

Assouad–Nagata dimension and gap for ordered metric spaces

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Abstract. We prove that all spaces of finite Assouad–Nagata dimension admit a good order for the travelling salesman problem, and provide sufficient conditions under which the converse is true. We formulate a conjectural characterization of spaces of finite AN-dimension, which would yield a gap statement for the efficiency of orders on metric spaces. Under the assumption of doubling, we prove a stronger gap phenomenon about all orders on a given metric space.

1. Introduction

Given a metric space (M, d) , we consider a finite subset X of M . We denote by $l_{\text{opt}}(X)$ the minimal length of a path which visits all points of X . Now assume that T is a total order on M . For a finite subset $X \subset M$ we consider the restriction of the order T on X , and enumerate the points of X accordingly:

$$x_1 \leq_T x_2 \leq_T x_3 \leq_T \cdots \leq_T x_k,$$

where $k = \#X$. Here and in the sequel $\#X$ denotes the cardinality of the set X . We denote by $l_T(X)$ the length of the corresponding path

$$l_T(X) := d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k).$$

Given an ordered metric space $M = (M, d, T)$ containing at least two points and $k \geq 1$, we define the *order ratio function*

$$\text{OR}_{M,T}(k) := \sup_{X \subset M \mid 2 \leq \#X \leq k+1} \frac{l_T(X)}{l_{\text{opt}}(X)}.$$

See Definition 2.1 and Section 2 for more on this definition.

We say that a metric space M is *uniformly discrete*, if there exists $\delta > 0$ such that for all pairs of points $x \neq y$ the distance between x and y is at least δ .

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The travelling salesman problem aims to construct a cycle of minimal total length that visits each of the k given points. Bartholdi and Platzman introduced the idea to order all points of a metric space and then, given a k -point subset, visit its points in the corresponding order [3, 41]. Such an approach is called the *universal travelling salesman problem*. (One of the motivations of Bartholdi and Platzman was that this approach works fast for subsets of a 2-dimensional plane.) Their argument implies a logarithmic upper bound for the function $\text{OR}_{\mathbb{R}^2}(k)$.

For some metric spaces the function OR is even better. Our result in [15, Theorem B] shows that the best possible situation, when $\text{OR}(k)$ is bounded from above by a constant, holds true for uniformly discrete δ -hyperbolic spaces.

While in the original paper of Bartholdi and Platzman it was suggested that such an efficient behavior (with bounded ratio) holds for \mathbb{Z}^d , it is known that it is not the case (unless $d = 1$). The initial argument of Bartholdi and Platzman shows the existence of an order with $\text{OR}(k) \leq \text{Const} \ln k$. Orders with at most logarithmic OR also exist on spaces with doubling property, by the result of Jia et al. [28]. It seems natural to conjecture that this upper bound is optimal for \mathbb{Z}^d , $d \geq 2$. A question of existence of an order on \mathbb{Z}^2 that violates a logarithmic lower bound is asked in [11]. Under the additional assumption that the order is hierarchical, this logarithmic lower bound (for a unit square, and hence for \mathbb{Z}^2) is proved in [14] and [13, Theorem 1, Section 3.3].

Observe that among finitely generated groups only those that are virtually nilpotent do satisfy the doubling property (see Section 3.1 for definitions and background). We show that many groups of exponential growth also admit a logarithmic upper bound for an appropriate choice of the order, as we explain below.

The notion of Assouad–Nagata dimension goes back to [37], and the term *Nagata dimension* is used in [2]. This notion provides a control of both local and global properties of the space. If the space is uniformly discrete, only large scales (and thus global properties) matter, and this notion coincides with the *linearly controlled metric dimension* (also called linearly controlled asymptotic dimension). For uniformly discrete spaces, the AN-dimension is a quasi-isometric invariant. We recall the definition of Assouad–Nagata dimension in Section 4. Here we mention that the following spaces have finite AN-dimension. Spaces with doubling property [32], wreath products of groups of linear growth with finite ones [8], polycyclic groups [24], trees (see, e.g., [43]), hyperbolic groups (and δ -hyperbolic spaces that are “doubling in the small” [32]), and more generally, graphs and groups admitting quasi-isometric imbeddings into finite products of trees (for imbedding of hyperbolic groups, see [9]), such as Coxeter groups [12] (and more generally for virtually special groups, see [22]). Groups relatively hyperbolic with respect to groups of finite Assouad–Nagata dimension also have finite Assouad–Nagata dimension [26]. Not necessarily finitely presented $C'(1/6)$ small cancellation groups have AN-dimension at most 2 [46].

In fact, the argument of [28] for $OR(k)$ in the case of doubling spaces can be adapted to prove a logarithmic upper bound more generally for spaces of finite Assouad–Nagata dimension. In [28] the authors study the notion of (σ, I) -partitioning schemes, the existence of which can be shown to be equivalent to the finiteness of Assouad–Nagata dimension. In the proof of the theorem below we argue directly in terms of the AN-dimension. The second claim of the theorem below provides not only the asymptotic bound on OR , but also discusses the value of the order breakpoint $Br(M, T)$, which is defined as the smallest integer s such that $OR_{M,T}(s) < s$ (Definition 2.2). For some basic examples of Br , see also [15], where this notion was introduced.

Theorem I (Theorem 4.4). *If M is a metric space of finite Assouad–Nagata dimension m with m -dimensional control function at most Kr , then there exists an order T such that for all $k \geq 2$:*

- (1) $OR_{M,T}(k) \leq C \ln k$, where a positive constant C can be chosen depending on m and K only.
- (2) $Br(M, T) \leq 2m + 2$. Moreover, the elongations of snakes on $2m + 3$ points are bounded by some constant depending on m and K only.

This theorem gives us upper estimation on invariants of M : $OR_M(k) = O(\log k)$ and $Br(M) \leq 2m + 2$, where

$$OR_M(k) := \inf_T OR_{M,T}(k).$$

The “snakes” mentioned in the second claim of the theorem are order-increasing sequences of points which oscillate between neighborhoods of two points (We discuss the notion of snakes in more detail in Section 2).

In view of Lemma 2.6, it is sufficient to prove the statement of Theorem 4.4 for finite metric spaces. We make a more general assumption that M is uniformly discrete. We choose an appropriate constant λ , defined by the linearity constant K for the control function (a possible choice is $\lambda = 4K$), and consider coverings from the definition of Assouad–Nagata dimension with $r = \lambda^n, n \in \mathbb{N}$. In Lemma 4.6 we modify this family of coverings to enforce a certain hierarchical structure on the sets of these coverings. This hierarchical structure guarantees that the sets of the coverings satisfy the assumption of Lemma 2.7, which provides a sufficient condition for the existence of an order, such that given sets on some space are “convex” with respect to this order. We discuss this notion at the end of Section 2. In Lemma 4.7 we show that such orders satisfy an upper bound for the order ratio function in the claim of the theorem.

The worst possible case for solving the universal travelling salesman problem are spaces with linear $OR(k)$. An example of a sequence of finite graphs with linear $OR(k)$ is constructed in Gorodezky et al. [17], see also Bhalgat et al. [5] who show

that a sequence of Ramanujan graphs of large girth and of bounded diameter-by-girth ratio has this property. The above mentioned paper considered both the question of the dependence of the number of required points k and the dependence on the cardinality of a finite graph n . As we have mentioned, in this paper we consider the dependence on k , the question that makes both sense for finite and infinite spaces.

Since a result of Osajda [39] allows us to imbed subsequences of graphs with large girth into Cayley graphs, combining his result with that of [17] one can conclude that there exist groups with linear $\text{OR}(k)$. While the above mentioned argument uses both a remarkable construction of Ramanujan graphs of large girth and a recent graphical small cancellation technique, we prove that there is a large class of metric spaces and sequences of metric spaces with infinite order breakpoint (and thus claiming that $\text{OR}(k) = k$, not only that this function is linear). Easy examples of groups of this kind can be obtained from Theorem III.

Before stating this theorem, we formulate Theorem II, the first claim of which gives a sufficient spectral condition for infinite Br. Informally speaking, infinite Br means that whatever orders we choose on vertices of our graphs, there are arbitrary large subsets of vertices on which this order is extremely far from optimal for the travelling salesman problem. The second claim of Theorem II provides additional information about snakes for a sequence of expander graphs. In this theorem we use the notion of the order ratio function and the order breakpoint not only for a given space, but also for a sequence of graphs, which can be defined analogously, see the beginning of Section 5.

Theorem II (Theorem 5.1 (1), (3)). *Let Γ_i be a sequence of finite graphs of degree $d_i \geq 3$ on n_i vertices. Let T_i be an order on Γ_i .*

(1) *Assume that the normalized spectral gap $\delta_i = (\lambda_1^{\Gamma_i} - \lambda_2^{\Gamma_i})/d_i$ satisfies*

$$1/\delta_i = o\left(\frac{\log_{d_i} n_i}{\ln \log_{d_i} n_i}\right).$$

Then the order breakpoint of the sequence (Γ_i, T_i) is infinite.

(2) *For a sequence of bounded degree expander graphs the following holds. If $d_i = d \geq 3$ and $\delta = \inf_i \delta_i > 0$, then for each k the graphs (Γ_i, T_i) admit snakes on k points of bounded width C_k and of length at least $\log_{d-1} n_i - C'_k$, for some $C_k, C'_k > 0$.*

In Theorem II above we have formulated claims (1) and (3) of Theorem 5.1. In claim (2) of Theorem 5.1 we will also give an estimate on possible length and width of a snake, in terms of a spectral gap of a graph.

In general, if the decay of δ_i is quick, one cannot understand whether Br is infinite or not, given a sequence of graphs and knowing their cardinality n_i , degree d_i and

spectral gap δ_i , see Remark 5.8. For Claim (2) of the theorem, the assumption of expansion cannot be weakened if we want a criterion in terms of n_i , d_i and δ_i , see Remark 5.9. One can ask whether $o(\log_{d_i}^2(n_i))$ (which is a sufficient condition for infinite AN-dimension of bounded degree graphs by [27], see Remark 5.4) is sufficient for infiniteness of Br. If $1/\delta_i \sim (\log_{d_i} n_i)^2$, then Br can be finite (see Remark 5.5). Since a sequence of expander graphs can have a diameter close to $\log_{d-1} n_i$, the lower bound on the length of the snake in the claim (2) cannot be significantly improved (see also Remark 5.7).

