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Boundary singularities in mean curvature flow and total curvature of minimal surface boundaries

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Abstract. For hypersurfaces moving by standard mean curvature flow with fixed boundary, we show that if a tangent flow at a boundary singularity is given by a smoothly embedded shrinker, then the shrinker must be non-orientable. We also show that there is an initially smooth surface in $\mathbf{R}^3$ that develops a boundary singularity for which the shrinker is smoothly embedded (and therefore non-orientable). Indeed, we show that there is a non-empty open set of such initial surfaces.

Let κ be the largest number with the following property: if M is a minimal surface in $\mathbf{R}^3$ bounded by a smooth simple closed curve of total curvature $< \kappa$, then M is a disk. Examples show that $\kappa < 4\pi$. In this paper, we use mean curvature flow to show that $\kappa \geq 3\pi$. We get a slightly larger lower bound for orientable surfaces.

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1. Introduction

Suppose $M \subset \mathbf{R}^{n+1}$ is a compact, smoothly embedded n-manifold with boundary Γ. According to [14], there is a standard Brakke flow

$$t \in [0, \infty) \mapsto M(t)$$

with (fixed) boundary Γ and with initial surface $M(0) = M$. Furthermore, if Γ lies on the boundary of the convex hull of M, then the flow is regular at the boundary for all times. (See [14, Sections 5.1 and 13.2] for the definitions of Brakke flow with boundary and standard Brakke flow with boundary. Briefly, standard Brakke flows are those that are unit-regular – which prevents certain gratuitous vanishing – and that take their boundaries as mod 2 chains. In particular, triple junction singularities do not occur in standard Brakke flow.)

In this paper, we explore boundary regularity of such a flow $M(\cdot)$ without assuming that Γ lies on the boundary of the convex hull of M.

In particular, we show that if a shrinker corresponding to a boundary singularity is smooth and embedded, then it must be non-orientable. We also show that smooth, non-orientable shrinkers do arise as boundary singularities of certain smooth initial surfaces in $\mathbf{R}^3$. Indeed, we show that there is a non-empty open set of such initial surfaces; see Theorem 16.

We also apply mean curvature flow to questions in minimal surface theory:

(1) What is the largest number κ such that if M is a smooth minimal surface in $\mathbf{R}^3$ bounded by a smooth simple closed curve of total curvature $< \kappa$, then M must be a disk?

(2) What is the largest number κ' such that if M is an orientable smooth minimal surface in $\mathbf{R}^3$ bounded by a smooth simple closed curve of total curvature $< \kappa'$, then M must be a disk?

Examples show that $\kappa < 4\pi$. Here, we show (Theorem 17) that $\kappa > 3\pi$.

Examples of Almgren–Thurston show that $\kappa' \leq 4\pi$. It is conjectured that $\kappa' = 4\pi$. We show (Theorem 19) that $\kappa' > (2\pi)^{3/2} e^{-1/2} \sim 3\pi(1.014)$.

For the existence results in this paper, we work with standard Brakke flows of 2-dimensional surfaces with entropy less than two. Such flows are rather well-behaved: they are smooth at almost all times, and the shrinkers corresponding to tangent flows are smoothly embedded and have multiplicity one; see Theorem 9.

2. A Bernstein theorem for orientable boundary shrinkers

Theorem 1. *Let $M \subset \mathbf{R}^{n+1}$ be a smoothly embedded, oriented shrinker bounded by an $(n-1)$-dimensional linear subspace L. Then M is a half-plane.*

Proof. We can assume that $L = \{x : x_1 = x_2 = 0\}$. Consider the 1-form

$$d\theta = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2}$$

on $\mathbf{R}^{n+1} \setminus L$.

Let C be an oriented closed curve in $M \setminus L$. The winding number of C about L is equal to the intersection number of C and M. Since we can move C slightly in the direction of the unit normal to M to get a curve C' disjoint from M, we see that the winding number of C about L is equal to 0. Thus,

$$\int_C d\theta = 0.$$

Since this holds for any closed curve in $M \setminus L$, we see that $d\theta$ is exact on M. Thus, there is a single-valued function

$$\theta\colon M \setminus L \to \mathbf{R}$$

such that for $x \in M \setminus L$,

$$x_1 = (x_1^2 + x_2^2)^{1/2} \cos\theta(x),$$
$$x_2 = (x_1^2 + x_2^2)^{1/2} \sin\theta(x).$$

Note that $\theta(\cdot)$ extends continuously to L.

Let M' be a connected component of M.

Claim 1. *$\theta|M'$ attains a maximum.*

Proof of Claim 1. Let g be the shrinker metric on $\mathbf{R}^{n+1}$. Choose an R large enough that the g-mean curvature vector of $\partial\mathbf{B}(0, R)$ points outward. (That is, choose $R > \sqrt{2n}$.)

Let α be the maximum of θ on $\{x \in M' : |x| \le 3R\}$. By rotating, we can assume that $\alpha = 0$.

Suppose the claim is not true. Then

$$\Sigma := \{x \in M' : 0 < \theta(x) < \pi\}$$

is a non-empty hypersurface in $\{x_2 > 0\}$ whose boundary lies in the hyperplane $\{x_2 = 0\}$.

Consider n-dimensional surfaces S in $\{x : x_2 \ge 0\}$ such that:

(1) $\partial S = \partial\mathbf{B}(0, 2R) \cap \{x_2 = 0\}$.

(2) S is rotationally invariant about the x_2-axis. That is, if $\rho \in O(n+1)$ and $\rho(\mathbf{e}_2) = \mathbf{e}_2$, then $\rho(S) = S$.

(3) The region U bounded by S and $\mathbf{B}(0, 2R) \cap \{x_2 = 0\}$ contains $\mathbf{B}(0, R) \cap \{x_2 \ge 0\}$ and is disjoint from Σ.

Choose a surface S that minimizes g-area with among all such surfaces. Then S is smooth, g-minimal, and g-stable.

The restriction that S be rotationally invariant ensures that S is smooth; otherwise, S might have an $n-7$-dimensional singular set. If the smoothness is not clear, note that the rotational symmetry implies that the quotient set

$$\big\{\big(|(x_1, 0, x_3, \ldots, x_{n+1})|, x_2\big) : x \in S\big\}$$

is a curve; that curve must be a geodesic with respect to a certain Riemannian metric on $(0, \infty) \times \mathbf{R}$.

