Homological norms on nonpositively curved manifolds

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Abstract. We relate the Gromov norm on homology classes to the harmonic norm on the dual cohomology and obtain double sided bounds in terms of the volume and other geometric quantities of the underlying manifold. Along the way, we provide comparisons to other related norms and quantities as well.

1. Introduction and results

The vector space $C_p(X, \mathbb{R})$ of singular p-chains of a topological space X comes equipped with a natural choice of basis consisting of the set of all continuous maps from the p-dimensional Euclidean simplex into X. The ℓ^1-norm on $C_p(X, \mathbb{R})$ associated to this basis descends to a semi-norm $\|\cdot\|_1$ on the singular homology $H_p(X, \mathbb{R})$ by taking the infimum of the norm within each equivalence class, or more precisely, we have the following definition.

Definition 1.1. The Gromov norm of a p-homology class α is

$$\|\alpha\|_1 := \inf\left\{\sum_i |a_i| : \left[\sum_i a_i \sigma_i\right] = \alpha \in H_p(X, \mathbb{R})\right\}.$$

This norm was introduced by Thurston [30, Chapter 6], and later generalized by Gromov [21], and an important special case is the *simplicial volume*, written $\|M\|$, of a closed oriented connected n-manifold M which is defined to be $\|[M]\|_1$, where $[M]$ is the fundamental class in $H_n(M, \mathbb{R})$ – the image of a preferred generator in $H_n(M, \mathbb{Z})$ under the change of coefficient map. More concretely,

$$\|M\| = \inf\left\{\sum_i |a_i| : \left[\sum_i a_i \sigma_i\right] = [M] \in H_n(M, \mathbb{R})\right\},$$

where the infimum is taken over all singular cycles with real coefficients representing the fundamental class in the top homology group of M. This invariant is multiplicative under finite covers, so the definition can be extended to closed connected

non-orientable manifolds as well. In what follows, we will always assume that our manifolds are connected.

For a cohomology class $\beta \in H^p(M, \mathbb{R})$ we denote the *harmonic norm* by $\|\beta\|_H$ and define it to be the L^2-norm of the unique harmonic form representing the class β in the de Rham cohomology. Since the harmonic representative minimizes the L^2-norm among all forms, we have the following definition.

Definition 1.2. The harmonic norm of a cohomology class $\beta \in H^p(M, \mathbb{R})$ is

$$\|\beta\|_H := \inf_{[\eta]=\beta} \|\eta\|_2 := \inf_{[\eta]=\beta} \left(\int_M \eta \wedge *\eta \right)^{1/2},$$

where the infimum is taken over all closed differential forms η representing β. The infimum is achieved by the harmonic representative ω satisfying $\Delta\omega = 0$, i.e. $\|\omega\|_2 = \|\beta\|_H$.

For M a closed hyperbolic 3-manifold, Bergeron, Şegün and Venkatesh established the following result.

Theorem 1.3 ([2, Proposition 4.2]). *For any closed orientable hyperbolic 3-manifold M_0 there are constants C_1, C_2 depending only on M_0 such that for any finite cover M of M_0 and all $\beta \in H^1(M, \mathbb{R})$ with Poincaré dual $\beta^* \in H_2(M, \mathbb{R})$, we have*

$$\frac{C_1}{\mathrm{vol}(M)} \|\beta^*\|_1 \le \|\beta\|_H \le C_2 \|\beta^*\|_1.$$

In the same setting, Brock and Dunfield significantly tightened the relationship between these two norms.

Theorem 1.4 ([5, Theorem 1.2]). *For all closed orientable hyperbolic 3-manifolds M and all $\beta \in H^1(M, \mathbb{R})$ with Poincaré dual $\beta^* \in H_2(M, \mathbb{R})$, we have*

$$\frac{\pi}{2\sqrt{\mathrm{vol}(M)}} \|\beta^*\|_1 \le \|\beta\|_H \le \frac{5\pi}{\sqrt{\mathrm{Inj}(M)}} \|\beta^*\|_1.$$

Remark 1. The above theorems were originally stated in terms of the Thurston norm $\|\beta\|_{Th}$, which by a result of Gabai [18, Corollary 6.18] is just half the Gromov norm $\|\beta^*\|_1$.

In the case of Theorem 1.4, the same authors showed through examples that their bounds are optimal up to a multiplicative constant on the lower side and optimal up to a sublinear function of $\mathrm{Inj}(M)$ on the upper side. In particular, they provide an example of a sequence of hyperbolic 3-manifolds M_i with uniformly bounded volumes and $\mathrm{Inj}(M_i) \to 0$ and $\beta_i \in H^1(M_i, \mathbb{R})$ with

$$\frac{\|\beta_i\|_H}{\|\beta_i^*\|_1} \ge C \sqrt{-\log \mathrm{Inj}(M_i)}$$

for some constant $C > 0$.

The main purpose of our paper is to extend these results to all dimensions and to well-known classes of nonpositively curved manifolds. In fact, the lower bound does not even require nonpositive curvature, and it is a consequence of the duality principle and Gromov's comass inequality.

Theorem 1.5. *Let M be a closed oriented manifold of dimension n normalized so that* $\mathrm{Ric} \geq -(n-1)$. *If $\beta \in H^p(M, \mathbb{R})$ is any cohomology class, then the Gromov norm of its Poincaré dual has an upper bound*

$$\|\beta^*\|_1 \leq (n-p)!(n-1)^{n-p}\sqrt{\mathrm{vol}(M)}\,\|\beta\|_H.$$

Remark 2. Distinct from the Poincaré dual, we may define a *pairing dual* $\widehat{\beta} \in H_p(M, \mathbb{R})$ of a class $\beta \in H^p(M, \mathbb{R})$ to be any element such that $(\beta, \widehat{\beta}) = 1$ and

$$\|\widehat{\beta}\|_1 = \inf\{\|\gamma\|_1 : \gamma \in H_p(M, \mathbb{R}) \text{ with } (\beta, \gamma) = 1\}.$$

Similarly for $\alpha \in H_p(M, \mathbb{R})$, a pairing dual $\widehat{\alpha}$ is defined by $(\widehat{\alpha}, \alpha) = 1$ and

$$\|\widehat{\alpha}\|_\infty = \inf\{\|\beta\|_\infty : \beta \in H^p(M, \mathbb{R}) \text{ with } (\beta, \alpha) = 1\}.$$

By the Hahn–Banach theorem and properties of the norms (see [1, Proposition F.2.2]), pairing duals exist (though not necessarily unique). Moreover,

$$\|\beta\|_\infty \cdot \|\widehat{\beta}\|_1 = 1 = \|\widehat{\alpha}\|_\infty \cdot \|\alpha\|_1$$

for all $\alpha \in H_p(M, \mathbb{R})$ and $\beta \in H^p(M, \mathbb{R})$. This duality principle is essential to the proof of the above theorem; see Section 4.

On the other hand, having an upper bound like Theorem 1.4 will provide positivity of the Gromov norm on corresponding non-trivial classes. In nonpositively curved manifolds, such results are only known in certain degrees and in certain cases such as the negatively curved manifolds, higher rank locally symmetric ones, and rank one manifolds with $\mathrm{Ric}_k < 0$. We show in all these cases:

Theorem 1.6. *Let $n \geq 3$. There is a constant $C(n) > 0$, with the following property. Let M be a closed oriented nonpositively curved manifold of dimension n, and for $0 \leq p \leq n-2$, let $\beta \in H^p(M, \mathbb{R})$ be any cohomology class. Denote by $\beta^* \in H_{n-p}(M, \mathbb{R})$ the Poincaré dual of β.*

(a) *If M has maximum sectional curvature $-a^2$ with $a > 0$, then*

$$\|\beta\|_H \leq C(n)b_p^{(n-1)/2}a^{p-n}\max\left\{1, \frac{1}{(a\,\mathrm{Inj}(M))^{1/2}}\right\}\|\beta^*\|_1,$$

where $b_p > 0$ is given by

$$-b_p^2 = \begin{cases} (n-1)^{-1}(\text{lower bound of Ricci curvature}) & p \leq 1, \\ \text{lower bound of the curvature operator} & p \geq 2. \end{cases}$$

(b) *If M is locally symmetric of real rank $r \geq 2$ normalized to have sectional curvatures $-1 \leq K \leq 0$, whose universal cover has no direct factor of $\mathbb{R}$, $\mathbb{H}^2$, $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, $\mathrm{Sp}(2, \mathbb{R})/\mathrm{U}(2)$, $G_2^2/\mathrm{SO}(4)$, or $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$, and $p \leq n - 2 - \mathrm{srk}(\tilde{M})$, then*

$$\|\beta\|_H \leq C(n) \max\left\{1, \frac{1}{(\mathrm{Inj}(M))^{r/2}}\right\} \|\beta^*\|_1.$$

(c) *If M is geometric rank one satisfying $\mathrm{Ric}_{k+1} < 0$ for some $k \leq \lfloor \frac{n}{4} \rfloor$, then there exists a constant $C(\tilde{M})$ that depends only on $\tilde{M}$ such that when $p \leq n - 4k$,*

$$\|\beta\|_H \leq C(\tilde{M}) \max\left\{1, \frac{1}{\mathrm{Inj}(M)^{k/2}}\right\} \|\beta^*\|_1.$$

In the statement of the above theorem, the definition of the *splitting rank*, srk, and k-*Ricci curvature*, Ric_k, are as follows.

