Comment. Math. Helv. 97 (2022), 611–633
DOI 10.4171/CMH/535

Commentarii Mathematici Helvetici

Cycle integrals of the j-function on Markov geodesics

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Abstract. We give asymptotic upper and lower bounds for the real and imaginary parts of cycle integrals of the classical modular j-function along geodesics that correspond to Markov irrationalities.

Mathematics Subject Classification (2020). 11F03; 11E16.

Keywords. Modular forms, cycle integrals, j-function, Markov quadratic irrationalities, binary quadratic forms.

1. Introduction

Let w be a real quadratic irrationality, so a root of an equation

$$ax^2 + bx + c = 0 \quad (a, b, c \in \mathbb{Z},\ (a, b, c) = 1)$$

with positive non-square discriminant $D = b^2 - 4ac$. Let Q be the quadratic form $[a, b, c]$. The geodesic S_Q in the upper half plane $\mathcal{H}$ joining w and its Galois conjugate is given by the equation

$$a|z|^2 + b\,\mathrm{Re}\, z + c = 0 \quad (z \in \mathcal{H}).$$

We orientate S_Q counterclockwise if $a > 0$ and clockwise if $a < 0$. There is an infinite cyclic group Γ_Q in $\mathrm{SL}(2, \mathbb{Z})$, corresponding to the group of totally positive units in $\mathbb{Q}(\sqrt{D})$, that preserves Q, and hence S_Q. The smallest positive unit in $\mathbb{Q}(\sqrt{D})$ is given by $\varepsilon = (t + u\sqrt{D})/2$, where (t, u) is the smallest positive integral solution to Pell's equation $t^2 - Du^2 = 4$. The induced geodesic $C_Q = \Gamma_Q \backslash S_Q$ on the modular surface is closed, primitive, positively-oriented with length

$$\mathrm{length}(C_Q) = \int_{C_Q} \frac{\sqrt{D}}{Q(z, 1)}\, dz = 2 \log \varepsilon.$$

*Research supported by SNF grant 173976.

Here $(\sqrt{D}/Q(z,1))\,dz$ is the hyperbolic arc length on S_Q. As usual, we denote by $j(z)$ the modular function for $\mathrm{SL}(2,\mathbb{Z})$ introduced by Klein as an invariant of elliptic curves. The group Γ_Q preserves $j(z)(\sqrt{D}/Q(z,1))\,dz$, and so we can consider the integral

$$\int_{C_Q} j(z)\frac{\sqrt{D}}{Q(z,1)}\,dz$$

and define the '*value*' (also called *cycle integral*) of j at w, as the complex number:

$$j(w) = j(Q) := \frac{1}{2\log\varepsilon}\int_{C_Q} j(z)\frac{\sqrt{D}}{Q(z,1)}\,dz. \tag{1.1}$$

Note that the integral above is $\mathrm{SL}(2,\mathbb{Z})$-invariant.

Cycle integrals have been related to mock modular forms [10], to modular knots [12] and to class numbers of real quadratic fields [13]. Moreover, cycle integrals of the Klein invariant j share several analogies with singular moduli (the values of the j-function at imaginary quadratic irrationalities) when both are gathered in 'traces' (see [9–11, 18]). By analogy to the traces of singular moduli we define

$$\mathrm{Tr}_D\, j := \sum j(w_Q),$$

where the sum is over $\mathrm{SL}(2,\mathbb{Z})$-classes of indefinite binary quadratic forms Q of fundamental discriminant $D > 0$. The asymptotic distribution of the traces was studied in [8], and [9, 18] for negative and positive discriminants, respectively. As fundamental discriminants $D \to +\infty$, it was shown in [9, 18] that

$$\frac{\mathrm{Tr}_D(j)}{\mathrm{Tr}_D(1)} \to 720. \tag{1.2}$$

Here 720 is an 'average' value of the j function. The individual values remain very much unknown. In [4], we gave the first bounds for the real parts of the values $j(w)$ (in fact, j could be replaced by any modular function f which is real valued on the geodesic arc $\{e^{i\theta} : \pi/3 \le \theta \le 2\pi/3\}$). We showed that

$$\mathrm{Re}(j(w)) \le 744$$

(we recall that 744 is the constant term in the Fourier expansion of j). For quadratic irrationalities w satisfying a 'quite strong' diophantine condition, we proved that

$$\mathrm{Re}(j(w)) \ge j\left(\frac{1+\sqrt{5}}{2}\right) \approx 706.3248.$$

These bounds are optimal and were conjectured earlier by Kaneko in [14] based on numerical evidence. The imaginary parts of $j(w)$ are conjectured to lie in $(-1,1)$, but to our knowledge nothing is known yet.

The values $j(w)$ are particularly interesting at Markov irrationalities w. Markov irrationalities are important in diophantine approximation and in the search of positive minima of indefinite binary quadratic forms. They are structured on a tree called the Markov–Hurwitz tree and there is a very rich theory attached to them in the interplay of number theory, diophantine approximation, hyperbolic geometry, dynamics and graph theory. Kaneko published in [14] the first numerical data on values $j(w)$ at Markov irrationalities, together with some conjectures based on his data. Namely, he conjectured that each $j(w)$ is between the values of j at two Markov irrationalities that lie above w on the tree. We call these two irrationalities the predecessors of w and we call this property 'interlacing property'. In [3], we proved that for every branch of the tree, the interlacing property holds after some level that depends on the branch. One can think of that result as a 'local' asymptotic interlacing property (an asymptotic property that holds for each branch).

In this paper we prove the 'global' asymptotic interlacing property, namely:

Theorem 1. *Let $w = w_n$ be a Markov irrationality and n be the level of w in the Markov–Hurwitz tree. There is a unique path on the Markov–Hurwitz tree that ends with w_n; let w_{n-1} be the Markov irrationality above w_n on the path, and $w_{n'-1}$ be the Markov irrationality above the last irrationality where the path turns. There exists an integer n_0 such that, if $n > n_0$, then $j(w_n)$ lies between $j(w_{n-1})$ and $j(w_{n'-1})$.*

The irrationalities w_{n-1} and $w_{n'-1}$ are the 'predecessors' of w_n in Theorem 1. The interlacing property is expected to hold for $n \geq 2$, hence we expect the bounds:

$$706.32481 \approx j((1+\sqrt{5})/2) \leq \mathrm{Re}(j(w)) \leq j(\sqrt{2}) \approx 709.8929,$$
$$-0.26703\ldots \leq \mathrm{Im}(j(w)) \leq 0.26703\ldots.$$

By making explicit most of the estimations in Theorem 1, we obtain:

Theorem 2. *Let w be a Markov irrationality and n be the level of w in the Markov–Hurwitz tree. We have that*

$$681.50081 \leq \mathrm{Re}(j(w)) \leq 742.03641 \quad \text{as } n \to \infty,$$
$$-0.93637 \leq \mathrm{Im}(j(w)) \leq 0.67396 \quad \text{as } n \to \infty.$$

Although the bounds in Theorem 2 are not optimal, they are the first lower bounds for the real parts of the values $j(w)$ at Markov irrationalities and the first bounds for the imaginary parts that we have until now.

One reason why we obtained only a 'local' interlacing property in [3] is that we lacked of a convenient order on the tree. A new key approach here is to order the Markov–Hurwitz tree by Farey fractions (which can probably be more exploited in the future). We also exploit more accurately the diophantine properties of the Markov–Hurwitz tree as well as its relations with the Farey tree, and use an asymptotic formula from [19] on Markov numbers. The arguments used in this paper apply to any modular function f which is real valued on the geodesic arc $\{e^{i\theta} : \pi/3 \leq \theta \leq 2\pi/3\}$. The values (7.4), (10.2), and (10.3) will change when we replace j by f, and so will the subsequent calculations.

