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$\mathbb{R}$-covered foliations and transverse pseudo-Anosov flows in atoroidal pieces

Sergio R. Fenley

Abstract. We study the transverse geometric behavior of 2-dimensional foliations in 3-manifolds. We show that an $\mathbb{R}$-covered, transversely orientable foliation with Gromov hyperbolic leaves in a closed 3-manifold admits a regulating, transverse pseudo-Anosov flow (in the appropriate sense) in each atoroidal piece of the manifold. The flow is a blow up of a one prong pseudo-Anosov flow. In addition we show that there is a regulating flow for the whole foliation. We also determine how deck transformations act on the universal circle of the foliation.

1. Introduction

This article studies the transverse geometric behavior of 2-dimensional foliations in 3-manifolds. We will assume that the foliation is transversely orientable, so there is a transverse flow. Any such flow can be used to understand how the geometry of leaves varies transversely – at least locally. By change in geometry in this article we mean the following: consider a geodesic arc in a leaf of the foliation and use the chosen transverse flow to push this arc to an arc in a nearby leaf. Does the length increase or decrease and by how much? Notice that there are other important ways to consider changes in transverse geometry: for example consider how the spacing between distinct leaves varies, which leads to the study of contracting and expanding directions, or the existence of holonomy invariant transverse measures.

To illustrate what can happen in terms of transverse change in geometry, the obvious first example to analyze is when the foliation is a fibration of a closed 3-manifold. The foliation is encoded by the monodromy, which is a homeomorphism of a closed surface. By the Nielsen–Thurston theory [10, 35], the monodromy is, up to isotopy, either periodic, reducible or pseudo-Anosov. Reducible means that there is a simple closed curve and a power preserves this curve up to free homotopy. Pseudo-Anosov

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means that, up to isotopy, the homeomorphism preserves a pair of singular, transverse 1-dimensional foliations, whose leaves are either contracted by the map (stable) or expanded (unstable). The singularities are p-prong type with $p \geq 3$. The pseudo-Anosov option very strongly describes how the geometry of leaves is changing by some appropriate transverse suspension flow, describing the directions of maximal contraction and expansion. More specifically the leaves of the foliations are quasi-geodesics and they are uniformly efficient in measuring length in relative homotopy classes. In rough terms iterating the map stretches length unless you are in the stable contracting direction. This geometric information was crucial to geometrize such manifolds [32, 33].

In this article we study this problem of transverse change in geometry for more general foliations. One initial difficulty is the use of a transverse flow. In the case of fibrations, any transverse flow induces homeomorphisms between leaves, so we can use it to see the transverse effect on the geometry of leaves. Whenever there is non-trivial holonomy of closed curves [9], the transverse flow cannot even take a closed curve to a closed curve. This leads to the first adjustment: the transverse change of geometry is best understood in the universal cover and for Reebless foliations: then leaves in the universal cover are simply connected [29], so any compact set in a leaf can be pushed by the transverse flow to nearby leaves. But in general this cannot be accomplished for entire leaves. In fact, one necessary condition for the transverse flow in the universal cover to be a homeomorphism between arbitrary leaves is that the foliation is what is called $\mathbb{R}$-covered [12]: the leaf space of the foliation in the universal cover is homeomorphic to the reals $\mathbb{R}$. In this article we will prove results about $\mathbb{R}$-covered foliations.

Fibrations are $\mathbb{R}$-covered. But even in this situation the pseudo-Anosov case is best understood by looking at the action on the ideal boundary of the universal cover as follows. Restrict to the case that the fiber is negatively curved, which is the generic case, so one can assume that the fiber is a hyperbolic surface. The universal cover is the hyperbolic plane, compactified with an ideal circle [2]. Any lift of the monodromy to the universal cover induces a homeomorphism of this ideal circle. Thurston [10, 35], following ideas of Nielsen, did a very thorough study of this action on a circle, yielding (in the non-periodic, irreducible case) invariant geodesic laminations on the surface, which blow down to the singular foliations associated with the pseudo-Anosov monodromy. This yields a pseudo-Anosov flow in M which is transverse to the fibration. The geodesic laminations give the directions of maximal contraction and maximal expansion transverse to the foliation. These laminations blow down to foliations by quasigeodesic curves, which still have these maximal expansion and maximal contraction properties.

The subject of this article is to analyze the existence of transverse pseudo-Anosov flows for more general foliations. The stable and unstable foliations of the flows should be the directions of maximal contraction and maximal stretch transverse to the foliation.

This program of seeing all ideal circles of leaves in the universal circle as one single object can be carried out to a certain extent for any foliation with hyperbolic leaves: this is the theory of the universal circle of foliations [6,7,36,37], which introduces a powerful way to collate all circles at infinity into a single circle, called the universal circle of the foliation. This has some powerful consequences for the geometry of the foliation and the manifold [6, 7]. The general expectation is that either geometry does not change very much transversally – usually yielding a Seifert fibered space structure (see details later); or there is some region with unbounded distortion, yielding to some pseudo-Anosov behavior in at least part of the manifold. This expectation comes from the works of Thurston [33,35,36] which show exactly this behavior in some cases.

This strategy has been carried out very successfully when the foliation is $\mathbb{R}$-covered (again the case of Gromov hyperbolic leaves is the generic case) [5, 12], and M is atoroidal. The atoroidal case is the most common one as the manifold is then hyperbolic by Perelman's results. In this case it was proved in [5, 12] that (when the foliation is transversely orientable) there is a pseudo-Anosov flow transverse to the foliation and regulating. *Regulating* means that in the universal cover every lifted flow line intersects every leaf of the foliation. One of the important consequences is that this provided a proof of the weak hyperbolization conjecture: either there is a $\mathbb{Z}^2$ subgroup or the fundamental group of the manifold is Gromov hyperbolic [20]. This result on Gromov hyperbolicity was of course superseded by the full proof of the geometrization conjecture by Perelman.

What was left unanswered in [5,12] is the question of what happens in the intermediate case: that is, when the foliation $\mathcal{F}$ is $\mathbb{R}$-covered and transversely oriented and in addition M is neither Seifert fibered nor atoroidal. In particular, the JSJ decomposition of M is not trivial. The purpose of this article is to analyze the transverse geometry of $\mathbb{R}$-covered foliations in this intermediate situation.

Before stating our main result we have to rule out one special case. This case happens when the manifold is not atoroidal, but it can be cut by say a torus into a manifold that is a product. This happens for example if the manifold fibers over the circle with torus fibers and monodromy which is Anosov. The JSJ decomposition in this case is given by a torus, and after cutting, the resulting manifold is a product. By definition this piece of the JSJ decomposition is atoroidal. But obviously there cannot be a pseudo-Anosov flow in such a piece. This piece of the JSJ decomposition is exceptional in the sense that it is both atoroidal and Seifert. We call an atoroidal piece of the JSJ decomposition *truly atoroidal* if it is not a product.

Our main result is the following:

Theorem 1.1 (Main theorem). *Let $\mathcal{F}$ be a 2-dimensional foliation in M^3 closed so that $\mathcal{F}$ is transversely oriented, $\mathbb{R}$-covered, and has Gromov hyperbolic leaves. Suppose that there is a truly atoroidal piece P of the JSJ decomposition of the manifold. Then there is a flow Φ in a representative of P which is a blow up of a one prong pseudo-Anosov flow, so that Φ is transverse to $\mathcal{F}$ restricted to P and it is regulating for $\mathcal{F}$ restricted to P. The union of the regular periodic orbits of Φ is dense in P.*

We stress that a one prong pseudo-Anosov flow is a generalized pseudo-Anosov flow where one allows the existence of one prong orbits. In addition notice that the atoroidal piece P is only defined up to isotopy. Given a boundary torus T of P, then in general one can isotope it to be either a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ [17, 31]. It cannot be a leaf since the leaves of $\mathcal{F}$ are Gromov hyperbolic. If it is transverse to $\mathcal{F}$ then it is regulating, that is, a lift to the universal cover intersects all leaves of $\widetilde{\mathcal{F}}$. We choose the tori and Klein bottles in the boundary of P to have this property. The tori and Klein bottles will be in what we call "good position"; see Section 2.

Unless otherwise stated, when we consider an atoroidal piece P of the JSJ decomposition of M we are assuming that P is a truly atoroidal piece of M.

Once the Main theorem is proved it is not very hard to obtain the following consequences. The first involves the action of $\pi_1(M)$ in the universal circle.

Corollary 1.2. *Let $\mathcal{F}$ as in the Main theorem, and let γ an element in $\pi_1(M)$ associated with a periodic orbit of Φ in the interior of P. Then up to a finite iterate the action of γ on the universal circle of $\mathcal{F}$ has finitely many fixed points which are alternatively attracting and repelling. If the orbit is regular there are exactly 4 fixed points up to an iterate.*

In Section 2 we provide the necessary background on the universal circle of foliations.

We also prove the following, which extends well known results for $\mathbb{R}$-covered foliations in atoroidal manifolds [5, 12]:

Corollary 1.3. *Suppose that $\mathcal{F}$ is a transversely oriented foliation which is $\mathbb{R}$-covered. Then there is a flow transverse to $\mathcal{F}$ which is regulating for $\mathcal{F}$.*

In fact, as the constructions in this article will show, there is a regulating flow so that in the atoroidal pieces it is pseudo-Anosov and in the Seifert pieces it is essentially an isometry between the metrics of the leaves in those pieces. In the pseudo-Anosov pieces the stable and unstable foliations induced in the leaves are those of maximal contraction and expansion by the transversal pseudo-Anosov flow.

The Main theorem also helps us to understand in general the action of $\pi_1(M)$ on the universal circle of the foliation $\mathcal{F}$.

Besides their intrinsic interest, the results of this article, particularly the Main theorem have some uses in other situations. For example in partially hyperbolic dynamics in dimension 3 the properties of 2-dimensional foliations are essential due to the results of Burago–Ivanov [4] who produced some "branching foliations" associated with the dynamics. In many cases these foliations are $\mathbb{R}$-covered and the change in transverse geometry can give important information. In particular, in [15] we use the results of this article on transverse pseudo-Anosov flows on atoroidal pieces and group actions on the universal cover to obtain some geometric results about the invariant foliations associated with some partially hyperbolic diffeomorphisms in 3-manifolds.

1.1. Ideas of the proof of the Main theorem

Let $\widetilde{\mathcal{F}}$ be the lift of the foliation $\mathcal{F}$ to the universal cover $\widetilde{M}$. As indicated above the main idea is to use the universal circle of the foliation. In the case of an $\mathbb{R}$-covered foliation the universal circle is canonically homeomorphic to the circle at infinity of any leaf of $\widetilde{\mathcal{F}}$. This is described more carefully in the next section. To understand the change of geometry across leaves of $\widetilde{\mathcal{F}}$, one uses geometric shapes in the leaves of $\widetilde{\mathcal{F}}$ determined by ideal points of the leaves. Any pair of geodesics in the hyperbolic plane are isometric and so are any ideal triangles. To see distortion one has to look at ideal quadrilaterals in leaves of $\widetilde{\mathcal{F}}$ determined by four ideal points in these leaves. Now change the leaves and move the ideal points according to the universal circle identifications. The ideal quadrilaterals change and may become thinner in one direction or in the opposite direction. In [12] we used this distortion of quadrilaterals to produce a pair of 2-dimensional immersed laminations transverse to $\mathcal{F}$ and intersecting leaves of $\mathcal{F}$ in geodesics. These laminations capture some of the distortion in the transverse direction. In particular, we stress that up to this point in [12] the fact that M was atoroidal was not used, except for the fact that M was not Seifert fibered. The atoroidal property was then used heavily in [12] to show that these immersed laminations fill M and they produce embedded laminations, which lead to the construction of a transverse pseudo-Anosov flow.

We continue this analysis in the toroidal case (but M not Seifert). It is expected that the immersed laminations constructed in [12] do not fill M. The constructions and proofs in the later sections of [12] are to a certain extent specific to the atoroidal case. In particular, Theorem 5.1 of [12] produces what is called a leafwise geodesic embedded lamination. The passage from an immersed to an embedded lamination is fundamental in the understanding of the problem. The process is done by a convex hull procedure. The convex hull is done in each leaf L of $\widetilde{\mathcal{F}}$: take the convex hull of a component of the intersection of the immersed leafwise geodesic lamination with L. Suppose that the convex hull is not all of L. Then each boundary component of this convex hull in L is a geodesic in L. This varies continuously with L and produces an *embedded* leafwise geodesic lamination transverse to $\mathcal{F}$. In [12] we prove that this lamination has a torus leaf. This is disallowed in case M is atoroidal.

In the toroidal case the embedded leafwise geodesic lamination obtained by this convex hull process theoretically can well be a cutting torus in the JSJ decomposition of the manifold. We definitely do not want such a lamination (a torus), as it would not describe the most extreme transversal change of geometry, which is what we are looking for. At this point the proofs in the atoroidal and toroidal case diverge.

Recall that we explained above that except in the case that M is Seifert fibered, there are distortion quadrilaterals. In the toroidal case we study in more detail the actual immersed laminations which are the limits of the distortion quadrilaterals described above. We show that components of these laminations are contained in the interior of the atoroidal pieces (appropriately adjusted). In particular, they cannot be any of the tori of the JSJ decomposition. This is the hardest fact to prove and it uses a lot the definition of the distortion quadrilaterals. This is done in two steps: We first show that the geodesics produced by the limiting process cannot cross the tori of the JSJ decomposition. Then we show that the geodesics cannot be contained in the union of the tori either. The second property is harder to obtain and involves manipulating the foliation we start with.

After this is done, we show that the a priori immersed laminations produce an embedded sublamination. We will also show that the two laminations we obtain satisfy the following: (1) they are not the same, (2) they are transverse to each other, and (3) they fill an atoroidal piece P. Then there is a blow down process to produce a pseudo-Anosov like flow. In the blow down process to produce a flow, it is possible that one prongs may be created. So we generalize the notion of a pseudo-Anosov flow to a one prong pseudo-Anosov flow. Finally, an appropriate blow up of this flow produces a flow in P which is transverse to $\mathcal{F}$ in P and regulating.

Once this is done the construction of the regulating flow for the whole foliation is not so complicated.

2. $\mathbb{R}$-covered foliations with Gromov hyperbolic leaves

Here we explain the basics about these foliations and review the information we need about them. The details are in [12]. Let $\mathcal{F}$ be an $\mathbb{R}$-covered foliation with Gromov hyperbolic leaves. By Candel's theorem [8], there is a metric in M making every leaf of $\mathcal{F}$ into a hyperbolic surface (notice that $\mathbb{R}$-covered is not necessary for Candel's theorem). Let $\widetilde{\mathcal{F}}$ be the lifted foliation to $\widetilde{M}$. Each leaf F of $\widetilde{\mathcal{F}}$ with its induced Riemannian metric is isometric to the hyperbolic plane and is compactified with an ideal circle $S^1(F)$. This ideal circle and its topology is independent of the metric in M.