Definition 1.7. Let X be any symmetric space without compact factors, the splitting rank of X is

$$\mathrm{srk}(X) = \max\{\dim(Y \times \mathbb{R}) \mid \text{where } Y \times \mathbb{R} \subset X \text{ is totally geodesic}\}.$$

This number is explicitly computed in [31, Table 1] for all irreducible symmetric spaces. For example, if $X = \mathrm{SL}(m, \mathbb{R})/\mathrm{SO}(m)$, then $\mathrm{srk}(X) = \dim(X) - \mathrm{rank}(X)$, and if $X = \mathrm{SL}(m, \mathbb{C})/\mathrm{SO}(m)$, then $\mathrm{srk}(X) = \dim(X) - 2\,\mathrm{rank}(X)$.

Definition 1.8. For $u, v \in T_x M$, the k-Ricci tensor is defined to be

$$\mathrm{Ric}_k(u, v) = \sup_{\substack{V \subset T_x M \\ \dim V = k}} \mathrm{Tr}\, R(u, \cdot, v, \cdot)|_V,$$

where $R(u, \cdot, v, \cdot)|_V$ is the restriction of the curvature tensor to $V \times V$, thus the trace is with respect to any orthonormal basis of V. We set $\mathrm{Ric}_k = \sup_{v \in T^1 M} \mathrm{Ric}_k(v, v)$.

Remark 3. The range of p in Theorem 1.6 (a) is sharp as the Gromov norm of any 1-class is zero. It is almost sharp in case (b) due to a recent construction of the second author [32], which shows that there is a non-trivial homology class with zero Gromov norm at degree $\mathrm{srk}(\tilde{M})$.

We also note that for a Riemannian manifold with sectional curvatures between $-b^2 \leq K \leq 0$, then by Proposition 3.8 of [4], we have $b_p \leq (3 + \frac{2n}{3})b$. So up to constants depending only on dimension, we may replace b_p by b everywhere.

In the case when the injectivity radius is large we obtain a better bound, in most cases exponentially decaying in the injectivity radius, but for a restricted range of p.

Theorem 1.9. *Let $n \geq 3$. There is a constant $C(n) > 0$, with the following property. Let M be a closed oriented nonpositively curved manifold of dimension n, and for $0 \leq p \leq n - 2$, let $\beta \in H^p(M, \mathbb{R})$ be any cohomology class. Denote by $\beta^* \in H_{n-p}(M, \mathbb{R})$ the Poincaré dual of β.*

(a) *If M has sectional curvatures in $[-b^2, -1]$ and $p < (n-1)/2b$, and $\mathrm{Inj}(M) > 1 + \ln(2)/((n-1) - 2pb)$, then*

$$\|\beta\|_H \leq \frac{C(n)\sqrt{b}}{\sqrt{n-1-2pb}} e^{-((n-1)/2 - pb)\,\mathrm{Inj}(M)} \|\beta^*\|_1.$$

Moreover, if $b \neq 1$ and p is a non-negative integer such that $p = (n-1)/2b$ and $\mathrm{Inj}(M) > 2$, then

$$\|\beta\|_H \leq C(n)(\mathrm{Inj}(M))^{-1/2}\|\beta^*\|_1.$$

(b) *If M is locally symmetric of real rank $r \geq 2$ whose universal cover has no direct factor of $\mathbb{R}$, $\mathbb{H}^2$, $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, $\mathrm{Sp}(2,\mathbb{R})/\mathrm{U}(2)$, $G_2^2/\mathrm{SO}(4)$, or $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$, normalized to have $-1 \leq K \leq 0$ with Ricci curvature bound $\mathrm{Ric} \leq -\delta^2 g$ and $p < \delta/2$, then*

$$\|\beta\|_H \leq \frac{C(n)}{\sqrt{\delta - 2p}} e^{-(\delta/2 - p)\,\mathrm{Inj}(M)}\|\beta^*\|_1.$$

(c) *If M is geometric rank one satisfying $-b^2 \leq K$, $\mathrm{Ric} < -\delta^2 b^2$, $\mathrm{Ric}_{k+1} < 0$ for some $k \leq \lfloor \frac{n}{4} \rfloor$, $0 < p \leq \min\{n - 4k, \delta/2\}$ and $\mathrm{Inj}(M) > 1 + p\log(2)/(\delta - 2p)$, then there exists a constant $C(\tilde{M})$ that depends only on $\tilde{M}$ such that*

$$\|\beta\|_H \leq C(\tilde{M}) e^{-(\delta/2 - p)b\,\mathrm{Inj}(M)}\|\beta^*\|_1.$$

Remarks 4. In the case when $n = 3$, $p = 1$ and M is hyperbolic, part (a) of Theorems 1.6 and 1.9 recover the estimate in Theorem 1.4 up to uniform multiplicative constants. However, their numerical constants are sharper on both sides.

The irreducible symmetric spaces arising in part (b) of Theorem 1.9 are Einstein with explicit Ricci constant $-\delta^2 = -\operatorname{tr}(\mathrm{ad}_v)^2$ for any self-adjoint v in the Lie-algebra $\mathfrak{g}$ with unit norm, with respect to the curvature normalized metric; see e.g. [17, Section 2.14] for details.

In the same setting, as explained in [5, Section 5], an equivalent form of Theorem 2 of [24] states that for any closed oriented irreducible 3-manifold M and any $\beta \in H^1(M; \mathbb{R})$,

$$2\pi\|\beta^*\|_1 = \inf_g \|s(g)\|_{H_g}\|\beta\|_{H_g},$$

where the inf is over all Riemannian metrics g on M with scalar curvature $s(g)$ and harmonic norm $\|\cdot\|_{H_g}$. Theorems 1.5, 1.6 and 1.9 thus provide bounds of how close to and how far from the above infimum general negatively curved metrics on M can become.

In [21], Gromov showed that if Ric $\geq -(n-1)$ then for all $\alpha \in H_p(M, \mathbb{R})$ and $\beta \in H^p(M, \mathbb{R})$, we have

$$\|\alpha\|_1 \leq p!(n-1)^p \operatorname{mass}(\alpha), \quad \operatorname{comass} \beta \leq p!(n-1)^p \|\beta\|_\infty,$$

and if the sectional curvatures of M satisfy $K \leq -a^2$ with $a > 0$, then

$$\operatorname{mass}(\alpha) \leq \frac{\pi a^{-p}}{(p-1)!} \|\alpha\|_1, \quad \|\beta\|_\infty \leq \frac{\pi a^{-p}}{(p-1)!} \operatorname{comass}(\beta).$$

It follows immediately from the above remarks that we obtain equivalences for the other norms in this context. Specifically, we may replace $\|\beta^*\|_1$ by the quantities $\|\widehat{\beta^*}\|_\infty^{-1}$, $\operatorname{comass}(\widehat{\beta^*})^{-1}$ or $\operatorname{mass}(\beta^*)$ in Theorems 1.5, 1.6 and 1.9. However, the constants change by a multiplicative constant in n, b_p and a; see the definition of comass in Section 2.

When passing to a finite cover, neither volume nor injectivity radius of a manifold decreases. Thus our theorems immediately generalize the result of Bergeron, Şegün and Venkatesh.