The paper is organized as follows. In the next two sections, we quickly review the main key points of Markov's theory and other related facts that will be useful. Concretely, in Section 2, we introduce Markov numbers and Markov irrationalities. In Section 3, we introduce the Markov–Hurwitz and the Farey trees as well as some of their properties and interrelations. In Section 4, we give an asymptotic formula for $\log \varepsilon$ in terms of the Farey tree by using Zagier's asymptotic formula for Markov numbers in [19]. In Section 5, we write the values $j(w)$ in terms of cycles of certain reduced binary quadratic forms, usually known as 'simple forms' after Zagier. Simple forms can be written themselves in terms of cycles of quadratic irrationalities produced by a certain continued fraction algorithm. We work with the cycles of continued fraction expansions in Sections 6 and 7 to obtain a 'local' formula for $(2\log \varepsilon) j(w)$ that depends on two neighbours of w on the Markov–Hurwitz tree (the two predecessors of w). In Section 8, we deduce a 'global' formula for $(2\log \varepsilon) j(w)$ from the local formula obtained in Section 7. We prove Theorem 1 in Section 9 using the global formula from Section 8 and the asymptotic formula for $\log \varepsilon$ given in Section 4. We prove Theorem 2 in Section 10. We give some numerical data in the appendix.

2. Markov's theory

Markov's work (1880) [16, 17] establishes very beautiful connections between positive integral minima for indefinite binary quadratic forms and the Lagrange–Hurwitz problem in diophantine approximation. The Lagrange–Hurwitz problem consists in describing the Lagrange constants

$$L(x) = \big(\liminf_{q\to\infty} q\|qx\|\big),$$

where x runs through the real numbers and $\|\cdot\|$ denotes the distance to a closest integer. The quantity $L(x)$ provides a 'measure' of how well x can be approximated by the rationals. For almost all $x \in \mathbb{R}$ we have $L(x) = 0$, and when $L(x) > 0$ we call x *badly approximable*. A well known theorem of Hurwitz states that $L(x) < 1/\sqrt{5}$.

The *Lagrange spectrum* $Ł := \{L(x)^{-1}\}_{x\in\mathbb{R}} \subseteq [\sqrt{5}, \infty]$ is structured in three parts: $Ł \cap [\sqrt{5}, 3)$ is discrete with 3 as the only accumulation point, $Ł \cap (3, F]$ is fractal, where $F \approx 4.528$ is Freiman's constant, and $Ł \cap (F, \infty]$ is continuous.

Markov discovered that on the interval $(0, 3)$ the Lagrange spectrum and the spectrum M of integral minima

$$M(Q) = \frac{\sqrt{D}}{\inf\limits_{(x,y)\in\mathbb{Z}^2,\,(x,y)\neq(0,0)} |Q(x, y)|}$$

of binary quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2 \quad (A, B, C \in \mathbb{R})$$

with positive discriminant $D = B^2 - 4AC$, are the same.

The unifying thread is the diophantine equation

$$a^2 + b^2 + c^2 = 3abc \quad (a, b, c \in \mathbb{N}), \tag{2.1}$$

and its integer solutions. The integer solutions (a, b, c) are obtained by starting with $(1, 1, 1)$, $(1, 1, 2)$, $(2, 1, 5)$ and then proceeding recursively going from (a, b, c) to the new triples obtained by Vieta involutions $(c, b, 3bc - a)$ and $(a, c, 3ac - b)$. *Markov numbers* are the greatest coordinates of each solution (a, b, c). They form the *Markov sequence*

$$\{c_i\}_{i=1}^{\infty} = \{1, 2, 5, 13, 29, 34, 89, 169, 194, \ldots\}. \tag{2.2}$$

(We count multiplicities if the unicity conjecture is false.) To each Markov number c, there is associated a *Markov irrationality*

$$w = \frac{3c - 2k + \sqrt{9c^2 - 4}}{2c},$$

where k is an integer that satisfies $ak \equiv b \pmod{c}$, $0 \le k < c$, and (a, b, c) is a solution to (2.1) with c maximal. The quadratic w is a root of the *Markov form*

$$Q(x, y) = cx^2 + (3c - 2k)xy + (\ell - 3k)y^2,$$

where $\ell = (k^2 + 1)/c \in \mathbb{Z}$. Then

$$M(Q) = L(w)^{-1} = \sqrt{9 - 4/c^2}.$$

Moreover,

$$M \cap (0, 3) = Ł \cap (0, 3) = \left\{ \sqrt{9 - 4/c_i^2} \right\}_{i \ge 1}$$

and any $x \in \mathbb{R}$ with $L(x)^{-1} \in (0, 3)$ or any Q with $M(Q) \in (0, 3)$ is $\mathrm{GL}(2, \mathbb{Z})$-equivalent to a Markov quadratic or a Markov form, respectively.

3. Trees related to Markov's theory

3.1. The Markov–Hurwitz tree. The solutions of the diophantine equation (2.1) inherit a tree structure with two bifurcations from the two Vieta involutions described earlier; see Figure 1.

Naturally, we can consider the parallel tree of Markov irrationalities, where each vertex is a quadratic irrationality that corresponds to a Markov number. What

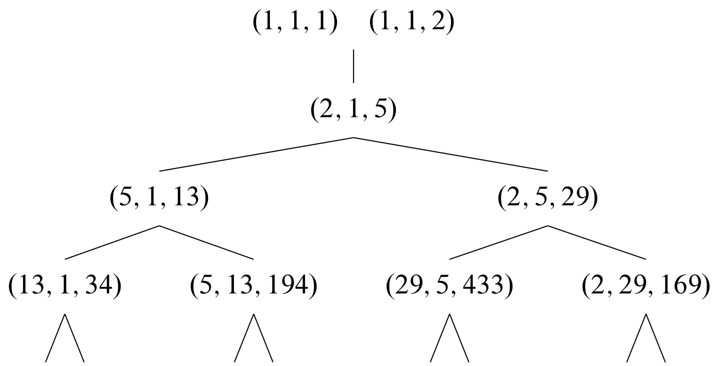


Figure 1. The Markov tree.

is more interesting is that each Markov irrationality can be constructed from two predecessors on the tree by a simple conjunction operation on the continued fraction expansions. A formal description of the procedure using '+' continued fraction expansions can be found in [5]. It will be more convenient for us to work with the '−' continued fraction since it corresponds to transformations by the modular group (whereas the '+' continued fraction corresponds to transformations by $\mathrm{GL}(2,\mathbb{Z})$). It was observed in [3] that roughly the same conjunction operation works with the '−' continued fraction. We denote by

$$(a_1, a_2, a_3, \dots) = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots}}}$$

the '−' continued fraction expansion, with $a_i \in \mathbb{Z}$ and $a_i \geq 2$ for $i \geq 2$.

We define the *conjunction operation* of two periods as

$$(\overline{a_1, \dots, a_r}) \odot (\overline{b_1, \dots, b_s}) = (\overline{a_1, \dots, a_r, b_1, \dots, b_s}). \tag{3.1}$$

Each Markov irrationality not on the most left branch is the result of the conjunction operation of two predecessors: its immediate predecessor on the same branch and the immediate predecessor of the tip of the branch. We call the *right* predecessor the one on the right of w on the tree, and the *left* predecessor the one on the left. With this terminology, w is the result of the conjunction operation of the right predecessor with the left predecessor. For example,

$$(\overline{2, 4, (2, 3, 4)_2}) = (\overline{2, 4, 2, 3, 4}) \odot (\overline{2, 3, 4});$$

the right and left predecessors are $(\overline{2, 4, 2, 3, 4})$ and $(\overline{2, 3, 4})$, respectively. On the leftmost branch, the Markov irrationality at level n is $(\overline{2, 3_n, 4})$. We call $(\overline{3})$ the *left* predecessor and $(\overline{2, 3_{n-1}, 4})$ the *right* predecessor.