In Section 6 we provide another sufficient condition of a different nature for infinite Br. It can be deduced from Lusternik–Schnirelmann theorem that any order on an ε -net of a sphere S^k admits snakes on $k + 2$ points, alternating between ε -neighborhoods of antipodal points (see Lemma 6.1). Combining it with the control of order ratio function for weak imbeddings of cubes we get

Theorem III (Corollary 6.6). *If a metric space M weakly contains arbitrarily large cubes of dimension d , then for any order T on M it holds*

$$\text{OR}_{M,T}(d) = d.$$

In particular, if a metric space M weakly contains a sequence of arbitrarily large cubes, then for any order T on M the order breakpoint of (M, T) is infinite.

We recall again that the informal meaning of infinite order breakpoint is that whatever order we choose on M , this order behaves extremely badly (for the travelling salesman problem) on some arbitrary large subsets.

In Section 6 we will define for metric spaces the property of containing a sequence of arbitrarily large cubes. Here we mention that the class of such spaces includes spaces admitting uniform imbeddings of \mathbb{Z}^d (or \mathbb{Z}_+^d) for all d . In particular, this condition holds for any finitely generated group G that contains the direct sum \mathbb{Z}^∞ as a subgroup. These include many classes of solvable groups and many other classes of amenable groups. Here we mention that Grigorchuk groups (and moreover all known constructions of groups of intermediate growth) admit uniform imbeddings of \mathbb{Z}_+^d , for all d . These also include many examples of not amenable groups, as well as some groups such as Thompson group (where famous amenability question remains open). Further examples of spaces that weakly contain sequences of arbitrarily large cubes are $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$ and more generally $B \wr A$, where B is an infinite group of not linear growth and A is any finite or infinite group containing at least two elements (this statement is inspired by the argument in [8], see Lemma 6.7 where we study weak imbeddings of cubes in wreath products). We also mention that imbeddings of cubes appear naturally as lower estimates for AN-dimension in various classes of groups and spaces, see [23, 45].

In view of Theorems II and III mentioned above we ask the following question.

Question. Let M be a metric space of infinite Assouad–Nagata dimension. Is it true that the order breakpoint of M is infinite?

We recall that there are various known examples of groups of finite asymptotic dimension which have infinite AN-dimension (see Nowak [38] and Brodskiy–Dydak–Lang [8]). As we have already mentioned, some of them satisfy the assumption of our Theorem III. In view of Theorem I, if the answer to the question is positive, this would provide an equivalent characterization of spaces of finite AN-dimension. Taking into account the above mentioned examples, for a question of a possible characterization it is essential to speak about AN-dimension and not about asymptotic dimension.

Observe also that if the answer to the above mentioned question is positive, in view of Theorem 4.4 this would provide a positive answer to the following

Question (Gap problem for existence of orders). Let M be a metric space. Is it true that either for any order T on M and all $k \geq 1$, it holds that $\text{OR}_{M,T}(k) = k$ or there exists an order T such that for all $k \geq 2$ it holds $\text{OR}_{M,T}(k) \leq \text{Const} \ln k$?

Among classes of spaces where we obtain a positive answer to this question are Cayley graphs of wreath products of groups (see Corollary 6.8).

Given a metric space, one can formulate a stronger Gap problem, which describes the behavior of all orders (rather than searches an order on the space). Our next result below solves this problem for spaces with doubling property. As we have already mentioned, the argument of Bartholdi and Platzman for Euclidean plane (and generalizations of their argument for the spaces with doubling property) provides orders with logarithmic upper bound for the function $\text{OR}(k)$. This is in contrast with the lexicographic order on \mathbb{R}^2 ($(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$), where it is easy to see that $\text{OR}(s) = s$ for all s . Our theorem below shows that any order on a space with doubling property (in particular, any order on \mathbb{R}^d) satisfies the same dichotomy.

Theorem IV (Gap for order ratio functions on spaces with doubling property (Theorem 3.2)). *Let M be a metric space with doubling property and T be an order on M . Then either for all s it holds*

$$\text{OR}_{M,T}(s) = s,$$

or there exists C (depending only on the doubling constant of M , s and ε such that $\text{OR}_{M,T}(s) \leq s - \varepsilon$) such that for all $k \geq 1$,

$$\text{OR}_{M,T}(k) \leq C \ln k.$$

2. Preliminaries and basic properties

Below we recall some basic properties of the order ratio function and order break-point, discussed in [15].

Definition 2.1 (Order ratio function). Given an ordered metric space $M = (M, d, T)$ containing at least two points and $k \geq 1$, we define the *order ratio function*

$$\text{OR}_{M,T}(k) := \sup_{X \subset M \mid 2 \leq \#X \leq k+1} \frac{l_T(X)}{l_{\text{opt}}(X)}.$$

If M consists of a single point, then the supremum in the definition above is taken over an empty set. We use in this case the convention that $\text{OR}_{M,T}(k) = 1$ for all $k \geq 1$.

Given an (unordered) metric space (M, d) , we also define the *order ratio function* as

$$\text{OR}_M(k) := \inf_T \text{OR}_{M,T}(k).$$

Denote by L the diameter of X . It is clear that $L \leq l_{\text{opt}}(X) \leq l_T(X)$ and $l_T(X) \leq L(\#X - 1)$. Hence, $1 \leq \text{OR}_{M,T}(k) \leq k$ for any M, T and $k \geq 1$.

Definition 2.2 (Order breakpoint). Let M be a metric space, containing at least two points, and let T be an order on M . We say that the order breakpoint $\text{Br}(M, T) = s$ if s is the smallest integer such that $\text{OR}_{M,T}(s) < s$. If such s does not exist, we say that $\text{Br}(M, T) = \infty$. A possible convention for a one-point space M is to define $\text{Br}(M, T) = 1$.

Given an (unordered) metric space M , we define $\text{Br}(M)$ as the minimum over all orders T on M :

$$\text{Br}(M) = \min_T \text{Br}(M, T).$$

It is clear that for any $k > \text{Br}(M, T)$ it holds $\text{OR}_{M,T}(k) < k$.

The property $\text{Br}_{M,T} \geq k$ means that on some k -point subsets of M the order T behaves extremely non-optimal as a universal order for the universal travelling salesman problem.

We will give an idea of such non-optimal subsets. Consider a sequence of s points:

$$x_1 <_T x_2 <_T \cdots <_T x_s,$$

let a be the diameter of this set and let b be the maximal distance $d(x_i, x_j)$, where i and j are of the same parity. Then a is called the *length* of the *snake* (x_i) , b is called its *width* and the ratio a/b is the *elongation* of the snake (x_i) . We say that elongation is ∞ if $b = 0$.

If a set X consists of $k + 1$ points and the value of $l_T(X)$ is close to $kl_{\text{opt}}(X)$, then it can be shown that any two points with indices of the same parity are relatively close

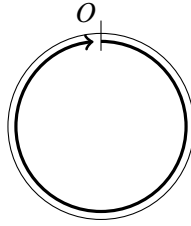


Figure 1. A clockwise order on the circle.

to each other, compared with the diameter of X . Hence, one can show the following lemma.

Lemma 2.3 ([15, Lemma 2.11]). *Let (M, T) be an ordered metric space. Given $s \geq 2$, we have $\text{OR}_{M,T}(s) = s$ if and only if in (M, T) there exist snakes of arbitrary large elongation on $s + 1$ points.*

If we allow a metric to assume the value ∞ , then a natural way to define OR is to consider the ratio $l_T(X)/l_{\text{opt}}(X)$ for subsets X with finite diameter. In this setting we can speak about OR and Br of disjoint unions of metric spaces.

We mention a simple example (which we discussed already in [15, Lemma 3.1]) when the metric space M is a circle S^1 with its inner metric (Figure 1). It is not difficult to show, that $\text{OR}_{S^1}(k) = 2$ for all $k \geq 2$. A natural order (a clockwise order) provides the estimation $\text{OR}_{S^1}(k) \leq 2$, and the lower estimation follows from the following lemma.

Lemma 2.4. *For any order T of S^1 and any $\varepsilon > 0$ there exist two antipodal points x, y of the circle and a snake $z_1 <_T z_2 <_T z_3$ on 3 points such that z_1 and z_3 belong to the ε -neighborhood of x and z_2 is in the ε -neighborhood of y .*

Proof. We call a point $x \in S^1$ T -small if $x <_T \bar{x}$, where \bar{x} denotes the antipodal point to x . Otherwise, we call it T -large. Since (S^1, T) contains both T -small and T -large points, we can find two points s_1, s_2 of different types that are ε -close to each other. Assume $s_1 <_T \bar{s}_1$ and $s_2 >_T \bar{s}_2$. If $\bar{s}_1 <_T s_2$, we take the snake $s_1 <_T \bar{s}_1 <_T s_2$. Otherwise, we take the snake $\bar{s}_2 <_T s_2 <_T \bar{s}_1$. ■

We recall the notion of quasi-isometric imbedding. Given metric spaces N and M , a map α from N to M is a *quasi-isometric imbedding* if there exist C_1, C_2 such that for any $x_1, x_2 \in N$ it holds

$$\frac{1}{C_1}(d_N(x_1, x_2)) - C_2 \leq d_M(\alpha(x_1), \alpha(x_2)) \leq C_1(d_N(x_1, x_2)) + C_2.$$

If M is at bounded distance from $\alpha(N)$, this map is called a *quasi-isometry*, and the spaces X and Y are called *quasi-isometric*. If α is bijective and $C_2 = 0$, then the spaces are said to be *bi-Lipschitz equivalent*.

We also recall a weaker condition of uniform imbeddings. Given metric spaces N and M , a map α from N to M is a *uniform imbedding* (also called *coarse imbedding*) if there exist two non-decreasing functions $\rho_1, \rho_2: [0, +\infty) \rightarrow [0, +\infty)$, with $\lim_{r \rightarrow \infty} \rho_1(r) = \infty$, such that

$$\rho_1(d_X(x_1, x_2)) \leq d_M(\alpha(x_1), \alpha(x_2)) \leq \rho_2(d_X(x_1, x_2)).$$

Given $\varepsilon, \delta > 0$, a subset U in a metric set M is said to be an (ε, δ) -*net* if any point of M is at distance at most ε from U and the distance between any two distinct points of U is at least δ . It is easy to show that for any metric space M and any $\varepsilon > 0$ there exists an $(\varepsilon, \varepsilon)$ -net of M .

Definition 2.5. We say that two functions $f_1, f_2(r): \mathbb{N} \rightarrow \mathbb{R}$ are equivalent up to a multiplicative constant if there exists $K_1, K_2 > 0$ such that $f_1(r) \leq K_1 f_2(r)$ and $f_2(r) \leq K_2 f_1(r)$ for all $r \geq 1$.