First we introduce the ideal annulus $\mathcal{A}$. As a set $\mathcal{A}$ is the union of $S^1(E)$ where E are the leaves of $\widetilde{\mathcal{F}}$; see [12, Definition 3.1]. The topology is as follows: consider a transversal τ to $\widetilde{\mathcal{F}}$. For each point x in τ with x in E leaf of $\widetilde{\mathcal{F}}$, consider the unit tangent bundle of E at x which is a circle. Each unit vector determines a geodesic ray in E starting at x with that direction. This determines an ideal point in E, hence a point in $S^1(E)$. The map between directions and $S^1(E)$ is a bijection for each E. As x varies in τ this provides a bijection between the unit tangent bundle of $\widetilde{\mathcal{F}}$ restricted to τ and the union of the circles at infinity of the leaves intersecting τ. The union of the unit tangent bundles to $\widetilde{\mathcal{F}}$ along τ has a natural topology coming from the geometry of $\widetilde{M}$. We put a topology in $\mathcal{A}$ induced by these local bijections. In [12, Lemma 3.2], we proved that this topology is well defined, and deck transformations act by homeomorphisms on $\mathcal{A}$. Notice that this topology in $\mathcal{A}$ clearly induces the natural topology in each $S^1(E)$. One important continuity property is the following: let α, β be continuous curves in $\mathcal{A}$ transverse to the foliation of $\mathcal{A}$, which is the foliation by the circles of infinity of leaves of $\widetilde{\mathcal{F}}$. Suppose that for each E in $\widetilde{\mathcal{F}}$ the intersections of α, β with $S^1(E)$, denoted respectively by α_E, β_E, are distinct points in $S^1(E)$. As E varies let ℓ_E be the geodesic in E with ideal points α_E, β_E. Then the geodesics ℓ_E vary continuously in $\widetilde{M}$ with E.

We now describe the universal circle $\mathcal{U}$ of $\mathcal{F}$. There are two possibilities:

• The *uniform case* – Here for any two leaves E, F of $\widetilde{\mathcal{F}}$, the Hausdorff distance between them (as subsets of $\widetilde{M}$) is finite [36]. The bound obviously depends on the pair of leaves. For any pair of leaves E, F there is a map $\tau\colon E \to F$ which is a quasi-isometry. This quasi-isometry is coarsely well defined and induces a homeomorphism τ_∞ between $S^1(E)$ and $S^1(F)$. The map τ_∞ is as follows: given p in $S^1(E)$ there is a unique q in $S^1(F)$ so that if r is a geodesic ray in E with ideal point p and r' is a geodesic ray in F with ideal point q then r, r' are a finite Hausdorff distance from each other in $\widetilde{M}$. The homeomorphisms between ideal circles satisfy a cocycle property and they are obviously equivariant under the action of deck transformations. This is proved in [12, Proposition 3.4].

• The *non-uniform case* – In particular, there are no compact leaves. In this case $\mathcal{F}$ has a unique minimal sublamination $\mathcal{F}'$. The complementary regions of $\mathcal{F}'$ are I-bundles over non-compact surfaces. One can then collapse the complementary regions to produce a new $\mathbb{R}$-covered foliation which is minimal; see [12, Proposition 2.6]. All the results proved for this induced minimal foliation pull back to $\mathcal{F}$. So when necessary we assume in this case that $\mathcal{F}$ is minimal. Under this condition and $\mathcal{F}$ not uniform then for any leaves E, F of $\widetilde{\mathcal{F}}$ there is a dense set of directions in E (and in F too) so that if r is a ray in E with one of these directions, then r is asymptotic to F. In fact, it is asymptotic to a geodesic ray in F; see [12, Proposition 3.22]. This gives a way to identify a dense set of points in $S^1(E)$ with a dense set of points in $S^1(F)$.

This extends to a unique homeomorphism τ_∞ between $S^1(E)$ and $S^1(F)$. Again these homeomorphisms satisfy a cocycle condition and are equivariant under deck transformations. This is proved in [12, Proposition 3.22].

The universal circle $\mathcal{U}$ of $\mathcal{F}$ is the quotient of $\mathcal{A}$ by these identifications: that is, x in $S^1(E)$ is identified with y in $S^1(F)$ if $\tau_\infty(x) = y$, where τ_∞ is the map described above.

In both the uniform and non-uniform cases the universal circle induces a vertical foliation in $\mathcal{A}$: two points in $\mathcal{A}$ are in the same leaf of the vertical foliation if they represent the same point of the universal circle. This foliation is by continuous curves in $\mathcal{A}$.

Notice that in the non-uniform case one first collapses the foliation to produce a minimal foliation. The reverse process blows up an at most countable number of leaves, each into a foliated I-bundle. In the universal cover this is a product and all leaves in the interval are a bounded distance from each other. There is an induced vertical foliation in the union of the corresponding circles at infinity which is the one given by the analysis of the uniform case restricted to an interval of leaves. The enlarged object is still a foliation in $\mathcal{A}$ with leaves which are transverse to the foliation of $\mathcal{A}$ by circles at infinity of leaves of $\widetilde{\mathcal{F}}$.

Remark 2.1. The construction of the universal circle done here for $\mathbb{R}$-covered foliations is different from the construction of the universal circle for general foliations done in [7], using markers. It can be shown that in the case that the $\mathbb{R}$-covered foliation is not uniform, then the universal circles are the same here and in [7], even though this is not used in this article. The situation is different in the uniform case, in general the universal circles are not the same. In this case the "leftmost" and "rightmost" universal circles as constructed in [7] can be quite different from each other, and obviously one cannot have that both are isomorphic to the universal circle constructed here.

We now introduce the ideal quadrilaterals and parallelepipeds in $\widetilde{M}$. Let a, b, c, d be 4 distinct points in $\mathcal{U}$, which are circularly ordered. Let J be a compact interval in the leaf space of $\widetilde{\mathcal{F}}$. For each F leaf of $\widetilde{\mathcal{F}}$ in J let Q_F be the ideal quadrilateral in F with ideal points a_F, b_F, c_F, d_F which are the representatives of a, b, c, d in $S^1(F)$. By the properties of the universal circle, the quadrilaterals Q_F vary continuously with F. Let $\mathcal{P}$ be the union of Q_F over all F in J. We call $\mathcal{P}$ a *parallelepiped*.

The case of uniformly quasi-symmetric action. In [12, Section 4], we studied the case that the action on $\pi_1(M)$ on $\mathcal{U}$ is uniformly quasi-symmetric. In [12, p. 453, after Claim 3], we arrive at two possibilities for M:

(1) M is Seifert fibered manifold,

(2) M is a torus bundle over $\mathbf{S}^1$ with Anosov monodromy.

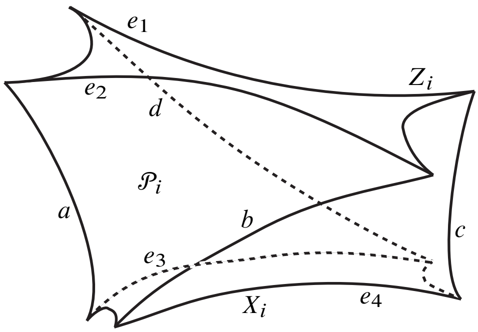

Figure 1. A parallelepiped $\mathcal{P}_i$: this is a 3-dimensional set in $\widetilde{M}$ made up of ideal quadrilaterals in an interval of leaves of $\widetilde{\mathcal{F}}$. The top quadrilateral is Z_i and the bottom one is X_i. The curves a, b, c, d in the figure are not made up of points in $\widetilde{M}$, rather they are curves of ideal points of leaves of $\widetilde{\mathcal{F}}$. Each of these curves denotes ideal points of different leaves, but corresponding to the same point in the universal circle. On the top the ideal quadrilateral Z_i is thin in one direction: the geodesic sides e_1, e_2 are very close to each other in the respective leaf of $\widetilde{\mathcal{F}}$. In the bottom, the ideal quadrilateral X_i is thin in the opposite direction: the geodesics e_3, e_4 are now close to each other in the respective leaf of $\widetilde{\mathcal{F}}$.

Case (1) is ruled out by our assumption that M has an atoroidal piece. In case (2) the manifold is neither Seifert nor atoroidal. Since the monodromy is Anosov, it follows that up to isotopy, the only incompressible tori are the fibers of the fibration. The JSJ decomposition is made up of a single torus, producing an atoroidal piece which is not truly atoroidal. In the Main theorem we are assuming that M has a truly atoroidal piece so this cannot happen. Notice that in case (2) the manifold M admits $\mathbb{R}$-covered foliations with Gromov hyperbolic leaves. For example, the weak stable foliation of the suspension Anosov flow satisfies these properties.

The case of non-uniformly quasi-symmetric action. In [12, Section 5], we proved the following: if the action of $\pi_1(M)$ on $\mathcal{U}$ is not uniformly quasi-symmetric, then there is a sequence of parallelepipeds $\mathcal{P}_i$ with tops Z_i and bottoms X_i so that Z_i is very thin in one direction and X_i is very thin in the opposite direction; see [12, Lemma 5.3]. The thinness of a quadrilateral is the minimum distance in the corresponding leaf of $\widetilde{\mathcal{F}}$ between opposites sides of the quadrilateral. An ideal quadrilateral is regular if both such minimum distances are equal; see Figure 1 for a depiction of such a parallelepiped. The thinness of an ideal quadrilateral is measured by the cross ratio of the 4 ideal points; where in [12] we identified $\mathcal{U}$ with the unit circle in the complex plane. Thinness in one direction is the same as the cross ratio very close to 0, while thinness in the other direction is the same as cross ratio very close to 1.

In [12, Lemma 5.3], we constructed these parallelepipeds so that thinness of Z_i converges to 0 in one direction while thinness of X_i converges to 0 in the other direction. These parallelepipeds are measuring the distortion of the geometry transversally to $\widetilde{\mathcal{F}}$. Pick a height F where Q_F is a regular quadrilateral. Going up, to the top Z_i, makes the quadrilateral very thin – in other words, stretching the leaf in the direction of the side of the quadrilateral which are very near and contracting the other direction. Going down, the opposite happens.

Since the thinness of Z_i converges to 0 they are getting closer and closer to geodesics. Project to M and consider such a limit geodesic ℓ_0. Lift to a geodesic ℓ in a leaf F of $\widetilde{\mathcal{F}}$. Now saturate ℓ by the universal circle, that is, for any E leaf of $\widetilde{\mathcal{F}}$ take the geodesic in E so that its ideal points correspond to the same points in the universal circle $\mathcal{U}$ as the ideal points of ℓ. The union of all of these is a closed set in $\widetilde{M}$. The projection is the a priori immersed lamination $\mathcal{L}^u$, with lift $\widetilde{\mathcal{L}}^u$. Each leaf L of $\mathcal{L}^u$ intersects the leaves of $\widetilde{\mathcal{F}}$ in geodesics. We call $\mathcal{L}^u$ a *leafwise geodesic lamination*. In the same way considering limits of the bottoms of the parallelepipeds produces the immersed lamination $\mathcal{L}^s$.

We state these results formally for future reference: Given a geodesic ℓ in a leaf F of $\widetilde{\mathcal{F}}$ the *saturation* of ℓ is the union over all E leaves of $\widetilde{\mathcal{F}}$ of the geodesic ℓ_E in E, so that the ideal points of ℓ_E in $S^1(E)$ and the ideal points of ℓ in $S^1(F)$ are the same points under the universal circle identification. We also call this the saturation of ℓ by the universal circle. In [12, Definition 5.4], it is explained that this is a properly embedded plane in $\widetilde{M}$. We call it a *wall*.

Proposition 2.2. *Suppose that the action of $\pi_1(M)$ on M is not uniformly quasi-symmetric, in particular this happens if M is not Seifert fibered and has a truly atoroidal piece. Then there is a sequence of parallelepipeds $\mathcal{P}_i$ in $\widetilde{M}$ satisfying the following: the tops Z_i of $\mathcal{P}_i$ have thinness converging to zero (as $i \to \infty$) in one direction, and the bottoms X_i of $\mathcal{P}_i$ have thinness converging to zero in the opposite direction.*

Proposition 2.3. *Suppose that $\mathcal{P}_i$ is a sequence of parallelepipeds in $\widetilde{M}$ satisfying the properties in Proposition 2.2. Let Z_i be the tops of $\mathcal{P}_i$ and X_i be the bottoms of $\mathcal{P}_i$. Then there is an immersed leafwise geodesic lamination $\mathcal{L}^u$ in M obtained as follows: consider all limits (as $i \to \infty$) of deck translates of Z_i. These form a collection of geodesics in leaves of $\widetilde{\mathcal{F}}$. Saturate these geodesics by the universal circle producing a collection of walls in $\widetilde{M}$. The immersed lamination $\mathcal{L}^u$ is the projection of this collection of saturated walls to M. Let $\mathcal{L}^s$ be the immersed leafwise geodesic lamination obtained by doing the same procedure with the bottoms X_i.*

Notice that since different walls may intersect, we keep track of the leaves of $\mathcal{L}^u$ and not just the set.

2.1. JSJ decomposition and $\mathbb{R}$-covered foliations

Here we review some results from [14]. The JSJ decomposition of an irreducible manifold splits it into Seifert and atoroidal pieces [22–24]. Since we are considering non-orientable manifolds we allow Klein bottles amongst the cutting surfaces. Let M be a 3-manifold with a Reebless, $\mathbb{R}$-covered foliation with hyperbolic leaves. Suppose that M does not fiber over the circle with Anosov monodromy. Per standard practice the collection of cutting surfaces is minimal, hence no two such surfaces are isotopic.

Let T be a torus or Klein bottle in the JSJ decomposition and $\widetilde{T}$ be a lift to $\widetilde{M}$. Then $\widetilde{T}$ with its path metric is quasi-isometrically embedded in $\widetilde{M}$. This follows from [25, Theorem 1.1], see also [28, Section 3.1]. Then one can isotope T so that $\widetilde{T}$ intersects each leaf F of $\widetilde{\mathcal{F}}$ in a single component which is a quasigeodesic in F. One can furthermore pull tight these quasigeodesics, and assume that T satisfies that $\widetilde{T}$ intersects leaves of $\widetilde{\mathcal{F}}$ in geodesics. We always assume this is the case for any T a cutting surface of the JSJ decomposition. We say that T is in *good position*.

In addition we have the following very important fact ([14, Proposition 4.4]): for any F leaf of $\widetilde{\mathcal{F}}$ let $\ell_F = \widetilde{T} \cap F$, a geodesic in F with ideal points a_F, b_F in $S^1(F)$. Then the set of b_F as F varies in $\widetilde{\mathcal{F}}$ is a leaf of the vertical foliation in $\mathcal{A}$. In other words the set $\{b_F \in S^1(F), F \in \widetilde{\mathcal{F}}\}$ corresponds to a single point in the universal circle $\mathcal{U}$ of $\mathcal{F}$. Obviously the same holds for the points a_F, $F \in \widetilde{\mathcal{F}}$.