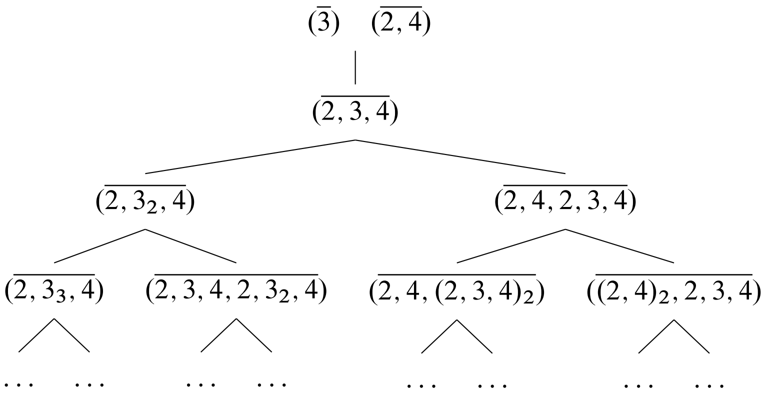


Figure 2. The Markov–Hurwitz tree $\mathcal{MH}$.

3.2. The Farey tree. There is a natural parametrisation of Markov numbers by Farey fractions which goes back to Frobenius [1]. The Farey tree is constructed by following a very similar procedure as described in Section 3.1 for the Markov–Hurwitz tree. We start with $0/1$, $1/2$ and $1/3$, and construct each following fraction from the two predecessors a/b, c/d that are in the same position as in the Markov–Hurwitz tree (a/b is the right predecessor and c/d is the left predecessor) by taking the Farey median

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d}.$$

Hence, each vertex of the tree is a Farey fraction p/q that is in correspondence with a Markov quadratic $w = w(p/q)$ and a Markov number $c = m(p/q)$. We denote the Farey tree by $\mathcal{F}$.

By the construction of the Markov–Hurwitz and the Farey trees, the denominators in $\mathcal{F}$ correspond to the lengths of the periods of the continued fractions that occupy the same corresponding position in $\mathcal{MH}$.

The operation giving the denominators in the Farey tree corresponds to the well known Euclidean algorithm which, starting with $(0, 1, 1)$, gives from a triple (s, t, u) two new triples $(s, u, s + u)$ and $(t, u, t + u)$, thus obtaining all solutions to the equation

$$s + t = u, \quad 0 \le s \le t \le u, \ (s, t) = 1. \tag{3.2}$$

The denominators in the Farey tree are the maximum terms of the solutions (s, t, u).

4. Asymptotic formula for $\log \varepsilon$ in terms of Farey denominators

We order the denominators of the Farey fractions in $\mathcal{F}$ as real increasing numbers $(q_n)_{n\ge1}$. If two Farey fractions p/q, p'/q' satisfy $q = q'$, then we write $q = q_n$,

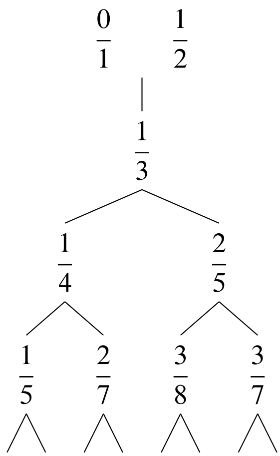


Figure 3. The Farey tree $\mathcal{F}$.

$q' = q_m$ with $n < m$ if $m(\frac{p}{q}) < m(p'/q')$. Hence, we obtain a sequence:

$$q_1 = 1, \quad q_2 = 2, \quad q_3 = 3, \quad q_4 = 4, \quad q_5 = 5, \quad q_6 = 5, \dots \tag{4.1}$$

that is in bijective correspondence with the sequence of Markov numbers (2.2).

The number of denominators q_n less than x (with multiplicity) is given by

$$\#\{q_n \leq x\}_{n \geq 1} = \#\{(s,t,u) \text{ solution to (3.2)} : u \leq x\} = 1 + \frac{1}{2}\sum_{u \leq x} \varphi(u),$$

where φ is the Euler function. From the asymptotic formula

$$1 + \frac{1}{2}\sum_{u \leq x} \varphi(u) \sim \frac{3}{2\pi^2}x^2,$$

it follows that

$$q_n \sim \pi\sqrt{\frac{2}{3}n}. \tag{4.2}$$

Let $w = w(p/q)$ be a Markov irrationality and p/q be its corresponding Farey fraction. Set $q = q_n$, where q_n is a Farey denominator ordered as in (4.1), so that $m(p/q) = c_n$. The discriminant of w is $9c_n^2 - 4$, so Pell's equation is

$$t^2 - (9c_n^2 - 4)u^2 = 4,$$

with $(3c_n, 1)$ being the smallest positive solution. Thus, we have

$$\varepsilon = \frac{3c_n + \sqrt{9c_n^2 - 4}}{2}.$$

By a result of Zagier [19], we have that

$$c_n \sim e^{\sqrt{n/C}}, \tag{4.3}$$

where $C \approx 0.18071704711507$. Using (4.2), we have

$$c_n \sim e^{\frac{q_n}{\pi}\sqrt{3/2C}}. \tag{4.4}$$

Hence,

$$\log \varepsilon \sim \frac{\sqrt{3}}{\sqrt{2C}\pi} q_n + \log 3. \tag{4.5}$$

5. Simple forms and continued fractions

Any indefinite binary quadratic form $[a,b,c]$ is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to one satisfying

$$a > 0 > c. \tag{5.1}$$

There is a finite number of forms satisfying (5.1) with fixed discriminant. Such forms are commonly called 'simple' after Zagier [21] and play an important role in the theory of rational periods and period functions of modular forms (see [2,6,15,20]).

It is shown in [7] that all simple forms $\mathrm{SL}(2,\mathbb{Z})$ equivalent to a form $[a,b,c]$ are obtained by applying iteratively the following continued fraction algorithm. Let w be the period in the '$-$' continued fraction expansion of the root

$$\frac{-b+\sqrt{b^2-4ac}}{2a};$$

we define

$$w^{(1)} = w - 1, \quad w^{(k+1)} = \begin{cases} w^{(k)} - 1 = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(w^{(k)}\right) & \text{if } w^{(k)} \geq 1, \\ \frac{w^{(k)}}{1-w^{(k)}} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(w^{(k)}\right) & \text{otherwise.} \end{cases} \tag{5.2}$$

This algorithm is cyclic because the continued fraction of w is purely periodic; we denote by $\ell = \ell_w$ the length of the cycle, so that $w^{(\ell+1)} = w^{(1)}$. The cycle $w^{(1)}, \ldots, w^{(\ell)}$ corresponds to the cycle of simple forms in the $\mathrm{SL}(2,\mathbb{Z})$-equivalence class of the quadratic form $[a,b,c]$.

The value $j(w)$ is written in terms of the cycle (5.2) when w is a Markov irrationality in [3, Lemma 4.1] and when w is an arbitrary quadratic irrationality in [4]. More concretely, in terms of simple forms, we have the following lemma.

Lemma 5.1. *For any real quadratic irrationality w, we have*

$$j(w) = \frac{1}{2\log\varepsilon} \int_{e^{\pi i/3}}^{e^{2\pi i/3}} j(z) \sum_{\substack{[a,b,c] \text{ simple} \\ [a,b,c] \in \mathcal{A}}} \frac{\sqrt{D}}{az^2+bz+c}\, dz, \tag{5.3}$$

where $\mathcal{A}$ is the $\mathrm{SL}(2,\mathbb{Z})$-equivalence class of w and D its discriminant.

