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An upper bound on the revised first Betti number and a torus stability result for RCD spaces

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Abstract. We prove an upper bound on the rank of the abelianised revised fundamental group (called "revised first Betti number") of a compact $\mathsf{RCD}^*(K,N)$ space, in the same spirit of the celebrated Gromov–Gallot upper bound on the first Betti number for a smooth compact Riemannian manifold with Ricci curvature bounded below. When the synthetic lower Ricci bound is close enough to (negative) zero and the aforementioned upper bound on the revised first Betti number is saturated (i.e. equal to the integer part of N, denoted by $\lfloor N \rfloor$), then we establish a torus stability result stating that the space is $\lfloor N \rfloor$-rectifiable as a metric measure space, and a finite cover must be mGH-close to an $\lfloor N \rfloor$-dimensional flat torus; moreover, in case N is an integer, we prove that the space itself is bi-Hölder homeomorphic to a flat torus. This second result extends to the class of non-smooth $\mathsf{RCD}^*(-\delta,N)$ spaces a celebrated torus stability theorem by Colding (later refined by Cheeger–Colding).

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1. Introduction

Let us start by recalling that an $\mathsf{RCD}^*(K,N)$ space is a (possibly non-smooth) metric measure space $(X,\mathsf{d},\mathfrak{m})$ with dimension bounded above by $N \in [1,\infty)$ and Ricci curvature bounded below by $K \in \mathbb{R}$, in a synthetic sense (see Section 2.3 for the precise notions and the corresponding bibliography). The class of $\mathsf{RCD}^*(K,N)$ spaces is a natural non-smooth extension of the class of smooth Riemannian manifolds of dimension $\leq N$ and Ricci curvature bounded below by $K \in \mathbb{R}$, indeed:

- It contains the class of smooth Riemannian manifolds of dimension $\leq N$ and Ricci curvature bounded below by $K \in \mathbb{R}$;
- It is closed under pointed measured Gromov–Hausdorff convergence, so Ricci limit spaces are examples of $\mathsf{RCD}^*(K,N)$ spaces;
- It includes the class of $\lfloor N \rfloor$-dimensional Alexandrov spaces with curvature bounded below by $K/(\lfloor N \rfloor - 1)$, the latter being the synthetic extension of the

class of smooth $\lfloor N \rfloor$-dimensional Riemannian manifolds with sectional curvature bounded below by $K/(\lfloor N \rfloor - 1)$;

- In contrast to the class of smooth Riemannian manifolds, it is closed under natural geometric operations such as quotients, foliations, conical and warped product constructions (provided natural assumptions are met);
- Several fundamental comparison and structural results known for smooth Riemannian manifolds with Ricci curvature bounded below and for Ricci limits have been extended to $\mathsf{RCD}^*(K,N)$ spaces.

It was proved by Wei and the second named author [39] (after Sormani–Wei [43,45]) that an $\mathsf{RCD}^*(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ admits a universal cover $(\widetilde{X},\mathsf{d}_{\widetilde{X}},\mathfrak{m}_{\widetilde{X}})$, which is an $\mathsf{RCD}^*(K,N)$ space as well. The group of deck transformations on the universal cover is called *revised fundamental group of* X and denoted by $\bar{\pi}_1(X)$ (see Section 2.6 for the precise definitions and basic properties).

We next discuss the main results of the present paper. Let $(X,\mathsf{d},\mathfrak{m})$ be a compact $\mathsf{RCD}^*(K,N)$ space and let $\bar{\pi}_1(X)$ be its revised fundamental group. Set

$$H := [\bar{\pi}_1(X), \bar{\pi}_1(X)] \quad \text{and} \quad \Gamma := \bar{\pi}_1(X)/H,$$

respectively the commutator and the abelianised revised fundamental group. As a consequence of Bishop–Gromov volume comparison, Γ is finitely generated (see Proposition 2.25, after Sormani–Wei [44]), and thus it can be written as

$$\Gamma = \mathbb{Z}^s \times \mathbb{Z}_{p_1}^{s_1} \times \cdots \times \mathbb{Z}_{p_l}^{s_l}.$$

We define the *revised first Betti number* of $(X,\mathsf{d},\mathfrak{m})$ as

$$\mathrm{b}_1(X) := \mathrm{rank}(\Gamma) = s.$$

The goal of the paper is two-fold:

- First, we prove an upper bound for the revised first Betti number of a compact $\mathsf{RCD}^*(K,N)$ space, generalising to the non-smooth metric measure setting a classical result of M. Gromov [30] and S. Gallot [24] originally proved for smooth Riemannian manifolds with Ricci curvature bounded below.
- Second, we prove a torus stability/almost rigidity result, roughly stating that if $(X,\mathsf{d},\mathfrak{m})$ is a compact $\mathsf{RCD}^*(-\varepsilon,N)$ space with $\mathrm{b}_1(X) = \lfloor N \rfloor$, then a finite cover must be measured Gromov–Hausdorff close to a flat $\lfloor N \rfloor$-dimensional torus; if moreover N is an integer, then (X,d) is bi-Hölder homeomorphic to a flat N-dimensional torus and $\mathfrak{m}$ is a constant multiple of the N-dimensional Hausdorff measure. This extends to the non-smooth RCD setting a celebrated result by T. Colding originally established for smooth Riemannian manifolds with Ricci curvature bounded below [16, Theorem 0.2] and later refined by Cheeger–Colding [12, Theorem A.1.13]; this proved an earlier conjecture by M. Gromov.

More precisely, the first main result is the following upper bound on $\mathrm{b}_1(X)$:

Theorem 1.1 (An upper bound on $\mathrm{b}_1(X)$ for $\mathsf{RCD}^*(K,N)$ spaces). *There exists a positive function*

$$C(N,t) > 0$$

with $\lim_{t\to 0} C(N,t) = \lfloor N \rfloor$ *such that for any compact* $\mathsf{RCD}^*(K,N)$ *space* $(X,\mathsf{d},\mathfrak{m})$ *with*

$$\mathrm{supp}(\mathfrak{m}) = X, \quad \mathrm{diam}(X) \le D$$

for some $K \in \mathbb{R}$, $N \in [1,\infty)$, $D > 0$*, the revised first Betti number satisfies*

$$\mathrm{b}_1(X) \le C(N, KD^2).$$

In particular, for any $N \in [1,\infty)$ *there exists* $\varepsilon(N) > 0$ *such that if* $(X,\mathsf{d},\mathfrak{m})$ *is a compact* $\mathsf{RCD}^*(K,N)$ *space with* $\mathrm{diam}(X) \le D$, $KD^2 \ge -\varepsilon(N)$*, then*

$$\mathrm{b}_1(X) \le \lfloor N \rfloor.$$

The upper bound of Theorem 1.1 is sharp, as a flat $\lfloor N \rfloor$-dimensional torus $\mathbb{T}^{\lfloor N \rfloor}$, is an example of an $\mathsf{RCD}^*(0,\lfloor N \rfloor)$ space (thus, of an $\mathsf{RCD}^*(-\varepsilon,N)$ space for any $\varepsilon > 0$) saturating the upper bound

$$\mathrm{b}_1\big(\mathbb{T}^{\lfloor N \rfloor}\big) = \lfloor N \rfloor.$$

In order to state the second main result, let us adopt the standard notation $\varepsilon(\delta|N)$ to denote a real valued function of δ and N satisfying that

$$\lim_{\delta\to 0} \varepsilon(\delta \mid N) = 0,$$

for every fixed N. Let us also recall that (see Section 2.5 for more details and for the relevant bibliography):

- We say that $(X,\mathsf{d},\mathfrak{m})$ has essential dimension equal to $N \in \mathbb{N}$ if $\mathfrak{m}$-a.e. x has a unique tangent space isometric to the N-dimensional Euclidean space $\mathbb{R}^N$;
- We say that $(X,\mathsf{d},\mathfrak{m})$ is N-rectifiable as a metric measure space for some $N \in \mathbb{N}$ if there exists a family of Borel subsets $U_\alpha \subset X$ and charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^N$, which are bi-Lipschitz on their image such that

$$\mathfrak{m}\Big(X \setminus \bigcup_\alpha U_\alpha\Big) = 0 \quad \text{and} \quad \mathfrak{m}\llcorner U_\alpha \ll \mathcal{H}^N \llcorner U_\alpha,$$

where $\mathcal{H}^N$ denotes the N-dimensional Hausdorff measure.

Theorem 1.2 (Torus stability for $\mathsf{RCD}^*(K,N)$ spaces). *For every* $N \in [1,\infty)$*, there exists*

$$\delta(N) > 0$$

with the following property. Let $(X,\mathsf{d},\mathfrak{m})$ *be a compact* $\mathsf{RCD}^*(K,N)$ *space with*

$$K\,\mathrm{diam}(X)^2 > -\delta(N) \quad \textit{and} \quad \mathrm{b}_1(X) = \lfloor N \rfloor.$$

(1) *Then* $(X, \mathsf{d}, \mathfrak{m})$ *has essential dimension equal to* $\lfloor N \rfloor$ *and it is* $\lfloor N \rfloor$*-rectifiable as a metric measure space.*

(2) *There exists a finite cover* $(X', \mathsf{d}_{X'}, \mathfrak{m}_{X'})$ *of* $(X, \mathsf{d}, \mathfrak{m})$ *which is* $\varepsilon(\delta|N)$*-mGH-close to a flat torus of dimension* $\lfloor N \rfloor$*.*

(3) *If in addition* $N \in \mathbb{N}$*, then* $\mathfrak{m} = c\mathcal{H}^N$ *for some constant* $c > 0$ *and* (X, d) *is bi-Hölder homeomorphic to an* N*-dimensional flat torus.*

The *torus stability* above should be compared with the *torus rigidity* below, proved by Wei and the second named author [39], extending to the non-smooth $\mathsf{RCD}^*(0, N)$ setting a classical result of Cheeger–Gromoll [15]. See also Gigli–Rigoni [29] for a related torus rigidity result, where the maximality assumption on the rank of the revised fundamental group is replaced by the maximality of the rank of harmonic one forms (recall that the rank of the space of harmonic one forms coincides with the first Betti number in the smooth setting).

Theorem 1.3 ([39], after [15]). *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathsf{RCD}^*(0, N)$ *space for some* $N \in [1, \infty)$*. If the revised fundamental group* $\overline{\pi}_1(X)$ *contains* $\lfloor N \rfloor$ *independent generators of infinite order, then* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic as a metric measure space to a flat torus*

$$\mathbb{T}^{\lfloor N \rfloor} = \mathbb{R}^{\lfloor N \rfloor}/\Gamma$$

for some lattice $\Gamma \subset \mathbb{R}^{\lfloor N \rfloor}$*.*

1.1. Outline of the arguments and organisation of the paper. Our first goal will be to establish the Gromov–Gallot's upper bound on $\mathrm{b}_1(X)$ stated in Theorem 1.1. To that aim:

- Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathsf{RCD}^*(K, N)$ space. If $N = 1$ then all the results hold trivially (see Remark 2.7.1). So we assume that $N \in (1, \infty)$;
- Let $(\widetilde{X}, \mathsf{d}_{\widetilde{X}}, \mathfrak{m}_{\widetilde{X}})$ be the universal cover of $(X, \mathsf{d}, \mathfrak{m})$. Recall that $(\widetilde{X}, \mathsf{d}_{\widetilde{X}}, \mathfrak{m}_{\widetilde{X}})$ is an $\mathsf{RCD}^*(K, N)$ space as well, and the revised fundamental group $\overline{\pi}_1(X)$ acts on $(\widetilde{X}, \mathsf{d}_{\widetilde{X}}, \mathfrak{m}_{\widetilde{X}})$ by deck transformations (actually $\overline{\pi}_1(X)$ can be identified with the group of deck transformations on $\widetilde{X}$);
- Let $H = [\overline{\pi}_1(X), \overline{\pi}_1(X)]$ be the commutator of $\overline{\pi}_1(X)$ and consider the quotient space $\overline{X} = \widetilde{X}/H$. Then $\overline{X}$ inherits a natural quotient metric measure structure from $\widetilde{X}$, denoted by $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$, which satisfies the $\mathsf{RCD}^*(K, N)$ condition as well (see Corollary 2.26). Moreover, $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ is a covering space for $(X, \mathsf{d}, \mathfrak{m})$, with fibres of countable cardinality (corresponding to $\Gamma := \overline{\pi}_1(X)/H$);
- We will also consider $X' := \overline{X}/\Gamma'$, where $\Gamma' \cong \mathbb{Z}^{\mathrm{b}_1(X)}$ is a suitable subgroup of Γ. More precisely, fix a point $\bar{x} \in \overline{X}$; extending a classical argument of Gromov to the non-smooth RCD setting, one can construct $\Gamma' < \Gamma$ isomorphic to $\mathbb{Z}^{\mathrm{b}_1(X)}$ such that the distance between $\bar{x}$ and any element in $\Gamma'\bar{x}$ is bounded above and below uniformly in terms of $\mathrm{diam}(X)$ (see Lemma 3.2 for the precise statement).

The quotient space $(X', \mathsf{d}_{X'}, \mathfrak{m}_{X'})$ still satisfies the $\mathsf{RCD}^*(K, N)$ condition, it is a covering space for $(X, \mathsf{d}, \mathfrak{m})$, with fibres of finite cardinality (corresponding to the index of Γ' in Γ).

After the above constructions, a counting argument combined with Bishop–Gromov's volume comparison Theorem in $(\bar{X}, \mathsf{d}_{\bar{X}}, \mathfrak{m}_{\bar{X}})$ will give Theorem 1.1 at the end of Section 3.

In order to show Colding's torus stability for $\mathsf{RCD}^*(-\delta, N)$ spaces (i.e. Theorem 1.2), in Section 4 we will construct ε-mGH approximations from large balls in $\bar{X}$ to balls of the same radius in the Euclidean space $\mathbb{R}^{\lfloor N \rfloor}$ (see Theorem 4.1 for the precise statement).

This is achieved by an inductive argument with $\lfloor N \rfloor$ steps: in each step we obtain that a ball in $\bar{X}$ is mGH-close to a ball in a product $\mathbb{R}^n \times Y$, where Y is an $\mathsf{RCD}^*(0, N-n)$ space. In order to prove the inductive step and pass from n to $n+1$, we show that for $\delta > 0$ small enough, Y must have large diameter, so that the almost splitting theorem applies to Y. Therefore, we get an mGH approximation from a ball in $\bar{X}$ into $\mathbb{R}^{n+1} \times Y'$. The diameter estimate for Y relies on the volume counting argument described in the previous paragraph and contained in Section 3.

The approach above is inspired by Colding's paper [16], however there are some substantial differences: indeed Colding's inductive argument is based on the construction of what are now known as δ-splitting maps, while we only use ε-mGH approximations and the almost splitting theorem; moreover the non-smooth RCD^* setting, in contrast to the smooth Riemannian framework, poses some challenges at the level of regularity, of global/local structure, and of topology. Below we briefly sketch the main lines of arguments; the expert will recognise the differences from [16].

The existence of ε-mGH approximations into the Euclidean space yields the first claim of Theorem 1.2: for $\delta > 0$ small enough, $(X, \mathsf{d}, \mathfrak{m})$ has essential dimension equal to $\lfloor N \rfloor$, it is $\lfloor N \rfloor$-rectifiable as a metric measure space and moreover, if N is an integer, the measure coincides with the Hausdorff measure $\mathcal{H}^N$, up to a positive constant. This will be proved in Theorem 5.1 by combining Theorem 4.1 with an ε-regularity result by Naber and the second named author [38], revisited in the light of the constancy of dimension of $\mathsf{RCD}^*(K, N)$ spaces by Brué–Semola [6] and a measure-rigidity result by Honda [33] for non-collapsed $\mathsf{RCD}^*(K, N)$ spaces.

When $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i)\}_{i \in \mathbb{N}}$ is a sequence of spaces as in the assumptions of Theorem 1.2 with $\delta_i \downarrow 0$, Theorem 4.1 yields pmGH convergence for $(\bar{X}_i, \mathsf{d}_{\bar{X}_i}, \mathfrak{m}_{\bar{X}_i})$ to the Euclidean space of dimension $\lfloor N \rfloor$. Then by taking the subgroups

$$\mathbb{Z}^{\lfloor N \rfloor} \cong \Gamma_i' < \Gamma_i := \bar{\pi}_1(X_i)/H_i$$

already considered above (i.e. the ones constructed in Lemma 3.2, with $k = 3$) and using equivariant Gromov–Hausdorff convergence (introduced by Fukaya [22] and further developed by Fukaya–Yamaguchi [23]), we deduce GH convergence of (a non-relabelled subsequence of)

$$X_i' := \bar{X}_i / \Gamma_i'$$

to a flat torus of dimension $\lfloor N \rfloor$. This will show the second claim of Theorem 1.2 (see Proposition 6.2 for more details).

When N is an integer, the measure of X_i' coincides with $\mathcal{H}^N$ (up to a constant), thanks to the aforementioned result by Honda [33]. This fact allows to apply Colding's volume convergence for RCD spaces proved by De Philippis–Gigli [18] and get that the GH convergence obtained above can be promoted to mGH convergence of X_i' to a flat torus. A recent result by Kapovitch and the second named author [34] (which builds on top of Cheeger–Colding's metric Reifenberg theorem [12]) states that for $N \in \mathbb{N}$, if a non-collapsed $\mathsf{RCD}^*(K, N)$ space is mGH-close enough to a compact smooth N-manifold M, then it is bi-Hölder homeomorphic to M. This implies that for $\delta > 0$ small enough as in Theorem 1.2,

$$X' := \bar{X} / \Gamma'$$

is bi-Hölder homeomorphic to a flat torus, and thus $\bar{X}$ is locally (on arbitrarily large compact subsets) bi-Hölder homeomorphic to $\mathbb{R}^N$. In order to conclude the proof of the third claim of Theorem 1.2, we show that Γ is torsion free, yielding that $\Gamma \cong \mathbb{Z}^N$, and thus

$$X = \bar{X} / \Gamma$$

is bi-Hölder homeomorphic to a flat torus. This last step uses the classical Smith's theory of groups of transformations with finite period.

The paper is organised as follows. Section 2 is devoted to recall previous results about RCD spaces, covering spaces and pointed Gromov–Hausdorff convergence (measured and equivariant) that are used in the rest of the paper. In particular, we show that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is $\mathsf{RCD}^*(K, N)$ if and only if any of its regular coverings with countable fibre is an $\mathsf{RCD}^*(K, N)$ space as well. This is essential since in our proofs we often use properties of RCD^* spaces on the coverings $\tilde{X}, \bar{X}$ and X' of X. Section 3 contains the proof of the upper bound for the revised first Betti number and its consequences. In Section 4, we construct by induction ε-mGH approximations between large balls in the covering $\bar{X}$ and balls in Euclidean space of dimension $\mathrm{b}_1(X) = \lfloor N \rfloor$. Section 5 is devoted to proving the $\lfloor N \rfloor$ rectifiability, i.e. the first claim of Theorem 1.2. In Section 6, we conclude the proof of Theorem 1.2 by first showing that X' is GH-close to a flat torus $\mathbb{T}^N$ and then obtaining that, for integer N, X' is bi-Hölder homeomorphic to $\mathbb{T}^N$ and $X = X'$. In the appendix we construct two explicit mGH approximations that are used in Section 4.

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2. Background

In this section we recall some fundamental notions about convergence of metric measure spaces and about metric measures spaces with a synthetic lower bound on the Ricci curvature which will be used in the paper.

2.1. Metric measure spaces and pointed metric measure spaces. A *metric measure space* (m.m.s. for short) is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite non-negative complete Borel measure on X, with $X = \operatorname{supp}(\mathfrak{m})$ and $\mathfrak{m}(X) > 0$. A *pointed metric measure space* (p.m.m.s. for short) is a quadruple $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ where $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s. and $\bar{x} \in X$ is a given reference point. Two p.m.m.s. $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$ and $(X', \mathsf{d}', \mathfrak{m}', \bar{x}')$ are said to be *isomorphic* if there exists an isometry

$$\varphi\colon (X, \mathsf{d}) \to (X', \mathsf{d}') \text{ such that } \varphi_\sharp \mathfrak{m} = \mathfrak{m}' \text{ and } \varphi(\bar{x}) = \bar{x}'.$$

Recall that (X, d) is said to be:

- *proper* if closed bounded sets are compact;
- *geodesic* if for every pair of points $x, y \in X$ there exists a length minimising geodesic from x to y.

As we will recall later in this section, the synthetic Ricci curvature lower bounds used in the paper (i.e. $\mathsf{CD}^*(K, N)$ for some $K \in \mathbb{R}$, $N \in [1, \infty)$) imply that (X, d) is proper and geodesic (see Remark 2.8.1).

2.2. Gromov–Hausdorff convergence. We first define pointed measured Gromov–Hausdorff (pmGH) convergence of p.m.m.s. which will be used in Section 4. For details, see [7, 27, 49]. Then we define equivariant pointed Gromov–Hausdorff (EpGH) convergence and state some results by Fukaya and Fukaya–Yamaguchi which will be employed in Section 6. For details, see [22, 23].

Definition 2.1 (Definition of pmGH convergence via pmGH approximations). Let

$$(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n), \quad n \in \mathbb{N} \cup \{\infty\}$$

be a sequence of p.m.m.s. We say $(X_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ converges to $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ in the pmGH sense if for any $\varepsilon, R > 0$, there exists $N(\varepsilon, R) \in \mathbb{N}$ such that, for each $n \geq N(\varepsilon, R)$, there exists a Borel map

$$f_n^{R,\varepsilon}\colon B_R(\bar{x}_n) \to X_\infty$$

satisfying:

- $f_n^{R,\varepsilon}(\bar{x}_n) = \bar{x}_\infty$;
- $\sup_{x,y \in B_R(\bar{x}_n)} |\mathsf{d}_n(x,y) - \mathsf{d}_\infty(f_n^{R,\varepsilon}(x), f_n^{R,\varepsilon}(y))| \leq \varepsilon$;
- the ε-neighbourhood of $f_n^{R,\varepsilon}(B_R(\bar{x}_n))$ contains $B_{R-\varepsilon}(\bar{x}_\infty)$,
- $(f_n^{R,\varepsilon})_\sharp(\mathfrak{m}_n \llcorner B_R(\bar{x}_n))$ weakly converges to $\mathfrak{m}_\infty \llcorner B_R(x_\infty)$ as $n \to \infty$, for a.e. $R > 0$.

The maps $f_n^{R,\varepsilon}\colon B_R(\bar{x}_n) \to X_\infty$ are called ε-pmGH approximations. If we do not require the maps $f_n^{R,\varepsilon}$ to be Borel, nor the last item to hold, we say that the maps $f_n^{R,\varepsilon}$ are ε-pGH approximations and that the sequence converges in pointed Gromov–Hausdorff (pGH) sense.

We next define equivariant pointed Gromov–Hausdorff (EpGH) convergence. To this aim, given a metric space (X, d), we endow its group of isometries $\mathrm{Iso}(X)$ with the compact-open topology. In this case, it is known that the compact-open topology is equivalent to the topology induced by uniform convergence on compact sets (see for example [40, Theorem 46.8]). When X is proper, a sequence $(f_n)_{n\in\mathbb{N}}$ of isometries of X converges to f in the compact-open topology if and only if $(f_n)_{n\in\mathbb{N}}$ converges to f point-wise on X.

