Ulrich bundles on cubic fourfolds

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Abstract. We show the existence of rank 6 Ulrich bundles on a smooth cubic fourfold. First, we construct a simple sheaf $\mathcal{E}$ of rank 6 as an elementary modification of an ACM bundle of rank 6 on a smooth cubic fourfold. Such an $\mathcal{E}$ appears as an extension of two Lehn–Lehn–Sorger–van Straten sheaves. Then we prove that a general deformation of $\mathcal{E}(1)$ becomes Ulrich. In particular, this says that general cubic fourfolds have Ulrich complexity 6.

Introduction

An Ulrich sheaf on a closed subscheme X of $\mathbb{P}^N$ of dimension n and degree d is a non-zero coherent sheaf $\mathcal{F}$ on X satisfying

$$\mathrm{H}^*(X, \mathcal{F}(-j)) = 0 \quad \text{for } 1 \leq j \leq n.$$

In particular, the cohomology table $\{h^i(X, \mathcal{F}(j))\}$ of $\mathcal{F}$ is a multiple of the cohomology table of $\mathbb{P}^n$. It turns out that the reduced Hilbert polynomial $\mathrm{p}_{\mathcal{F}}(t) = \chi(\mathcal{F}(t))/\mathrm{rk}(\mathcal{F})$ of an Ulrich sheaf $\mathcal{F}$ must be

$$\mathrm{u}(t) := \frac{d}{n!} \prod_{i=1}^{n} (t + i).$$

Ulrich sheaves first appeared in commutative algebra in the 1980s, namely, in the form of maximally generated maximal Cohen–Macaulay modules [33]. Pioneering work of Eisenbud and Schreyer [14] popularized them in algebraic geometry in view

of their many connections and applications. Eisenbud and Schreyer asked whether every projective scheme supports an Ulrich sheaf. That this should be the case is now called a conjecture of Eisenbud-Schreyer (see also [15]). They also proposed another question about what is the smallest possible rank of an Ulrich sheaf on X. This is called the *Ulrich complexity* $\mathrm{uc}(X)$ of X (cf. [5]).

Both the Ulrich existence problem and the Ulrich complexity problem have been elucidated only for a few cases. We focus here on the case when X is a hypersurface in $\mathbb{P}^{n+1}$ over an algebraically closed field $\Bbbk$ of characteristic different from 2. Using the generalized Clifford algebra, Backelin and Herzog proved in [1] that any hypersurface X has an Ulrich sheaf (even in characteristic 2). However, their construction yields an Ulrich sheaf of rank $d^{\tau(X)-1}$, where $\tau(X)$ is the Chow rank of X (i.e. the smallest length of an expression of the defining equation of X as sums of products of d linear forms), often much bigger than $\mathrm{uc}(X)$.

Looking in more detail at the Ulrich complexity problem for smooth hypersurfaces of degree d in $\mathbb{P}^{n+1}$, the situation is well-understood for arbitrary n only for $d = 2$. Indeed, in this case the only indecomposable Ulrich bundles on X are spinor bundles, which have rank $2^{\lfloor(n-1)/2\rfloor}$ [7]. On the other hand, for $d \geq 3$, the Ulrich complexity problem is wide open except for a very few small-dimensional cases. For instance, any smooth cubic curve or surface X satisfies $\mathrm{uc}(X) = 1$, while for smooth cubic threefolds X we have $\mathrm{uc}(X) = 2$ (cf. [3,4,27]).

The main goal of this paper is to prove existence of Ulrich bundles $\mathcal{U}$ defined on any smooth cubic fourfold X. In particular, the Chern character $\mathrm{ch}(\mathcal{U})$ should lie in $\mathbb{Q}[H_X]$. More precisely, $\mathrm{ch}(\mathcal{U}(-1)) = k\gamma$, with $\gamma = 3 - H_X^2 + \frac{1}{4}H_X^4$, for some integer $k > 1$. Our main result is as follows.

Theorem 1. *Any smooth cubic fourfold X admits an Ulrich bundle $\mathcal{U}$ of rank 6 with*

$$\mathrm{ch}(\mathcal{U}(-1)) = 2\gamma.$$

Hence, there is $M\colon 18\mathcal{O}_{\mathbb{P}^5}(-1) \to 18\mathcal{O}_{\mathbb{P}^5}$ with $\det(M) = f^6$, where f is an equation of X.

This allows to settle the Ulrich complexity problem for very general cubic fourfolds. We know of no pair (n, d) with $n \geq 5$, $d \geq 4$ such that the Ulrich complexity problem of very general hypersurfaces of degree d in $\mathbb{P}^n$ is solved; even for $n \in \{3, 4\}$ the problem is open for large d.

Corollary. *If X is a very general cubic fourfold, then $\mathrm{uc}(X) = 6$.*

To explain this, first note that when X is a smooth cubic fourfold then X does not support Ulrich bundles of rank 1, but some X can have an Ulrich bundle $\mathcal{F}$ of rank 2, namely the pfaffian cubic fourfolds; their moduli space forms a divisor $\mathcal{C}_{14}$ of the moduli space $\mathcal{C}$ of cubic fourfolds, so a general cubic fourfold X has $\mathrm{uc}(X) \geq 3$. A few more cubic fourfolds which have an Ulrich bundle of rank 3 or 4 have been reported very recently by Troung and Yen [32]. However, all these cases are special

cubic fourfolds which contain a surface not homologous to a complete intersection. Indeed, it turns out that the Ulrich complexity of a very general cubic fourfold is divisible by 3 and at least 6, see [23]. On the other hand, a general cubic fourfold has a rank 9 Ulrich bundle (cf. [22, 23, 30]).

Let us sketch briefly the strategy of the proof of Theorem 1. As a warm-up it, let us review a construction of a rank-2 Ulrich bundle on a smooth cubic threefold. First, starting from a line L contained in the threefold, one constructs an ACM bundle of rank 2 having $(c_1, c_2) = (0, L)$. Such a bundle is unstable since it has a unique global section which vanishes along L. By choosing a line L' disjoint from L, we may take an elementary modification of it so that we have a simple and semistable sheaf $\mathcal{E}$ of $(c_1, c_2) = (0, 2L)$. The sheaf $\mathcal{E}$ is not Ulrich, but one can show that its general deformation becomes Ulrich. A similar argument is used to prove the existence of rank 2 Ulrich bundles on K3 surfaces [16] and prime Fano threefolds [6].

For fourfolds, twisted cubics play a central role in the construction, rather than lines. Note that twisted cubics in X form a 10-dimensional family. For each twisted cubic $C \subset X$, its linear span $V = \langle C \rangle$ defines a linear section $Y \subset X$ which is a cubic surface. When Y is smooth, the rank-3 sheaf $\mathcal{G} = \ker[3\mathcal{O}_X \to \mathcal{O}_Y(C)]$ is stable. The family of such stable sheaves of rank 3 forms an 8-dimensional moduli space, which is indeed a very well studied smooth hyperkähler manifold [26, 28]. These sheaves have been used extensively in [26, 29]. We will call them Lehn–Lehn–Sorger–van Straten sheaves and the Lehn–Lehn–Sorger–van Straten eightfold.

To construct an Ulrich bundle of rank 6, we start from a twisted cubic $C \subset X$ and consider a rank 6 vector bundle $\mathcal{S}$ obtained as fourth syzygy of $\mathcal{O}_C(5)$. Then we take a modification of $\mathcal{S}$ along the cubic surface Y obtained cutting X with the span of C, which one achieves upon choosing a second twisted cubic D in Y that cuts C at 2 points. This affords a sheaf $\mathcal{E}$ which is certainly not reflexive but has Chern character 2γ and enjoys almost all cohomology vanishing needed to be an Ulrich sheaf.

As it turns out, $\mathcal{E}$ is a simple extension of the two Lehn–Lehn–Sorger–van Straten sheaves of rank 3 associated with the two twisted cubics D and C – or rather its *transpose* C^{t}, namely the residual of C in $2H_Y$. The key point is that the sheaf $\mathcal{E}$ lies in the Kuznetsov category Ku(X) of X ([25]). This allows to obtain an Ulrich sheaf by taking a generic deformation $\mathcal{F}$ of $\mathcal{E}$ in the moduli space of simple sheaves over X and using that the cohomology vanishing of $\mathcal{E}$ propagates to $\mathcal{F}$ by semicontinuity. This step relies on deformation-obstruction theory of the sheaf as developed in [2, 24] and makes substantial use of the fact that Ku(X) is a K3 category.

Next, we argue about stability of our Ulrich bundles. It is well known that Ulrich bundles are semistable. Here we prove that the ones we construct are stable, provided some generality assumption is made on X. Also, combining our bundles with the ones arising from [22], we may construct stable Ulrich bundles of arbitrarily high rank. This provides a higher-dimensional version of the main results of [11, 12], in the

sense that X is strictly Ulrich wild and thus verifies [17, Conjecture 1]. We do not know of other examples of hypersurfaces of dimension $n \geq 4$ where this conjecture is known to hold true.

Theorem 2. *Given a smooth cubic fourfold, there is a* 26-*dimensional symplectic family of stable Ulrich bundles of rank* 6. *If X is general enough, then for any $k > 1$ there is a $(6k^2 + 2)$-dimensional symplectic family of stable Ulrich bundles $\mathcal{U}$ on X with* $\mathrm{ch}(\mathcal{U}(-1)) = k\gamma$.

The above results can be thought of in terms of moduli of stable objects of $\mathrm{Ku}(X)$ with respect to a Bridgeland stability condition σ on $\mathrm{Ku}(X)$. Indeed, in the Mukai lattice of $\mathrm{Ku}(X)$, if we denote by v_0 the Mukai vector of the object $\mathcal{G}$ arising from a twisted cubic as above, then the Mukai vector of our Ulrich sheaves is $2v_0$. Then, our result implies that the moduli space of σ-stable objects $\mathrm{M}_\sigma(2v_0)$ has an irreducible component whose generic point is a stable Ulrich bundle of rank 6. It is likely that the spaces $\mathrm{M}_\sigma(kv_0)$ are actually irreducible for all $k > 1$. However, we are not aware of a proof of this fact, nor of the answer to the following question.

Question 1. *Let X be a smooth cubic fourfold, take $k > 1$ and consider the Maruyama moduli space $\mathrm{M}_X(k\gamma)$ of semistable sheaves of rank k with Chern character $k\gamma$. Is the open piece of $\mathrm{M}_X(k\gamma)$ consisting of Ulrich bundles irreducible?*

An intriguing question arises when looking at the space $\mathrm{M}_X(3\gamma)$, with X very general. Indeed, the construction of [22] gives rise to Ulrich bundles of rank 9 by realizing X as a $\mathbb{P}^5$-linear section of the Cartan cubic in $\mathbb{P}^{26}$, which is equipped with an E_6-equivariant Ulrich sheaf of rank 9. The choice of the linear section is the equivalence class of a point of $\mathbb{G}(6, 27)$ up the action of the 78-dimensional group E_6. On a sufficiently general cubic fourfold X, this affords a 28-dimensional family of stable Ulrich bundles $\mathcal{U}$ such that $\mathcal{U}(-1)$ lies in $\mathrm{M}_X(3\gamma)$. On the other hand, the family of stable Ulrich bundles $\mathcal{U}$ with $\mathcal{U}(-1)$ in $\mathrm{M}_X(3\gamma)$ is symplectic of dimension 56. We ask whether the Cartan bundles from [22] form a Lagrangian subvariety of this family. Perhaps this can be shown using the fact that rationally connected varieties admit no non-zero 2-form, but we have not been able to prove this rigorously. Anyway, we do not know how to find a Lagrangian subvariety of $\mathrm{M}_X(k\gamma)$ for $k > 3$.

The structure of this paper is as follows. In Section 1, we recall basic notions and develop some background mainly on Ulrich bundles, syzygies and matrix factorizations. In Section 2, we introduce an ACM bundle of rank 6 which arises as a (higher) syzygy sheaf of a twisted cubic and review some material on Lehn–Lehn–Sorger–van Straten sheaves as syzygy sheaves. Then we take an elementary modification to define a strictly semistable sheaf $\mathcal{E}$ of rank 6 whose reduced Hilbert polynomial is $\mathrm{u}(t)$. In Section 3, we show that a general deformation of $\mathcal{E}$ is Ulrich. We first prove this claim for cubic fourfolds which do not contain surfaces of small degrees other than linear sections. Then we extend it for every smooth cubic fourfold. Finally, in Section 4, we prove Theorem 2.

1. Background

Let us collect here some basic material. We work over an algebraically closed field $\Bbbk$ of characteristic other than 2.

1.1. Background definitions and notation.   Consider a smooth connected n-dimensional projective subvariety $X \subseteq \mathbb{P}^N$ and denote by H_X the hyperplane divisor on X and $\mathcal{O}_X(1) = \mathcal{O}_X(H_X)$. Given a coherent sheaf $\mathcal{F}$ on X and $t \in \mathbb{Z}$, write $\mathcal{F}(t)$ for $\mathcal{F} \otimes \mathcal{O}_X(tH_X)$. Let $\mathcal{F}$ be a torsion-free sheaf on X. The reduced Hilbert polynomial of $\mathcal{F}$ is defined as

$$p_{\mathcal{F}}(t) := \frac{1}{\mathrm{rk}(\mathcal{F})} \chi(\mathcal{F}(t)) \in \mathbb{Q}[t].$$

Let $\mathcal{F}, \mathcal{G}$ be torsion-free sheaves on X. We say that $p_{\mathcal{F}} < p_{\mathcal{G}}$ if $p_{\mathcal{F}}(t) < p_{\mathcal{G}}(t)$ for $t \gg 0$. The slope of $\mathcal{F}$ is defined as

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H_X^{n-1}}{\mathrm{rk}(\mathcal{F})}.$$

A torsion-free sheaf $\mathcal{F}$ on X is *stable* (respectively, *semistable, μ-stable, μ-semistable*) if, for any subsheaf $0 \neq \mathcal{F}' \subsetneq \mathcal{F}$, we have

$$p_{\mathcal{F}'} < p_{\mathcal{F}}, \quad \text{(respectively, } p_{\mathcal{F}'} \leq p_{\mathcal{F}}, \mu(\mathcal{F}') < \mu(\mathcal{F}), \mu(\mathcal{F}') \leq \mu(\mathcal{F})\text{)}.$$

A polystable sheaf is a direct sum of stable sheaves having the same reduced Hilbert polynomial.

1.2. ACM and Ulrich sheaves.   We are mostly interested in coherent sheaves on X which admit nice minimal free resolutions over $\mathbb{P}^N$, namely ACM and Ulrich sheaves. Equivalently, such properties are characterized by cohomology vanishing conditions as follows.

Definition 1.1. Let $X \subseteq \mathbb{P}^N$ be as above, and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is:

(i) *ACM* if it is locally Cohen–Macaulay and $\mathrm{H}^i(X, \mathcal{F}(j)) = 0$ for $0 < i < n$ and $j \in \mathbb{Z}$.

(ii) *Ulrich* if $H^i(X, \mathcal{F}(-j)) = 0$ for $i \in \mathbb{Z}$ and $1 \le j \le n$.

We refer to [14, Proposition 2.1] for several equivalent definitions for Ulrich sheaves. In particular, every Ulrich sheaf is ACM. If X is smooth, then a coherent sheaf is locally Cohen–Macaulay if and only if is locally free. Moreover, for Ulrich sheaves, (semi)stability is equivalent to μ-(semi)stability, see [12].