If w has a purely periodic '−' continued fraction expansion, we can write (5.3) as follows:

$$j(w) = \frac{1}{2\log\varepsilon} \int_{\pi/3}^{2\pi/3} j(e^{i\theta}) i e^{i\theta} \sum_{k=1}^{\ell_w} \left(\frac{1}{e^{i\theta} - w^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(k)}} \right) d\theta, \tag{5.4}$$

where $w^{(k)}$ are defined in (5.2) and $\widetilde{w}^{(k)}$ are the Galois conjugates of $w^{(k)}$. Each term $w^{(k)}$ is of the form

$$w^{(k)} = (a_0, \overline{a_1, \dots, a_n}) = a_0 - \frac{1}{(\overline{a_1, \dots, a_n})} \qquad (1 \le a_0 \le a_n - 1).$$

It is a well known fact (see, for example, [21]) that the conjugate of $1/(\overline{a_1, \dots, a_n})$ is $(\overline{a_n, \dots, a_1})$, and hence

$$\widetilde{w}^{(k)} = -(a_n - a_0, \overline{a_{n-1}, a_{n-2}, \dots, a_1, a_n}).$$

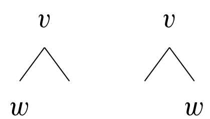
If w is a Markov irrationality, then the structure of the Markov–Hurwitz tree implies that

$$\frac{3}{8} = (1,2,3,2) \le w^{(k)} \le (3,2,4,2) = \frac{29}{12}, \tag{5.5}$$

$$-\frac{21}{8} = -(3,3,3) \le \widetilde{w}^{(k)} \le -(1,2,3) = -\frac{2}{5}. \tag{5.6}$$

6. Strategy for the proof of Theorem 1

Let w be a Markov irrationality on the n-th level of the Markov–Hurwitz tree, the first level corresponding to the Markov irrationality $(\overline{2,3,4})$. Let v be the 'immediate' predecessor of w, which is on the same branch as w. Then we have one of the following two configurations:



Let $u = (\overline{a_1, \dots, a_s})$ be the other predecessor of w as explained in Section 3.1. In the second configuration, u is the right predecessor of w and v. Then let $(\overline{b}) = (\overline{b_1, \dots, b_t})$ be the left predecessor of u. We have $v = (\overline{u_r, b})$ for some $r \ge 1$ and $w = (\overline{u_{r+1}, b})$. In the first configuration, u is the left predecessor of w and v. If w is not on the most left branch of the tree, then we let $(\overline{b}) = (\overline{b_1, \dots, b_t})$ be the right predecessor of u, so that $v = (\overline{b, u_r})$ for some $r \ge 1$ and $w = (\overline{b, u_{r+1}})$. The cycles are the same if we consider $(\overline{u_r, b})$ and $(\overline{u_{r+1}, b})$, and we will refer to these two situations as 'case 1'. Note that in this case, $rs > n$.

The situation where w is on the most left branch of the tree will be 'case 2'. In this case, $u = (\overline{3})$, $v = (\overline{2, 3_r, 4})$ for some $r \geq 1$ and $w = (\overline{2, 3_{r+1}, 4})$. Note that in this case, $r = n - 1$ and

$$\ell_u = 2, \quad \ell_v = 2r + 4, \quad \ell_w = \ell_u + \ell_v = 2(r + 1) + 4.$$

Let

$$J(w) := (2 \log \varepsilon) j(w).$$

In the next section, we will write $J(w)$ as

$$J(w) = J(u) + J(v) + \delta_w, \tag{6.1}$$

where δ_w is the error term that we will explicitly bound in terms of n. We see the identity (6.1) as a 'local' formula for $J(w)$ on the Markov–Hurwitz tree. We will deduce a recursive formula for $J(w)$ and q_n in Section 8 (see (8.7) and (8.9)–(8.11)) that we see as a 'global' formula and that will give the global interlacing property in Section 9.

In order to obtain the local formula (6.1), we will compare the cycle of w in the algorithm (5.2) with the cycles of u and v and their J values in Section 7.1. In Sections 7.2–7.4, we bound the error term δ_w.

7. Local formula for $(2 \log \varepsilon) j(w)$

7.1. Comparing cycles of w, u, v. We will compare the cycle of w in the algorithm (5.2) with the cycles of u and v. We define a sum S_u and a sum S_v that contain all the terms in the cycle of u and v respectively. The definitions of S_u and S_v are different for case 1 and case 2. In case 1, the sum S_u below compares the first terms $w^{(1)}, \dots, w^{(\ell_u)}$ and the conjugates $\widetilde{w}^{(r\ell_u+1)}, \dots, \widetilde{w}^{((r+1)\ell_u)}$ with the cycle of u and conjugates, while the sum S_v compares the remaining terms with the cycle of v and conjugates.

Case 1. We define

$$S_u(\theta) = \sum_{k=1}^{\ell_u} \frac{1}{e^{i\theta} - w^{(k)}} - \frac{1}{e^{i\theta} - u^{(k)}} + \sum_{k=1}^{\ell_u} \frac{1}{e^{i\theta} - \widetilde{u}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(r\ell_u+k)}},$$

$$S_v(\theta) = \sum_{k=1}^{\ell_v} \frac{1}{e^{i\theta} - w^{(k+\ell_u)}} - \frac{1}{e^{i\theta} - v^{(k)}} + \sum_{k=1}^{r\ell_u} \frac{1}{e^{i\theta} - \widetilde{v}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(k)}}$$
$$+ \sum_{k=r\ell_u+1}^{\ell_v} \frac{1}{e^{i\theta} - \widetilde{v}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(\ell_u+k)}}.$$

In case 2 we arrange the terms $u^{(k)}, v^{(k)}, w^{(k)}$ and the conjugates in a somewhat different way.

Case 2. We define

$$S_u(\theta) = \sum_{k=1}^{2} \frac{1}{e^{i\theta} - w^{(1+k)}} - \frac{1}{e^{i\theta} - u^{(k)}} + \sum_{k=1}^{2} \frac{1}{e^{i\theta} - \widetilde{u}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(1+2r+k)}},$$

$$S_v(\theta) = \frac{1}{e^{i\theta} - w^{(1)}} - \frac{1}{e^{i\theta} - v^{(1)}} + \sum_{k=2}^{4+2r} \frac{1}{e^{i\theta} - w^{(2+k)}} - \frac{1}{e^{i\theta} - v^{(k)}}$$
$$+ \sum_{k=1}^{1+2r} \frac{1}{e^{i\theta} - \widetilde{v}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(k)}} + \sum_{k=2+2r}^{4+2r} \frac{1}{e^{i\theta} - \widetilde{v}^{(k)}} - \frac{1}{e^{i\theta} - \widetilde{w}^{(2+k)}}.$$

Using (5.4), we have that

$$J(w) - J(u) - J(v) = \int_{\pi/3}^{2\pi/3} j(e^{i\theta}) i e^{i\theta} \big(S_u(\theta) + S_v(\theta)\big)\, d\theta. \tag{7.1}$$

Our next goal is to bound the integral in (7.1); we will give bounds for the real and imaginary parts. For $\alpha \in \{u, v\}$, let

$$\varepsilon_\alpha(\theta) = \cos\theta \operatorname{Im}(S_\alpha(\theta)) + \sin\theta \operatorname{Re}(S_\alpha(\theta)),$$
$$\varepsilon'_\alpha(\theta) = \cos\theta \operatorname{Re}(S_\alpha(\theta)) - \sin\theta \operatorname{Im}(S_\alpha(\theta)).$$

The real and imaginary parts of the integral in (7.1) are respectively

$$\operatorname{Re}(7.1) = -\int_{\pi/3}^{2\pi/3} j(e^{i\theta})(\varepsilon_u(\theta) + \varepsilon_v(\theta))\, d\theta, \tag{7.2}$$

$$\operatorname{Im}(7.1) = \int_{\pi/3}^{2\pi/3} j(e^{i\theta})(\varepsilon'_u(\theta) + \varepsilon'_v(\theta))\, d\theta. \tag{7.3}$$

Our goal is to bound (7.2) and (7.3). It will be useful to define the following real functions with three variables $(x, y, \theta) \in \mathbb{R}^2 \times [\frac{\pi}{3}, \frac{2\pi}{3}]$:

$$g(x,y,\theta) = \frac{1}{x-y}\left(\cos\theta \operatorname{Im}\left(\frac{1}{e^{i\theta}-x} - \frac{1}{e^{i\theta}-y}\right) + \sin\theta \operatorname{Re}\left(\frac{1}{e^{i\theta}-x} - \frac{1}{e^{i\theta}-y}\right)\right),$$

$$g'(x,y,\theta) = \frac{1}{x-y}\left(\cos\theta \operatorname{Re}\left(\frac{1}{e^{i\theta}-x} - \frac{1}{e^{i\theta}-y}\right) - \sin\theta \operatorname{Im}\left(\frac{1}{e^{i\theta}-x} - \frac{1}{e^{i\theta}-y}\right)\right).$$