It is easy to observe ([15, Lemma 2.9]) that if two metric spaces M and N are quasi-isometric, M' is an (ε, δ) -net of M and N' is an (ε, δ) -net of N , then the order ratio functions $OR_{M'}$ and $OR_{N'}$ are equivalent up to a multiplicative constant. In particular, the asymptotic class of OR is a quasi-isometric invariant of uniformly discrete metric spaces. Moreover, if a metric space N is quasi-isometrically imbedded into a space M , then OR_M is asymptotically larger or equal to OR_N . We mention in this context that given (a not necessary injective) map $\phi: N \rightarrow M$ and an order T_M on M , one can construct an order T_N on N such that $\phi(x) <_{T_M} \phi(y)$ always implies $x <_{T_N} y$. We call any such order a *pullback of T_M* .

While the asymptotic class of the order ratio function is invariant under quasi-isometries, specific values of OR can change under quasi-isometries, as can be easily seen already from finite space examples. But for any s the equality $OR(s) = s$ is preserved under quasi-isometries of uniformly discrete spaces ([15, Lemma 2.12]). To show this one can consider a pullback and argue that snakes of large elongation map under quasi-isometries to snakes of large elongation. Moreover, a uniformly discrete metric space cannot be quasi-isometrically imbedded to a metric space with smaller Br.

From the definition it is clear that if M has at least two points then $Br(M, T) \geq 2$ for any order T . A result of M. Kapovich [30] about (R, ε) -tripods can be used to characterize uniformly discrete metric spaces with $Br \leq 2$ (see [15, Lemma 4.1]): such spaces are either bounded or quasi-isometric to a ray or to a line. It is not difficult to see that virtually free groups have $Br \leq 3$, and [15, Theorem A] shows the converse.

Let G be a finitely generated group. Then G admits an order T with $\text{Br}(G, T) \leq 3$ if and only if G is virtually free.

While the property $\text{OR}_M(s) = s$ is not preserved under uniform mappings of uniformly discrete spaces, there is a similar property which is inherited by such imbeddings. Below we recall this property, studied in [15]. We say that an ordered metric space (M, T) admits a *sequence of snakes of bounded width* on s points if there is a sequence of snakes on s points of uniformly bounded width and with diameters tending to infinity. It is not difficult to see that if N admits a uniform imbedding to a metric space M and if N admits a sequence of snakes of bounded width on s points for any order on this space, then the same happens for M ; and in particular $\text{OR}_M(s - 1) = s - 1$ (see [15, Lemma 2.14]).

Using a compactness argument one obtains the following.

Lemma 2.6 ([15, Lemma 2.15]). *Let M be a metric space. We consider a function $F: \mathbb{Z}_+ \rightarrow \mathbb{R}_+$, and assume that for any finite subset $M' \subset M$ there exists an order T' satisfying $\text{OR}_{M', T'}(k) \leq F(k)$ for all $k \geq 1$. Then there exists an order T on M satisfying $\text{OR}_{M, T}(k) \leq F(k)$ for all $k \geq 1$.*

Using an induction on the cordiality of \mathcal{A} in case when this family is finite and a compactness argument for the general case, one also obtains the following lemma.

Lemma 2.7 ([15, Lemma 2.17]). *Let \mathcal{A} be a family of subsets of M such that for any two sets $A_1, A_2 \in \mathcal{A}$ either their intersection is empty, or one of the sets is contained in the other one. Then there exists an order T on M such that for any set $V \in \mathcal{A}$ and any three points $x, y, z \in M$ such that $x <_T y <_T z$, the condition $x, z \in V$ implies $y \in V$.*

We say that the subset V of M is *convex with respect to the order T* if V and T satisfy the property described in Lemma 2.7.

This elementary lemma can be used to construct orders with given properties in various spaces. We will use it several times in the proof of Theorem I about spaces of finite AN-dimension and of Theorem IV about spaces with doubling property.

3. Gap for the order ratio functions. Nilpotent groups and spaces with doubling property

We have mentioned that a crucial observation for using orders for the travelling salesman problem is the result of Bartholdi and Platzman, who have shown existence of orders (related to space-filling curves) such that for any finite set X in the plane the order provides a tour with length at most $\text{Const}(\ln \#X)$ times longer than the optimal one.

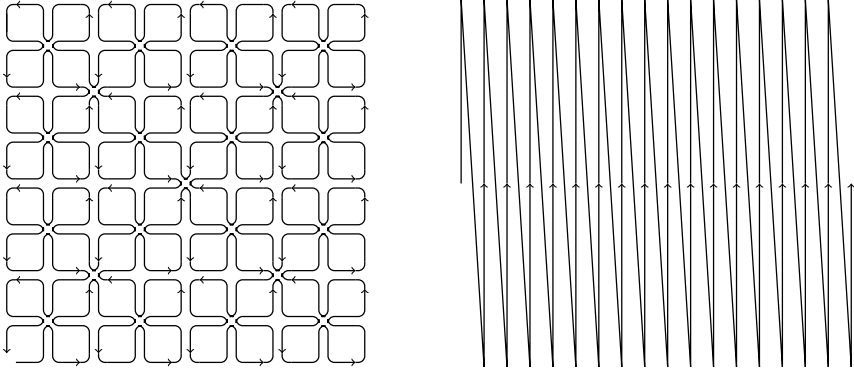


Figure 2. Two orders on the square.

Consider two orders on the square $K = [0; 1]^2$ (Figure 2). The first order T_{BP} is the one considered by Bartholdi and Platzman in [3] and their estimate (combined with obtained later lower bound in [4]) implies that $OR_{K,T_{BP}}(k) \sim \log k$.

The second order T_{lex} is the lexicographical order. We define this order by saying $(x_1, y_1) <_{T_{lex}} (x_2, y_2)$ if $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. It is clear that (K, T_{lex}) contains snakes of diameter 1 and arbitrary small width, and hence $OR_{K,T_{lex}}(k) = k$ for all k .

Our result below will prove that neither on K nor on any other space with doubling property it is possible to construct an order T such that $OR_{K,T}(k)$ is sub-linear but grows faster than a logarithmic function in k .

3.1. Spaces with doubling property

Definition 3.1. A metric space (M, d) is said to satisfy *doubling property* if there exists a doubling constant $D > 0$ such that for any $x \in M$ and $r > 0$ the ball $B(x, 2r)$ of radius $2r$ can be covered by at most D balls of radius r .

Such spaces are also called doubling metric spaces.

In Section 3.2 we will prove the following theorem.

Theorem 3.2 (Gap for order ratio functions on doubling spaces). *Let X be a doubling metric space and T be an order on X . Then either for all k it holds that*

$$OR_{X,T}(k) = k,$$

or there exists C , depending only on the doubling constant of X , on the order breakpoint s of (X, T) and on ε such that $OR_{X,T}(s) \leq s - \varepsilon$, such that for all $k \geq 2$,

$$OR_{X,T}(k) \leq C \ln k.$$

Spaces \mathbb{R}^d and their subsets are examples of doubling spaces. It is easy to see that any group of polynomial growth has doubling property for infinitely many r . It is also known in fact that any group of polynomial growth has doubling property for all r , but known proofs use that these groups are virtually nilpotent (by the polynomial growth theorem [20]).

Below we recall the definition and state these remarks more precisely.

Definition 3.3. A *growth function* of a finitely generated group G with respect to a finite generating set S is the number of elements of G of the word length at most n :

$$v_{G,S}(n) = \#B_{G,S}(n) = \#\{g : l_{G,S}(g) \leq n\}.$$

Suppose that a group G has polynomial growth, that is $v_{G,S}(n) \leq C_S n^d$ for some (and hence for all) generating set S . It is clear directly from the definition that there exists infinitely many n such that $v_{G,S}(2n) \leq C v_{G,S}(n)$ for some constant C . Using the Polynomial growth theorem, we can claim that G is virtually nilpotent, and in this case

$$C_1 n^d \leq v_{G,S}(n) \leq C_2 n^d$$

holds for some integer d (and some positive constants C_1 and C_2 , depending on S ; for this basic fact about growth see, e.g., [34, Chapter 4]). In particular, there exists a positive constant C depending on S such that

$$v_{G,S}(2n) \leq C v_{G,S}(n)$$

for all n .

Remark 3.4. Let G be a finitely generated group, $R > 0$. Let U be a (R, R) -net of the ball of radius $2R$ in G . Consider the balls of radius R centered at points of the nets. The family of these balls covers the ball of radius $2R$, and the number of the balls is at most $v(2.5R)/v(R/2)$. In particular, if $v(2.5R)/v(R/2) \leq v(2^3 R/2)/v(R/2) \leq C^3$, then G has doubling property on the scale R with constant C^3 .

3.2. Proof of Theorem 3.2

We assume that there exists s_0 such that $\text{OR}_{M,T}(s_0) = s_0 - \varepsilon$ for some $\varepsilon > 0$. Observe that there exists a number λ , depending only on s_0 and ε , such that any snake on $s_0 + 1$ points in (X, T) has elongation (the ratio of its diameter and its width) at most λ .

Lemma 3.5. *Let M be a doubling metric space with a doubling constant D , let T be an order on M and let $\lambda > 1$ be such that all snakes on $s_0 + 1$ points in (M, T) have elongations at most λ . Then there exists a constant $N(D, s_0, \lambda)$ such that whatever $x \in M$ and $R > 0$ we choose, there are no sequences $x_1 <_T x_2 <_T \dots <_T x_{2N}$ satisfying:*

- (1) $x_i \in B(x, 4R)$ for all i ;
- (2) $d(x_{2i-1}, x_{2i}) \geq R$ for all i .

Proof. Arguing by induction on k and using the definition of doubling spaces, we observe that for any k , any x and any $R > 0$ the ball $B(x, 4R)$ can be covered by D^{k+3} balls of radius at most $2^{-k-1}R$, and thus of diameter not greater than $2^{-k}R$. Choose k such that $2^k > \lambda$. Put $N = s_0(D^{k+3})^2 + 1$. Assume that there exist points

$$x_1 <_T x_2 <_T \cdots <_T x_{2N}$$

satisfying properties (1) and (2) in the formulation of the lemma. Cover the ball $B(x, 4R)$ by $m \leq D^{k+3}$ balls B_1, \dots, B_m of diameters at most $4R/2^{k+2} < R/\lambda$. We have N pairs (x_i, x_{i+1}) of points at distance at least R . Each pair (x_i, x_{i+1}) has x_i in B_j and x_{i+1} in B_l for some $j \neq l$.