Let us review some previous works on the existence of Ulrich bundles on a smooth cubic fourfold X, possibly of small rank. In terms of Hilbert polynomial, an Ulrich bundle $\mathcal{U}$ satisfies:

$$p_{\mathcal{U}}(t) = u(t) := \frac{1}{8}(t + 4)(t + 3)(t + 2)(t + 1).$$

Note that X carries an Ulrich line bundle if and only if it is linearly determinantal, which is impossible since a determinantal hypersurface is singular along a locus of codimension 3. X carries a rank 2 Ulrich bundle if and only if it is linearly pfaffian. Equivalently, such an X contains a quintic del Pezzo surface [3]. Note that a pfaffian cubic fourfold also carries a rank 5 Ulrich bundle [30]. For rank 3 and 4, Truong and Yen provided computer-aided construction of a rank 3 Ulrich bundle on a general element in the moduli of special cubic fourfolds $\mathcal{C}_{18}$ of discriminant 18, and of a rank 4 Ulrich bundle on a general element in $\mathcal{C}_8$ [32].

All the above cases were made over special cubic fourfolds, i.e., they contain a surface which is not homologous to a complete intersection. Such cubic fourfolds form a countable union of irreducible divisors in the moduli space of smooth cubic fourfolds $\mathcal{C}$. We refer to [19] for the convention and more details. On a very general cubic fourfold X (so that any surface contained in X is homologous to a complete intersection), it is easy to find the following necessary condition on Chern classes of a coherent sheaf to be Ulrich.

Proposition 1.2 ([23, Proposition 2.5]). *Let $\mathcal{E}$ be an Ulrich bundle of rank r on a very general cubic fourfold $X \subset \mathbb{P}^5$. Let $c_i := c_i(\mathcal{E}(-1))$. Then r is divisible by 3, $r \ge 6$, and*

$$c_1 = 0, \quad c_2 = \frac{1}{3}rH^2, \quad c_3 = 0, \quad c_4 = \frac{1}{6}r(r - 9).$$

The existence of rank 9 Ulrich bundles on a general cubic fourfold X is known according to [22, 23, 30]. Therefore, the Ulrich complexity of a very general cubic fourfold is either 6 or 9. It is thus natural to ask the question: *Does a smooth cubic fourfold carry an Ulrich bundle of rank* 6? The main goal of this paper is to give a positive answer to this question. In particular, the Ulrich complexity $uc(X)$ of a (very) general cubic fourfold X is 6.

1.3. Reflexive sheaves. Let $\mathcal{E}$ be a torsion-free sheaf on a smooth connected projective n-dimensional variety X. The following lemma is standard.

Lemma 1.3. *For each $k \in \{0, \ldots, n-2\}$, there is $p_k \in \mathbb{Q}[t]$ with $\deg(p_k) \leq k$ such that*

$$\mathrm{h}^{k+1}(\mathcal{E}(-t)) = p_k(t), \quad \mathrm{h}^0(\mathcal{E}(-t)) = 0 \qquad \text{for } t \gg 0,$$

Assume $\mathcal{E}$ is reflexive. Then $\forall k \in \{0, \ldots, n-3\}$ there is $q_k \in \mathbb{Q}[t]$, with $\deg(q_k) \leq k$ such that

$$\mathrm{h}^{k+2}(\mathcal{E}(-t)) = q_k(t), \quad \mathrm{h}^0(\mathcal{E}(-t)) = \mathrm{h}^1(\mathcal{E}(-t)) = 0 \qquad \text{for } t \gg 0,$$

Moreover, $\mathcal{E}$ is locally free if and only if $p_k = 0$ for all $k \in \{0, \ldots, n-2\}$, equivalently if $q_k = 0$ for all $k \in \{0, \ldots, n-3\}$.

Proof. Given positive integers p, q with $p + q \leq n$, Serre duality and the local-global spectral sequence, for all $t \in \mathbb{Z}$, we have

$$\mathrm{H}^{n-p-q}(\mathcal{E}(-t))^\vee \simeq \mathrm{Ext}_X^{p+q}(\mathcal{E}, \omega_X(t))$$
$$\Leftarrow \mathrm{H}^p(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X) \otimes \mathcal{O}_X(t)) = E_2^{p,q}. \quad (1)$$

For $t \gg 0$ and $p > 0$, we have $\mathrm{H}^p(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X) \otimes \mathcal{O}_X(t)) = 0$ by Serre vanishing. Then,

$$\mathrm{h}^{n-q}(\mathcal{E}(-t)) = \mathrm{h}^0(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X) \otimes \mathcal{O}_X(t)) \quad \text{for } t \gg 0.$$

Hence, $\mathrm{h}^{n-q}(\mathcal{E}(-t))$ is a rational polynomial function of t for $t \gg 0$. By [20, Proposition 1.1.10], since $\mathcal{E}$ is torsion-free, we have

$$\mathrm{codim}(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X)) \geq q + 1 \quad \text{for } q \geq 1,$$

while when $\mathcal{E}$ is reflexive,

$$\mathrm{codim}(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X)) \geq q + 2 \quad \text{for } q \geq 1.$$

Thus, for $t \gg 0$, the degree of the polynomial function $\mathrm{h}^{n-q}(\mathcal{E}(-t))$ is at most $n - q - 1$, actually at most $n - q - 2$ if $\mathcal{E}$ is reflexive.

Finally, $\mathcal{E}$ is locally free if and only if

$$\mathcal{E}xt_X^q(\mathcal{E}, \omega_X) = 0 \quad \text{for all } q > 1.$$

Since this happens if and only if

$$\mathrm{h}^0(\mathcal{E}xt_X^q(\mathcal{E}, \omega_X) \otimes \mathcal{O}_X(t)) = 0 \quad \text{for } t \gg 0,$$

the last statement follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

1.4. Minimal resolutions and syzygies. We recall some notions from commutative algebra. Let $R = \Bbbk[x_0, \ldots, x_N]$ be a polynomial ring over a field $\Bbbk$ with the standard grading, and let $R_X = R/I_X$ be the homogeneous coordinate ring of X where I_X is the ideal of X. Let Γ be a finitely generated graded R_X-module. The minimal free resolution of Γ over R_X is constructed by choosing minimal generators of Γ of degrees $(a_{0,0}, \ldots, a_{0,r_0})$ so that there is a surjection

$$F_0 = \bigoplus_{j=0}^{r_0} R_X(-a_{0,j}) \twoheadrightarrow \Gamma.$$

Taking a minimal set of generators of degrees $(a_{1,0}, \ldots, a_{1,r_1})$ of its kernel we get a minimal presentation of Γ of the form

$$F_1 = \bigoplus_{j=0}^{r_1} R_X(-a_{1,j}) \to F_0.$$

Repeating this process, we have a free resolution of Γ:

$$F_\bullet(\Gamma) \colon \cdots \to F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \to F_1 \xrightarrow{d_1} F_0 \to \Gamma \to 0,$$

with

$$F_i = \bigoplus_{j=0}^{r_i} R_X(-a_{i,j}).$$

Note that the resolution obtained this way is minimal, i.e., $d_i \otimes_{R_X} \Bbbk = 0$ for every i, and is unique up to homotopy; see [13, Corollary 1.4]. In general, it has infinitely many terms.

We define the *minimal resolution* of a coherent sheaf $\mathcal{F}$ on X as the sheafification of the minimal graded free resolution of its module of global sections

$$\Gamma_*(\mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} \Gamma(X, \mathcal{F}(j)),$$

provided that this is finitely generated. In this case, for $i \in \mathbb{N}$, we call the *i-th syzygy* of $\mathcal{F}$ the sheafification of $\mathrm{Im}(d_i)$, and we denote this by $\Sigma_i^X(\mathcal{F})$. Of course for positive j, we have

$$\Sigma_{i+j}^X(\mathcal{F}) \simeq \Sigma_j^X \Sigma_i^X(\mathcal{F}).$$

1.5. Matrix factorizations and ACM/Ulrich sheaves. We recall the notion of matrix factorization which is introduced by Eisenbud [13] to study free resolutions over hypersurfaces.

Definition 1.4. Let $X \subseteq \mathbb{P}^N$ be a hypersurface defined by a homogeneous polynomial f of degree d, and let $\mathcal{F}$ and $\mathcal{G}$ be two finite direct sums of line bundles. A pair of morphisms $\varphi \colon \mathcal{F} \to \mathcal{G}$ and $\psi \colon \mathcal{G}(-d) \to \mathcal{F}$ is called a *matrix factorization* of f (of X) if

$$\varphi \circ \psi = f \cdot \mathrm{id}_{\mathcal{G}(-d)}, \quad \psi(d) \circ \varphi = f \cdot \mathrm{id}_{\mathcal{F}}.$$

Matrix factorizations have a powerful application to ACM/Ulrich bundles as follows.

Proposition 1.5 ([13, Corollary 6.3]). *The association*

$$(\varphi, \psi) \mapsto M_{(\varphi, \psi)} := \mathrm{coker}\, \varphi$$

induces a bijection between the set of equivalence classes of reduced matrix factorizations of f and the set of isomorphism classes of indecomposable ACM sheaves. In particular, when (φ, ψ) is completely linear, that is,

$$\varphi \colon \mathcal{O}_{\mathbb{P}^N}(-1)^{\oplus t} \to \mathcal{O}_{\mathbb{P}^N}^{\oplus t}$$

for some $t \in \mathbb{Z}$ then the corresponding sheaf is Ulrich.

1.6. Twisted cubics and Lehn–Lehn–Sorger–van Straten eightfold. Let us briefly recall how can we construct a rank 2 Ulrich bundle on a cubic threefold X via deformation theory. If there is such an Ulrich bundle $\mathcal{F}$, then $\mathcal{F}(-1)$ must have the Chern classes $(c_1, c_2) = (0, 2)$ by Riemann–Roch. Note that X has an ACM bundle $\mathcal{F}_1$ of rank 2 with $(c_1, c_2) = (0, 1)$ which fits into the following short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F}_1 \to \mathcal{I}_\ell \to 0,$$

where $\ell \subset X$ is a line. We see that $\mathcal{F}_1$ is unstable due to its unique global section. We can take an elementary modification with respect to $\mathcal{O}_{\ell'}$ where $\ell' \subset X$ is a line disjoint to ℓ. The resulting sheaf $\mathcal{F}_2 := \ker[\mathcal{F}_1 \to \mathcal{O}_{\ell'}]$ is simple, strictly semistable, and non-reflexive. One can check that its general deformation is stable and locally free, and becomes Ulrich after twisting by $\mathcal{O}_X(1)$. One major difference between the case of cubic threefolds and fourfolds is that not lines but twisted cubics play a significant role both in finding an ACM bundle (of same c_1 as Ulrich) and taking an elementary modification.

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold which does not contain a plane, and let $M_3(X)$ be the irreducible component of the Hilbert scheme of X containing the twisted cubics. Then $M_3(X)$ is a smooth irreducible projective variety of dimension 10 [28, Theorem A]. Let C be a twisted cubic contained in X, and $V \simeq \mathbb{P}^3$ be its linear span. According to [28], the natural morphism $C \mapsto V \in Gr(4, 6)$ factors through a smooth projective eightfold Z' so that $M_3(X) \to Z'$ is a $\mathbb{P}^2$-fibration. In Z', there is an effective divisor coming from non-CM twisted cubics on X which induces a further contraction $Z' \to Z$ so that Z is a smooth hyperkähler

eightfold which contains X as a Lagrangian submanifold, and the map $\mathcal{Z}' \to \mathcal{Z}$ is the blow-up along X [28, Theorem B]. The variety $\mathcal{Z}$ is called the Lehn–Lehn–Sorger–van Straten eightfold.

We are interested in a moduli description of $\mathcal{Z}'$. Let $Y := V \cap X$ be a cubic surface containing C. The sheaf $\mathcal{I}_{C/Y}(2)$ is indeed an Ulrich line bundle on Y, and hence it fits into the following short exact sequence

$$0 \to \mathcal{G}_C \to 3\mathcal{O}_X \to \mathcal{I}_{C/Y}(2) \to 0.$$

Lahoz, Lehn, Macrì and Stellari showed that the sheaf $\mathcal{G}_C$ is stable, and the moduli space of Gieseker stable sheaves with the same Chern character is isomorphic to $\mathcal{Z}'$ (see [26]). Since we are only interested in general CM twisted cubics and corresponding Lehn–Lehn–Sorger–van Straten sheaves, we may regard that a general point of the Lehn–Lehn–Sorger–van Straten eightfold $\mathcal{Z}$ corresponds to a rank-3 sheaf $\mathcal{G}_C$, where C is a CM twisted cubic on X, even when X potentially contains a plane.

2. Syzygies of twisted cubics

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold. Here we construct and study the (twisted) rank-6 fourth syzygy $\mathcal{S}$ of a twisted cubic $C \subset X$ and its modification along an Ulrich line bundle supported on the cubic surface Y with $C \subset Y \subset X$.

2.1. Twisted cubics and 6-bundles.
Here we show that taking the fourth syzygy of the structure sheaf of a twisted cubic C is a vector bundle of rank 6 which admits a trivial subbundle of rank 3. Factoring out this quotient gives back the second syzygy of C, with a degree shift. We will use this filtration later on.

Proposition 2.1. *Let $C \subset X$ be a twisted cubic, V its linear span and set $Y = X \cap V$. Put*

$$\mathcal{S} = \Sigma_4^X(\mathcal{O}_C(5)), \quad \mathcal{G}_C = \Sigma_1^X(\mathcal{I}_{C/Y}(2)).$$

Then $\mathcal{S}$ is an ACM sheaf of rank 6 on X with

$$\mathrm{p}_{\mathcal{S}}(t) = \frac{1}{8}(t+2)^2(t+1)^2, \quad \mathrm{H}^*(\mathcal{S}(-1)) = \mathrm{H}^*(\mathcal{S}(-2)) = 0.$$

Moreover, $\mathrm{h}^0(X, \mathcal{S}) = 3$ and there is an exact sequence

$$0 \to 3\mathcal{O}_X \to \mathcal{S} \to \mathcal{G}_C \to 0. \tag{2}$$

In particular, we have

$$\mathrm{ch}(\mathcal{G}_C) = \gamma = 3 - H_X^2 + \frac{1}{4}H_X^4, \quad \mathrm{ch}(\mathcal{S}) = 6 - H_X^2 + \frac{1}{4}H_X^4.$$

To keep notation lighter, we remove the subscript C from $\mathcal{G}_C$ so we just write $\mathcal{G}$, as soon as no confusion occurs, i.e., until Section 2.2.