Remark 2.1.1. Given any $x_0 \in X$, denote

$$\mathsf{d}_{x_0}(f,g) = \sup\big\{\exp\big(-\mathsf{d}(x_0,x)\big)\underline{\mathsf{d}}\big(f(x), g(x)\big) \mid x \in X\big\},$$

where $\underline{\mathsf{d}}(x,y) = \min\{\mathsf{d}(x,y), 1\}$. If (X, d) is proper, one can check that d_{x_0} induces the compact-open topology and that the group $(\mathrm{Iso}(X), \mathsf{d}_{x_0})$ is a proper metric space.

Let $\mathcal{M}_{eq}^p$ be the set of quadruples $(X, \mathsf{d}, \bar{x}, \Gamma)$, where $(X, \mathsf{d}, \bar{x})$ is a proper pointed metric space and $\Gamma \subset \mathrm{Iso}(X)$ is a closed subgroup of isometries. Define the set

$$\Gamma(r) = \big\{\gamma \in \Gamma \mid \gamma(\bar{x}) \in B_r(\bar{x})\big\}.$$

We are now in position to define equivariant pointed Gromov–Hausdorff convergence for elements of $\mathcal{M}_{eq}^p$.

Definition 2.2. Let $(X_n, \mathsf{d}_n, \bar{x}_n, \Gamma_n) \in \mathcal{M}_{eq}^p$, $n = 1, 2$. An ε-equivariant pGH approximation is a triple of functions (f, ϕ, ψ):

$$f\colon B_{\varepsilon^{-1}}(\bar{x}_1) \to X_2, \quad \phi\colon \Gamma_1(\varepsilon^{-1}) \to \Gamma_2, \quad \psi\colon \Gamma_2(\varepsilon^{-1}) \to \Gamma_1,$$

that satisfy

(1) $f(\overline{x}_1) = \overline{x}_2$;
(2) The ε-neighbourhood of $f(B_{\varepsilon^{-1}}(\overline{x}_1))$ contains $B_{\varepsilon^{-1}}(\overline{x}_2)$;
(3) For all $x, y \in B_{\varepsilon^{-1}}(\overline{x}_1)$, it holds that

$$|\mathsf{d}_1(x,y) - \mathsf{d}_2(f(x), f(y))| < \varepsilon;$$

(4) For all $\gamma_1 \in \Gamma_1(\varepsilon^{-1})$ such that $x, \gamma_1 x \in B_{\varepsilon^{-1}}(\overline{x}_1)$, it holds that

$$\mathsf{d}_2(f(\gamma_1 x), \phi(\gamma_1) f(x)) < \varepsilon;$$

(5) For all $\gamma_2 \in \Gamma_2(\varepsilon^{-1})$ such that $x, \psi(\gamma_2)x \in B_{\varepsilon^{-1}}(\overline{x}_1)$, it holds that

$$\mathsf{d}_2(f(\psi(\gamma_2)x), \gamma_2 f(x)) < \varepsilon.$$

Note that we do not assume f to be continuous, nor ϕ and ψ to be homeomorphisms.

Definition 2.3. A sequence $\{(X_n, \mathsf{d}_n, \overline{x}_n, \Gamma_n)\}_{n\in\mathbb{N}}$ of spaces in $\mathcal{M}^p_{eq}$ converges in the equivariant pointed Gromov–Hausdorff (EpGH for short) sense to

$$(X_\infty, \mathsf{d}_\infty, \overline{x}_\infty, \Gamma_\infty) \in \mathcal{M}^p_{eq}$$

if there exist ε_n-equivariant pGH approximations between

$$(X_n, \mathsf{d}_n, \overline{x}_n, \Gamma_n) \quad \text{and} \quad (X_\infty, \mathsf{d}_\infty, \overline{x}_\infty, \Gamma_\infty)$$

such that $\varepsilon_n \to 0$ as $n \to \infty$.

Theorem 2.4 (Fukaya–Yamaguchi [23, Proposition 3.6]). *Let $\{(X_n, \mathsf{d}_n, \overline{x}_n, \Gamma_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^p_{eq}$ such that $\{(X_n, \mathsf{d}_n, \overline{x}_n)\}_{n\in N}$ converges in the pointed Gromov–Hausdorff sense to $(X_\infty, \mathsf{d}_\infty, \overline{x}_\infty)$. Then there exist Γ_∞ a closed subgroup of isometries of X_∞ and a subsequence $\{(X_{n_j}, \mathsf{d}_{n_j}, \overline{x}_{n_j}, \Gamma_{n_j})\}_j \in \mathcal{M}^p_{eq}$ that converges in the equivariant pointed Gromov–Hausdorff sense to $(X_\infty, \mathsf{d}_\infty, \overline{x}_\infty, \Gamma_\infty) \in \mathcal{M}^p_{eq}$.*

For a closed subgroup Γ in $\mathrm{Iso}(X)$ and $x \in X$, let $\Gamma x \subset X$ denote the orbit of x under the action of Γ. The space of orbits is denoted by X/Γ. Let

$$\mathsf{d}_{X/\Gamma}(\Gamma x, \Gamma x') = \inf\{\mathsf{d}_X(z, z') \mid z \in \Gamma x,\ z' \in \Gamma x'\}. \tag{2.1}$$

It is a standard fact that $\mathsf{d}_{X/\Gamma}$ defines a distance on X/Γ. Indeed, the equivalence between convergence in compact-open topology and point-wise convergence in X implies that the orbits of Γ are closed in x. Then consider $\Gamma x \neq \Gamma x'$ and assume by contradiction that

$$\mathsf{d}_{X/\Gamma}(\Gamma x, \Gamma x') = 0.$$

Then there exists a sequence of points in Γx converging to a point y in $\Gamma x'$, and since orbits are closed, y belongs to Γx too. Therefore the two orbits coincide, which we assumed not. As a consequence, whenever $\Gamma x \neq \Gamma x'$, we have

$$\mathsf{d}_{X/\Gamma}(\Gamma x, \Gamma x') > 0.$$

Theorem 2.5 (Fukaya [22, Theorem 2.1]). *Let $\{(X_n, \mathsf{d}_n, \bar{x}_n, \Gamma_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^p_{eq}$ that converges in the equivariant pointed Gromov–Hausdorff sense to $(X_\infty, \mathsf{d}_\infty, \bar{x}_\infty, \Gamma_\infty) \in \mathcal{M}^p_{eq}$. Then $\{(X_n/\Gamma_n, \mathsf{d}_{X_n/\Gamma_n}, \Gamma_n \cdot \bar{x}_n)\}_{n\in N}$ converges in the pointed Gromov–Hausdorff sense to $(X_\infty/\Gamma_\infty, \mathsf{d}_{X_\infty/\Gamma_\infty}, \Gamma_\infty \cdot \bar{x}_\infty)$.*

2.3. Synthetic Ricci curvature lower bounds. We briefly recall here the definition of RCD^* spaces, and we refer to [1–4, 21, 26, 37, 47, 48] for more details about synthetic curvature-dimension conditions and calculus on metric measure spaces. There are different ways to define the curvature-dimension condition, that are now known to be equivalent in the case of infinitesimally Hilbertian m.m.s. (see, for example, [21, Theorem 7]). We chose to give here only the definitions of the $\mathsf{CD}^*(K,N)$ condition and infinitesimally Hilbertian m.m.s., since this will be the framework of the paper. For $\kappa, s \in \mathbb{R}$, we introduce the generalised sine function

$$\sin_\kappa(s) = \begin{cases} \frac{\sin(\sqrt{\kappa}s)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ s & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}s)}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases}$$

For $(t,\theta) \in [0,1] \times \mathbb{R}_+$ and $\kappa \in \mathbb{R}$, the distortion coefficients are defined by

$$\sigma_\kappa^{(t)}(\theta) = \begin{cases} \frac{\sin_\kappa(t\theta)}{\sin_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2. \end{cases}$$

For a metric space (X, d), let $\mathcal{P}_2(X)$ be the space of Borel probability measures μ over X with finite second moment, i.e. satisfying

$$\int_X \mathsf{d}(x_0, x)^2 \, d\mu(x) < \infty$$

for some (and thus, for every) $x_0 \in X$. The L^2-*Wasserstein distance* between $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ is defined by

$$W_2(\mu_0, \mu_1)^2 = \inf_{\mathrm{q}} \int_{X\times X} \mathsf{d}(x,y)^2 \, d\mathrm{q}(x,y), \tag{2.2}$$

where q is a Borel probability measure on $X \times X$ with marginals μ_0, μ_1. A measure $\mathrm{q} \in \mathcal{P}(X^2)$ achieving the minimum in (2.2) is called an *optimal coupling*. The L^2-Wasserstein space $(\mathcal{P}_2(X), W_2)$ is a complete and separable space, provided (X, d) is so. Let

$$\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m}) \subset \mathcal{P}_2(X)$$

denote the subspace of $\mathfrak{m}$-absolutely continuous measures and $\mathcal{P}_\infty(X, \mathsf{d}, \mathfrak{m})$ the set of measures in $\mathcal{P}_2(X, \mathsf{d}, \mathfrak{m})$ with bounded support.

Definition 2.6. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the curvature-dimension condition $\mathsf{CD}^*(K, N)$ if and only if for each $\mu_0, \mu_1 \in \mathscr{P}_\infty(X, \mathsf{d}, \mathfrak{m})$ with $\mu_i = \rho_i m$, $i = 0, 1$, there exists an optimal coupling q and a W_2-geodesic

$$(\mu_t)_{t \in [0,1]} \subset \mathscr{P}_\infty(X, \mathsf{d}, \mathfrak{m})$$

between μ_0 and μ_1 such that for all $t \in [0, 1]$ and $N' \geq N$, we have

$$\begin{aligned} \int_X \rho_t^{-1/N'} d\mu_t \geq \int_{X \times X} \big(&\sigma_{K/N'}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0(x_0)^{-1/N'} \\ &+ \sigma_{K/N'}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1(x_1)^{-1/N'}\big)\, d\mathsf{q}(x_0, x_1). \end{aligned} \tag{2.3}$$

Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is by definition the space of $L^2(X, \mathfrak{m})$ functions having finite Cheeger energy, and it is endowed with the natural norm

$$\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + 2\mathrm{Ch}(f),$$

which makes it a Banach space. Here, the Cheeger energy is given by the formula

$$\mathrm{Ch}(f) := \frac{1}{2} \int_X |Df|_w^2 \, \mathrm{d}\mathfrak{m},$$

where $|Df|_w$ denotes the weak upper differential of f.

The metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be *infinitesimally Hilbertian* if the Cheeger energy is a quadratic form (i.e. it satisfies the parallelogram identity) or, equivalently, if the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Hilbert space.

Definition 2.7. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(K, N)$ space if it is infinitesimally Hilbertian and it satisfies the $\mathsf{CD}^*(K, N)$ condition.

Remark 2.7.1 (The case $N = 1$). If $(X, \mathsf{d}, \mathfrak{m})$ is a compact $\mathsf{RCD}^*(K, N)$ space with $N = 1$, then by Kitabeppu–Lakzian [36], we know that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic either to a point, or a segment, or a circle. Hence, all the statements of this paper will hold trivially. For instance:

- The revised first Betti number upper bound $\mathrm{b}_1(X) \leq 1$ holds trivially;
- The torus stability holds trivially since $\mathrm{b}_1(X) = 1$ only if $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to a circle.

Without loss of generality, we will thus assume $N \in (1, \infty)$ throughout the paper to avoid trivial cases.

Remark 2.7.2 (Other synthetic notions: $\mathsf{CD}(K, N)$, $\mathsf{CD}_{\mathrm{loc}}(K, N)$, $\mathsf{RCD}(K, N)$). For $K, N \in \mathbb{R}$, $N \geq 1$, one can consider the τ-distortion coefficients

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K/(N-1)}^{(t)}(\theta)^{(N-1)/N}.$$

Replacing the σ-distortion coefficients with the τ-distortion coefficients in (2.3), one obtains the $\mathsf{CD}(K,N)$ condition. Since

$$\tau^{(t)}_{K,N}(\theta) \geq \sigma^{(t)}_{K/N}(\theta),$$

the $\mathsf{CD}(K,N)$ condition implies $\mathsf{CD}^*(K,N)$. Conversely, the $\mathsf{CD}^*(K,N)$ condition implies $\mathsf{CD}(K^*,N)$ for $K^* = K(N-1)/N$, see [4, Proposition 2.5 (ii)].

Analogously to Definition 2.7, one can define the class of $\mathsf{RCD}(K,N)$ spaces as those $\mathsf{CD}(K,N)$ spaces which in addition are infinitesimally Hilbertian. It is clear from the above discussion that $\mathsf{RCD}(K,N)$ implies $\mathsf{RCD}^*(K,N)$, and that $\mathsf{RCD}^*(K,N)$ implies $\mathsf{RCD}(K^*,N)$. An important property of $\mathsf{RCD}^*(K,N)$ spaces is the *essential non-branching* [42], roughly stating that every W_2-geodesic with endpoints in $\mathscr{P}_2(X,\mathsf{d},\mathfrak{m})$ is concentrated on a set of non-branching geodesics. This has been recently pushed to full non-branching in [20].

The local version of $\mathsf{CD}(K,N)$, called $\mathsf{CD}_{\mathrm{loc}}(K,N)$, amounts to require that every point $x \in X$ admits a neighbourhood $U(x)$ such that for each pair $\mu_0,\mu_1 \in \mathscr{P}_\infty(X,\mathsf{d},\mathfrak{m})$ supported in $U(x)$ there exists a W_2-geodesic from μ_0 to μ_1 (not necessarily supported in $U(x)$) satisfying the $\mathsf{CD}(K,N)$ concavity condition. For essentially non-branching spaces, it is not hard to see that $\mathsf{CD}^*(K,N)$ is equivalent to $\mathsf{CD}_{\mathrm{loc}}(K,N)$. It is much harder to establish the equivalence in turn with $\mathsf{CD}(K,N)$. This was proved for essentially non-branching spaces with finite total measure in [8]. In particular, it follows that for spaces of finite total measure, the conditions $\mathsf{RCD}_{\mathrm{loc}}(K,N)$, $\mathsf{RCD}(K,N)$ and $\mathsf{RCD}^*(K,N)$ are all equivalent.

We state here some well-known properties of $\mathsf{RCD}^*(K,N)$ spaces that we are going to use throughout the paper. First of all, we have the following natural scaling properties: if $(X,\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(K,N)$ space, then

- for any $c>0$, $(X,\mathsf{d},c\mathfrak{m})$ is an $\mathsf{RCD}^*(K,N)$ space,
- for any $\lambda>0$, $(X,\lambda\mathsf{d},\mathfrak{m})$ is an $\mathsf{RCD}^*(\lambda^{-2}K,N)$ space.

The following sharp Bishop–Gromov volume comparison was proved in [48] for $\mathsf{CD}(K,N)$ spaces, then generalised to non-branching $\mathsf{CD}_{\mathrm{loc}}(K,N)$ spaces in [10], and to essentially non-branching $\mathsf{CD}_{\mathrm{loc}}(K,N)$ spaces in [9]. In particular, it holds for $\mathsf{RCD}^*(K,N)$ spaces. It will be useful in proving the appropriate upper bound for the revised first Betti number $\mathrm{b}_1(X)$.

Theorem 2.8 (Bishop–Gromov volume comparison). *Let $K \in \mathbb{R}$ and $N \in (1,\infty)$. If $K<0$, then for any $\mathsf{RCD}^*(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$, all $x \in X$ and all $r \leq R$, we have*

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \geq \frac{\int_0^r \sinh^{N-1}(\sqrt{-K/(N-1)}t)\,dt}{\int_0^R \sinh^{N-1}(\sqrt{-K/(N-1)}t)\,dt}.$$

If $K \geq 0$, then

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_R(x))} \geq \left(\frac{r}{R}\right)^N.$$

Remark 2.8.1. The Bishop–Gromov volume comparison implies that $\mathsf{RCD}^*(K,N)$ spaces are locally doubling and thus proper. It is also not hard to check directly from the Definition 2.6 that $\operatorname{supp}\mathfrak{m}$ (and thus X, since we are assuming throughout that $X = \operatorname{supp}\mathfrak{m}$) is a length space. Since a proper length space is geodesic, we have that $\mathsf{RCD}^*(K,N)$ spaces are proper and geodesic. Thus, without loss of generality, we will assume that all the metric spaces in the paper are proper and geodesic.

The set of $\mathsf{RCD}^*(K,N)$ spaces is compact when endowed with the pointed measured Gromov–Hausdorff topology ([2, 21, 27, 37, 47]):

Theorem 2.9 (Stability with respect to pmGH convergence). *Let $K \in \mathbb{R}$, $N \in [1,\infty)$ and $C > 1$. The set*

$$\Big\{(X,\mathsf{d},\mathfrak{m},\bar{x}) \text{ p.m.m.s. such that } (X,\mathsf{d},\mathfrak{m}) \text{ is an } \mathsf{RCD}^*(K,N) \text{ space} \\ \text{and } C^{-1} \le \mathfrak{m}(B_1(\bar{x})) \le C\Big\}$$

endowed with the pmGH topology is compact.

Following the terminology of De Philippis–Gigli [18] (after Cheeger–Colding [12]), recall that an $\mathsf{RCD}^*(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$ is said to be:

- *non-collapsed* if $\mathfrak{m} = \mathcal{H}^N$ up to a positive constant;
- *weakly non-collapsed* if $\mathfrak{m} \ll \mathcal{H}^N$.

It follows from [18, Theorem 1.12] that whenever $(X,\mathsf{d},\mathfrak{m})$ is a weakly non-collapsed $\mathsf{RCD}^*(K,N)$ space, N is necessarily an integer. Honda [33, Corollary 1.3] proved the following additional property of compact weakly non-collapsed spaces:

Theorem 2.10. *Let $K \in \mathbb{R}$ and $N \in \mathbb{N}$. For any compact weakly non-collapsed $\mathsf{RCD}^*(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$, there exists $c > 0$ such that $m = c\mathcal{H}^N$.*

2.4. Almost splitting. We recall some results from [38] that we will use in the proofs, starting from an Abresh–Gromoll inequality on the excess function. For a metric measure space $(X,\mathsf{d},\mathfrak{m})$ we consider two points p,q and define the excess function as

$$e_{p,q}(x) := \mathsf{d}(p,x) + \mathsf{d}(x,q) - \mathsf{d}(p,q).$$

For radii $0 < r_0 < r_1$, let $A_{r_0,r_1}(\{p,q\})$ be the annulus around p and q:

$$A_{r_0,r_1}\big(\{p,q\}\big) = \big\{x \in X \mid r_0 < \mathsf{d}(p,x) < r_1 \vee r_0 < \mathsf{d}(q,x) < r_1\big\}.$$

We will use the following estimates, contained in [38, Theorem 3.7, Corollary 3.8 and Theorem 3.9].

Theorem 2.11. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}^*(K,N)$ space for some $K \in \mathbb{R}$ and $N \in (1,\infty)$, and let $p,q \in X$ with*

$$\mathsf{d}_{p,q} := \mathsf{d}(p,q) \le 1.$$

For any $\varepsilon_0 \in (0,1)$, *there exists*

$$\overline{r} = \overline{r}(K, N, \varepsilon_0) \in (0,1]$$

such that if $x \in A_{\varepsilon_0 \mathsf{d}_{p,q}, 2\mathsf{d}_{p,q}}(\{p,q\})$ *satisfies* $e_{p,q}(x) \le r^2 \,\mathsf{d}_{p,q}$ *for some* $r \in (0, \overline{r}]$, *then*

(i) *The following integral estimate holds:*

$$\fint_{B_{r\mathsf{d}_{p,q}}(x)} e_{p,q}(y)\, d\mathfrak{m}(y) \le C(K,N,\varepsilon_0) r^2 \,\mathsf{d}_{p,q}.$$

(ii) *There exists* $\alpha = \alpha(N) \in (0,1)$ *such that*

$$\sup_{y \in B_{r\mathsf{d}_{p,q}}(x)} e_{p,q}(y) \le C(K,N,\varepsilon_0) r^{1+\alpha} \mathsf{d}_{p,q}. \tag{2.4}$$

(iii) *If, moreover,* x *is such that the ball* $B_{2r\mathsf{d}_{p,q}}(x)$ *is contained in the annulus* $A_{\varepsilon_0 \mathsf{d}_{p,q}, 2\mathsf{d}_{p,q}}(\{p,q\})$, *then there exists* $\alpha = \alpha(N) \in (0,1)$ *such that*

$$\fint_{B_{r\mathsf{d}_{p,q}}(x)} |De_{p,q}|^2 d\mathfrak{m} \le C(K,N,\varepsilon_0) r^{1+\alpha}. \tag{2.5}$$

The almost splitting theorem for RCD^* spaces states that if there exist k points in $(X, \mathsf{d}, \mathfrak{m})$ that are far enough, and whose excess function and derivatives satisfy the appropriate smallness condition, then the space almost splits k Euclidean factors, meaning that $(X, \mathsf{d}, \mathfrak{m})$ is mGH-close to a product $\mathbb{R}^k \times Y$, for an appropriate RCD^* metric measure space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$. More precisely, we follow the notation of [38, Theorem 5.1], where $p_i + p_j$ denotes a point and d^p is the distance function $\mathsf{d}^p(\cdot) = \mathsf{d}(p, \cdot)$.

Theorem 2.12. *Let* $\varepsilon > 0$, $N \in (1,\infty)$ *and* $\beta > 2$. *Then there exists* $\delta(\varepsilon, N) > 0$ *with the following property. Assume that, for some* $\delta \le \delta(\varepsilon, N)$, *the following holds:*

(i) (X, d, m) *is an* $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ *space;*

(ii) *there exist points* x, $\{p_i, q_i, p_i + p_j\}_{1 \le i < j \le k}$ *in* X *for some* $k \le N$, *such that*

$$\mathsf{d}(p_i, x), \quad \mathsf{d}(q_i, x), \quad \mathsf{d}(p_i + p_j, x) \ge \delta^{-\beta} \quad \text{for } 1 \le i < j \le k,$$

and for all $r \in [1, \delta^{-1}]$, *we have*

$$\sum_{i=1}^k \sup_{B_r(x)} e_{p_i,q_i} + \sum_{i=1}^k \fint_{B_r(x)} |De_{p_i,q_i}|^2 \, d\mathfrak{m} + \sum_{1 \le i < j \le k} \fint_{B_r(x)} \left| D\left(\frac{\mathsf{d}^{p_i} + \mathsf{d}^{p_j}}{\sqrt{2}} - \mathsf{d}^{p_i + p_j} \right) \right|^2 d\mathfrak{m} \le \delta.$$

Then there exists a p.m.m.s. $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$ *such that*

$$\mathsf{d}_{mGH}\big(B^X_{\varepsilon^{-1}}(x), B^{\mathbb{R}^k\times Y}_{\varepsilon^{-1}}((0^k, y))\big) < \varepsilon.$$

More precisely,

(1) *if* $N - k < 1$*, then* $Y = \{y\}$ *is a singleton;*

(2) *if* $N - k \in [1, +\infty)$*, then* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *is an* $\mathsf{RCD}^*(0, N - k)$*-space, there exist maps*

$$u: X \supset B_{\varepsilon^{-1}}(x) \to \mathbb{R}^k \quad \text{and} \quad v: X \supset B_{\varepsilon^{-1}}(x) \to Y,$$

where $u^i = \mathsf{d}(p_i, \cdot) - \mathsf{d}(p_i, x)$*, such that the product map*

$$(u, v): X \supset B_{\varepsilon^{-1}}(x) \to \mathbb{R}^k \times Y$$

is an ε*-mGH approximation on its image.*

Theorem 2.12 was proved in [38] by Naber and the second named author, building on top of Gigli's proof of the Splitting theorem for $\mathsf{RCD}^*(0, N)$ spaces [25], after Cheeger–Gromoll's Splitting theorem [15] and Cheeger–Colding's Almost splitting theorem [11].