For given j and l , it is clear that there is at most s_0 such pairs are in $B_j \cup B_l$, since otherwise we find a snake on $s_0 + 1$ points with diameter at least R and width smaller than R/λ . This is in a contradiction with the assumption of the lemma. ■

Claim 3.6. *Let M be a metric space and let $(x_0, x_1, x_2, \dots, x_n)$ be a sequence of points such that*

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}) = L.$$

Let $L = a_1 + a_2 + \dots + a_m$, all $a_i > 0$. Then the set $\{x_i\}$ can be partitioned into m subsets A_1, \dots, A_m such that $\text{diam}(A_j) \leq a_j$ for all j .

Proof. Induction on m . The base of induction: if $m = 1$, then the claim of the lemma is straightforward. Induction step: find the maximal $l \geq 0$ such that

$$\sum_{i=0}^{l-1} d(x_i, x_{i+1}) \leq a_1$$

and put $A_1 := \{x_0, x_1, \dots, x_l\}$. Then

$$\sum_{i=l}^{n-1} d(x_i, x_{i+1}) \leq a_2 + \dots + a_m$$

and $\text{diam}(A_1) \leq a_1$. ■

Lemma 3.7. *Let M be a doubling metric space with a doubling constant D and let T be an order on M such that all snakes on $s_0 + 1$ points in (M, T) have elongation at most λ . Then there exists a constant $N(D, s_0, \lambda)$ such that the following holds.*

Take a sequence of points (x_i) in M , $i : 1 \leq i \leq m$ satisfying

$$x_1 <_T x_2 <_T \cdots <_T x_m.$$

Assume that the minimal length of a path visiting these points is L . Then for all $k \geq 0$, it holds that

$$\#\left\{i \in (1, \dots, m - 1) \mid \frac{L}{2^{k+1}} \leq d(x_i, x_{i+1}) \leq \frac{L}{2^k}\right\} \leq N2^k.$$

Proof. Applying Claim 3.6 for $m = 2^k$, $a_1 = a_2 = \cdots = a_m = L/2^k$ and the reordering of $\{x_i\}$ to minimal length, we conclude that there exist 2^k points y_1, \dots, y_{2^k} such that

$$\{x_i\}_{i=1}^m \subset B\left(y_1, \frac{L}{2^k}\right) \cup \cdots \cup B\left(y_{2^k}, \frac{L}{2^k}\right).$$

Then for each i such that $d(x_i, x_{i+1}) \leq L/2^k$ we can find y_j such that

$$x_i, x_{i+1} \in B\left(y_j, \frac{L}{2^{k-1}}\right).$$

Put $R := 2^{-k-1}L$ and apply Lemma 3.5 for each y_i and R . We can take $N(D, s_0, \lambda)$ twice as big as in Lemma 3.5. ■

Now we are ready to complete the proof of Theorem 3.2. Consider a finite subset $X \subset M$ of cardinality n and enumerate its points $x_1 <_T \cdots <_T x_n$. Let $2^{k-1} < n \leq 2^k$. As we have already mentioned, there exists λ , depending on s_0 and $\varepsilon > 0$ satisfying $\text{OR}(s_0) = s_0 - \varepsilon$, such that all snakes on $(s_0 + 1)$ points in (M, T) have elongation at most λ . Consider $N = N(D, s_0, \lambda)$ from the claim of Lemma 3.7. Let us show that

$$l_T(X) = d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) < Nl_{\text{opt}}(Y)k.$$

Denote $l_{\text{opt}}(X)$ by L .

We consider the following $n - 1$ numbers: the distances $d(x_i, x_{i+1})$ for $1 \leq i < n$. For any $i, i = 0, \dots, k - 1$ let D_i be the number of distances between $L/2^{i+1}$ and $L/2^i$, and denote by D_k the number of distances that are smaller than $2^{-k}L$. Observe that each distance $d(x_i, x_{i+1})$ is at most L .

Note that $D_i \leq N2^i$ for any i . For D_k it is clear because $n < N2^k$, and for all other D_i it follows from Lemma 3.7. Hence, the sum of numbers in each group is not greater than NL , and the total sum $l_T(X)$ is not greater than

$$NLk = Nl_{\text{opt}}(X) \lceil \log_2(n) \rceil.$$

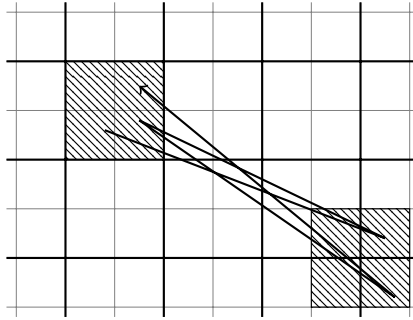


Figure 3. The vertices of the snake are inside the shaded squares.

3.3. Examples of orders with finite order breakpoint

Consider some order T on $X = \mathbb{R}^d$ and suppose that we want to prove that

$$\text{OR}_{X,T}(k) = O(\log(k)).$$

From Theorem 3.2 it follows that its enough to find s such that $\text{OR}(s) < s$, i.e., such that the elongations of snakes on $s + 1$ points in (X, T) are bounded.

Here we give an example of a simple order with this property, and together with Theorem 3.2 this will give us an alternative proof that $\text{OR}_{\mathbb{R}^d} = O(\log k)$.

From Lemma 2.6 it follows that it is enough to provide an order on a unit cube $K = [0; 1)^d$. For each point $x = (x_1, \dots, x_d) \in K$, we construct an infinite binary sequence $a_0 a_1 \dots$ as follows: for any $i \in (1, 2, \dots, d)$ and $j \in (0, 1, \dots)$, we put

$$a_{jd+i-1} = \text{the } j\text{-th digit in the (lex-minimal) binary representation of } x_i.$$

Let T be the lexicographical order on the corresponding sequences.

Lemma 3.8. *In (K, T) there are no snakes on $2^{d+1} + 1$ points with elongation greater than $8\sqrt{d}$.*

Proof. For any j the cube K is divided into 2^{jd} cubes of sizes $2^{-j} \times \dots \times 2^{-j}$, we call these cubes the *base cubes of level j* . Note that each base cube is convex with respect to T .

Suppose $s > 2^{d+1}$ and there is a snake $x_1 <_T \dots <_T x_s$ of width b and diameter $a > 8\sqrt{d}b$. Find k such that $2^{-k-1} \leq b < 2^{-k}$. Points of the snake with odd indices can be covered by no more than 2^d base cubes of level k , the same holds for the points with even indices (Figure 3).

By the pigeonhole principle there are two points of the snake with odd indices in one base cube A of level k . Convexity means that there must be a point with even

index in the same cube A , but since $a - 2b > \sqrt{d}2^{-k}$, none among the cubes of level k can contain both odd points and even points. ■

We will see later in the second claim of Lemma 4.7 that the constant in Lemma 3.8 above is far from being optimal, indeed the number of the points of snakes of large elongation, under assumption of Lemma 3.8 is at most linear in d .

4. Spaces of finite Assouad–Nagata dimension

We have already mentioned that the spaces with doubling property admit orders with at most logarithmic order ratio function $OR_{M,T}(k)$, and that among finitely generated groups only groups of polynomial growth (virtually nilpotent ones) satisfy the doubling property. In this section we are going to show that any space of finite Assouad–Nagata dimension admits an order with $OR(k) \leq \text{Const} \ln k$ and provide an upper bound for the order breakpoint. We have mentioned that any group of polynomial growth and many groups of exponential growth have finite Assouad–Nagata dimension (wreath products with a base group of linear growth [8], Coxeter groups [12], relatively hyperbolic groups [26], polycyclic groups [24]; see also further classes of groups mentioned in the introduction).

Assouad–Nagata dimension can be viewed as a (more strong, linearly controlled) version of the notion of asymptotic dimension introduced later by Gromov in [21].

Definition 4.1. Assouad–Nagata dimension of a metric space M is defined as follows. Let M be a metric space, let m be a positive integer and $K \geq 1$. Suppose that for all $r > 0$ there is a family U_r of subsets, such that their union is equal to M and such that the following holds:

- (1) The diameter of any set $A \in U_r$ is at most Kr .
- (2) Any closed (equivalently open) ball of radius $\leq r$ is contained in some set of U_r .
- (3) Any point of M belongs to at most $m + 1$ sets of U_r .

We say in this case that Assouad–Nagata dimension of M is at most m . To shorten the notation, we will also call Assouad–Nagata dimension AN-dimension.

If we weaken our assumption on the covering, and instead of property (1) require that there exists some bound (not necessarily linear) for diameters of sets $A \in U_r$ in terms of r , we obtain a definition of spaces of finite *asymptotic dimension*. Such upper bound for the diameters of $A \in U_r$ (in definition of AN-dimension, this bound is $\leq Kr$) is called *m-dimensional control function*.

Definition 4.2 (Equivalent definition of AN-dimension). Let M be a metric space. Let us say that M has AN-dimension at most m if the following holds. There exists $K \geq 1$ such that for any r there exists a partition W_r of the space M such that all sets from W_r have diameters at most Kr , and any ball of radius r , centered at some point $x \in M$, intersects at most $m + 1$ sets from W_r .

Although the dimension m does not depend on the definition, the constants K in Definitions 4.1 and 4.2 may differ. In both cases we call Kr , considered as a function of r , to be m -dimensional control function.

For the convenience of the reader we explain why the definitions are equivalent (with an appropriate choice K for each of them).

Proof. Suppose that M has AN-dimension at most d with respect to the first definition. To explain the claim of the second definition, consider a covering U_r in the first definition of Assouad–Nagata dimension. For any subset A from U_r , replace this subset by a set A' , which we obtain from A by removing the open r -neighborhood of its complement.

To prove that the union of A' is equal to M observe the following. For any point x choose and fix a subset A from U_r containing a ball of radius r , centered at x . We consider the corresponding A' . Since for any x we have a set A' containing x , we conclude that A' cover M .

Now observe that if a ball of radius r centered at y intersects some A' , then y belongs to A . This implies that any ball of radius r intersects at most $m + 1$ among subsets A' . Finally, observe that we can ensure that the sets are disjoint, replacing the sets by appropriate subsets.

Now observe that if M admits a partition satisfying the claim of Definition 4.2, then we can consider open r -neighborhoods of the sets of the partition in this definition. Let K be a constant from Definition 4.2. It is clear that the obtained sets satisfy the assumption of Definition 4.1 for the same r and the constant of the m -dimensional control function in the sense of this definition $K' = K + 2$. ■

As an example of a space of finite Assouad–Nagata dimension recall the example of [8] which shows that Assouad–Nagata dimension of $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ is one. We recall that the wreath product $A \wr B$ of groups A and B is a semi-direct product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts. Elements of the wreath product are pairs (a, f) , where $a \in A$ and $f: A \rightarrow B$ is a function with $f(x) = e_B$ for all but finitely many $x \in A$. We consider a standard generating set of $A \wr B$, which corresponds to the union of S_A , and S_B , where $S_A \subset A \subset A \wr B$ and $S_B \subset B$ is imbedded to $\sum_A B \subset A \wr B$ by sending B to the copy of B in $\sum_A B$ indexed by e_A . In the example below we consider the word metric on the Cayley graph with respect to this standard generating set (edges are not included). The argument [8] uses the fact that the kernel of the map

to \mathbb{Z} is zero-dimensional, and then uses a Hurewicz-type theorem for AN-dimension for group extensions. In the example below we describe explicitly partitions of the infinite wreath product, as well as of a sequence of finite wreath products. These partitions uniformly satisfy the assumption of Definition 4.2 and show that the spaces have AN-dimension equal to one.