According to [3, Proposition 5], S is flat with respect to the Euclidean metric. (One lets $k \to \infty$ in the statement of that proposition.) But that is impossible since $S \setminus \partial S \subset \{x_2 > 0\}$ and since ∂S is an $(n-1)$-sphere in $\{x_2 = 0\}$. This completes the proof of the claim. □

By Claim 1 and the strong maximum principle, θ is constant on M'. Thus M' is a half-plane with boundary L. Consequently, M is a union of such half-planes. Since M is embedded, it is a single half-plane. □

Remark. The proof of Theorem 1 was inspired by the proof of Theorem 11.1 in the Hardt–Simon boundary regularity paper [6].

3. A Bernstein theorem for orientable boundary shrinkers with singularities

In this section, we extend Theorem 1 to possibly singular shrinkers. The proof is not longer, but it does use more machinery. This section is not used in the rest of the paper.

Theorem 2. *Let $M \subset \mathbf{R}^{n+1}$ be a shrinker bounded by an $(n-1)$-dimensional linear subspace L. Suppose that*

$$\mathcal{H}^{n-1}(\operatorname{sing} M) = 0 \tag{1}$$

and that the regular part

$$\operatorname{reg} M := M \setminus \operatorname{sing} M$$

is orientable. Then M is a half-plane.

By definition, $\operatorname{reg} M$ is a smooth, properly embedded manifold-with-boundary in $\mathbf{R}^{n+1} \setminus (\operatorname{sing} M)$, the boundary being $L \setminus \operatorname{sing} M$.

Proof. As in the proof of Theorem 1, we assume that L is the subspace $\{x_1 = x_2 = 0\}$. As in that proof, there is a smooth function

$$\theta \colon \operatorname{reg} M \to \mathbf{R}$$

such that

$$x_1 = (x_1^2 + x_2^2)^{1/2} \cos\theta(x),$$
$$x_2 = (x_1^2 + x_2^2)^{1/2} \sin\theta(x).$$

Claim 2. *Let $\mathcal{H}$ be the half-space $ax_1 + bx_2 > 0$, where a and b are not both zero. Then $\operatorname{reg}(M) \cap \mathcal{H}$ is connected.*

This is stated for shrinkers without boundary in [4, Corollary 3.12]. But the same proof gives Claim 2.

Let M' be a connected component of $\operatorname{reg}(M)$. Thus, $\theta(M')$ is an interval.

First, we claim that I has length at most π. For if not, by rotating we can assume that

$$\inf_{M'} \theta < 0$$

and that

$$\sup_{M'} \theta > \pi.$$

But then $\{x \in M' : \theta(x) \in (-\pi, 0)\}$ and $\{x \in M' : \theta(x) \in (\pi, 2\pi)\}$ are non-empty and lie in different components of $M' \cap \{x_2 < 0\}$, contradicting Claim 2.

Next, we claim that M' is a half-plane. For suppose not, then $\theta|M'$ does not attain a maximum or a minimum by the strong maximum principle. Thus, the interval $I = \theta(M')$ is an open interval of length at most π. By rotating, we can assume that

$$\theta(M') = (-\alpha, \alpha)$$

for some α with $0 < \alpha \leq \pi/2$. Now rotate M' by π about L to get M''. Then $M^* := \overline{M' \cup M''}$ is an embedded shrinker without boundary. Furthermore,

$$\operatorname{reg}(M^*) \cap \{x_2 > 0\}$$

is not connected, which is impossible by Claim 2 (applied to M^*).

We have shown that each component of $\operatorname{reg}(M)$ is a half-plane bounded by L. By (1), there can be at most one such half-plane. □

4. Basic properties of entropy and total curvature

Suppose that Γ is an $(m-1)$-dimensional submanifold of $\mathbf{R}^n$ and that $v \in \mathbf{R}^n$. We define the cone $C_{\Gamma,v}$ and the exterior cone $E_{\Gamma,v}$ over Γ with vertex v by

$$\begin{aligned} C_{\Gamma,v} &= \big\{v + s(x - v) : x \in \Gamma,\ s \in [0, \infty)\big\}, \\ E_{\Gamma,v} &= \big\{v + s(x - v) : x \in \Gamma,\ s \in [1, \infty)\big\}. \end{aligned}$$

We will also use $C_{\Gamma,v}$ and $E_{\Gamma,v}$ to denote the corresponding Radon measures on $\mathbf{R}^n$ (counting multiplicity). Thus,

$$\begin{aligned} C_{\Gamma,v} f &= \int_{p \in \mathbf{R}^n} f(p) \mathcal{H}^0 \big\{(x, s) \in \Gamma \times [0, \infty) : v + s(x - v) = p\big\}\, d\mathcal{H}^m p, \\ E_{\Gamma,v} f &= \int_{p \in \mathbf{R}^n} f(p) \mathcal{H}^0 \big\{(x, s) \in \Gamma \times [1, \infty) : v + s(x - v) = p\big\}\, d\mathcal{H}^m p. \end{aligned}$$

Now $C_{\Gamma,v}$ and $E_{\Gamma,v}$ depend continuously on v for $v \in \mathbf{R}^n \setminus \Gamma$. However, the dependence is not continuous at $v \in \Gamma$. For suppose $v_i \notin \Gamma$ converges to $v \in \Gamma$. Then, after passing to a subsequence, C_{Γ,v_i} converges to the union of $C_{\Gamma,v}$ and a half-plane bounded by $\operatorname{Tan}(\Gamma, v)$, and similarly for E_{Γ,v_i}.

Consequently,

$$v \in \mathbf{R}^n \mapsto \frac{C_{\Gamma,v} \mathbf{B}(v, r)}{\omega_m r^m} + \frac{1}{2} 1_\Gamma(v) \tag{2}$$

is continuous in v. (Since $C_{\Gamma,v}$ is conical, the quantity (2) is independent of r.) Here $1_\Gamma(\cdot)$ is the characteristic function (or indicator function) of Γ.

We define the *vision number* $\mathrm{vis}(\Gamma)$ of Γ to be the supremum of the quantity (2) over $v \in \mathbf{R}^n$. If Γ is compact, then the supremum is attained by a v in the convex hull of Γ.