Proof. For the sake of this proof, for any integer i we omit writing $\mathcal{O}_C$ from expressions of the form $\Sigma_i^X(\mathcal{O}_C)$ and $\Sigma_i^Y(\mathcal{O}_C)$ so that, for instance,

$$\Sigma_1^X \simeq \mathcal{I}_{C/X}, \quad \Sigma_1^Y \simeq \mathcal{I}_{C/Y}. \tag{3}$$

The curve C is Cohen–Macaulay of degree 3 and arithmetic genus 0, its linear span V is a $\mathbb{P}^3$, and the linear section Y is a cubic surface equipped with the Ulrich line bundle $\mathcal{I}_{C/Y}(2)$. Hence, we have a linear resolution on V as follows:

$$0 \to 3\mathcal{O}_V(-3) \xrightarrow{M} 3\mathcal{O}_V(-2) \to \mathcal{I}_{C/Y} \to 0, \tag{4}$$

where M is a matrix of linear forms whose determinant is an equation of Y in V. Put $G_1 = 3\mathcal{O}_Y(-2)$ and $G_2 = 3\mathcal{O}_Y(-3)$. Thanks to [13, Theorem 6.1], taking the adjugate matrix M' of M forms a matrix factorization (M, M') of Y which provides the following 2-periodic resolution on Y (we still denote by M, M' the reduction of M and M' modulo Y):

$$\cdots \xrightarrow{M'} G_2(-3) \xrightarrow{M} G_1(-3) \xrightarrow{M'} G_2 \xrightarrow{M} G_1 \to \mathcal{I}_{C/Y} \to 0. \tag{5}$$

For all $i \in \mathbb{N}$, this gives

$$\Sigma_{2i+1}^Y \simeq \mathcal{I}_{C/Y}(-3i). \tag{6}$$

Next, set $K_0 = \mathcal{O}_X$, $K_1 = 2\mathcal{O}_X(-1)$, $K_2 = \mathcal{O}_X(-2)$ and write the Kozsul resolution

$$0 \to K_2 \to K_1 \to K_0 \to \mathcal{O}_Y \to 0. \tag{7}$$

Now we look at the exact sequence

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{I}_{C/X} \to \mathcal{I}_{C/Y} \to 0. \tag{8}$$

Set $F_1 = 3\mathcal{O}_X(-2)$, $F_2 = 3\mathcal{O}_X(-3)$. Now we proceed in two directions. On one hand, the composition $F_1 \to G_1 \to \Sigma_1^Y$ lifts to $F_1 \to \Sigma_1^X$ to give a diagram (we omit zeroes all around for brevity):

$$
\begin{array}{ccccc}
K_2 & \longrightarrow & K_1 & \longrightarrow & \mathcal{I}_{Y/X} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_2^X & \longrightarrow & F_1 \oplus K_1 & \longrightarrow & \Sigma_1^X \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_1^X \Sigma_1^Y & \longrightarrow & F_1 & \longrightarrow & \Sigma_1^Y .
\end{array}
$$

The leftmost column of the above diagram says that $\Sigma_1^X \Sigma_1^Y$ and Σ_2^X agree up to the free factor K_2 and therefore their higher syzygies are the same. More precisely,

we have $\Gamma_*(K_2) \simeq R_X(-2)$, so the sheafified minimal resolutions of Σ_2^X and $\Sigma_1^X \Sigma_1^Y$ over X differ only by the term $\mathcal{O}_X(-2)$ in degree 0, so that $\Sigma_3^X \simeq \Sigma_2^X \Sigma_1^Y$, and in turn $\Sigma_{i+1}^X \simeq \Sigma_i^X \Sigma_1^Y$ for all $i \geq 2$. Summing up, we have

$$0 \to K_2 \to \Sigma_2^X \to \Sigma_1^X \Sigma_1^Y \to 0, \quad \Sigma_{i+1}^X \simeq \Sigma_i^X \Sigma_1^Y \qquad \forall i \geq 2. \qquad (9)$$

Next, (8), (3) and (5) induce a diagram

$$\begin{array}{ccccc}
F_1 \otimes \mathcal{I}_{Y/X} & \longrightarrow & \Sigma_1^X \Sigma_1^Y & \longrightarrow & \Sigma_2^Y \\
\| & & \downarrow & & \downarrow \\
F_1 \otimes \mathcal{I}_{Y/X} & \longrightarrow & F_1 & \longrightarrow & G_1 \\
& & \downarrow & & \downarrow \\
& & \Sigma_1^Y & = & \Sigma_1^Y .
\end{array}$$

This in turn gives the exact sequence

$$0 \to F_1 \otimes \mathcal{I}_{Y/X} \to \Sigma_1^X \Sigma_1^Y \to \Sigma_2^Y \to 0. \qquad (10)$$

Lifting $F_2 \to \Sigma_2^Y$ to $F_2 \to \Sigma_1^X \Sigma_1^Y$, we get the exact diagram

$$\begin{array}{ccccc}
F_1 \otimes K_2 & \longrightarrow & F_1 \otimes K_1 & \longrightarrow & F_1 \otimes \mathcal{I}_{Y/X} \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_2^X \Sigma_1^Y & \longrightarrow & F_1 \otimes K_1 \oplus F_2 & \longrightarrow & \Sigma_1^X \Sigma_1^Y \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_1^X \Sigma_2^Y & \longrightarrow & F_2 & \longrightarrow & \Sigma_2^Y .
\end{array}$$

Using the diagram and the fact that $\Gamma_*(F_1 \otimes K_2)$ is free, we get

$$0 \to F_1 \otimes K_2 \to \Sigma_2^X \Sigma_1^Y \to \Sigma_1^X \Sigma_2^Y \to 0, \quad \Sigma_{i+1}^X \Sigma_1^Y \simeq \Sigma_i^X \Sigma_2^Y \qquad \forall i \geq 2. \qquad (11)$$

Repeating once more this procedure and using the periodicity of (5), we get

$$0 \to F_2 \otimes \mathcal{I}_{Y/X} \to \Sigma_1^X \Sigma_2^Y \to \Sigma_3^Y \to 0.$$

Then, using (6) and lifting $F_1(-3) \to \mathcal{I}_{C/Y}(-3) \simeq \Sigma_3^Y$ to $F_1(-3) \to \Sigma_1^X \Sigma_2^Y$, we have the exact sequence

$$0 \to F_2 \otimes K_2 \to \Sigma_2^X \Sigma_2^Y \to \Sigma_1^X \Sigma_3^Y \to 0.$$

Summing up, (9) and (11) give $\Sigma_4^X \simeq \Sigma_3^X \Sigma_1^Y \simeq \Sigma_2^X \Sigma_2^Y$, so that the above sequence tensored with $\mathcal{O}_X(5)$ becomes

$$0 \to 3\mathcal{O}_X \to \mathcal{S} \to \Sigma_1^X(\mathcal{I}_{C/Y}(2)) \to 0,$$

which is the sequence appearing in the statement. The fact that $h^0(X, \mathcal{S}) = 3$ is clear from the sequence. Since X is smooth and $C \subset X$ is arithmetically Cohen–Macaulay of codimension 3, the syzygy sheaf Σ_4^X is ACM and hence locally free. Looking at the above resolution, we compute the following invariants of $\mathcal{S}$:

$$\mathrm{rk}(\mathcal{S}) = 6, \quad c_1(\mathcal{S}) = 0, \quad c_2(\mathcal{S}) = H^2, \quad \mathrm{p}_{\mathcal{S}}(t) = \frac{1}{8}(t+1)^2(t+2)^2.$$

It remains to prove that $\mathrm{H}^*(\mathcal{S}(-1)) = \mathrm{H}^*(\mathcal{S}(-2)) = 0$. By (2), it suffices to show $\mathrm{H}^*(\mathcal{G}(-1)) = \mathrm{H}^*(\mathcal{G}(-2)) = 0$. By definition, we have

$$0 \to \mathcal{G} \to 3\mathcal{O}_X \to \mathcal{I}_{C/Y}(2) \to 0, \tag{12}$$

and $\mathcal{I}_{C/Y}(2)$ is Ulrich on Y, so

$$\mathrm{H}^*(\mathcal{I}_{C/Y}(1)) = \mathrm{H}^*(\mathcal{I}_{C/Y}) = 0.$$

We conclude that $\mathrm{H}^*(\mathcal{G}(-1)) = \mathrm{H}^*(\mathcal{G}(-2)) = 0$. The Chern characters for $\mathcal{G}_C$ and $\mathcal{S}$ can be computed immediately from (12) and (2), see also [26, Section 2.2]. $\quad\square$

Along the way we found the following minimal free resolution of $\mathcal{O}_C$ over X:

$$
\cdots \to
\begin{matrix} 3\mathcal{O}_X(-5) \\ \oplus \\ 9\mathcal{O}_X(-6) \end{matrix}
\overset{d_4}{\to}
\begin{matrix} 9\mathcal{O}_X(-4) \\ \oplus \\ 3\mathcal{O}_X(-5) \end{matrix}
\overset{d_3}{\to}
\begin{matrix} \mathcal{O}_X(-2) \\ \oplus \\ 9\mathcal{O}_X(-3) \end{matrix}
\overset{d_2}{\to}
\begin{matrix} 2\mathcal{O}_X(-1) \\ \oplus \\ 3\mathcal{O}_X(-2) \end{matrix}
\overset{d_1}{\to}
\mathcal{O}_X \to \mathcal{O}_C \to 0.
$$

This is an instance of Shamash's resolution. It becomes periodic after three steps. We record that $\mathcal{S}$ fits into:

$$\cdots \to 9\mathcal{O}_X(-2) \oplus 3\mathcal{O}_X(-3) \xrightarrow{\ d_5\ } 3\mathcal{O}_X \oplus 9\mathcal{O}_X(-1) \xrightarrow{\ d_4\ } 9\mathcal{O}_X(1) \oplus 3\mathcal{O}_X \to \cdots$$

with $\Sigma_5^X(\mathcal{O}_C(5))$ and $\mathcal{S}$ appearing as the intermediate cohomology/image sheaves.

Recall that the matrix M of linear forms presents $\mathcal{I}_{C/Y}$. We see it as a map of $\mathcal{O}_Y$-modules with target in $3\mathcal{O}_Y$ and denote its image by $\mathcal{R}$:

$$\mathcal{R} = \Sigma_1^Y(\mathcal{I}_{C/Y}(2)) \simeq \mathrm{Im}(M) \quad \text{with } M \colon 3\mathcal{O}_Y(-1) \to 3\mathcal{O}_Y. \tag{13}$$

From (5), we get the exact sequences

$$0 \to \mathcal{I}_{C/Y}(-1) \to 3\mathcal{O}_Y(-1) \to \mathcal{R} \to 0, \quad 0 \to \mathcal{R} \to 3\mathcal{O}_Y \to \mathcal{I}_{C/Y}(2) \to 0. \tag{14}$$

The following lemma is essentially [26, Proposition 2.5], we reproduce it here for self-containedness. In fact, given a Cohen–Macaulay twisted cubic $C \subset X$, the sheaf $\mathcal{G} = \mathcal{G}_C$ represents uniquely a point of the Lehn–Lehn–Sorger–van Straten eightfold $\mathcal{Z}$ associated with the cubic fourfold X.

Lemma 2.2. *Assume that Y is integral. Then the sheaf $\mathcal{G}$ is stable with*

$$p_\mathcal{G}(t) = u(t-1) = \frac{1}{8}(t+3)(t+2)(t+1)t, \quad H^*(X, \mathcal{G}(-t)) = 0 \quad \text{for } t = 0, 1, 2.$$

Finally, we have $\mathcal{E}xt_X^i(\mathcal{G}, \mathcal{O}_X) = 0$ except for $i = 0, 1$, in which case

$$\mathcal{G}^\vee \simeq 3\mathcal{O}_X, \quad \mathcal{E}xt_X^1(\mathcal{G}, \mathcal{O}_X) \simeq \mathcal{H}om_Y(\mathcal{I}_{C/Y}, \mathcal{O}_Y) = \mathcal{O}_Y(C).$$

Proof. The Hilbert polynomial of $\mathcal{G}$ is computed directly from the previous proposition. Next, we use the sheaf $\mathcal{R}$ which satisfies $\mathcal{R} \simeq \Sigma_2^Y(\mathcal{O}_C(2))$. Recall from the proof of the previous proposition the sequence (10) that we rewrite as

$$0 \to 3\mathcal{I}_{Y/X} \to \mathcal{G} \to \mathcal{R} \to 0. \tag{15}$$

By definition of $\mathcal{G} = \Sigma_1^X(\mathcal{I}_{C/Y}(2))$, the map $3\mathcal{O}_X \to \mathcal{I}_{C/Y}(2)$ in (12) induces an isomorphism on global sections, hence $H^*(X, \mathcal{G}) = 0$. The vanishing

$$H^*(X, \mathcal{G}(-1)) = H^*(X, \mathcal{G}(-2)) = 0$$

was proved in the previous proposition.

Next, we show first that $\mathcal{G}$ is simple. Applying $\mathrm{Hom}_X(-, \mathcal{G})$ to (12), we get

$$\mathrm{End}_X(\mathcal{G}) \simeq \mathrm{Ext}_X^1(\mathcal{I}_{C/Y}(2), \mathcal{G}).$$

We note that $\mathcal{I}_{C/Y}$ is simple, $\mathrm{Hom}_X(\mathcal{I}_{C/Y}(2), \mathcal{O}_X) = 0$ as $\mathcal{I}_{C/Y}$ is torsion and

$$\mathrm{Ext}_X^1(\mathcal{I}_{C/Y}(2), \mathcal{O}_X) \simeq H^3(\mathcal{I}_{C/Y}(-1))^\vee = 0$$

since $\dim(Y) = 2$. Hence, applying $\mathrm{Hom}_X(\mathcal{I}_{C/Y}(2), -)$ to (12), we observe that $\mathcal{G}$ is simple

$$\mathrm{End}_X(\mathcal{G}) \simeq \mathrm{Ext}_X^1(\mathcal{I}_{C/Y}(2), \mathcal{G}) \simeq \mathrm{End}_X(\mathcal{I}_{C/Y}) \simeq \Bbbk.$$

Suppose that $\mathcal{G}$ is not stable. Consider a saturated destabilizing subsheaf $\mathcal{K}$ of $\mathcal{G}$ so $\mathrm{rk}(\mathcal{K}) \in \{1, 2\}$ and $p_\mathcal{K} \geq p_\mathcal{G}$, so that $\mathcal{Q} = \mathcal{G}/\mathcal{K}$ is torsion-free with

$$\mathrm{rk}(\mathcal{Q}) = 3 - \mathrm{rk}(\mathcal{K}).$$

Since $\mathcal{K} \subset \mathcal{G} \subset 3\mathcal{O}_X$, we have $\mu(\mathcal{K}) \leq 0$. From $p_\mathcal{K} \geq p_\mathcal{G}$ we deduce that

$$c_1(\mathcal{K}) = c_1(\mathcal{Q}) = 0.$$

We look at the two possibilities for $\mathrm{rk}(\mathcal{K})$. If $\mathrm{rk}(\mathcal{K}) = 1$, then $\mathcal{K}$ is torsion-free with $c_1(\mathcal{K}) = 0$ so there is a closed subscheme $Z \subset X$ of codimension at least 2 such that $\mathcal{K} \simeq \mathcal{I}_{Z/X}$. If $Z = \emptyset$ then $\mathcal{K} \simeq \mathcal{O}_X$, which is impossible as $H^0(X, \mathcal{G}) = 0$. Now, for $Z \neq \emptyset$, consider the inclusion $\mathcal{I}_{Z/X} \subset \mathcal{G} \subset 3\mathcal{O}_X$. Taking reflexive hulls, we see that this factors through a single copy of $\mathcal{O}_X$ in $3\mathcal{O}_X$. Looking at (12), we get

that the quotient $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_{Z/X}$ inherits a non-zero map to $\mathcal{I}_{C/Y}(2)$. The image of this map is $\mathcal{O}_Y$ itself because $\mathcal{I}_{C/Y}(2)$ is torsion-free of rank 1 over Y as Y is integral.

Note that $\mathrm{p}_{\mathcal{I}_{Z/X}} = \mathrm{p}_{\mathcal{G}}$ precisely when Z is a linear subspace $\mathbb{P}^2$ contained in X, and that $\mathrm{p}_{\mathcal{I}_{Z/X}} < \mathrm{p}_{\mathcal{G}}$ if $\deg(Z) \geq 2$ and $\dim(Z) = 2$. Hence, the image of

$$\mathcal{O}_Z \to \mathcal{I}_{C/Y}(2)$$

cannot be the whole $\mathcal{O}_Y$ as then $Y \subseteq Z$, so we have $\dim(Z) = 2$ and $\deg(Z) \geq 3$, whilst assuming $\mathrm{p}_{\mathcal{I}_{Z/X}} \geq \mathrm{p}_{\mathcal{G}}$. Therefore, the possibility $\mathrm{rk}(\mathcal{K}) = 1$ is ruled out.