3. Properties of the immersed leafwise geodesic laminations

In this section we prove the main ingredients to produce the pseudo-Anosov flow in an atoroidal piece P.

Proposition 3.1. *Let $\mathcal{L}^u, \mathcal{L}^s$ be the immersed leafwise geodesic laminations transverse to $\mathcal{F}$, as in Proposition 2.3. Then no leaf of $\mathcal{L}^u$ transversely intersects a torus or Klein bottle of the JSJ decomposition.*

Proof. Suppose that some leaf L in $\widetilde{\mathcal{L}}^u$ transversely intersects a lift $\widetilde{T}$ for T one of the tori or Klein bottles of the JSJ decomposition. If necessary lift to a double cover and we can assume that M is orientable. Hence we can assume that T is a torus.

Recall that both $\widetilde{T}$ and L intersect leaves of $\widetilde{\mathcal{F}}$ in geodesics so that endpoints are constant under the universal circle identification. Therefore this transverse intersection of $\widetilde{T}$ and L is seen in every leaf of $\widetilde{\mathcal{F}}$. Since L is in $\widetilde{\mathcal{L}}^u$, then in particular up to deck transformations there is a sequence of distortion parallelepipeds denoted by $\mathcal{P}_i$ so that the top quadrilaterals of $\mathcal{P}_i$, call them Z_i, converge to a geodesic in L. Let F be the leaf of $\widetilde{\mathcal{F}}$ containing this limit geodesic, in other words Z_i converges to $L \cap F$; see the construction of $\mathcal{L}^u$ in Section 2. Let F_i be the leaf of $\widetilde{\mathcal{F}}$ containing Z_i. Since F_i

is converging to F we can slightly adjust the top of the parallelepipeds so that now every Z_i is contained in F. Again since F_i is converging to F, it also follows that the thinness of the adjusted Z_i is going to zero in the same direction as the original Z_i. This is because the vertical foliation in $\mathcal{A}$ implies that the geodesics which are the sides of the quadrilaterals vary continuously.

Let $\mu_i = Z_i \cap \widetilde{T}$. Since L intersects $\widetilde{T}$ transversely, it follows that the segments μ_i have length converging to zero. In particular, μ_i converges to a single point which is $L \cap F \cap \widetilde{T}$ and is denoted by p. Let $\eta = F \cap \widetilde{T}$.

The bottoms of the parallelepipeds $\mathcal{P}_i$, call them X_i, are quadrilaterals that are very thin in the other direction. For each i, the quadrilateral X_i still intersects $\widetilde{T}$ in a geodesic arc. This is because the parallelepiped intersects leaves of $\widetilde{\mathcal{F}}$ in ideal quadrilaterals with ideal points constant when identified with the universal circle. Now this geodesic arc is not very short, as was the case for the top quadrilaterals, but rather very long. Call these geodesic arcs ζ_i. These project in M into T. Using again continuity of geodesics defined by pairs of points in $\mathcal{U}$, then after adjusting the bottoms of $\mathcal{P}_i$, we can assume that $\pi(\zeta_i)$ contains $\pi(\mu_i)$ and distance of $\pi(p)$ from both endpoints of $\pi(\zeta_i)$ along $\pi(\zeta_i)$ goes to infinity with i.

Since $\pi(\zeta_i)$ contains $\pi(\mu_i)$ and satisfies the property in the previous paragraph, it follows that there are deck transformations γ_i satisfying the following: γ_i sends ζ_i to segments in η containing μ_i and so that distance along η from p to endpoints of $\gamma_i(\zeta_i)$ goes to infinity. In particular, γ_i is also a deck transformation of $\widetilde{T}$. This is the crucial property: we are using that T is compact so we can always use deck transformations to bring long segments of the lifted foliation $\widetilde{\mathcal{F}} \cap \widetilde{T}$ to intersect a compact set.

We now analyze the action of γ_i on the universal circle $\mathcal{U}$. We use the identification of $\mathcal{U}$ with $S^1(F)$. Recall that $\widetilde{T} \cap F$ is the geodesic η in F. Let I be a complementary interval in $S^1(F)$ of the ideal points of η. We can parametrize I as follows: for each q in I there is a unique q' in η so that the geodesic ray in F from q' with ideal point q is perpendicular to η. In this way I is parametrized by η. The action of $\pi_1(T)$ on $\mathcal{U}$ preserves the points of $\mathcal{U}$ corresponding to the ideal points of η in F. In other words, when expressing this action in terms of $S^1(F)$, it follows that $\pi_1(T)$ preserves I. This uses that $\mathcal{F}$ is transversely orientable and M is orientable.

We analyze the action of $\pi_1(T)$ on I. Since I is canonically identified with η, this induces an action of $\pi_1(T)$ on η. Let this action be denoted by ρ.

Let the endpoints of μ_i be x_i, x'_i and the endpoints of $\gamma_i(\zeta_i)$ be y_i, y'_i. These are points in η. By renaming we assume that y'_i, x'_i, x_i, y_i are always linearly ordered in η. Therefore, we have the following property:

Property 1. We have points x_i in η converging to p which are taken by $\rho(\gamma_i)$ to y_i which escapes in η as $i \to \infty$.

We use the following result. This result almost surely has more hypothesis than what is needed to get a global fixed point of a $\mathbb{Z}^2$ action on $\mathbb{R}$, but it suffices for our needs.

Lemma 3.2. *Let $G \cong \mathbb{Z}^2$ acting on $\mathbb{R} \cong \eta$ so that there are points x_i in $\mathbb{R}$ in a compact set of $\mathbb{R}$, and let g_i in G with $g_i(x_i) \to \infty$ and g_i has a fixed point $< x_i$. For each i let $z_i = \lim_{n \to -\infty} g_i^n(x_i)$. Suppose that $d(z_i, x_i)$ converges to 0 as $i \to \infty$ and z_i converges to z_0 in $\mathbb{R}$. Then z_0 is a global fixed point of G.*

Proof. Suppose that the action has an orientation reversing homeomorphism, with unique fixed point w. Since $G \cong \mathbb{Z}^2$ it is easy to see that w is a global fixed point of the action and also that $z_0 = w$., From now on assume that the action is by orientation preserving homeomorphisms.

Let β in G and suppose that β does not fix z_0. Up to taking an inverse we assume that $z_0 < \beta(z_0)$. Now x_i, z_i converge to z_0 as $i \to \infty$. We claim that for any i big enough, then $\beta(z_i) > x_i$, and $\beta(z_i) < g_i(x_i)$. Since z_i converges to z_0 as $i \to \infty$, it follows that $\beta(z_i)$ is bounded. Since $g_i(x_i) \to \infty$, then for any i big enough $\beta(z_i) < g_i(x_i)$. On the other hand, x_i converges to z_0 as $i \to \infty$, z_i converges to z_0, as well and $\beta(z_i)$ converges to $\beta(z_0)$. Hence, for i big enough $x_i < \beta(z_i)$. This proves the claim.

Then,

$$g_i \beta(z_i) = \beta g_i(z_i) = \beta(z_i) > x_i.$$

In other words, $x_i < \beta(z_i) < g_i(x_i)$, and $\beta(z_i)$ fixed by g_i. This is a contradiction and finishes the proof of the lemma. ∎

Conclusion of the proof of Proposition 3.1. Let $g_i = \rho(\gamma_i)$ acting on $\mathbb{R} \cong \eta$. By Property 1, we have x_i in $\mathbb{R}$ with $g_i(x_i) = y_i$ and y_i converging to ∞. In addition, g_i has a fixed point z_i in $[x'_i, x_i]$ and $[x'_i, x_i]$ converges to p. So g_i, z_i, x_i satisfy the properties of Lemma 3.2 with $z_0 = p$. The lemma shows that $\pi_1(T^2)$ has a global fixed point.

Consider the set A of $\widetilde{M}$ which in each leaf E of $\widetilde{\mathcal{F}}$ is the geodesic ray in E satisfying

- The starting point of r is in $\widetilde{T} \cap E$ and r is perpendicular in E to $\widetilde{T} \cap E$.
- The ideal point of r in $S^1(E)$ and the global fixed point of the action of $\pi_1(T^2)$ on $I \subset S^1(F)$ correspond to the same point in the universal circle $\mathcal{U}$.

Since $\widetilde{T}$ is a properly embedded plane, the union of the geodesic rays r as above forms an embedding of a closed half plane in $\widetilde{M}$.

We claim that $H = \pi_1(T^2)$ leaves A invariant. Any $\gamma \in \pi_1(T^2)$ leaves $\widetilde{T}$ invariant. When acting on the universal circle $\mathcal{U}$ then $\pi_1(T^2)$ fixes the point corresponding to the global fixed point of $\pi_1(T^2)$ acting on I. Let E in $\widetilde{\mathcal{F}}$ and let $\nu = E \cap A$. Then $\gamma(\nu)$ is contained in a leaf D of $\widetilde{\mathcal{F}}$, and since γ is an isometry then $\gamma(\nu)$ is a geodesic ray in D, $\gamma(\nu)$ starts in $\widetilde{T}$ and $\gamma(\nu)$ is perpendicular to $\widetilde{T} \cap D$ in D. Finally, $\gamma(\nu)$ has ideal point in $S^1(D)$ which is the same point as the global fixed point of $\pi_1(T^2)$ acting on I under the universal circle identification. It follows that $\gamma(\nu) = D \cap A$, hence γ preserves A.

In particular, $\pi_1(T^2)$ leaves invariant the infinite curve ∂A. This is impossible since $\pi_1(T^2)$ has to act freely and properly discontinuously on ∂A.

This finishes the proof of Proposition 3.1. ∎

We now prove a further property:

Proposition 3.3. *Let P be an atoroidal piece of M. Then there is an immersed leafwise geodesic lamination $\mathcal{L}^u$ (as in Proposition 2.3) contained in the interior of P and no leaf of $\mathcal{L}^u$ is isotopic to a component of ∂P. In addition there are distortion parallelepipeds in $\widetilde{M}$ which produce $\mathcal{L}^u$ in the limit, as in Section 2. Similarly for $\mathcal{L}^s$.*

Proof. We do the proof for $\mathcal{L}^u$. In order to do this we will use a doubling trick. First of all we do a preliminary step. For any torus or Klein bottle element B of the JSJ decomposing set of M, we explained in Section 2.1 that it can be put in good position. Once it is put in good position it intersects leaves of $\mathcal{F}$ in geodesics and when lifted to the universal cover, the endpoints are constant when seen in the universal circle. Let now B, B' be distinct elements of the collection of cutting surfaces, and suppose both are put in good position resulting to T, T'. Since the original surfaces are disjoint then the ones in good position can intersect but not cross. Let $\widetilde{T}, \widetilde{T}'$ be lifts to $\widetilde{M}$. In each leaf F of $\widetilde{\mathcal{F}}$ the intersections of $\widetilde{T}, \widetilde{T}'$ are geodesics. If they intersect in F, then $\widetilde{T} \cap F = \widetilde{T}' \cap F$, because of no crossing. Since the ideal points of both $\widetilde{T}, \widetilde{T}'$ are constant when seen in the universal circle $\mathcal{U}$, it follows that $\widetilde{T} = \widetilde{T}'$. Hence, their stabilizers are the same and the original elements B, B' in the JSJ cutting surfaces are isotopic. This contradicts the assumption of minimality as in Section 2.1.

From now on we assume that all the cutting surfaces in the JSJ decomposition are in good position.

We do the following doubling trick. Cut M along the components of the boundary of P and double P along the boundary. Let this be the manifold N, which we think of $P \cup P'$, where P' is a copy of P. Now do the whole analysis for N. The foliation $\mathcal{F}'$ is the double of $\mathcal{F}_{|P}$.

Claim. The foliation $\mathcal{F}'$ in N is $\mathbb{R}$-covered.

Let $\widetilde{\mathcal{F}}'$ be the lift of $\mathcal{F}'$ to $\widetilde{N}$. We also think of P as a subset of N. Fix T a boundary component of P, fix lifts $\widetilde{P}, \widetilde{T}$ of P, T respectively to $\widetilde{N}$ with $\widetilde{T}$ a boundary component of $\widetilde{P}$.

First we use a fact about $\mathcal{F}$. We can think of $\widetilde{P}, \widetilde{T}$ as a subset of $\widetilde{M}$ as well. The foliation $\mathcal{F}$ is $\mathbb{R}$-covered and since T is in good position, then we proved in

Section 2.1 that every leaf of $\widetilde{\mathcal{F}}$ intersects $\widetilde{T}$, and in particular every leaf of $\widetilde{\mathcal{F}}|_{\widetilde{P}}$ intersects $\widetilde{T}$. The same holds for any other boundary component of $\widetilde{P}$. Hence, if Z is another such boundary component of $\widetilde{P}$, and ℓ is a leaf of $\widetilde{\mathcal{F}} \cap Z$, then the leaf L of $\widetilde{\mathcal{F}}|_{\widetilde{P}}$ intersecting Z in ℓ also intersects $\widetilde{T}$.

Now we go back to $\widetilde{N}$. The lifts of P, P' to $\widetilde{N}$ are the vertices of a graph, where the edges are the lifts of the cutting JSJ surfaces. This graph is a tree $\mathcal{T}$.

Let L be an arbitrary leaf of $\widetilde{\mathcal{F}}'$. It intersects a lift V of either P or P' to $\widetilde{N}$. Let $e_0 = V, e_1, \dots, e_j = \widetilde{P}$ be the vertices in the path in $\mathcal{T}$ from V to $\widetilde{P}$. Let A_i be the edges between e_i and e_{i+1}. If necessary we add one more element A_j which is equal to $\widetilde{T}$. We stress that the e_i's are lifts of either P or P' and the A_i's are lifts of cutting surfaces. The previous paragraph shows that L intersects A_0. Then inductively apply the previous paragraph to the sets e_i and obtain inductively that L intersects A_i. It follows that L intersects $A_j = \widetilde{T}$. We proved that every leaf of $\widetilde{\mathcal{F}}'$ intersects the fixed lift $\widetilde{T}$. By Section 2.1, we know that the leaf space of $\widetilde{\mathcal{F}}$ restricted to $\widetilde{T}$ is homeomorphic to the reals. Hence, the same is true for $\widetilde{\mathcal{F}}'$. Since any two distinct leaves of $\widetilde{\mathcal{F}}'|_{\widetilde{T}}$ are joined by a transversal in $\widetilde{T}$ these cannot be in the same leaf of $\widetilde{\mathcal{F}}'$.

This proves that the leaf space of $\widetilde{\mathcal{F}}'$ is homeomorphic to $\mathbb{R}$ and proves the claim.

As in the proof of Proposition 3.1 we initially lift to a double cover if necessary and assume that M is orientable, hence N is also orientable.