Corollary 1.10. *For any closed oriented manifold M_0 of dimension $n \geq 3$ and integer p satisfying the hypotheses of Theorems 1.5 and 1.6, there exists constants C_1 and C_2 depending only on M_0 such that for any finite cover M of M_0 and all $\beta \in H^p(M, \mathbb{R})$ with Poincaré dual $\beta^* \in H_{n-p}(M, \mathbb{R})$, we have*

$$\frac{C_1}{\sqrt{\operatorname{vol}(M)}} \|\beta^*\|_1 \leq \|\beta\|_H \leq C_2 \|\beta^*\|_1.$$

We structure our paper as follows. In Section 2, we establish an upper bound on the comass of the harmonic form by its harmonic norm in terms of the injectivity radius. This uses the Margulis lemma together with a Moser type inequality for a certain elliptic equation. In Section 3, we use the straightening method to relate the harmonic norm and the comass with the Gromov norm of the Poincaré dual. As a result, we prove our Theorems 1.6 and 1.9. The latter theorem also employs the more refined elliptic estimates from [15]. In Section 4, we use the duality principle to prove Theorem 1.5. And lastly in Section 5, we give an alternative approach to Section 2, relating the comass and the harmonic norm with other geometric quantities such as the Sobolev constants, isoperimetric type constants and others.

Dictionary of notations. Throughout this paper, all manifolds are oriented. We denote by $\Delta = \delta d + d\delta$ the Hodge Laplacian. We present here a short summary of our notations. Apart from the self-evident ones, and those already defined, these definitions will mostly be explained in Section 2 and 5.

$\|\cdot\|_H$	harmonic norm on $H^p(M, \mathbb{R})$
$\|\cdot\|_\infty$	ℓ^∞-norm on $H^p(M, \mathbb{R})$
$\|\cdot\|_1$	Gromov norm on $H_p(M, \mathbb{R})$
comass	supremum of $|\cdot|_\infty$
$|\cdot|_2$	pointwise ℓ^2-norm
$|\cdot|_\infty$	pointwise ℓ^∞-norm
β^*	Poincaré dual of β
α, γ	homology classes
β, φ	cohomology classes
η, ϕ	differential forms
ω	an harmonic form
C_0	L^1-Sobolev constant
C_1	isoperimetric type constant
λ_1	smallest positive eigenvalue for Laplacian
h	Cheeger constant
C_s	L^2-Sobolev constant
d_M	diameter of M
Inj	injectivity radius
Ric_k	k-Ricci
b_p, K_p	curvature operator (lower) bounds
a, b	sectional curvature bounds
srk	splitting rank

Table 1. Dictionary of notations.

2. Comparing harmonic norm with comass

2.1. Pointwise ℓ^2-norm and ℓ^∞-norm. For manifolds, the singular cohomology is isomorphic to the de Rham cohomology. Denote $\Omega^p(M)$ the space of differential p-forms on M. For any point $x \in M$, choose linearly independent local sections

$\{e_1, \ldots, e_n\}$ of TM over a neighborhood of x that form an orthonormal basis of $T_x M$, and let $\{e_1^*, \ldots, e_n^*\}$ be the canonical dual sections on T^*M over the same neighborhood. For any $\phi \in \Omega^p(M)$, ϕ can be locally expressed as

$$\phi = \sum_{i_1 < i_2 < \cdots < i_p} a_{i_1 i_2 \ldots i_p} \, e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots \wedge e_{i_p}^*.$$

We define the *pointwise ℓ^2-norm* at x by

$$|\phi|_2(x) = \left(\sum_{i_1 < i_2 < \cdots < i_p} a_{i_1 i_2 \ldots i_p}^2(x) \right)^{1/2}.$$

Note that the definition is independent of the choice of local sections, provided they are orthonormal at x, and it is easy to check that

$$|\phi|_2^2(x) = \frac{\phi \wedge *\phi}{d \, \mathrm{vol}}(x).$$

Thus, if ω is a harmonic form representing α, then

$$\|\alpha\|_H = \left(\int_M |\omega|_2^2 \, d \, \mathrm{vol} \right)^{1/2} = L^2\text{-norm of } |\omega|_2.$$

Similarly, we define the *pointwise ℓ^∞-norm*

$$|\phi|_\infty(x) = \sup\{\phi_x(e_1, \ldots, e_p) : \{e_1, \ldots, e_p\} \text{ is an orthonormal frame of } T_x M\},$$

and for $\phi \in \Omega^p(M)$, we define

$$\mathrm{comass}\, \phi = \sup_{x \in M} \big(|\phi|_\infty(x) \big).$$

For a class $\alpha \in H^p(M, \mathbb{R})$, define

$$\mathrm{comass}(\alpha) = \inf\{\mathrm{comass}(\phi) : \phi \in \Omega^p(M), \ [\phi] = \alpha\}.$$

It is not difficult to see that the pointwise ℓ^2, ℓ^∞-norm are equivalent up to a constant.

Lemma 2.1. *If $\phi \in \Omega^p(M)$ is a differential p-form on an n-dimensional closed Riemannian manifold M, then*

$$|\phi|_\infty(x) \leq |\phi|_2(x) \leq \binom{n}{p}^{1/2} |\phi|_\infty(x).$$

Thus,

$$\mathrm{comass}(\phi) \leq \sup |\phi|_2 \leq \binom{n}{p}^{1/2} \mathrm{comass}(\phi).$$

Proof. Suppose at x, ϕ evaluates to its supremum on the orthonormal p-frame $\{e_1, \ldots, e_p\}$, we extend the set to an orthonormal n-frame $\{e_1, \ldots, e_p, \ldots, e_n\}$, and write ϕ (at the point x) as

$$\phi_x = \sum_{i_1 < i_2 < \cdots < i_p} a_{i_1 i_2 \ldots i_p} \, e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots \wedge e_{i_p}^*.$$

By assumption, $a_{12\ldots p} = |\phi|_\infty(x)$ and $|a_{i_1 i_2 \ldots i_p}| \leq |\phi|_\infty(x)$ for all $i_1 < i_2 < \cdots < i_p$, so we obtain

$$|\phi|_\infty(x) = a_{12\ldots p} \leq \left(\sum_{i_1 < i_2 < \cdots < i_p} a_{i_1 i_2 \ldots i_p}^2 \right)^{1/2}$$

$$= |\phi|_2(x) \leq \binom{n}{p}^{1/2} |\phi|_\infty(x). \qquad \square$$

2.2. Relating harmonic norm with comass. Given a closed Riemannian manifold M, we define K_p for any integer $p \geq 0$ to be

$$K_p = \begin{cases} 0 & p = 0, \\ (n-1)^{-1}(\text{lower bound of Ricci curvature}) & p = 1, \\ \text{lower bound of the curvature operator} & p \geq 2, \end{cases} \qquad (1)$$

where the curvature operator is viewed as the linear extension of the curvature tensor to all of $\Lambda^2(T_x M)$ as opposed to the subset $\cup_{v,w \in T_x M} v \wedge w$. We recall from Theorem 1.6 the definition of b_p. In particular, if M is nonpositively curved, then $-b_p^2 \leq K_p \leq 0$.

We will use the following lemma.

Lemma 2.2 (cf. [27, Lemma 8]). *Let M be a closed Riemannian manifold of dimension n, and ω be a harmonic p-form. If the function f is the square pointwise ℓ^2-norm $|\omega|_2^2(x)$, then*

$$\Delta f \leq \lambda f,$$

where $\lambda = -2p(n-p)K_p \geq 0$.

Proof. By Bochner's formula,

$$\frac{1}{2}\Delta\big(|\omega|_2^2\big) = \langle \Delta\omega, \omega \rangle - |\nabla\omega|^2 - F(\omega), \qquad (2)$$

where $\Delta = d\delta + \delta d$ is the Hodge Laplacian and $F(\omega) \geq p(n-p)K_p|\omega|_2^2$ by [19, p. 264]. Since ω is harmonic, we obtain from equation (2) that

$$\Delta f \leq 2\big(-|\nabla\omega|^2 - p(n-p)K_p f\big) \leq \lambda f, \qquad (3)$$

where $\lambda = -2p(n-p)K_p$. $\qquad \square$

Remark 5. Note that the pointwise ℓ^2-norm $|\omega|_2(x)$ is *not* necessarily differentiable at points where it is zero, but its square is always a smooth function.