2.5. Structure of $\mathsf{RCD}^*(K, N)$ spaces and rectifiability. We collect here some known results about the structure of $\mathsf{RCD}^*(K, N)$ spaces, which extended to the $\mathsf{RCD}^*(K, N)$ setting previous work on Ricci limit spaces [12–14, 16, 17]. They will be used in order to prove that for $\varepsilon > 0$ small enough, a compact $\mathsf{RCD}^*(-\varepsilon, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with

$$\mathrm{b}_1(X) = \lfloor N \rfloor \quad \text{and} \quad \mathrm{diam}(X) = 1$$

is $\lfloor N \rfloor$-rectifiable and the measure $\mathfrak{m}$ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{\lfloor N \rfloor}$.

We first recall the notion of k-rectifiability for metric and metric measure spaces.

Definition 2.13 (k-rectifiability). Let $k \in \mathbb{N}$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be $(\mathfrak{m}, k)$*-rectifiable as a metric space* if there exists a countable collection of Borel subsets $\{A_i\}_{i\in I}$ such that

$$\mathfrak{m}\Big(X \setminus \bigcup_{i\in I} A_i\Big) = 0$$

and there exist bi-Lipschitz maps between A_i and Borel subsets of $\mathbb{R}^k$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be k*-rectifiable as a metric measure space* if, additionally, the measure $\mathfrak{m}$ is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^k$.

We next recall the definitions of tangent space and of k-regular set $\mathcal{R}^k$.

Definition 2.14. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(K, N)$ space for $N \in (1, \infty)$ and $K \in \mathbb{R}$, and let $x \in X$. A metric measure space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, \bar{y})$ is a tangent space of $(X, \mathsf{d}, \mathfrak{m})$ at x if there exists a sequence $r_i \in (0, +\infty)$, $r_i \downarrow 0$ such that $(X, r_i^{-1}\mathsf{d}, \mathfrak{m}^x_{r_i}, x)$ converges in the pmGH topology to $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, \bar{y})$, where

$$\mathfrak{m}^x_r = \left(\int_{B_r(x)} \left(1 - \frac{\mathsf{d}(x, y)}{r}\right) d\,\mathfrak{m}(y)\right)^{-1} m.$$

The set of all tangent spaces of $(X, \mathsf{d}, \mathfrak{m})$ at x is denoted by $\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x)$.

Definition 2.15. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(K, N)$ space for $N \in (1, \infty)$ and $K \in \mathbb{R}$. For any $k \in \mathbb{N}$, the k-th regular set $\mathcal{R}_k$ is given by the set of points $x \in X$ such that tangent space at x is unique and equal to the Euclidean space $(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, c_k \mathcal{H}^k, 0^k)$, with

$$c_k = \left(\int_{B_1(0^k)} (1 - |y|)\, d\,\mathcal{L}^k(y)\right)^{-1}.$$

In [38, Theorem 1.1] it was proved that for any $\mathsf{RCD}^*(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$, the k-regular sets $\mathcal{R}_k$ for $k = 1, \dots, \lfloor N \rfloor$ are $(\mathfrak{m}, k)$-rectifiable as a metric spaces and form an essential decomposition of X, i.e.

$$\mathfrak{m}\Big(X \setminus \bigcup_{k=0}^{\lfloor N \rfloor} \mathcal{R}_k\Big) = 0.$$

A subsequent refinement by the independent works [19, 28, 35] showed that the measure $\mathfrak{m}$ restricted to $\mathcal{R}^k$ is absolutely continuous with respect to $\mathcal{H}^k$. Moreover, in [6], E. Bruè and D. Semola showed that there exists exactly one regular set $\mathcal{R}_k$ having positive measure. It is then possible to define the essential dimension of an $\mathsf{RCD}^*(K, N)$ space as follows.

Definition 2.16 (Essential dimension). Let $K \in \mathbb{R}$, $N \in (1, \infty)$ and let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(K, N)$ space. The *essential dimension* of X is the unique integer $k \in \{1, \dots, \lfloor N \rfloor\}$ such that $\mathfrak{m}(\mathcal{R}_k) > 0$.

Observe that, as a consequence, any $\mathsf{RCD}^*(K, N)$ space of essential dimension equal to k is k-rectifiable as a metric measure space.

We finally state two theorems that will be used in the final part of the paper, to show that an $\mathsf{RCD}^*(K, N)$ space with

$$\mathrm{b}_1(X) = N \in \mathbb{N} \quad \text{and} \quad \mathrm{diam}(X)^2 K \geq -\varepsilon$$

is mGH-close and bi-Hölder homeomorphic to a flat torus $\mathbb{T}^N$.

Theorem 2.17 ([18, Theorem 1.2]). *Let $N \in \mathbb{N}$, $N > 1$ and let $(X_i, \mathsf{d}_i, \mathcal{H}^N, x_i)$ be a sequence of non-collapsed* $\mathsf{RCD}^*(K, N)$ *spaces such that (X_i, d_i, x_i) converges to (X, d, x) in the pointed Gromov–Hausdorff sense. Then one of the following holds:*

(i) *If*

$$\limsup_i \mathcal{H}^N(B_1(x_i)) > 0,$$

then $\mathcal{H}^N(B_1(x_i))$ *converges to* $\mathcal{H}^N(B_1(x))$ *and* $(X_i, \mathsf{d}_i, \mathcal{H}^N, x_i)$ *converges in the pmGH sense to* $(X, \mathsf{d}, \mathcal{H}^N, x)$.

(ii) *If*

$$\lim_{i\to\infty} \mathcal{H}^N(B_1(x_i)) = 0,$$

then $\dim_{\mathcal{H}}(X) \leq N - 1$.

In the following statement, we rephrase Theorem 1.10 of [34]:

Theorem 2.18 ([34, Theorem 1.10]). *Let* (M, g) *be a compact manifold of dimension* N *(without boundary). There exists*

$$\varepsilon = \varepsilon(M) > 0$$

such that the following holds. If $(X, \mathsf{d}, \mathfrak{m}, x)$ *is a pointed* $\mathsf{RCD}^*(K, N)$ *space for some* $K \in \mathbb{R}$ *satisfying*

$$\mathsf{d}_{pmGH}(X, M) < \varepsilon,$$

then $\mathfrak{m} = c_X \mathcal{H}^N$ *for some* $c_X > 0$ *and* (X, d) *is bi-Hölder homeomorphic to* M.

2.6. Covering spaces, universal cover and revised fundamental group. We first discuss the definition of covering spaces, universal cover, revised fundamental group, and actions of groups of homeomorphisms over topological spaces. Then we focus on length metric measure spaces and see that the $\mathsf{RCD}^*(K, N)$ condition can be lifted to the total space of an $\mathsf{RCD}^*(K, N)$ base, when having a covering map.

2.6.1. Covering spaces. Let us provide some definitions and results related to coverings spaces from [32, 46]. In particular, we state the notion of a group acting properly discontinuously as it appears in these references. Note that sometimes this is defined differently.

We say that a topological space Y is a *covering space* for a topological space X if there exists a continuous map

$$p_{Y,X}: Y \to X,$$

called a *covering map*, with the property that for every point $x \in X$ there exists a neighbourhood $U \subset X$ of x such that $p_{Y,X}^{-1}(U)$ is the disjoint union of open subsets of Y and so that the restriction of $p_{Y,X}$ to each of these subsets is homeomorphic to U. By definition, the covering map is a local homeomorphism. Two covering spaces Y, Y' of X are said to be equivalent if there exists a homeomorphism between them,

$$h: Y \to Y',$$

so that
$$p_{Y',X} \circ h = p_{Y,X}.$$

If X is path-connected then the cardinality of $p_{Y,X}^{-1}(x)$ does not depend on $x \in X$. We recall that given a topological space Z and $z \in Z$, the *fundamental group* of Z, $\pi_1(Z, z)$, is the group of the equivalence classes under based homotopy of the set of closed curves from $[0, 1]$ to Z with endpoints equal to z. Any covering map $p_{Y,X}$ induces a monomorphism

$$p_{Y,X\sharp}\colon \pi_1(Y, y_0) \to \pi_1(X, p_{Y,X}(y_0));$$

moreover, when both Y and X are path-connected, the cardinality of $p_{Y,X}^{-1}(x)$ agrees with the index of $p_{Y,X\sharp}(\pi_1(Y, y_0))$ in $\pi_1(X, p_{Y,X}(y_0))$. For Y path-connected, the covering map $p_{Y,X}$ is called *regular* if $p_{Y,X\sharp}(\pi_1(Y, y_0))$ is a normal subgroup of $\pi_1(X, p_{Y,X}(y_0))$.

Before defining the group of deck transformations of a covering space, we introduce some terminology of group actions.

Definition 2.19. A group of homeomorphisms G of a topological space Y is said to *act effectively* or *faithfully* if

$$\bigcap_{y \in Y} \{g \mid g(y) = y\} = \{e\},$$

where e denotes the identity element of G. It acts *without fixed points* or *freely* if the only element of G that fixes some point of Y is the identity element. We say that G acts *discontinuously* if the orbits of G in Y are discrete subsets of Y and we say that G acts *properly discontinuously* if every $y \in Y$ has a neighbourhood $U \subset Y$, so that

$$U \cap gU = \emptyset$$

for all $g \in G \setminus \{e\}$[1].

So, acting properly discontinuously implies acting discontinuously and without fixed points, and every free action is effective.

The group of deck transformations of a covering space Y of X is the group of self-equivalences of Y:

$$G(Y \mid X) := \{h\colon Y \to Y \mid h \text{ is a homeomorphism and } p_{Y,X} \circ h = p_{Y,X}\}.$$

By the unique lifting property, $G(Y \mid X)$ acts without fixed points. Combining this fact with the definition of covering map, we see that $G(Y \mid X)$ also acts properly discontinuously on Y.

If Y is connected and locally path-connected, then $p_{Y,X}$ is regular if and only if the group $G(Y \mid X)$ acts transitively on each fibre of $p_{Y,X}$. In this case, for any $y_0 \in Y$, we have

[1]This is sometimes defined differently, i.e. G acts properly discontinuously if every $y \in Y$ has a neighbourhood $U \subset Y$, so that $U \cap gU \neq \emptyset$ for finitely many $g \in G$

- an isomorphism of groups:

$$G(Y \mid X) \cong \pi_1(X, p_{Y,X}(y_0)) \big/ p_{Y,X\sharp}(\pi_1(Y, y_0));$$

- a bijection between any fibre of $p_{Y,X}$ and $G(Y \mid X)$;
- a homeomorphism of spaces:

$$X \cong Y/G(Y \mid X).$$

Definition 2.20 (Universal cover of a connected space). Given a connected topological space X, a *universal covering space* $\widetilde{X}$ for X is a connected covering space for X such that for any other connected covering space Y of X there exists a map

$$f\colon \widetilde{X} \to Y$$

that forms a commutative triangle with the corresponding covering maps, i.e.

$$p_{Y,X} \circ f = p_{\widetilde{X},X}.$$

Since we do not require X to be semi-locally simply connected, then $\widetilde{X}$ might not be simply connected. Thus, the group $G(\widetilde{X} \mid X)$ of deck transformations of $\widetilde{X}$ might not be isomorphic to the fundamental group of X. However, $G(\widetilde{X} \mid X)$ acts properly discontinuously on $\widetilde{X}$, transitively on each fibre of $p_{Y,X}$; thus, $p_{\widetilde{X},X}$ is regular. Moreover, any (connected) covering space of X is covered by $\widetilde{X}$. In particular, universal covering spaces of a connected and locally path-connected space are equivalent.

Recall also that for a connected topological space Y and a group G of homeomorphisms of Y acting properly discontinuously on Y, the projection map $Y \to Y/G$ is a regular covering whose group of deck transformations coincides with G, i.e.

$$G(Y \mid Y/G) = G.$$

We conclude this subsection summarising some results that will be used later.

Proposition 2.21. *Let $p\colon Y \to X$ be a regular covering and let $H \leq G(Y \mid X)$.*

- *If Y is connected, then the projection map $Y \to Y/H$ is a regular covering map and*

$$G(Y \mid Y/H) = H;$$

- *If Y is path connected and locally path connected and H is a normal subgroup of $G(Y \mid X)$, then the projection map $Y/H \to X$ is a regular covering map and*

$$G(Y/H \mid X) = G(Y \mid X)/H.$$

Proof. For a covering map, the group of deck transformations acts properly discontinuously on the total space. Hence, H also acts properly discontinuously on Y and so the first item holds by the paragraph above this proposition. The second item can be proved in a similar way: first observe that Y/H is connected because it is the image of the projection map which is continuous, then note that $G(Y \mid X)/H$ acts properly discontinuously on Y/H (see also [32, Chapter 1, Section 1.3, Exercise 24]). □

2.6.2. Coverings of metric spaces and $\mathsf{RCD}^*(K,N)$ spaces. We now discuss some definitions and results related to coverings of metric spaces. For more details we refer to [44] and [39].

Let (X, d_X) be a length metric space and $p_{Y,X}\colon Y \to X$ be a covering map. The length and metric structure of X can be lifted to Y so that the covering map becomes a local isometry. Explicitly, denoting by L_X the length structure of X, define the metric $\mathsf{d}_Y\colon Y \times Y \to \mathbb{R}$ as

$$\mathsf{d}_Y(y,y') := \inf\big\{L_X(p_{Y,X}\circ\gamma) \mid \gamma\colon[0,1]\to Y,\ p_{Y,X}\circ\gamma \text{ is Lipschitz} \\ \text{and } \gamma(0)=y, \gamma(1)=y'\big\}. \quad (2.6)$$

This lifting process implies that Y is complete whenever X is so. In particular, if X is compact, then Y will be a complete, locally compact length space, and thus proper [7, Proposition 2.5.22].

If X is locally compact and $\mathfrak{m}_X$ is a Borel measure on it, we can lift $\mathfrak{m}_X$ to a Borel measure $\mathfrak{m}_Y$ on Y that is locally isomorphic to $\mathfrak{m}_X$. In order to define $\mathfrak{m}_Y$, denote by $\mathcal{B}(Y)$ the family of Borel subsets of Y and consider the following collection of subsets of Y:

$$\Sigma := \big\{E \subset Y \mid p_{Y,X}|_E\colon E \to p_{Y,X}(E) \text{ is an isometry}\big\}.$$

Note that Σ is stable under intersections and that Y is locally compact given that $p_{Y,X}$ is a local isometry. Thus, the smallest σ-algebra that contains Σ equals $\mathcal{B}(Y)$. For $E \in \Sigma$, define

$$\mathfrak{m}_Y(E) := \mathfrak{m}_X\big(p_{Y,X}(E)\big)$$

and then extend it to all $\mathcal{B}(Y)$.

From now on, all the covering spaces will be endowed with this metric and measure. The following result was proved in [39].

Theorem 2.22. *For any $K \in \mathbb{R}$ and any $N \in (1,\infty)$, any $\mathsf{RCD}^*(K,N)$ space admits a universal cover space $(\widetilde{X}, \mathsf{d}_{\widetilde{X}}, \mathfrak{m}_{\widetilde{X}})$ which is itself an $\mathsf{RCD}^*(K,N)$ space.*

We now state Sormani–Wei's definition of revised fundamental group [44].

Definition 2.23. (Revised fundamental group) Given a complete length metric space (X, d_X) that admits a universal cover $(\widetilde{X}, \mathsf{d}_{\widetilde{X}})$, the revised fundamental group of X, denoted by $\bar{\pi}_1(X)$, is defined to be the group of deck transformations $G(\widetilde{X} \mid X)$.

Recall that the covering map $p_{\widetilde{X},X}$ associated to the universal cover space of X is regular and thus $\overline{\pi}_1(X)$ acts transitively on each fibre of $p_{\widetilde{X},X}$ and properly discontinuously on $\widetilde{X}$ by homeomorphisms; such homeomorphisms are measure-preserving isometries on $\widetilde{X}$, provided $\widetilde{X}$ is endowed with the lifted distance and measure of X, as described above.

We conclude this subsection by mentioning two properties that will be used later. First, for a covering map $p_{Y,X}: Y \to X$, one can prove (by lifting geodesics of X to Y) that for any $x, x' \in X$ and $y \in Y$ with $y \in p_{Y,X}^{-1}(x)$ there exists $y' \in p_{Y,X}^{-1}(x')$ such that

$$\mathsf{d}_Y(y, y') = \mathsf{d}_X(x, x').$$

It follows that if $p_{Y,X}$ is regular, and thus $G(Y \mid X)$ acts transitively on its fibres, then for any $y, y'' \in Y$ there exists $h \in G(Y \mid X)$ such that

$$\mathsf{d}_Y(y, h(y'')) \le \operatorname{diam}(X). \tag{2.7}$$

The second property is that a quotient space Y/H as in Proposition 2.21 is an $\mathsf{RCD}^*(K,N)$ space provided either X or Y is an $\mathsf{RCD}^*(K,N)$ space. We give more details below. In Theorem 1.1 we will use this fact to get an upper bound on the revised first Betti number of an $\mathsf{RCD}^*(K,N)$ by passing to a quotient space ($\widetilde{X}/H$ for $H = [\overline{\pi}_1(X), \overline{\pi}_1(X)]$); this fact will be also useful in Lemma 6.3 to infer that the GH convergence of a sequence of quotient spaces can be promoted to mGH convergence.

Lemma 2.24. *Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ be a compact m.m.s with a regular covering map $p_{Y,X}: Y \to X$. Assume that $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ has the structure of m.m.s. so that $p_{Y,X}$ is a surjective local isomorphism of m.m.s. and that $p_{Y,X}^{-1}(x)$ is at most countable. Let $K \in \mathbb{R}$ and $N \in (1,\infty)$. Then $(X, \mathsf{d}_X, \mathfrak{m}_X)$ is an $\mathsf{RCD}^*(K,N)$ space if and only if $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is so.*

Proof. We argue along the lines of [39, Lemma 2.18].

Assume that $(X, \mathsf{d}_X, \mathfrak{m}_X)$ is an $\mathsf{RCD}^*(K,N)$ space. Then (X, d_X) is complete, separable, proper and geodesic. Since $p_{Y,X}$ is a regular covering map, we can apply [7, Proposition 3.4.16] stating that the length metrics on X are in 1-1 correspondence with the $G(Y \mid X)$-invariant length metrics on Y; thus (Y, d_Y) is a length metric space. Since $p_{Y,X}$ is a local isometry, we automatically get that (Y, d_Y) is a complete and locally compact space. Moreover, by our assumption on $p_{Y,X}^{-1}(x)$, (Y, d_Y) is separable. Now every complete locally compact length space is geodesic [7, Theorem 2.5.23]. Hence, (Y, d_Y) is a complete, separable and geodesic space.

In order to prove that $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is an $\mathsf{RCD}^*(K,N)$, first recall that by [21, Theorem 3.17] we know that $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ is $\mathsf{RCD}^*(K,N)$ if and only if it is infinitesimally Hilbertian and it satisfies the strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ condition, defined as in [21, Definition 3.1]. Since $(X, \mathsf{d}_X, \mathfrak{m}_X)$ is an $\mathsf{RCD}^*(K,N)$ space, by [21, Theorem 3.17, Remark 3.18] we infer that $(X, \mathsf{d}_X, \mathfrak{m}_X)$ satisfies the strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ condition.

Now [21, Theorem 3.14] says that on a geodesic m.m.s. the strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ condition is equivalent to the strong local $\mathrm{CD}^{\mathrm{e}}_{\mathrm{loc}}(K,N)$ condition, thus in particular $(X,\mathsf{d}_X,\mathfrak{m}_X)$ satisfies the strong local $\mathrm{CD}^{\mathrm{e}}_{\mathrm{loc}}(K,N)$ condition. Now each point $y\in Y$ has a compact neighbourhood U_y such that

$$\big(U_y,\mathsf{d}_Y|_{U_y\times U_y},\mathfrak{m}_Y\llcorner_{U_y}\big)$$

is isomorphic as metric measure space to

$$\big(p_{Y,X}(U_y),\mathsf{d}_X|_{p_{Y,X}(U_y)\times p_{Y,X}(U_y)},\mathfrak{m}_X\llcorner_{p_{Y,X}(U_y)}\big).$$

It follows that the strong local $\mathrm{CD}^{\mathrm{e}}_{\mathrm{loc}}(K,N)$ condition satisfied by $(X,\mathsf{d}_X,\mathfrak{m}_X)$ passes to the covering $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$. Since Y is geodesic, then by [21, Theorem 3.14] it also satisfies the strong $\mathrm{CD}^{\mathrm{e}}(K,N)$ condition.

It remains to show that $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ is infinitesimally Hilbertian. This follows by a partition of unity on Y made by Lipschitz functions with compact support contained in small metric balls isomorphic to metric balls in X, using the fact that the Cheeger energy is a local object (see [2,26]). Indeed, the validity of the parallelogram identity for the Cheeger energy on Y can be checked locally (on each small ball) using a partition of unity. Since such small balls in Y are isomorphic to small balls of X where the Cheeger energy satisfies the parallelogram identity, we conclude that the Cheeger energy on Y satisfies the parallelogram identity as well.

Thus, $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ is infinitesimally Hilbertian, satisfies the $\mathrm{CD}^{\mathrm{e}}(K,N)$ condition and

$$\mathrm{supp}(\mathfrak{m}_Y)=Y.$$

It follows by [21, Theorem 3.17] that $(Y,\mathsf{d}_Y,\mathfrak{m}_Y)$ is an $\mathsf{RCD}^*(K,N)$ space.