The manifold N has an $\mathbb{R}$-covered foliation $\mathcal{F}'$ which is the double of $\mathcal{F}_{|P}$. This foliation $\mathcal{F}'$ has hyperbolic leaves. Let $\mathcal{V}$ be the universal circle of $\mathcal{F}'$. We consider first the case that the action of $\pi_1(N)$ on $\mathcal{V}$ is uniformly quasi-symmetric. We will rule this out. As explained in Section 2 (see proofs in [12, Section 4]), there are two possibilities:

(1) N is Seifert fibered, or

(2) N is a torus bundle over $\mathbf{S}^1$ with Anosov monodromy.

In the first case it follows that P is Seifert fibered, contrary to the assumption on P. In the second case, as explained in Section 2 any torus in N is isotopic to a fiber. Then P is not truly atoroidal, and this is ruled out by assumption. Therefore this case cannot happen.

Therefore the action of $\pi_1(N)$ on $\mathcal{V}$ is not uniformly quasi-symmetric.

By Propositions 2.2 and 2.3 there are distortion parallelepipeds $\mathcal{P}_i$ producing laminations in N, still denoted by $\mathcal{L}^u$, $\mathcal{L}^s$. We will prove that they induce laminations in M as required.

By the previous proposition we know that no leaf $\mathcal{L}^u$ intersects a boundary component of P transversely. If we prove that no boundary component of P is a leaf of $\mathcal{L}^u$, then we get a sublamination of $\mathcal{L}^u$ contained in the interior of either P or P'. If it is contained in P' then the mirror image is contained in P. This will prove the proposition.

We use the setup of the proof of the previous proposition. Suppose that some leaf of $\mathcal{L}^u$ is a torus T of the JSJ decomposition of N. Let $\widetilde{T}$ be a lift. As in the proof of the previous proposition there are distortion parallelepipeds denoted by $\mathcal{P}_i'$ so that the tops are in a fixed leaf F of $\widetilde{\mathcal{F}}'$ and converge to $\eta = F \cap \widetilde{T}$. Let the ideal points of η in $S^1(F)$ be x_1, x_2. We denote the tops by Z_i'.

We will adjust the tops Z_i' and produce a new set of parallelepipeds $\mathcal{P}_i$, still satisfying the thin conditions as before. The new set of parallelepipeds will be symmetric with respect to $\widetilde{T}$.

We first describe a reflection map in $\widetilde{N}$ with respect to $\widetilde{T}$. The manifold N is the double of P. Let f be the isometry of N associated with the doubling. It is an involution so that $f(P) = P'$. We choose the lift $\widetilde{f}$ of f to $\widetilde{N}$ so that $\widetilde{f}$ is the identity in $\widetilde{T}$. Notice that $\widetilde{f}$ leaves invariant every leaf L of $\widetilde{\mathcal{F}}'$, and in L it is the reflection around $L \cap \widetilde{T}$.

Let a_i', b_i', c_i', d_i' be the ideal points of Z_i' in $S^1(F)$, so that the geodesics (a_i', b_i') and (c_i', d_i') are very close in F to η. Up to renaming the points, assume that (a_i', d_i') are very close in $F \cup S^1(F)$ to x_1 and (b_i', c_i') are very close to x_2. Now we do the symmetrization of Z_i' with respect to η. There is a reflection in the universal circle $\mathcal{V}$ of $\mathcal{F}'$ with respect to the ideal points of η (seen as points in $\mathcal{V}$). This reflection is induced by $\widetilde{f}|_F$. Denote this reflection map by $\xi\colon \mathcal{V} \to \mathcal{V}$. Replace Z_i' by an ideal quadrilateral Z_i in F with ideal points a_i, b_i, c_i, d_i as follows. Consider the pair b_i', c_i'. Both are very close to x_2 and are distinct from each other, so at least one is distinct from x_2. If $\xi(b_i') = c_i'$ (we are identifying $S^1(F)$ with $\mathcal{V}$), we choose $b_i = b_i'$ and $c_i = c_i'$. Otherwise one of b_i' or c_i' is farther from x_2 – use the reflection ξ to compare them if on opposite sides of x_2. Suppose the farthest point is b_i'. Then let $b_i = b_i'$ and choose $c_i = \xi(b_i)$. Do the same for the pair a_i', d_i'. The resulting quadrilateral with ideal points (a_i, b_i, c_i, d_i) is denoted by Z_i. It has ideal points still very close to x_1, x_2 respectively, so it is very thin in the same direction that Z_i' is. In particular, by construction the thinness in this direction goes to 0 as $i \to \infty$. But Z_i is less thin in this direction than Z_i' since we may have pushed a pair of endpoints slightly farther away from x_1, x_2 respectively.

For each i let $\mathcal{P}_i$ be the parallelepiped intersecting the same set of leaves of $\widetilde{\mathcal{F}}$ that $\mathcal{P}_i'$ intersects, but the top is now Z_i instead of Z_i'. Let X_i denote the bottoms of $\mathcal{P}_i$. The tops are very thin in the direction very close to η. Since we made the tops slightly thinner in the opposite direction – they are still very thick in that direction, but slightly less thick. This implies that the bottoms X_i still have thinness converging to 0 in the opposite direction. We explain a bit more: when moving the ideal quadrilaterals across leaves of $\widetilde{\mathcal{F}}'$ using the universal circle to move the ideal points, the following happens: the top quadrilateral Z_i' moves to X_i' which is very thin in the opposite direction. Since Z_i is slightly thinner than Z_i' in the opposite direction then Z_i moves to even thinner quadrilaterals X_i in the opposite direction. This is depicted in Figure 2.

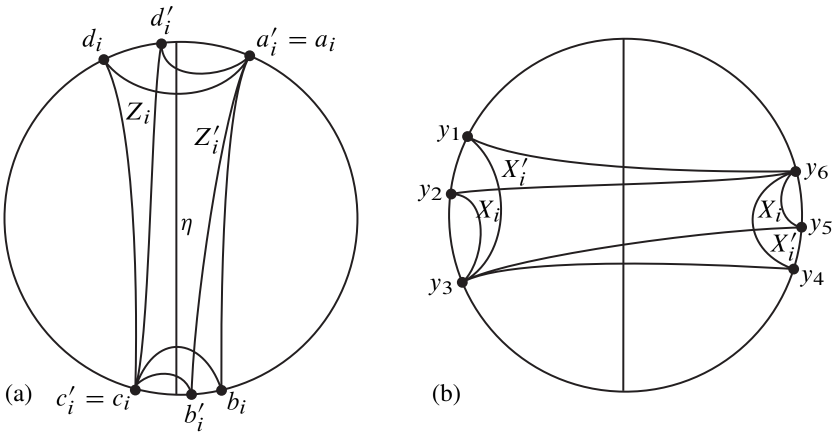

Figure 2. Part (a) depicts the situation in the top leaves of the parallelepipeds $\mathcal{P}_i', \mathcal{P}_i$, part (b) depicts the situation in the bottom leaves of $\mathcal{P}_i', \mathcal{P}_i$. The quadrilaterals on the top Z_i', Z_i are thin in the horizontal direction, and Z_i' is thinner in this direction than Z_i is. Therefore Z_i' is thicker than Z_i in the vertical direction. Pushing down, the Z_i', Z_i push to X_i', X_i respectively. The previous fact implies that X_i is thinner than X_i' in the vertical direction. Since X_i' is very thin in the vertical direction, then so is X_i. In part (b) the ideal points of X_i' are y_1, y_3, y_4, y_6, and the ideal points of X_i are y_2, y_3, y_5, y_6.

We use this sequence of parallelepipeds $\mathcal{P}_i$. The isometry $\widetilde{f}$ of $\widetilde{N}$ induces reflections on every leaf L of $\widetilde{\mathcal{F}}'$ around $L \cap \widetilde{T}$. Since the top quadrilateral Z_i of $\mathcal{P}_i$ is symmetric with respect to η, this implies the following: for any leaf E of $\widetilde{\mathcal{F}}$ intersecting $\mathcal{P}_i$, the quadrilateral $Q^i_E = \mathcal{P}_i \cap E$ in E is symmetric with respect to $\widetilde{T} \cap E$. In particular, the bottom X_i of $\mathcal{P}_i$ is also symmetric with respect to the intersection of $\widetilde{T}$ with that leaf. In addition any deck translate of $\mathcal{P}_i$ under an element of $\pi_1(T)$ is still symmetric with respect to $\widetilde{T}$.

We will use the setup of Proposition 3.1. We recall that I is a component of $S^1(F) - \{x_1, x_2\}$, which is associated with an open interval of $\mathcal{V}$. Recall that X_i are the bottoms of the parallelepipeds $\mathcal{P}_i$. Consider $X_i \cap \widetilde{T}$ which is a compact segment denoted by ζ_i of the foliation induced by $\mathcal{F}'$ in T. Orient the 1-dimensional foliation $\mathcal{F}' \cap T$. Let v_i be the positive endpoint of ζ_i with respect to the orientation of $\widetilde{\mathcal{F}}' \cap \widetilde{T}$. Up to subsequence, assume that $\pi(v_i)$ converges. Up to a modification of the bottoms X_i we can assume that $\pi(v_i)$ are always in the same local leaf of the foliation in T. So there are g_i in $\pi_1(T)$ with $g_i(v_i)$ in a fixed leaf of $\widetilde{\mathcal{F}}$ which we can assume is F, and suppose that $g_i(v_i)$ converges to a point v_0. In other words $g_i(v_i)$ are all in the curve $\eta = F \cap \widetilde{T}$.

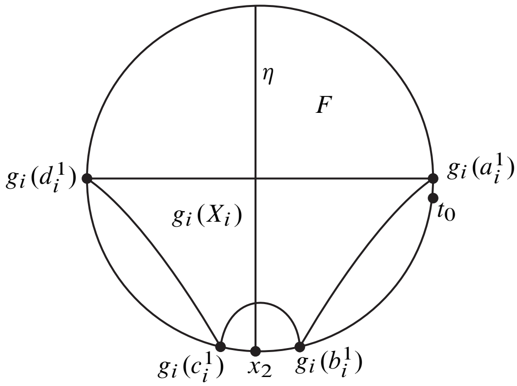

Figure 3. An example of action on the double $N = P \cup P'$.

We first consider the case that the lengths of $g_i(\zeta_i)$ converge to zero. Then the induced action of g_i on $\eta \cong I$ has big intervals which are contracted to a bounded subcompact interval. This brings us to a setup very similar to that of Lemma 3.2. Then a proof very similar to the proof of Lemma 3.2 produces a global fixed point of the action of $\pi_1(T)$ on I. As seen in the proof of Proposition 3.1 this leads to a contradiction.

Next we consider the case that the lengths of $g_i(\zeta_i)$ do not converge to zero. We will show that this is impossible as well. Up to a subsequence assume that lengths of $g_i(\zeta_i)$ are always bigger than a positive number ε_0. We refer to Figure 3. By symmetry the $g_i(\zeta_i)$ realize the minimum distance between the sides with ideal points $g_i(d_i), g_i(a_i)$ and ideal points $g_i(b_i), g_i(c_i)$.

This means that the sides $(g_i(d_i^1), g_i(a_i^1))$ and $(g_i(c_i^1), g^i(b_i^1))$ of $g^i(X_i)$ are not getting close to each other in F. Hence, (d_i^1, a_i^1) and (c_i^1, b_i^1) are not getting closer to each other in E_i as well. This is a contradiction, by construction of the parallelepipeds $\mathcal{P}_i$: the bottoms X_i are thin in the other direction.

This proves that T cannot be a leaf of $\mathcal{L}^u$. In fact, we proved the following: for any choice of distortion parallelepipeds $\mathcal{P}_i$ in $\widetilde{N}$ so that the tops Z_i converge to a geodesic ℓ in any leaf F of $\widetilde{\mathcal{F}}'$ then ℓ cannot cross any lift of a JSJ torus in N, nor be contained in any such lift.

Dealing with a subtle point. We obtained an immersed leafwise geodesic lamination $\mathcal{L}^u$ in N which is contained in P or P' and it is obtained by taking limits of distortion parallelepipeds $\mathcal{P}_i$. In fact, $\mathcal{L}^u$ is contained in the interior of P or P' and so induces an immersed leafwise geodesic lamination in M. Up to taking the image under the symmetry f, we may assume that it is contained in the interior of P. The subtle point is that the distortion parallelepipeds $\mathcal{P}_i$ are contained in $\widetilde{N}$, but do not necessarily generate distortion parallelepipeds in $\widetilde{M}$ which will generate $\mathcal{L}^u$ in M.

Why do we care about the distortion parallelepipeds since we already obtained the leafwise immersed laminations in M? The reason is that the distortion parallelepipeds are needed to obtain the following further properties of these laminations which are used later in the article: (1) the laminations are embedded, (2) the laminations have the properties of stable and unstable laminations. Hence, we need to have distortion parallelepipeds in $\widetilde{M}$.

We will adjust our construction of the distortion parallelepipeds. For notational reasons we will rename our parallelepipeds $\mathcal{P}'_i$. We will adjust the $\mathcal{P}'_i$ to obtain new parallelepipeds (to be denoted by $\mathcal{P}_i$) with the property we need. What we want is that the distortion parallelepipeds can be chosen contained in a fixed lift $\widetilde{P}$ of P to $\widetilde{N}$.

First we prove a preliminary fact. Consider the leaf F of $\widetilde{\mathcal{F}}'$ as in the beginning of the proof of the proposition. The intersection $F \cap \widetilde{P}$ is a hyperbolic surface with geodesic boundary. The tops Z'_i of the parallelepipeds $\mathcal{P}'_i$ converge to a geodesic ℓ in F. We proved that ℓ is contained in the interior of $F \cap \widetilde{P}$.

We claim that ℓ is not asymptotic to a geodesic g in F which is the intersection of $\widetilde{T}$ with F for some JSJ surface T of N and lift $\widetilde{T}$ to $\widetilde{N}$ (notice that here we are not taking a double cover of N to make it orientable). Suppose by way of contradiction that there is such ℓ. Since we are taking all $\pi_1(N)$ translates and closures of the limits this means that some geodesic $E \cap \widetilde{T}$ in $\widetilde{T}$ (E leaf of $\widetilde{\mathcal{F}}'$) is contained in all the limits. But we just proved that this it is impossible. This proves the claim.

In addition, any ideal point p of ℓ in F is accumulated in $F \cup S^1(F)$ by geodesics g_i which are intersections of lifts of JSJ surfaces with F. Otherwise a half plane in F does not intersecting such a lift, and hence taking deck translates and limits, it follows that a full leaf of $\widetilde{\mathcal{F}}$ does not intersect such a lift, contradiction.