Example 4.3 (Partitions of wreath products). We give two examples.

- (1) For each $r > 0$ there exists a partition \mathcal{A}_r of $G = \mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ such that the diameter of every set $A \in \mathcal{A}_r$ is at most $6r$ and any ball of radius $r/2$ intersects at most 2 sets.
- (2) For each $r > 0$ there exists a partition \mathcal{A}_r^i of $G = \mathbb{Z}/i\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ such that the diameter of every set $A \in \mathcal{A}_r^i$ is at most $9r$ and any ball of radius $r/2$ intersects at most 2 sets.

Proof. (1) The proof is reminiscent of a possible argument to show that asymptotic dimension, as well as AN-dimension of a free group, is 1 (see, e.g., [43, Proposition 9.8]).

We partition \mathbb{Z} into disjoint intervals of length r and call the corresponding subset $\{(x, f) \mid kr \leq x < (k + 1)r\}$ the *layer* L_k . It is clear that if $x \in L_k$ and $y \in L_m$ and $m - k \geq 2$ then the distance between (x, f) and (y, g) is $\geq r$. Now we subdivide each layer L_k into sets, saying that (x, f) and (y, g) ($kr \leq x, y < (k + 1)r$) are in the same set if $f(z) = g(z)$ for any $z \notin [rk - r/2, (r + 1)k + r/2]$. Observe that if (x, f) and (y, g) are in the same layer but not in the same set, then the distance between them is $\geq r$ since, starting from a point inside $[kr, (k + 1)r]$ one needs to switch the value of the function at (at least) one point of \mathbb{Z} which is either $\leq kr - r/2$ or $\geq (k + 1)r + r/2$. Observe also that diameter of any set is at most $6r$, since to go from (x, f) to any point (y, g) in the same set it is sufficient to start in x , end at y , visit all points of $[kr - r/2, (k + 1)r + r/2]$ (since the length of this interval is $2r$, and hence $4r$ steps suffices) and make at most $2r$ switches.

(2) Similarly one can construct a partition of a finite wreath product $\mathbb{Z}/i\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$. If $r \geq i$, we consider the partition consisting of one set. Now we assume that $r < i$. We subdivide $\mathbb{Z}/i\mathbb{Z}$ into several intervals of length r and possibly one interval of length $\geq r$ and $< 2r$. Each layer L_k corresponding to an interval $[I_k, J_k]$ is subdivided into sets, saying that (x, f) and (y, g) are in the same set if f and g coincide on the complement of the interval $[I_k - r/2, J_k + r/2]$. As before, distinct sets of the same layer are at distance at least r and the sets of the layers L_k and L_m , for $k - m \geq 2$ are at distance at least r . The diameter of each set is at most $3r + 6r$. ■

We will return to (2) of the example above in the next section, when we discuss the relation between infinite Br and spectral properties of a sequence of graphs. In

that context we will discuss an order, provided by Theorem 4.4 for a disjoint union of graphs from (2) of Example 4.3.

Theorem 4.4. *If M is a metric space of finite Assouad–Nagata dimension m with m -dimensional control function at most Kr , then there exist an order T on M such that*

- (1) *for all $k \geq 2$ it holds $\text{OR}_{M,T}(k) \leq C \ln k$, where a positive constant C can be chosen depending on m and K only;*
- (2) *$\text{Br}(M, T) \leq 2m + 2$. Moreover, the elongations of snakes in (M, T) on $2m + 3$ points are bounded by some constant depending on m and K only.*

In view of Lemma 2.6 it is sufficient to prove the statement of Theorem 4.4 for finite metric spaces. In the lemmas below we will make a more general assumption that M is uniformly discrete. We recall that this means that there exists $c > 0$ such that for all $x \neq y$ it holds $d(x, y) \geq c$.

Definition 4.5 (AN-filtrations). Given a metric space M , we say that a sequence of partitions of it $\mathcal{V}_j, j \in \mathbb{Z}$ is AN-filtration with coefficients m, λ, D, δ , if

- (1) the diameter of any set in $A \in \mathcal{V}_j$ is at most $\lambda^j D$;
- (2) for any $x \in M$ the ball $B(x, \lambda^j \delta)$ can be covered by a union of $m + 1$ sets from \mathcal{V}_j ;
- (3) $A \in \mathcal{V}_j$ and $B \in \mathcal{V}_{j'}, j < j'$, then either the intersection $A \cap B$ is empty, or $A \subseteq B$.

In the definition above we assume that $m \in \mathbb{Z}_{\geq 0}, \lambda, D, \delta \in \mathbb{R}_+$ and $\lambda > 1$.

In the definition above, j takes all integer values. Observe that if the metric space M is uniformly discrete, then for all sufficiently large j the sets in \mathcal{V}_{-j} are one-point sets.

Lemma 4.6. *Let M be a uniformly discrete metric space. Assume that Assouad–Nagata dimension of M is at most m and that M satisfies Definition 4.2 with a linear constant of m -dimensional control function at most K . Then M admits an AN-filtration with parameters $m, \lambda = 4K, \delta = 1/2, D = 2K$.*

Proof. Since our space is uniformly discrete, there exists $c > 0$ such that for any distinct x_1, x_2 in M it holds $d(x_1, x_2) \geq c$. Observe that the constant K in the definition of Assouad–Nagata dimension satisfies $K \geq 1$, and in particular $\lambda = 4K \geq 4 > 1$. Take the maximal $k_0 \in \mathbb{Z}$ such that $2K\lambda^{k_0} < c$. For all integers $k \leq k_0$, define \mathcal{V}_j to be a partition of M into one-point sets. Observe that properties (1), (2), (3) in the definition of AN-filtration are verified for $j, j' \leq k_0$.

Now suppose that we have already constructed \mathcal{V}_j for all $j \leq k, k \in \mathbb{Z}$, such that the properties of AN-filtration are verified for all $j, j' \leq k$. Now we explain how to construct \mathcal{V}_{k+1} such that the properties of AN-filtration are verified for all $j, j' \leq k + 1$.

Each set from \mathcal{V}_{k+1} will be a union of some sets from \mathcal{V}_k .

By our assumption on M , we know that for any r there exists a partition W_r of M (into disjoint sets) such that

- (1) the diameters of the sets from W_r are at most Kr ;
- (2) any ball of radius r , centered at some point $x \in M$, intersects at most $m + 1$ sets from W_r .

Consider $r = \lambda^{k+1}$ and the partition W_r . We will group sets from \mathcal{V}_k together using this partition. For any set from W_r there will be one corresponding set in \mathcal{V}_{k+1} . For any set A in \mathcal{V}_k we choose one of the subsets of W_r with a non-empty intersection with A , we denote this subset by $f(A)$. To give an idea of the size of sets $A \in \mathcal{V}_k$, we mention that by induction hypothesis the diameter of A is at most $2K(4K)^k = r/2$.

Now define subsets of \mathcal{V}_{k+1} as the union of subsets A of \mathcal{V}_k with the same associated subset $f(A)$ (of W_r). Since subsets of \mathcal{V}_k are disjoint, it is clear that subsets of \mathcal{V}_{k+1} are also disjoint. It is also clear that for any subset $A \in \mathcal{V}_{k+1}$ and any subset $B \in \mathcal{V}_j$, with $j \leq k$, either B is contained in A or they have empty intersection.

Now we also observe that there is an upper bound on diameters of all subsets $A \in \mathcal{V}_{k+1}$, that guarantees that property (1) in the definition of AN-filtration is verified with parameters $D = 2K, \lambda = 4K$:

$$\begin{aligned} \sup_{A \in \mathcal{V}_{k+1}} \text{diam}(A) &\leq \sup_{A \in W_{\lambda^{k+1}}} \text{diam}(A) + 2 \sup_{A \in \mathcal{V}_k} \text{diam}(A) \\ &\leq K\lambda^{k+1} + 2K\lambda^k \leq 2K\lambda^{k+1}. \end{aligned}$$

Finally, observe that any ball of radius λ^{k+1} intersects at most $m + 1$ subsets of W_r , and hence any ball of radius $\lambda^{k+1} - \sup_{A \in \mathcal{V}_k} \text{diam}(A)$ intersects at most $m + 1$ subsets of \mathcal{V}_{k+1} .

Since

$$\lambda^{k+1} - \sup_{A \in \mathcal{V}_k} \text{diam}(A) \geq \lambda^{k+1} - 2K\lambda^k = \frac{1}{2}\lambda^{k+1},$$

we can conclude that any ball of radius $\frac{1}{2}\lambda^{k+1}$ intersects at most $m + 1$ subsets of \mathcal{V}_{k+1} . Hence, property (2) on the definition of AN-filtration is satisfied with parameters $\lambda = 4K$ and $\delta = 1/2$. ■

Lemma 4.7. *Suppose that a metric space M admits an AN-filtration (\mathcal{V}_j) with coefficients m, λ, D, δ , and for some order T all the subsets from this AN-filtration are convex with respect to this order T . Then*

- (1) $\text{OR}_{M,T}(k) \leq C \ln k$, where C is a constant depending on m, λ, D and δ .
- (2) Elongations of snakes on $2m + 3$ points are bounded by some constant C' , depending on m, λ, D and δ .

Proof. Consider a finite subset $X \subset M$, of cardinality $N + 1$ and enumerate its points

$$x_1 <_T \cdots <_T x_{N+1}.$$

Denote by L the length $l_{\text{opt}}(X)$ of a shortest path visiting all points of X . Consider some $j \in \mathbb{Z}$. Observe that the set X can be partitioned into at most $\lambda^{-j}L/\delta + 1$ subsets of diameter at most $\lambda^j\delta$.

In view of property (2) in the definition of AN-filtration, each of these sets can be covered by union of some $m + 1$ sets from \mathcal{V}_j . We can therefore conclude that X can be covered by at most $(m + 1)(\lambda^{-j}L/\delta + 1)$ sets of the partition \mathcal{V}_j . We call these sets of \mathcal{V}_j parts.