Note that

$$\frac{1}{e^{i\theta}-x} - \frac{1}{e^{i\theta}-y} = \frac{(x-y)\big(\operatorname{Re}((e^{i\theta}-x)(e^{i\theta}-y)) - i \operatorname{Im}((e^{i\theta}-x)(e^{i\theta}-y))\big)}{|(e^{i\theta}-x)(e^{i\theta}-y)|^2}. \tag{7.4}$$

We have that

$$\mathrm{Re}((e^{i\theta}-x)(e^{i\theta}-y)) = \cos^2\theta - \sin^2\theta - \cos\theta(x+y) + xy,$$
$$\mathrm{Im}((e^{i\theta}-x)(e^{i\theta}-y)) = 2\sin\theta\cos\theta - \sin\theta(x+y),$$

so

$$g(x,y,\theta) = -\frac{\sin\theta(1-xy)}{((\cos\theta-x)^2+\sin^2\theta)((\cos\theta-y)^2+\sin^2\theta)}, \tag{7.5}$$
$$g'(x,y,\theta) = \frac{-x-y+\cos\theta(1+xy)}{((\cos\theta-x)^2+\sin^2\theta)((\cos\theta-y)^2+\sin^2\theta)}. \tag{7.6}$$

For x, y satisfying (5.5), one can easily compute

$$\begin{aligned} -1.26964 \le g(x,y,\theta) \le 0.354112, \\ -1.10636 \le g'(x,y,\theta) \le -0.07222. \end{aligned} \tag{7.7}$$

For x, y bounded as in (5.6), we have

$$\begin{aligned} -1.25946 \le g(x,y,\theta) \le 0.354112, \\ 0.04705 \le g'(x,y,\theta) \le 1.10636. \end{aligned} \tag{7.8}$$

The following lemma (see [3, Lemma 2.1] for a proof) will also be useful.

Lemma 7.1. *If the '$-$' continued fraction expansions of two Markov quadratics u and v coincide in the first r partial quotients, then*

$$|u-v| \le 10\Big(\frac{2}{1+\sqrt{5}}\Big)^{2(r-1)}.$$

Let

$$b(x) = 10\Big(\frac{2}{1+\sqrt{5}}\Big)^{2(x-1)}$$

be the bound of Lemma 7.1. Note that

$$\sum_{k=k_0}^{\infty} b(k) \le 10\Big(\frac{2}{1+\sqrt{5}}\Big)^{2k_0-3}. \tag{7.9}$$

7.2. Bounds for $\varepsilon_u, \varepsilon'_u$.

Case 1. We write ε_u using the function $g(x,y,\theta)$:

$$\begin{aligned} \varepsilon_u(\theta) = &\sum_{k=1}^{\ell_u}(w^{(k)}-u^{(k)})g(w^{(k)},u^{(k)},\theta) \\ &+ \sum_{k=1}^{\ell_u}(\widetilde{u}^{(k)}-\widetilde{w}^{(r\ell_u+k)})g(\widetilde{u}^{(k)},\widetilde{w}^{(r\ell_u+k)},\theta). \end{aligned}$$

We have a similar expression for $\varepsilon'_u(\theta)$ replacing g by g'. We apply Lemma 7.1 to bound $|w^{(k)} - u^{(k)}|$ and $|\widetilde{u}^{(k)} - \widetilde{w}^{(r\ell_u+k)}|$ and use (7.7) and (7.8) to bound the rest.

In the first sum of $\varepsilon_u(\theta)$, the first $a_1 - 1$ terms $w^{(k)}$ and $u^{(k)}$ share the same first $rs + s$ partial quotients, the next $a_2 - 1$ terms share the first $rs + s - 1$ partial quotients, etc., and the last $a_s - 1$ terms coincide in at least the first $rs + 1$ partial quotients.

In the second sum $\varepsilon_u(\theta)$, we have a similar reversed situation, where the first $a_1 - 1$ terms $\widetilde{w}^{(r\ell_u+k)}$ and $\widetilde{u}^{(k)}$ share the same first $rs + 1$ partial quotients, the next $a_2 - 1$ terms share the same $rs + 2$ first partial quotients, etc., and the last $a_s - 1$ terms share the same $rs + s$ partial quotients.

Using Lemma 7.1, (7.7), (7.8) and that $a_k - 1 \le 3$ for $k = 1, \dots, s$, we have

$$|\varepsilon_u(\theta)| \le 3(1.26964 + 1.25946) \sum_{k=1}^{s} b(rs + k),$$

$$|\varepsilon'_u(\theta)| \le 1.10636 \cdot 6 \sum_{k=1}^{s} b(rs + k).$$

Case 2. We proceed analogously to case 1. The two terms in each sum of S_u share the same first $r + 1$ partial quotients. By Lemma 7.1, for $k = 1, 2$, we have

$$|w^{(1+k)} - u^{(k)}|, |\widetilde{w}^{(k+1+2r)} - \widetilde{u}^{(k)}| \le b(r + 1). \tag{7.10}$$

Using (7.7), (7.8) and (7.10), we have that

$$|\varepsilon_u(\theta)| \le 2 \cdot 2.5291\, b(r + 1), \quad |\varepsilon'_u(\theta)| \le 4 \cdot 1.10636\, b(r + 1).$$

7.3. Bounds for $\boldsymbol{\varepsilon_v, \varepsilon'_v}$. We bound $\varepsilon_v, \varepsilon'_v$ in a similar way to $\varepsilon_u, \varepsilon'_u$.

Case 1. In the first sum of S_v, the first $a_1 - 1$ terms $w^{(\ell_u+k)}$ and $v^{(k)}$ coincide in the first $2rs + t$ partial quotients, the next $a_2 - 1$ terms share the same first $2rs + t - 1$ partial quotients, etc., and the last $b_t - 1$ terms coincide in the first $rs + 1$ partial quotients.

In the second sum of S_v, the first $a_1 - 1$ terms $\widetilde{w}^{(k)}$ and $\widetilde{v}^{(k)}$ share the same first $rs + t + 1$ partial quotients, the next $a_2 - 1$ terms coincide in the first $rs + t + 2$ partial quotients, etc., and the last $a_s - 1$ terms coincide in the first $2rs + t$ partial quotients.

In the third sum of S_v, the first $b_1 - 1$ terms $\widetilde{v}^{(k)}$ and $\widetilde{w}^{(\ell_u+k)}$ coincide in the first $rs + 1$ partial quotients, the next $b_2 - 1$ terms coincide in the first $rs + 2$ partial quotients, etc., and the last $b_t - 1$ partial quotients coincide in the first $rs + t$ partial quotients.

Using again Lemma 7.1, (7.7), (7.8) and that $a_k - 1 \leq 3$, $b_j - 1 \leq 3$ for $k \in \{1, \dots, s\}$, $j \in \{1, \dots, t\}$, we have

$$|\varepsilon_v(\theta)| \leq 3 \cdot 2.5291 \sum_{k=1}^{rs+t} b(rs+k), \quad |\varepsilon_v'(\theta)| \leq 1.10636 \cdot 6 \sum_{k=1}^{rs+t} b(rs+k).$$

Case 2. The terms $w^{(1)}$ and $v^{(1)}$ share the first same $r+1$ partial quotients. In the first sum of S_v, the first two terms coincide in $2r+2$ partial quotients, each next two terms coincide in $2r+1, \dots, r+3$ partial quotients, and the last three terms coincide in $2+r$ partial quotients. In the second sum, the first terms share the same $1+r$ first partial quotients, and each next block of two terms $\widetilde{w}^{(k)}, \widetilde{v}^{(k)}$ coincide in $2+r, 3+r$, etc., $2+2r$ first partial quotients. In the third sum, the terms coincide in $r+1$ partial quotients.