The Margulis lemma states that for any complete simply connected n-dimensional Riemannian manifold X, with sectional curvatures $-b^2 \leq K_X \leq 0$, there are constants $N \in \mathbb{N}$ and $\varepsilon > 0$, depending only on b and n, such that for any discrete group of isometries $\Gamma < \text{Isom}(X)$, and any $x \in X$, the group Γ_ε generated by

$$\{\gamma \in \Gamma : d(x, \gamma x) < \varepsilon\}$$

has a finite index nilpotent subgroup of index at most N. The largest value of ε for a given b and n is called the *Margulis constant* $\mu = \mu(b, n)$. Moreover, if $M = X/\Gamma$ is a compact manifold then by [20], Γ_μ has a normal crystallographic subgroup Σ_μ of index at most N which preserves and acts cocompactly on a totally geodesic flat submanifold $A \cong \mathbb{R}^k \subset X$. If X is negatively curved then Γ_μ is infinite cyclic and A is a geodesic.

Define the *thin/thick decomposition* of $M = M_{\text{thin}} \cup M_{\text{thick}}$ to be that given by the disjoint subsets,

$$M_{\text{thin}} = \{x \in M : \text{Inj}(x) < \mu\} \quad \text{and} \quad M_{\text{thick}} = \{x \in M : \text{Inj}(x) \geq \mu\}.$$

If M is compact and negatively curved then M_{thin} is diffeomorphic to a finite disjoint union of copies of the product of an $(n-1)$-disk with a closed geodesic corresponding to A (possibly together with some twisted product components if M is non-orientable) and $N(n, b) = 1$ for $b > 0$. Even for non-flat closed nonpositively curved M, we may have $M_{\text{thin}} = M$, as is the case for some graph manifolds.

The following lemma indicates a useful scale to apply to subsequent results.

Lemma 2.3. *Suppose X is an n-dimensional Hadamard manifold with $n > 1$ and Margulis constant μ and $\Gamma < \text{Isom}(X)$ is a cocompact discrete group of isometries. Set $M = X/\Gamma$ and let $\varepsilon = \min\{\text{Inj}(M), \mu/2\}$, and $B = B(x, \mu/2) \subset X$ be a metric ball of radius $\mu/2$ centered at some point $x \in X$. If*

$$D = \{y \in X : d(x, y) \leq d(x, \gamma y) \text{ for all } \gamma \in \Gamma\}$$

is the Dirichlet fundamental domain of Γ, centered at x, then in each case we have

(a) *If X is negatively curved then $\#\{\gamma \in \Gamma : \gamma D \cap B \neq \emptyset\} \leq \mu/\varepsilon$;*

(b) *If X has $\text{Ric}_k < 0$, then $\#\{\gamma \in \Gamma : \gamma D \cap B \neq \emptyset\} \leq C(\mu/\varepsilon)^{k-1}$;*

(c) *If X is a rank $k > 1$ symmetric space then $\#\{\gamma \in \Gamma : \gamma D \cap B \neq \emptyset\} \leq C(\mu/\varepsilon)^k$;*

(d) *If X is any Hadamard space, then $\#\{\gamma \in \Gamma : \gamma D \cap B \neq \emptyset\} \leq C(\mu/\varepsilon)^n$,*

where in each case C is a constant only depending on n.

Proof. We may assume $\varepsilon = \text{Inj}(M) < \mu/2$, otherwise $B \subset D$ and the left-hand side in each case is 1 while the right-hand side in each case is at least 2. Similarly,

we may assume $\text{Inj}(x) \leq \mu/2$, otherwise again $B \subset D$. Hence, $B \subset \widetilde{M}_{\text{thin}}$, where $\widetilde{M}_{\text{thin}} \subset X$ is preimage of M_{thin} under the covering map.

The elements $\gamma \in \Gamma$ which translate $x \in B$ a distance less than the diameter μ of B lie in Γ_μ. Hence, we can estimate the number of translates of D that intersect B as follows:

$$\#\{\gamma \in \Gamma \mid \gamma D \cap B \neq \emptyset\} = \#\{\gamma \in \Gamma_\mu \mid \gamma D_{\Gamma_\mu} \cap B \neq \emptyset\}$$
$$\leq N \times \#\{\sigma \in \Sigma_\mu \mid \sigma D_{\Sigma_\mu} \cap B \neq \emptyset\},$$

where D_{Γ_μ} and D_{Σ_μ} denote the Dirichlet fundamental domains centered at x associated with the groups Γ_μ and Σ_μ, respectively. (Recall $\Sigma_\mu < \Gamma_\mu$ of index at most N.) In particular, it can at most be N times the number of translates of a Σ_μ-fundamental domain that intersect B.

Moreover, recall that for any flat $A \subset X$ on which Σ_μ acts cocompactly and for any $y \in A$ and $x \in X$, $d(y, \gamma y) \leq d(x, \gamma x)$ holds for all $\gamma \in \Sigma_\mu$ by nonpositive curvature and since Σ_μ stabilizes the flat A. Hence, the number of points in the orbit of $x \in B$ lying in B is no more than the number of points in the orbit of $y \in A$ in $B(y, \mu/2) \cap A$. Since the translation length of each $\gamma \in \Gamma$ is at least 2ε, there can be at most $C(n)(\mu/\varepsilon)^k$ Σ_μ-orbit points in B, where k is the rank of Σ_μ which is the dimension of A. Here $C(n)$ a priori depends on the crystallographic group Σ_μ, but this lattice density constant (for any shortest translation length ε) is universally bounded from above for each n.

In each of the cases of the lemma we observe that the dimension of A is at most 1, $k - 1$, k, and n, respectively. The constant C in the lemma is $C(n)N$, where N is the index of Σ_μ. Moreover, when $K_X < 0$, both $N = 1$ and $C(n) = 1$ since $\Gamma_\mu \cong \mathbb{Z}$. $\qquad\square$

Lastly, we will need the following Moser type inequality which represents one of the two directions needed for the Harnack inequality for solutions to general parabolic operators.

Theorem 2.4 ([29, Theorem 5.1]). *Let M be a complete Riemannian manifold of dimension n. If for any $x \in M$ and $r > 0$, the Ricci curvatures restricted to the ball $B(x, 2r)$ satisfy $\text{Ric} \geq -(n-1)K$, then for any $u \in C^\infty(B(x, r))$ satisfying $\Delta u \leq \lambda u$ for some $\lambda \geq 0$, we have*

$$\sup_{B(x,r/2)} u \leq C_1 (1 + \lambda r^2)^{1+n/2} \frac{e^{C_2 \sqrt{(n-1)K} r}}{\text{vol}(B(x,r))} \|u|_{B(x,r)}\|_1,$$

where C_1 and C_2 are constants only depending on the dimension n.

Proof. This is a special case of [29, Theorem 5.1] of Saloff-Coste applied to the restricted setting of the elliptic operator $\mathcal{L} = \Delta - \lambda \, \mathrm{I}$ for $\lambda \geq 0$, instead of the full parabolic setting. In the notation of [29], we are specifically setting $m = 1$, $\mathcal{A} = \mathrm{I}$, $\mathcal{X} = 0 = \mathcal{Y}$, $b = -\lambda$, $\beta = \lambda$, $Q = (0, r^2) \times B(x, r)$, $p = 1$, and $\delta = 1/2$.

It follows that $\alpha = \mu = 1$. Finally, we have integrated out the time factor $(0, r^2)$ in $\|f\|_Q\|_2$. $\qquad\square$

Now we are ready to relate the comass of a harmonic form with its harmonic norm. (Recall the definition of b_p from Theorem 1.6 (a), and note Remark 3.)

Theorem 2.5. *Let $n \geq 2$. There exists a constant $C(n)$ such that if M is any n-dimensional closed nonpositively curved Riemannian manifold and ω is any harmonic p-form, then*

$$\text{comass}(\omega) \leq C(n) \max\left\{ b_p^{n/2}, \frac{b_p^{(n-\ell)/2}}{\text{Inj}(M)^{\ell/2}} \right\} \|\omega\|_2,$$

where

(a) *if M is negatively curved, then $\ell = 1$,*

(b) *if M has $\text{Ric}_k < 0$, then $\ell = k - 1$,*

(c) *if M is locally a rank $k > 1$ symmetric space, then $\ell = k$, and*

(d) *in all other cases $\ell = n$.*

Proof. Set $\bar{f}(x) = |\omega|_2(x)$ and let $f : \widetilde{M} \to \mathbb{R}$ be the lift of $\bar{f}$ to $\widetilde{M}$. Choose $x_0 \in \widetilde{M}$ to be a point where

$$f(x_0) = \sup_{x \in \widetilde{M}} f(x) = \text{comass}(\omega).$$

We set $u = f^2$, a smooth function on $\widetilde{M}$.