The converse implication can be proved with analogous arguments. □

Proposition 2.25. *Let $(X,\mathsf{d},\mathfrak{m})$ be a compact $\mathsf{RCD}^*(K,N)$ space for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Then the revised fundamental group $\bar{\pi}_1(X)$ is finitely generated.*

Proof. By [44, Proposition 6.4 and Lemma 6.2] and Bishop–Gromov volume comparison theorem, for any compact $\mathsf{RCD}^*(K,N)$ space $(X,\mathsf{d},\mathfrak{m})$, its revised fundamental group $\bar{\pi}_1(X)$ can be generated by a set of cardinality at most

$$N(\delta_0,\mathrm{diam}(X))<\infty,$$

where δ_0 corresponds to the δ_0-cover of X so that $\widetilde{X}=X^{\delta_0}$ and $N(\delta_0,\mathrm{diam}(X))$ is the maximal number of balls in $\widetilde{X}$ of radius δ_0 in a ball of radius $\mathrm{diam}(X)$. □

Corollary 2.26. *Let $(X,\mathsf{d},\mathfrak{m})$ be a compact $\mathsf{RCD}^*(K,N)$ space, for some $K\in\mathbb{R}$ and $N\in(1,\infty)$. Then for any normal subgroup H of the revised fundamental group $\bar{\pi}_1(X)$, the metric measure space*

$$\big(\widetilde{X}/H,\mathsf{d}_{\widetilde{X}/H},\mathfrak{m}_{\widetilde{X}/H}\big)$$

is an $\mathsf{RCD}^*(K,N)$ space which is covered by $\widetilde{X}$ and covers X.

Proof. Since from Proposition 2.25 we know that $\overline{\pi}_1(X)$ is finitely generated, then it is at most countable. Thus the cardinality of each fibre of the covering map is at most countable. We can thus conclude using Proposition 2.21 and Lemma 2.24. □

Remark 2.26.1. Since the group of deck transformations $G(Y \mid X)$ acts properly discontinuously on Y, the semi-metric

$$\mathsf{d}_{Y/H}(Hy, Hy') = \inf\{\mathsf{d}_Y(z, z') \mid z \in Hy,\ z' \in Hy'\} \tag{2.8}$$

defined on the quotient space

$$Y/H = \{Hy \mid y \in Y\}$$

is actually a metric (this can be seen, for example, using Section 2.2). We also observe that, under the assumptions of Proposition 2.21, since Y/H is a cover of X it can also be endowed with the lifted metric of X as defined in (2.6). We note that this metric coincides with (2.8), so we will use the quotient metric of Y/H whenever it is convenient. Notice that, in particular, all the covering maps appearing in Proposition 2.21 are local isometries.

3. Upper bound on the revised first Betti number: $\mathsf{b}_1 \leq \lfloor N \rfloor$

In this section we obtain an upper bound for the revised first Betti number of an $\mathsf{RCD}^*(K, N)$ space with $K \leq 0$ and $N \in (1, \infty)$. In the case of smooth manifolds, the estimate is due to M. Gromov [30] and S. Gallot [24] (compare also [41, Section 9.2]).

We consider a compact geodesic space admitting a universal cover and define its revised first Betti number as the rank of the abelianisation of the revised fundamental group, whenever the abelianisation is finitely generated. Indeed, the fundamental theorem of finitely generated abelian groups states that for any finitely generated abelian group G there exist a rank $s \in \mathbb{N}$, prime numbers p_i and integers s_i such that G is isomorphic to $\mathbb{Z}^s \times \mathbb{Z}_{p_1}^{s_1} \times \cdots \times \mathbb{Z}_{p_l}^{s_l}$.

Definition 3.1. Let (X, d) be a compact geodesic space admitting a universal cover. Let $\overline{\pi}_1(X)$ be its revised fundamental group, set

$$H := [\overline{\pi}_1(X), \overline{\pi}_1(X)]$$

the commutator and

$$\Gamma := \overline{\pi}_1(X)/H.$$

Then we define the revised first Betti number of X as

$$\mathsf{b}_1(X) := \begin{cases} s & \text{if } \Gamma \text{ is finitely generated, } \Gamma = \mathbb{Z}^s \times \mathbb{Z}_{p_1}^{s_1} \times \cdots \times \mathbb{Z}_{p_l}^{s_l}, \\ \infty & \text{otherwise.} \end{cases}$$

From now on, we denote by $\bar{X}$ the quotient space $\tilde{X}/H$:

$$(\bar{X}, \mathsf{d}_{\bar{X}}, \mathfrak{m}_{\bar{X}}) := (\tilde{X}/H, \mathsf{d}_{\tilde{X}/H}, \mathfrak{m}_{\tilde{X}/H}). \tag{3.1}$$

By Proposition 2.21, we know that $\bar{X}$ is a cover of X; moreover, Γ acts on $\bar{X}$ as an abelian group of isometries. Since the action of $\bar{\pi}_1(X)$ is properly discontinuous, the same is true for Γ. In particular, the action is discontinuous and all the orbits Γx, $x \in X$, are discrete.

The first step in proving the upper bound on the revised first Betti number consists in showing the appropriate analog of a result of Gromov [31, Lemma 5.19]. In the case of smooth manifolds, compare with [41, Lemma 2.1, Section 9.2] for $k = 1$ and for general $k \in \mathbb{N}$ with [16, Lemma 3.1].

Lemma 3.2. *Let (X, d) be a compact geodesic space that admits a universal cover $(\tilde{X}, \mathsf{d}_{\tilde{X}})$, assume that $\Gamma := \bar{\pi}_1(X)/H$ is finitely generated and let $(\bar{X}, \mathsf{d}_{\bar{X}})$ be as in (3.1). Then for any $k \in \mathbb{N}$ and $x \in \bar{X}$, there exists a finite index subgroup*

$$\Gamma' = \langle \tilde{\gamma}_1, \dots, \tilde{\gamma}_{\mathrm{b}_1(X)} \rangle$$

of Γ isomorphic to $\mathbb{Z}^{\mathrm{b}_1(X)}$ such that for any non-trivial element $\tilde{\gamma} \in \Gamma'$

$$k \operatorname{diam}(X) < \mathsf{d}_{\bar{X}}\big(\tilde{\gamma}(x), x\big), \tag{3.2}$$

and for all $i = 1, \dots, \mathrm{b}_1(X)$, we have

$$\mathsf{d}_{\bar{X}}\big(\tilde{\gamma}_i(x), x\big) \le 2k \operatorname{diam}(X). \tag{3.3}$$

Proof. We first find a subgroup $\Gamma'' \le \Gamma$ of finite index and generated by elements that satisfy (3.3) for $k = 1$. For any $\varepsilon > 0$, set

$$r_\varepsilon = 2 \operatorname{diam}(X) + \varepsilon$$

and let Γ_ε be the subgroup of Γ generated by the set

$$\overline{\Gamma(r_\varepsilon)} := \big\{\gamma \in \Gamma \mid \mathsf{d}_{\bar{X}}\big(\gamma(x), x\big) \le 2 \operatorname{diam}(X) + \varepsilon\big\}.$$

Observe that the previous set is not empty since, because of (2.7), there exists $\gamma \in \Gamma$ such that

$$\mathsf{d}_{\bar{X}}\big(\gamma(x), x\big) \le \operatorname{diam}(X).$$

Endow $\bar{X}/\Gamma_\varepsilon$ with the quotient topology and the distance $\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}$ induced by $\mathsf{d}_{\bar{X}}$, c.f. (2.1). Let $\pi_\varepsilon\colon \bar{X} \to \bar{X}/\Gamma_\varepsilon$ be the covering map.

Step 1. We claim that $\bar{X}/\Gamma_\varepsilon \subset \bar{B}_{\operatorname{diam}(X)+\varepsilon}(\pi_\varepsilon(x))$, i.e. that for each $z \in \bar{X}$ it holds

$$\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(z)\big) \le \operatorname{diam}(X) + \varepsilon. \tag{3.4}$$

By contradiction, assume that there is $z \in \bar{X}$ such that

$$\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(z)\big) > \operatorname{diam}(X) + \varepsilon.$$

Since $\bar{X}/\Gamma_\varepsilon$ is a geodesic space, we can take a point in the geodesic connecting $\pi_\varepsilon(x)$ to $\pi_\varepsilon(z)$, that we denote again by $\pi_\varepsilon(z)$, so that

$$\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(z)\big) = \operatorname{diam}(X) + \varepsilon.$$

Since the action of Γ in $\bar{X}$ is discontinuous and hence the same is true for the action of Γ_ε, there exist representatives $x, z \in \bar{X}$ that achieve $\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}(\pi_\varepsilon(x), \pi_\varepsilon(z))$, c.f. (2.1), we have

$$\begin{aligned}\mathsf{d}_{\bar{X}}(x, z) &= \mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(z)\big)\\ &= \operatorname{diam}(X) + \varepsilon.\end{aligned}$$

Now, there is $\gamma \in \Gamma$ such that $\mathsf{d}_{\bar{X}}(\gamma(x), z) \le \operatorname{diam}(X)$, c.f. (2.7). Then,

$$\begin{aligned}\mathsf{d}_{\bar{X}}\big(\gamma(x), x\big) &\le \mathsf{d}_{\bar{X}}\big(\gamma(x), z\big) + \mathsf{d}_{\bar{X}}(z, x)\\ &\le 2\operatorname{diam}(X) + \varepsilon.\end{aligned}$$

This implies that $\gamma \in \Gamma_\varepsilon$. Thus, $\pi_\varepsilon(\gamma(x)) = \pi_\varepsilon(x)$ and

$$\begin{aligned}0 &= \mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(\gamma(x))\big)\\ &\ge \mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(x), \pi_\varepsilon(z)\big) - \mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(z), \pi_\varepsilon(\gamma(x))\big) \ge \varepsilon,\end{aligned}$$

where we used that by definition of the quotient distance $\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}$ and the choice of γ we have

$$\begin{aligned}\mathsf{d}_{\bar{X}/\Gamma_\varepsilon}\big(\pi_\varepsilon(z), \pi_\varepsilon(\gamma(x))\big) &\le \mathsf{d}_{\bar{X}}\big(z, \gamma(x)\big)\\ &\le \operatorname{diam}(X).\end{aligned}$$

This is a contradiction, and thus claim (3.4) is proved.

Step 2. Proof of (3.3) in case $k = 1$.

From step 1 we know that

$$\bar{X}/\Gamma_\varepsilon \subset \bar{B}_{\operatorname{diam}(X)+\varepsilon}\big(\pi_\varepsilon(x)\big).$$

Since $\bar{X}/\Gamma_\varepsilon$ is proper, we infer that $\bar{X}/\Gamma_\varepsilon$ is compact and thus the index of Γ_ε in Γ is finite.

Since the action of Γ in $\bar{X}$ is discontinuous, the set

$$\Gamma(3\operatorname{diam}(X)) := \big\{\gamma \in \Gamma \mid \mathsf{d}\big(x, \gamma(x)\big) < 3\operatorname{diam}(X)\big\}$$

is finite. Thus, there exists some $\varepsilon_1 < \operatorname{diam}(X)$ such that for all $\varepsilon \leq \varepsilon_1$ the sets $\overline{\Gamma(r_\varepsilon)}$ have bounded cardinality. Since their intersection is not empty, we get that it coincides with some finite set

$$\overline{\Gamma(r_{\varepsilon_0})} = \{\gamma_1, \dots, \gamma_m\}$$

for ε_0 small enough, i.e.

$$\overline{\Gamma(r_{\varepsilon_0})} = \bigcap_{\varepsilon>0} \overline{\Gamma(r_\varepsilon)} = \overline{\Gamma(2\operatorname{diam}(X))} := \big\{\gamma \in \Gamma \mid \mathsf{d}\big(x, \gamma(x)\big) \leq 2\operatorname{diam}(X)\big\}.$$

Hence, for every element of $\overline{\Gamma(r_{\varepsilon_0})}$ inequality (3.3) holds with $k = 1$.

Step 3. Conclusion by induction.

We are going to select b_1 elements of $\overline{\Gamma(r_{\varepsilon_0})}$, say $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\mathrm{b}_1}\}$, in such a way that the subgroup Γ' generated by $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\mathrm{b}_1}\}$ satisfies the conclusions of the lemma. First observe that, since the rank of Γ_{ε_0} equals b_1, by possibly discarding some elements we can choose a linearly independent subset of $\overline{\Gamma(r_{\varepsilon_0})}$ with cardinality b_1. For simplicity, let us denote this subset by $\{\gamma_1, \dots, \gamma_{\mathrm{b}_1}\}$. Consider $\Gamma'' \subset \Gamma_{\varepsilon_0}$ the subgroup generated by $\{\gamma_1, \dots, \gamma_{\mathrm{b}_1}\}$. For fixed $k \in \mathbb{N}$, we are going to choose

$$\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_{\mathrm{b}_1}\} \quad \text{in } \Gamma'' \cap \overline{\Gamma(2\operatorname{diam}(X))}$$

and such that both (3.2) and (3.3) hold. In order to do that, we proceed by induction on $j = 1, \dots, \mathrm{b}_1$ and we choose

$$\tilde{\gamma}_1, \dots, \tilde{\gamma}_j \quad \text{in } \Gamma'' \cap \overline{\Gamma(2\operatorname{diam}(X))}$$

such that:

(a) the subgroup $\langle \tilde{\gamma}_1, \dots, \tilde{\gamma}_j \rangle$ has finite index in $\langle \gamma_1, \dots, \gamma_j \rangle$;

(b) $\tilde{\gamma}_j = \gamma = \ell_{1j}\tilde{\gamma}_1 + \dots + \ell_{j-1,j}\tilde{\gamma}_{j-1} + \ell_{jj}\gamma_j$ is chosen so that

$$\begin{aligned} l_{jj} = \max\big\{|k|,\ k \in \mathbb{Z} \text{ such that } &\exists\, \ell_{1j}, \dots, \ell_{j-1,j} \in \mathbb{Z} \\ &\text{such that if } \gamma = \ell_{1j}\tilde{\gamma}_1 + \dots + \ell_{j-1,j}\tilde{\gamma}_{j-1} + k\gamma_j, \\ &\text{then } \mathsf{d}_{\overline{X}}(\gamma(x), x) \leq 2k\operatorname{diam}(X)\big\}. \end{aligned}$$

Notice that condition (b) ensures that $\Gamma' = \langle \tilde{\gamma}_1, \dots, \tilde{\gamma}_{\mathrm{b}_1} \rangle$ satisfies (3.3). We now show that Γ' also satisfies (3.2). For any element $\gamma \in \Gamma'$, there exists $j \in \{1, \dots, \mathrm{b}_1\}$ such that γ can be written as

$$\gamma = m_1\tilde{\gamma}_1 + \dots + m_j\tilde{\gamma}_j$$

with $m_j \neq 0$. Assume by contradiction that

$$\mathsf{d}\big(x, \gamma(x)\big) \leq k\operatorname{diam}(X)$$

and consider

$$2\gamma(x) = 2m_1\widetilde{\gamma}_1 + \cdots + 2m_j\widetilde{\gamma}_j = \sum_{i=1}^{j-1}(2m_i + 2m_j\ell_{ij})\widetilde{\gamma}_i + 2m_j\ell_{jj}\gamma_j.$$

Then by using the triangle inequality and (3.3), we obtain that

$$\begin{aligned}\mathsf{d}_{\overline{X}}\big(x, 2\gamma(x)\big) &\le \mathsf{d}_{\overline{X}}\big(x, \gamma(x)\big) + \mathsf{d}_{\overline{X}}\big(\gamma(x), 2\gamma(x)\big)\\ &= 2\mathsf{d}_{\overline{X}}\big(x, \gamma(x)\big)\\ &\le 2k \operatorname{diam}(X).\end{aligned}$$

Then 2γ satisfies

$$\mathsf{d}_{\overline{X}}\big(2\gamma(x), x\big) \le 2k \operatorname{diam}(X) \quad \text{and} \quad |2m_j\ell_{jj}| > |\ell_{jj}|,$$

contradicting the way we chose ℓ_{jj} and $\widetilde{\gamma}_j$ in (b). This concludes the proof. □

Remark 3.2.1. With a more careful analysis, a similar version of Lemma 3.2 holds true if one replaces geodesic space by length spaces. c.f. [31]. Furthermore, the same conclusion holds if we consider $\overline{X}/T$ instead of $\overline{X}$, where T is any torsion subgroup of Γ.

Remark 3.2.2. Note that $\overline{X}$ is not compact. Indeed, for any $\overline{x} \in \overline{X}$ and corresponding Γ' given by Lemma 3.2, the orbit

$$\Gamma'\overline{x} = \{\gamma(\overline{x}) \mid \gamma \in \Gamma'\}$$

is countable, since Γ' acts properly discontinuously on $\overline{X}$ and Γ' is isomorphic to $\mathbb{Z}^{\mathrm{b}_1}$. Now, if by contradiction $\overline{X}$ is compact, then $\Gamma'\overline{x}$ has a converging subsequence $\{\gamma_i(\overline{x})\}$. By using either that the action is properly discontinuous or property (3.2), it is not difficult to show that $\{\gamma_i(\overline{x})\}$ must be a constant sequence starting from i large enough, giving a contradiction.

Remark 3.2.3. It is not difficult to see that Γ' is a closed discrete group in the compact-open topology. Recall if a sequence of isometries γ_i in Γ' converges to γ in the compact-open topology, then it converges uniformly on every compact subset and in particular for any $\overline{x} \in \overline{X}$, we have

$$\gamma_i(\overline{x}) \to \gamma(\overline{x}).$$

We know that for any fixed $\overline{x} \in \overline{X}$, the only converging sequences in the orbit $\Gamma'\overline{x}$ are (definitely) constant sequences. Thus there exist $\gamma \in \Gamma'$ and $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$, we have

$$\gamma_i(\overline{x}) = \gamma(\overline{x}).$$

Therefore, any converging sequence γ_i in Γ' is constantly equal to an element γ of Γ', yielding that Γ' is closed and discrete.

In the following, we consider a compact $\mathsf{RCD}^*(K,N)$ space, $(X,\mathsf{d},\mathfrak{m})$ with $N \in (1,\infty)$. By Theorem 2.22, we know that it admits a universal cover space, $(\tilde{X},\tilde{\mathsf{d}},\tilde{\mathfrak{m}})$, that satisfies the $\mathsf{RCD}^*(K,N)$ condition. Using the same notation as in Lemma 3.2, by Corollary 2.26, the quotient m.m.s. $(\bar{X},\bar{\mathsf{d}},\bar{\mathfrak{m}})$ is also an $\mathsf{RCD}^*(K,N)$ space. Since by Proposition 2.25 we know that the revised fundamental group $\bar{\pi}_1(X)$ is finitely generated, we infer that the revised first Betti number of $(X,\mathsf{d},\mathfrak{m})$ is finite.

We are now ready to prove the first main result of the paper, namely the desired upper bound for $\mathrm{b}_1(X)$. This is done by combining Lemma 3.2 with Theorem 2.8 for $(\bar{X},\bar{\mathsf{d}},\bar{\mathfrak{m}})$, and generalises to the non-smooth RCD setting the celebrated upper bound proved in the smooth setting by M. Gromov [30] and S. Gallot [24] (see also [41, Theorem 2.2, Section 9.2] and [31, Theorem 5.21]).

Proof of Theorem 1.1. Let $(X,\mathsf{d},\mathfrak{m})$ be a compact $\mathsf{RCD}^*(K,N)$ space with $K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathrm{supp}(\mathfrak{m}) = X$ and $\mathrm{diam}(X) \le D$. If $N = 1$, the claim holds trivially (see Remark 2.7.1); thus, we can assume $N \in (1,\infty)$ without loss of generality. By [39, Theorem 1.2], if $K > 0$, then $\bar{\pi}_1(X)$ is finite. Hence,

$$\mathrm{b}_1(X) = \mathrm{rank}(\Gamma) = 0.$$

Thus, the claim holds trivially.

Assume that $K \le 0$ and take $x \in \bar{X} = \tilde{X}/H$. Recall that both X and $\bar{X}$ are geodesic spaces (see Section 2), thus we can apply Lemma 3.2 with $k = 1$. Therefore, there exists a subgroup of deck transformations

$$\Gamma' = \langle \gamma_1,\dots,\gamma_{\mathrm{b}_1}\rangle \subset \Gamma$$

such that

$$\mathrm{diam}(X) < \mathsf{d}_{\bar{X}}(\gamma(x),x) \quad \text{for all non-trivial } \gamma \in \Gamma', \tag{3.5}$$

$$\mathsf{d}_{\bar{X}}(\gamma_i(x),x) \le 2\,\mathrm{diam}(X) \quad \text{for all } i = 1,\dots,\mathrm{b}_1. \tag{3.6}$$

By (3.5) all the open balls $B_{\mathrm{diam}(X)/2}(\gamma(x))$, $\gamma \in \Gamma'$, are mutually disjoint. Now set

$$I_r = \big\{\gamma = l_1\gamma_1 + \dots + l_{\mathrm{b}_1}\gamma_{\mathrm{b}_1} \in \Gamma' : |l_1| + \dots + |l_{\mathrm{b}_1}| \le r,\ l_i \in \mathbb{Z}\big\}.$$

By (3.6), since each element of Γ is an isometry and applying the triangle inequality, for all $\gamma \in I_r$, we have

$$\mathsf{d}_{\bar{X}}\big(\gamma(x),x\big) \le 2r\,\mathrm{diam}(X).$$

Hence, for all $\gamma \in I_r$, we have

$$B_{\mathrm{diam}(X)/2}\big(\gamma(x)\big) \subset B_{2r\,\mathrm{diam}(X)+\mathrm{diam}(X)/2}(x).$$

Since each element of Γ preserves the measure, i.e.

$$\gamma_\sharp \mathfrak{m}_{\bar{X}} = \mathfrak{m}_{\bar{X}},$$

then all the balls $B_{\mathrm{diam}(X)/2}(\gamma(x))$ have the same $\mathfrak{m}_{\overline{X}}$-measure, and thus

$$|I_r|\,\mathfrak{m}_{\overline{X}}\big(B_{\mathrm{diam}(X)/2}(\gamma(x))\big) \le \mathfrak{m}_{\overline{X}}\big(B_{2r\,\mathrm{diam}(X)+\mathrm{diam}(X)/2}(x)\big).$$

By the definition of I_r, $|I_r|$ is non-decreasing with respect to r. If $r = 1$, then $\{\gamma_i\}_{i=1}^{\mathrm{b}_1} \subset I_r$, and thus

$$\mathrm{b}_1 \le |I_r| \quad \text{for } r \ge 1. \tag{3.7}$$

For arbitrary $r \in \mathbb{N}$, it is easy to check that

$$|I_r| = (2r+1)^{\mathrm{b}_1}. \tag{3.8}$$

Now we apply the relative volume comparison theorem, Theorem 2.8, to obtain an upper bound on the cardinality of I_r. Since the right-hand sides of both equations in Theorem 2.8 are non-increasing as a function of K, we can assume that $K < 0$. Thus,

$$\begin{aligned}
|I_r| &\le \frac{\mathfrak{m}_{\overline{X}}\big(B_{2r\,\mathrm{diam}(X)+\mathrm{diam}(X)/2}(x)\big)}{\mathfrak{m}_{\overline{X}}\big(B_{\mathrm{diam}(X)/2}(\gamma(x))\big)} \\
&\le \frac{\int_0^{2r\,\mathrm{diam}(X)+\mathrm{diam}(X)/2} \sinh^{N-1}\big(\sqrt{-K/(N-1)}s\big)\,ds}{\int_0^{\mathrm{diam}(X)/2} \sinh^{N-1}\big(\sqrt{-K/(N-1)}s\big)\,ds} \\
&= \frac{\int_0^{(2r+1/2)\sqrt{-K\,\mathrm{diam}(X)^2/(N-1)}} \sinh^{N-1}(s)\,ds}{\int_0^{\sqrt{-K\,\mathrm{diam}(X)^2/(N-1)}/2} \sinh^{N-1}(s)\,ds} \\
&=: C_r\big(N, -K\,\mathrm{diam}(X)^2/(N-1)\big).
\end{aligned} \tag{3.9}$$

That is, $C_r(N,\cdot)\colon [0, -KD^2/(N-1)] \to \mathbb{R}$ is the function given by

$$C_r(N,t) = \frac{\int_0^{(2r+1/2)\sqrt{t}} \sinh^{N-1}(s)\,ds}{\int_0^{\sqrt{t}/2} \sinh^{N-1}(s)\,ds}.$$

By (3.7) and since $C_r(N,t)$ is non-decreasing as a function of t, we have

$$\mathrm{b}_1(X) \le C_r\big(N, -KD^2/(N-1)\big).$$

By using the Taylor expansion of sinh, we calculate that

$$\lim_{t\to 0} C_r(N,t) = \left(\frac{(2r+1/2)}{1/2}\right)^N.$$

Thus, for small t, we have

$$C_r(N,t) < 2^N\left(2r+\frac{1}{2}\right)^N + \delta. \tag{3.10}$$

Now assume by contradiction that there exists a sequence $\varepsilon_i \downarrow 0$ and $\mathsf{RCD}^*(K_i, N)$ metric measure spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ such that

$$-K_i \operatorname{diam}(X_i)^2 \le \varepsilon_i, \quad \mathrm{b}_1(X_i) > N.$$

Thanks to (3.8) and (3.9), we know that for any integer $r \ge 1$, we have

$$(2r+1)^{\mathrm{b}_1(X_i)} \le C_r(N, -K_i \operatorname{diam}(X_i)^2/(N-1)).$$

For ε_i small enough, we can apply (3.10), so that for all $r \in \mathbb{N}$, $r \ge 1$, we have

$$(2r+1)^{\mathrm{b}_1(X_i)} \le C_r(N, \varepsilon_i) < 2^N(2r+1/2)^N + \delta.$$

Thus, for r large enough

$$(2r+1)^{\mathrm{b}_1(X_i)} \le 5^N r^N.$$

Now, if $\mathrm{b}_1(X_i) > N$, it is easily seen that the last inequality fails for $r = r(N)$ large enough. Hence, we have shown that there exists $\varepsilon(N) > 0$ such that if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(K, N)$ m.m.s. with $-K \operatorname{diam}(X)^2 \le \varepsilon(N)$, then $\mathrm{b}_1(X) \le N$. Since by definition $\mathrm{b}_1(X)$ is integer, the last bound is actually equivalent to

$$\mathrm{b}_1(X) \le \lfloor N \rfloor.$$

This concludes the proof of the second assertion.