Using Lemma 7.1, (7.7), (7.8), we have that

$$\begin{aligned} |\varepsilon_v(\theta)| &\leq 1.26964(b(r+1) + b(r+2)) \\ &\qquad + 2.5291 \cdot 2 \sum_{k=2}^{r+2} b(r+k) + 1.25946 \cdot 4b(r+1), \\ |\varepsilon_v'(\theta)| &\leq 1.10636\Big(5b(r+1) + b(r+2) + 4\sum_{k=2}^{r+2} b(r+k)\Big). \end{aligned}$$

7.4. Conclusion. Recall that we denote by n the level on the tree where the Markov irrationality w lies. In case 1, using that $rs > n$, we have

$$|\varepsilon_u(\theta)| + |\varepsilon_v(\theta)| \leq 151.7460\Big(\frac{2}{1+\sqrt{5}}\Big)^{2n+1}, \tag{7.11}$$

$$|\varepsilon_u'(\theta)| + |\varepsilon_v'(\theta)| \leq 132.7632\Big(\frac{2}{1+\sqrt{5}}\Big)^{2n+1}. \tag{7.12}$$

In case 2, using that $r = n-1$, we have

$$\begin{aligned} &|\varepsilon_u(\theta)| + |\varepsilon_v(\theta)| \\ &\quad \leq 113.6564\Big(\frac{2}{1+\sqrt{5}}\Big)^{2(n-1)} + 63.278\Big(\frac{2}{1+\sqrt{5}}\Big)^{2n-1}, \end{aligned} \tag{7.13}$$

$$\begin{aligned} &|\varepsilon_u'(\theta)| + |\varepsilon_v'(\theta)| \\ &\quad \leq 99.5724\Big(\frac{2}{1+\sqrt{5}}\Big)^{2(n-1)} + 55.318\Big(\frac{2}{1+\sqrt{5}}\Big)^{2n-1}. \end{aligned} \tag{7.14}$$

Using (7.11)–(7.14) and that

$$\int_{\pi/3}^{2\pi/3} j(e^{i\theta})\, d\theta = 753.982,$$

we have

$$J(w) = J(u) + J(v) + \delta_w, \tag{7.15}$$

with

$$|\operatorname{Re}(\delta_w)| \le 115181.57371 \left(\frac{2}{1+\sqrt{5}}\right)^{2(n-1)}, \tag{7.16}$$

$$|\operatorname{Im}(\delta_w)| \le 100853.23866 \left(\frac{2}{1+\sqrt{5}}\right)^{2(n-1)}. \tag{7.17}$$

8. Global formula for $(2\log\varepsilon)\, j(w)$

Consider a path on the Markov–Hurwitz tree given by Markov irrationalities w_0, w_1, $w_2, \ldots, w_n$, $n \ge 2$. The quadratic w_n is on the n-th level of the tree, the first level corresponding to $w_1 = (\overline{2,3,4})$. If w_n is on the left half of the tree, $w_0 = (\overline{3})$, otherwise $w_0 = (\overline{2,4})$. Let $r_1, \ldots, r_m$ be the successive levels where the path turns, starting from $r_1 = 1$, and let $r_{m+1} = n$. For convenience, we set $r_{-1} = 1$. We write q_i for the denominator of the Farey fraction associated to w_i. If w_n is on the left half of the tree, then $q_0 = 1$, otherwise $q_0 = 2$. We write $\delta_i = \delta_{w_i}$ for the error in (7.15) associated to w_i.

It follows from the relation (7.15) applied to $J(w_n)$ that

$$J(w_n) = J(w_{n-1}) + J(w_{r_m-1}) + \delta_n. \tag{8.1}$$

By the construction of the Farey tree, we also have

$$q_n = q_{n-1} + q_{r_m-1}. \tag{8.2}$$

If we apply (7.15) recursively to $J(w_n), \ldots, J(w_{r_m})$ and $q_n, \ldots, q_{r_m}$, and if $m \ge 2$, we have

$$J(w_n) = (n - r_m + 1) J(w_{r_m-1}) + J(w_{r_{m-1}-1}) + \sum_{i=r_m}^{n} \delta_i, \tag{8.3}$$

$$q_n = (n - r_m + 1) q_{r_m-1} + q_{r_{m-1}-1}. \tag{8.4}$$

If we again apply (7.15) recurrently to $J(w_{r_k-1}), \ldots, J(w_{r_{k-1}})$ and $q_{r_k-1}, \ldots, q_{r_{k-1}}$, then for $3 \le k \le m$, we obtain

$$J(w_{r_k-1}) = (r_k - r_{k-1}) J(w_{r_{k-1}-1}) + J(w_{r_{k-2}-1}) + \sum_{i=r_{k-1}}^{r_k-1} \delta_i, \tag{8.5}$$

$$q_k = (r_k - r_{k-1}) q_{r_{k-1}-1} + q_{r_{k-2}-1}. \tag{8.6}$$

Let k_0 be a positive constant 'large enough' that will indicate a level 'down enough' on the Markov–Hurwitz tree. We will choose a specific value for k_0 later. We denote by k and s the positive integers such that

$$k_0 = r_{k-1} + s, \quad 1 \le s \le r_k - r_{k-1}.$$

If $m \ge 2$, the recursive formulas (8.5) and (8.3) give

$$\begin{pmatrix} J(w_n) \\ J(w_{r_m-1}) \end{pmatrix} = \begin{pmatrix} n - r_m + 1 & 1 \\ 1 & 0 \end{pmatrix} \Bigg(\prod_{i=k+1}^{m} \begin{pmatrix} r_i - r_{i-1} & 1 \\ 1 & 0 \end{pmatrix} \Bigg) \cdot \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J(w_{r_{k-1}-1}) \\ J(w_{r_{k-2}-1}) \end{pmatrix} + \begin{pmatrix} \delta \\ \delta' \end{pmatrix} \tag{8.7}$$

where

$$\delta = \sum_{i=r_m}^{n} \delta_i + \Bigg(\sum_{j=k+1}^{m} \lambda_j \sum_{i=r_{j-1}}^{r_j-1} \delta_i \Bigg) + \lambda_k \sum_{i=k_0}^{r_k-1} \delta_i \tag{8.8}$$

and λ_j is the coefficient of $J(w_{r_{j-1}})$ in the recursive formula (8.7) for $k \le j \le m$.

If $m = 1$, then

$$J(w_n) = (n - k_0) J(w_0) + J(w_{k_0}) + \sum_{i=k_0-1}^{n} \delta_i. \tag{8.9}$$

We obtain similar formulas for q_n from (8.6) and (8.4): if $m \ge 2$, then

$$\begin{pmatrix} q_n \\ q_{r_m-1} \end{pmatrix} = \begin{pmatrix} n - r_m + 1 & 1 \\ 1 & 0 \end{pmatrix} \Bigg(\prod_{i=k+1}^{m} \begin{pmatrix} r_i - r_{i-1} & 1 \\ 1 & 0 \end{pmatrix} \Bigg) \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{r_{k-1}-1} \\ q_{r_{k-2}-1} \end{pmatrix}, \tag{8.10}$$

and if $m = 1$, then

$$q_n = (n - k_0) q_0 + q_{k_0}. \tag{8.11}$$

The coefficients of $q_{r_{j-1}}$ in (8.10) are the same as the coefficients λ_j for $J(w_{r_{j-1}})$ in (8.7), so for all $j \ge 2$, we have

$$q_n \ge \lambda_j q_{r_{j-1}} \tag{8.12}$$

and

$$\frac{J(w_n)}{q_n} = \frac{\alpha_n J(w_{r_{k-2}-1}) + \beta_n J(w_{r_{k-1}-1})}{\alpha_n q_{r_{k-2}-1} + \beta_n q_{r_{k-1}-1}} + \frac{\delta}{q_n}, \tag{8.13}$$

where α_n, β_n are some positive constants given by (8.7) that depend on n, (8.10), and δ is given by (8.8).