By Lemma 2.2, we have that u is a subsolution of the operator $\Delta - \lambda I$, where Δ is the Laplacian of $\widetilde{M}$ and $\lambda = -2p(n-p)K_p \geq 0$. Since $\text{Ric}_M \geq -(n-1)b_p^2$, by Theorem 2.4 applied to $u = f^2$, and taking the square root on both sides, for any $r > 0$, we have

$$\sup_{B(x_0, r/2)} f \leq C_1 (1 + \lambda r^2)^{\frac{1}{2} + \frac{n}{4}} \frac{e^{C_2 \sqrt{(n-1)} b_p r}}{\text{vol}(B(x_0, r))^{1/2}} \|f\|_{B(x_0,r)}\|_2,$$

where C_1 and C_2 only depend on n.

We note that the ratio of the right-hand side with $\|\omega\|_2$ grows exponentially in r for large r, since in that case

$$\frac{\|f\|_{B(x_0,r)}\|_2}{\text{vol}(B(x_0,r))^{1/2}} \approx \|\omega\|_2.$$

On the other hand, at least when $\sup f \gg \|\bar{f}\|_2$, the estimate is likewise poor for small r since then

$$\frac{\|f\|_{B(x_0,r)}\|_2}{\text{vol}(B(x_0,r))^{1/2}} \approx f(x_0).$$

However, for $r = \mu/2$, where μ is the Margulis constant, the term

$$C(r, b_p, n) = C_1(1 + \lambda r^2)^{\frac{1}{2} + \frac{n}{4}} \frac{e^{C_2\sqrt{(n-1)}b_p r}}{\text{vol}(B(x_0, r))^{1/2}}$$

depends only on b_p and n since μ does. Moreover, varying b_p by scaling the metric implies that $2r = \mu = C_3/b_p$, where C_3 depends only on n and p. Recalling that

$$\lambda r^2 = -2p(n - p)K_p \frac{C_3}{4b_p^2},$$

we obtain $C(r, b_p, n) = C_4 b_p^{n/2}$ for some C_4 depending only on n and p.

Lemma 2.3 states that $B(x, \mu/2)$ is contained in a union of at most $C_5(\mu/\varepsilon)^\ell$, for each case of ℓ, fundamental domains $D \subset \tilde{M}$ of M, where $\varepsilon = \min\{\text{Inj}(M), \mu/2\}$ and C_5 depends only on n. Hence,

$$\text{comass}(\omega) \leq C_4 b_p^{n/2}\|f|_{B(x,\mu/2)}\|_2 \leq C_4 b_p^{n/2}\sqrt{C_5(\mu/\varepsilon)^\ell}\|f|_D\|_2$$
$$= C_4 b_p^{n/2}\sqrt{C_5}(\mu/\varepsilon)^{\ell/2}\|\omega\|_2.$$

The statement of the theorem follows from

$$\left(\frac{\mu}{\varepsilon}\right)^{\ell/2} \leq \max\left\{2^{\ell/2}, \left(\frac{C_3}{b_p\,\text{Inj}(M)}\right)^{\ell/2}\right\}$$

and that $p \leq n$. $\qquad\qquad\square$

3. Producing lower bounds on the Gromov norm by simplex straightening

The idea of simplex straightening was first developed by Gromov [21] and Thurston [30] in negative curvatures as a tool for obtaining lower bounds on the Gromov norm (also see [22]). This method has been extended by many authors in different contexts including higher rank symmetric spaces [23, 25, 26, 31], certain nonpositively curved manifolds [12, 13] and others [28]. Below we give a brief overview on some of these results.

We begin with the definition of a straightening.

Definition 3.1 (cf. [25]). Let $\tilde{M}$ be the universal cover of a connected n-dimensional manifold M. We denote by Γ the fundamental group of M, and by $C_*(\tilde{M})$ the real singular chain complex of $\tilde{M}$. Equivalently, $C_k(\tilde{M})$ is the free $\mathbb{R}$-module generated by $C^0(\Delta^k, \tilde{M})$, the set of singular k-simplices in $\tilde{M}$, where Δ^k is equipped with some fixed Riemannian metric. We say a collection of maps

$$st_k : C^0(\Delta^k, \tilde{M}) \to C^0(\Delta^k, \tilde{M})$$

is a straightening if it satisfies the following conditions:

(a) the maps st_k are Γ-equivariant,

(b) the maps st_* induce a chain map $st_*\colon C_*(\widetilde{M}, \mathbb{R}) \to C_*(\widetilde{M}, \mathbb{R})$ that is Γ-equivariantly chain homotopic to the identity,

(c) the image of st_n lies in $C^1(\Delta^n, \widetilde{M})$, that is, the top dimensional straightened simplices are C^1. (Hence, all the straightened k-simplices are C^1.)

In addition, if all the straightened k-simplices have uniformly bounded volume, we say the straightening is *k-bounded*.

When M is negatively curved, Gromov and Thurston used the "geodesic straightening" and showed it is k-bounded for $k \geq 2$. More specifically, given any k-simplex with ordered vertices $\{v_0, v_1, \ldots, v_k\}$, we connect v_k with $st_{k-1}(\{v_0, v_1, \ldots, v_{k-1}\})$ by geodesics with the obvious parametrization, where $st_{k-1}(\{v_0, v_1, \ldots, v_{k-1}\})$ is the inductively defined straightened $(k-1)$-simplex. This geodesic coning procedure defines a straightening that has explicit volume control on each straightened simplex.

When M is higher rank locally symmetric, Lafont and Schmidt [25] defined the "barycentric straightening." Using this procedure and a previous estimate of Connell and Farb [10, 11], they showed that apart from a few exceptional cases, all top dimensional straightened simplices have uniformly bounded volume. This has been extended to all k-simplices when $k \geq \mathrm{srk}(X) + 2$ in [26, 31], where *the splitting rank*, denoted by $\mathrm{srk}(X)$, is defined to be the maximal dimension among all totally geodesic submanifolds in X that split off a direct $\mathbb{R}$-factor; see explicitly [31, Table 1]. The idea of barycentric straightening is based on the barycenter method originally developed by Besson, Courtois, and Gallot [3].

When M is a nonpositively curved rank one manifold, the barycentric straightening is also well defined. If the manifold satisfies additional curvature conditions, then the straightening is also k-bounded when k is large enough. We summarize the above discussion into the following proposition.

Theorem 3.2. *Let M be a closed nonpositively curved manifold of dimension n.*

(a) ([21]) *If M has sectional curvatures $K \leq -a^2$ with $a > 0$, then the geodesic straightening is k-bounded for $k \geq 2$. Moreover, the volume of the straightened simplices satisfy*

$$\mathrm{vol}(st_k(f)) \leq \frac{\pi a^{-k}}{(k-1)!} \quad \forall f \in C^0(\Delta^k, \widetilde{M}), \ k \geq 2.$$

(b) ([26, 31]) *If M is higher rank locally symmetric whose universal cover $\widetilde{M}$ has no direct factor of $\mathbb{R}$, $\mathbb{H}^2$, $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, $\mathrm{Sp}(2, \mathbb{R})/\mathrm{U}(2)$, $G_2^2/\mathrm{SO}(4)$, and $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$, then the barycentric straightening is k-bounded for $k \geq \mathrm{srk}(\widetilde{M}) + 2$.*

(c) ([12]) *If M is geometric rank one with $\mathrm{Ric}_{l+1} < 0$ for some $l \leq \lfloor \frac{n}{4} \rfloor$, then the barycentric straightening is k-bounded for $k \geq 4l$.*

Now we illustrate how the straightening gives a lower bound on the Gromov norm.