Observe that if $A \in \mathcal{V}_j, x_i \in A$, and $d(x_i, x_{i+1}) > \lambda^j D$, then $x_{i+1} \notin A$. Since A is convex with respect to T , all points $x_{i'}, i' > i$, do not belong to A . Therefore the number of indices i such that $d(x_i, x_{i+1}) > \lambda^j D$ is not greater than the total number of parts, that is, not greater than

$$(m + 1)(\lambda^{-j}L/\delta + 1).$$

Consider n and l such that $\lambda^{n-1} < N + 1 \leq \lambda^n$ and $\lambda^{l-1} < L \leq \lambda^l$. Let D_0 be the number of indices i such that $\lambda^{-1}L < d(x_i, x_{i+1}) \leq L$. Let D_1 be the number of indices i such that $\lambda^{-2}L < d(x_i, x_{i+1}) \leq \lambda^{-1}L$, and so on until D_{n-1} . Finally, we define D_n as the number of indices i for which $d(x_i, x_{i+1}) \leq \lambda^{-n}L$.

The total length of the path with respect to our order T can be estimated as

$$l_T(X) = \sum_{i=1}^N d(x_i, x_{i+1}) \leq L \cdot \sum_{k=0}^n D_k \lambda^{-k}.$$

For any $k < n, D_k$ is not greater than the number of indices i such that

$$d(x_i, x_{i+1}) > L\lambda^{-k-1}.$$

We have $L\lambda^{-k-1} > \lambda^{l-k-2} > \lambda^{l-k-\lceil \log_\lambda D \rceil - 3} D$. Hence, we can estimate D_k as

$$D_k \leq (m + 1) \left(\frac{\lambda^{-(l-k-\lceil \log_\lambda D \rceil - 3)} L}{\delta} + 1 \right) \leq (m + 1) \left(\frac{\lambda^{k+\lceil \log_\lambda D \rceil + 3}}{\delta} + 1 \right);$$

$$D_k \lambda^{-k} \leq (m + 1)(\lambda^{\lceil \log_\lambda D + 3 \rceil} / \delta + 1).$$

Note that this number does not depend on k . We also estimate $D_n \lambda^{-n}$ as

$$(N + 1)\lambda^{-n} \leq 1.$$

Finally,

$$l_T(X) \leq L(n + 1)(m + 1)(\lambda^{\lceil \log_\lambda D+3 \rceil} / \delta + 1).$$

The first claim of the lemma follows with

$$C = 2(1/\ln \lambda + 1)(m + 1)(\lambda^{\lceil \log_\lambda D+3 \rceil} / \delta + 1).$$

To prove the second claim, consider in (M, T) a snake with diameter a and width b . Let k be such that $\delta\lambda^{k-1} < b \leq \delta\lambda^k$. Consider all points with odd indices of this snake. Since they form a set of diameter at most $\delta\lambda^k$, by the definition of AN-filtration this set can be covered by $m + 1$ subsets of \mathcal{V}_k . With the same argument we know the analogous fact about all even points of our snake.

Since $b > \delta\lambda^{k-1}$, we know the following. If the elongation a/b of the snake is larger than $\lambda D/\delta + 2$, then $a - 2b > D\lambda^k$ and no two points of the snake with different parity belong to the same set from \mathcal{V}_k . In this case any of $2m + 2$ covering sets from \mathcal{V}_k contains at most 1 point of the snake. Hence, elongations of snakes on $2m + 3$ points are bounded by $\lambda(D/\delta + 2)$. ■

Now we have all the ingredients to prove Theorem 4.4. As we mentioned above, we can assume that the metric space M is uniformly discrete. From Lemma 4.6 it follows that M admits an AN-filtration with parameters depending only on the m -dimensional control function. Lemma 2.7 implies that for some order T all the sets from this family are convex with respect to T , and Lemma 4.7 gives us needed bounds on $\text{OR}_{M,T}$ and elongations of snakes on $2m + 3$ points in (M, T) .

4.1. Product of two binary trees

In this subsection we will illustrate Theorem 4.4 by giving an explicit example of an order on a product of two trees. We encode vertices of an infinite binary tree M by finite binary words $u_i \in \{0, 1\}^*$, the empty word is denoted by ε . Points of the product P of two binary trees M_1 and M_2 can be coded by pairs of finite binary words (u_1, u_2) . The distance between points (u_1, u_2) and (u_3, u_4) in P is defined as $\max(d_{M_1}(u_1, u_3), d_{M_2}(u_2, u_4))$.

For given r consider the following partition W_r of M . Points u and v of M belong to the same set $A \in W_r$ if and only if for some non-negative integer k it holds that

$$kr \leq |u|, |v| < (k + 1)r$$

and the largest common prefix of u and v has length $\geq (k - 1/2)r$. Here by $|x|$ we denote the length of a word x , i.e., the distance to the root in M .

This is a well-known construction of a partition of a metric tree with 1-dimensional control function $6r$, in the sense of Definition 4.2 (we referred already to this result

in the proof of Example 4.3). If we take partitions W_r for all integer powers of 2, we obtain an AN-filtration \mathcal{V}_k of M with parameters $\lambda = 2, m = 1, D = 3$ and $\delta = 1/2$.

We obtain an AN-filtration on P with parameters $\lambda = 2, m = 3, D = 3, \delta = 1/2$ if for each k we take the partition

$$\mathcal{W}_k = \{A \times B \mid A, B \in \mathcal{V}_k\}.$$

(Note that AN-dimension of P is 2, hence from Lemma 4.6 there exists an AN-filtration of P with $m = 2$, but explicit description of sets of the partitions for such filtrations would require extra work.)

Now we can describe an order T on P for which all sets of \mathcal{W}_k are convex and consequently $\text{OR}_{P,T}(k)$ is logarithmic in k by Lemma 4.7.

For integers $l \geq 0$ and $k > 0$, we use notation $f(l, k) = \max([\frac{l}{2^{k+1}}]2^{k+1} - 2^k, 0)$. We also use notation $f(l, 0) = l$. For a point $x = (u_1, u_2) \in P$ we construct the following sequence of finite words $(v_i(x))$. For any k , let $v_{2k}(x)$ be the prefix of u_1 of length $f(|u_1|, k)$ and let $v_{2k+1}(x)$ be the prefix of u_2 of length $f(|u_2|, k)$. Here $|u|$ denotes the length of the word u . It is clear that for $x = (u_1, u_2)$, we have

$$v_0(x) = u_1, \quad v_1(x) = u_2, \quad \text{and} \quad v_k(x) = \varepsilon$$

for all large enough k . Hence, for any two points $x_1, x_2 \in P, x_1 \neq x_2$ the sequences $(v_i(x_1))$ and $(v_i(x_2))$ are distinct but coincide at all but finitely many positions. Finally, we put $x_1 <_T x_2$ if $v_i(x_1) <_{\text{lex}} v_i(x_2)$ and i is the largest integer such that

$$v_i(x_1) \neq v_i(x_2).$$

5. Weak expansion and infinite order breakpoint

Given a sequence of spaces $(M_\alpha, T_\alpha), \alpha \in \mathcal{A}$, we can speak about order ratio function of this sequence by defining

$$\text{OR}(k) = \sup_{\alpha} \text{OR}_{M_\alpha, T_\alpha}(k).$$

Then we can also speak about order breakpoint of a sequence, considering minimal k such that $\text{OR}(k) < k$.

Another way to formulate the definitions above is to consider a disjoint union

$$M = \bigcup M_\alpha.$$

Then one can define the function OR for metric spaces with infinite distance allowed by considering only subsets with finite pairwise distances. In this case OR and Br of the sequence are equal to OR_M and $\text{Br}(M)$.

Given a finite d -regular graph Γ on n vertices, we consider its adjacency matrix A , and eigenvalues $d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. It is clear that $\tilde{\lambda}_i = \lambda_i/d$ are eigenvalues of the normalized adjacency matrix, which is a matrix of transition probabilities of the simple random walk on Γ . We recall that a connected d -regular graph is a Ramanujan graph if

$$\max_{i:|\lambda_i|<d} \lambda_i \leq 2\sqrt{d-1}.$$

Gorodezky et al. show in [17] (see also Bhalgat–Chakrabarty–Khanna [5, Theorem 2]) the existence of a sequence of bounded degree graphs Γ_i which have linear lower bound, in terms of cardinality of subsets, for the competitive ratio of the universal travelling salesman problem. In our terminology, they have provided a linear lower bound for the order ratio function of the sequence Γ_i . Both [17] and [5] use a remarkable construction of Ramanujan graphs (constructed by Lubotzky–Philips–Sarnak [33], see also [35]).

Now we prove that for linearity of competitive ratio in k , and moreover for an a priori stronger claim that the order breakpoint is infinite, a milder assumption than Ramanujan condition for graphs is sufficient.

Since we consider a sequence of graphs in the theorem below, we use the name of the graph as an upper index for λ , denoting $\lambda_j^{\Gamma_i}$ the eigenvalue λ_j of Γ_i .

Theorem 5.1. *Let Γ_i be a sequence of finite graphs of degree $d_i \geq 3$ on n_i vertices. Let T_i be an order on Γ_i .*

- (1) *Let $\delta_i = (\lambda_1^{\Gamma_i} - \lambda_2^{\Gamma_i})/d_i$. Assume that*

$$1/\delta_i = o\left(\frac{\log_{d_i} n_i}{\ln \log_{d_i} n_i}\right).$$

Then the order breakpoint of the sequence (Γ_i, T_i) is infinite.

- (2) *Moreover, for any $k \geq 1$ any ordered graph (Γ_i, T_i) admits snakes on k points of width at most*

$$t_i = \left\lceil \frac{24k(\ln k + \ln(2/\delta_i))}{\delta_i} \right\rceil$$

and length at least

$$L_i = \left\lfloor \frac{\ln n_i - 4 - 2k \ln(2k) - 2k \ln(2/\delta_i)}{\ln(2d_i - 1)} \right\rfloor - 1.$$

- (3) *In particular, for a sequence of bounded degree expander graphs the following holds. If $d_i = d \geq 3$ and $\delta = \inf_i \delta_i > 0$, then for each k the graphs (Γ_i, T_i) admit snakes on k points of bounded width C_k and of length at least $\log_{d-1} n_i - C'_k$ for some $C_k, C'_k > 0$.*

Proof. First observe that we can easily modify our graphs to ensure that the spectrum is non-negative. Let Γ_i^+ be a graph obtained from Γ_i by adding d_i loops in all vertices of Γ_i . Then Γ_i^+ is a regular graph of degree $2d_i$ (we use the convention that each loop contributes one edge to its vertex). Observe that it is sufficient to prove that the order breakpoint of a sequence Γ_i^+ is infinite. We will also prove other claims of the theorem under the assumption that the spectrum is non-negative, and then the general case will follow. Let

$$\lambda_1^{\Gamma_i^+} = 2d_i \geq \lambda_2^{\Gamma_i^+} \geq \dots \geq \lambda_n^{\Gamma_i^+}$$

be eigenvalues of the adjacency matrix of Γ_i^+ . It is clear that

$$\lambda_j^{\Gamma_i^+} = \lambda_j^{\Gamma_i} + d_i \geq 0$$

for all j . In particular, $\max(\lambda_2, |\lambda_n|) = \lambda_2$. Observe also that

$$\lambda_2^{\Gamma_i^+} / (2d_i) = (d_i + \lambda_2^{\Gamma_i}) / (2d_i)$$

and that the spectral gap $(\lambda_1^{\Gamma_i^+} - \lambda_2^{\Gamma_i^+}) / (2d_i)$ is equal to $\delta_i / 2$.