Now we may assume $\mathrm{rk}(\mathcal{K}) = 2$. Arguing as in the previous case, we deduce that there is a closed subscheme $Z \subset X$ of codimension at least 2 such that $\mathcal{Q} \simeq \mathcal{I}_{Z/X}$. Using (15) and noting that $3\mathcal{I}_{Y/X}$ cannot be contained in $\mathcal{K}$ for $\mathrm{rk}(\mathcal{K}) = 2$, we get a non-zero map

$$3\mathcal{I}_{Y/X} \to \mathcal{I}_{Z/X}$$

by composing $3\mathcal{I}_{Y/X} \hookrightarrow \mathcal{G}$ with $\mathcal{G} \twoheadrightarrow \mathcal{I}_{Z/X}$. The image of this map is of the form $\mathcal{I}_{Z'/X} \subset \mathcal{I}_{Z/X}$ for some closed subscheme $Z' \supseteq Z$ of X. Since $3\mathcal{I}_{Y/X}$ is polystable and $3\mathcal{I}_{Y/X} \twoheadrightarrow \mathcal{I}_{Z'/X}$, we have

$$\mathcal{I}_{Z'/X} \simeq \mathcal{I}_{Y/X},$$

so $Z' = Y$. In particular, we have $Z \subseteq Y$.

Again, we use that $\mathrm{p}_{\mathcal{I}_{Z/X}} > \mathrm{p}_{\mathcal{G}}$ as soon as $\dim(Z) \leq 1$, so the assumption that $\mathcal{K}$ destabilizes $\mathcal{G}$ forces $\dim(Z) \geq 2$. Hence, Z is a surface contained in Y so that $Z = Y$ since Y is integral. Then $\mathcal{I}_{Y/X}$ is a direct summand of $\mathcal{G}$ which therefore splits as $\mathcal{G} = \mathcal{K} \oplus \mathcal{I}_{Y/X}$. But this contradicts the fact that $\mathcal{G}$ is simple. We conclude that $\mathcal{G}$ must be stable.

Finally, we apply $\mathcal{H}om_X(-, \omega_X)$ to (12) and use Grothendieck duality to compute $\mathcal{E}xt^i_X(\mathcal{G}, \mathcal{O}_X)$ using that $\mathcal{I}_{C/Y}$ is reflexive on Y to get

$$\mathcal{E}xt^1_X(\mathcal{G}, \omega_X) \simeq \mathcal{E}xt^2_X(\mathcal{I}_{C/Y}(2), \omega_X) \simeq \mathcal{H}om_Y(\mathcal{I}_{C/Y}(2), \omega_Y).$$

Since $\omega_X \simeq \mathcal{O}_X(-3)$ and $\omega_Y \simeq \mathcal{O}_Y(-1)$, the conclusion follows. $\qquad\square$

The next lemma analyzes the restriction of $\mathcal{S}$ onto Y.

Lemma 2.3. *There is a surjection* $\xi \colon \mathcal{S}|_Y \to \mathcal{R}$ *whose kernel fits into*

$$0 \to \mathcal{R}(1) \to \ker(\xi) \to 2\mathcal{I}_{C/Y}(1) \to 0. \tag{16}$$

Proof. First of all, restricting the Koszul resolution (7) to Y, we find

$$\mathcal{T}or^X_1(\mathcal{I}_{C/Y}, \mathcal{O}_Y) \simeq 2\mathcal{I}_{C/Y}(-1), \quad \mathcal{T}or^X_2(\mathcal{I}_{C/Y}, \mathcal{O}_Y) \simeq \mathcal{I}_{C/Y}(-2).$$

Therefore, restricting (12) to Y, we get

$$0 \to 2\mathcal{I}_{C/Y}(1) \to \mathcal{G}|_Y \to 3\mathcal{O}_Y \to \mathcal{I}_{C/Y}(2) \to 0, \tag{17}$$

and hence

$$0 \to 2\mathcal{I}_{C/Y}(1) \to \mathcal{G}|_Y \to \mathcal{R} \to 0. \tag{18}$$

We also get

$$\mathcal{T}or_1^X(\mathcal{G}, \mathcal{O}_Y) \simeq \mathcal{T}or_2^X(\mathcal{I}_{C/Y}(2), \mathcal{O}_Y) \simeq \mathcal{I}_{C/Y}.$$

Therefore, by the previous display, restricting (2) to Y, we obtain the exact sequence

$$0 \to \mathcal{I}_{C/Y} \to 3\mathcal{O}_Y \to \mathcal{S}|_Y \to \mathcal{G}|_Y \to 0.$$

Then, comparing (14) and the leftmost part of the above sequence, we see that the image of the middle map is $\mathcal{R}(1)$. Hence. we extract from (17) the exact sequence

$$0 \to \mathcal{R}(1) \to \mathcal{S}|_Y \to \mathcal{G}|_Y \to 0. \tag{19}$$

Composing $\mathcal{S}|_Y \to \mathcal{G}|_Y$ with the surjection appearing in (18), we get the surjection ξ. Using (18) and (19) we get the desired filtration for $\ker(\xi)$. $\qquad \square$

2.2. Elementary modification along a cubic surface.
In Section 2.1 we constructed an ACM bundle $\mathcal{S}$ of rank 6. Recall that $h^0(X, \mathcal{S}) = 3$, and these three global sections of $\mathcal{S}$ make it unstable. Hence, it is natural to consider an elementary modification of $\mathcal{S}$ by a sheaf $\mathcal{A}$ such that

$$H^0(\mathcal{S}) \xrightarrow{\sim} H^0(\mathcal{A}).$$

Moreover, Proposition 1.2 suggests a good candidate for $\mathcal{A}$ to get closer to an Ulrich bundle on X. Indeed, we should have

$$\chi_{\mathcal{A}}(t) = 6p_{\mathcal{S}}(t) - 6u(t-1) = \frac{3}{2}(t+2)(t+1).$$

A natural choice for $\mathcal{A}$ would thus be an Ulrich line bundle on Y. In terms of Chern classes (as a coherent sheaf on X), we should have

$$c_1(\mathcal{A}) = 0, \quad c_2(\mathcal{A}) = -H_X^2.$$

Since an Ulrich line bundle on a cubic surface comes from a twisted cubic, we need to choose another twisted cubic D in Y, construct a surjection $\mathcal{S} \to \mathcal{O}_Y(D)$ so that the induced map on H^0 is an isomorphism, and take the kernel to perform an elementary modification. To do this, from now on in this section, we assume that Y is the blow-up of $\mathbb{P}^2$ at the six points $p_1, \ldots, p_6$ in general position and that the blow-down map $\pi \colon Y \to \mathbb{P}^2$ is associated with the linear system $|\mathcal{O}_Y(C)|$. Write L for the class of a line in $\mathbb{P}^2$ and denote by $E_1, \ldots, E_6$ the exceptional divisors of π, so that $C = \pi^*L$ and $H_Y = 3C - E_1 - \cdots - E_6$.

We will relate the pulled-back cotangent bundle of $\mathbb{P}^2$ to the sheaf $\mathcal{R}$ defined in (13) by proving that $\mathcal{R}(1) \simeq \pi^*(\Omega_{\mathbb{P}^2}(2))$; see Lemma 2.5.

Lemma 2.4. *Let* $Z = \{p_1, p_2, p_3\}$. *Then we have*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \Omega_{\mathbb{P}^2}(1) \to \mathcal{I}_{Z/\mathbb{P}^2}(1) \to 0.$$

Proof. By assumption Z is contained in no line, hence by the Cayley–Bacharach property (see, for instance, [20, Theorem 5.1.1]) there is a vector bundle $\mathcal{F}$ of rank 2 fitting into

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{F} \to \mathcal{I}_{Z/\mathbb{P}^2}(1) \to 0.$$

Note that $c_1(\mathcal{F}) = -L$ and $c_2(\mathcal{F}) = L^2$. By the above sequence $\mathrm{H}^0(\mathcal{F}) = 0$, so $\mathcal{F}$ is stable. But the only stable bundle on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = -L$ and $c_2(\mathcal{F}) = L^2$ is $\Omega_{\mathbb{P}^2}(1)$. $\qquad\square$

Set $D = 2C - E_1 - E_2 - E_3$. This is a class of a twisted cubic in Y with

$$D \cdot C = 2.$$

Lemma 2.5. *There is a surjection* $\eta \colon \mathcal{R}(1) \to \mathcal{O}_Y(D)$ *such that the induced map on global sections* $\mathrm{H}^0(\mathcal{R}(1)) \to \mathrm{H}^0(\mathcal{O}_Y(D))$ *is an isomorphism.*

Proof. From (14), we recall the exact sequence

$$0 \to \mathcal{O}_Y(-C) \to 3\mathcal{O}_Y \to \mathcal{R}(1) \to 0.$$

Comparing this with the Euler sequence, since $C = \pi^* L$, we get $\mathcal{R}(1) \simeq \pi^*(\Omega_{\mathbb{P}^2}(2))$. It follows from the projection formula that

$$\mathrm{H}^0(\mathcal{R}) = 0. \tag{20}$$

Next, we relate the pull-back to Y of the ideal sheaf of Z to $\mathcal{O}_Y(D)$. To do this we note that, given a smooth point p of a surface T, the blow-up $\mu \colon \widetilde{T} \to T$ at p gives an exceptional divisor E and an exact sequence

$$0 \to \mathcal{O}_E(-1) \to \mu^*(\mathcal{I}_{p/T}) \to \mathcal{O}_{\widetilde{T}}(-E) \to 0. \tag{21}$$

Indeed, after localizing at p, the Kozsul resolution of p in T is given by two linear equations f_1 and f_2 vanishing at p and reads

$$0 \to \mathcal{O}_T \to 2\mathcal{O}_T \to \mathcal{I}_{p/T} \to 0.$$

Pulling back via μ, we get

$$0 \to \mathcal{O}_{\widetilde{T}} \to 2\mathcal{O}_{\widetilde{T}} \to \mu^*(\mathcal{I}_{p/T}) \to 0.$$

The pull-back of f_1 and f_2 vanish at E, so $\mathcal{O}_{\widetilde{T}} \to 2\mathcal{O}_{\widetilde{T}}$ factors through

$$\mathcal{O}_{\widetilde{T}} \to \mathcal{O}_{\widetilde{T}}(E) \to 2\mathcal{O}_{\widetilde{T}},$$

hence we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\widetilde{T}} & \longrightarrow & 2\mathcal{O}_{\widetilde{T}} & \longrightarrow & \mu^*(\mathcal{I}_{p/T}) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\widetilde{T}}(E) & \longrightarrow & 2\mathcal{O}_{\widetilde{T}} & \longrightarrow & \mathcal{O}_{\widetilde{T}}(-E) & \longrightarrow & 0.
\end{array}
$$

By the Snake lemma, we thus get (21). Applying this subsequently to the blow-up of $\mathbb{P}^2$ at p_1, p_2, p_3, we get the exact sequence

$$
0 \to \bigoplus_{i=1}^{3} \mathcal{O}_{E_i}(-1) \to \pi^*(\mathcal{I}_{Z/\mathbb{P}^2}(2)) \to \mathcal{O}_Y(D) \to 0. \tag{22}
$$

By the previous lemma, we have

$$
0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \Omega_{\mathbb{P}^2}(2) \to \mathcal{I}_{Z/\mathbb{P}^2}(2) \to 0, \tag{23}
$$

and thus via π^* an exact sequence

$$
0 \to \mathcal{O}_Y(-C) \to \mathcal{R}(1) \to \pi^*(\mathcal{I}_{Z/\mathbb{P}^2}(2)) \to 0.
$$

Composing $\mathcal{R}(1) \to \pi^*(\mathcal{I}_{Z/\mathbb{P}^2}(2))$ with the surjection appearing in (22), we get the following

$$
0 \to \mathcal{O}_Y(-C + E_1 + E_2 + E_3) \to \mathcal{R}(1) \to \mathcal{O}_Y(D) \to 0.
$$

The map on global sections $\mathrm{H}^0(\mathcal{R}(1)) \to \mathrm{H}^0(\mathcal{O}_Y(D))$ is induced by the map

$$
\mathrm{H}^0(\Omega_{\mathbb{P}^2}(2)) \to \mathrm{H}^0(\mathcal{I}_{Z/\mathbb{P}^2}(2))
$$

arising from (23) and as such it is an isomorphism since $\mathrm{H}^*(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0$. $\square$

Given the class of a twisted cubic C in Y, we observe that $C^{\mathrm{t}} = 2H_Y - C$ is also the class of a twisted cubic. We denote

$$
C^{\mathrm{t}} = 2H_Y - C.
$$

This notation is justified by the fact that $\mathcal{I}_{C^{\mathrm{t}}/Y}$ is presented by the transpose matrix M^{t} of M. We have

$$
C^{\mathrm{t}} \cdot D = C \cdot D^{\mathrm{t}} = 4.
$$

Lemma 2.6. *There is a surjection* $\zeta \colon \mathcal{S} \to \mathcal{O}_Y(D)$ *inducing an isomorphism*

$$
\mathrm{H}^0(X, \mathcal{S}) \to \mathrm{H}^0(\mathcal{O}_Y(D)).
$$

Proof. According to the previous lemma, we have $\eta \colon \mathcal{R}(1) \to \mathcal{O}_Y(D)$ inducing an isomorphism on global sections. We would like to use Lemma 2.3 to lift η to a surjection $\mathcal{S}|_Y \to \mathcal{O}_Y(D)$ and compose this lift with the restriction $\mathcal{S} \to \mathcal{S}|_Y$ preserving the isomorphism on global sections.

So in the notation of Lemma 2.3, we first lift η to $\ker(\xi)$. To do this, we apply $\mathrm{Hom}_Y(-, \mathcal{O}_Y(D))$ to (16) and get

$$\cdots \to \mathrm{Hom}_Y(\ker(\xi), \mathcal{O}_Y(D)) \to \mathrm{Hom}_Y(\mathcal{R}(1), \mathcal{O}_Y(D))$$
$$\to 2\mathrm{H}^1(\mathcal{O}_Y(C + D - H_Y)) \to \cdots$$

Now, $C + D - H_Y = E_4 + E_5 + E_6$, so

$$\mathrm{H}^1(\mathcal{O}_Y(C + D - H_Y)) = 0.$$

Therefore, η lifts to $\widehat{\eta} \colon \ker(\xi) \to \mathcal{O}_Y(D)$. Note that by (16) the map $\mathcal{R}(1) \to \ker(\xi)$ induces an isomorphism on global sections, so $\widehat{\eta}$ gives an isomorphism

$$\mathrm{H}^0(\ker(\xi)) \simeq \mathrm{H}^0(\mathcal{O}_Y(D)).$$

Next, write

$$0 \to \ker(\xi) \to \mathcal{S}|_Y \to \mathcal{R} \to 0,$$

and apply $\mathrm{Hom}_Y(-, \mathcal{O}_Y(D))$. We get an exact sequence

$$\cdots \to \mathrm{Hom}_Y(\mathcal{S}|_Y, \mathcal{O}_Y(D)) \to \mathrm{Hom}_Y(\ker(\xi), \mathcal{O}_Y(D)) \to \mathrm{Ext}^1_Y(\mathcal{R}, \mathcal{O}_Y(D)) \to \cdots$$

So $\widehat{\eta}$ lifts to $\mathcal{S}|_Y \to \mathcal{O}_Y(D)$ if we prove $\mathrm{Ext}^1_Y(\mathcal{R}, \mathcal{O}_Y(D)) = 0$. To do this, write again the defining sequence of $\mathcal{R}$ as

$$0 \to \mathcal{R} \to 3\mathcal{O}_Y \to \mathcal{O}_Y(C^{\mathrm{t}}) \to 0.$$

Applying $\mathrm{Hom}_Y(-, \mathcal{O}_Y(D))$ to this sequence, we get

$$\cdots \to 3\mathrm{H}^1(\mathcal{O}_Y(D)) \to \mathrm{Ext}^1_Y(\mathcal{R}, \mathcal{O}_Y(D)) \to \mathrm{H}^2(\mathcal{O}_Y(C + D - 2H_Y)) \to \cdots$$

Now, $\mathcal{O}_Y(D)$ is Ulrich, so $\mathrm{H}^1(\mathcal{O}_Y(D)) = 0$ and $H_Y - C - D = -E_4 - E_5 - E_6$, then

$$\mathrm{h}^2(\mathcal{O}_Y(C + D - 2H_Y)) = \mathrm{h}^0(\mathcal{O}_Y(H_Y - C - D)) = 0.$$

This provides a lift $\widetilde{\eta} \colon \mathcal{S}|_Y \to \mathcal{O}_Y(D)$ of $\widehat{\eta}$ and again $\ker(\xi) \hookrightarrow \mathcal{S}|_Y$ induces an isomorphism on global sections, hence so does $\widetilde{\eta}$.