9. Asymptotic interlacing property

Suppose $n > k_0$. Below we bound δ/q_n. We consider first the case when $m \geq 2$. Using (7.16), (8.12) and $q_{r_{k-1}} \geq 4$, we have that

$$\frac{\lambda_k}{q_n}\sum_{i=k_0}^{r_k-1}|\operatorname{Re}(\delta_i)| \leq \frac{115181.57371}{4}\sum_{i=k_0}^{r_k-1}\left(\frac{2}{1+\sqrt{5}}\right)^{2(k-1)}$$
$$\leq \frac{115181.57371}{4}\left(\frac{2}{1+\sqrt{5}}\right)^{2k_0-3}. \tag{9.1}$$

Similarly, for $j \geq k+1$, using (8.12) and $q_{r_{j-1}} \geq r_{j-1} \geq k_0$, we have

$$\frac{1}{q_n}\sum_{j=k+1}^{m}\lambda_j\sum_{i=r_{j-1}}^{r_j-1}|\operatorname{Re}(\delta_i)| \leq \sum_{j=k+1}^{m}\frac{1}{q_{r_{j-1}}}\sum_{i=r_{j-1}}^{r_j-1}|\operatorname{Re}(\delta_i)|$$
$$\leq \frac{115181.57371}{k_0}\left(\frac{2}{1+\sqrt{5}}\right)^{2k_0-1}. \tag{9.2}$$

By (7.16) and $q_n > k_0$, we also have that

$$\frac{|\operatorname{Re}(\delta_n)|}{q_n} \leq \frac{115181.57371}{k_0+1}\left(\frac{2}{1+\sqrt{5}}\right)^{2k_0}. \tag{9.3}$$

If $m = 1$, then $q_n \geq n+2 \geq k_0+3$, and so

$$\frac{|\operatorname{Re}(\delta)|}{q_n} = \frac{1}{q_n}\sum_{i=k_0-1}^{n}|\operatorname{Re}(\delta_i)| \leq \frac{115181.57371}{k_0+3}\left(\frac{2}{1+\sqrt{5}}\right)^{2k_0-5}. \tag{9.4}$$

Similar bounds for the imaginary parts can be found by using (7.17). Thus, letting $k_0 \to \infty$ as $n \to \infty$, we have

$$\frac{\delta}{q_n} \to 0 \quad \text{as } n \to \infty. \tag{9.5}$$

By (8.1) and (8.2), if

$$\frac{\operatorname{Re}(J(w_{r_m-1}))}{q_{r_m-1}} \leq \frac{\operatorname{Re}(J(w_{n-1}))}{q_{n-1}},$$

then

$$\frac{\operatorname{Re}(J(w_{r_m-1}))}{q_{r_m-1}} \leq \frac{\operatorname{Re}(J(w_n))}{q_n} \leq \frac{\operatorname{Re}(J(w_{n-1}))}{q_{n-1}} \quad \text{as } n \to \infty,$$

and otherwise the inequalities are reversed. The same inequalities for the imaginary parts hold. By (4.2), we conclude that $j(w_n)$ lies between $j(w_{n-1})$ and $j(w_{r_m-1})$, and hence Theorem 1 is proved.

10. Asymptotic bounds for $j(w)$

By (8.13), we have

$$\frac{\operatorname{Re}(J(w_{r_{k-1}-1}))}{q_{r_{k-1}-1}} \leq \frac{\alpha_n J(w_{r_{k-2}-1}) + \beta_n J(w_{r_{k-1}-1})}{\alpha_n q_{r_{k-2}-1} + \beta_n q_{r_{k-1}-1}} \leq \frac{\operatorname{Re}(J(w_{r_{k-2}-1}))}{q_{r_{k-2}-1}} \tag{10.1}$$

if

$$\frac{\operatorname{Re}(J(w_{r_{k-2}-1}))}{q_{r_{k-2}-1}} \geq \frac{\operatorname{Re}(J(w_{r_{k-1}-1}))}{q_{r_{k-1}-1}},$$

otherwise the inequalities are reversed in (10.1). We have similar bounds for the imaginary parts.

Let us choose $k_0 = 12$. We checked computationally for all $\ell \leq 12$ that

$$1251.36168 \leq \frac{\operatorname{Re}(J(w_\ell))}{q_\ell} \leq 1359.5674, \tag{10.2}$$

$$-0.4813 \leq \frac{\operatorname{Im}(J(w_\ell))}{q_\ell} \leq 0. \tag{10.3}$$

By (10.1) and (8.13),

$$1251.36168 + \frac{\operatorname{Re}(\delta)}{q_n} \leq \frac{\operatorname{Re}(J(w_n))}{q_n} \leq 1359.5674 + \frac{\operatorname{Re}(\delta)}{q_n},$$

$$-0.4813 + \frac{\operatorname{Im}(\delta)}{q_n} \leq \frac{\operatorname{Im}(J(w_n))}{q_n} \leq \frac{\operatorname{Im}(\delta)}{q_n}.$$

From the bounds (9.1)–(9.4) with $k_0 = 12$, we obtain

$$\frac{|\operatorname{Re}(\delta)|}{q_n} \leq 1.41173, \tag{10.4}$$

$$\frac{|\operatorname{Im}(\delta)|}{q_n} \leq 1.23611. \tag{10.5}$$

By (4.2), (7.16) and (7.17), we have

$$3206.24623 \leq \frac{\operatorname{Re}(J(w_n))}{\sqrt{n}} \leq 3491.04708 \quad \text{as } n \to \infty, \tag{10.6}$$

$$-4.40533 \leq \frac{\operatorname{Im}(J(w_n))}{\sqrt{n}} \leq 3.170734 \quad \text{as } n \to \infty, \tag{10.7}$$

and by (4.5),

$$681.50081 \leq \operatorname{Re}(j(w_n)) \leq 742.03641 \quad \text{as } n \to \infty, \tag{10.8}$$

$$-0.93637 \leq \operatorname{Im}(j(w_n)) \leq 0.67396 \quad \text{as } n \to \infty. \tag{10.9}$$

Remark 10.1. The condition $n \to \infty$ in (10.6)–(10.9) could be replaced by $n \geq n_0$ for an explicit value of n_0 if the asymptotic relations (4.2) and (4.3) are made explicit. Then one could compute the values $J(w_n)$, $j(w_n)$ for $n < n_0$, and hence obtain upper and lower bounds for all $n \geq 1$.

Acknowledgements. We thank T. Matsusaka for his comments on the paper.

A. Appendix

The first table below shows the values of $J(w(p/q))/q$ and $j(w(p/q))$ for the first 40 Farey fractions p/q (ordered as real numbers) among all Farey fractions up to the level $2^{12} = 4096$ on the Farey tree. The second table shows the last 40 values. The programs used for the computations were done in collaboration with Don Zagier.

p/q	$J(w(p/q))/q$	$j(w(p/q))$
0	1359.56741044	706.324813541
1/14	1341.67984291 – 0.122490502636 ∗ I	706.858789119 – 0.0645336432753 ∗ I
1/13	1340.30387617 – 0.131912848914 ∗ I	706.900488474 – 0.0695732206634 ∗ I
2/25	1339.53333481 – 0.137189362590 ∗ I	706.923879686 – 0.0724001664859 ∗ I
1/12	1338.69858166 – 0.142905585739 ∗ I	706.949252302 – 0.0754665750541 ∗ I
3/35	1338.10232944 – 0.146988601101 ∗ I	706.967396097 – 0.0776593436020 ∗ I
2/23	1337.79124132 – 0.149118869986 ∗ I	706.976869216 – 0.0788042174199 ∗ I
3/34	1337.47100355 – 0.151311793837 ∗ I	706.986625824 – 0.0799833523780 ∗ I
1/11	1336.80141549 – 0.155896998255 ∗ I	707.007041981 – 0.0824507472184 ∗ I
4/43	1336.27197388 – 0.159522502181 ∗ I	707.023200307 – 0.0844035587194 ∗ I
3/32	1336.08997833 – 0.160768769155 ∗ I	707.028757860 – 0.0850752157429 ∗ I
5/53	1335.94232156 – 0.161779891418 ∗ I	707.033267994 – 0.0856202872524 ∗ I
2/21	1335.71732077 – 0.163320649151 ∗ I	707.040142609 – 0.0864511175163 ∗ I
5/52	1335.48799304 – 0.164891036840 ∗ I	707.047151949 – 0.0872982300007 ∗ I
3/31	1335.33264200 – 0.165954847856 ∗ I	707.051901658 – 0.0878722552532 ∗ I
4/41	1335.13561141 – 0.167304071583 ∗ I	707.057927362 – 0.0886004905281 ∗ I
1/10	1334.52481658 – 0.171486665136 ∗ I	707.076619003 – 0.0908594653933 ∗ I
5/49	1334.01374406 – 0.174986346915 ∗ I	707.092272845 – 0.0927512885887 ∗ I
4/39	1333.88269983 – 0.175883701217 ∗ I	707.096288695 – 0.0932366184737 ∗ I
7/68	1333.78827089 – 0.176530324170 ∗ I	707.099182986 – 0.0935864040170 ∗ I
3/29	1333.66128026 – 0.177399920555 ∗ I	707.103075992 – 0.0940568878788 ∗ I
8/77	1333.54913268 – 0.178167875804 ∗ I	707.106514624 – 0.0944724589373 ∗ I
5/48	1333.48137685 – 0.178631848767 ∗ I	707.108592428 – 0.0947235689505 ∗ I
7/67	1333.40350822 – 0.179165071426 ∗ I	707.110980627 – 0.0950121914106 ∗ I