In order to prove the first assertion, set

$$C(N,t) = \sup\big\{\mathrm{b}_1(X) : (X, \mathsf{d}, \mathfrak{m}) \text{ is an } \mathsf{RCD}^*(K,N) \text{ m.m.s.} \\ \text{with } -K \operatorname{diam}(X)^2 = t\big\}$$

and observe that, thanks to (3.9), $C(N,t)$ is bounded by $C_r(N, t/(N-1))$. Since it is a bounded supremum of integer numbers, $C(N,t)$ is an integer. Moreover, the flat torus $\mathbb{T}^{\lfloor N \rfloor}$ is an $\mathsf{RCD}^*(0,N)$ space with $\mathrm{b}_1(\mathbb{T}^N) = \lfloor N \rfloor$, hence

$$C(N,t) \ge \lfloor N \rfloor.$$

The previous argument also shows that for $t \le \varepsilon(N)$, it holds that $\mathrm{b}_1(X) \le N$, thus for any $t \le \varepsilon(N)$, we have

$$\lfloor N \rfloor \le C(N,t) \le N.$$

This implies that $C(N,t) = \lfloor N \rfloor$ for any $t \le \varepsilon(N)$. As a consequence, $C(N,t)$ is the desired function tending to $\lfloor N \rfloor$ as $t \to 0$. □

Remark 3.2.4. In the case of n-dimensional manifolds, Gallot proved an optimal bound for $\mathrm{b}_1(M)$ and expressed the function $C(n,t)$ as $\xi(n,t)n$, where $\xi(n,t)$ is an explicit function tending to one as t tends to zero [24, Section 3].

Corollary 3.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathsf{RCD}^*(K, N)$ *space with* $N \in (1, \infty)$ *and* $\operatorname{diam}(X) = 1$. *Let* $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ *be as in* (3.1). *Then for any* $\overline{x} \in \overline{X}$, $k \in \mathbb{N}$ *and* $R > 1$, *the open ball* $B_R(\overline{x})$ *contains at least* $\lfloor R/k \rfloor^{\mathrm{b}_1(X)}$ *disjoint balls of radius* $k/2$.

Proof. By Lemma 3.2, there exists a subgroup of deck transformations

$$\Gamma' = \langle \gamma_1, \dots, \gamma_{\mathrm{b}_1} \rangle \subset \Gamma$$

such that

$$\begin{aligned} k \operatorname{diam}(X) &< \mathsf{d}_{\overline{X}}(\gamma(x), x) && \text{for all non-trivial } \gamma \in \Gamma', \\ \mathsf{d}_{\overline{X}}(\gamma_i(x), x) &\leq 2k \operatorname{diam}(X) && \text{for all } i = 1, \dots, \mathrm{b}_1. \end{aligned}$$

Arguing as in the proof of Theorem 1.1 we get that, for any $r \in \mathbb{N}$, the number of disjoint balls of radius $k/2$ in $B_{k/2+2kr}(\overline{x})$ is larger than or equal to the number of elements in

$$I_r = \{\gamma \in \Gamma' \mid \gamma = \ell_1\gamma_1 + \dots + \ell_{\mathrm{b}_1(X)}\gamma_{\mathrm{b}_1(X)},\ |\ell_1| + \dots + |\ell_{\mathrm{b}_1(X)}| < r\}.$$

The cardinality of I_r equals $(2r+1)^{\mathrm{b}_1(X)}$. Then for $R > 1$ write $\lfloor R \rfloor = k(2r+1)$ and get that $B_R(\overline{x})$ contains at least $(2r+1)^{\mathrm{b}_1(X)} = \lfloor R/k \rfloor^{\mathrm{b}_1(X)}$ disjoint balls of radius $k/2$. □

4. Construction of mGH approximations in the Euclidean space

This section is devoted to proving Theorem 4.1, which corresponds to the non-smooth RCD version of [16, Lemma 3.5]. The main goal is to show that if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(-\delta, N)$ space with $\delta = \delta(\varepsilon, N)$ small enough, $\operatorname{diam}(X) = 1$ and $\mathrm{b}_1(X) = \lfloor N \rfloor$, then the covering space $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ (defined in (3.1)) is locally (on suitably large metric balls) mGH-close to the Euclidean space $\mathbb{R}^{\lfloor N \rfloor}$.

The proof consists in applying inductively the almost splitting theorem. More precisely, we show that $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ is locally (on suitably large metric balls) mGH-close to a product $\mathbb{R}^k \times Y_k$ by induction on $k = 1, \dots, \lfloor N \rfloor$. Since the diameter of the covering space $\overline{X}$ is infinite (see Remark 3.2.2), the base case of induction $k = 1$ will follow by carefully applying the almost splitting theorem. As for the inductive step, thanks to Corollary 3.3 we will prove a diameter estimate on Y_k that allows us to apply the almost splitting theorem on Y_k. We will conclude by deducing the almost splitting of an additional Euclidean factor by constructing an ε-mGH approximation into $\mathbb{R}^{k+1} \times Y_{k+1}$.

Theorem 4.1. *Fix* $N \in (1, \infty)$ *and* $\beta > (2+\alpha)/\alpha$, *where* $\alpha = \alpha(N)$ *is given by Theorem 2.11. For any* $\varepsilon \in (0, 1)$, *there exists*

$$\delta(\varepsilon, N) > 0$$

such that the following holds. Let $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ *space with* $\delta \in (0, \delta(\varepsilon, N)]$, $\mathrm{b}_1(X) = \lfloor N \rfloor$ *and* $\mathrm{diam}(X) = 1$, *and let* $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ *be the covering space as in* (3.1)*. Then there exists* $\overline{x} \in \overline{X}$ *such that*

$$\mathsf{d}_{mGH}\big(B^{\overline{X}}_{\varepsilon^{-1}}(\overline{x}), B^{\mathbb{R}^{\lfloor N \rfloor}}_{\varepsilon^{-1}}(0^{\lfloor N \rfloor})\big) \le \varepsilon.$$

Remark 4.1.1. From Theorem 4.1 it directly follows that the point $\overline{x} \in \overline{X}$ has a $\lfloor N \rfloor$-dimensional Euclidean tangent cone and it belongs to the $\lfloor N \rfloor$-regular set $\mathcal{R}_{\lfloor N \rfloor}$.

Indeed, if $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ is a sequence of $\mathsf{RCD}^*(-\delta_i^{2\beta}, N)$ spaces with $\delta_i \to 0$, $\mathrm{b}_1(X_i) = \lfloor N \rfloor$, $\mathrm{diam}(X_i) = 1$ and $\overline{x}_i$ are as in Theorem 4.1, then the covering spaces $(\overline{X}_i, \mathsf{d}_{\overline{X}_i}, \mathfrak{m}_{\overline{X}_i}, \overline{x}_i)$ converge in the pointed measured Gromov–Hausdorff sense to the Euclidean space

$$\big(\mathbb{R}^{\lfloor N \rfloor}, \mathsf{d}_{\mathbb{R}^{\lfloor N \rfloor}}, \mathcal{L}^{\lfloor N \rfloor}, 0^{\lfloor N \rfloor}\big).$$

The claim follows by applying this observation to a sequence of blow-ups of $\overline{X}$ centred at $\overline{x}$.

In order to prove the base case of induction $k = 1$, we start by showing the almost splitting of a line for $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$. This will be a direct consequence of the next proposition, which in turn will follow by combining Theorems 2.11 and 2.12 with suitable blow-up arguments.

Proposition 4.2. *Fix* $N \in (1, \infty)$ *and* $\beta > (2 + \alpha)/\alpha$, *where* $\alpha = \alpha(N)$ *is given by Theorem 2.11. For any* $\varepsilon > 0$, *there exists*

$$\delta_1 = \delta_1(\varepsilon, N) > 0,$$

where $\delta_1(\varepsilon, N) \to 0$ *as* ε *goes to zero, such that for any* $\delta \in (0, \delta_1]$ *the following holds. Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ *m.m.s. such that*

$$\mathrm{diam}(X, \mathsf{d}) \ge 2\delta^{-\beta}.$$

Then there exist $x_\varepsilon \in X$, *a pointed* $\mathsf{RCD}^*(0, N-1)$ *metric measure space* $(Y_\varepsilon, \mathsf{d}_{Y_\varepsilon}, \mathfrak{m}_{Y_\varepsilon}, y_\varepsilon)$, *such that*

$$\mathsf{d}_{mGH}\big(B^X_{\varepsilon^{-1}}(x_\varepsilon), B^{\mathbb{R}\times Y_\varepsilon}_{\varepsilon^{-1}}(0, y_\varepsilon)\big) \le \varepsilon.$$

Proof. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ space. Because of the assumption on the diameter, there exist points $p, q \in X$ such that

$$\mathsf{d}(p, q) = 2\delta^{-\beta}.$$

Define x_ε as the midpoint of a geodesic connecting p and q. Consider the rescaled metric

$$\mathsf{d}_\delta = (\delta^\beta/2)\,\mathsf{d}.$$

Since X is an $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ space, $(X, \mathsf{d}_\delta, \mathfrak{m})$ is an $\mathsf{RCD}^*(-4, N)$ space. Observe that $\mathsf{d}_\delta(p, q) = 1$. With respect to the metric d_δ, we have that $x_\varepsilon \in A_{1/4,2}(\{p, q\})$ and $e^\delta_{p,q}(x_\varepsilon) = 0$.

Step 1. Estimate on the sup of the excess.

We can apply Theorem 2.11 and infer that there exist

$$\overline{r} = \overline{r}(N) > 0, \quad C = C(N) > 0, \quad \alpha = \alpha(N) \in (0,1)$$

such that the estimate (2.4) centred at x_ε holds for d_δ. By scaling back to the metric d, such an estimate can be written as follows:

$$\sup_{y \in B_r(x_\varepsilon)} e_{p,q}(y) \le C(N)\, \delta^{\alpha\beta}\, r^{1+\alpha} \quad \text{for all } r \in (0, 2\delta^{-\beta}\overline{r}(N)]. \tag{4.1}$$

We aim to choose $\delta > 0$ such that (4.1) can be turned into the following:

$$\sup_{y \in B_r(x_\varepsilon)} e_{p,q}(y) < \delta/2 \quad \text{for all } r \in [1, \delta^{-1}]. \tag{4.2}$$

Hence, we first require

$$1 \le \delta^{-1} \le 2\delta^{-\beta}\overline{r}(N), \tag{a}$$

so that (4.1) applies to all radii $r \in [1, \delta^{-1}]$. Secondly, we need

$$C(N)\, \delta^{\alpha\beta} (\delta^{-1})^{1+\alpha} < \delta/2,$$

so the right-hand side of (4.1) is bounded above by $\delta/2$. That means

$$\delta^{\beta\alpha - 2 - \alpha} < 1/(2C(N)). \tag{b}$$

Such a choice is possible since the assumption $\beta > (2+\alpha)/\alpha$ ensures that the exponent on the left-hand side is strictly positive. By choosing $\delta > 0$ sufficiently small so that both conditions (a) and (b) are satisfied, we obtain from (4.1) that estimate (4.2) holds.

Step 2. L^2-estimate on the gradient of the excess.

Consider again the rescaled metric $\mathsf{d}_\delta = (\delta^\beta/2)\,\mathsf{d}$ and choose $r > 0$, so that

$$r \le \min\{\overline{r}(N), 1/4\}.$$

Then $B^X_{2r}(x_\varepsilon) \subset A_{1/4,2}(\{p,q\})$ and estimate (2.5) of Theorem 2.11 holds as well. By scaling back to the metric d, we have:

$$⨍_{B^X_r(x_\varepsilon)} |De_{p,q}|^2 d\mathfrak{m} \le C(N)\, \delta^{\beta(1+\alpha)} r^{1+\alpha} \quad \text{for all } r \le 2\delta^{-\beta} \min\{\overline{r}(N), 1/4\}. \tag{4.3}$$

As in step 1, we aim to choose $\delta > 0$ so that (4.3) implies the following:

$$⨍_{B^X_r(x_\varepsilon)} |De_{p,q}|^2 d\mathfrak{m} \le \delta/2 \quad \text{for all } r \in [1, \delta^{-1}]. \tag{4.4}$$

Hence, we require

$$1 \le \delta^{-1} \le 2\delta^{-\beta} \min\{\overline{r}(N), 1/4\}. \tag{c}$$

In order for the right-hand side of (4.3) to be less than or equal to $\delta/2$ we need

$$C(N)\delta^{\beta(1+\alpha)}\delta^{-(1+\alpha)} < \delta/2,$$

that is,

$$\delta^{(1+\alpha)(\beta-1)-1} < 1/(2C(N)). \tag{d}$$

Note that since $\beta > 2$, the exponent on the left-hand side is strictly positive.

Step 3. Conclusion.

Fix $\delta_0 = \delta_0(N) > 0$ satisfying inequalities (a), (b), (c) and (d). Then for all $\delta \in (0, \delta_0(N)]$, inequalities (a), (b), (c) and (d) hold as well. Now, for any $\varepsilon > 0$ let $\delta(\varepsilon, N) > 0$ be as in Theorem 2.12. We define

$$\delta_1 = \delta_1(\varepsilon, N) = \min\{\delta_0(N), \delta(\varepsilon, N)\}.$$

Then for all $\delta \in (0, \delta_1]$, inequalities (4.2) and (4.4) hold and we have

$$\sup_{y \in B_r^X(x_\varepsilon)} e_{p,q}(y) + \fint_{B_r^X(x_\varepsilon)} |De_{p,q}|^2 d\mathfrak{m} \le \delta \quad \text{for all } r \in [1, \delta^{-1}].$$

Since $\delta \le \delta_1 \le \delta(\varepsilon, N)$, we can apply Theorem 2.12 and conclude the proof. □

Proposition 4.2 can be in particular applied to the covering space $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$. Indeed, thanks to Corollary 2.26 it is an RCD^* space and it is not compact (thus it must have infinite diameter, since it is proper), as it was pointed out in Remark 3.2.2. This gives the base case of induction, $k = 1$.

Corollary 4.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *and* $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ *be as in Theorem 4.1. Then there exist* $\overline{x}_{1,\varepsilon} \in \overline{X}$ *and a pointed* $\mathsf{RCD}^*(0, N-1)$ *space* $(Y_{1,\varepsilon}, \mathsf{d}_{Y_{1,\varepsilon}}, m_{Y_{1,\varepsilon}}, y_{1,\varepsilon})$ *such that*

$$\mathsf{d}_{mGH}\big(B^{\overline{X}}_{\varepsilon^{-1}}(\overline{x}_{1,\varepsilon}), B^{\mathbb{R}\times Y_{1,\varepsilon}}_{\varepsilon^{-1}}(0, y_{1,\varepsilon})\big) \le \varepsilon.$$

Observe that for the base case of induction (i.e. in Corollary 4.3), we did not use the assumptions on the diameter and revised first Betti number. These assumptions will play a key role in the following, instead. Let us state the induction hypothesis.

Assumption $\mathbf{A_k}$. Fix $N \in (1, \infty)$ and let $k \in \mathbb{N}$ with $k < \lfloor N \rfloor$. For all $\eta \in (0, 1)$, there exists $\delta_k = \delta_k(\eta, N) > 0$ such that for all $\delta \in (0, \delta_k]$, the following holds: if $(X, \mathsf{d}, \mathfrak{m})$ and $(\overline{X}, \mathsf{d}_{\overline{X}}, \mathfrak{m}_{\overline{X}})$ are as in Theorem 4.1, then there exists $\overline{x}_{k,\eta} \in \overline{X}$ and a pointed $\mathsf{RCD}^*(0, N-k)$ space $(Y_{k,\eta}, \mathsf{d}_{Y_{k,\eta}}, m_{Y_{k,\eta}}, y_{k,\eta})$ such that

$$\mathsf{d}_{mGH}\big(B^{\overline{X}}_{\eta^{-1}}(\overline{x}_{k,\eta}), B^{\mathbb{R}^k\times Y_{k,\eta}}_{\eta^{-1}}(0^k, y_{k,\eta})\big) \le \eta.$$

In order to prove $\mathbf{A_{k+1}}$ given $\mathbf{A_k}$, we aim to apply Proposition 4.2 to the space $Y_{k,\eta}$: in this way, $Y_{k,\eta}$ will almost split a line, thus $\bar{X}$ will almost split an additional Euclidean factor, yielding $\mathbf{A_{k+1}}$. To this aim, the following diameter estimate will be key.

Lemma 4.4. *Assume that* $\mathbf{A_k}$ *holds. For any* $\eta \in (0,1)$*, let* $\delta_k(\eta, N) > 0$*,* $(X, \mathsf{d}, \mathfrak{m})$ *and* $(\bar{X}, \mathsf{d}_{\bar{X}}, \mathfrak{m}_{\bar{X}})$ *be as in* $\mathbf{A_k}$ *and* $(Y_{k,\eta}, \mathsf{d}_{Y_{k,\eta}}, m_{Y_{k,\eta}}, y_{k,\eta})$ *be the corresponding* $\mathsf{RCD}^*(0, N-k)$ *p.m.m.s. Then there exist* $c_N \in (0,1)$ *and* $\eta_0(N) > 0$ *such that for all* $\eta \in (0, \eta_0(N)]$ *and for all* $\delta \in (0, \delta_k(\eta, N)]$*, it holds that*

$$\operatorname{diam}\big(B^{Y_{k,\eta}}_{\eta^{-1}}(y_{k,\eta})\big) > c_N \eta^{-1}.$$

Proof. We argue by contradiction. Assume there exists a sequence $\eta_i \downarrow 0$, corresponding $\delta_i = \delta_k(\eta_i, N) \to 0$ and pointed $\mathsf{RCD}^*(-\delta_i^{2\beta}, N)$ spaces $(\bar{X}_i, \mathsf{d}_{\bar{X}_i}, \mathfrak{m}_{\bar{X}_i}, \bar{x}_i)$ for which there exists pointed $\mathsf{RCD}^*(0, N-k)$ spaces $(Y_i, \mathsf{d}_{Y_i}, \mathfrak{m}_{Y_i}, y_i)$ such that

$$\mathsf{d}_{mGH}\big(B^{\bar{X}_i}_{\eta_i^{-1}}(\bar{x}_i), B^{\mathbb{R}^k\times Y_i}_{\eta_i^{-1}}((0^k, y_i))\big) \le \eta_i \quad \text{and} \quad \lim_{i\to+\infty} \eta_i \operatorname{diam}\big(B^{Y_i}_{\eta_i^{-1}}(y_i)\big) = 0. \tag{4.5}$$

Let i be sufficiently large so that $\eta_i < 1$. By Corollary 3.3 we know that $B^{\bar{X}_i}_{\eta_i^{-1}}(\bar{x}_i)$ contains at least $(\lfloor \eta_i^{-1} \rfloor)^b$ disjoint balls of radius $1/2$, at positive mutual distance. Using (4.5) we infer that, for i large enough, the ball

$$B^{\mathbb{R}^k\times Y_i}_{\eta_i^{-1}}((0^k, y_i))$$

in $\mathbb{R}^k \times Y_i$ also contains at least $(\lfloor \eta_i^{-1} \rfloor)^b$ disjoint balls of radius $1/2$.

Rescale the metric of $\mathbb{R}^k \times Y_i$ by a factor η_i and denote the resulting space as $(\mathbb{R}^k \times Y_i)^{\eta_i}$. Then for large enough i the ball of radius 1 in $(\mathbb{R}^k \times Y_i)^{\eta_i}$ centred at $(0^k, y_i)$ contains at least $(\lfloor \eta_i^{-1} \rfloor)^b$ disjoint balls of radius $\eta_i/2$ at positive mutual distance. Furthermore, since

$$\eta_i \operatorname{diam}\big(B^{Y_i}_{\eta_i^{-1}}(y_i)\big)$$

tends to zero as i tends to infinity, when taking the Gromov–Hausdorff limit of such balls we obtain

$$\lim_{i\to\infty} \mathsf{d}_{GH}\big(B^{(\mathbb{R}^k\times Y_i)^{\eta_i}}_1(0^k, y_i), B^{(\mathbb{R}^k)^{\eta_i}}_1(0)\big) = 0.$$

As a consequence, for large enough i, $B^{(\mathbb{R}^k)^{\eta_i}}_1(0)$ contains at least $(\lfloor \eta_i^{-1} \rfloor)^b$ disjoint balls of radius $\eta_i/2$. Denote by ω_k the volume of $B^{\mathbb{R}^k}_1(0)$. Since we only rescaled the metric of $\mathbb{R}^k \times Y_i$ by a factor η_i, then the mass of $B^{(\mathbb{R}^k)^{\eta_i}}_1(0)$ equals $\omega_k(\eta_i^{-1})^k$ and the mass of a ball of radius $\eta_i/2$ in $B^{(\mathbb{R}^k)^{\eta_i}}_1(0)$ equals $\omega_k(1/2)^k$. Hence,

$$\omega_k\big(\lfloor \eta_i^{-1} \rfloor\big)^b (1/2)^k \le \omega_k(\eta_i^{-1})^k. \tag{4.6}$$

However, since $1 \le k < \lfloor N \rfloor = b$ and $\eta_i \to 0$, the estimate (4.6) cannot hold for i sufficiently large. □

Remark 4.4.1. Notice that we used the hypotheses

$$\mathrm{diam}(X) = 1 \quad \text{and} \quad b := \mathrm{b}_1(X) = \lfloor N \rfloor$$

to have a given number of disjoint balls of radius $1/2$ in a ball in $\bar{X}$ of radius larger than 1.

