limit $\rho_n \to \rho$ (where ρ_n is the representation corresponding to V_n and ρ corresponds to Y) and a strong limit $\xi_i \to \xi$ (where ξ_n is the representation corresponding to W_n and ξ corresponds to Z).

Let $\gamma \subset Z$ be a closed geodesic satisfying the length bound of the theorem. Then Proposition 4.1 implies that for $n \gg 0$, the approximating manifold W_n contains a closed geodesic γ_n, where the sequence $\{\gamma_n\}$ converges to γ as $n \to \infty$. Consequently, $\mathcal{L}_{W_n}(\gamma_n) \to \mathcal{L}_Z(\gamma)$. In particular, for all $n \gg 0$, we have

$$m_n = \mathrm{len}_{W_n}(\gamma_n) \leq 0.056.$$

Let $\mathbf{s}$ be the tuple of slopes in Z corresponding to the meridians of Σ, and let $\mathbf{s}^n$ be the tuple of slopes in W_n corresponding to the meridians of Σ. Then, as in the proof of Theorem 7.2, we have $L(\mathbf{s}^n) \to L(\mathbf{s}) > 512$ as $n \to \infty$. In the double-doubled

manifold $DD(W_n^\circ)$, we obtain a tuple of slopes $DD(\mathbf{s}^n)$, where each coordinate of $\mathbf{s}^n$ appears four times, once per copy of W_n°. Thus, just as in (7.2), we get

$$\frac{1}{L(DD(\mathbf{s}^n))^2} = \frac{4}{L(\mathbf{s}^n)^2} \leq \frac{4}{512} = \frac{1}{128},$$

where the inequality holds for $n \gg 0$. Thus, $DD(W_n^\circ)$ satisfies the hypotheses of Theorem 8.8. By Theorem 7.2, filling $DD(W_n^\circ)$ along the tuple of slopes $DD(\mathbf{s}^n)$ produces the finite-volume hyperbolic manifold $DD(V_n^\circ)$.

By Theorem 8.8, the closed geodesic $\gamma_n \subset DD(W_n^\circ)$ is isotopic to a geodesic in the filled manifold $DD(V_n^\circ)$. Furthermore, since γ_n can be isotoped to be disjoint from the red and blue totally geodesic surfaces that partition the copies of V_n°, the geodesic representative of γ_n must be entirely contained in one copy of V_n°. Applying Theorem 8.8 to $DD(W_n^\circ)$ and $DD(V_n^\circ)$, we obtain

$$d_{\text{hyp}}(\mathcal{L}_{V_n^\circ}(\gamma_n), \mathcal{L}_{W_n^\circ}(\gamma_n)) \leq 4\pi^2 F\left(z_{\min}^n, \frac{2\pi}{(L(\mathbf{s}^n)/2)^2 - 14.7}\right),$$

where

$$\begin{aligned} z_{\min}^n &= \text{haze}^{-1}\left(\frac{(2\pi)^2}{L(DD(\mathbf{s}^n))^2 - 14.7} + 2\pi \cdot 1.656\, m_n\right) \\ &= \text{haze}^{-1}\left(\frac{(2\pi)^2}{(L(\mathbf{s}^n)/2)^2 - 14.7} + 2\pi \cdot 1.656\, m_n\right). \end{aligned}$$

Since we are using Theorem 8.8 with $z_{\min}^n \geq 0.624$ and $(L(\mathbf{s}^n)/2)^2 - 14.7 \geq 113.3$, we can substitute $F(0.642, 2\pi/113.3) \leq 0.128$ in the above bound on complex length. Thus, Lemma 8.3 implies $\text{len}_{V_n}(\gamma_n) < 1.66 m_n < 0.1$, enabling us to apply Proposition 4.1.

Now, recall the strong limit $\rho_n \to \rho$. By Proposition 4.1, the geodesics $\gamma_n \subset V_n$ converge to a geodesic $\gamma \subset Y$. In particular, $\mathcal{L}_{V_n}(\gamma_n) \to \mathcal{L}_Y(\gamma)$. Taking limits of the above bound on $d_{\text{hyp}}(\mathcal{L}_{V_n^\circ}(\gamma_n), \mathcal{L}_{W_n^\circ}(\gamma_n))$, we have

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\gamma)) \leq 4\pi^2 F\left(z_{\min}, \frac{2\pi}{(L(\mathbf{s}^n)/2)^2 - 14.7}\right),$$

as desired. □

As a corollary of Theorem 8.7, we obtain

Corollary 8.9. *Let Y be a tame hyperbolic 3-manifold and Σ a geodesic link in Y. Suppose that $Y - \Sigma$ admits a hyperbolic structure Z with the same end invariants as those of Y, and such that the total normalized length of the meridians of Σ satisfies $L^2 > 512$. Let $\gamma \subset Z$ be a closed geodesic of complex length $\text{len}_Z(\gamma) + i\tau_Z(\gamma)$, with $\text{len}_Z(\gamma) < 0.056$. Then γ is isotopic to a closed geodesic in Y, and furthermore*

$$1.657^{-1} \leq \frac{\text{len}_Z(\gamma)}{\text{len}_Y(\gamma)} \leq 1.657 \quad \textit{and} \quad |\tau_Z(\gamma) - \tau_Y(\gamma)| \leq 0.0295.$$

Proof. The hypotheses of this corollary match those of Theorem 8.7. The assumption $L^2 > 512$ is equivalent to $(L/2)^2 - 14.7 \geq 113.3$, hence

$$\begin{aligned} z_{\min} &= \operatorname{haze}^{-1}\left(\frac{(2\pi)^2}{(L/2)^2 - 14.7} + 2\pi \cdot 1.656\, m\right) \\ &\geq \operatorname{haze}^{-1}\left(\frac{(2\pi)^2}{113.3} + 2\pi \cdot 1.656 \cdot 0.056\right) \geq 0.624. \end{aligned}$$

Plugging $z = z_{\min} \geq 0.624$ and $\ell = 2\pi/((L/2)^2 - 14.7) \leq 2\pi/113.3$ into Theorem 8.7, we obtain

$$d_{\text{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\gamma)) \leq 4\pi^2 F(z, \ell) \leq 4\pi^2 F\left(0.624, \frac{2\pi}{113.3}\right) \leq 0.5045,$$

where the second inequality uses the monotonicity of F and the third inequality comes from evaluating Definition 8.4. Finally, Lemma 8.3 converts the bound on $d_{\text{hyp}}(\mathcal{L}_Y(\gamma), \mathcal{L}_Z(\gamma))$ into the desired upper bounds on the distance between the real and imaginary parts of $\mathcal{L}_Y(\gamma)$ and $\mathcal{L}_Z(\gamma)$. □

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D. Futer, Department of Mathematics, Temple University,
Philadelphia, PA 19122, USA
E-mail: dfuter@temple.edu

J. S. Purcell, School of Mathematics, Monash University,
Clayton, VIC 3800, Australia
E-mail: jessica.purcell@monash.edu

S. Schleimer, Department of Mathematics, University of Warwick,
Coventry CV4 7AL, UK
E-mail: s.schleimer@warwick.ac.uk