Remark 3. Because the quantity (2) depends continuously on v, we have

$$\begin{aligned}\sup_v \left(\frac{C_{\Gamma,v}\mathbf{B}(v,r)}{\omega_m r^m} + \frac{1}{2} 1_\Gamma(v) \right) &= \sup_{v\notin\Gamma} \left(\frac{C_{\Gamma,v}\mathbf{B}(v,r)}{\omega_m r^m} + \frac{1}{2} 1_\Gamma(v) \right) \\ &= \sup_{v\notin\Gamma} \frac{C_{\Gamma,v}\mathbf{B}(v,r)}{\omega_m r^m} \\ &\le \sup_v \frac{C_{\Gamma,v}\mathbf{B}(v,r)}{\omega_m r^m} \\ &\le \sup_v \left(\frac{C_{\Gamma,v}\mathbf{B}(v,r)}{\omega_m r^m} + \frac{1}{2} 1_\Gamma(v) \right).\end{aligned}$$

Since the first and last expressions are the same, in fact equality holds. Thus, in the definition of vision number, it does not matter whether or not we include the term $\frac{1}{2}1_\Gamma(p)$.

Definition 4. If M is a Radon measure on $\mathbf{R}^n$ and if Γ is a smooth $(m-1)$-dimensional manifold in $\mathbf{R}^n$, then the *entropy* of the pair (M, Γ) is

$$e(M;\Gamma) = \sup_{v\in\mathbf{R}^n,\lambda>0} (M + E_{v,\Gamma})\psi_{v,\lambda},$$

where

$$\psi_{v,\lambda}(x) = \frac{1}{(4\pi\lambda)^{m/2}} \exp\left(-\frac{|x-v|^2}{4\lambda} \right).$$

Theorem 5 ([14, Theorem 8.1]). *Suppose that*

$$t \in I \mapsto M(t)$$

is a Brakke flow with boundary Γ *in* $\mathbf{R}^n$. *Then* $e(M(t);\Gamma)$ *is a decreasing function of* t.

Theorem 6. *Suppose that* Γ *is a smooth* $(m-1)$*-dimensional manifold in* $\mathbf{R}^n$ *and that* $t \in [T_0, T] \mapsto M(t)$ *is an* m*-dimensional Brakke flow with boundary* Γ. *Then,*

$$e(M(t);\Gamma) \le \frac{\mathrm{area}(M(T_0))}{(4\pi(t-T_0))^{m/2}} + \mathrm{vis}(\Gamma).$$

Proof. This is an immediate consequence of [14, Corollary 7.3]. □

Corollary 7. *In Theorem 6, if $M(\cdot)$ is an ancient flow and if the area of $M(t)$ is bounded above as $t \to -\infty$, then*

$$e(M(t);\Gamma) \le \mathrm{vis}(\Gamma).$$

Theorem 8 ([5, Theorem 1.1]). *Suppose that Γ is a simple closed curve in $\mathbf{R}^n$. Then*

$$\mathrm{vis}(\Gamma) \le \frac{1}{2\pi}\,\mathrm{tc}(\Gamma),$$

where $\mathrm{tc}(\Gamma)$ is the total curvature of Γ. Equality holds if and only if Γ is a convex planar curve.

Theorem 9. *Suppose that Γ is a smooth, simple closed curve in $\mathbf{R}^n$. Suppose that*

$$(a,\infty) \in \mathbf{R} \mapsto M(t)$$

is a standard Brakke flow with boundary Γ such that

$$A := \sup_{t\in(a,\infty)} \mathrm{area}(M(t)) = \lim_{t\to a} \mathrm{area}(M(t)) < \infty,$$

and that

$$\sup_t e(M(t);\Gamma) < 2.$$

Then:

(1) *At each spacetime point, each tangent flow is given by a smooth, multiplicity 1 shrinker.*

(2) *The flow $M(\cdot)$ is regular at almost all times.*

(3) *If $n = 3$, then for each tangent flow at a boundary singularity, the corresponding shrinker is non-orientable.*

(4) *Every sequence of times tending to ∞ has a subsequence $t(i)$ for which $M(t(i))$ converges smoothly to an embedded minimal surface.*

(5) *As $t \to \infty$, $M(t)$ converges smoothly to an embedded minimal surface $M(\infty)$.*

(6) *If $a = -\infty$, then as $t \to -\infty$, $M(t)$ converges smoothly to an embedded minimal surface $M(-\infty)$.*

Proof. See Appendix A for the proofs of assertions (1) and (2). In particular, Lemmas 20 and 21 give assertion (1), and Proposition 23 gives assertion (2).

In assertion (3), the non-orientability of Σ follows immediately from assertion (1) and Theorem 1.

To prove assertion (4), suppose $t(i) \to \infty$. By passing to a subsequence, we may assume that the flows

$$t \mapsto M(t(i) + t)$$

converge to a standard limit Brakke flow

$$t \in \mathbf{R} \mapsto M'(t)$$

with boundary Γ. (This is by the compactness theory for standard Brakke flows with boundary; see Theorems 10.2 and 13.1 and Definition 13.2 in [14].) Since the area of $M'(t)$ is independent of t (it is equal to $\lim_{t\to\infty} \text{area}(M(t))$), it follows that $M'(t)$ is a stationary integral varifold V independent of t. By assertion (2) applied to the flow $M'(\cdot)$, the surface $M'(t)$ is smoothly embedded for almost all t. Thus, V is smoothly embedded.

The uniqueness in assertion (5) follows from assertion (4) and from the Lojasiewicz–Simon inequality [11, Theorem 3]; see Theorem 29 below.

The proof of assertion (6) is the same as the proof of assertion (5). □

5. A general existence theorem for eternal flows

In the following two theorems, $\mathcal{F}$ is a set of C^1 compact manifolds-with-boundary in $\mathbf{R}^3$ with the following property: if $M \in \mathcal{F}$ and if M' is isotopic to M, then $M' \in \mathcal{F}$. For example, $\mathcal{F}$ might be the set of non-orientable surfaces, or the set of genus-one orientable surfaces, etc.