We next combine Lemma 4.4 and Proposition 4.2 in order to prove that the space $Y_{k,\eta}$ almost splits a line, for $\eta > 0$ small enough depending on $\varepsilon > 0$.

Proposition 4.5. *Assume that* $\mathbf{A_k}$ *is satisfied. Then for any* $\varepsilon \in (0,1)$, *there exists*

$$\eta(\varepsilon, N) > 0$$

such that the following holds. For any $\eta \in (0, \eta(\varepsilon, N)]$, *let* $(Y_{k,\eta}, \mathsf{d}_{Y_{k,\eta}}, \mathfrak{m}_{Y_{k,\eta}}, y_{k,\eta})$ *be the pointed m.m.s. given by* $\mathbf{A_k}$. *Then there exist* $y \in B^{Y_{k,\eta}}_{\eta^{-1}/2}(y_{k,\eta})$ *and a pointed* $\mathsf{RCD}^*(0, N-k-1)$ *space* $(Y', \mathsf{d}_{Y'}, \mathfrak{m}_{Y'}, y')$ *such that*

$$B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y) \subseteq B^{Y_{k,\eta}}_{\eta^{-1}}(y_{k,\eta}) \quad \textit{and} \quad \mathsf{d}_{mGH}\big(B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y), B^{\mathbb{R}\times Y'}_{\varepsilon^{-1}}(0, y')\big) \le \varepsilon.$$

Proof. Define

$$\eta(\varepsilon, N) = \min\Big\{\frac{\varepsilon}{2}, \eta_0(N), \frac{c_N}{2}\delta_1(\varepsilon, N)^{\beta}\Big\},$$

where $\delta_1(\varepsilon, N) > 0$ is the quantity given by Proposition 4.2 and $c_N, \eta_0(N)$ are defined in Lemma 4.4. Then by assumption $\mathbf{A_k}$ and Lemma 4.4, for any $\eta \in (0, \eta(\varepsilon, N)]$ and for all $\delta \in (0, \delta_k(\eta, N)]$, if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ space as in assumption $\mathbf{A_k}$, then there exist $\bar{x}_{k,\eta} \in \bar{X}$ and a pointed $\mathsf{RCD}^*(0, N-k)$ space $(Y_{k,\eta}, \mathsf{d}_{Y_{k,\eta}}, \mathfrak{m}_{Y_{k,\eta}}, y_{k,\eta})$ such that

$$\begin{aligned} \mathsf{d}_{mGH}\big(B^{\bar{X}}_{\eta^{-1}}(\bar{x}_{k,\eta}), B^{\mathbb{R}^k\times Y_{k,\eta}}_{\eta^{-1}}(0^k, y_{k,\eta})\big) &\le \eta \\ \mathrm{diam}\big(B^{Y_{k,\eta}}_{\eta^{-1}}(y_{k,\eta})\big) &> c_N\, \eta^{-1}. \end{aligned} \tag{4.7}$$

Let $\xi > 0$ be such that $c_N\, \eta^{-1} = 2\xi^{-\beta}$. Our choice of $\eta(\varepsilon, N)$ ensures that for any $\eta \in (0, \eta(\varepsilon, N)]$, we have $\xi \in (0, \delta_1(\varepsilon, N)]$. Therefore, we can apply Proposition 4.2 to $Y_{k,\eta}$ and get that there exist $y \in Y_{k,\eta}$ and a pointed $\mathsf{RCD}^*(0, N-k-1)$ space $(Y', \mathsf{d}_{Y'}, \mathfrak{m}_{Y'}, y')$ such that

$$\mathsf{d}_{mGH}\big(B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y), B^{\mathbb{R}\times Y'}_{\varepsilon^{-1}}(0, y')\big) \le \varepsilon.$$

It remains to show that $y \in B^{Y_{k,\eta}}_{\eta^{-1}/2}(y_{k,\eta})$ and that $B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y) \subseteq B^{Y_{k,\eta}}_{\eta^{-1}}(y_{k,\eta})$.

From the proof of Proposition 4.2, we know that y is a midpoint of a geodesic between two points p, q at distance equal to $c_N \eta^{-1}$. Since $Y_{k,\eta}$ is a geodesic space and $c_N \in (0,1)$, it is easily seen that (4.7) implies that there exists a point $q \in B^{Y_{k,\eta}}_{\eta^{-1}}(y_{k,\eta})$ such that

$$\mathsf{d}_{Y_{k,\eta}}(q, y_{k,\eta}) = c_N \eta^{-1}.$$

Then, in the proof of Proposition 4.2 we can chose $p = y_{k,\eta}$, $q \in Y_{k,\eta}$ with

$$\mathsf{d}_{Y_{k,\eta}}(q, y_{k,\eta}) = c_N \eta^{-1}$$

and y a midpoint of a geodesic between p and q. Therefore,

$$\mathsf{d}_{Y_{k,\eta}}(y, y_{k,\eta}) = c_N \eta^{-1}/2.$$

Now, for any point $z \in B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y)$, we have

$$\begin{aligned}\mathsf{d}_{Y_{k,\eta}}(z, y_{k,\eta}) &\le \mathsf{d}_{Y_{k,\eta}}(z, y) + \mathsf{d}_{Y_{k,\eta}}(y, y_{k,\eta}) \\ &< \varepsilon^{-1} + c_N \eta^{-1}/2.\end{aligned}$$

Moreover, our choice of $\eta \le \eta(\varepsilon, N) \le \varepsilon/2$ ensures that $\varepsilon^{-1} \le \eta^{-1}/2$. Therefore, for any $z \in B^{Y_{k,\eta}}_{\varepsilon^{-1}}(y)$, we have

$$\mathsf{d}_{Y_{k,\eta}}(z, y_{k,\eta}) < \eta^{-1},$$

as desired. □

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. We proceed by induction. For $k = 1$, $\mathbf{A_1}$ follows from Corollary 4.3. Now assume that $\mathbf{A_k}$ holds for some $1 \le k < \lfloor N \rfloor$ and let us show $\mathbf{A_{k+1}}$. Denote by $C_1, C_2 > 0$ the constants appearing in Propositions A.1 and A.2, respectively, and define

$$C := \max\{C_1 C_2, 2C_1\}.$$

Fix $\varepsilon \in (0,1)$ and let

$$\varepsilon_1 := \min\{1/2, 1/C\}\,\varepsilon, \quad \eta_1 := \min\{\varepsilon_1/4, \eta(\varepsilon_1, N)\},$$

where $\eta(\varepsilon_1, N) > 0$ is given by Proposition 4.5.

With these choices, if $\delta \in (0, \delta_k(\eta, N))$ and $(X, \mathsf{d}_X, \mathfrak{m}_X)$ is an $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ space that satisfies $\mathbf{A_k}$, then there exist $\bar{x}_{k,\eta_1} \in \bar{X}$, a pointed $\mathsf{RCD}^*(0, N-k)$ space $(Y_{k,\eta_1}, \mathsf{d}_{Y_{k,\eta_1}}, \mathfrak{m}_{Y_{k,\eta_1}}, y_{k,\eta_1})$ and an η_1-mGH approximation

$$\phi\colon B^{\bar{X}}_{\eta_1^{-1}}(\bar{x}_{k,\eta_1}) \to B^{\mathbb{R}^k \times Y_{k,\eta_1}}_{\eta_1^{-1}}\big(0^k, y_{k,\eta_1}\big).$$

Moreover, by Proposition 4.5, there exist

$$y \in B^{Y_{k,\eta_1}}_{\eta_1^{-1}/2}(y_{k,\eta_1}) \quad \text{with } B^{Y_{k,\eta_1}}_{\varepsilon_1^{-1}}(y) \subset B^{Y_{k,\eta_1}}_{\eta_1^{-1}}(y_{k,\eta_1}),$$

an $\mathsf{RCD}^*(0, N-k-1)$ space $(Y', \mathsf{d}_{Y'}, \mathfrak{m}_{Y'}, y')$ and an ε_1-mGH approximation

$$\varphi'\colon B^{Y_{k,\eta_1}}_{\varepsilon_1^{-1}}(y) \to B^{\mathbb{R}\times Y'}_{\varepsilon_1^{-1}}(0, y').$$

Since $\eta_1 \in (0, \varepsilon_1)$, the inclusion

$$B^{Y_{k,\eta_1}}_{\varepsilon_1^{-1}}(y) \subset B^{Y_{k,\eta_1}}_{\eta_1^{-1}}(y_{k,\eta_1})$$

ensures that

$$B^{\mathbb{R}^k \times Y_{k,\eta_1}}_{\varepsilon_1^{-1}}((0^k, y)) \subset B^{\mathbb{R}^k \times Y_{k,\eta_1}}_{\eta_1^{-1}}((0^k, y_{k,\eta_1})).$$

Therefore, there exists $\bar{x}_{k+1,\eta_1}$ in $B^{\bar{X}}_{\eta_1^{-1}}(\bar{x}_{k,\eta_1})$ such that

$$\mathsf{d}_{\mathbb{R}^k \times Y_{k,\eta_1}}\big((0^k, y), \phi(\bar{x}_{k+1,\eta_1})\big) \leq \eta_1.$$

We aim to show that

$$\mathsf{d}_{mGH}\big(B^{\bar{X}}_{\varepsilon^{-1}}(\bar{x}_{k+1,\eta_1}), B^{\mathbb{R}^{k+1} \times Y'}_{\varepsilon^{-1}}(0^{k+1}, y')\big) \leq \varepsilon. \tag{4.8}$$

We first claim that

$$B^{\bar{X}}_{\eta_1^{-1}/4}(\bar{x}_{k+1,\eta_1}) \subset B^{\bar{X}}_{\eta_1^{-1}}(\bar{x}_{k,\eta_1}). \tag{4.9}$$

Indeed, since ϕ is an η_1-mGH approximation, by the definition of $\bar{x}_{k+1,\eta_1}$ and y, we have

$$\begin{aligned}
\mathsf{d}_{\bar{X}}(\bar{x}_{k+1,\eta_1}, \bar{x}_{k,\eta_1}) &\leq \mathsf{d}_{\mathbb{R}^k \times Y_{k,\eta_1}}\big(\phi(\bar{x}_{k+1,\eta_1}), (0^k, y_{k,\eta_1})\big) + \eta_1 \\
&\leq \mathsf{d}_{\mathbb{R}^k \times Y_{k,\eta_1}}\big(\phi(\bar{x}_{k+1,\eta_1}), (0^k, y)\big) + \mathsf{d}_{\mathbb{R}^k \times Y_{k,\eta_1}}\big((0^k, y), (0^k, y_{k,\eta_1})\big) + \eta_1 \\
&\leq 2\eta_1 + \frac{1}{2}\eta_1^{-1}.
\end{aligned}$$

The claim (4.9) follows by the triangle inequality.

As a consequence, since $\varepsilon^{-1} \leq \eta_1^{-1}/4$, by Proposition A.1 we can construct a $(C_1\eta_1)$-mGH approximation out of ϕ as follows:

$$\phi_1 \colon B^{\bar{X}}_{\varepsilon^{-1}}(\bar{x}_{k+1,\eta_1}) \to B^{\mathbb{R}^k \times Y_{k,\eta_1}}_{\varepsilon^{-1}}((0^k, y)).$$

Thanks to Proposition A.2, there exists a $(C_2\varepsilon_1)$-mGH approximation

$$\varphi \colon B^{\mathbb{R}^k}_{\varepsilon_1^{-1}}(0^k) \times B^{Y_{k,\eta_1}}_{\varepsilon_1^{-1}}(y) \to B^{\mathbb{R}^k}_{\varepsilon_1^{-1}}(0^k) \times B^{\mathbb{R} \times Y'}_{\varepsilon_1^{-1}}((0, y')).$$

Since the ball centred at $(0^k, y)$ of radius $\varepsilon^{-1} \leq \varepsilon_1^{-1}/\sqrt{2}$ is included in the previous product of balls, we can use again Proposition A.1 to construct a $(C_1C_2\varepsilon_1)$-mGH approximation out of φ as follows:

$$\varphi_1 \colon B^{\mathbb{R}^k \times Y_{k,\eta_1}}_{\varepsilon^{-1}}((0^k, y)) \to B^{\mathbb{R}^{k+1} \times Y'}_{\varepsilon^{-1}}((0^{k+1}, y')).$$

The composition of φ_1 with ϕ_1 then gives a $(2C_1\eta_1 + C_1C_2\varepsilon_1)$-mGH approximation:

$$f = \varphi_1 \circ \phi_1 \colon B_{\varepsilon^{-1}}(\bar{x}_{k+1,\eta_1}) \to B^{\mathbb{R}^{k+1} \times Y'}_{\varepsilon^{-1}}((0^{k+1}, y')).$$

Thanks to our choices of C, ε_1 and η_1, the map f is an ε-mGH approximation and the claim (4.8) is proved.

Finally, set

$$Y_{k+1,\varepsilon} := Y' \quad \text{and} \quad y_{k+1,\varepsilon} := y'.$$

We have proved that given $\mathbf{A_k}$, for any $\varepsilon \in (0, \varepsilon(N)]$, there exists

$$\delta_{k+1} := \delta_k(\varepsilon, N)$$

such that for any $\delta \in (0, \delta_{k+1}(\varepsilon, N))$ and any $\mathsf{RCD}^*(-\delta^{-2\beta}, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with

$$\operatorname{diam}(X) = 1 \quad \text{and} \quad \mathrm{b}_1(X) = \lfloor N \rfloor,$$

there exist $\bar{x}_{k+1,\varepsilon} \in \bar{X}$ and a pointed $\mathsf{RCD}^*(0, N-k-1)$ space

$$\big(Y_{k+1,\varepsilon}, \mathsf{d}_{Y_{k+1,\varepsilon}}, \mathfrak{m}_{Y_{k+1,\varepsilon}}, y_{k+1,\varepsilon}\big)$$

such that

$$\mathsf{d}_{mGH}\big(B^{\bar{X}}_{\varepsilon^{-1}}(\bar{x}_{k+1,\varepsilon}), B^{\mathbb{R}^{k+1}\times Y_{k+1,\varepsilon}}_{\varepsilon^{-1}}((0^{k+1}, y_{k+1,\varepsilon}))\big) \le \varepsilon.$$

This shows that for any integer $0 < k < \lfloor N \rfloor$, $\mathbf{A_k}$ implies $\mathbf{A_{k+1}}$. □

5. Proof of Theorem 1.2, first claim

In this section we prove the first part of the main Theorem 1.2, by combining Theorem 4.1 with the structure theory of $\mathsf{RCD}^*(K, N)$ spaces [6, 28, 35, 38]. More precisely we show the following result, which in turn immediately implies the first claim of Theorem 1.2 by a standard scaling argument.

Theorem 5.1. *For any $\varepsilon \in (0,1)$ and $N \in (1, \infty)$, there exists*

$$\delta(\varepsilon, N) > 0$$

such that for all $\delta \in (0, \delta(\varepsilon, N)]$, any $\mathsf{RCD}^*(-\delta, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with

$$\mathrm{b}_1(X) = \lfloor N \rfloor \quad \textit{and} \quad \operatorname{diam}(X) = 1$$

has essential dimension equal to $\lfloor N \rfloor$ and it is $\lfloor N \rfloor$-rectifiable as a metric measure space. Moreover, if $N \in \mathbb{N}$, there exists $c > 0$ such that $\mathfrak{m} = c\mathscr{H}^{\lfloor N \rfloor}$.

In [38, Theorem 6.8], the authors proved that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(-\delta, N)$ space and a ball of radius δ^{-1} is δ-mGH-close to a Euclidean ball of the same radius in $\mathbb{R}^{\lfloor N \rfloor}$, then there exists a subset (of large measure) U_ε of the unit ball which is $(1+\varepsilon)$ bi-Lipschitz to a subset of $\mathbb{R}^{\lfloor N \rfloor}$. In order to construct U_ε and the bi-Lipschitz map into $\mathbb{R}^{\lfloor N \rfloor}$, they showed the existence of a function u on the unit ball which, restricted to any ball $B_s(x)$ centred at a point x of U_ε, is an (εs)-mGH isometry. We summarise these results in the following statement.

Theorem 5.2 ([38, Theorem 6.8]). *For every $N \in (1, \infty)$, there exists*

$$\delta_0 = \delta_0(N) > 0$$

with the following property. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(-\delta, N)$ space for some $\delta \in [0, \delta_0)$ and assume that for some $x_0 \in X$ it holds that

$$\mathsf{d}_{mGH}\big(B^X_{\delta^{-1}}(x_0), B^{\mathbb{R}^{\lfloor N \rfloor}}_{\delta^{-1}}(0^{\lfloor N \rfloor})\big) \le \delta.$$

Then there exists a Borel subset $U_\varepsilon \subset B_1(\bar{x})$ such that

(1) $\mathfrak{m}(B_1(x_0) \setminus U_\varepsilon) \le \varepsilon$;

(2) *U_ε is $(1+\varepsilon)$ bi-Lipschitz to a subset of $\mathbb{R}^{\lfloor N \rfloor}$;*

(3) *For all $x \in U_\varepsilon$ and for all $r \in (0, 1]$ such that $B^X_r(x) \subset B^X_1(x)$, we have*

$$\mathsf{d}_{mGH}\big(B^X_r(x), B^{\mathbb{R}^{\lfloor N \rfloor}}_r(0^{\lfloor N \rfloor})\big) \le \varepsilon r.$$

In particular, for any $x \in U_\varepsilon$ and for any tangent cone $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ at x, we have

$$\mathsf{d}_{mGH}\big(B^Y_1(x), B^{\mathbb{R}^{\lfloor N \rfloor}}_1(0^{\lfloor N \rfloor})\big) \le \varepsilon.$$

The third property is contained in the proof of [38, Theorem 6.8]. Thanks to the constancy of the dimension of $\mathsf{RCD}^*(K, N)$ spaces proved by Brué–Semola [6], the following holds.

Corollary 5.3. *For every $N \in (1, \infty)$, there exists*

$$\delta_0 = \delta_0(N) > 0$$

with the following property. Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}^*(-\delta, N)$ space for some $\delta \in [0, \delta_0)$ and assume that for some $x_0 \in X$, the following holds:

$$\mathsf{d}_{GH}\big(B^X_{\delta^{-1}}(x_0), B^{\mathbb{R}^{\lfloor N \rfloor}}_{\delta^{-1}}(0^{\lfloor N \rfloor})\big) \le \delta. \tag{5.1}$$

Then the essential dimension of X is equal to $\lfloor N \rfloor$ and $(X, \mathsf{d}, \mathfrak{m})$ is $\lfloor N \rfloor$-rectifiable as a metric measure space.

Proof. By the definition of the dimension of RCD spaces, we know that there exists a unique $n \in \mathbb{N}$, with $n \le \lfloor N \rfloor$, such that the n-th regular stratum $\mathcal{R}_n$ has positive measure. Therefore, by definition of $\mathcal{R}_n$ for $\mathfrak{m}$-a.e. $x \in X$, tangent cones at x are unique and equal to the Euclidean space $(\mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, \mathcal{L}^n)$. Now assume by contradiction that $n < \lfloor N \rfloor$. Because of Theorem 5.2, (5.1) implies the existence of a set U_ε satisfying properties (1) to (3), with $\mathfrak{m}(U_\varepsilon) > 0$. As a consequence, there exists $x \in U_\varepsilon$ with unique tangent cone equal to $\mathbb{R}^n$. Property (3) then implies that the unit ball in $\mathbb{R}^n$ is ε-GH-close to the unit ball in $\mathbb{R}^{\lfloor N \rfloor}$, which is impossible for $n < \lfloor N \rfloor$ and $\varepsilon > 0$ sufficiently small. Therefore, $\lfloor N \rfloor$ is the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$ and $(X, \mathsf{d}, \mathfrak{m})$ is $\lfloor N \rfloor$-rectifiable as a metric measure space. □

The combination of Corollary 5.3 and Theorem 4.1 yields the following result.

Corollary 5.4. *For any $\varepsilon \in (0,1)$ and $N \in (1,\infty)$, there exists*

$$\delta(\varepsilon, N) > 0$$

with the following property. If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}^*(-\delta, N)$ space with

$$\mathrm{b}_1(X) = \lfloor N \rfloor \quad \textit{and} \quad \mathrm{diam}(X) = 1,$$

then the covering space $(\bar{X}, \mathsf{d}_{\bar{X}}, \mathfrak{m}_{\bar{X}})$ has essential dimension equal to $\lfloor N \rfloor$ and it is $\lfloor N \rfloor$-rectifiable as a metric measure space.