Theorem 3.3. *Let M be a closed nonpositively curved manifold of dimension n, and ω be the harmonic representative of $\beta \in H^p(M, \mathbb{R})$, where $p \leq n - 2$. Denote $\beta^* \in H_{n-p}(M, \mathbb{R})$ the Poincaré dual of β.*

(a) *If M is negatively curved and has sectional curvatures $K \leq -a^2$ with $a > 0$, then*

$$\|\omega\|_2^2 \leq \frac{\pi a^{p-n}}{(n - p - 1)!} \operatorname{comass}(\omega) \|\beta^*\|_1.$$

(b) *If M is higher rank locally symmetric whose universal cover has no direct factor of $\mathbb{R}$, $\mathbb{H}^2$, $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, $\mathrm{Sp}(2, \mathbb{R})/\mathrm{U}(2)$, $G_2^2/\mathrm{SO}(4)$, and $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$, and $p \leq n - 2 - \operatorname{srk}(\tilde{M})$, then there exists a constant $C(\tilde{M})$ that depends only on $\tilde{M}$ such that*

$$\|\omega\|_2^2 \leq C(\tilde{M}) \operatorname{comass}(\omega) \|\beta^*\|_1.$$

(c) *If M is geometric rank one satisfying $\operatorname{Ric}_{k+1} < 0$ for some $k \leq \lfloor \frac{n}{4} \rfloor$, and $p \leq n - 4k$, then there exists a constant $C(\tilde{M})$ that depends only on $\tilde{M}$ such that*

$$\|\omega\|_2^2 \leq C(\tilde{M}) \operatorname{comass}(\omega) \|\beta^*\|_1.$$

Proof. (a)   By definition, the square of the harmonic norm is given by

$$\|\omega\|_2^2 = \int_M \omega \wedge *\omega = \int_{\beta^*} *\omega,$$

where $\beta^* \in H_{n-p}(M, \mathbb{R})$ denotes the Poincaré dual of β. Suppose $\sum_i a_i \sigma_i$ is a chain that represents β^*, since the straightening is Γ-equivariant it descents to M, and by using (b) of Definition 3.1 we can replace each σ_i by geodesically straightened simplices $st_{n-p}(\sigma_i)$ so that

$$\int_{\beta^*} *\omega = \int_{[\sum_i a_i \sigma_i]} *\omega = \int_{[\sum_i a_i st(\sigma_i)]} *\omega.$$

By lifting to the universal cover $\tilde{M}$, we have

$$\int_{[\sum_i a_i st(\sigma_i)]} *\omega \leq \sum_i |a_i| \left| \int_{st(\sigma_i)} *\omega \right| \leq \sum_i |a_i| \operatorname{comass}(*\omega) \operatorname{vol}(st(\tilde{\sigma}_i)).$$

By (a) of Theorem 3.2 together with the fact $\operatorname{comass}(*\omega) = \operatorname{comass}(\omega)$, we obtain

$$\sum_i |a_i| \operatorname{comass}(*\omega) \operatorname{vol}(st(\tilde{\sigma}_i)) \leq \frac{\pi a^{p-n}}{(n - p - 1)!} \operatorname{comass}(\omega) \left(\sum_i |a_i| \right).$$

Finally, by taking the infimum among all chains representing β^*, and combining the above inequalities, we conclude

$$\|\omega\|_2^2 \leq \frac{\pi a^{p-n}}{(n - p - 1)!} \operatorname{comass}(\omega) \|\beta^*\|_1.$$

(b) and (c) follows similarly by simply replacing geodesic straightening with barycentric straightening, and (a) of Theorem 3.2 with (b) and (c). □

Combining Theorem 2.5 with Theorem 3.3 directly yields Theorem 1.6.

Next we consider the small injectivity radius case. For this, we employ the sharp estimates of Di Cerbo and Stern ([15]). We summarize their results into the form we will use below.

Theorem 3.4 ([15, Theorems 39, 87 and 96]). *Let $n \geq 3$. There is a constant $C(n) > 0$, with the following property. Let M be a closed oriented nonpositively curved manifold of dimension n, and for $0 \leq p \leq n - 2$, let ω be any harmonic differential p-form on M.*

(a) *If M has curvatures in $[-b^2, -1]$ with $b \geq 1$, $p < (n-1)/2b$, and $\mathrm{Inj}(M) > 1 + (\ln(2)/((n-1) - 2pb))$, then for any $x \in M$,*

$$\int_{B_1(x)} |\omega|_2^2 \, d \operatorname{vol} \leq \frac{C(n)b}{n - 1 - 2pb} e^{-((n-1) - 2pb) \operatorname{Inj}(M)} \|\omega\|_{L^2(M,g)}^2.$$

Moreover, if $b \neq 1$ and p is a non-negative integer such that $p = (n-1)/2b$ and $\mathrm{Inj}(M) > 2$, then

$$\int_{B_1(x)} |\omega|_2^2 \, d \operatorname{vol} \leq C(n) e^{2b} (\operatorname{Inj}(M))^{-1} \|\omega\|_{L^2(M,g)}^2.$$

(b) *If $n > 5$, M has curvatures $-1 \leq K \leq 0$, $-\operatorname{Ric} \geq \delta^2 g$, $p < \delta/2$ and $\mathrm{Inj}(M) > 1 + (p \log(2)/(\delta - 2p))$, then for any $x \in M$,*

$$\int_{B_1(x)} |\omega|_2^2 \, d \operatorname{vol} \leq \frac{C(n)}{\delta - 2p} e^{-(\delta - 2p) \operatorname{Inj}(M)} \|\omega\|_{L^2(M,g)}^2.$$

Proof. The two cases in part (a) correspond to Theorems 87 and 96 of [15], respectively, with the choices $k = p$, $\sigma = 1$, $\tau = \operatorname{Inj}(M)$, $\varepsilon = b - 1$. (Note that under the hypotheses $e^{-(\tau - \sigma)(n-1-2pb)} \leq 1/2$.) This is a similar statement to Theorem 23 of [16].

For part (b) we employ Theorem 39 of [15] with $\kappa = 0$, $\sigma = 1$, $\tau = \operatorname{Inj}(M)$, and see the last line of the proof of Theorem 39 regarding why we can choose their r_0 to be ∞ in the $\kappa = 0$ case. We also renormalize for arbitrary curvature bounds, simplify the expressions with dependent constants by adding in the appropriate multiple of the integral over $B_1(x)$ to both sides of the inequalities. □

Before we prove Theorem 1.9 we will need the following lemma.

Lemma 3.5. *For any of the symmetric spaces $\widetilde{M}$ in case* (b) *of Theorem 1.9, scaled to have sectional curvatures $-1 \leq K \leq 0$, if the Ricci curvature is bounded by $-\operatorname{Ric} \geq \delta^2 g$, then*

$$\left\lfloor \frac{\delta}{2} \right\rfloor \leq n - 2 - \operatorname{srk}(\widetilde{M}).$$

Proof. According to the definition of $\widetilde{M}$, there exists a unit tangent vector $v \in T_o^1 \widetilde{M}$ at some point $o \in \widetilde{M}$ and a totally geodesic submanifold $Y \times \mathbb{R} \subset \widetilde{M}$ where v is tangent to the $\mathbb{R}$-direction such that $\dim(Y \times \mathbb{R}) = \mathrm{srk}(\widetilde{M})$. We compute the Ricci curvature in direction v with respect to a specific orthonormal frame $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ such that $e_1 = v, e_2, \ldots, e_k \in T_o Y$ and $e_{k+1}, \ldots, e_n \in (T_o Y)^\perp$, where $k = \mathrm{srk}(\widetilde{M})$:

$$\mathrm{Ric}(v) = \sum_{i=1}^{n} \langle R(v, e_i)e_i, v \rangle = \sum_{i=1}^{k} \langle R(v, e_i)e_i, v \rangle + \sum_{i=k+1}^{n} \langle R(v, e_i)e_i, v \rangle$$
$$\geq 0 + (n - k) \cdot (-1)$$
$$= -(n - \mathrm{srk}(\widetilde{M})),$$

where the inequality uses the sectional curvature bound together with the fact that $Y \times \mathbb{R}$ is a totally geodesic Riemannian product. Thus, we have $\delta^2 \leq -\mathrm{Ric}(v) \leq n - \mathrm{srk}(\widetilde{M})$, and it follows that

$$\delta \leq \sqrt{n - \mathrm{srk}(\widetilde{M})}. \tag{1}$$

Note that the following inequality holds

$$\frac{\sqrt{n - \mathrm{srk}(\widetilde{M})}}{2} \leq n - 2 - \mathrm{srk}(\widetilde{M}),$$

provided $\mathrm{srk}(\widetilde{M}) \leq n - 3$, hence the lemma follows in this case. If $\mathrm{srk}(\widetilde{M}) = n - 2$, then inequality (1) implies that $\delta \leq \sqrt{2}$, so the inequality also holds. In the remaining cases, $\mathrm{srk}(\widetilde{M}) = n$ would imply that $\widetilde{M}$ has an $\mathbb{R}$-factor, and $\mathrm{srk}(\widetilde{M}) = n - 1$ would imply $\widetilde{M}$ has an $\mathbb{H}^2$-factor, which are excluded by the assumption. $\qquad \square$

Proof of Theorem 1.9. Set $f(x) = |\omega|_2(x)$ and set $u = f^2$. Choose $x_0 \in \widetilde{M}$ to be a point where $f(x_0) = \sup_{x \in \widetilde{M}} f(x) = \mathrm{comass}(\omega)$.