Theorem 10. *Suppose that Γ is a smooth, simple closed curve in $\mathbf{R}^3$ of total curvature at most 4π. Suppose that Γ bounds a minimal surface in $\mathcal{F}$ and that Γ is a smooth limit of curves Γ_i such that Γ_i does not bound a minimal surface in $\mathcal{F}$.*

Then there is an eternal standard Brakke flow

$$t \in \mathbf{R} \mapsto M'(t)$$

with boundary Γ such that:

(i) $\sup_t e(M'(t); \Gamma) < \frac{1}{2\pi} \text{tc}(\Gamma) \le 2$.

(ii) *$M'(t)$ converges smoothly as $t \to -\infty$ to a minimal surface $M'(-\infty)$ in $\mathcal{F}$.*

(iii) *$M'(-\infty)$ has the least area among all minimal surfaces of type $\mathcal{F}$ bounded by Γ.*

(iv) *$M'(t)$ converges smoothly as $t \to \infty$ to a minimal surface $M'(\infty)$ that is not in $\mathcal{F}$.*

Proof. The set of minimal surfaces in $\mathcal{F}$ bounded by Γ is compact (by Theorem 25), so there is a surface M that attains the least area A among all such surfaces.

Note that $t \in \mathbf{R} \mapsto M$ is a standard Brakke flow, so

$$e(M; \Gamma) \le \text{vis}(\Gamma) < \frac{1}{2\pi} \text{tc}(\Gamma) \le 2$$

by Corollary 7 and Theorem 8.

For each i, let M_i be a surface diffeomorphic to M and bounded by Γ_i such that M_i converges smoothly to M as $i \to \infty$. Then $e(M_i;\Gamma_i) \to e(M;\Gamma)$, so by passing to a subsequence, we can assume that

$$e(M_i;\Gamma_i) < 2.$$

for all i.

Let

$$t \in [0,\infty) \mapsto M_i(t)$$

be a standard Brakke flow with boundary Γ_i and with initial surface $M_i(0) = M_i$.

Of course,

$$\sup_t e(M_i(t);\Gamma_i) = e(M_i;\Gamma_i) < 2, \tag{3}$$

and

$$\sup_t \operatorname{area}(M_i(t)) = \operatorname{area}(M_i(0)) = \operatorname{area}(M_i) \to \operatorname{area} M = A. \tag{4}$$

As $t \to \infty$, we can assume, by passing to a subsequence, that the flow $M_i(\cdot)$ converges to a standard Brakke flow $M(\cdot)$ with $M(0) = M$. Since M is smooth and minimal, the only such flow is the constant flow $M(t) \equiv M$.

As $t \to \infty$, $M_i(t)$ converges smoothly to an embedded minimal surface $M_i(\infty)$ (by (3) and Theorem 9). By hypothesis on Γ_i, the surface $M_i(\infty)$ is not in the collection $\mathcal{F}$. Thus for all sufficiently large t, $M_i(t)$ is not in $\mathcal{F}$.

Since $M_i(0)$ is in $\mathcal{F}$ and since $M_i(t)$ is not in $\mathcal{F}$ for large t, we see that there must be singularities in the flow. Let T_i be the first singular time. Since the flow $t \mapsto M_i(t)$ converges to the constant flow $t \mapsto M$, we see that

$$T_i \to \infty.$$

By passing to a subsequence, we can assume that the time-shifted flows

$$t \in [-T_i,\infty) \mapsto M_i'(t) := M_i(T_i + t)$$

converge to an eternal standard Brakke flow

$$t \in \mathbf{R} \mapsto M'(t)$$

with boundary Γ. By Corollary 7 and Theorem 8, the flow satisfies the entropy bound (i).

By Theorem 9, there is a $c \in (0,\infty)$ such that

$$M'(t) \text{ is smooth for } |t| > c, \tag{5}$$

$$M'(t) \text{ converges smoothly to a minimal surface } M'(-\infty) \text{ as } t \to -\infty, \text{ and} \tag{6}$$

$$M'(t) \text{ converges smoothly to a minimal surface } M'(\infty) \text{ as } t \to \infty. \tag{7}$$

By choice of T_i, $M_i'(t)$ in $\mathcal{F}$ for $t < 0$. By local regularity [13], $M_i'(t)$ converges smoothly to $M'(t)$ for $t < -c$. Thus $M'(t)$ is in $\mathcal{F}$ for $t < -c$, and hence $M'(-\infty)$ is in $\mathcal{F}$.

Since 0 is a singular time of the flow $M_i'(\cdot)$, it follows (again by local regularity [13]) that 0 is a singular time of the flow $M'(\cdot)$. Hence, the flow $M'(\cdot)$ is not constant, so

$$\text{area}(M'(\infty)) < \text{area}(M'(-\infty)) \leq A. \tag{8}$$

Since A is the least area of any minimal $\mathcal{F}$-type surface bounded by Γ, it follows that area $M'(-\infty) = A$ and that $M'(\infty)$ is not of type $\mathcal{F}$. □

Theorem 11. *Suppose that the family $\mathcal{F}$ does not include disk-type surfaces. Suppose there is a smooth simple closed curve Γ_0 of total curvature $< 4\pi$ that bounds a minimal surface of type $\mathcal{F}$. Then there is a curve Γ such that $\text{tc}(\Gamma) \leq \text{tc}(\Gamma_0)$ and such that Γ satisfies the hypotheses of Theorem 10.*

Proof. By Theorem 28, there is a smooth one-parameter family

$$s \in [0, 1] \mapsto \Gamma_s$$

of simple closed curves (starting from the given curve Γ_0) for which Γ_1 is a round circle and for which each curve has total curvature $\leq \text{tc}(\Gamma_0)$.

Let S be the set of $s \in [0, 1]$ such that Γ_s bounds a minimal surface of type $\mathcal{F}$.

By Theorem 25, S is closed. And S is non-empty since $0 \in S$. Thus, S has a maximum $\hat{s}$. Note that $\hat{s} < 1$. Hence $\Gamma_{\hat{s}}$ bounds a minimal surface of type $\mathcal{F}$, and $\Gamma_{\hat{s}}$ is a smooth limit of the curves Γ_s, $s > \hat{s}$, that do not bound minimal surfaces of type $\mathcal{F}$. □

6. Generalized Möbius strips

Suppose that $M \subset \mathbf{R}^3$ is a smoothly embedded, compact surface with exactly one boundary component. More generally, we can allow M to have self-intersections and singularities, provided they occur away from the boundary. That is, we only require that ∂M is a smooth, simple closed curve and that M is a smoothly embedded manifold-with-boundary (the boundary being ∂M) near ∂M.