Now we adjust the parallelepipeds $\mathcal{P}'_i$. Let ℓ have ideal points y_1, y_2. Let the tops of $\mathcal{P}'_i$ be Z'_i with ideal points a'_i, b'_i, c'_i, d'_i so that a'_i, d'_i are very close to y_1 and b'_i, c'_i very close to y_2. We will enlarge the ideal quadrilateral Z'_i still keeping it very thin in the direction close to the geodesic ℓ. Let I_i be the interval in $S^1(F)$ with ideal points a'_i, d'_i and very close to y_1. Recall that there are sequences of endpoints of $\widetilde{T}' \cap F$ for $\widetilde{T}'$ lifts of JSJ surfaces converging to y_1. Hence, for i big we can choose a_i arbitrarily close to a'_i, a_i not in the interior of I_i and a_i an ideal point of $\widetilde{T}' \cap F$ for some lift $\widetilde{T}'$ of a JSJ surface. Do the same for d'_i, b'_i, c'_i, producing a_i, b_i, c_i, d_i. These are distinct and define an ideal quadrilateral Z_i.

Let $\mathcal{P}_i$ the parallelepipeds intersecting the same leaves of $\widetilde{\mathcal{F}}'$ that $\mathcal{P}'_i$ does but defined by the ideal points a_i, b_i, c_i, d_i. Now we prove properties of $\mathcal{P}_i$. The geodesics (a_i, d_i) are very close to (a'_i, d'_i) and similarly (b_i, c_i) are very close to (b'_i, c'_i). So Z_i is very thin in the same direction that Z'_i is. Let X_i be the bottoms of the parallelepipeds $\mathcal{P}_i$. The Z_i are very thin in the ℓ direction, but less thin than Z'_i in this direction. This implies that the Z_i are a little bit thinner than the Z'_i in the other direction. This implies that the bottoms X_i of $\mathcal{P}_i$ are even thinner than the X'_i in the opposite direction. This argument has already been employed in the proof of this proposition.

We conclude that the $\mathcal{P}_i$ have thinness in the top (Z_i) in one direction converging to zero, and in the bottom (X_i) thinness in the other direction converging to zero. Hence, the $\mathcal{P}_i$ are distortion parallelepipeds which yield immersed leafwise geodesic laminations $\mathcal{L}^s$ and $\mathcal{L}^u$.

We finally prove the crucial property we want. By choice of the points a_i, b_i, c_i and d_i, they are ideal points of the hyperbolic surface $F \cap \widetilde{P}$. In particular, the geodesics $(a_i, b_i), (b_i, c_i), (c_i, d_i)$ and (d_i, a_i) are contained in $F \cap \widetilde{P}$. Hence, the tops Z_i are entirely contained in $\widetilde{P}$. Since the quadrilaterals in $\mathcal{P}_i$ are obtained by following the universal circle and so is $\widetilde{P}$, it follows that $\mathcal{P}_i \cap E$ is contained in $\widetilde{P}$ for any leaf E of $\widetilde{\mathcal{F}}'$ that it intersects. In particular, $\mathcal{P}_i$ is entirely contained in $\widetilde{P}$. This is the fact we wanted to prove.

Since $\mathcal{P}_i$ is entirely contained in $\widetilde{P}$ then we can think of them also as contained in $\widetilde{M}$. The quadrilaterals in $\mathcal{P}_i$ have the same leafwise geometry whether seen in $\widetilde{N}$ or in $\widetilde{M}$. It follows that $\mathcal{P}_i$ are distortion quadrilaterals in $\widetilde{M}$, and of course they generate the immersed leafwise geodesic laminations $\mathcal{L}^s$ and $\mathcal{L}^u$ in M. At this point we can completely forget N and consider all objects $\mathcal{P}_i, \mathcal{L}^s, \mathcal{L}^u$ in $\widetilde{M}$ or in M.

This finishes the proof of Proposition 3.3. ∎

By the above proposition there is an immersed lamination $\mathcal{L}^u$ contained in the interior of P, and an immersed lamination $\mathcal{L}^s$ contained in the interior of P.

Remark 3.4. This proof used the auxiliary manifold N, the double of P. We do not know how to prove Proposition 3.3 using only M. In other words, a priori some other construction using only distortion parallelepipeds in $\widetilde{M}$ (notice that this is in $\widetilde{M}$ and not in $\widetilde{N}$) could yield a lamination so that a component of ∂P is a leaf of this lamination. We strongly believe this is not possible, but as remarked, we are not able to prove this.

3.1. Analysis of the laminations $\mathcal{L}^u, \mathcal{L}^s$

Unfortunately, a priori the immersed laminations $\mathcal{L}^u, \mathcal{L}^s$ may have self intersections. This problem also happens in [12]. We will extensively use the analysis of [12]. The geometric arguments in [12] carry over to this article: in particular the facts that quadrilaterals are thin in one direction or the other, and what this implies to the laminations $\mathcal{L}^u$, $\mathcal{L}^s$, will carry over completely. The embedded laminations produced in [12] are the same as the ones we work with here. The difference is that some arguments in [12] produce tori that track some boundary leaves of the laminations. In [12] with the hypothesis of M being atoroidal, this implies that these tori bound solid tori. We do not have that here, and have to be more careful with the additional possibilities.

In [12, pp. 458–459], we showed that there are 3 options for the immersed leafwise geodesic laminations $\mathcal{L}^u$, $\mathcal{L}^s$:

Option A. No leaf of $\mathcal{L}^u$ transversely intersects another leaf of $\mathcal{L}^u$. In other words, $\mathcal{L}^u$ is an embedded leafwise geodesic lamination. There is an analogous statement for $\mathcal{L}^s$.

Option B. No leaf of $\mathcal{L}^u$ transversely intersects a leaf of $\mathcal{L}^s$.

Option C. There is a leaf of $\mathcal{L}^u$ transversely intersecting a leaf of $\mathcal{L}^s$.

In [12, pp. 459–464], it is proved that Option C implies Option A for both $\mathcal{L}^s$ and $\mathcal{L}^u$. This is explicitly stated in [12, Lemma 5.6]. The argument in [12, pp. 458–464] does not assume that M is atoroidal. So in order to obtain some embedded leafwise geodesic lamination, the problematic case is Option B. We will eventually rule out Option B.

If Option C happens, then $\mathcal{L}^u$, $\mathcal{L}^s$ are embedded. In this case let $\mathcal{L}^u_m$, $\mathcal{L}^s_m$ be minimal sublaminations of $\mathcal{L}^u$, $\mathcal{L}^s$ respectively. In this case these are leafwise quasi-geodesic laminations disjoint from ∂P.

We consider now the case that Option C does not happen. In particular, Option B happens. Under Option B we will, as in [12], use the convex hull operation.

Again, if $\mathcal{L}^u$ is embedded let $\mathcal{L}^u_m$ be a minimal sublamination of $\mathcal{L}^u$, and likewise $\mathcal{L}^s_m$ for $\mathcal{L}^s$. As before $\mathcal{L}^u_m$, $\mathcal{L}^s_m$ are disjoint from ∂P.

Fix a lift $\widetilde{P}$ of P.

Suppose now that one of the laminations $\mathcal{L}^u$, $\mathcal{L}^s$ is not embedded, without loss of generality suppose that $\mathcal{L}^u$ self intersects transversely. By hypothesis there are leaves L, L' of $\widetilde{\mathcal{L}}^u$ contained in $\widetilde{P}$ which intersect transversely. Fix one such leaf L'. We consider all leaves L of $\widetilde{\mathcal{L}}^u$ in $\widetilde{P}$ so that there is a sequence

$$L_0 = L', L_1, \ldots, L_k = L$$

of leaves of $\widetilde{\mathcal{L}}^u$ in $\widetilde{P}$ with L_i intersecting L_{i-1} transversely for all $1 \leq i \leq k$. This forms a subset $\mathcal{B}$ of leaves of $\widetilde{\mathcal{L}}^u$ in $\widetilde{P}$. We also think of $\mathcal{B}$ as a subset of $\widetilde{P}$.

We consider the convex hull envelope of $\mathcal{B}$. Let $\mathcal{CH}$ denote this convex hull envelope. Each component C of the boundary of $\mathcal{CH}$ intersects each leaf of $\widetilde{\mathcal{F}}$ in a geodesic. When one varies the leaf in $\widetilde{\mathcal{F}}$ then the intersection of C with that leaf of $\widetilde{\mathcal{F}}$ varies according to the universal circle of $\mathcal{F}$. That is the endpoints of $C \cap F$ are constant when seen in $\mathcal{U}$ as F varies in $\widetilde{\mathcal{F}}$. In addition there are no transverse self intersections when projecting C to M. This is because $\mathcal{CH}$ is the convex hull of these chains of consecutively intersecting leaves. Now take all $\pi_1(P)$ translates of all boundary component of $\mathcal{CH}$. Any two of these sets do not intersect transversely, and if they intersect they are exactly the same. So this projects to an embedded collection of surfaces in P. The closure of this is an embedded leafwise geodesic lamination, and we denote it by $\mathcal{G}^u$. Similarly, for $\mathcal{L}^s$ defining $\mathcal{G}^s$ if there are transverse self intersections of $\mathcal{L}^s$.

We stress that even though all leaves of $\mathcal{L}^u$ are in the interior of P, this is not a priori true for $\mathcal{G}^u$.

Let $\pi\colon \widetilde{M} \to M$ be the universal covering map.

Lemma 3.5. *$\mathcal{G}^u, \mathcal{G}^s$ are not contained in ∂P, and any leaf of $\mathcal{G}^u$ or $\mathcal{G}^s$ contained in ∂P is an isolated leaf.*

Proof. Suppose that T is a leaf of $\mathcal{G}^u$ which is a component of ∂P. Lift T to $\widetilde{T}$ in the fixed lift $\widetilde{P}$ of P. There is an embedded path α in a leaf F of $\widetilde{\mathcal{F}}$ from $\widetilde{T}$ until a point in a leaf in $\mathcal{B}$, hitting $\mathcal{B}$ only in this boundary point, which we denote by p. Let L_0 be this leaf of $\mathcal{B}$. The intersection $L_0 \cap F$ is a geodesic in F. Follow along this geodesic, starting at p. Then on one side of p along $L_0 \cap F$ the leaf L_0 eventually first hits another leaf L_1 which is in $\mathcal{B}$. We can continue on both sides, producing L_n, $n \in \mathbf{Z}$ leaves in $\mathcal{B}$. There is no leaf of $\mathcal{B}$ between this collection and $\widetilde{T}$. It follows that T is isolated in $\mathcal{G}^u$.

In addition, suppose that the only leaves in $\mathcal{G}^u$ are contained in ∂P. It follows that $\pi(\mathcal{B})$ completely fills P: the complementary regions of $\mathcal{B}$ in P are either simply connected, solid tori, solid Klein bottles, or they are peripheral. But since $\mathcal{L}^s$ does not intersect $\pi(\mathcal{B})$ then the leaves in $\mathcal{L}^s$ have to be peripheral. The only possibility is that they would be components of ∂P, but this is impossible.

This proves the lemma. ∎

Since $\mathcal{G}^u$ is not contained in ∂P, there are leaves of $\mathcal{G}^u$ contained in the interior of P. By the previous lemma, the closure of the union of these leaves is also contained in the interior of P. Therefore there is a minimal sublamination of $\mathcal{G}^u$ which is contained in the interior of P. Choose one such and denote it by $\mathcal{L}^u_m$. If necessary do the same for $\mathcal{L}^s$ generating $\mathcal{L}^s_m$.

Notice that if $\mathcal{G}^u$ has a compact leaf, then it is a torus or Klein bottle, since it has a foliation induced by intersection with leaves of $\widetilde{\mathcal{F}}$. But as P is atoroidal this leaf is homotopic into the boundary and has to be a component of ∂P. Therefore no leaf of $\mathcal{L}^u_m$, $\mathcal{L}^s_m$ is a compact leaf, and in particular it is not isolated.

We have a final property to prove. Notice that under Option B with $\mathcal{L}^u$ not an embedded lamination, it follows that every leaf of $\mathcal{G}^u$ is disjoint from $\pi(\mathcal{B})$, by definition. $\mathcal{B}$ is the set defined before in this discussion. However since every leaf L of $\mathcal{L}^u_m$ is not isolated, it is a limit of distinct leaves L_n of $\mathcal{G}^u$. Notice that all of these leaves are projections by π of the $\pi_1(P)$ saturation of the boundary of the convex hull envelope of $\mathcal{B}$. Hence, it follows that there are leaves E_n in $\pi(\mathcal{B})$ which converge to L as well. In particular, L is in $\mathcal{L}^u$. It follows that $\mathcal{L}^u_m$ is a subset of the leaves of $\mathcal{L}^u$, and likewise for $\mathcal{L}^s_m$. These subsets form embedded leafwise geodesic laminations.

Conclusion. Under any option (A, B or C), we obtain $\mathcal{L}^u_m$, $\mathcal{L}^s_m$ which are minimal embedded leafwise geodesic laminations contained in the interior of P. These are contained in $\mathcal{L}^u$, $\mathcal{L}^s$ respectively.

We will obtain properties of $\mathcal{L}^u_m$, $\mathcal{L}^s_m$. In fact, most of the properties are proved in [12]. A *crown* is a hyperbolic surface which is a half open annulus: its completion has one boundary component which is a closed geodesic. There are finitely many boundary components. The other boundary components are infinite geodesics, which are consecutively asymptotic; see [10].

Lemma 3.6. *The complementary regions of $\mathcal{L}^u_m$ (or $\mathcal{L}^s_m$) in P are either S^1 bundles over open finite sided ideal polygons (generating open solid tori or solid Klein bottles) or S^1 bundles over crowns, generating sets homeomorphic to torus* $\times[0,1)$ *or Klein bottle* $\times[0,1)$.