Finally, we define $\zeta \colon \mathcal{S} \to \mathcal{O}_Y(D)$ as composition of the restriction $\mathcal{S} \to \mathcal{S}|_Y$ and $\widetilde{\eta}$. Since

$$H^*(\mathcal{S}(-1)) = H^*(\mathcal{S}(-2)) = 0$$

by Proposition 2.1, tensoring the Koszul resolution (7) by $\mathcal{S}$ we see that $\mathcal{S} \to \mathcal{S}|_Y$ induces an isomorphism on global sections. Therefore, so does ζ and the lemma is proved. $\qquad\square$

Consider $D^{\mathrm{t}} = 2H_Y - D$ and $\mathcal{G}_{D^{\mathrm{t}}} = \ker(3\mathcal{O}_X \to \mathcal{O}_Y(D))$. Let $\mathcal{E} = \ker(\zeta)$, so we have

$$0 \to \mathcal{E} \to \mathcal{S} \to \mathcal{O}_Y(D) \to 0. \tag{24}$$

Lemma 2.7. *The sheaf $\mathcal{E}$ is simple and has a Jordan–Hölder filtration*

$$0 \to \mathcal{G}_{D^{\mathrm{t}}} \to \mathcal{E} \to \mathcal{G}_C \to 0. \tag{25}$$

Also, we have

$$\mathcal{E}^\vee \simeq \mathcal{S}^\vee, \quad \mathrm{p}_{\mathcal{E}}(t) = \mathrm{u}(t-1), \quad \mathrm{ch}(\mathcal{E}) = 2\gamma, \quad \mathrm{H}^*(\mathcal{E}(-t)) = 0 \quad \textit{for } t = 0,1,2.$$

Proof. The sheaves $\mathcal{G}_{D^{\mathrm{t}}}$ and $\mathcal{G}_C$ are stable by Lemma 2.2 and the reduced Hilbert polynomial of both of them is $\mathrm{u}(t-1)$. Also, they are not isomorphic, so

$$\mathrm{Hom}_X(\mathcal{G}_{D^{\mathrm{t}}}, \mathcal{G}_C) = \mathrm{Hom}_X(\mathcal{G}_C, \mathcal{G}_{D^{\mathrm{t}}}) = 0.$$

The sheaf $\mathcal{E}$ is semistable and has reduced Hilbert polynomial $\mathrm{u}(t-1)$ as soon as it fits in (25). Also, $\mathcal{E}$ is simple if this sequence is non-split. To see this, first apply $\mathrm{Hom}_X(-, \mathcal{G}_{D^{\mathrm{t}}})$ to (25) and note that the identity of $\mathcal{G}_{D^{\mathrm{t}}}$ is carried to the non-split extension represented by $\mathcal{E}$. Hence, using that $\mathcal{G}_{D^{\mathrm{t}}}$ is simple and $\mathrm{Hom}_X(\mathcal{G}_C, \mathcal{G}_{D^{\mathrm{t}}}) = 0$, we get

$$\mathrm{Hom}_X(\mathcal{E}, \mathcal{G}_{D^{\mathrm{t}}}) = 0.$$

Then, apply $\mathrm{Hom}_X(\mathcal{G}_C, -)$ to (25) and use that $\mathrm{Hom}_X(\mathcal{G}_C, \mathcal{G}_{D^{\mathrm{t}}}) = 0$ and that $\mathcal{G}_C$ is simple to get that $\mathrm{Hom}_X(\mathcal{G}_C, \mathcal{E})$ is at most 1-dimensional. Finally, apply $\mathrm{Hom}_X(-, \mathcal{E})$ to (25) to deduce that $\mathrm{End}_X(\mathcal{E})$ is included $\mathrm{Hom}_X(\mathcal{G}_C, \mathcal{E})$ so that $\mathrm{End}_X(\mathcal{E})$ is 1-dimensional and $\mathcal{E}$ is simple.

The Chern character of $\mathcal{E}$ is also computed from (25) – recall the notation

$$\gamma = 3 - H_X^2 + \frac{1}{4} H_X^4$$

from the introduction. Moreover, by Lemma 2.2, we get $\mathrm{H}^*(\mathcal{E}(-t)) = 0$ for $t = 0, 1, 2$ as well by (25).

Summing up, it suffices to prove that $\mathcal{E}$ fits in (25) and that this sequence is non-split. To do it, use the previous lemma to show that the evaluation of global sections gives an exact commutative diagram:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{G}_{D^{\mathrm{t}}} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G}_C & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & 3\mathcal{O}_X & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{G}_C & \longrightarrow & 0. \\
& & \downarrow & & \downarrow & & & & \\
& & \mathcal{O}_Y(D) & = & \mathcal{O}_Y(D) & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0 & & & &
\end{array}
$$

We thus have (25). By contradiction, assume that it splits as $\mathcal{E} \simeq \mathcal{G}_C \oplus \mathcal{G}_{D^{\mathfrak{t}}}$. Note that, since $\mathcal{S}$ is locally free, (24) gives

$$\mathcal{E}^{\vee} \simeq \mathcal{S}^{\vee}, \quad \mathcal{E}xt_X^1(\mathcal{E}, \mathcal{O}_X) \simeq \mathcal{E}xt_X^2(\mathcal{O}_Y(D), \mathcal{O}_X) \simeq \mathcal{O}_Y(D^{\mathfrak{t}}).$$

On the other hand, if $\mathcal{E} \simeq \mathcal{G}_C \oplus \mathcal{G}_{D^{\mathfrak{t}}}$, then by Lemma 2.2 we would have $\mathcal{E}^{\vee} \simeq 6\mathcal{O}_X$ and $\mathcal{E}xt_X^1(\mathcal{E}, \mathcal{O}_X) \simeq \mathcal{O}_Y(C) \oplus \mathcal{O}_Y(D^{\mathfrak{t}})$, which is not the case. $\qquad\square$

3. Smoothing the modified sheaves

In the previous section we constructed a simple and semistable sheaf $\mathcal{E}$ with $p_{\mathcal{E}}(t) = u(t-1)$. In particular, the sheaf $\mathcal{E}(1)$ has the same reduced Hilbert polynomial as an Ulrich bundle $\mathcal{U}$. However, $\mathcal{E}(1)$ itself cannot be Ulrich: for instance it is not locally free since $\mathcal{E}xt_X^1(\mathcal{E}, \mathcal{O}_X) \simeq \mathcal{O}_Y(D^{\mathfrak{t}})$. The goal of this section is to show that $\mathcal{E}(1)$ admits a flat deformation to an Ulrich bundle.

3.1. The Kuznetsov category. The bounded derived category $D(X)$ of coherent sheaves on X has the semiorthogonal decomposition:

$$\langle \mathrm{Ku}(X), \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle,$$

where $\mathrm{Ku}(X)$ is a K3 category. Indeed, $\mathrm{Ku}(X)$ equips with the K3-type Serre duality

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{2-i}(\mathcal{G}, \mathcal{F})^{\vee}$$

for any $\mathcal{F}, \mathcal{G} \in \mathrm{Ku}(X)$ [25]. We have

$$\mathrm{H}^*(X, \mathcal{E}) = \mathrm{H}^*(X, \mathcal{E}(-1)) = \mathrm{H}^*(X, \mathcal{E}(-2)) = 0,$$

and therefore

$$\mathcal{E} \in \mathrm{Ku}(X).$$

Lemma 2.2 says that for a Cohen–Macaulay twisted cubic $C \subset X$ spanning an irreducible cubic surface we have that $\mathcal{G}_C$ is stable and

$$\mathcal{G}_C \in \mathrm{Ku}(X).$$

We also know that $\mathcal{G}_C$ represents a point of the Lehn–Lehn–Sorger–van Straten irreducible symplectic eightfold $\mathbb{Z}$ and that $\mathbb{Z}$ contains a Zariski-open dense subset $\mathbb{Z}^{\circ}$ whose points are in bijection with the sheaves of the form $\mathcal{G}_C$ [26, 28].

Lemma 3.1. *We have* $\mathrm{h}^3(\mathcal{E}(-3)) = \mathrm{h}^4(\mathcal{E}(-3)) = 3$, $\mathrm{ext}_X^i(\mathcal{E}, \mathcal{E}) = 0$ *for* $i \geq 3$, *and*

$$\mathrm{ext}_X^1(\mathcal{E}, \mathcal{E}) = 26, \quad \mathrm{ext}_X^2(\mathcal{E}, \mathcal{E}) = 1.$$

Proof. First note that

$$h^3(\mathcal{E}(-3)) = h^2(\mathcal{O}_Y(D - 3H_Y)) = 3$$

since $\mathcal{O}_Y(D)$ is an Ulrich line bundle on a cubic surface Y. We also have

$$h^4(\mathcal{E}(-3)) = h^4(\mathcal{S}(-3)) = 3,$$

since $\chi(\mathcal{S}(-3)) = 6p_{\mathcal{S}}(-3) = 3$ and $h^i(\mathcal{S}(-3)) = 0$ for $i < 4$.

Recall that $\mathcal{E}$ is a simple sheaf, $\mathrm{ch}(\mathcal{E}) = 2\gamma$, and that $\mathcal{E}$ lies in $\mathrm{Ku}(X)$. Since $\mathrm{Ku}(X)$ is a K3 category, we have

$$\mathrm{ext}_X^2(\mathcal{E}, \mathcal{E}) = \hom_X(\mathcal{E}, \mathcal{E}) = 1 \quad \text{and} \quad \mathrm{ext}_X^i(\mathcal{E}, \mathcal{E}) = 0 \quad \text{for } i \geq 3.$$

Now the equality $\mathrm{ext}_X^1(\mathcal{E}, \mathcal{E}) = 26$ follows from Riemann–Roch:

$$\chi(\mathcal{E} \otimes \mathcal{E}^\vee) = \left[(36 - 24H_X^2 + 10H_X^4)\left(1 + \frac{3}{2}H_X + \frac{5}{4}H_X^2 + \frac{3}{4}H_X^3 + \frac{1}{3}H_X^4\right) \right]_4$$
$$= -24. \qquad \square$$

3.2. Deforming to Ulrich bundles.

We assume in this subsection that X does not contain an integral surface of degree up to 3 other than linear sections. In other words, X does not contain a plane (equivalently, a quadric surface) nor a smooth cubic scroll, nor a cone over a rational normal cubic curve.

The content of our main result is that there is a smooth connected quasi-projective variety T° of dimension 26 and a sheaf $\mathcal{F}$ on $T^\circ \times X$, flat over T°, together with a distinguished point $s_0 \in T^\circ$ such that $\mathcal{F}_{s_0} \simeq \mathcal{E}$, and such that $\mathcal{F}_s(1)$ is an Ulrich bundle on X for all s in $T^\circ \setminus \{s_0\}$. Here, we write $\mathcal{F}_s = \mathcal{F}|_{\{s\} \times X}$ for all $s \in T^\circ$. Stated in short form this gives the next result.

Theorem 3.2. *If Y is smooth, then the sheaf $\mathcal{E}(1)$ deforms to an Ulrich bundle on X.*

Proof. We divide the proof into several steps.

Step 1. Compute negative cohomology of $\mathcal{E}$, i.e., $h^k(\mathcal{E}(-t))$ for $t \gg 0$ and $k \in \{0, 1, 2, 3\}$.

Let $C \subset Y \subset X$ be a twisted cubic with Y smooth. The sheaf $\mathcal{G}_C$ is stable and lies in $\mathrm{Ku}(X)$ by Lemma 2.2. We note that $h^1(\mathcal{O}_Y(D + tH_Y)) = 0$ for $t \in \mathbb{Z}$, while $h^2(\mathcal{O}_Y(D - tH_Y)) = 0$ for $t \leq 1$, and while

$$h^2(\mathcal{O}_Y(D - tH_Y)) = \frac{3}{2}(t - 1)(t - 2) \quad \text{for } t \geq 2.$$

Also, $H^0(\mathcal{E}) = H^1(\mathcal{E}) = 0$ since the surjection (24) induces an isomorphism on global sections. This also implies that, since $H^0(\mathcal{O}_Y(D)) \otimes H^0(\mathcal{O}_X(t))$ generates $H^0(\mathcal{O}_Y(D + tH_Y))$ for all $t \geq 0$, the map

$$H^0(\mathcal{S}(t)) \to H^0(\mathcal{O}_Y(D + tH_Y))$$

induced by (24) is surjective. Since $\mathrm{H}^1(\mathcal{S}(t)) = 0$ for all $t \in \mathbb{Z}$, we obtain $\mathrm{H}^1(\mathcal{E}(t)) = 0$ for $t \in \mathbb{Z}$. By (24), we have

$$\begin{cases} h^0(\mathcal{E}(-t)) = 0, & t \geq 0, \\ h^1(\mathcal{E}(t)) = 0, & t \in \mathbb{Z}, \\ h^2(\mathcal{E}(t)) = 0, & t \in \mathbb{Z}, \\ h^3(\mathcal{E}(-t)) = \frac{3}{2}(t-1)(t-2), & t \geq 2. \end{cases}$$

Step 2. Argue that $\mathcal{E}$ is unobstructed.

This follows from the argument of [2, §31], which applies to the sheaf $\mathcal{E}$ as it is simple and lies in $\mathrm{Ku}(X)$. To sketch this, recall that the framework is based on a combination of Mukai's unobstructedness theorem [31] and Buchweitz–Flenner's approach to the deformation theory of $\mathcal{E}$; see [8, 9]. To achieve this step, we use the proof of [2, Theorem 31.1] which goes as follows. Let $\mathrm{At}(\mathcal{E}) \in \mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E} \otimes \Omega_X)$ be the Atiyah class of $\mathcal{E}$.

- Via a standard use of the infinitesimal lifting criterion, one reduces to show that $\mathcal{E}$ has a formally smooth deformation space.

- We show that the deformation space of $\mathcal{E}$ is formally smooth by observing that $\mathcal{E}$ extends over any square-zero thickening of X, conditionally to the vanishing of the product of the Atiyah class $\mathrm{At}(\mathcal{E})$ and the Kodaira–Spencer class κ of the thickening, see [21] – note that this holds in arbitrary characteristic.

- We use [24] in order to show $\kappa \cdot \mathrm{At}(\mathcal{E}) = 0$. Indeed, in view of [24, Theorem 4.3], this takes place if the trace $\mathrm{Tr}(\kappa \cdot \mathrm{At}^2(\mathcal{E}))$ vanishes as element of $\mathrm{H}^3(\Omega_X)$.

- We use that $\mathrm{Tr}(\kappa \cdot \mathrm{At}(\mathcal{E})^2) = 2\kappa \cdot \mathrm{ch}(\mathcal{E})$, (cf. the proof of [2, Theorem 31.1]) and note that this vanishes as the Chern character $\mathrm{ch}(\mathcal{E})$ remains algebraic under any deformation of X. This holds because all components of $\mathrm{ch}(\mathcal{E})$ are multiples of powers of the hyperplane class, while $\kappa \cdot \mathrm{ch}(\mathcal{E})$ is the obstruction to algebraicity of $\mathrm{ch}(\mathcal{E})$ along the thickening of X – again, cf. the proof of [2, Theorem 31.1]. Note that the assumption that $\Bbbk$ has characteristic other than 2 is needed to use the formula $\mathrm{Tr}(\kappa \cdot \mathrm{At}(\mathcal{E})^2) = 2\mathrm{Tr}(\kappa \cdot \exp(\mathrm{At}(\mathcal{E}))) = 2\kappa \cdot \mathrm{ch}(\mathcal{E})$.