Below we therefore assume that the graph Γ_i has non-negative spectrum. Given i , consider Γ_i and choose m_i starting points in Γ_i independently with respect to the uniform distribution on the vertices of Γ_i . Consider m_i trajectories of independent random walks with t_i steps, starting in the chosen starting points. The values m_i and t_i will be specified later in the text. We will use the following observation.

Claim 5.2. *Let T be an order on a graph G . Suppose that we have a partition of G into several convex sets with respect T (we will call them intervals). Consider several t -step trajectories of a simple random walk on G , and assume that the distance between starting points of any two trajectories is at least L . If there are at least k intervals with non-empty intersection with two among our trajectories, then (G, T) admits a snake on $k + 1$ points, of width at most t and length at least $L - 2t$.*

Proof. First observe that the distance between any two trajectories is clearly at least $L - 2t$.

For each among k intervals choose a point A_i from the first trajectory and B_i from the second. Let $A_1 <_T \dots <_T A_k$ and $B_1 <_T \dots <_T B_k$ be points of these trajectories. Without loss of generality, $A_1 <_T B_1$. Then we can choose a snake

$$(A_1 <_T B_1 <_T A_2 <_T B_3 <_T \dots)$$

on $k + 1$ points that satisfies our lemma. ■

We will use Claim 5.2 above for a fixed number k , $G = G_i$ and numbers $t = t_i$ and $L = L_i$, for appropriately chosen sequence t_i and L_i . We will explain the proof of (1) and (2) of the theorem, where for the proof of (1) we take any $k \geq 1$ and fix it for the argument below. And in (2) k is the number in the formulation of this claim.

We consider the partition of the elements of Γ_i into N_i sets of equal cardinality (almost equal in case N_i does not divide n_i), convex with respect to the order T_i . The numbers N_i will be specified later. In other words, we number the points of Γ_i with respect to T_i : $g_1 \leq_{T_i} g_2 \leq_{T_i} \dots \leq_{T_i} g_{n_i}$; put $r_i = \lfloor n_i/N_i \rfloor$ and consider sets

$$\Omega_1^i = \{g_1, g_2, \dots, g_{r_i}\}, \quad \Omega_2^i = \{g_{r_i+1}, g_{r_i+2}, \dots, g_{2r_i}\},$$

$$\Omega_3^i = \{g_{2r_i+1}, g_{2r_i+2}, \dots, g_{3r_i}\} \text{ etc.}$$

Before we apply Claim 5.2 and specify a sequence m_i of numbers of the trajectories, we start with a straightforward observation. If we choose independently m points, with respect to uniform distribution on vertices of a graph of cardinality n , of degree d , then the probability that for at least two among the chosen points the distance is $\leq L$ is at most

$$\frac{m^2 d(d-1)^L}{(d-2)n}. \tag{1}$$

Indeed, the number of points at distance L from a given point is at most

$$1 + d + d(d-1) + \dots + d(d-1)^{L-1},$$

and there are m^2 possible pairs of points chosen among m points.

We want to choose m_i and L_i in such a way that

$$\frac{m_i^2 d_i (d_i - 1)^{L_i}}{(d_i - 2)n_i} \tag{2}$$

is small enough. In view of the upper bound (1) on the distance between a pair of points among m points chosen at random in a graph of degree d , under this assumption we will have a lower bound for the probability that the maximum of pairwise distance between m_i starting points of trajectories (in Γ_i) will be at least L_i . In order to use Claim 5.2, to find snakes of arbitrarily large elongation (and thus to prove (1) of the theorem) we need

$$\frac{L_i - 2t_i}{t_i} \rightarrow \infty.$$

We will need another condition, to estimate the number of intervals a trajectory intersects. We put

$$m_i = C_{N_i}^k + 1.$$

We use here the notation for binomial coefficients $C_n^k = \frac{n!}{k!(n-k)!}$.

Since $m_i > C_{N_i}^k$, if each of the trajectories intersects at least k convex intervals, then there exist two among our m_i trajectories and k among our N_i intervals with non-empty intersection with both of these trajectories.

The probability P_i that one of our m_i trajectories intersects less than k intervals satisfies

$$P_i \leq m_i C_{N_i}^k (\beta_i + \alpha_i)^{t_i},$$

for $\beta_i = k/N_i$ being the density of a set which is the union of k convex intervals and $\alpha_i = \lambda_2^{\Gamma_i}/d_i = 1 - \delta_i$ (since we assume that the spectrum is non-negative). Indeed, for each among our m_i trajectories (with starting points chosen according to a uniform distribution on the vertices of Γ_i) and fixed k among our convex intervals, the probability to stay inside the union of these intervals is at most

$$(\beta_i + \alpha_i)^{t_i},$$

as follows from [25, Theorem 3.6]. This result, that goes back to [1], states the following. Let G be a d -regular graph on n vertices and spectral values of the normalized adjacency matrix satisfy $|\tilde{\lambda}_2|, |\tilde{\lambda}_n| \leq \alpha$. Let B be a subset of vertices of G of cardinality $\leq \beta n$. Then the probability that a t step trajectory of a random walk, starting at a point chosen with respect to the uniform distribution on G , stays inside B is at most $(\alpha + \beta)^t$. As we have already mentioned, for the proof of the theorem we can assume that the spectrum of Γ_i is non-negative, so that in this case

$$\max(|\lambda_2^{\Gamma_i}/d_i|, |\lambda_{n_i}^{\Gamma_i}/d_i|) = 1 - \delta_i.$$

We return to the proof of the theorem. To get a desired bound, we choose β_i of the same order as $1 - \alpha_i$, putting $\beta_i = k/N_i = \delta_i/2$. More precisely, to ensure that N_i is an integer, we choose

$$N_i = \lceil 2k/\delta_i \rceil. \tag{3}$$

Then, since we can assume that $k \geq 2$ and hence $N_i \geq 4$, it holds

$$\begin{aligned} P_i &\leq (C_{N_i}^k + 1) C_{N_i}^k (\beta_i + \alpha_i)^{t_i} \leq 2(C_{\lceil 2k/\delta_i \rceil}^k)^2 (1 - \delta_i/2)^{t_i} \\ &\leq 2((2k/\delta_i)^k)^2 \exp(-\delta_i t_i/2) = 2(2k/\delta_i)^{2k} \exp(-\delta_i t_i/2), \end{aligned}$$

since $(1 - 1/x)^x \leq e^{-1}$ for any $x > 1$.

Take the logarithm of the previous expression. We want to choose t_i in such a way that

$$\delta_i t_i/2 - 2k(\ln k + \ln(1/\delta_i) + \ln 2) - \ln 2 \geq 1,$$

this guarantees that the probability P_i that one of the trajectories intersects less than k intervals is at most $\exp(-1)$. In particular, we can take

$$t_i = \left\lceil \frac{12k(\ln k + \ln(1/\delta_i))}{\delta_i} \right\rceil. \tag{4}$$

We rewrite the upper bound in the formula (1) for the probability, that two among our m_i trajectories are at distance smaller than L_i . We take into account the choice of m_i in the formula (2), the choice of N_i in the formula (3) and the estimate $m_i^2 \leq 2(2k/\delta_i)^{2k}$. We also want to assume that the obtained upper bound satisfies

$$Q_i = 2(2k/\delta_i)^{2k} \frac{d_i(d_i - 1)^{L_i}}{(d_i - 2)n_i} \leq e^{-1}. \tag{5}$$

Since $2e^{-1} < 1$, the inequality (5) on Q_i combined with our assumption on P_i guarantees that our argument, with probability $\geq 1 - 2e^{-1}$, provides us snakes of width at most t_i and of diameter at least L_i in the ordered space (Γ_i, T_i) .

We take the logarithm of the expression in the formula (5) and use that

$$\ln(d_i/(d_i - 2)) \leq \ln 3 \leq 2.$$

We introduce the notation $w_i = L_i/t_i$, and we want therefore that t_i and w_i satisfy

$$2k(\ln(2k) + \ln(1/\delta_i)) + t_i w_i \ln(d_i - 1) \leq \ln n_i - 4.$$

In order to have the inequality above, to assure that L_i is an integer and that we have snakes of length L_i , we can take w_i such that

$$w_i = \frac{1}{t_i} \left\lceil \frac{\ln n_i - 4 - 2k \ln(2k) - 2k \ln(1/\delta_i)}{\ln(d_i - 1)} \right\rceil.$$

Thus for ordered graphs with positive spectrum we have shown existence of snake of width

$$t_i = \left\lceil \frac{12k(\ln k + \ln(1/\delta_i))}{\delta_i} \right\rceil$$

and of length at least

$$L_i = \left\lceil \frac{\ln n_i - 4 - 2k \ln(2k) - 2k \ln(1/\delta_i)}{\ln(d_i - 1)} \right\rceil - 1.$$

As we have mentioned it is easy to modify the graph to ensure that the spectrum is positive. Since by this modification δ_i is replaced by $\delta_i/2$, d_i by $2d_i$ and cardinality of the balls in Γ_i does not change, this implies claim (2) of the theorem for a general graph.

In particular, if we have a sequence of expander graphs of fixed degree, then $1/\delta_i$ is bounded from above, hence there exist snakes of width at most C and of length greater or equal to $\log_{d-1} n_i - C'$, where constants C' and C depends on k and the spectral gap $\delta > 0$, $\delta = \inf_i \delta_i$. We have proved the third claim.