Although M may not be orientable, we can choose an orientation for ∂M. Now push ∂M slightly into M to get another smooth embedded curve C. For example, if $\varepsilon > 0$ is sufficiently small, we can let

$$C = \{p \in M : \text{dist}(p, \partial M) = \varepsilon\}.$$

We let $\lambda(M)$ be the linking number of ∂M and C. This can be defined in various (equivalent) ways. For example, let Σ be a compact oriented surface (not necessarily embedded) with boundary ∂M. By perturbing Σ slightly, we can assume that C

intersects Σ transversely. Then $\lambda(M)$ is the intersection number (in $\mathbf{R}^3 \setminus \partial M$) of C and Σ. Alternatively, we can let Σ be a compact oriented surface with boundary C. Then $\lambda(M)$ is the intersection number of ∂M and Σ in $\mathbf{R}^3 \setminus C$.

We began by choosing an orientation for ∂M, but the resulting value of $\lambda(M)$ does not depend on that choice, since reversing the orientation of ∂M also reverses the orientations of C and of Σ, thus leaving the value of $\lambda(M)$ the same.

Of course, if M is smoothly embedded and orientable, then $\lambda(M) = 0$, since we can let Σ be the portion of M bounded by C. Note that Σ is an oriented surface with boundary C and that Σ is disjoint from ∂M, so the linking number of C and ∂M is 0.

Proposition 12. *If M is a smoothly embedded Möbius strip, then $\lambda(M) \neq 0$. Indeed, $\lambda(M)/2$ is an odd integer.*

Proof. Let S be an embedded, orientation-reversing path in the interior of M. Note that we can perturb S slightly to get a curve S' that intersects M transversely and in exactly one point. Thus the mod 2 linking number of ∂M and S is 1, and therefore the integer linking number of ∂M and S (whichever way we orient those curves) is an odd integer.

Now push ∂M into M to get an embedded curve C as in the definition of $\lambda(M)$. Then C is homologous in $M \setminus \partial M$ (and therefore in $\mathbf{R}^3 \setminus \partial M$) to S traversed twice, so $\lambda(M)$, the linking number of ∂M and C, is equal to the twice the linking number of ∂M and S. □

Definition 13. A *generalized Möbius strip* in $\mathbf{R}^3$ is a smoothly embedded, compact (not necessarily connected) surface M in $\mathbf{R}^3$ such that ∂M has exactly one component, and such that $\lambda(M)$ is non-zero.

Every generalized Möbius strip is non-orientable, but not every non-orientable surface in $\mathbf{R}^3$ is a generalized Möbius strip. For example, if we attach a handle to a flat disk to make a non-orientable surface M (topologically a Klein bottle with a disk removed), then $\lambda(M) = 0$.

Note that if a generalized Möbius strip in Euclidean space is a minimal surface, then it has no components without boundary and thus must be connected.

Lemma 14. *Suppose Γ is a smooth, simple closed curve in $\mathbf{R}^3$, suppose $t \in [a, b] \mapsto M(t)$ is a Brakke flow with boundary Γ such that a and b are regular times, and suppose the flow has no boundary singularities. Then $t \in [a, b] \mapsto \lambda(M(t))$ is constant. In particular, $M(a)$ is a generalized Möbius strip if and only if $M(b)$ is a generalized Möbius strip.*

Proof. The lemma is true because $\lambda(M(t))$ only depends on the behavior of $M(t)$ in an arbitrarily small neighborhood of the boundary. □

Theorem 15. *Let κ_{gm} be the infimum of the total curvature $\mathrm{tc}(\Gamma)$ among smooth, simple closed curves Γ in $\mathbf{R}^3$ that bound generalized minimal Möbius strips. Then*

$$\kappa_{gm} < 4\pi. \tag{9}$$

If $\kappa_{gm} < \kappa < 4\pi$, then there exists a smooth, simple closed curve Γ in $\mathbf{R}^3$ of total curvature $< \kappa$ and an eternal standard Brakke flow $t \in \mathbf{R} \mapsto M(t)$ with boundary Γ such that:

(1) $\sup_t e(M(t);\Gamma) < \frac{\mathrm{tc}(\Gamma)}{2\pi}$.

(2) *$M(t)$ converges smoothly as $t \to -\infty$ to a generalized Möbius strip $M(-\infty)$.*

(3) *The surface $M(-\infty)$ has the least area of any minimal, generalized Möbius strip bounded by Γ.*

(4) *$M(t)$ converges smoothly as $t \to \infty$ to a minimal surface $M(\infty)$ that is not a generalized Möbius strip.*

Furthermore, such a flow must have a boundary singularity, and the shrinker Σ corresponding to a tangent flow at any such boundary singularity is a smoothly embedded, non-orientable surface with straight line boundary.

Proof. By [5, Section 5], there is a smooth, simple closed curve of total curvature $<$ 4π such that the curve bounds a minimal Möbius strip. Thus, $\kappa_{gm} < 4\pi$.

By Theorem 11, there exists a curve Γ and an eternal standard Brakke flow $M(\cdot)$ with boundary Γ and having properties (1)–(4). (One lets $\mathcal{F}$ be the family of all generalized Möbius strips.)

By Lemma 14, there must be a boundary singularity. By Theorem 9, the tangent flow must have the indicated properties. □

7. Boundary singularities are unavoidable

Theorem 16. *Let $\mathcal{C}$ be the set of smoothly embedded surfaces M in $\mathbf{R}^3$ such that:*

(i) *the boundary curve Γ has total curvature less than 4π,*

(ii) *the entropy $e(M;\Gamma)$ is less than 2, and*

(iii) *any standard Brakke flow with boundary Γ and with initial surface M must develop a boundary singularity for which the corresponding shrinker is a smoothly embedded, multiplicity-one, non-orientable surface with straight line boundary.*

Then $\mathcal{C}$ has non-empty interior.

Proof. Let Γ and

$$t \in \mathbf{R} \mapsto M(t)$$

be as in Theorem 15. Let T be a regular time at which $M(T)$ is a smoothly embedded generalized Möbius strip. (For example, T could be any time before the first singularity.) We will show that $M(T)$ lies in the interior of $\mathcal{C}$.