According to the above deformation argument, there is a smooth quasi-projective scheme T representing an open piece of the moduli space of simple sheaves over X containing $\mathcal{E}$. In other words, there is a point $s_0 \in T$, together with a coherent sheaf $\mathcal{F}$ over $T \times X$, such that $\mathcal{F}_{s_0} \simeq \mathcal{E}$, and the Zariski tangent space of T at s_0 is identified with $\mathrm{Ext}^1_X(\mathcal{E}, \mathcal{E})$. Note that all the sheaves $\mathcal{F}_s$ are simple. By the openness of semistability and torsion-freeness, there is a connected open dense subset $T_0 \subset T$, with $s_0 \in T_0$, such that $\mathcal{F}_s$ is simple, semistable and torsion-free for all $s \in T_0$.

Step 3. Compute the cohomology of the reflexive hull $\mathcal{F}_s^{\vee\vee}$ of $\mathcal{F}_s$ and of $\mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$.

For $s \in T_0$, let us consider the reflexive hull $\mathcal{F}_s^{\vee\vee}$ and the torsion sheaf $\mathcal{Q}_s = \mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$. Let us write the reflexive hull sequence:

$$0 \to \mathcal{F}_s \to \mathcal{F}_s^{\vee\vee} \to \mathcal{Q}_s \to 0. \tag{26}$$

By the upper-semicontinuity of cohomology, there is a Zariski-open dense subset $s_0 \in T_1$ of T_0 such that for all $s \in T_1$, we have

$$\mathrm{H}^*(\mathcal{F}_s) = \mathrm{H}^*(\mathcal{F}_s(-1)) = \mathrm{H}^*(\mathcal{F}_s(-2)) = 0.$$

In particular, for $s \in T_1$ and $t \geq 0$:

$$\mathrm{h}^k(\mathcal{F}_s(-t)) = 0 \quad \text{for } k \leq 2.$$

By Lemma 1.3, since $\mathcal{F}_s$ is torsion-free there is a polynomial $q_2 \in \mathbb{Q}[t]$, with $\deg(q_2) \leq 2$, such that

$$\mathrm{h}^3(\mathcal{F}_s(-t)) = q_2(t) \quad \text{for } t \gg 0.$$

By semicontinuity, there is a Zariski-open dense subset T_2 of T_1, with $s_0 \in T_2$, such that for all $s \in T_2$ we have $q_2(t) \leq \frac{3}{2}(t-1)(t-2)$.

Since $\mathrm{codim}(\mathcal{Q}_s) \geq 2$, we get $\mathrm{H}^k(\mathcal{Q}_s(t)) = 0$ for each $k \geq 3$ and $t \in \mathbb{Z}$. Using Lemma 1.3, we get the vanishing

$$\mathrm{H}^0(\mathcal{F}_s^{\vee\vee}(-t)) = \mathrm{H}^1(\mathcal{F}_s^{\vee\vee}(-t)) = 0 \quad \text{for } t \gg 0,$$

and the existence of polynomials $q_0, q_1 \in \mathbb{Q}[t]$ with $\deg(q_k) \leq k$ such that

$$\mathrm{h}^{k+2}(\mathcal{F}_s^{\vee\vee}(-t)) = q_k(t) \quad \text{for } t \gg 0.$$

By (26), for $t \gg 0$ and $k \neq 2$, we have

$$\begin{cases} \mathrm{h}^2(\mathcal{Q}_s(-t)) = q_2(t) - q_1(t) + q_0, \\ \mathrm{h}^k(\mathcal{Q}_s(-t)) = 0. \end{cases}$$

Next, we use again the local-global spectral sequence

$$\mathrm{Ext}_X^{p+q}(\mathcal{Q}_s, \mathcal{O}_X(t-3)) \Leftarrow \mathrm{H}^p\big(\mathcal{E}xt_X^q(\mathcal{Q}_s, \mathcal{O}_X(t-3))\big) = E_2^{p,q}.$$

Via Serre vanishing for $t \gg 0$ and Serre duality this gives

$$\mathrm{h}^2(\mathcal{Q}_s(-t)) = \mathrm{h}^0\big(\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_X(t-3))\big),$$

$$\mathcal{E}xt_X^k(\mathcal{Q}_s, \mathcal{O}_X) = 0 \quad \text{for } k \neq 2. \tag{27}$$

Assume $\mathcal{Q}_s \neq 0$. By the above discussion, $\mathcal{Q}_s$ is a non-zero reflexive sheaf supported on a codimension 2 subvariety Y_s of X, in which case $\mathrm{h}^2(\mathcal{Q}_s(-t))$ must agree with a polynomial function of degree 2 of t for $t \gg 0$. Hence, the sheaf

$$\hat{\mathcal{Q}}_s = \mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_X(-3))$$

supported on Y_s satisfies

$$\chi(\hat{\mathcal{Q}}_s(t)) = h^2(\mathcal{Q}_s(-t)) = q_2(t) - q_1(t) + q_0$$
$$\leq \frac{3}{2}(t-1)(t-2),$$
$$\deg(\chi(\hat{\mathcal{Q}}_s(t))) = 2. \tag{28}$$

Note that (27) and [20, Proposition 1.1.10] imply $\mathcal{E}xt_X^k(\mathcal{F}_s^{\vee\vee}, \mathcal{O}_X) = 0$ for $k \geq 3$ and therefore, via (26), also $\mathcal{E}xt_X^k(\mathcal{F}_s, \mathcal{O}_X) = 0$ for $k \geq 3$. We prove along the way that

$$\mathrm{H}^1(\mathcal{E}xt_X^2(\mathcal{F}_s(t), \mathcal{O}_X)) = 0 \quad \text{for } t \gg 0. \tag{29}$$

Indeed, dualizing (26) and using (27), we get an epimorphism

$$\mathcal{E}xt_X^2(\mathcal{F}_s^{\vee\vee}(t), \mathcal{O}_X) \twoheadrightarrow \mathcal{E}xt_X^2(\mathcal{F}_s(t), \mathcal{O}_X) \quad \text{for } t \in \mathbb{Z}.$$

By [20, Proposition 1.1.10], if the sheaf $\mathcal{E}xt_X^2(\mathcal{F}_s^{\vee\vee}, \mathcal{O}_X)$ is non-zero then it is supported on a 0-dimensional subscheme of X, hence the same happens to $\mathcal{E}xt_X^2(\mathcal{F}_s, \mathcal{O}_X(-t))$ by the above epimorphism. Therefore,

$$\mathrm{H}^1(\mathcal{E}xt_X^2(\mathcal{F}_s, \mathcal{O}_X(-t))) = 0 \quad \text{for } t \gg 0.$$

Step 4. Show that, if $\mathcal{F}_s$ is not reflexive, then it is an extension of sheaves coming from $\mathcal{Z}^\circ$.

We have proved that, if $\mathcal{F}_s$ is not reflexive, the support Y_s of $\hat{\mathcal{Q}}_s$ is a surface of degree at most 3. But then, since X contains no integral surface of degree up to 3 other than complete intersections, the reduced structure of each primary component of Y_s must be a cubic surface contained in X, and hence Y_s itself must be a cubic surface. So the open subset $T_2 \subset T_1$ provides a family of cubic surfaces $\mathcal{Y} \to T_2$ whose fibre over T_2 is the cubic surface Y_s, where $Y_{s_0} = Y$ is smooth. Since smoothness is an open condition, there is a Zariski-open dense subset T_3 of T_2, with $s_0 \in T_3$, such that Y_s is smooth for all $s \in T_3$.

It follows again by (28) that $\mathcal{Q}_s$ is reflexive of rank 1 over Y_s, i.e., $\mathcal{Q}_s$ is a line bundle on Y_s since Y_s is smooth. Hence, we have a family of sheaves $\{\mathcal{Q}_s \mid s \in T_3\}$ where $\mathcal{Q}_s$ is a line bundle over Y_s and $\mathcal{Q}_{s_0} \simeq \mathcal{O}_Y(D)$. Since the Picard group of Y_s is discrete, this family must be locally constant. In other words, for each $s \in T_3$ there is a divisor class D_s on Y_s corresponding to a twisted cubic contained in Y_s such that $\mathcal{Q}_s \simeq \mathcal{O}_{Y_s}(D_s)$ and $D_{s_0} \equiv D$.

Since $\mathrm{H}^0(\mathcal{F}_s) = \mathrm{H}^1(\mathcal{F}_s) = 0$, the evaluation map of global sections

$$3\mathcal{O}_X \to \mathcal{O}_{Y_s}(D_s)$$

lifts to a non-zero map $\beta_s : 3\mathcal{O}_X \to \mathcal{F}_s^{\vee\vee}$. The snake lemma yields an exact sequence

$$0 \to \ker(\beta_s) \to \mathcal{G}_{D_s^t} \to \mathcal{F}_s \to \mathrm{coker}(\beta_s) \to 0.$$

Since the sheaves $\mathcal{F}_s$ and $\mathcal{G}_{D_s^t}$ share the same reduced Hilbert polynomial, with $\mathcal{F}_s$ semistable and $\mathcal{G}_{D_s^t}$ stable, we must have $\ker(\beta_s) = 0$. By semistability of $\mathcal{F}_s$, we note that $\mathcal{D}_s = \operatorname{coker}(\beta_s)$ is torsion-free, since otherwise the reduced Hilbert polynomial of the torsion-free part of $\mathcal{D}_s$ would be strictly smaller than $p_{\mathcal{D}_s} = p_{\mathcal{F}_s}$.

Therefore, we have a flat family of sheaves over T_3 whose fibre over s is $\mathcal{D}_s$, with $\mathcal{D}_{s_0} \simeq \mathcal{G}_C$. Hence, for all $s \in T_3$, the sheaf $\mathcal{D}_s$ corresponds to a point of the open part $\mathcal{Z}^\circ$ of the Lehn–Lehn–Sorger–van Straten eightfold, i.e., $\mathcal{D}_{s_0} \simeq \mathcal{G}_{C_s}$ for some twisted cubic $C_s \subset X$. We take a further Zariski-open dense subset T_4 of T_3 such that C_s is Cohen–Macaulay and spans a smooth cubic surface, for all $s \in T_4$.

Summing up, in a Zariski-open neighborhood T_4 of s_0, dense in T, the hypothesis $\mathcal{Q}_s \neq 0$ for $s \in T_4$ implies the existence of twisted cubics D_s, C_s in X such that $\mathcal{F}_s$ fits into:

$$0 \to \mathcal{G}_{D_s^t} \to \mathcal{F}_s \to \mathcal{G}_{C_s} \to 0,$$

where the twisted cubic C_s is Cohen–Macaulay, so that $\mathcal{G}_{C_s}$ lies in $\operatorname{Ku}(X)$. Therefore, the sheaves $\mathcal{G}_{C_s}$ and $\mathcal{G}_{D_s^t}$ correspond uniquely to well-defined points of $\mathcal{Z}^\circ$.

Step 5. Conclude that $\mathcal{F}_s(1)$ is Ulrich for generic $s \in T$.

We compute the dimension of the family $\mathcal{W}$ of sheaves $\mathcal{F}_s$ fitting into extensions as in the previous display. Indeed, $\mathcal{W}$ is equipped with a regular map $\mathcal{W} \to \mathcal{Z}^\circ \times \mathcal{Z}^\circ$ defined by

$$\mathcal{F}_s \mapsto (\mathcal{G}_{D_s^t}, \mathcal{G}_{C_s}),$$

whose fibre is $\mathbb{P}(\operatorname{Ext}_X^1(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}))$. Since $D^t = D_{s_0}^t$ and $C = C_{s_0}$ are contained in Y and satisfy $C \cdot D^t = 4$, $C \cdot C^t = 5$, we have $C \not\equiv D^t$, so

$$\mathcal{G}_{D_{s_0}^t} \not\simeq \mathcal{G}_{C_{s_0}}.$$

Therefore, $\mathcal{G}_{D_s^t} \not\simeq \mathcal{G}_{C_s}$ for all s in a Zariski-open dense subset $T_5 \subset T_4$, with $s_0 \in T_5$. Since $\mathcal{G}_{D_s^t}$ and $\mathcal{G}_{C_s}$ lie in $\operatorname{Ku}(X)$ and represent stable non-isomorphic sheaves, we have

$$\hom_X(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) = 0, \quad \operatorname{ext}_X^2(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) \simeq \hom_X(\mathcal{G}_{D_s^t}, \mathcal{G}_{C_s}) = 0.$$

Also, $\operatorname{ext}_X^k(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) = 0$ for $k \geq 3$, hence

$$\operatorname{ext}_X^1(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) = -\chi(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) = 6.$$

Therefore, the fibre of $\mathcal{W} \to \mathcal{Z}^\circ \times \mathcal{Z}^\circ$ is 5-dimensional and

$$\dim(\mathcal{W}) = 2 \cdot \dim(\mathcal{Z}^\circ) + \operatorname{ext}_X^1(\mathcal{G}_{C_s}, \mathcal{G}_{D_s^t}) - 1 = 21.$$

So there is a Zariski-open dense subset $T_6 \subset T_5$ with $s_0 \in T_6$, such that $\mathcal{Q}_s = 0$ for all $s \in T_6 \setminus \{s_0\}$. Hence, $\mathcal{F}_s$ is reflexive for all $s \in T_6 \setminus \{s_0\}$. Then $\mathcal{F}_s^\vee$ is also semistable and shares the same reduced Hilbert polynomial as $\mathcal{F}_s$, hence we have

$$h^4(\mathcal{F}_s(-3)) = \operatorname{ext}_X^4(\mathcal{O}_X(3), \mathcal{F}_s) = h^0(\mathcal{F}_s^\vee) = 0.$$

Since $\mathrm{h}^k(\mathcal{F}_s(-3)) = 0$ for $k \leq 2$, by Riemann–Roch we obtain $\mathrm{h}^3(\mathcal{F}_s(-3)) = 0$, i.e., $\mathrm{H}^*(\mathcal{F}_s(-3)) = 0$. Now we have proved that $\mathcal{F}_s(1)$ is Ulrich for $s \in T_6 \setminus \{s_0\}$.

Put $T^\circ = T_6$. We have proved that, for all $s \in T^\circ \setminus \{s_0\}$, the sheaf $\mathcal{F}_s(1)$ is an Ulrich bundle of rank 6. $\qquad\square$

3.3. Fourfolds containing planes or cubic scrolls.
Now we turn our attention to the case of smooth cubic fourfolds X containing a surface of degree up to three, other than linear sections.

The goal is to prove our main theorem from the introduction, in other words, we would like to extend Theorem 3.2 to these fourfolds. Note that Steps 1, 2 and 3 of the proof of Theorem 3.2 are still valid. Also, the argument of Step 5 holds once Step 4 is established. Summing up, it remains to work out Step 4. Recalling the base scheme T_2 introduced in Step 3, we are done as soon as we prove the following result.

Proposition 3.3. *There is a Zariski-open neighborhood of s_0 in T_2° such that, for all $s \in T_2^\circ$, the sheaf $\mathcal{Q}_s = \mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$ is either zero or it is supported on a linear section surface of X.*

Proof. We proved in Step 3 that, for $s \in T_2$, the sheaf $\mathcal{Q}_s$ is zero or it is a locally Cohen–Macaulay sheaf supported on a projective surface $Y_s \subset X$ with $\deg(Y_s) \leq 3$. Assuming $\mathcal{Q}_s \neq 0$, we would like to show that Y_s does not contain any surface Z other than linear sections of X. Passing to the purely 2-dimensional part of the reduced structure of each primary component of Z, we may assume without loss of generality that Z is integral, still of degree at most 3 and not a linear section: we must then seek a contradiction. The surface Z is either a plane, or a quadric surface, or a smooth cubic scroll, or a cone over a rational normal cubic. The Hilbert polynomial of $\mathcal{O}_Z$ is thus either $r_1 = (t+1)(t+2)/2$, $r_2 = (t+1)^2$, or $r_3 = (t+1)(3t+2)/2$, and Z is locally a complete intersection in any case.