Table 1. First 40 values of $j(w)$ (cont. on next page).

p/q	$J(w(p/q))/q$	$j(w(p/q))$
2/19	1333.20678745 − 0.180512160249 ∗ I	707.117015286 − 0.0957415017533 ∗ I
7/66	1333.00708607 − 0.181879659509 ∗ I	707.123143307 − 0.0964820951663 ∗ I
5/47	1332.92635572 − 0.182432478358 ∗ I	707.125621143 − 0.0967815506251 ∗ I
8/75	1332.85531302 − 0.182918958946 ∗ I	707.127801902 − 0.0970451031725 ∗ I
3/28	1332.73606276 − 0.183735551361 ∗ I	707.131463014 − 0.0974875617700 ∗ I
7/65	1332.59846631 − 0.184677773378 ∗ I	707.135688236 − 0.0979981950158 ∗ I
4/37	1332.49433927 − 0.185390806256 ∗ I	707.138886314 − 0.0983846943323 ∗ I
5/46	1332.34720323 − 0.186398352714 ∗ I	707.143406240 − 0.0989309436276 ∗ I
1/9	1331.74231064 − 0.190540488152 ∗ I	707.161999257 − 0.101177976749 ∗ I
6/53	1331.21731397 − 0.194135299692 ∗ I	707.178150980 − 0.103129850277 ∗ I
5/44	1331.10992829 − 0.194870602053 ∗ I	707.181456402 − 0.103529297799 ∗ I
9/79	1331.03788473 − 0.195363906168 ∗ I	707.183674280 − 0.103797319799 ∗ I
4/35	1330.94731569 − 0.195984059913 ∗ I	707.186462829 − 0.104134305293 ∗ I
11/96	1330.87278491 − 0.196494394766 ∗ I	707.188757875 − 0.104411652668 ∗ I
7/61	1330.83002135 − 0.196787209845 ∗ I	707.190074826 − 0.104570801246 ∗ I
10/87	1330.78283397 − 0.197110316139 ∗ I	707.191528119 − 0.104746426028 ∗ I

Table 1. First 40 values of $j(w)$ (cont. from previous page).

p/q	$J(w(p/q))/q$	$j(w(p/q))$
67/144	1256.80214081 − 0.102528424246 ∗ I	709.686203382 − 0.0578953566189 ∗ I
47/101	1256.79136764 − 0.102325397664 ∗ I	709.686610778 − 0.0577812408124 ∗ I
74/159	1256.78161080 − 0.102141524533 ∗ I	709.686979746 − 0.0576778888468 ∗ I
27/58	1256.76462045 − 0.101821331667 ∗ I	709.687622275 − 0.0574979097853 ∗ I
61/131	1256.74399856 − 0.101432700632 ∗ I	709.688402163 − 0.0572794549412 ∗ I
34/73	1256.72761405 − 0.101123925288 ∗ I	709.689021820 − 0.0571058826255 ∗ I
41/88	1256.70322348 − 0.100664271084 ∗ I	709.689944295 − 0.0568474876195 ∗ I
7/15	1256.58452267 − 0.0984272872898 ∗ I	709.694434219 − 0.0555898124674 ∗ I
36/77	1256.44886461 − 0.0958707343734 ∗ I	709.699566670 − 0.0541521589596 ∗ I
29/62	1256.41604412 − 0.0952522135066 ∗ I	709.700808570 − 0.0538042897974 ∗ I
51/109	1256.39285899 − 0.0948152767474 ∗ I	709.701685917 − 0.0535585356735 ∗ I
22/47	1256.36227436 − 0.0942388920864 ∗ I	709.702843321 − 0.0532343345784 ∗ I
59/126	1256.33581623 − 0.0937402736098 ∗ I	709.703844616 − 0.0529538613138 ∗ I
37/79	1256.32007532 − 0.0934436271744 ∗ I	709.704440344 − 0.0527869914925 ∗ I
52/111	1256.30220725 − 0.0931068933828 ∗ I	709.705116596 − 0.0525975662884 ∗ I
15/32	1256.25809547 − 0.0922755818348 ∗ I	709.706786180 − 0.0521298981978 ∗ I
53/113	1256.21476442 − 0.0914589837656 ∗ I	709.708426336 − 0.0516704732988 ∗ I
38/81	1256.19764598 − 0.0911363771209 ∗ I	709.709074332 − 0.0514889627848 ∗ I
61/130	1256.18276611 − 0.0908559574990 ∗ I	709.709637605 − 0.0513311839730 ∗ I

Table 2. Last 40 values of $j(w)$ (cont. on next page).

p/q	$J(w(p/q))/q$	$j(w(p/q))$
23/49	1256.15816877 − 0.0903924066955 ∗ I	709.710568762 − 0.0510703571911 ∗ I
54/115	1256.13036308 − 0.0898683927437 ∗ I	709.711621420 − 0.0507754964001 ∗ I
31/66	1256.10971946 − 0.0894793520826 ∗ I	709.712402972 − 0.0505565755912 ∗ I
39/83	1256.08111686 − 0.0889403198412 ∗ I	709.713485890 − 0.0502532388899 ∗ I
8/17	1255.97007146 − 0.0868476064334 ∗ I	709.717690658 − 0.0490754390393 ∗ I
33/70	1255.83840334 − 0.0843662462496 ∗ I	709.722677340 − 0.0476786169351 ∗ I
25/53	1255.79617017 − 0.0835703382661 ∗ I	709.724277078 − 0.0472305134542 ∗ I
42/89	1255.76295307 − 0.0829443432229 ∗ I	709.725535379 − 0.0468780499190 ∗ I
17/36	1255.71405012 − 0.0820227394093 ∗ I	709.727388008 − 0.0463591090605 ∗ I
43/91	1255.66622196 − 0.0811213906246 ∗ I	709.729200069 − 0.0458515317763 ∗ I
26/55	1255.63491625 − 0.0805314168746 ∗ I	709.730386225 − 0.0455192770302 ∗ I
35/74	1255.59641869 − 0.0798059086145 ∗ I	709.731844962 − 0.0451106692538 ∗ I
9/19	1255.48497839 − 0.0777057531246 ∗ I	709.736068162 − 0.0439277065400 ∗ I
28/59	1255.34520582 − 0.0750716597984 ∗ I	709.741366188 − 0.0424436737723 ∗ I
19/40	1255.27881384 − 0.0738204654684 ∗ I	709.743883191 − 0.0417386346705 ∗ I
29/61	1255.21459865 − 0.0726102939033 ∗ I	709.746317939 − 0.0410566358914 ∗ I
10/21	1255.09228401 − 0.0703052052080 ∗ I	709.750956292 − 0.0397573845879 ∗ I
21/44	1254.92271143 − 0.0671095140622 ∗ I	709.757388330 − 0.0379557027688 ∗ I
11/23	1254.76788430 − 0.0641917091029 ∗ I	709.763262682 − 0.0363102351122 ∗ I
12/25	1254.49538854 − 0.0590563723747 ∗ I	709.773605298 − 0.0334131593620 ∗ I
1/2	1251.36168734	709.892890920

Table 2. Last 40 values of $j(w)$ (cont. from previous page).

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Received 28 May 2021

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