By time translation, we can assume that $T = 0$. Let M_i be a sequence of surfaces that converge smoothly to $M(0)$. Let $\Gamma_i = \partial M_i$. Let $t \in [0,\infty) \mapsto M_i(t)$ be a standard Brakke flow with boundary Γ_i and with initial surface $M_i(0) = M_i$.

Note that

$$\sup_t e(M_i(t);\Gamma_i) = (e(M_i(0);\Gamma_i) \to e(M(0);\Gamma) < 2. \tag{10}$$

Consequently, $M_i(t)$ converges smoothly as $t \to \infty$ to an embedded minimal surface $M_i(\infty)$. By passing to a subsequence, we can assume (see Theorem 25) that $M_i(\infty)$ converges smoothly to a minimal surface M' bounded by Γ. Now,

$$\operatorname{area}(M') = \lim_i \operatorname{area}(M_i(\infty)) \le \lim_i \operatorname{area} M_i(0) = \operatorname{area} M(0) < \operatorname{area}(M(-\infty)).$$

Since $M(-\infty)$ achieves the least area of minimal generalized Möbius strips bounded by Γ, it follows that M' is not a generalized Möbius strip, and hence neither is $M_i(\infty)$ for i large. For each such i, $M_i(t)$ is not a generalized Möbius strip for all sufficiently large t.

Thus, for all sufficiently large i,

$$M_i(0) \text{ is a generalized Möbius strip, and} \tag{11}$$

$$M_i(t) \text{ is not a generalized Möbius strip for all sufficiently large } t. \tag{12}$$

By (11), (12), and Lemma 14, $M_i(\cdot)$ must have a boundary singularity. By (10) and Theorem 9, the corresponding shrinker must be smoothly embedded and non-orientable, and must have multiplicity one. □

8. The three pi theorem

Theorem 17. *Suppose that Γ' is a smoothly embedded, simple closed curve in $\mathbf{R}^3$ and that Γ' bounds a smooth minimal surface that is not a disk. Then,* $\operatorname{tc}(\Gamma') > 3\pi$.

Proof. We may assume that $\operatorname{tc}(\Gamma') < 4\pi$.

By Theorem 11 (applied to the family $\mathcal{F}$ of non-disk surfaces), there is a smooth, simple closed curve Γ with

$$\operatorname{tc}(\Gamma) \le \operatorname{tc}(\Gamma')$$

and a there is a standard Brakke flow $t \in \mathbf{R} \mapsto M(t)$ with boundary Γ for which $M(-\infty)$ is not a disk and $M(\infty)$ is a disk.

Thus, the flow must have at least one singularity. Consider the shrinker Σ corresponding to a tangent flow at the first singular time T.

If the singularity is a boundary singularity, $e(M(\cdot);\Gamma) > \frac{3}{2}$ by Theorem 18 below.

Now suppose the singularity is an interior singularity. Since $M(t)$ is connected and with non-empty boundary for $t < T$, we see that Σ is non-compact. By [2, Corollary 1.2], the entropy of Σ is greater than or equal to the entropy

$$\sigma_1 = (2\pi/e)^{1/2} \cong 1.52$$

of a round cylinder.

Thus, in either case (boundary singularity or interior singularity),

$$e(M(T);\Gamma) \geq \frac{3}{2}.$$

On the other hand,

$$e(M(T);\Gamma) \leq \frac{\operatorname{tc}(\Gamma)}{2\pi}.$$

Thus,

$$\frac{\operatorname{tc}(\Gamma)}{2\pi} > \frac{3}{2},$$

so

$$\operatorname{tc}(\Gamma') \geq \operatorname{tc}(\Gamma) > 3\pi. \qquad \square$$

Theorem 18. *Let $\Sigma \subset \mathbf{R}^{n+1}$ be a smooth, non-orientable shrinker whose boundary is an $(n-1)$-dimensional linear subspace L. Then*

$$e(\Sigma; L) > \frac{3}{2}.$$

Proof. Rotate Σ by π about L to get Σ'. Then $M := \Sigma \cup \Sigma'$ is a smoothly immersed shrinker without boundary. Since it is non-orientable, it must have a point p of self-intersection. Thus the entropy of M is ≥ 2. In fact, the entropy must be > 2, since otherwise M would be a cone centered at p. Since M is smooth, that means M would be planar, which is impossible since it is non-orientable.

Thus,

$$\int_\Sigma \frac{1}{(4\pi)^{1/2}} e^{-|x|^2/4}\,dx = \frac{1}{2}\int_M \frac{1}{(4\pi)^{1/2}} e^{-|x|^2/4}\,dx = \frac{1}{2} e(M) > 1,$$

where $e(M) := e(M;\emptyset)$ is the entropy of M, so

$$e(\Sigma; L) \geq \frac{1}{2} + \int_\Sigma \frac{1}{(4\pi)^{1/2}} e^{-|x|^2/4}\,dx > \frac{3}{2}. \qquad \square$$

9. Oriented surfaces

We can improve Theorem 17 slightly in the case of oriented surfaces.

Theorem 19. *Suppose that Γ' is a smoothly embedded, simple closed curve in $\mathbf{R}^3$ and that Γ' bounds a smooth, oriented minimal surface that is not a disk. Then*

$$\mathrm{tc}(\Gamma') > 2\pi\sigma_1,$$

where $\sigma_1 = (2\pi/e)^{1/2}$ is the entropy of $\mathbf{S}^1 \times \mathbf{R}$.

This is a slight improvement over the 3π in Theorem 17 because

$$\frac{2\pi\sigma_1}{3\pi} = 1.01356\ldots.$$

It is conjectured that Theorem 19 holds with 4π in place of $2\pi\sigma_1$. The constant 4π would be sharp since (by work of Almgren–Thurston [1] or the simplified version by Hubbard [8]), for every $\varepsilon > 0$ and g, there is a smooth simple closed curve of total curvature $< 4\pi + \varepsilon$ that bounds no embedded minimal surface of genus $\leq g$. Such a curve bounds immersed minimal surfaces (the Douglas solutions) of each genus $\leq g$, and an embedded minimal surface (the least area integral current) of genus $> g$; see the discussion in the introduction of [5].

Proof of Theorem 19. We may suppose that $\mathrm{tc}(\Gamma') < 4\pi$.