We denote by $\mathcal{H}$ union of primary components of $\mathrm{Hilb}_r(X)$ containing integral subschemes $Z \subset X$ having Hilbert polynomial r, with $r \in \{r_1, r_2, r_3\}$. Note that $\mathrm{Hilb}_{r_1}(X)$ is a finite reduced scheme consisting of planes contained in X. For $r = r_2$ or $r = r_3$, a priori a surface in $\mathrm{Hilb}_r(X)$ might be badly singular. However, we have the following claim.

Claim 1. Each surface of $\mathrm{Hilb}_{r_2}(X)$ is a reduced quadric. For $r = r_3$, all the surfaces of $\mathcal{H}$ are purely 2-dimensional Cohen–Macaulay. For $r = r_2$ or $r = r_3$, each component of $\mathcal{H}$ is a projective plane.

Proof. Take a surface $Z = Z_h$ in $\mathrm{Hilb}_{r_2}(X)$. If Z is reduced, then Z is a quadric. Otherwise, the reduced structure of a component of Z must be a plane $L \subset X$. By computing the Hilbert polynomial of $\mathcal{I}_{L/Z}$, we see that this sheaf must be supported on a plane $L' \subset Z$ and have rank one over L'. Hence, its $\mathcal{O}_{L'}$-torsion-free part is of the form $\mathcal{I}_{B/L'}(b)$ for a subscheme $B \subset L'$ and some $b \in \mathbb{Z}$. Note that

$$\mathcal{I}_{L/Z} \simeq \mathcal{I}_{L/X}/\mathcal{I}_{Z/X},$$

so the surjection $3\mathcal{O}_X(-1) \to \mathcal{I}_{L/X}$ induces an epimorphism

$$3\mathcal{O}_{L'}(-1) \to \mathcal{I}_{B/L'}(b),$$

whence $b \in \{-1, 0\}$.

Computing Hilbert polynomials and arguing that the leading term of the Hilbert polynomial of the possible L'-torsion part of $\mathcal{I}_{L/Z}$ must be non-negative, we see that actually $b = -1$. This in turn implies $B = \emptyset$, i.e., $\mathcal{I}_{L/Z} \simeq \mathcal{O}_{L'}(-1)$. This says that Z is a quadric surface. A direct computation shows that Z must be reduced, for a cubic fourfold containing a non-reduced quadric surface is singular at least along a subscheme of length 4.

All surfaces of a component of $\mathrm{Hilb}_{r_2}(X)$ are residual to the same plane in X so each component of $\mathrm{Hilb}_{r_2}(X)$ is the projective plane of linear sections of X containing a given plane.

Now assume $r = r_3$ and let $Z = Z_h \subset X$ be an integral surface, so that Z is a smooth cubic scroll or a cone over a rational normal cubic. We work roughly like in Proposition 2.1. The linear span V of Z is a $\mathbb{P}^4$ that cuts X along a cubic threefold W and $\mathcal{I}_{Z/W}(2)$ is an Ulrich sheaf of rank 1 over W, so we have a presentation

$$0 \to 3\mathcal{O}_V(-1) \xrightarrow{M} 3\mathcal{O}_V \to \mathcal{I}_{Z/W}(2) \to 0. \tag{30}$$

Note that the threefold W can have only finitely many singular points as if W had a 1-dimensional family of singular points then X would be singular along the intersection of this family and a quadric in V.

The idea is to prove that, on one hand, denoting by $\mathcal{N}_{Z/X}$ the normal sheaf of Z in X, we have $\mathrm{h}^0(\mathcal{N}_{Z/X}) = 2$. On the other hand, inspired by [19, Section 4.1.2], we describe an explicit projective plane parametrizing elements of $\mathcal{H}$ by proving that each global section of $\mathcal{I}_{Z/W}(2)$ gives an element of $\mathcal{H}$ and that all elements obtained this way are Cohen–Macaulay and indeed contained in W.

First, let us accomplish the second task. By (30), the projectivization $P = \mathbb{P}(\mathcal{I}_{Z/W}(2))$ is embedded into $V \times \mathbb{P}^2 = \mathbb{P}(3\mathcal{O}_V)$ and the subscheme P is cut in $V \times \mathbb{P}^2$ by 3 linear equations defined by the columns of M. Write π and σ for the projections to V and $\mathbb{P}^2$ from $V \times \mathbb{P}^2$ and by $\mathfrak{h}, \mathfrak{l}$ the pull-back to $V \times \mathbb{P}^2$ of the hyperplane divisors of V and $\mathbb{P}^2$ via π and σ. Use the same letters upon restriction to P. From the Koszul resolution, we obtain

$$
\begin{aligned}
0 \to \mathcal{O}_{\mathbb{P}^2 \times V}(-2\mathfrak{h} - 3\mathfrak{l}) &\to 3\mathcal{O}_{\mathbb{P}^2 \times V}(-\mathfrak{h} - 2\mathfrak{l}) \\
&\to 3\mathcal{O}_{\mathbb{P}^2 \times V}(-\mathfrak{l}) \to \mathcal{O}_{\mathbb{P}^2 \times V}(\mathfrak{h}) \to \mathcal{O}_P(\mathfrak{h}) \to 0.
\end{aligned}
$$

Taking σ_*, we get that the sheaf $\mathcal{V} = \sigma_*(\mathcal{O}_P(\mathfrak{h}))$ fits into

$$0 \to 3\mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{N} 5\mathcal{O}_{\mathbb{P}^2} \to \mathcal{V} \to 0.$$

Observe that $H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ is naturally identified with $H^0(\mathcal{I}_{Z/W}(2))$. The choice of a line $\ell \subset \mathbb{P}^2$ corresponds uniquely to surjection $\ell \colon H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \twoheadrightarrow 2\Bbbk$, and thus to an epimorphism $3\mathcal{O}_V \to 2\mathcal{O}_V$. Composing with M, the line ℓ gives uniquely a matrix $M_\ell \colon 3\mathcal{O}_V(-1) \to 2\mathcal{O}_V$.

We have $\mathbb{P}(\mathcal{V}) \simeq P$. Note that the map $\pi \colon P \to W$ is birational since $\mathcal{I}_{Z/W}(2)$ has rank 1 over W and $\mathcal{O}_W$ has the same Hilbert polynomial as $\mathcal{O}_P(\mathfrak{h})$. Therefore, P is irreducible and thus $\mathcal{V}$ is torsion-free. In particular, for any line $\ell \subset \mathbb{P}^2$, setting N_ℓ for the restriction of N to ℓ, we get that N_ℓ is injective and, by restriction of π to ℓ, yields

$$\pi_\ell \colon \mathbb{P}(\mathcal{V}|_\ell) \to Z_\ell \subset W,$$

where $Z_\ell = \mathrm{Im}(\pi_\ell)$ is a surface in W. The scheme $\mathbb{P}(\mathcal{V}|_\ell)$ is equipped with two divisor classes inherited from P, which we still denote by $\mathfrak{l}$ and $\mathfrak{h}$. The surface Z_ℓ is the image of $\mathbb{P}(\mathcal{V}|_\ell)$ by the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{V}|_\ell)}(\mathfrak{h})|$.

Now $\mathcal{V}|_\ell \simeq \mathrm{coker}(N_\ell)$ is of the form

$$\mathcal{O}_\ell(a_1) \oplus \mathcal{O}_\ell(a_2) \oplus \mathcal{B},$$

where $0 \le a_1 \le a_2 \le 3$, and $\mathcal{B}$ is a torsion sheaf on ℓ of length b, with $a_1 + a_2 + b = 3$. The direct sum decomposition of $\mathrm{coker}(N_\ell)$ corresponds to a decomposition of N_ℓ into block-diagonal matrices. In turn, each block is classified by Kronecker–Weierstrass theory, see for instance according to [10, Chapter 19.1]. In view of this, once chosen homogeneous coordinates $(y_0 : y_1)$ on ℓ, a block of N_ℓ corresponding to $\mathcal{O}_\ell(a)$ for some $a \ge 1$, after transposition, takes the form of the following matrix with a rows and $a + 1$ columns:

$$\begin{pmatrix} y_0 & y_1 & 0 & \cdots & \\ 0 & y_0 & y_1 & 0 & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_0 & y_1 \end{pmatrix}.$$

On the other hand, $\mathcal{B}$ is presented as cokernel of an injective matrix of size b, where b is the length of $\mathcal{B}$ and the rank of the matrix drops at the points of the support of $\mathcal{B}$. Up to choosing the coordinates $(y_0 : y_1)$ suitably, since b is at most 3, we may assume that such support is contained in set-theoretically in $\{(0 : 1), (1 : 0), (1 : 1)\}$. Also, non-reduced points in the support of $\mathcal{B}$ are described in terms of Jordan blocks of N_ℓ. However, the presence of non-reduced points in the support of $\mathcal{B}$ is incompatible with the smoothness of X. Indeed the matrices N_ℓ would take one of the following normal forms for $b = 3$:

$$N_\ell^{\mathrm{t}} = \begin{pmatrix} y_0 & 0 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 & 0 \\ 0 & 0 & y_0 & 0 & 0 \end{pmatrix}, \quad N_\ell^{\mathrm{t}} = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 & 0 \\ 0 & 0 & y_0 & 0 & 0 \end{pmatrix},$$

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & y_0 & y_1 & 0 & 0 \\ 0 & 0 & y_0 & 0 & 0 \end{pmatrix},$$

or for $b = 2$:

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 & 0 & 0 \\ 0 & 0 & 0 & y_0 & 0 \end{pmatrix}, \quad N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 & y_1 & 0 \\ 0 & 0 & 0 & y_0 & 0 \end{pmatrix}.$$

One checks that any cubic fourfold X containing a surface arising from a matrix of this form has at least one singular point. Indeed, the first, second and fourth cases above do not even give rise to surfaces in $\mathbb{P}^5$, while in the third and the fifth cases the surface contains respectively a triple plane or a double plane and this forces X to be singular.

Summing up, after removing the cases forbidden by the smoothness of X, in a suitable basis of $H^0(\mathcal{V})$ and $H^1(\mathcal{V}(-1))$ and coordinates $(y_0 : y_1)$ on ℓ, the possibilities for N_ℓ are:

- $(a_1, a_2, b) = (1, 2, 0)$: Z_ℓ is a smooth cubic scroll and

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 & y_1 & 0 \\ 0 & 0 & 0 & y_0 & y_1 \end{pmatrix}.$$

- $(a_1, a_2, b) = (0, 3, 0)$: Z_ℓ is a cone over a rational normal cubic curve and

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & y_0 & y_1 & 0 & 0 \\ 0 & 0 & y_0 & y_1 & 0 \end{pmatrix}.$$

- $(a_1, a_2, b) = (1, 1, 1)$: Z_ℓ is the union of a $\mathbb{P}^2$ and a smooth quadric meeting along a line.

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & y_0 \end{pmatrix}.$$

- $(a_1, a_2, b) = (0, 2, 1)$: Z_ℓ is the cone over the union of a smooth conic and a line meeting at a single point, spanning a $\mathbb{P}^3 \subset V$ and having apex at a point outside V.

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & y_0 & y_1 & 0 & 0 \\ 0 & 0 & 0 & y_0 & 0 \end{pmatrix}.$$

- $(a_1, a_2, b) = (0, 1, 2)$: Z_ℓ is the cone over the union of a line and reducible conic, meeting at a length-two subscheme of the line, spanning a $\mathbb{P}^3 \subset V$. The apex of the cone is a point outside V.

$$N_\ell^t = \begin{pmatrix} y_0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 \end{pmatrix}.$$

- $(a_1, a_2, b) = (0, 0, 3)$: Z_ℓ is a cone over a non-colinear subscheme of length 3 in $\mathbb{P}^2 \subset V$, having a skew $\mathbb{P}^1 \subset V$ as apex.

$$N_\ell^t = \begin{pmatrix} y_0 & 0 & 0 & 0 & 0 \\ 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_0 - y_1 & 0 & 0 \end{pmatrix}.$$

In all these cases the resulting subscheme Z_ℓ lies in $\mathcal{H}$ and has projective dimension 2 with a Hilbert–Burch resolution given by M_ℓ^t. Then the dual plane parametrizing lines $\ell \subset \mathbb{P}^2$ describes an explicit projective plane of arithmetically Cohen–Macaulay surfaces in $\mathcal{H}$.

Finally, we have to show that $h^0(\mathcal{N}_{Z/X}) = 2$. We have an exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{I}_{Z/X} \to \mathcal{I}_{Z/W} \to 0.$$

Applying $\mathcal{H}om_X(-, \mathcal{O}_Z)$, we get

$$0 \to \mathcal{N}_{Z/W} \to \mathcal{N}_{Z/X} \to \mathcal{O}_Z(1)$$
$$\xrightarrow{\delta} \mathcal{E}xt_X^1(\mathcal{I}_{Z/W}, \mathcal{O}_Z) \to \mathcal{E}xt_X^1(\mathcal{I}_{Z/X}, \mathcal{O}_Z) \to 0.$$

Since the surfaces Z under consideration are locally complete intersection in X, we get that

$$\mathcal{E}xt_X^1(\mathcal{I}_{Z/X}, \mathcal{O}_Z) \simeq \mathcal{E}xt_X^2(\mathcal{O}_Z, \mathcal{O}_Z)$$

is the determinant of the normal bundle $\mathcal{N}_{Z/X}$ and is thus identified with the line bundle $\mathcal{N}_{Z/W}(1)$. On the other hand, using (30) we see that the sheaf $\mathcal{E}xt_X^1(\mathcal{I}_{Z/W}, \mathcal{O}_Z)$ fails to be locally free of rank 1 at the subscheme $\Upsilon \subset W$ defined by the 2-minors of M. Since Υ is contained in (though sometimes not equal to) the singular locus of W, we have $\dim(\Upsilon) = 0$ so the resolution of Υ is obtained by the Gulliksen–Negard complex (see [18]):

$$0 \to \mathcal{O}_V(-6) \to 9\mathcal{O}_V(-4) \to 16\mathcal{O}_V(-3) \to 9\mathcal{O}_V(-2) \to \mathcal{I}_{\Upsilon/V} \to 0.$$

Thus, Υ has length 6 and $H^0(\mathcal{I}_{\Upsilon/V}(1)) = 0$, which in turn implies

$$H^0(\mathcal{I}_{\Upsilon/Z}(1)) = 0.$$

Therefore, $\ker(\delta) \subset \mathcal{I}_{\Upsilon/Z}(1)$ gives $\mathrm{H}^0(\ker(\delta)) = 0$. In turn we get

$$\mathrm{H}^0(\mathcal{N}_{Z/X}) \simeq \mathrm{H}^0(\mathcal{N}_{Z/W}),$$

so it only remains to show $\mathrm{h}^0(\mathcal{N}_{Z/W}) = 2$. To get this, since Z and W are locally complete intersection, we may use adjunction to the effect that

$$\mathcal{N}_{Z/W} \simeq \mathcal{H}om_W(\omega_W, \omega_Z)/\mathcal{O}_W.$$

Therefore, using $\omega_W \simeq \mathcal{O}_W(-2)$ and restricting (30) to W, we get

$$0 \to \mathcal{I}_{Z/W}(-1) \to 3\mathcal{O}_W(-1) \to 2\mathcal{O}_W \to \mathcal{N}_{Z/W} \to 0.$$

Taking cohomology we obtain $\mathrm{h}^0(\mathcal{N}_{Z/W}) = 2$ as desired. $\square$

Write $\mathcal{Z} \subset X \times \mathcal{H}$ for the tautological surface. For each point $h \in \mathcal{H}$, we denote by $Z_h = \mathcal{Z} \cap X \times \{h\}$ the corresponding surface. Consider $\mathcal{X} = X \times T_2 \times \mathcal{H}$ and write $\pi_{1,2}, \pi_{1,3}$ and $\pi_{2,3}$ for the projections from $\mathcal{X}$ onto $X \times T_2$, $X \times \mathcal{H}$ and $T_2 \times \mathcal{H}$. We have the following claim.