Proof. We do the proof for $\mathcal{L}^u_m$. First of all $\mathcal{L}^u_m$ is not a foliation in P, since it does not intersect ∂P. For each complementary region V of $\mathcal{L}^u_m$ in P, consider the boundary S of the set in V which is ε near $\mathcal{L}^s_m$, for some ε sufficiently small. In [12, Lemma 6.3], it is proved that each component of S is either a torus or a Klein bottle. If this complementary region is not peripheral, then the torus or Klein bottle is compressible and bounds a solid torus or solid Klein bottle. Proposition 6.1 of [12] further shows that V is a S^1 bundle over a finite sided ideal polygon. If the region is peripheral then S is isotopic to a component S' of the boundary. S' is either a torus or Klein bottle and V is homeomorphic to $S' \times [0,1)$. The other boundary components of V are in annular or Möbius band leaves of $\mathcal{L}^s_m$. As in [12, Proposition 6.1], there are finitely many of them, they are asymptotic, leading to V being an S^1 bundle over a crown surface. This finishes the proof of the lemma. ∎

Lemma 3.7. *$\mathcal{L}^u_m$, $\mathcal{L}^s_m$ are distinct and intersect transversely. The interior complementary regions of $A = \mathcal{L}^u_m \cup \mathcal{L}^s_m$ in P are either finite sided polygons (with compact completion) times* $\mathbb{R}$ *or an S^1 bundle over a finite sided polygon (with compact completion). A peripheral complementary region V of A has completion which is either a torus or Klein bottle times* $[0,1]$. *The boundary of V is made up of a component of ∂P and the union of compact annuli or Möbius bands contained in leaves of $\mathcal{L}^u_m$ or $\mathcal{L}^s_m$.*

Proof. The proof that they are distinct is exactly as in [12]. Let L' be a boundary leaf of (say) $\mathcal{L}^u_m$. This means it is isolated on one side. Then it is an annulus or Möbius band with $\pi_1(L')$ generated by a deck transformation γ. Let L be the lift of L' to $\widetilde{M}$

with $\gamma(L) = L$. This is analyzed in detail in [12, Lemma 6.6, pp. 471–473]. The pair of laminations $\mathcal{G}^m_-, \mathcal{G}^m_+$ corresponds to the pair to $\mathcal{L}^u_m, \mathcal{L}^s_m$. In [12, Lemma 6.6], the following is proved: Let $\theta(\gamma)$ be the action of γ on the universal circle $\mathcal{U}$ of $\mathcal{F}$, and suppose that γ is monotone increasing on the leaf space of $\widetilde{\mathcal{F}}$. Let I be the open interval in $\mathcal{U}$ determined by the ideal points of $L \cap F$ for some fixed leaf F of $\widetilde{\mathcal{F}}$, under the identification $S^1(F) \cong \mathcal{U}$, and so that I satisfies the following: there are leaves E_n in $\widetilde{\mathcal{L}}^u_m$ converging to L and with ideal points of $E_n \cap F$ in $\overline{I}$, again under the identification $S^1(F) \cong \mathcal{U}$. Recall that L is isolated on one side, but not on the other. This is because $\mathcal{L}^u_m$ is minimal, but not a compact leaf. Then [12, Proposition 6.6] proves that the action of $\theta(\gamma)$ in I is a contraction with a single fixed point. In the case of the lamination $\mathcal{L}^s_m$ the same proposition shows that $\theta(\gamma)$ acts as an expansion in I. These are contradictory and show that $\mathcal{L}^u_m$ and $\mathcal{L}^s_m$ are not the same lamination.

Since both $\mathcal{L}^u_m, \mathcal{L}^s_m$ are minimal it follows that they do not share any leaf. By the properties of the complementary regions of $\mathcal{L}^s_m$ and $\mathcal{L}^u_m$ (separately) it now follows that they have to intersect transversely. The components of the intersection of complementary regions of A with any leaf of $\mathcal{F}$ have to have compact completion. The description of the interior complementary regions of A is done in [12, Proposition 6.11]. The description of the peripheral components follows from the description of the peripheral complementary components of $\mathcal{L}^u_m$ and $\mathcal{L}^s_m$ in P separately, done in the previous lemma. ■

Remark 3.8. Lemma 3.7 needed embedded laminations as opposed to only immersed laminations. In particular, the embedded laminations have some leaves with non-trivial fundamental group, which was essential to the argument. A priori, an immersed lamination may have only simply connected leaves, so one could not apply the argument above.

Remark 3.9. Since $\mathcal{L}^u_m, \mathcal{L}^s_m$ intersect transversely and they are contained respectively in $\mathcal{L}^u, \mathcal{L}^s$, then the last pair intersects transversely. It follows that Option C occurs, and as a consequence, Option B does not occur.

4. One prong pseudo-Anosov flows and blow ups in atoroidal pieces

We generalize the notion of pseudo-Anosov flows to include one prongs:

Definition 4.1 (One prong pseudo-Anosov flows). A flow φ in a closed 3-manifold Q^3 is a one prong topological pseudo-Anosov flow if there are no point orbits of φ and orbits of φ are contained in a pair of (possibly singular) 2-dimensional foliation $\mathcal{E}^s, \mathcal{E}^u$ weak stable and weak unstable of φ, satisfying:

- All flow lines in a leaf of $\mathcal{E}^s$ are forward asymptotic. In the backwards direction, the orbits diverge from each other in the intrinsic metric of the 2-dimensional leaves. Similarly for $\mathcal{E}^u$ with the reversed direction.
- The (topological) singularities of $\mathcal{E}^s$, $\mathcal{E}^u$ are all of p-prong type, where p is a positive integer which can be equal to one. The singular locus is a finite union of periodic orbits of φ. The singular locus of $\mathcal{E}^s$ is the same as the singular locus of $\mathcal{E}^u$.
- The foliations $\mathcal{E}^s$, $\mathcal{E}^u$ are (topologically) transverse to each other and intersect exactly along the flow lines of φ.

Theorem 4.2. *Let $\mathcal{F}$ be a transversely oriented $\mathbb{R}$-covered foliation with hyperbolic leaves in a 3-manifold M. Suppose that there is a truly atoroidal piece P in the JSJ decomposition of M. Then there is a one prong pseudo-Anosov flow in a closed manifold P_* obtained from P by collapsing each boundary component of P to a circle.*

Proof. A part of this is done carefully in [12, Section 7], which itself just follows the constructions of Mosher [26, 27]. There is a problem with the collapsing near each component of the boundary of P which we will explain how to adjust.

In the previous section we constructed the embedded minimal leafwise geodesic laminations $\mathcal{L}^s_m$, $\mathcal{L}^u_m$ contained in the interior of P. Lemmas 3.6 and 3.7 give the properties of the laminations, including properties of the complementary regions. Let $A = \mathcal{L}^s_m \cup \mathcal{L}^u_m$. For each leaf F of $\mathcal{F}$ restricted to P we collapse every closure of a component of

$$F - (\mathcal{L}^s_m \cup \mathcal{L}^u_m)$$

not intersecting the boundary of P to a point. The laminations $\mathcal{L}^s_m$ and $\mathcal{L}^u_m$ collapse to 2-dimensional foliations $\mathcal{E}^s$, $\mathcal{E}^u$ in the collapsed set. Most of these closures of complementary regions in F are compact quadrilaterals. Some may have closures which are in the interior of P, and are finite sided polygons with compact closure having $2p$ sides. Here $p \geq 3$. The p boundary leaves of $\mathcal{L}^s_m$ associated with this complementary region collapse to a p-prong singularity of $\mathcal{E}^s$. Proposition 6.11 of [12] states that for every such complementary region of $\mathcal{L}^s_m$ there is also a complementary region of $\mathcal{L}^u_m$ which intersects the leaf F in a p-sided ideal polygon. So the same complementary region of A also generates a p-prong singularity of $\mathcal{E}^u$. There are finitely many such complementary regions of A.

But there is a problem with the peripheral complementary components: the leaves of $\mathcal{F}$ may not intersect them in sets which are compact. Since $\mathcal{L}^s_m$, $\mathcal{L}^u_m$ fill P, then the interior components of $P - A$ are either simply connected, solid tori or solid Klein bottles. A leaf F of $\mathcal{F}$ intersects the boundary of such a complementary component locally in a finite sided ideal polygon with compact closure in P. That is, if you look at a local leaf F intersecting the boundary of this complementary region, and go around the boundary, the intersection with F closes up and bounds a disk in F in the complementary component. But for a peripheral complementary component W of A in P, look at how a leaf F intersects the boundary of W contained in $\mathcal{L}^s_m \cup \mathcal{L}^u_m$: when you go around it may not close up. In fact, if you continue going around maybe it will be dense in this boundary component of W and the collapsing of points in F will be an awful topological space.

In order to deal with this we do the following for each peripheral complementary region W of A in P: Let α be a closed curve which is an intersection of a leaf L of $\mathcal{L}^s_m$ intersecting the boundary of W with a leaf E of $\mathcal{L}^u_m$, and α contained in the boundary of W. Notice that the boundary component of W in the interior of P is two sided. It implies that α is two sided in both L and E, and that α is an orientation preserving curve in L and in E, even if L, E are Möbius bands. Consider a local annulus C in L with one boundary in α and entering $P - W$. We have a local annulus and not a Möbius strip because α is orientation preserving in L. We remove W from P to produce Q (this is done for all such W). Each boundary component of Q is made up of annuli and Möbius bands contained in leaves of $\mathcal{L}^u$ and $\mathcal{L}^s$.

Now cut Q along C for each peripheral component of $P - A$. The curve α splits into two closed curves in the boundary of Q. Now do the collapsing along closures of intersections of F leaf of $\mathcal{F}$ with the complement of $A = \mathcal{L}^s_m \cup \mathcal{L}^u_m$.

The laminations $\mathcal{L}^s_m, \mathcal{L}^u_m$ project to foliations in the collapsed set. The intersection of the laminations $\mathcal{L}^s_m$ and $\mathcal{L}^u_m$ is a 1-dimensional foliation in Q. It is orientable because the intersection of $\mathcal{L}^s_m$ and $\mathcal{L}^u_m$ was transverse to the foliation $\mathcal{F}$ before collapsing. We orient the flow going in the positive direction transverse to the foliation.

The cutting operation along C produces two annuli C_1, C_2 intersecting in both boundary components of the collapsed set, one of which corresponds to α. Both C_1, C_2 have flows with one closed orbit (corresponding to α) and the others spiraling towards this one. Glue C_1 with C_2 so that flow lines glue to flow lines. The resulting object is a closed manifold P_* with two induced 2-dimensional foliations $\mathcal{E}^s, \mathcal{E}^u$. Notice that $\mathcal{F}$ may not induce a 2-dimensional foliation in P^* because of the difficulty we mentioned above. The intersection of $\mathcal{E}^s, \mathcal{E}^u$ is a 1-dimensional foliation of a flow.

Recapping: we first do the collapsing of disks in leaves of $\mathcal{F}$, then in the final step we do the gluing along flow lines.

As proved in [12, Proposition 7.2] orbits in the same leaf of $\mathcal{E}^s$ are forward asymptotic and in a leaf of $\mathcal{E}^u$ they are backwards asymptotic. Hence, this flow is a one prong pseudo-Anosov flow. The reason for the possible one prongs is because of the peripheral components of P: let W be a component of $P - A$ and consider the boundary component of W contained in the interior of P. This boundary component may be made up of a single annulus or Möbius band in $\mathcal{L}^s_m$ and a single annulus or Möbius band in $\mathcal{L}^u_m$. Then the collapsing will fold over the two sides of the annulus or

Möbius band leaves producing a one pronged leaf for each of $\mathcal{E}^s$ and $\mathcal{E}^u$ in the quotient. This situation is in fact extremely common: for example consider the suspension case where $\mathcal{F}$ is a fibration over the circle and the monodromy has a pseudo-Anosov component. The pseudo-Anosov map associated with this monodromy may have one prongs when collapsing a boundary component to a point. This is exactly the same that happens here.

This finishes the proof of the theorem. ∎

4.1. Blow ups of one pronged pseudo-Anosov flows

We now do blow ups of one pronged pseudo-Anosov flows following Fried's method in [16]. Consider a periodic orbit α of a one pronged pseudo-Anosov flow in a manifold Q. The return map of flow lines in a local a cross section satisfies the following: in a prong of the weak stable leaf of α, the return map is conjugated to $x \mapsto \pm\frac{1}{2}x$. In a weak unstable leaf it is conjugated to $x \mapsto \pm 2x$. This obviously is only topological conjugation. Fried [16] blew up the orbit α to its projective tangent bundle. The derivative of the return map above induces a flow in the blown up set. This extends the flow in $Q - \alpha$ to the blown up set with boundary.

Now return to the situation of Theorem 4.2. Given the construction of the laminations $\mathcal{L}^s_m, \mathcal{L}^u_m$ and the collapsed one prong pseudo-Anosov flow in P^* the next result follows immediately.

Corollary 4.3. *Under the hypothesis of Theorem 4.2 the blow up of the one pronged pseudo-Anosov flow in P_* produces a flow Φ in P which is transverse to $\mathcal{F}$ restricted to P and which is regulating for $\mathcal{F}$ restricted to P.*

As remarked above the singular foliations $\mathcal{E}^s, \mathcal{E}^u$ in P_* may have one prong singularities. The blown up objects in P are denoted by $\mathcal{E}^s_b, \mathcal{E}^u_b$. We follow Fried [16] who did the blow up in terms of the projective tangent bundle, keeping track of directions. Hence the prongs of leaves of $\mathcal{E}^s, \mathcal{E}^u$ blow up to prongs of $\mathcal{E}^s_b, \mathcal{E}^u_b$ respectively. It follows that $\mathcal{E}^s_b, \mathcal{E}^u_b$ are foliations in the interior of P, but they are not foliations in the boundary. The only points in the boundary that are parts of leaves are those that come from the blow up of the leaves in P_* that originated from collapsing the boundary of P. There are only finitely many of these prongs in $\mathcal{E}^s_b$ and in $\mathcal{E}^u_b$. For example if $\mathcal{E}^s$ has a one prong in γ, then the blow up of γ will be a torus or Klein bottle intersecting leaves of $\mathcal{E}^s_b$ in a single component. This is illustrated in Figure 4. The figure shows the 3 steps in the process: Figure (a) depicts the laminations $\mathcal{L}^s_m, \mathcal{L}^u_m$. A local cross section intersects each of these foliations in a Cantor set. Figure (b) shows the singular foliations $\mathcal{E}^s, \mathcal{E}^u$ in P_*. Figure (c) shows the blow up of figure (b). The blow up of the foliations $\mathcal{E}^s, \mathcal{E}^u$ is a foliation in the interior of P, but not in the boundary.

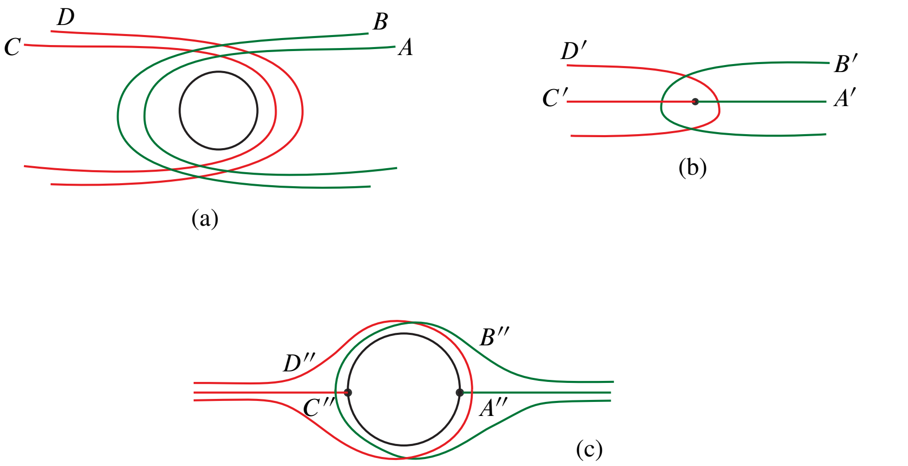

Figure 4. Figures of the objects in the construction. (a) Depicts leaves of the laminations $\mathcal{L}^s_m, \mathcal{L}^u_m$. The leaves A, C are extremal, there is a path from boundary of P to these leaves not intersecting any other leaf of the appropriate lamination. (b) The corresponding figure in P_*. The leaf A of the lamination collapses to a one prong leaf (this is the situation of a one prong), denoted by A' in this figure. The leaf C collapses to C', and A', C' intersect locally in the one prong depicted. (c) The blow up. The singularity blows up to a torus or Klein bottle. A' blows up to the leaf A'' intersecting the boundary in a circle, similarly C' blows to C''.