Let $\mathcal{F}$ be the family of connected, oriented surfaces of genus ≥ 1. By Theorem 11, there is smooth simple closed curve Γ with

$$\mathrm{tc}(\Gamma) \leq \mathrm{tc}(\Gamma')$$

and a standard Brakke flow

$$t \in \mathbf{R} \mapsto M(t)$$

with boundary Γ such that $M(\infty)$ is a smooth disk and such that $M(-\infty)$ is smooth, orientable minimal surface that is not a disk.

Thus, the flow must have a singularity. Consider the shrinker Σ corresponding to a singularity at the first singular time T. Since $M(t)$ is orientable for $t < T$, Σ must be orientable. Thus, the singularity is an interior singularity (by Theorem 9.)

As in the proof of Theorem 17, the entropy of Σ is greater than or equal to the entropy σ_1 of a cylinder, and thus

$$\sigma_1 \leq e(M(T); \Gamma) < \frac{\mathrm{tc}(\Gamma)}{2\pi}. \qquad \square$$

A. Regularity and compactness

Lemma 20. *Suppose that Σ is a 2-dimensional integral varifold in $\mathbf{R}^n$, that*

$$t \in (-\infty, 0) \mapsto |t|^{1/2}\Sigma \tag{13}$$

is a standard Brakke flow without boundary, and that Σ has entropy < 2. Then Σ is a smoothly embedded surface.

Proof. First, we prove that

$$\text{if } \Sigma \text{ is a cone, then it is a multiplicity-one plane.} \tag{14}$$

If Σ is a cone, it must be a stationary cone, so its intersection with the unit sphere is a geodesic network. Since the entropy is < 2, each geodesic arc occurs with multiplicity 1. Also, at each vertex of the network, 3 or fewer arcs meet. But 3 arcs cannot meet at a point because the flow is standard and therefore has no triple junctions.

Thus, there are no vertices. That is, $\Sigma \cap \partial\mathbf{B}$ consists of disjoint, multiplicity-one geodesics. Because the entropy is < 2, there can only be one such geodesic. Thus, Σ is a multiplicity-one plane. This proves (14).

In the general case, note that Σ is a stationary integral varifold for the shrinker metric on $\mathbf{R}^n$. Let C be a tangent cone to Σ at a point p. Then

$$t \in (-\infty, 0) \mapsto C = |t|^{1/2}C$$

is a tangent flow to the flow (13) at the spacetime point $(p, -1)$. By (14), C is a multiplicity-one plane. Hence, (by Allard regularity) p is a regular point of Σ. □

Lemma 21. *Suppose that Σ is a 2-dimensional integral varifold in $\mathbf{R}^n$, that*

$$t \in (-\infty, 0) \mapsto |t|^{1/2}\Sigma \tag{15}$$

is a standard Brakke flow with boundary L, where L is a straight line through the origin, and that $e(\Sigma; L) < 2$ (see Definition 4). Then Σ is a smoothly embedded manifold with boundary L.

Proof. First, we prove that

$$\text{if } \Sigma \text{ is a cone, then it is a multiplicity-one half-plane.} \tag{16}$$

Note that if Σ is a cone, then $\Sigma \cap \partial\mathbf{B}$ is a geodesic network. Each geodesic arc occurs with multiplicity 1. Exactly as in the proof of Lemma 20, there can be no vertices of the network, except the two points of $L \cap \partial\mathbf{B}$.

Thus, $\Sigma \cap \partial\mathbf{B}$ consists of j geodesic semicircles with endpoints $L \cap \partial\mathbf{B}$, together with k geodesic circles (for some integers j and k). Consequently, Σ consists of j half-planes (each with boundary L) together with k planes. The extended entropy of Σ is $(j/2) + k + 1/2$, so

$$\frac{j+1}{2} + k < 2, \tag{17}$$

and therefore $j < 3$. By standardness, the mod 2 boundary of Σ is L, so j is odd. Thus, $j = 1$. By (17), $k = 0$. This proves (16).

Now we consider the general case. Exactly as in Lemma 20, Σ is smooth and embedded except perhaps along L. Let C be a tangent cone to Σ at a point $p \in L$. Then

$$t \in (-\infty, 0) \mapsto C = |t|^{1/2} C$$

is a tangent flow to the flow (15) at the spacetime point $(p, -1)$, so C is a multiplicity-one half-plane by (16). Thus, p is a regular point of Σ by Allard regularity. □

Corollary 22. *Let Σ be as in Lemma 21. If Σ is invariant under translations in some direction, then Σ is a half-plane.*

Proof. The direction of translational invariance would have to be L. Let P be the tangent half-plane to Σ at 0. Then P and Σ are g-minimal surfaces that are tangent along L (where g is the shrinker metric). Thus, P and Σ coincide. □

Proposition 23. *Suppose that Γ is a smoothly embedded curve in $\mathbf{R}^n$ and suppose that*

$$t \in I \mapsto M(t)$$

is a standard Brakke flow with boundary Γ satisfying the entropy bound

$$e(M(t); \Gamma) < 2 \quad \textit{for all } t.$$

Then the set of singular times has measure 0.

Proof. Let Σ be the shrinker for the tangent flow at a singularity. Then Σ is smoothly embedded, so if it had a direction of translational invariance, then by Corollary 22, the singularity would not be at a boundary point. Thus, Σ would be a cylinder. Regularity at almost all times follows from the standard stratification theory [12]. □

Remark 24. From the proof, we see that we do not really need to assume that the surfaces have entropy < 2; it is enough to assume that all the tangent flows have entropy < 2.

Theorem 25. *Suppose that Γ is a smooth, simple closed curve in $\mathbf{R}^n$ of total curvature $\leq 4\pi$ and that Γ_i converges smoothly to Γ. Suppose that M_i is a smooth embedded minimal surface bounded by Γ_i. Then, after passing to a subsequence, M_i converges smoothly to a minimal surface bounded by Γ.*

Proof. This is not hard to prove by minimal surface techniques (using extended monotonicity [5, Theorem 9.1] and arguments analogous to the proofs of Lemmas 20 and 21). Here we prove it using mean curvature flow since we have already established all the necessary ingredients.