Proof. Fix $\varepsilon \in (0,1)$ and let $\beta > 0$ be as in Theorem 4.1. Let $\eta(\varepsilon, N)$ be given in Corollary 5.3 and set $\varepsilon_1 = \eta(\varepsilon, N)$. Then by Theorem 4.1, there exists

$$\delta_1(\varepsilon_1, N) > 0$$

such that for any $\delta \in (0, \delta_1(\varepsilon_1, N)]$ and for any $\mathsf{RCD}^*(-\delta^{2\beta}, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ with $\mathrm{b}_1(X) = \lfloor N \rfloor$ and $\mathrm{diam}(X) = 1$, there exists $\bar{x} \in \bar{X}$ such that

$$\mathsf{d}_{mGH}\big(B^{\bar{X}}_{\varepsilon_1^{-1}}(\bar{x}), B^{\mathbb{R}^{\lfloor N \rfloor}}_{\varepsilon_1^{-1}}(0^{\lfloor N \rfloor})\big) \le \varepsilon_1.$$

As a consequence, $(\bar{X}, \mathsf{d}_{\bar{X}}, \mathfrak{m}_{\bar{X}})$ satisfies the assumptions of Corollary 5.3, thus it has essential dimension equal to $\lfloor N \rfloor$ and it is $\lfloor N \rfloor$-rectifiable as a metric measure space. It suffices then to choose $\delta(\varepsilon, N) = \delta_1(\varepsilon_1, N)^{1/2\beta}$. □

We are now in position to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\bar{p}\colon \bar{X} \to X$ be the covering map and denote by $\mathcal{R}_{\lfloor N \rfloor}(\bar{X})$ the $\lfloor N \rfloor$-th regular set of $\bar{X}$. Recall that

$$\mathfrak{m}_{\bar{X}}\big(\bar{X} \setminus \mathcal{R}_{\lfloor N \rfloor}(\bar{X})\big) = 0.$$

Let $B^{\bar{X}}_r(\bar{x})$ be a sufficiently small ball in $\bar{X}$ such that

$$\bar{p}|_{B^{\bar{X}}_r(\bar{x})}\colon B^{\bar{X}}_r(\bar{x}) \to B^X_r\big(\bar{p}(\bar{x})\big)$$

is an isomorphism of metric measure spaces. Since for $\mathfrak{m}_{\bar{X}}$-a.e. $\bar{x}' \in B^{\bar{X}}_r(\bar{x})$ the tangent cone is unique and equal to $\mathbb{R}^{\lfloor N \rfloor}$, the same is true for $\mathfrak{m}$-a.e. $x' \in B^X_r(\bar{p}(\bar{x}))$, and thus the regular set $\mathcal{R}^{\lfloor N \rfloor}$ of X has positive $\mathfrak{m}$-measure. Therefore, $(X, \mathsf{d}, \mathfrak{m})$ has essential dimension equal to $\lfloor N \rfloor$ and it is $\lfloor N \rfloor$-rectifiable as a metric measure space. In particular,

$$\mathfrak{m} \ll \mathcal{H}^{\lfloor N \rfloor}.$$

If N is an integer, then $\mathfrak{m} \ll \mathcal{H}^N$ and $(X, \mathsf{d}, \mathfrak{m})$ is a compact weakly non-collapsed $\mathsf{RCD}^*(-\delta, N)$ space. Corollary 1.3 in [33] ensures that for any compact weakly non-collapsed $\mathsf{RCD}^*(-\delta, N)$ space, there exists $c > 0$ such that

$$\mathfrak{m} = c\mathcal{H}^N,$$

thus concluding the proof. □

6. Proof of Theorem 1.2, second and third claims

Now we are in position to conclude the proof of Theorem 1.2. Given a sequence of $\mathsf{RCD}^*(-K_i, N)$ spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ with $K_i < 0$ tending to zero, $\operatorname{diam}(X_i) = 1$ and $\mathrm{b}_1(X_i) = \lfloor N \rfloor$, the proof consists in applying the results of equivariant pointed Gromov–Hausdorff convergence as in Section 2.2 to the sequence $(\bar{X}_i, \mathsf{d}_i, \bar{x}_i)$ and subgroups Γ_i' as in Lemma 3.2, in order to obtain equivariant convergence (up to a subsequence) to $(\mathbb{R}^b, \mathsf{d}_{\mathbb{R}^b}, 0, \mathbb{Z}^b)$. Then we will conclude that the quotients $\bar{X}_i/\Gamma_i'$ mGH-converge to a flat torus, which, by applying Theorem 2.18, will imply that for large i the quotients are bi-Hölder homeomorphic to this torus. In the last step we show that

$$\bar{X}_i/\Gamma_i' = X_i.$$

We start with the following lemma.

Lemma 6.1. *Let $(X_i, \mathsf{d}_i, x_i, \Gamma_i) \in \mathcal{M}^p_{eq}$ be a sequence of spaces that converge in the equivariant pGH sense to $(X_\infty, \mathsf{d}_\infty, x_\infty, \Gamma_\infty) \in \mathcal{M}^p_{eq}$. Assume Γ_i is an abelian group for each $i \in \mathbb{N}$. Then Γ_∞ is an abelian group as well.*

Proof. Given arbitrary $\gamma_{\infty 1}, \gamma_{\infty 2} \in \Gamma_\infty$, we will show that they commute. By hypothesis, there exist ε_i-equivariant pGH approximations (f_i, ϕ_i, ψ_i):

$$f_i \colon B^{X_\infty}_{\varepsilon_i^{-1}}(x_\infty) \to X_i, \quad \phi_i \colon \Gamma_\infty(\varepsilon_i^{-1}) \to \Gamma_i, \quad \psi_i \colon \Gamma_i(\varepsilon_i^{-1}) \to \Gamma_\infty,$$

satisfying the conditions of Definition 2.2 and so that $\varepsilon_i \to 0$.

Take an arbitrary point $z_\infty \in X_\infty$. By the triangle inequality and for i large enough such that $z_\infty, \gamma_{\infty 1} z_\infty, \gamma_{\infty 1}\gamma_{\infty 2} z_\infty \in B^{X_\infty}_{\varepsilon_i^{-1}}(x_\infty)$ and $\gamma_{\infty 1}\gamma_{\infty 2} \in \Gamma_\infty(\varepsilon_i^{-1})$, we get

$$\begin{aligned} &\mathsf{d}_i\big(f_i(\gamma_{\infty 1}\gamma_{\infty 2} z_\infty), \phi_i(\gamma_{\infty 1})\phi_i(\gamma_{\infty 2}) f_i(z_\infty)\big) \\ &\quad \le \mathsf{d}_i\big(f_i(\gamma_{\infty 1}\gamma_{\infty 2} z_\infty), \phi_i(\gamma_{\infty 1}) f_i(\gamma_{\infty 2} z_\infty)\big) \\ &\qquad + \mathsf{d}_i\big(\phi_i(\gamma_{\infty 1}) f_i(\gamma_{\infty 2} z_\infty), \phi_i(\gamma_{\infty 1})\phi_i(\gamma_{\infty 2}) f_i(z_\infty)\big). \end{aligned}$$

Applying (4) of Definition 2.2 and that $\phi_i(\gamma_{\infty 1})$ is an isometry, we see that each term in the right-hand side of the previous inequality is bounded above by ε_i. We conclude that

$$\mathsf{d}_i\big(f_i(\gamma_{\infty 1}\gamma_{\infty 2} z_\infty), \phi_i(\gamma_{\infty 1})\phi_i(\gamma_{\infty 2}) f_i(z_\infty)\big) \le 2\varepsilon_i.$$

The same estimate holds reversing the roles of $\gamma_{\infty 1}$ and $\gamma_{\infty 2}$, that is:

$$\mathsf{d}_i\big(f_i(\gamma_{\infty 2}\gamma_{\infty 1} z_\infty), \phi_i(\gamma_{\infty 2})\phi_i(\gamma_{\infty 1}) f_i(z_\infty)\big) \le 2\varepsilon_i.$$

By the triangle inequality and using that Γ_i is abelian, so that

$$\phi_i(\gamma_{\infty 2})\phi_i(\gamma_{\infty 1}) = \phi_i(\gamma_{\infty 1})\phi_i(\gamma_{\infty 2}),$$

we get

$$\mathsf{d}_i\big(f_i(\gamma_{\infty 1}\gamma_{\infty 2}z_\infty), f_i(\gamma_{\infty 2}\gamma_{\infty 1}z_\infty)\big) \le 4\varepsilon_i.$$

From (3) of Definition 2.2, we also have

$$\big|\mathsf{d}_\infty\big(\gamma_{\infty 1}\gamma_{\infty 2}z_\infty, \gamma_{\infty 2}\gamma_{\infty 1}z_\infty\big) - \mathsf{d}_i\big(f_i(\gamma_{\infty 1}\gamma_{\infty 2}z_\infty), f_i(\gamma_{\infty 2}\gamma_{\infty 1}z_\infty)\big)\big| < \varepsilon_i.$$

Therefore, when taking the limit as $i \to \infty$, we obtain

$$\mathsf{d}_\infty\big(\gamma_{\infty 1}\gamma_{\infty 2}z_\infty, \gamma_{\infty 2}\gamma_{\infty 1}z_\infty\big) = 0.$$

Since $z_\infty \in X_\infty$ is an arbitrary point, we conclude that $\gamma_{\infty 1}$ and $\gamma_{\infty 2}$ commute. □

We are now ready to prove the key result of this section, which directly gives the second claim of Theorem 1.2 by a standard compactness/contradiction argument.

Proposition 6.2. *Let $N \in (1,\infty)$ and let $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ be a sequence of $\mathsf{RCD}^*(-K_i, N)$ spaces with*

$$\mathrm{b}_1(X_i) = \lfloor N \rfloor, \quad \mathrm{diam}(X_i) = 1,$$

and $K_i > 0$ such that $K_i \downarrow 0$. Fix some $\bar{x}_i \in \bar{X}_i$ and let Γ_i' be as in Lemma 3.2, for $k = 3$. Then any Gromov–Hausdorff limit of $X_i' = \bar{X}_i / \Gamma_i'$ is isometric to an $\lfloor N \rfloor$-dimensional flat torus.

Remark 6.2.1. In Proposition 6.2 we require $\mathrm{diam}(X_i) = 1$ instead of the bound $K_i \,\mathrm{diam}(X_i)^2 \downarrow 0$. To show that the latter condition is *not* enough, consider a sequence X_i of manifolds with

$$K_i = i \quad \text{and} \quad \mathrm{diam}(X_i) = i^{-1}.$$

Then $K_i \,\mathrm{diam}(X_i)^2 \downarrow 0$, but any GH limit of this sequence collapses due to $\mathrm{diam}(X_i) \to 0$. We could also consider manifolds X_i with

$$K_i = i^{-3} \quad \text{and} \quad \mathrm{diam}(X_i) = i,$$

then $K_i \,\mathrm{diam}(X_i)^2 \downarrow 0$ and any GH-converging subsequence has a limit space with infinite diameter. Hence, it is necessary to have two sided uniform bounds on $\mathrm{diam}(X_i)$ and for simplicity we set them equal to 1.

Proof of Proposition 6.2. Set $b := \lfloor N \rfloor = \mathrm{b}_1(X_i)$. For simplicity of notation, we will not relabel subsequences. By Theorem 4.1 and Remark 4.1.1, the sequence $(\bar{X}_i, \mathsf{d}_{\bar{X}_i}, \bar{x}_i)$ converges in the pointed Gromov–Hausdorff sense to

$$\big(\mathbb{R}^b, \mathsf{d}_{\mathbb{R}^b}, 0^b\big).$$

By Gromov's Compactness theorem and stability of the $\mathsf{RCD}^*(0, N)$ condition, there exists an $\mathsf{RCD}^*(0, N)$ space $(X, \mathsf{d}_X, \mathfrak{m}_X)$ with

$$\mathrm{diam}(X) = 1$$

such that $X_i \to X$ in the mGH sense, up to a subsequence. From Remark 3.2.3, we know that for any $i \in \mathbb{N}$, the groups Γ_i' given by Lemma 3.2 are closed. Thus, by Theorem 2.4, there exist a group of isometries of $\mathbb{R}^b$, Γ_∞', and a subsequence $(\bar{X}_i, \mathsf{d}_{\bar{X}_i}, \bar{x}_i, \Gamma_i')$ that converges in the equivariant pointed Gromov–Hausdorff sense to

$$\big(\mathbb{R}^b, \mathsf{d}_{\mathbb{R}^b}, 0^b, \Gamma_\infty'\big).$$

Moreover, $\mathbb{R}^b$ is the universal cover of X, and Γ_∞' is contained in the corresponding group of deck transformations.

We will show that $\mathbb{R}^b/\Gamma_\infty'$ is a flat torus. To this aim, we prove that Γ_∞' is isomorphic to $\mathbb{Z}^b$.

Step 1. We claim that

$$\mathsf{d}_{\mathbb{R}^b}(\gamma_\infty y_\infty, y_\infty) \geq 1 \quad \text{for all } y_\infty \in \mathbb{R}^b \text{ and for all } \gamma_\infty \in \Gamma_\infty',\ \gamma_\infty \neq \mathrm{id}. \tag{6.1}$$

Let (f_i, ϕ_i, ψ_i) be equivariant ε_i-pGH approximations, $\varepsilon_i \to 0$ as in Definition 2.2:

$$f_i \colon B^{\mathbb{R}^b}_{\varepsilon_i^{-1}}(0^b) \to \bar{X}_i, \quad \phi_i \colon \Gamma_\infty'(\varepsilon_i^{-1}) \to \Gamma_i' \quad \psi_i \colon \Gamma_i'(\varepsilon_i^{-1}) \to \Gamma_\infty'.$$

To prove (6.1), we first show that the claim holds for all non-trivial $\gamma_i \in \Gamma_i'$ and all $y_i \in \bar{X}_i$, $i \in \mathbb{N}$. Then a convergence argument will show that the claim holds.

Since $\operatorname{diam}(X_i) = 1$ for all $i \in \mathbb{N}$ and $y_i \in \bar{X}_i$, there exists $\gamma \in \Gamma_i'$ such that

$$\mathsf{d}_{\bar{X}_i}(\gamma \bar{x}_i, y_i) \leq 1.$$

Moreover, by Lemma 3.2 for any $\gamma' \in \Gamma_i' \setminus \{\mathrm{id}\}$, we have

$$3 < \mathsf{d}_{\bar{X}_i}(\gamma' \bar{x}_i, \bar{x}_i).$$

Then, by the triangle inequality, we have

$$\begin{aligned} 3 &< \mathsf{d}_{\bar{X}_i}(\gamma' \bar{x}_i, \bar{x}_i) \\ &= \mathsf{d}_{\bar{X}_i}(\gamma' \gamma \bar{x}_i, \gamma \bar{x}_i) \\ &\leq \mathsf{d}_{\bar{X}_i}(\gamma' \gamma \bar{x}_i, \gamma' y_i) + \mathsf{d}_{\bar{X}_i}(\gamma' y_i, y_i) + \mathsf{d}_{\bar{X}_i}(y_i, \gamma \bar{x}_i) \\ &\leq 2 + \mathsf{d}_{\bar{X}_i}(\gamma' y_i, y_i). \end{aligned}$$

Therefore,

$$\mathsf{d}_{\bar{X}_i}(\gamma' y_i, y_i) > 1 \quad \text{for all } \gamma' \in \Gamma_i' \setminus \{\mathrm{id}\} \text{ and } y_i \in \bar{X}_i. \tag{6.2}$$

Now let $\gamma_\infty \in \Gamma_\infty' \setminus \{\mathrm{id}\}$ and $y_\infty \in \mathbb{R}^b$. For i large enough, $\gamma_\infty y_\infty, y_\infty \in \Gamma_\infty'(\varepsilon_i^{-1})$ and then by (3) of Definition 2.2 it holds that

$$\mathsf{d}_{\mathbb{R}^b}(\gamma_\infty y_\infty, y_\infty) > -\varepsilon_i + \mathsf{d}_{\bar{X}_i}\big(f_i(\gamma_\infty y_\infty), f_i(y_\infty)\big). \tag{6.3}$$

By (4) of Definition 2.2, we also have

$$\mathsf{d}_{\bar{X}_i}\big(f_i(\gamma_\infty y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big) < \varepsilon_i. \tag{6.4}$$

Combining (6.3), the triangle inequality and (6.4), we get

$$\begin{aligned}
&\mathsf{d}_{\mathbb{R}^b}(\gamma_\infty y_\infty, y_\infty)\\
&\quad > -\varepsilon_i + \mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big) - \mathsf{d}_{\bar{X}_i}\big(f_i(\gamma_\infty y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big)\\
&\quad > \mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big) - 2\varepsilon_i.
\end{aligned} \tag{6.5}$$

If we show that $\phi_i(\gamma_\infty) \neq \mathrm{id}$, then we have that

$$\mathsf{d}_{\bar{X}_i}\big(\phi_i(\gamma_\infty) f_i(y_\infty), f_i(y_\infty)\big) > 1$$

and by passing to the limit we will be able to conclude the proof of the claim. We are going to prove that

$$\mathsf{d}_{\bar{X}_i}\big(\phi_i(\gamma_\infty) f_i(y_\infty), f_i(y_\infty)\big) > 0,$$

so that $\phi_i(\gamma_\infty) \neq \mathrm{id}$. By the triangle inequality, arguing as in (6.3) and using (6.4), we get

$$\begin{aligned}
&\mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big)\\
&\qquad \geq \mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), f_i(\gamma_\infty y_\infty)\big) - \mathsf{d}_{\bar{X}_i}\big(f_i(\gamma_\infty y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big)\\
&\qquad \geq \mathsf{d}_{\mathbb{R}^b}(y_\infty, \gamma_\infty y_\infty) - 2\varepsilon_i.
\end{aligned} \tag{6.6}$$

Since by hypothesis γ_∞ is a non-trivial isometry and elements in the deck transformations do not fix points, we have

$$\mathsf{d}_{\mathbb{R}^b}(y_\infty, \gamma_\infty y_\infty) > 0.$$

Thus, by (6.6) for sufficiently large i, we have

$$\mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big) > 0.$$

This shows that $\phi_i(\gamma_\infty)$ is non-trivial, and thus

$$\mathsf{d}_{\bar{X}_i}\big(f_i(y_\infty), \phi_i(\gamma_\infty) f_i(y_\infty)\big) > 1.$$

Therefore, as $i \to \infty$, inequality (6.5) implies the claim (6.1).

Step 2. We show that $\Gamma'_\infty \cong \mathbb{Z}^b$.

From Lemma 3.2, we know that

$$\Gamma'_i \cong \mathbb{Z}^b.$$

Let $\{\gamma_{ij}\}_{j=1}^b$ be a set of generators for Γ'_i.

By the Arzelá–Ascoli theorem, there exists a subsequence $(\bar{X}_{i_k}, \mathsf{d}_{\bar{X}_{i_k}}, \bar{x}_{i_k}, \Gamma'_{i_k})$ and corresponding subsequences of isometries

$$\{\gamma_{i_k 1}\}_{k=1}^{\infty}, \dots, \{\gamma_{i_k b}\}_{k=1}^{\infty}$$

that converge to $\gamma_{\infty 1}, \dots, \gamma_{\infty b} \in \Gamma'_{\infty}$, respectively. We are going to show that $\{\gamma_{\infty j}\}_{j=1}^{b}$ are independent generators of Γ'_{∞} and that they have infinite order.

To simplify notation consider that the whole sequence converges. Given $\gamma_\infty \in \Gamma'_\infty$, notice that $\phi_i(\gamma_\infty) \to \gamma_\infty$ in the Arzelá–Ascoli sense. Indeed, for all $z \in \mathbb{R}^b$ and $z_i \in \bar{X}_i$ such that $\mathsf{d}_{\bar{X}_i}(f_i(z), z_i) \to 0$, by using the triangle inequality and (4) in Definition 2.2, and since $\phi_i(\gamma_\infty)$ is an isometry, we have

$$\begin{aligned} &\mathsf{d}_{\bar{X}_i}\big(\phi_i(\gamma_\infty)z_i, f_i(\gamma_\infty z)\big) \\ &\quad \le \mathsf{d}_{\bar{X}_i}\big(\phi_i(\gamma_\infty)z_i, \phi_i(\gamma_\infty) f_i(z)\big) + \mathsf{d}_{\bar{X}_i}\big(\phi_i(\gamma_\infty) f_i(z), f_i(\gamma_\infty z)\big) \\ &\quad \le \mathsf{d}_{\bar{X}_i}\big(z_i, f_i(z)\big) + \varepsilon_i \to 0. \end{aligned}$$

Moreover, for any $\gamma_\infty \in \Gamma'_\infty$, there exist $s_1, \dots s_b \in \mathbb{Z}$ such that

$$\phi_i(\gamma_\infty) = \gamma_{i1}^{s_1} \cdots \gamma_{ib}^{s_b}.$$

Then we know that the left-hand side of the previous equation converges to γ_∞, while the right-hand side converges to $\gamma_{\infty 1}^{s_1} \cdots \gamma_{\infty b}^{s_b}$. Thus, any $\gamma_\infty \in \Gamma'_\infty$ can be written as a composition of elements in $\{\gamma_{\infty j}\}_{j=1}^{b}$.

We next show that $\{\gamma_{\infty j}\}_{j=1}^{b}$ are independent and have infinite order. Let $(s_1, \dots, s_b) \in \mathbb{Z}^b \setminus \{(0, \dots, 0)\}$. We claim that

$$\gamma_{\infty 1}^{s_1} \cdots \gamma_{\infty b}^{s_b} \neq \mathrm{id}.$$

From the previous arguments, we know that

$$\gamma_{i1}^{s_1} \cdots \gamma_{ib}^{s_b} \to \gamma_{\infty 1}^{s_1} \cdots \gamma_{\infty b}^{s_b} \quad \text{as } i \to \infty.$$

Since $\{\gamma_{ij}\}_{j=1}^{b}$ are independent generators of $\Gamma'_i \cong \mathbb{Z}^b$, we have that $\gamma_{i1}^{s_1} \cdots \gamma_{ib}^{s_b} \neq \mathrm{id}$. Hence, from (6.2) it follows that

$$1 < \mathsf{d}_{\bar{X}_i}\big(\gamma_{i1}^{s_1} \cdots \gamma_{ib}^{s_b} f_i(z), f_i(z)\big) \to \mathsf{d}_{\mathbb{R}^b}\big(\gamma_{\infty 1}^{s_1} \cdots \gamma_{\infty b}^{s_b} z, z\big) \quad \text{for all } z \in \mathbb{R}^b,$$

and thus $\gamma_{\infty 1}^{s_1} \cdots \gamma_{\infty b}^{s_b} \neq \mathrm{id}$.

In conclusion, by the fundamental theorem of finitely generated abelian groups, we infer that

$$\Gamma'_\infty \cong \mathbb{Z}^b.$$

Thus, $\mathbb{R}^b / \Gamma'_\infty$ is a b-dimensional flat torus. The proposition follows by Theorem 2.5. □

Corollary 6.3. *For all $N \in \mathbb{N}$, $N > 1$, there exists*

$$\varepsilon(N) > 0$$

with the following property. Let $(X, \mathsf{d}, \mathcal{H}^N)$ be a compact $\mathsf{RCD}^*(K, N)$ space with

$$K \operatorname{diam}^2(X) > -\varepsilon(N) \quad \text{and} \quad \mathrm{b}_1(X) = N.$$

Then $X' := \overline{X}/\Gamma'$ is bi-Hölder homeomorphic to an N-dimensional flat torus, where Γ' is given by Lemma 3.2 for $k = 3$.

Proof. Suppose by contradiction that there is no such $\varepsilon(N) > 0$. Then there exists a sequence of compact $\mathsf{RCD}^*(K, N)$ spaces $(X_i, \mathsf{d}_i, \mathcal{H}^N)$ with

$$K_i \operatorname{diam}^2(X_i) > -\varepsilon_i, \quad \mathrm{b}_1(X_i) = N,$$

and $\varepsilon_i \to 0$ such that none of the X_i is bi-Hölder homeomorphic to a flat torus of dimension N. Consider the rescaled spaces

$$\big(X_i', \mathsf{d}_i', \mathcal{H}^N\big) := \big(X_i, \operatorname{diam}(X_i)^{-1}\mathsf{d}_i, \mathcal{H}^N\big).$$

Clearly, X_i' has diameter equal to 1 and it is an $\mathsf{RCD}^*(K_i \operatorname{diam}^2(X_i, \mathsf{d}_i), N)$ space with

$$\mathrm{b}_1(X_i') = N.$$

Thus, we can apply Proposition 6.2 and infer that any GH limit is a flat torus $\mathbb{T}^N$.