Claim 2. For any $(s, h) \in T_2 \times \mathcal{H}$, the surfaces Z_h and Y_s share a component if and only if

$$\mathrm{H}^2(\mathcal{E}xt_X^1(\mathcal{F}_s(t), \mathcal{O}_{Z_h})) \neq 0 \quad \text{for } t \gg 0.$$

Proof. Take $(s, h) \in T_2 \times \mathcal{H}$ and set $Z = Z_h$. Since $\mathcal{F}_s$ is torsion-free and $\mathcal{F}_s^{\vee\vee}$ is reflexive, for any coherent sheaf $\mathcal{B}$ on X, we have

$$\mathcal{E}xt_X^q(\mathcal{F}_s^{\vee\vee}, \mathcal{B}) = \mathcal{E}xt_X^{q+1}(\mathcal{F}_s, \mathcal{B}) = 0 \quad \text{for } q \geq 3, \tag{31}$$

and

$$\dim(\mathcal{E}xt_X^q(\mathcal{F}_s^{\vee\vee}, \mathcal{B})) \leq 2-q, \quad \dim(\mathcal{E}xt_X^q(\mathcal{F}_s, \mathcal{B})) \leq 3-q \quad \text{for } q \in \{1, 2\}. \tag{32}$$

Indeed, this follows from [20, Proposition 1.1.10] if $\mathcal{B}$ is locally free. Then, (31) and (32) hold for an arbitrary coherent sheaf $\mathcal{B}$ as we see by applying $\mathcal{H}om_X(\mathcal{F}_s, -)$ and $\mathcal{H}om_X(\mathcal{F}_s^{\vee\vee}, -)$ to a finite locally resolution of $\mathcal{B}$ and using that (31) and (32) hold for the terms of the resolution.

Applying $\mathcal{H}om_X(-, \mathcal{O}_Z)$ to (26), for $q \geq 1$, we get

$$\cdots \to \mathcal{E}xt_X^q(\mathcal{F}_s^{\vee\vee}, \mathcal{O}_Z) \to \mathcal{E}xt_X^q(\mathcal{F}_s, \mathcal{O}_Z)$$
$$\to \mathcal{E}xt_X^{q+1}(\mathcal{Q}_s, \mathcal{O}_Z) \to \mathcal{E}xt_X^{q+1}(\mathcal{F}_s^{\vee\vee}, \mathcal{O}_Z) \to \cdots$$

We deduce that $\mathcal{E}xt_X^q(\mathcal{F}_s, \mathcal{O}_Z) = 0$ for $q \geq 3$, and

$$\dim(\mathcal{E}xt_X^1(\mathcal{F}_s, \mathcal{O}_Z)) = 2 \iff \dim(\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)) = 2.$$

Therefore,

$$\mathrm{H}^p(\mathcal{E}xt_X^1(\mathcal{F}_s(t), \mathcal{O}_Z)) = 0 \quad \text{for } p \geq 3 \text{ and all } t \in \mathbb{Z}, \tag{33}$$

and

$$\mathrm{H}^2(\mathcal{E}xt_X^1(\mathcal{F}_s(t), \mathcal{O}_Z)) \neq 0 \text{ for } t \gg 0 \iff \dim(\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)) = 2. \tag{34}$$

By Claim 1, we may assume that Z is a locally Cohen–Macaulay in X. So, since $\mathcal{O}_Z$ is a locally Cohen–Macaulay $\mathcal{O}_X$-module of projective dimension 2, by the Hilbert–Burch theorem, locally over X there exists a matrix M of size $p \times (p+1)$ whose p-minors cut Z as subscheme of X, namely we have a local presentation

$$0 \to p\mathcal{O}_X \to (p+1)\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Applying $\mathcal{H}om_X(\mathcal{Q}, -)$ to this sequence and using $\mathcal{E}xt_X^k(\mathcal{Q}_s, \mathcal{O}_X) = 0$ for $k \neq 2$ we get that the sheaf $\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)$ is locally presented as cokernel of the rightmost map in

$$0 \to p\widehat{\mathcal{Q}}_s \xrightarrow{\mathcal{H}om(\widehat{\mathcal{Q}}_s, M)} (p+1)\widehat{\mathcal{Q}}_s \xrightarrow{\mathcal{H}om(\widehat{\mathcal{Q}}_s, \wedge^p M)} \widehat{\mathcal{Q}}_s. \tag{35}$$

Now the p-minors of M vanish on an irreducible component of Y_s if and only if such component also lies in Z, in which case (35) shows that the support of $\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)$ is the whole component. Conversely, if Y_s and Z share no irreducible component so that the p-minors do not vanish identically on any component of Y_s, then again by (35) the sheaf $\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)$ is supported along a closed subset of Z having dimension at most 1. This shows that $\dim(\mathcal{E}xt_X^2(\mathcal{Q}_s, \mathcal{O}_Z)) = 2$ if and only if $\mathcal{Q}_s$ and Z share a common component. Together with (34), this finishes the proof.
□

Claim 3. For $t \in \mathbb{Z}$, put $\mathcal{B} = \mathcal{E}xt_{\mathcal{X}}^1(\pi_{12}^*(\mathcal{F}(-t)), \pi_{13}^*(\mathcal{O}_Z))$ and $\mathcal{P} = \mathrm{R}^2\pi_{23*}(\mathcal{B})$. For $(s,h) \in T_2 \times \mathcal{H}$, we have $\mathcal{P}_{(s,h)} \neq 0$ for $t \gg 0$ if and only if Z_h and Y_s have a common component.

Proof. Since $\mathcal{F}$ and $\mathcal{O}_Z$ are flat over $T_2 \times \mathcal{H}$, we have an identification $\mathcal{B}_{(s,h)} \simeq \mathcal{E}xt_X^1(\mathcal{F}_s(-t), \mathcal{O}_{Z_h})$ for all $t \in \mathbb{Z}$ and $(s,h) \in T_2 \times \mathcal{H}$. By the vanishing results of the previous paragraph and using the flattening stratification for $\mathcal{B}$ over $T_2 \times \mathcal{H}$ and working over each stratum, we get $\mathrm{R}^p\pi_{23*}(\mathcal{B}) = 0$ for all $p \geq 3$ and $t \in \mathbb{Z}$ so via base-change we obtain, for all $(s,h) \in T_2 \times \mathcal{H}$, we have

$$\mathcal{P}_{(s,h)} \simeq \mathrm{R}^2\pi_{23*}(\mathcal{B})_{(s,h)} \simeq \mathrm{H}^2\big(\mathcal{E}xt_X^1(\mathcal{F}_s(t), \mathcal{O}_{Z_h})\big)$$

for all $t \in \mathbb{Z}$. The conclusion follows from Claim 2.
□

Now we finish the proof of the proposition. Indeed, by Claim 1 for the special point $s_0 \in T_2$, the surface $Y = Y_{s_0}$ shares no component with any surface Z_h for $h \in \mathcal{H}$.

Indeed, since Z_h and Y are projective surfaces of degree 3 in $\mathbb{P}^5$, if Z_h contains Y then Z_h contains a further (possibly embedded) component of dimension at most 1, hence the surface Z_h would fail to be Cohen–Macaulay, a contradiction.

Therefore, by Claim 3 we have $\mathcal{P}_{(s_0,h)} = 0$ for all $h \in \mathcal{H}$. In other words, the support of $\mathcal{P}$ is disjoint from $\{s_0\} \times \mathcal{H}$. Since $\mathcal{H}$ is projective, the image of the support of $\mathcal{P}$ in T_2 is thus a closed subset of T_2, disjoint from s_0. Hence, there exists an open neighborhood T_2° of s_0 disjoint from this subset. Thus, the support of $\mathcal{P}$ does not intersect $T_2^\circ \times \mathcal{H}$. This implies that, for all $s \in T_2^\circ$, if $\mathcal{Q}_s$ is not zero then its support is a surface Y_s having degree at most 3 and containing no surface of $\mathcal{H}$ as a component, in other words Y_s must be a linear section of X. This completes the proof of Proposition 3.3 and consequently of the main theorem. $\qquad\square$

4. Stability of Ulrich bundles and the higher rank range

Our next goal is to prove Theorem 2. We know that $\mathcal{E}(1)$ deforms to a simple Ulrich bundle, and such Ulrich bundles constitute a family of dimension 26. We check that a sufficiently general deformation of $\mathcal{S}$ is stable, which will achieve the proof of the first statement.

Lemma 4.1. *The sheaf $\mathcal{E}(1)$ deforms flatly to a stable Ulrich bundle.*

Proof. We know that $\mathcal{E}(1)$ deforms to an Ulrich vector bundle $\mathcal{U}$. Also, $\mathcal{U}$ is necessarily semistable and any element of the graded objects associated to a Jordan–Hölder filtration of $\mathcal{U}$ must be a stable Ulrich bundle. Therefore it suffices to prove that, for a generic choice of $\mathcal{U}$, we have

$$\operatorname{Hom}_X(\mathcal{U}, \mathcal{B}) = \operatorname{Hom}_X(\mathcal{B}, \mathcal{U}) = 0,$$

where $\mathcal{B}$ is an Ulrich bundle of rank 2, and that $\mathcal{U}$ is not the extension of two Ulrich bundles of rank 3.

If X supports an Ulrich bundle $\mathcal{B}$ of rank 2, which is to say, if X is a pfaffian cubic fourfold, then we have $c_2(\mathcal{B}(-1)) \cdot H_X^2 = 2$ and $c_2(\mathcal{B}(-1))^2 = 6$, so

$$\chi(\mathcal{B}, \mathcal{B}) = 2.$$

Since $\mathcal{B}(-1)$ is stable and lies in $\operatorname{Ku}(X)$, this implies that $\operatorname{Ext}^1_X(\mathcal{B}, \mathcal{B}) = 0$, therefore there are finitely many Ulrich bundles $\mathcal{B}$ of rank 2 on X. For each of them and any twisted cubic $C \subset X$, we have

$$\operatorname{Hom}_X(\mathcal{G}_C, \mathcal{B}(-1)) = 0 = \operatorname{Hom}_X(\mathcal{B}(-1), \mathcal{G}_C)$$

for these sheaves are stable, not isomorphic, and share the same reduced Hilbert polynomial. Thus,

$$\mathrm{H}^4(X, \mathcal{B}^\vee \otimes \mathcal{E}(-2)) \simeq \operatorname{Hom}_X(\mathcal{E}, \mathcal{B}(-1))^\vee = 0$$

and

$$\mathrm{H}^0(X, \mathcal{B}^\vee \otimes \mathcal{E}(1)) \simeq \mathrm{Hom}_X(\mathcal{B}(-1), \mathcal{E}) = 0$$

by (25). Therefore,

$$\mathrm{Hom}_X(\mathcal{U}, \mathcal{B}) = 0 = \mathrm{Hom}_X(\mathcal{B}, \mathcal{U})$$

for a sufficiently general choice of $\mathcal{U}$ by semicontinuity of cohomology.

Next, consider Ulrich bundles $\mathcal{A}$ and $\mathcal{B}$ of rank 3 on X and assume that $\mathcal{U}$ contains $\mathcal{A}$ with $\mathcal{B} \simeq \mathcal{U}/\mathcal{A}$. Note that the degeneracy locus of two sufficiently general global sections of $\mathcal{A}$ is a smooth surface $A \subset X$ with $A \cdot H_X^2 = 12$ and $A \cdot A = 54$, which implies that $\chi(\mathcal{A}, \mathcal{A}) = 0$. Similarly, we have $\chi(\mathcal{B}, \mathcal{B}) = 0$ and the relation $\mathrm{ch}(\mathcal{U}) = \mathrm{ch}(\mathcal{A}) + \mathrm{ch}(\mathcal{B})$ gives

$$\chi(\mathcal{A}, \mathcal{B}) = \frac{1}{2}\chi(\mathcal{U}, \mathcal{U}) = 12.$$

It follows that the family of Ulrich bundles admitting a Jordan–Hölder filtration with graded object $\mathcal{A} \oplus \mathcal{B}$ has dimension 15. Hence a sufficiently general choice of $\mathcal{U}$ is stable.

To prove the second statement of Theorem 2, set $\mathcal{U}_1$ for a stable Ulrich bundle of rank 6 obtained as before and assume that X is general enough so that it contains no surface which is not homologous to mH_X^2 for any $m \in \mathbb{N}$ and that it supports a Cartan bundle $\mathcal{U}_2$ arising from [22]. We have

$$\mathrm{ch}(\mathcal{U}_i(-1)) = (i + 1)\gamma \quad \text{for } i \in \{1, 2\}$$

(cf. [23, Section 2] for the computation of Chern classes of Ulrich bundles on a (very) general cubic fourfold). Also, $\mathcal{U}_1$ and $\mathcal{U}_2$ are stable. Indeed they are semistable and may be only destabilized by Ulrich bundles, which is impossible, for Ulrich bundles must have rank $3m$ for some $m \in \mathbb{N}$, with $m \geq 2$, when X is very general according to [23].

Take i, j distinct in $\{1, 2\}$. From the observations above we deduce

$$\mathrm{Hom}_X(\mathcal{U}_i, \mathcal{U}_j) = 0$$

so, since $\mathcal{U}_1(-1)$ and $\mathcal{U}_2(-1)$ lie in $\mathrm{Ku}(X)$, we get $\mathrm{Ext}_X^\ell(\mathcal{U}_i, \mathcal{U}_j) = 0$ unless $\ell = 1$. By Riemann–Roch, we have

$$\mathrm{ext}_X^1(\mathcal{U}_i, \mathcal{U}_j) = 36.$$

Consider an integer $k > 1$. We may find integers $k_1, k_2 \in \mathbb{N}$ such that $k = 2k_1 + 3k_2$. Then we construct a deformation $\mathcal{U}$ of $k_1\mathcal{U}_1 \oplus k_2\mathcal{U}_2$ which is a stable Ulrich bundle. Indeed, we first consider a simple sheaf $\mathcal{U}_0$ as an extension of $k_1\mathcal{U}_1$ and $k_2\mathcal{U}_2$, which is possible for $\chi(k_1\mathcal{U}_1, k_2\mathcal{U}_2) = -36k_1k_2$. The sheaf $\mathcal{U}_0$ has a smooth deformation

space of dimension $6k^2 + 2$ and we take $\mathcal{U}$ to be a generic element of this space. Now, the summands of the graded object arising from a Jordan–Hölder filtration of $\mathcal{U}$ must be stable Ulrich bundles and thus, after twisting by $\mathcal{O}_X(-1)$, they must have Chern character $h_1\gamma, \ldots, h_r\gamma$ for some $r, h_1, \ldots, h_r \in \mathbb{N}^*$ with $h_1 + \cdots + h_r = k$. But the dimension of the family of sheaves admitting such a filtration is

$$6\left(k^2 - \sum_{1 \le i < j \le r} h_i h_j\right) + r + 1 < 6k^2 + 2,$$

as one can see by looking at the Jordan–Hölder filtration as r iterated extensions, the inequality being valid for all $r > 1$.

This proves that for any $k > 1$ there is a stable Ulrich bundle $\mathcal{U}$ on X with $\mathrm{ch}(\mathcal{U}(-1))$ $= k\gamma$ which is a generic flat deformation of $k_1\mathcal{U}_1 \oplus k_2\mathcal{U}_2$. The moduli space of stable sheaves $M_X(k\gamma)$ is smooth and symplectic at the points corresponding to these sheaves. $\qquad\square$

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