By Lemma 2.2 we have that u is a subsolution of the operator $\Delta - \lambda \, \mathrm{I}$, where Δ is the Laplacian of $\widetilde{M}$. Since in all cases $\mathrm{Ric}_M \geq -(n-1)b_p^2$, we may first apply Theorem 2.4 to $u = f^2$ with $r = 1$ to obtain

$$\mathrm{comass}(\omega)^2 = \sup_{x \in M} |\omega|_2^2(x) \leq C(b_p, n, p) \int_{B_1(x_0)} |\omega|_2^2(x) \, d\,\mathrm{vol}.$$

For case (a) we rescale the metric to have an upper curvature bound of -1 instead of $-a^2$. Under this curvature assumption and the stated Ricci curvature bounds, we may apply Theorem 3.4 in each case to obtain the bound on $\mathrm{comass}(\omega)$ analogous to Theorem 2.5 in each case.

For case (b) we apply Lemma 3.5 to obtain the stated bounds on p also imply $p \leq n - 2 - \mathrm{srk}(\widetilde{M})$. We also observe that in case (b) all of the symmetric spaces have dimension at least six. Finally, applying Theorem 3.3 completes the result. $\qquad \square$

4. Producing upper bounds on the Gromov norm

Consider the set of all singular p-simplices $\Sigma^p = \{\sigma : \Delta^p \to M\}$, a cochain $c \in C^p(M, \mathbb{R})$ has a natural ℓ^∞-norm,

$$\|c\|_\infty := \sup_{\sigma \in \Sigma^p} |c(\sigma)|.$$

Thus for cohomology class $\alpha \in H^p(M, \mathbb{R})$, the ℓ^∞-norm is defined to be

$$\|\alpha\|_\infty := \inf\{\|c\|_\infty : c \in C^p(M, \mathbb{R}),\ \delta c = 0,\ [c] = \alpha\}.$$

Note that by definition $\|\alpha\|_\infty$ can take infinite values, and we say α *is bounded* if $\|\alpha\|_\infty$ is finite, or equivalently, α has a bounded representative.

In the theory of bounded cohomology, the ℓ^∞-norm on the cohomology is considered dual to the ℓ^1-norm on the homology. In particular, surjectivity of the comparison map on top dimension is equivalent to the non-vanishing of the simplicial volume. More generally, we have duality principle

Proposition 4.1 ([21], [1, Proposition F.2.2]). *For any $\alpha \in H_k(M, \mathbb{R})$, the Gromov norm satisfies*

$$\|\alpha\|_1 = \frac{1}{\inf\{\|\varphi\|_\infty \mid (\varphi, \alpha) = 1,\ \varphi \in H^k(M, \mathbb{R})\}}.$$

Therefore, in order to produce an upper bound on the Gromov norm, it is equivalent to obtain a lower bound on the ℓ^∞-norm of certain cohomology classes. It is shown that whenever the manifold has Ricci curvatures bounded from below, the ℓ^∞-norm always bounds the comass.

Lemma 4.2 ([21, Corollary, p. 244]). *Let M be a complete Riemannian manifold of dimension n with $\mathrm{Ric} \geq -(n-1)$, then for any $\varphi \in H^p(M, \mathbb{R})$, we have*

$$\mathrm{comass}\,(\varphi) \leq p!(n-1)^p \|\varphi\|_\infty.$$

Remark 6. In [21], Gromov used a different normalization, namely $\mathrm{Ric} \geq -1/(n-1)$, but since $\mathrm{comass}(\varphi)$ scales by $(n-1)^p$ when the metric is scaled down by $n-1$, and $\|\varphi\|_\infty$ stays the same, the resulting inequality has an additional factor of $(n-1)^p$. In a recent preprint of Campagnolo and the second author [7], the above inequality is further sharpened by an additional factor of $1/p^{p/2}$, hence the upper bound of Theorem 1.5 can be improved accordingly by a factor of $1/(n-p)^{(n-p)/2}$.

Thus combining the above two results, we prove Theorem 1.5.

Theorem 4.3. *Let M be a closed manifold of dimension n with $\mathrm{Ric} \geq -(n-1)$, and ω be the harmonic representative of $\beta \in H^p(M, \mathbb{R})$, then the Gromov norm of the Poincaré dual of β has an upper bound*

$$\|\beta^*\|_1 \leq (n-p)!(n-1)^{n-p} \|\omega\|_2 \sqrt{\mathrm{vol}(M)}.$$

Proof. Let $\varphi \in H^{n-p}(M, \mathbb{R})$ be any class such that $(\varphi, \beta^*) = 1$ and η be any $(n-p)$-form representing φ, then we have

$$1 = (\beta^*, \varphi) = \int_{\beta^*} \varphi = \int_M \omega \wedge \eta \leq \text{comass}(\eta) \int_M |\omega|_\infty(x) \, d \text{vol}(x).$$

By taking the infimum of comass on all η representing φ, we get

$$1 \leq \text{comass}(\varphi) \int_M |\omega|_\infty(x) \, d \text{vol}(x).$$

Apply Lemma 2.1 and the Cauchy–Schwarz inequality, to further obtain

$$1 \leq \text{comass}(\varphi) \int_M |\omega|_2(x) \, d \text{vol}(x) \leq \text{comass}(\varphi) \|\omega\|_2 \sqrt{\text{vol}(M)}.$$

Applying Lemma 4.2, we have

$$1 \leq (n-p)!(n-1)^{n-p} \|\varphi\|_\infty \|\omega\|_2 \sqrt{\text{vol}(M)}.$$

Finally, we take the infimum of $\|\varphi\|_\infty$ over all φ satisfying $(\varphi, \beta^*) = 1$, and apply Proposition 4.1 to get

$$\|\beta^*\|_1 \leq (n-p)!(n-1)^{n-p} \|\omega\|_2 \sqrt{\text{vol}(M)}. \qquad \square$$

5. Alternative approach

In Section 2, we compared the comass of a harmonic p-form with its L^2-norm in terms of the injectivity radius of the manifold. We can also relate them by other geometric invariants such as the Sobolev constant. Such relation follows from a general theorem of Li.

5.1. Li's theorem. Let K_p be the constant defined in equation (1). In our case, K_p is always nonpositive but may be less than the lower Ricci bound. We let V be the volume of M and let C_S denote the Sobolev constant which will be defined in the next subsection. We will also relate it to other geometric quantities such as diameter, Cheeger's isoperimetric constant, and the bottom of the spectrum of the Laplacian.

Theorem 5.1 ([27, Theorem 7]). *There exists a constant $C(n)$ depending only on n such that if ϕ is a λ-eigenform for the Hodge Laplacian on a closed manifold M of dimension $n \geq 3$ and $\lambda \neq p(n-p)K_p$, then*

$$C(n) \left(\frac{\lambda - p(n-p)K_p}{C_S} \right)^{n/2} \exp\left[\frac{C(n)C_S}{V^{2/n}(\lambda - p(n-p)K_p)} \right] \|\phi\|_2^2 \geq \left(\sup|\phi|_2 \right)^2.$$

Moreover, if $\lambda = p(n-p)K_p$, then $\lambda = 0 = K_p$ and $|\phi|_2$ is pointwise constant. In particular, $\|\phi\|_2^2 = V(\sup|\phi|_2)^2$.

For our purpose, we only need to apply Li's theorem in the case of harmonic p-forms, that is, $\lambda = 0$.

Corollary 5.2. *If M is a closed manifold of dimension $n \geq 3$, then there exists a constant $C(n)$ depending only on n such that for any harmonic p-form ω with $1 \leq p \leq n - 1$,*

$$\text{comass}(\omega) \leq \sqrt{Q}\|\omega\|_2,$$

where Q is given by

$$Q = C(n)\left(\frac{-p(n-p)K_p}{C_S}\right)^{n/2} \exp\left[\frac{C(n)C_S}{-p(n-p)K_pV^{2/n}}\right] \quad \text{if } K_p \neq 0, \text{ or}$$

$$Q = V \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } K_p = 0.$$

Proof. Combine Lemma 2.1 with Theorem 5.1. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 7. In Theorem 7 of [27], the appendix shows that $C(n) = D(n)^{\beta/(\beta-1)}$ where $\beta = n/(n-2)$ and $D(n)$ is the constant from Lemma 2 of that paper. That constant is $D(n) = 2^\alpha$, where $\alpha = 0$ if $n = 3$ and $\alpha = (n-4)/(n-2)$ for $n \geq 4$. Therefore, in both the above Theorem 5.1 and Corollary 5.2, $C(n) \leq 1$ if $n = 3$ and otherwise $C(n) \leq 2^{(n^2-4n)/2}$.