4.2. Density of regular periodic orbits of Φ in P.

We denote by Φ_* the one prong pseudo-Anosov flow in P_*. The closed orbits of Φ_* which are obtained by collapsing components of ∂P are called the *boundary collapsed orbits*. These are exactly the orbits that may be one prong orbits of the one prong pseudo-Anosov flow Φ_*.

Proposition 4.4. *The flow Φ_* is transitive, that is, the set of periodic orbits of Φ_* is dense. The same is true of the blow up flow Φ in P. In fact, the set of regular periodic orbits of Φ in P is dense in P.*

Proof. If necessary lift to a double cover so that P^* (and hence P is orientable).

We will use a result about pseudo-Anosov flows. However, the flow P_* may have one prongs and in general many results do not hold for one prong pseudo-Anosov flows. To get around that we will do Fried's surgery. More specifically do Fried's surgery on the boundary collapsed orbits to obtain p prong orbits with $p > 1$. Fried's surgery [16] has two steps: blow up the orbit using the action on the tangent bundle, then blow down using a new meridian. The blow up is the procedure to go from the one prong pseudo-Anosov flow Φ_* in P_* to the blown up one prong pseudo-Anosov flow Φ in P. The blow down is chosen by a new choice of a meridian, call it m. If the intersection number of m with the collection of the blow ups of (say) the stable prongs of Φ_* is p, then the resulting orbit is a p-prong orbit. One can always do this so that $p \geq 2$, resulting in a pseudo-Anosov flow Φ_2 in a closed manifold P_2. The manifold P_2 is obtained by Dehn surgery on P_* on the curves associated with the boundary collapsed orbits of Φ_*.

What is important is that the flows Φ_* in P_* and Φ_2 in P_2 are what is called *almost equivalent* [11]: P_* minus the boundary collapsed orbits is the same as P_2 minus the union of the Dehn surgery orbits; and the flows restricted to these sets have the same orbits. Hence, if the set of periodic orbits of Φ_2 in P_2 is dense in P_2 the same is true for the flow Φ_* in P_*.

We prove the result for Φ_2 in P_2. Suppose that Φ_2 is not transitive. Mosher [26] proved that there is an incompressible torus T transverse to the flow Φ_2 and which separates basic sets of Φ_2. This transverse torus does not intersect periodic orbits of Φ_2. The blow up produces a torus T' in P transverse to the blown up one prong pseudo-Anosov flow Φ in P. Since P is atoroidal then T' is isotopic to a boundary component of P. Projecting to P_2, it follows that the torus T bounds a solid torus B in P_2 containing a periodic orbit α associated with the blow down of the corresponding boundary component of P. This orbit is obtained by a Dehn surgery of a possible one prong orbit of Φ_* in P_*.

For simplicity assume that the flow is outgoing from B along T. Then the weak stable leaf of α cannot intersect T – since the flow is outgoing from B, and so this stable leaf is entirely contained in B. A lift of B to the universal cover $\widetilde{P}_2$ of P_2 is a solid tube with bounded cross section. The lift E of the weak stable leaf of α is a p-prong leaf which is properly embedded in $\widetilde{P}_2$ [18, 26]. This uses that Φ_2 is a pseudo-Anosov flow. This is not true in general if there are one prongs. This is why we did the surgery in P_* to obtain P_2 and a pseudo-Anosov flow Φ_2 in P_2. But E is contained in a solid torus with compact cross sections, so this is impossible.

We conclude that this is impossible. Hence the flow Φ_2 is transitive and so is the flow Φ_* in P_*. In particular, Φ is also transitive in P. Since there are finitely many possibly singular orbits of Φ in P, then the union of the regular periodic orbits is dense in P.

This finishes the proof of the proposition. ∎

Remark 4.5. After this proposition one may ask the following: there is a lot of freedom in the initial collapsing map from P to P_* (collapsing the laminations $\mathcal{L}^s_m, \mathcal{L}^u_m$ to the singular foliations $\mathcal{E}^s, \mathcal{E}^u$ respectively). Topologically the type of the singular foliations is determined by the new meridian, which determines the topological type of the collapsing. Hence why consider one prong pseudo-Anosov flows in P_* instead of always choosing a collapsing that yields a true pseudo-Anosov flow? This is a valid question. Here is one important reason to consider one prong pseudo-Anosov flows:

Suppose that in each component D of ∂P the foliation $\mathcal{F}$ induces a foliation by circles in D. For example this happens if the foliation in $\mathcal{F}$ is a foliation by compact surfaces, that is $\mathcal{F}$ is a fibration over the circle in P. Then the preferred collapsing is the one that collapses each circle of $\mathcal{F}_{|D}$ to a point. This yields a foliation in P_* which is transverse to the induced flow Φ_*, so Φ_* is a suspension flow. But clearly the suspension flow Φ_* may have one prongs. In this way the theory generalizes the theory of pseudo-Anosov homeomorphisms of compact surfaces with boundary: it is well known that pseudo-Anosov homeomorphisms with one prongs in the boundary are extremely common and cannot be disregarded. For example $\mathbb{S}^2 \times \mathbb{S}^1$ has a one prong pseudo-Anosov flow which is a suspension of a one prong pseudo-Anosov homeomorphism of $\mathbb{S}^2$. Obviously $\mathbb{S}^2$ does not admit pseudo-Anosov homeomorphisms, but admits one prong pseudo-Anosov homeomorphisms, for example there is one with exactly 4 singularities, all one prongs. The blow up of the suspension is a sphere minus 4 disks times $\mathbb{S}^1$. There are also many other situations where the foliation restricted to each boundary component is by circles, but the foliation in P may not even have compact leaves. Whenever this happens, the natural collapsing produces a foliation in P_* transverse to the flow Φ_*.

4.3. Regulating flows transverse to $\mathbb{R}$-covered foliations

Theorem 4.6. *Suppose that $\mathcal{F}$ is a Reebless, $\mathbb{R}$-covered, transversely oriented foliation. Suppose that M does not fiber over the circle with Anosov monodromy. Then $\mathcal{F}$ admits a transverse regulating flow. There is such a flow which is essentially flowing along Seifert fibers in the Seifert pieces, and the flow is the blow up one prong pseudo-Anosov flow in the atoroidal pieces.*

Proof. We first deal with the case that $\mathcal{F}$ admits a holonomy invariant transverse measure; see of [9, Definition 9.2.14] for a definition of such a measure.

Suppose in addition that $\mathcal{F}$ has a compact leaf (hence clearly generating a holonomy invariant transverse measure supported on it). Goodman and Shields [19] proved that the compact leaf is a fiber of a fibration of M over the circle. This uses that $\mathcal{F}$ is transversely orientable. Then we proved in [12, Lemma 2.5] that there is a suspension flow transverse to the foliation which is regulating for $\mathcal{F}$.

Suppose that $\mathcal{F}$ does not have compact leaves. Then $\mathcal{F}$ has a unique minimal set and one can blow down complementary regions, to produce a minimal foliation; see [12, Proposition 2.6]. Since the new foliation is minimal the holonomy invariant measure has full support. By a result of Tischler [38] (see also [9, Theorem 9.4.2]), it follows that M fibers over the circle. In addition the foliation can be approximated arbitrarily well by a fibration over the circle. Suspension flows to the fibrations are also regulating for the foliation $\mathcal{F}$; see [34]. This finishes the proof in this case.

From now on assume that there is no holonomy invariant transverse measure. Hence, there are no compact leaves. As explained above there is a unique minimal set and the foliation can be blown down to a minimal foliation. If the blown down foliation has a transverse regulating flow, then it induces a transverse regulating flow for the original foliation, so we assume from now on that $\mathscr{F}$ is minimal. By Candel's theorem [8], there is a metric in M making all leaves of $\mathscr{F}$ hyperbolic, and we assume such a metric.

If the JSJ decomposition of M is trivial then M is either Seifert fibered or atoroidal. Consider first the case that M is Seifert fibered. By Brittenham's theorem [3] the foliation $\mathscr{F}$ has a sublamination which is either vertical or horizontal. But $\mathscr{F}$ is minimal, hence $\mathscr{F}$ is either vertical or horizontal. Suppose first that $\mathscr{F}$ is vertical, and up to an isotopy of the Seifert fibration, we may assume that each leaf of $\mathscr{F}$ is a union of Seifert fibers. But this contradicts that leaves of $\mathscr{F}$ are hyperbolic. It follows that the sublamination of $\mathscr{F}$ is horizontal and then so is $\mathscr{F}$. Therefore we can assume that the Seifert fibration is transverse to the foliation. Hence it is orientable, so its lift to $\widetilde{M}$ shows that the flow generated by the Seifert fibration is regulating. This finishes the analysis when M is Seifert fibered. If on the other hand M is atoroidal, the result is proved by the construction in [12]. The transverse pseudo-Anosov flow is the content of the Main theorem of [12]. For the regulating property of the pseudo-Anosov flow in [12], see [12, 5th paragraph, p. 420].

Assume from now on that the JSJ decomposition of M is not trivial. First we deal with the case that there is an atoroidal piece in the JSJ decomposition which is not truly atoroidal. Then the JSJ decomposition cuts up M into a product, and M fibers over either the torus or the Klein bottle. If the monodromy is either periodic or reducible, then the manifold is a nilmanifold. It follows that there is a holonomy invariant transverse measure and this reduces to a previous case.

The last possibility when there is a non-truly atoroidal piece in the JSJ decomposition is that the monodromy is Anosov. It implies that the fiber is a torus. Since $\mathscr{F}$ is minimal, [30, Corollary 4.3] (see also [21, Theorem B.6]) implies that $\mathscr{F}$ is topologically equivalent to the weak stable foliation $\mathscr{G}$ of a suspension Anosov flow. Topologically equivalent means that there is a homeomorphism sending one foliation to the other. The C^2 hypothesis in [30] is only used to eliminate exceptional minimal sets, which we do not have here by minimality. The foliation $\mathscr{G}$ has an obvious transverse regulating flow: it is given by the horocycle or unstable foliation of the flow, which is transverse to the weak stable foliation and regulating for it. By the topological equivalence this induces a transverse regulating flow to the original foliation $\mathscr{F}$.

Finally, assume that the JSJ decomposition of M is not trivial and all the atoroidal pieces are truly atoroidal. Using the results of Section 2 we assume that all decomposing tori and Klein bottles of the JSJ decomposition are in good position with respect with the foliation $\mathcal{F}$. Let P be a piece of the JSJ decomposition. If P is Seifert, then the Seifert fibration provides a transverse flow regulating for $\mathcal{F}_{|P}$. Notice that $\mathcal{F}$ is transversely orientable, so the Seifert fibration in P is orientable, as seen above.

If P is atoroidal Corollary 4.3 provides a blow up of a one prong pseudo-Anosov flow in P which is transverse to $\mathcal{F}_{|P}$ and regulating.

Now all one has to do is to match the flows in between the pieces. Suppose that T is a torus or Klein bottle of the JSJ decomposition. On either side of T there are pieces P_1, P_2 of the JSJ decomposition with flows transverse to the foliations $\mathcal{F}_{|P_1}$ and $\mathcal{F}_{|P_2}$ respectively. Hence there are two flows in T which are regulating for $\mathcal{F}_T$.

We do an isotopy between these flows in T through flows transverse to $\mathcal{F}_{|T}$ and regulating for this foliation. We do the case where T is a torus, which is the more complicated case. Choose a basis for the homology of T. Consider the three foliations in T: $\mathcal{F}_{|T}$, and the two foliations by flow lines coming from P_1 and P_2. None of these foliations have Reeb components. Hence given one of these foliations in T then all the leaves have the same slope. The slope of $\mathcal{F}_{|T}$ defines a half line of slopes positively transverse to $\mathcal{F}_{|T}$. Both the flows induced in T are in this half space of positive slopes. Then one can isotope one to the other keeping it in this half space of positive slopes and keeping it regulating for $\mathcal{F}_{|T}$. Then enlarge T to $T \times [0, 1]$ and interpolate the flows from one side to the other.

This constructs a flow transverse to $\mathcal{F}$ and regulating for $\mathcal{F}$. This finishes the proof of the theorem. ■

5. Action of the fundamental group on the universal circle

Let $\mathcal{F}$ be a transversely orientable, Reebless, $\mathbb{R}$-covered foliation in a closed 3-manifold M so that its leaves are Gromov hyperbolic. We obtain some information about the action of deck transformations on the universal circle $\mathcal{U}$ of $\mathcal{F}$.

This builds up on several prior works: for example work on actions of lifts of homeomorphisms of closed surfaces on the circle at infinity (see an account in [1, Appendix]). The case of atoroidal manifolds, has already been worked out previously; see [13]. The case that M fibers over the circle with Anosov monodromy has also been worked out previously. Hence, we can assume that any atoroidal piece is truly atoroidal.

We first work out one specific example which is the direct consequence of the results of this article. First assume that M has a non-trivial JSJ decomposition. Let φ be a regulating flow as constructed in Theorem 4.6. We can assume that φ preserves the tori and Klein bottles of the JSJ decomposition. If there is an atoroidal piece P we may assume that φ restricted to P is a blow up of a one prong pseudo-Anosov flow and it is denoted by Φ.

Let $\widetilde{\varphi}$ the lift to $\widetilde{M}$. Given any two leaves E, F of $\widetilde{\mathcal{F}}$ define

$$\tau_{E,F}: E \to F, \quad \tau_{E,F}(x) = \widetilde{\varphi}_{\mathbb{R}}(x) \cap F.$$

Since φ is regulating for $\mathcal{F}$ the maps $\tau_{E,F}$ are always homeomorphisms. They clearly satisfy a cocycle condition. Let $\widetilde{T}$ be a lift to $\widetilde{M}$ of a torus or Klein bottle in the JSJ decomposition. Clearly $\tau_{E,F}(\widetilde{T} \cap E) = \widetilde{T} \cap F$.

If M is Seifert or atoroidal there is a transverse regulating flow, so this defines a map $\tau_{E,F}$ for any E, F.

Lemma 5.1. *For any E, F, the homeomorphism $\tau_{E,F}$ extends to a homeomorphism,*

$$\zeta_{E,F}: E \cup S^1(E) \to F \cup S^1(F).$$

In addition, for any x in $S^1(E)$, x and $\zeta_{E,F}(x)$ define the same point in the universal circle $\mathcal{U}$ of $\mathcal{F}$.