By the convex hull property of minimal surfaces and the isoperimetric inequality, the M_i lie in a compact subset of $\mathbf{R}^n$ and their areas are uniformly bounded. Thus, after passing to a subsequence, the M_i converge weakly to a compactly supported integral varifold M. Note that the standard Brakke flows

$$t \in \mathbf{R} \mapsto M_i$$

with boundary Γ_i converge to the Brakke flow

$$t \in \mathbf{R} \mapsto M \tag{18}$$

with boundary Γ. Thus, $t \in \mathbf{R} \mapsto M$ is also standard [14, Theorem 13.1]. By Proposition 23, the flow (18) is smooth at almost all times. Thus, M is smooth. By Allard regularity, the convergence is smooth. □

B. Reducing total curvature

Theorem 26. *Let $\alpha < 4\pi$. Let $C(\alpha)$ be the collection of simple closed, polygonal curves in $\mathbf{R}^n$ with total curvature at most α. Then $C(\alpha)$ is connected.*

Proof. It suffices to show that any curve in $C(\alpha)$ can be deformed through curves in $C(\alpha)$ to a triangle.

Let Γ be a curve in $C(\alpha)$. According to Milnor's theorem [9], we can assume (by rotating and scaling) that the image of height function

$$h\colon x \in \Gamma \mapsto x \cdot \mathbf{e}_n$$

is $[0, 1]$, and that for each $y \in (0, 1)$, there are exactly two points $\gamma_1(y)$ and $\gamma_2(y)$ of Γ at which $h = y$.

We may also assume that $h = 1$ at exactly 1 point.

Let $\Gamma(0) = \Gamma$, and for $t \in (0, 1)$, let $\Gamma(t)$ be $\Gamma \cap \{h \geq t\}$ together with the segment joining $\gamma_1(t)$ and $\gamma_2(t)$.

The total curvature of $\Gamma(t)$ is a decreasing function of t, so $\Gamma(t) \in C(\alpha)$ for all $t \in [0, 1)$. Note that for t close to 1, $\Gamma(t)$ is a triangle. □

Lemma 27. *Let $\gamma\colon [0, 1] \to \mathbf{R}^n$ be a smooth simple closed curve. Then there exists a one-parameter family*

$$t \mapsto \gamma_t$$

such that:

(1) $\gamma_0 = \gamma$.

(2) γ_1 *is polygonal.*

(3) *Each* γ_t *is a piecewise-smooth simple closed curve.*

(4) *The total curvature of* γ_t *is a decreasing function of* t.

(5) *For each* $t \in [0,1]$ *and each* $s \in [0,1]$,

$$\gamma_t'(s_-) \cdot \gamma_t'(s_+) > 0.$$

Proof. Let $\gamma\colon [0,1] \to \Gamma$ be a smooth parametrization.

Fix a large integer N, and for each $t \in [0,1]$, let $\gamma_t\colon [0,1] \to \mathbf{R}^n$ be the closed curve formed from γ by replacing (for each $k = 0, \dots, N-1$)

$$\gamma | [k/N, (k+t)/N]$$

by the line segment with the same endpoints.

Then $\gamma_0 = \gamma$, the total curvature of γ_t is a decreasing function of t (by [9]), and γ_1 is polygonal.

Note also that if N is sufficiently large, then each γ_t will be an embedding. □

Theorem 28. *Let* Γ *be a smooth, simple closed curve of total curvature* $\leq \alpha < 4\pi$. *Then* Γ *can be deformed among such curves to a planar convex curve.*

Proof. If C is a simple closed curve in $\mathbf{R}^n$, let $\phi_t(C)$ be the result of flowing C for time t by curve-shortening flow. (We assume $t > 0$ is small enough that the flow is smooth on the time interval $(0, t]$.)

By Theorem 26 and Lemma 27, there is a one-parameter family

$$s \in [0,2] \mapsto \Gamma_s$$

of simple closed curves such that:

(1) $\Gamma_0 = \Gamma$.

(2) $s \in [0,1] \mapsto \Gamma_s$ are piecewise smooth curves as in Lemma 27.

(3) Γ_s is polygonal for $s \in [1,2]$.

(4) Γ_2 is a triangle.

(5) The total curvature of Γ_s is a decreasing function of s.

Choose $\varepsilon > 0$ small enough so that for each $s \in [0,2]$, then curve-shortening flow starting with Γ_s remains smooth and embedded on the time interval $(0, \varepsilon]$.

Now deform $\Gamma = \Gamma_0$ to $\phi_\varepsilon(\Gamma_0)$ by

$$t \in [0, \varepsilon] \mapsto \phi_t(\Gamma_0).$$

Then deform $\phi_\varepsilon(\Gamma_0)$ to the plane convex curve $\phi_\varepsilon(\Gamma_2)$ by

$$s \in [0,2] \mapsto \phi_\varepsilon(\Gamma_s).$$

Note that we have deformed Γ to the plane convex curve $\phi_\varepsilon(\Gamma)$ through smooth, simple closed curves, each of total curvature $\le \alpha$. (By [7, Lemma 3.4], curve shortening in $\mathbf{R}^n$ reduces total curvature.) $\square$

C. Unique limits

Theorem 29. *Suppose Σ is an m-dimensional compact, smoothly embedded minimal surface in $\mathbf{R}^n$ with smooth boundary Γ. Suppose*

$$t \in I \mapsto M(t)$$

is a standard mean curvature flow of m-dimensional surfaces in $\mathbf{R}^n$ with fixed boundary Γ.

(1) *If $I = [0,\infty)$ and if there is a sequence $t_i \to \infty$ such that $M(t_i)$ converges smoothly (with multiplicity 1) to Σ, then $M(t)$ converges smoothly as $t \to \infty$ to Σ.*

(2) *If $I = (-\infty, 0]$ and if there is a sequence $t_i \to -\infty$ such that $M(t_i)$ converges smoothly (with multiplicity 1) to Σ, then Then $M(t)$ converges smoothly as $t \to -\infty$ to Σ.*

Proof. If $t \in [0,\infty) \mapsto M(t)$ is a solution of the renormalized mean curvature flow

$$(\text{velocity at } x)^\perp = H + \frac{x^\perp}{2},$$

and if there is no boundary, then assertion (1) is proved in [10, Corollary 1.2]. In fact, the same proof works when M has boundary, when $t \to -\infty$, and when the flow is mean curvature flow rather than renormalized mean curvature flow. (For mean curvature flow, in the proofs in [10], one should let $\mathcal{E}$ be the ordinary area of the surface and ρ be the constant function 1.) $\square$

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