5.2. Sobolev constant, Cheeger's constant and bottom of the spectrum of the Laplacian.

Let C_0 be the classical L^1-Sobolev constant for the Sobolev space $W^{1,1}$ of functions on an n-dimensional closed manifold M with weak L^1 first derivatives, that is,

$$C_0 = \inf_{f \in W^{1,1}} \frac{\|\nabla f\|_1^n}{\inf_{a \in \mathbb{R}}\|f - a\|_{n/(n-1)}^n}.$$

This is related to a scale invariant isoperimetric type constant C_1:

$$C_1 = \inf_{N \subset M} \frac{(\text{vol}_{n-1}(N))^n}{\min\{\text{vol } M_1, \text{vol } M_2\}^{n-1}},$$

where the infimum is over all codimension one closed submanifolds $N \subset M$ whose complement in M consists of two components M_1 and M_2. The relation is

$$C_1 \leq C_0 \leq 2C_1. \tag{1}$$

Another well-known isoperimetric constant is the Cheeger's constant h, it is defined in a similar way as

$$h = \inf_{N \subset M} \frac{\text{vol}_{n-1}(N)}{\min\{\text{vol } M_1, \text{vol } M_2\}}.$$

Hence, it follows that

$$C_1 \leq \frac{h^n \text{vol}(M)}{2}. \tag{2}$$

Moreover, we define C_S by

$$C_S = \left(\frac{2(n-1)}{n-2}\right)^{2/n} C_0^{2/n}.$$

Lemma 1 of [27] shows

$$C_S \leq \inf_{f \in W^{1,2}} \frac{\|\nabla f\|_2^2}{\|f\|_{2n/(n-2)}^2},$$

where the right-hand side is the L^2 Sobolev constant.

If we denote λ_1 the smallest positive eigenvalue of the Laplacian on M, then it is bounded in terms of the Cheeger's constant h.

Theorem 5.3 ([6, 8]). *We have*

$$\lambda_1 \geq \frac{h^2}{4} \quad \text{(Cheeger's inequality)}.$$

If M^n has Ricci lower bound $-(n-1)$, then

$$\lambda_1 \leq 2(n-1)h + 10h^2 \quad \text{(Buser's inequality)}.$$

We attempt to give an upper bound on the constant Q in Corollary 5.2, in terms of these geometric quantities. First we observe that the exponential part of Q is bounded for non-flat nonpositively curved manifolds.

Lemma 5.4. *If M is a closed nonpositively curved manifold which is not flat, $C(n)$ is a constant that depends only on n, and $1 \leq p \leq n-1$, then*

$$\exp\left[\frac{C(n)C_S}{-p(n-p)K_p V^{2/n}}\right] \leq C(n, K_p),$$

where $C(n, K_p)$ is a constant that only depends on n and K_p.

Proof. Since $C(n)$ and p only depend on n, and $K_p < 0$ under the hypotheses, it suffices to bound $C_S/V^{2/n}$ from above. In what follows, let const_n represent a mutable constant depending only on n. According to the definition C_S and inequality (1),

$$C_S/V^{2/n} \leq \text{const}_n \left(\frac{C_0}{V}\right)^{2/n} \leq \text{const}_n \left(\frac{C_1}{V}\right)^{2/n}.$$

By inequality (2) and Cheeger's inequality (5.3),

$$\left(\frac{C_1}{V}\right)^{2/n} \leq \text{const}_n h^2 \leq \text{const}_n \lambda_1.$$

By Cheng's comparison theorem [9, Corollary 2.3],

$$\lambda_1 \leq \frac{(n-1)^2 K}{4} + \frac{\text{const}_n}{d_M^2},$$

where $-(n-1)K$ is the Ricci lower bound of M (hence, $K \leq -K_p$).

Consider a closed Dirichlet fundamental domain $D \subset \tilde{M}$ for $\pi_1(M)$ centered at some $x \in \tilde{M}$ as defined in Lemma 2.3. Since M is closed and not flat, $\pi_1(M)$ is not infra-nilpotent by [20]. Hence, in the generating set $S = \{\gamma \in \pi_1(M) : \gamma D \cap D \neq \emptyset\}$ there is at least one element $\gamma \in S$ with translation length greater than the Margulis constant μ. Therefore, the diameter of M satisfies $d_M \geq \mu$, and μ only depends on n and $b_p = \sqrt{-K_p}$. Therefore, we obtain an upper bound on λ_1 depending only on n and K_p. Hence, the same holds for the upper bound of $C_S / V^{2/n}$, which completes the proof. $\square$

Now contracting all constants in Corollary 5.2 that depend only on n and K_p, we obtain the following corollary.

Corollary 5.5. *If M is a closed nonpositively curved manifold of dimension $n \geq 3$ which is not flat, then there exists a constant $C(n, K_p)$ depending only on n and K_p such that for any harmonic p-form ω representing $\beta \in H^p(M, \mathbb{R})$ with $1 \leq p \leq n - 1$, we have*

$$\mathrm{comass}(\omega) \leq \frac{C(n, K_p)}{C_S^{n/4}} \|\omega\|_2.$$

Thus, in addition, if M and p satisfy any of the following conditions:

(a) *If M is negatively curved, and $p \leq n - 2$, or*

(b) *If M is rank $r \geq 2$ locally symmetric whose universal cover has no direct factor of $\mathbb{H}^2$, $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$, $\mathrm{Sp}(2, \mathbb{R})/\mathrm{U}(2)$, $G_2^2/\mathrm{SO}(4)$ and $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$, and $p \leq n - 2 - \mathrm{srk}(\tilde{M})$, or*

(c) *If M is geometric rank one satisfying $\mathrm{Ric}_{k+1} < 0$ for some $k \leq \lfloor \frac{n}{4} \rfloor$, and $p \leq n - 4k$,*

then there exists a constant $C(\tilde{M})$ that depends only on $\tilde{M}$ such that

$$\|\omega\|_2 \leq \frac{C(\tilde{M})}{C_S^{n/4}} \|\beta^*\|_1.$$

Proof. Combine Lemma 5.4, Corollary 5.2 and Theorem 3.3. $\square$

Remark 8. Note that since $C_S \simeq C_0^{2/n} \simeq C_1^{2/n}$ up to universal constants, the inequality in the above corollary can be replaced by

$$\mathrm{comass}(\omega) \leq \frac{C(n, K_p)}{\sqrt{C_0}} \|\omega\|_2 \quad \text{or} \quad \mathrm{comass}(\omega) \leq \frac{C(n, K_p)}{\sqrt{C_1}} \|\omega\|_2,$$

and also

$$\|\omega\|_2 \leq \frac{C(\tilde{M})}{\sqrt{C_0}} \|\beta^*\|_1 \quad \text{or} \quad \|\omega\|_2 \leq \frac{C(\tilde{M})}{\sqrt{C_1}} \|\beta^*\|_1.$$

We end the section by a result of Croke which provides a lower bound on C_1 (so equivalently on C_0 and C_S) in terms of the lower bound of Ricci curvature $(n-1)K$, upper bound of diameter d_M, lower bound of vol(M) and the dimension n.

Theorem 5.6 ([14, Theorem 13]). *For any closed Riemannian manifold with Ricci curvature bounded below by $(n-1)K$, there is a constant $C(n)$ depending only on n such that*

$$C_1 \geq C(n) \left(\frac{\text{vol}(M)}{\int_0^{d_M} (\sqrt{-1/K} \sinh \sqrt{-K}r)^{n-1} \, dr} \right)^{n+1},$$

where we use the convention that $(\sqrt{-1/K} \sinh \sqrt{-K}r)$ is interpreted as r if $\mathrm{K} = 0$ and as $\sqrt{1/\mathrm{K}} \sin(\mathrm{K}r)$ if $\mathrm{K} > 0$.

This provides an alternative way of relating the Gromov norm with the harmonic norm in terms of n, K_p, a lower bound on volume and an upper bound on diameter.

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