Moreover, from Theorem 2.17 (i), we have that $(X_i', \mathsf{d}_i', \mathcal{H}^N)$ converges in the mGH sense to $(\mathbb{T}^N, \mathsf{d}_{\mathbb{T}^N}, \mathcal{H}^N)$. For i large enough so that

$$\mathsf{d}_{mGH}\big(X_i', \mathbb{T}^N\big) \le \varepsilon\big(\mathbb{T}^N\big),$$

we can apply Theorem 2.18 and get that X_i' is bi-Hölder homeomorphic to $\mathbb{T}^N$. When scaling back to the original metric, the same conclusion holds. This is a contradiction. □

We can now conclude the proof of the main theorem.

Proof of the third claim of Theorem 1.2, i.e. when $N \in \mathbb{N}$. If $N = 1$, the claim holds trivially (see Remark 2.7.1); thus, we can assume $N \ge 2$ without loss of generality.

From Corollary 6.3, we know that $(\overline{X}, \mathsf{d}_{\overline{X}})$ is locally (on arbitrarily large compact subsets) bi-Hölder homeomorphic to $\mathbb{R}^N$ (thus, in particular, it has the integral homology of a point) and $\mathfrak{m}_{\overline{X}}$ is a constant multiple of the N-dimensional Hausdorff measure $\mathcal{H}^N$. By construction, we also know that the abelianised revised fundamental group

$$\Gamma := \overline{\pi}_1(X)/H$$

acts by deck transformations on $\bar{X} := \tilde{X}/H$ and that $X = \bar{X}/\Gamma$. Thus, summarising, we have that

$$(\bar{X}, \mathsf{d}_{\bar{X}}) \text{ is a topological manifold with the integral homology of a point and the action of } \Gamma \text{ on } \bar{X} \text{ has no fixed points.} \tag{6.7}$$

In order to prove that (X, d) is bi-Hölder homeomorphic to a flat torus and that $\mathfrak{m}$ is a constant multiple of $\mathcal{H}^N$, it is enough to prove that $\Gamma \cong \mathbb{Z}^N$. Since Γ is a finitely generated abelian group (recall Proposition 2.25), it is sufficient to show that Γ has no subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with p prime. This follows from (6.7): indeed, from Smith theory (see, for instance, [5, Chapter 3]), if $\mathbb{Z}/p\mathbb{Z}$, with p prime, acts on a topological manifold with the mod p homology of a point then the set of fixed points is non-empty. □

A. Some basic properties of mGH approximations

For the reader's convenience, in this appendix we recall some well-known properties of mGH approximations used in the paper.

Proposition A.1 (Restriction of mGH approximations). *Fix $K \in \mathbb{R}$, $N \in (1, \infty)$ and $V > 0$. Then there exists a constant*

$$C = C(K, N, V) > 0$$

with the following properties. Let $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ be $\mathsf{CD}^*(K, N)$ spaces. Assume that

$$V^{-1} \le \mathfrak{m}_X(B_R^X(x)) \le V$$

and that there exists an ε-mGH approximation

$$\phi\colon B_R^X(x) \to B_R^Y(y) \quad \text{with } \phi(x) = y.$$

Let $r \in (0, \varepsilon)$ and $y' \in Y$ with

$$\mathsf{d}_Y(y', y) \le R - r + 2\varepsilon,$$

so that $B_r^Y(y') \subset B_R^Y(y)$, and thus we can choose $x' \in B_R^X(x)$ such that

$$\mathsf{d}_Y\big(\phi(x'), y'\big) < \varepsilon \tag{A.1}$$

and $B_r^X(x') \subset B_R^X(x)$. Then the function $\varphi\colon B_r^X(x') \to B_r^Y(y')$ given by

$$\varphi(z) = \begin{cases} \phi(z) & \text{if } \phi(z) \in B_r^Y(y'), \\ w \text{ for some } w \in \partial B_r^Y(y') \text{ with} & \\ \mathsf{d}_Y(w, \phi(z)) = \mathsf{d}_Y(B_r^Y(y'), \phi(z)) & \text{otherwise}, \end{cases} \tag{A.2}$$

is a $C\varepsilon$-mGH approximation.

Proof. Before calculating the distortion of φ we see that for all $z \in B_r^X(x')$, we have

$$\mathsf{d}_Y\big(\varphi(z), \phi(z)\big) \le 2\varepsilon. \tag{A.3}$$

Indeed, for any $z \in B_r^X(x')$, using that ϕ is an ε-GH approximation and the definition of x' in (A.1), we get

$$\begin{aligned}\mathsf{d}_Y\big(\phi(z), y'\big) &\le \mathsf{d}_Y\big(\phi(z), \phi(x')\big) + \mathsf{d}_Y\big(\phi(x'), y'\big)\\ &\le \mathsf{d}_X(z, x') + 2\varepsilon < r + 2\varepsilon.\end{aligned}$$

Hence, if $\phi(z) \notin B_r^Y(y')$, then $\mathsf{d}_Y(\varphi(z), \phi(z)) \le 2\varepsilon$. The other case is trivial.

Step 1. Control of the distortion of φ.

Let $z, z' \in B_r^X(x')$ such that $\varphi(z) = w$ and $\varphi(z') = w'$. Then by (A.3) and using that ϕ is an ε-GH-approximation, we get

$$\begin{aligned}\mathsf{d}_Y\big(\varphi(z), \varphi(z')\big) &\le \mathsf{d}_Y\big(w, \phi(z)\big) + \mathsf{d}_Y\big(\phi(z), \phi(z')\big) + \mathsf{d}_Y\big(\phi(z'), w'\big)\\ &\le 2\varepsilon + \{\mathsf{d}_X(z, z') + \varepsilon\} + 2\varepsilon\\ &\le 5\varepsilon + \mathsf{d}_X(z, z').\end{aligned}$$

In a similar way, we can get

$$\mathsf{d}_X(z, z') \le 5\varepsilon + \mathsf{d}_Y\big(\varphi(z), \varphi(z')\big).$$

So, $\mathrm{dist}(\varphi) \le 5\varepsilon$.

Step 2. Almost surjectivity of φ.

Next, we show that for any $w \in B_r^Y(y')$, there exists $z' \in B_r^X(x')$ such that

$$\mathsf{d}_Y\big(w, \varphi(z')\big) \le 7\varepsilon.$$

Let $w \in B_r^Y(y')$. Since $B_r^Y(y') \subset B_R^Y(y)$ and ϕ is an ε-GH approximation, there exists $z \in B_R^X(x)$ such that

$$\mathsf{d}_Y\big(w, \phi(z)\big) \le \varepsilon.$$

If $z \in B_r^X(x')$, we set $z' = z$. By (A.3), we get

$$\begin{aligned}\mathsf{d}_Y\big(\varphi(z'), w\big) &\le \mathsf{d}_Y\big(\varphi(z'), \phi(z')\big) + \mathsf{d}_Y\big(\phi(z'), w\big)\\ &\le 2\varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

If $z \notin B_r^X(x')$, let $z' \in \partial B_r^X(x')$ be a closest point to z. Then, by (A.3) and using that ϕ is an ε-GH approximation, we get

$$\begin{aligned}\mathsf{d}_Y\big(\varphi(z'), w\big) &\le \mathsf{d}_Y\big(\varphi(z'), \phi(z')\big) + \mathsf{d}_Y\big(\phi(z'), \phi(z)\big) + \mathsf{d}_Y\big(\phi(z), w\big)\\ &\le 2\varepsilon + \{\mathsf{d}_X(z', z) + \varepsilon\} + \varepsilon.\end{aligned}$$

We next estimate $\mathsf{d}_X(z', z)$. For this, by the definition of z' it is enough to estimate $\mathsf{d}_X(z, x')$. We have

$$\begin{aligned} \mathsf{d}_X(z,x') &\le \mathsf{d}_Y\big(\phi(z),\phi(x')\big) + \varepsilon \\ &\le \big\{\mathsf{d}_Y\big(\phi(z),w\big) + \mathsf{d}_Y(w,y') + \mathsf{d}_Y\big(y',\phi(x')\big)\big\} + \varepsilon \\ &\le \{\varepsilon + r + \varepsilon\} + \varepsilon = 3\varepsilon + r. \end{aligned}$$

Thus, $\mathsf{d}_X(z', z) \le 3\varepsilon$ and $\mathsf{d}_Y(\varphi(z'), w) \le 7\varepsilon$.

Step 3. Control of the measure distortion.

Using that ϕ is an ε-GH approximation and the definition (A.2) of φ, it is clear that

$$\varphi \equiv \phi \text{ on } B_{r-2\varepsilon}(x). \tag{A.4}$$

From the Bishop–Gromov volume comparison, we have that there exists

$$\bar{C} = \bar{C}(K, N, V) > 0$$

such that

$$\mathfrak{m}_X\big(B_r^X(x) \setminus B_{r-2\varepsilon}^X(x)\big) \le \bar{C}(K,N,V)\,\varepsilon. \tag{A.5}$$

The combination of (A.4) and (A.5) with the fact that ϕ is an ε-GH approximation gives, together with steps 1 and 2, that ϕ is a $C\varepsilon$-GH approximation for some

$$C = C(K, N, V) > 0,$$

which concludes the proof. □

Remark A.1.1. Observe that the previous argument also shows that if

$$\phi\colon B_R^X(x) \to B_R^Y(y)$$

is an ε-GH approximation and $r < R$, the restriction

$$\varphi\colon B_r^X(x) \to B_r^Y(y)$$

defined in (A.2) is a 7ε-GH approximation. The dependence of C on K, N and V comes only in estimating the distortion of the measure.

Proposition A.2 (Product with a Euclidean factor). *There exists a universal constant $C > 0$ with the following properties. Let $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ and $(Y', \mathsf{d}_{Y'}, \mathfrak{m}_{Y'})$ be metric measure spaces. Let*

$$\phi\colon \bar{B}_r^Y(y) \to \bar{B}_r^{Y'}(y')$$

be an ε-mGH approximation with $\phi(y) = y'$ and $\varepsilon \in (0,1)$. Define

$$\varphi\colon \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^Y(y) \to \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y'}(y')$$

by

$$\varphi(a,z) = (a, \phi(z)) \quad \textit{for all } (a,z) \in \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^Y(y).$$

Then φ is a $C\varepsilon$-mGH approximation.

Proof. We break the proof into two steps.

Step 1. We first show that

$$\varphi\colon \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y}(y) \to \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y'}(y')$$

is a 3ε-GH approximation.

To this aim, note that since ϕ is an ε-GH approximation, we have

$$\begin{aligned} &\big|\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big)^2 - \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)^2\big| \\ &\quad = \big|\mathsf{d}^2_{Y'}\big(\phi(z_1),\phi(z_2)\big) - \mathsf{d}^2_Y(z_1,z_2)\big| \\ &\quad \le \varepsilon^2 + 2\varepsilon\,\mathsf{d}_Y(z_1,z_2). \end{aligned} \tag{A.6}$$

In case $\mathsf{d}_Y(z_1,z_2) \le \varepsilon$, by (A.6) and since ϕ is an ε-GH approximation, we have

$$\begin{aligned} &\big|\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big) - \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)\big| \\ &\quad = \frac{\big|\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big)^2 - \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)^2\big|}{\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big) + \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)} \\ &\quad \le \frac{\big|\mathsf{d}^2_{Y'}\big(\phi(z_1),\phi(z_2)\big) - \mathsf{d}^2_Y(z_1,z_2)\big|}{\mathsf{d}_{Y'}\big(\phi(z_1),\phi(z_2)\big) + \mathsf{d}_Y(z_1,z_2)} \\ &\quad \le \big|\mathsf{d}_{Y'}\big(\phi(z_1),\phi(z_2)\big) - \mathsf{d}_Y(z_1,z_2)\big| \le \varepsilon. \end{aligned} \tag{A.7}$$

If instead $\mathsf{d}_Y(z_1,z_2) \ge \varepsilon$, proceeding as in (A.7), we obtain

$$\begin{aligned} &\big|\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big) - \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)\big| \\ &\quad \le \frac{\varepsilon^2 + 2\varepsilon\,\mathsf{d}_Y(z_1,z_2)}{\mathsf{d}_{\mathbb{R}^k\times Y'}\big(\varphi(a_1,z_1),\varphi(a_2,z_2)\big) + \mathsf{d}_{\mathbb{R}^k\times Y}\big((a_1,z_1),(a_2,z_2)\big)} \\ &\quad \le \frac{\varepsilon^2 + 2\varepsilon\,\mathsf{d}_Y(z_1,z_2)}{\mathsf{d}_Y(z_1,z_2)} \\ &\quad \le 3\varepsilon. \end{aligned} \tag{A.8}$$

Combining (A.7) with (A.8), we obtain the claim.

Step 2. Control of the measure distortion.

In order to obtain the closeness of the measures

$$\varphi_\sharp\big(\mathcal{L}^k \otimes \mathfrak{m}_Y \llcorner \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y}(y)\big) \quad \text{and} \quad \mathcal{L}^k \otimes \mathfrak{m}_{Y'} \llcorner \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y'}(y'),$$

it is enough to notice that for each $\psi_1 \in C(\mathbb{R}^k)$, $\psi_2 \in C(Y')$ with

$$\int_{\bar{B}_r^{\mathbb{R}^k}(0^k)} \psi_1 \, d\mathcal{L}^k = 1$$

it holds that

$$\begin{aligned}\Big|\int \psi_1 \otimes \psi_2 \, d\varphi_\sharp\big(\mathcal{L}^k \otimes \mathfrak{m}_Y \llcorner \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^Y(y)\big) \\ - \int \psi_1 \otimes \psi_2 \, d\big(\mathcal{L}^k \otimes \mathfrak{m}_{Y'} \llcorner \bar{B}_r^{\mathbb{R}^k}(0^k) \times \bar{B}_r^{Y'}(y')\big)\Big| \\ = \Big|\int \psi_2 \, d\varphi_\sharp\big(m_Y \llcorner \times \bar{B}_r^Y(y)\big) - \int \psi_2 \, d\big(\mathfrak{m}_{Y'} \llcorner \bar{B}_r^{Y'}(y')\big)\Big|,\end{aligned}$$

where we used Fubini–Tonelli's theorem. □

References

[1] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure. *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 4661–4701 Zbl 1317.53060 MR 3335397

[2] L. Ambrosio, N. Gigli, and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below. *Duke Math. J.* **163** (2014), no. 7, 1405–1490 Zbl 1304.35310 MR 3205729

[3] L. Ambrosio, A. Mondino, and G. Savaré, Nonlinear diffusion equations and curvature conditions in metric measure spaces. *Mem. Amer. Math. Soc.* **262** (2019), no. 1270 Zbl 1477.49003 MR 4044464

[4] K. Bacher and K.-T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. *J. Funct. Anal.* **259** (2010), no. 1, 28–56 Zbl 1196.53027 MR 2610378

[5] G. E. Bredon, *Introduction to compact transformation groups*. Pure Appl. Math. 46, Academic Press, New York-London, 1972 Zbl 0246.57017 MR 0413144

[6] E. Brué and D. Semola, Constancy of the dimension for $\mathsf{RCD}(K,N)$ spaces via regularity of Lagrangian flows. *Comm. Pure Appl. Math.* **73** (2020), no. 6, 1141–1204 Zbl 1442.35054 MR 4156601

[7] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*. Grad. Stud. Math. 33, Amer. Math. Soc., Providence, RI, 2001 Zbl 0981.51016 MR 1835418

[8] F. Cavalletti and E. Milman, The globalization theorem for the curvature-dimension condition. *Invent. Math.* **226** (2021), no. 1, 1–137 Zbl 1479.53049 MR 4309491

[9] F. Cavalletti and A. Mondino, Almost Euclidean isoperimetric inequalities in spaces satisfying local Ricci curvature lower bounds. *Int. Math. Res. Not. IMRN* (2020), no. 5, 1481–1510 Zbl 1436.53022 MR 4073947

[10] F. Cavalletti and K.-T. Sturm, Local curvature-dimension condition implies measure-contraction property. *J. Funct. Anal.* **262** (2012), no. 12, 5110–5127 Zbl 1244.53050 MR 2916062

[11] J. Cheeger and T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)* **144** (1996), no. 1, 189–237 Zbl 0865.53037 MR 1405949

[12] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.* **46** (1997), no. 3, 406–480 Zbl 0902.53034 MR 1484888

[13] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. II. *J. Differential Geom.* **54** (2000), no. 1, 13–35 Zbl 1027.53042 MR 1815410

[14] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.* **54** (2000), no. 1, 37–74 Zbl 1027.53043 MR 1815411

[15] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Differential Geometry* **6** (1971/72), 119–128 Zbl 0223.53033 MR 303460

[16] T. H. Colding, Ricci curvature and volume convergence. *Ann. of Math. (2)* **145** (1997), no. 3, 477–501 Zbl 0879.53030 MR 1454700

[17] T. H. Colding and A. Naber, Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. *Ann. of Math. (2)* **176** (2012), no. 2, 1173–1229 Zbl 1260.53067 MR 2950772

[18] G. De Philippis and N. Gigli, Non-collapsed spaces with Ricci curvature bounded from below. *J. Éc. polytech. Math.* **5** (2018), 613–650 Zbl 1409.53038 MR 3852263

[19] G. De Philippis, A. Marchese, and F. Rindler, On a conjecture of Cheeger. In *Measure theory in non-smooth spaces*, pp. 145–155, Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017 Zbl 1485.53052 MR 3701738

[20] Q. Deng, Hölder continuity of tangent cones in $\mathsf{RCD}(K,N)$ spaces and applications to non-branching. 2020, arXiv:2009.07956

[21] M. Erbar, K. Kuwada, and K.-T. Sturm, On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. *Invent. Math.* **201** (2015), no. 3, 993–1071 Zbl 1329.53059 MR 3385639

[22] K. Fukaya, Theory of convergence for Riemannian orbifolds. *Japan. J. Math. (N.S.)* **12** (1986), no. 1, 121–160 Zbl 0654.53044 MR 914311

[23] K. Fukaya and T. Yamaguchi, The fundamental groups of almost non-negatively curved manifolds. *Ann. of Math. (2)* **136** (1992), no. 2, 253–333 Zbl 0770.53028 MR 1185120

[24] S. Gallot, Bornes universelles pour des invariants géométriques. In *Séminaire de théorie spectrale et géométrie. Vol. 1 (1982–1983)*, Talk no. 2, DOI: 10.5802/tsg.2. https://tsg.centre-mersenne.org/articles/10.5802/tsg.2/

[25] N. Gigli, The splitting theorem in non-smooth context. 2013, arXiv:1302.5555

[26] N. Gigli, On the differential structure of metric measure spaces and applications. *Mem. Amer. Math. Soc.* **236** (2015), no. 1113 Zbl 1325.53054 MR 3381131

[27] N. Gigli, A. Mondino, and G. Savaré, Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. *Proc. Lond. Math. Soc. (3)* **111** (2015), no. 5, 1071–1129 Zbl 1398.53044 MR 3477230

[28] N. Gigli and E. Pasqualetto, Behaviour of the reference measure on RCD spaces under charts. *Comm. Anal. Geom.* **29** (2021), no. 6, 1391–1414 Zbl 07473916 MR 4367429

[29] N. Gigli and C. Rigoni, Recognizing the flat torus among $\mathsf{RCD}^*(0,N)$ spaces via the study of the first cohomology group. *Calc. Var. Partial Differential Equations* **57** (2018), no. 4, Art. ID 104 Zbl 1404.53057 MR 3814057

[30] M. Gromov, *Structures métriques pour les variétés riemanniennes*. Textes Mathématiques 1, CEDIC, Paris, 1981 Zbl 0509.53034 MR 682063

[31] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*. English edn., Mod. Birkhäuser Class., Birkhäuser Boston, Boston, MA, 2007 Zbl 1113.53001 MR 2307192

[32] A. Hatcher, *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2002 Zbl 1044.55001 MR 1867354

[33] S. Honda, New differential operator and noncollapsed RCD spaces. *Geom. Topol.* **24** (2020), no. 4, 2127–2148 Zbl 1452.53041 MR 4173928

[34] V. Kapovitch and A. Mondino, On the topology and the boundary of N-dimensional $\mathsf{RCD}(K,N)$ spaces. *Geom. Topol.* **25** (2021), no. 1, 445–495 Zbl 1466.53050 MR 4226234

[35] M. Kell and A. Mondino, On the volume measure of non-smooth spaces with Ricci curvature bounded below. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **18** (2018), no. 2, 593–610 Zbl 1393.53034 MR 3801291

[36] Y. Kitabeppu and S. Lakzian, Characterization of low dimensional $\mathsf{RCD}^*(K,N)$ spaces. *Anal. Geom. Metr. Spaces* **4** (2016), no. 1, 187–215 Zbl 1348.53046 MR 3550295

[37] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)* **169** (2009), no. 3, 903–991 Zbl 1178.53038 MR 2480619

[38] A. Mondino and A. Naber, Structure theory of metric measure spaces with lower Ricci curvature bounds. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 6, 1809–1854 Zbl 1468.53039 MR 3945743

[39] A. Mondino and G. Wei, On the universal cover and the fundamental group of an $\mathsf{RCD}^*(K,N)$-space. *J. Reine Angew. Math.* **753** (2019), 211–237 Zbl 1422.53033 MR 3987869

[40] J. R. Munkres, *Topology*. Prentice Hall, Upper Saddle River, NJ, 2000 Zbl 0951.54001 MR 3728284

[41] P. Petersen, *Riemannian geometry*. Third edn., Grad. Texts in Math. 171, Springer, Cham, 2016 Zbl 1417.53001 MR 3469435

[42] T. Rajala and K.-T. Sturm, Non-branching geodesics and optimal maps in strong $\mathsf{CD}(K,\infty)$-spaces. *Calc. Var. Partial Differential Equations* **50** (2014), no. 3-4, 831–846 Zbl 1296.53088 MR 3216835

[43] C. Sormani and G. Wei, Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.* **353** (2001), no. 9, 3585–3602 Zbl 1005.53035 MR 1837249

[44] C. Sormani and G. Wei, The covering spectrum of a compact length space. *J. Differential Geom.* **67** (2004), no. 1, 35–77 Zbl 1106.58025 MR 2153481

[45] C. Sormani and G. Wei, Universal covers for Hausdorff limits of noncompact spaces. *Trans. Amer. Math. Soc.* **356** (2004), no. 3, 1233–1270 Zbl 1046.53027 MR 2021619

[46] E. H. Spanier, *Algebraic topology*. Springer, New York, 1966 Zbl 0145.43303 MR 1325242

[47] K.-T. Sturm, On the geometry of metric measure spaces. I. *Acta Math.* **196** (2006), no. 1, 65–131 Zbl 1105.53035 MR 2237206

[48] K.-T. Sturm, On the geometry of metric measure spaces. II. *Acta Math.* **196** (2006), no. 1, 133–177 Zbl 1106.53032 MR 2237207

[49] C. Villani, *Optimal transport*. Grundlehren Math. Wiss. 338, Springer, Berlin, 2009 Zbl 1156.53003 MR 2459454

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