Proof. Consider first the case that M is Seifert. Then we choose the transverse flow Φ so that it is an isometry between the leaves in the universal cover, and it extends as a homeomorphism between the compactifications. The foliation is uniform and the map $\tau_{E,F}$ sends a point to a point boundedly near it, so a geodesic ray to a geodesic ray boundedly near it. Hence the ideal points in E, F project to the same point in the universal circle. If M is atoroidal, then [5,12] proved that there is a pseudo-Anosov Φ transverse to F and regulating. This is constructed using the universal circle, with the distortion quadrilaterals. In particular, a leaf L of $\mathcal{L}^s_m$, $\mathcal{L}^u_m$ satisfies that the ideal points of $L \cap E$ as E varies in $\widetilde{\mathcal{F}}$ define the same point in $\mathcal{U}$. This implies the result in this case.

Finally, suppose that M has a non-trivial JSJ decomposition, so there is at least one torus or Klein bottle JSJ cutting surface T. Here we need more information from [14]. We assume that all such T are in good position. Let $\widetilde{T}$ be a lift of T to $\widetilde{M}$. Proposition 4.4 of [14] shows that $\widetilde{T} \cap E$ as E varies in $\widetilde{\mathcal{F}}$ defines a constant pair of points in $\mathcal{U}$. Given F in $\widetilde{\mathcal{F}}$ let $\mathcal{G}_F$ be the lamination in F obtained by intersecting all lifts $\widetilde{T}$ of JSJ tori or Klein bottles T with F. It is a lamination by geodesics. Lemma 4.8 of [14] states given F in $\widetilde{\mathcal{F}}$ then the set of ideal points of $\mathcal{G}_F$ is dense in $S^1(F)$ and for any non-degenerate interval J of $S^1(F)$ there are leaves of $\mathcal{G}_F$ with both ideal points in J.

Now fix E, F leaves of $\widetilde{\mathcal{F}}$. We know that for any $\widetilde{T}$ lift of a JSJ torus or Klein bottle then $\tau_{E,F}(\widetilde{T} \cap E) = \widetilde{T} \cap F$. No two leaves of $\mathcal{G}_E$ share an ideal point. In addition the circular order in $S^1(E)$ induced by the ideal points of leaves of $\mathcal{G}_E$ is preserved by $\tau_{E,F}$: the circular order induced in $S^1(F)$ by $\tau_{E,F}(\widetilde{T})$ as $\widetilde{T}$ varies over the lifts is the same as the circular order induced in $S^1(E)$, when one identifies $S^1(E)$ with $S^1(F)$ using the universal circle.

We know that for any ideal point q in $S^1(E)$ either it is an ideal point of some leaf of $\mathcal{G}_E$ or is accumulated by ideal points of leaves of $\mathcal{G}_E$ (so that both endpoints converge to q). These facts imply that $\tau_{E,F}$ extends to a homeomorphism from $E \cup S^1(E)$ to $F \cup S^1(F)$. Since the homeomorphism satisfies that $q, \tau_{E,F}(q)$ project to the same point in $\mathcal{U}$ for any q ideal point of leaf of $\mathcal{G}_E$, it follows that this is true for all q in $S^1(E)$. This proves the lemma. ■

We fix a transverse, regulating flow φ as above. Let γ in $\pi_1(M)$ be a deck transformation. For any E leaf of $\widetilde{\mathcal{F}}$ define

$$h_E = \gamma \circ \tau_{E,\gamma^{-1}(E)}.$$

This is a homeomorphism from E to itself. By Lemma 5.1 this induces a homeomorphism h_∞ from $S^1(E)$ to itself. Recall that $\tau_{E,\gamma^{-1}(E)}$ induces the identity map in the universal circle level. It follows that under the identification of $\mathcal{U}$ with $S^1(E)$, then h_∞ is the representation of the action of γ on the universal circle $\mathcal{U}$.

Suppose that there is at least one atoroidal piece P. By assumption P is truly atoroidal. Recall the "singular foliations" $\mathcal{E}^s_b$, $\mathcal{E}^u_b$ in P (they are singular foliations in the interior of P). Consider the lift of these to a lift $\widetilde{P}$ of P to $\widetilde{M}$. They induce foliations in $\widetilde{P}$ so that intersected with any leaf E of $\widetilde{\mathcal{F}}$ they are foliations by quasigeodesics. Some are p-prong leaves, each prong is a quasigeodesic. The transverse flow $\widetilde{\varphi}$ exponentially expands length along the unstable leaves and contracts along stable leaves.

Let γ be an arbitrary element of $\pi_1(M)$ and $\rho(\gamma)$ the induced action on the universal circle $\mathcal{U}$ of $\mathcal{F}$.

Proposition 5.2. *Suppose that M has a truly atoroidal piece P. Let γ be a deck transformation associated with an interior periodic orbit of the regulating pseudo-Anosov flow in P. Then up to a finite power, $\rho(\gamma)$ has finitely many fixed points in $\mathcal{U}$, alternating between attracting and repelling. In case γ is associated with a regular orbit, then $\rho(\gamma)$ (up to power) has exactly 4 fixed points.*

Finally, if E is in $\widetilde{\mathcal{F}}$ and q is an ideal point in $S^1(E)$ associated with a fixed point of $\rho(\gamma)$, the following happens: there is a neighborhood basis of q in $S^1(E)$ defined by geodesics ℓ_i in E so that for any x in ℓ_i and y in $\gamma \circ \tau_{E,\gamma^{-1}(E)}(\ell_i)$, then $d_E(x,y) \to \infty$ if $i \to \infty$.

Proof. Fix a leaf E of $\widetilde{\mathcal{F}}$, we do the analysis in E. Let α be the periodic orbit associated with γ and $\widetilde{\alpha}$ the lift of α fixed by γ. Up to a power, γ fixes the stable and unstable prongs of $\widetilde{\alpha}$. Let $z = \widetilde{\alpha} \cap E$. Let $\widetilde{P}$ be the lift of P containing $\widetilde{\alpha}$. Then the intersections of the stable and unstable prongs of $\widetilde{\alpha}$ with E are quasigeodesic rays in E entirely contained in $\widetilde{P}$. These prongs are contained in leaves of $\widetilde{\mathcal{E}}^s_b$, $\widetilde{\mathcal{E}}^u_b$, respectively.

Fix one unstable prong and denote it by r, that is, r is contained in $\mathcal{E}^u_b(\widetilde{\alpha}) \cap E$. Fix a regular stable leaf ζ (that is, a leaf of $\widetilde{\mathcal{E}}^s_b \cap E$) intersecting r. Recall that $h_E = \gamma \circ \tau_{E,\gamma^{-1}(E)}$. Consider $h^i_E(\zeta)$. The flow φ preserves $\mathcal{E}^s_b, \mathcal{E}^u_b$, contracts length exponentially along the stables and expands along the unstables. Hence, h_E preserves the foliations by stables and unstables in $\widetilde{P} \cap E$. Then $h^i_E(\zeta)$ converges to the stable leaf of z when $i \to -\infty$ (if z is a p-prong it converges to a properly embedded real line in this leaf). Let this limit be ζ'. In addition $h^i_E(r)$ escapes compact sets in E when $i \to \infty$. Notice that for any i, $h^i_E(\zeta)$ is entirely contained in $\widetilde{P}$. In addition, $h^i_E(\zeta)$ are uniform quasigeodesics, independent of i. As $i \to \infty$ they escape in E and are nested so they converge to a single ideal point, which is the ideal point of r.

Let a_1, a_2 be the ideal points in $S^1(E)$ of ζ' and b_1 the ideal point of r. Let I be the interval in $S^1(E)$ with endpoints a_1, a_2 and containing b_1. The above shows that h_∞ fixes a_1, a_2, b_1 and acts as a contraction on the interior of I with single fixed point b_1. This proves the first assertion of the proposition.

We now consider the last statement of the proposition. Consider the geodesics which are obtained by pulling $h^i_E(\zeta)$ tight. These are leaves of $\widetilde{\mathcal{L}}^s_m \cap E$. These form a neighborhood basis of the ideal point b_1 of r in $E \cup S^1(E)$. These geodesics are also a uniform bounded distance in E from $h^i_E(\zeta)$ (for the same i). The same happens for their images under h_E and the corresponding geodesics in E.

The angle between leaves of $\widetilde{\mathcal{L}}^s_m \cap E$ and leaves of $\widetilde{\mathcal{L}}^u_m \cap E$ is uniformly bounded below. Let ℓ_i be the geodesic obtained by pulling $h^i_E(\zeta)$ tight. Let r_g be the geodesic obtained by pulling r tight. The action of h_E on r expands length exponentially along unstable leaves. This follows because φ expands length exponentially. The geodesics ℓ_i are a bounded distance from $h^i_E(\zeta)$ and so is r_g from r. It follows that the distance from $\ell_i \cap r_g$ to $\ell_{i+1} \cap r_g$ converges to infinity as $i \to \infty$. The angle condition implies that for any x in ℓ_i and y in ℓ_{i+1} then $d_E(x, y) \to \infty$ as $i \to \infty$. Since the Hausdorff distance between ℓ_{i+1} and $h_E(\ell_i)$ is uniformly bounded, we obtain the bound desired.

If the ideal point is an ideal point of a stable prong, then we use inverses instead in the above argument.

This finishes the proof of the proposition. ∎

Remark 5.3. The last condition in the proposition is what is called *super attracting* in [15]. The super attracting definition is particularly useful in the case that $\mathcal{F}$ is uniform. Since we do not use that here, we do not define it formally, and we refer the interested reader to [15]. The specific result of this proposition is used in [15] to help analyze partially hyperbolic diffeomorphisms homotopic to the identity in 3-manifolds, and to obtain geometric results about their invariant foliations.

5.1. Some general properties of the action of $\pi_1(M)$ on the universal circle $\mathcal{U}$

We are now ready to obtain more general information about the action $\rho(\gamma)$, where $\gamma \in \pi_1(M)$ on the universal circle $\mathcal{U}$. There are too many cases to enumerate in the statement of a single result. Instead we, little-by-little, describe each individual case. The foliation $\mathcal{F}$ satisfies the properties announced in the beginning of this section.

(1) Suppose that M is Seifert. As explained above we can assume that the Seifert fibration is transverse to $\mathcal{F}$, and we can put a metric so that flowing along Seifert fibers is a local isometry between leaves of $\mathcal{F}$. Any deck transformation preserves the Seifert fibration so induces an isometry on the quotient of $\widetilde{M}$ by the lift of the Seifert fibration. This quotient R is isometric to the hyperbolic plane and the ideal circle of this plane is canonically identified to the universal circle $\mathcal{U}$. Let γ be a deck transformation, so it induces an isometry of R.

- If the isometry is elliptic then γ is associated with a fiber of the Seifert fibration. Then a finite power of γ is the identity on R. If γ preserves orientation then the action of $\rho(\gamma)$ on $\mathcal{U}$ is either free or fixes every point. If γ reverses orientation on R, then $\rho(\gamma)$ has two fixed points on $\mathcal{U}$.
- If the isometry induced on R is hyperbolic there are exactly two fixed points of $\rho(\gamma)$ on $\mathcal{U}$.
- The isometry on R cannot be parabolic, because M quotient the Seifert fibration is a closed orbifold surface.

(2) Suppose that M is atoroidal. Then the results of [12] imply that there is a pseudo-Anosov flow Φ transverse to $\mathcal{F}$ and regulating for $\mathcal{F}$. The action of elements of $\pi_1(M)$ on $\mathcal{U}$ was determined by [13, Proposition 5.3]. If γ fixes 3 or more points of $\mathcal{U}$ then γ is associated with a periodic orbit of Φ and $\rho(\gamma)$ has a finite even number of fixed points on $\mathcal{U}$, which are alternatively attracting and repelling. If γ is not associated with a periodic orbit of Φ, then $\rho(\gamma)$ has exactly two fixed points on $\mathcal{U}$, one attracting, one repelling. Finally, it could be that γ is associated with an orbit of Φ, but permutes the local prongs. In this case $\rho(\gamma)$ acts freely on $\mathcal{U}$ or fixes 2 points in $\mathcal{U}$, but a power of $\rho(\gamma)$ fixes at least 4 points on $\mathcal{U}$.

(3) Finally, suppose that the JSJ decomposition of M is non-trivial. As in Section 2, we put the JSJ tori and Klein bottles in good position with respect to the foliation $\mathcal{F}$. Let $\mathcal{T}$ be the JSJ tree of M: the vertices are lifts of pieces of the JSJ decomposition of M, an edge is a lift of a torus or Klein bottle of the JSJ decomposition which connects two lifts of pieces of the JSJ decomposition. This tree has a more or less canonical embedding into any leaf F of $\widetilde{\mathcal{F}}$ (modulo moving vertices in complementary regions, and isotoping edges) preserving the ordering; see details in [14, Section 4]. The universal circle is canonically homeomorphic to a quotient of the set of ends of this tree; see [14].

There are many possibilities.

(3.A) If γ acts freely on the tree $\mathcal{T}$, then γ does not fix any lift of a piece. Then $\rho(\gamma)$ has exactly two fixed points on $\mathcal{U}$.

(3.B) Suppose that γ fixes a lift $\widetilde{P}$ of a Seifert piece P. As in case (1) above γ induces an isometry of the quotient of $\widetilde{P}$ by the lift of the Seifert fibration, which is an isometry of a surface embedded in the hyperbolic plane, but with infinitely many geodesic boundaries. As in case (1), γ could be elliptic, with $\rho(\gamma)$ either not fixing any point on $\mathcal{U}$ or $\rho(\gamma)$ fixing at least a Cantor set of points in $\mathcal{U}$, or exactly 2 points on $\mathcal{U}$ (if $\rho(\gamma)$ reverses orientation); see a similar analysis in the appendix of [1]. If the action of γ on the quotient surface is hyperbolic then $\rho(\gamma)$ fixes exactly 2 points on $\mathcal{U}$.

(3.C) Finally, if P is atoroidal, look at the action of γ on the leaf spaces of the lifts of $\mathcal{E}^s_b, \mathcal{E}^u_b$ as constructed in Section 4. If these actions on the leaf spaces are free, then γ acts as a translation on these leaf spaces and $\rho(\gamma)$ fixes exactly two points on $\mathcal{U}$, one attracting, one repelling. If γ fixes some leaf of $\widetilde{\mathcal{E}}^s_b$ then it is associated with a periodic orbit of the blown up one prong pseudo-Anosov flow in P. If the orbit is in the interior of P (not peripheral), then a power of $\rho(\gamma)$ fixes finitely many (≥ 4) points in $\mathcal{U}$, which are alternatively attracting and repelling. If the associated orbit is peripheral, then $\rho(\gamma)$ fixes infinitely many points on $\mathcal{U}$. These properties are obtained in Proposition 5.2.

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Sergio R. Fenley
Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA;
fenley@math.fsu.edu