To prove claim (1) of the theorem, we want to guarantee that the elongation of snakes tends to ∞ , and thus we need to ensure that $w_i \rightarrow \infty$, as $i \rightarrow \infty$. This will follow from conditions

$$t_i \ln(d_i - 1) = o(\ln n_i) \tag{6}$$

and

$$1 + k \ln(1/\delta_i) = o(\ln n_i). \tag{7}$$

The assumption of claim (1) of the theorem implies that $1/\delta_i = o(\ln n_i)$, and hence condition (7) holds.

Since we can take t_i as in (4), to ensure (6) it is enough to show that

$$\frac{\ln(1/\delta_i)}{\delta_i} = o\left(\frac{\ln n_i}{\ln(d_i - 1)}\right). \tag{8}$$

The assumption of (1) of the theorem implies that $\log_{d_i} n_i$ tends to infinity; and that then

$$1/\delta_i = o\left(\frac{\log_{d_i} n_i}{\ln \log_{d_i} n_i}\right) = o\left(\frac{\log_{d_{i-1}} n_i}{\ln \log_{d_{i-1}} n_i}\right),$$

then

$$\begin{aligned} 1/\delta_i \ln(1/\delta_i) &= o\left(\frac{\log_{d_{i-1}} n_i}{\ln \log_{d_{i-1}} n_i}\right) \left(\ln \log_{d_{i-1}} n_i - \ln \ln \log_{d_{i-1}} n_i\right) \\ &= o(\log_{d_{i-1}} n_i), \end{aligned}$$

and hence (8) holds, and we have proved the first claim of the theorem. ■

Trajectories of random walks, for obtaining lower bound of the universal travelling salesman problem, appear in [17] (one trajectory) and in [5] (two trajectories). In the proof of Theorem 5.1 we consider several trajectories and an essential point of our argument is appropriate subdivision of Γ_i into convex intervals.

Remark 5.3. Recall that a discrete version of Cheeger’s inequality (see, e.g., [25, Theorem 2.4]) relates the expansion coefficient $h(\Gamma)$ of a d -regular finite graph with its spectral gap:

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

In particular, the assumption of Theorem 5.1 is verified for a sequence of graphs of fixed degree satisfying

$$\frac{1}{h(G_i)^2} = o\left(\frac{\ln n_i}{\ln \ln n_i}\right).$$

Remark 5.4. The work of Hume–Mackay–Tessera [27, Proposition 9.5] implies that for a union X of finite AN-dimension of graphs Γ_i of bounded degree, its Poincaré profile satisfies for some C ,

$$\Lambda_X^2(|\Gamma_i|) \leq \frac{C|\Gamma_i|}{\log |\Gamma_i|} + C.$$

Here [27, Proposition 9.5] is applied to X being a disjoint union of graphs, $\delta = 1$, and one uses the estimation $\gamma_n(t) \leq d^{Kt}$, which holds for graphs of degree d and n -dimensional control function Kt . From Bourdon [7, Proposition 1.2] for $p = 2$ it follows that

$$\Lambda_X^2(|\Gamma_i|) \geq |\Gamma_i|h^2(\Gamma_i) \sim |\Gamma_i|\lambda_{1,2}^{1/2}(\Gamma_i).$$

In notation of Theorem 5.1 this means if a sequence of bounded degree graphs has finite AN-dimension, then $\delta_i \leq \text{Const}(1/\ln n_i)^2$ for all i and some constant Const . In other words, if $1/\delta_i = o(\ln n_i)^2$, then AN-dimension of this sequence is infinite. We are grateful to David Hume for explanations on [27] and [7].

Remark 5.5 (Wreath products, lamplighter on cyclic groups). Let Γ_i be a Cayley graph of $\mathbb{Z}/i\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$, with respect to standard generators of this group. Then δ_i is asymptotically equivalent to $1/i^2$, and thus is equivalent to $1/\ln(n_i)^2$ (in fact, in this particular example all spectral values and the spectral measure for SWS random walk are calculated, see Grigorchuk–Leemann–Nagnibeda [19, Theorem 5.1] for a calculation of spectrum and spectral measure of de Bruijn graphs and [19, Theorem 6.1.3] for explanation that lamplighter graphs are a particular case of de Bruijn graphs). Thus, $1/\delta_i$ is equivalent to $(\ln n_i)^2$. Recall that the AN-dimension of the disjoint union of $\mathbb{Z}/i\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ is finite (see Example 4.3 in the previous section). Thus by Theorem 4.4 we know that Br of this union is finite, and claim (1) of Theorem 5.1 does not hold.

More generally, any non-trivial sequence of finite lamplighter graphs (and also many other sequences of wreath products) violates the assumption of Theorem 5.1, as we discuss in the following remark. However, in contrast with graphs from the previous remark, many such sequences satisfy claim (1) of the theorem. We will see it in the next section, as an application of the criterion of weak imbeddings of cubes.

Remark 5.6. There exists $C > 0$ such that for any finite group A , of cardinality at least 2, the spectral gap of $A \wr \mathbb{Z}/2\mathbb{Z}$ satisfies

$$1/\delta \geq C \ln n, \tag{9}$$

where n is the cardinality of $A \wr \mathbb{Z}/2\mathbb{Z}$. (It is clear $n = \#A2^{\#A}$). This follows from the result of Peres–Revelle [40]. Their result, under a more general assumption on the

Markov chain on the wreath product (including simple random walks) states that

$$t_{\text{rel}}(A \wr \mathbb{Z}/2\mathbb{Z}) \geq \frac{1}{8 \ln 2} t_{\text{hit}}(A),$$

where *relaxation time*

$$t_{\text{rel}} = \max_{j: \tilde{\lambda}_j < 1} \frac{1}{(1 - |\lambda_j/d|)}$$

(in particular for random walks with non-negative spectrum $t_{\text{rel}} = 1/(1 - \lambda_2/d)$) and $t_{\text{hit}}(A)$ is the *maximal hitting time* of a random walk on A , that is the maximum, over x, y in the graph of the expected time to hit y starting from x . Recall that for a simple random walk on a finite graph Γ , hitting time is $\geq \#\Gamma/2$. Indeed, if we fix $x \in \Gamma$, consider T_y to be the time of the first visit of y ; then for any trajectory p starting at x , we have

$$\sum_{y \in \Gamma} T_y \geq \#\Gamma(\#\Gamma - 1)/2,$$

and hence

$$\sum_{y \in \Gamma \setminus \{x\}} \mathbb{E}_x T_y \geq \#\Gamma(\#\Gamma - 1)/2.$$

(Another result of [40] also implies that for fixed $d \geq 3$ $A = (\mathbb{Z}/i\mathbb{Z})^d$ there is a matching upper bound for $1/\delta \leq \text{Const} \ln n$.)

Moreover, for many wreath products $A \wr B$ the inequality (9) holds, as can be deduced from [31, Theorem 1.3].

Remark 5.7. In Theorem 5.1 (3), we prove a logarithmic lower bound $\ln_{d-1} n_i - C_k$ for the length of some snakes on k points in Γ_i . These estimates cannot be significantly improved in some graphs, since this length cannot be greater than the diameter: recall that for any $d \geq 3$ and any $\varepsilon > 0$, the diameter of an a.e. random d -regular graph on n vertices is at most D , where D is the smallest integer satisfying

$$(d - 1)^D \geq (2 + \varepsilon)dn \ln n$$

(see [6]), in particular the diameter is close to the straightforward lower bound for the diameter (in terms of n and d)

$$D = \log_{d-1} n(1 + o(1)).$$

It is known that random d -regular graphs form a sequence of expander graphs, close to being Ramanujan [16]. For possible Cayley graph examples with a diameter close to $\log_{d-1} n$, see [42, Section 4.1] which provides numerical evidence that the diameter of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ with respect to a random generating set on d generators is close to $\ln_{d-1} n$ as $p \rightarrow \infty$.

Given a graph Γ and an integer $l \geq 1$, denote by $\text{Sc}(\Gamma, l)$ a graph obtained from Γ by replacing each edge by a chain of l edges. We will be interested in the metric on the vertices of such graphs, and therefore to make in a regular graph we can add loops for all new vertices, so that if Γ is a regular graph of degree d , then the scaled graph $\text{Sc}(\Gamma, l)$ is also regular of degree d .

Remark 5.8. Let $l_i \geq 1$ be a sequence of integer numbers. Observe that if Br is infinite for a sequence of graphs Γ_i , it is also infinite for the sequence $\text{Sc}(\Gamma_i, l_i)$. In particular, it is impossible to obtain a necessary and sufficient condition for $\text{Br} = \infty$ in terms of δ_i, n_i and d_i . Indeed, consider a sequence of expander graphs Γ_i and choose l_i to grow rapidly. We can obtain a sequence of graphs $\text{Sc}(\Gamma_i, l_i)$ with very quick decay of normalized spectral gap δ_i , but $\text{Br} = \infty$.

Remark 5.9. On the other hand, take any sequence of graphs Γ_i and choose l_i to grow very slow. We obtain a sequence of graphs $\Gamma'_i = \text{Sc}(\Gamma_i, l_i)$ with δ_i which tends to zero arbitrarily slow. It is not difficult to check that, whatever Γ_i we take and whatever l_i tending to ∞ , the obtained graphs Γ'_i admit orders T_i (a version of Star orders) such that the sequence (Γ'_i, T_i) does not contain snakes of bounded width. Thus, the assumption in (3) of the theorem cannot be weakened: no other condition on the decay of δ_i (unless δ_i is bounded away from zero) can guarantee the existence of a sequence of snakes of bounded width.

6. Weak imbeddings of cubes and infinite order breakpoint

In the previous section we have seen a spectral condition for a sequence of graphs that guarantees that the order breakpoint is infinite. In this section we prove another sufficient condition (for spaces or sequences of spaces) in terms of weak imbeddings of cubes.

The following lemma generalizes the claim of Lemma 2.4 about circles. We denote by S^d a unit d -sphere in \mathbb{R}^{d+1} , the metrics on S^d is the Euclidean metric induced from \mathbb{R}^{d+1} .

Lemma 6.1 (Snakes in the spheres). *Let $\varepsilon > 0$ and let X be a finite ε -net of the Euclidean sphere S^d . Let T be any order on X . Then there exist two antipodal points x and \dot{x} and a snake $(x_1 <_T \dots <_T x_{d+2})$, $x_i \in X$ for $1 \leq i \leq d + 2$, such that the following holds:*

$$\begin{aligned} d(x_i, x) &\leq \varepsilon \quad \text{if } i \text{ is odd,} \\ d(x_i, \dot{x}) &\leq \varepsilon \quad \text{if } i \text{ is even.} \end{aligned}$$

In particular, the diameter of this snake is at least $2 - 2\varepsilon$ and width $\leq 2\varepsilon$, and

$$\text{OR}_{S^d}(d + 1) = d + 1.$$

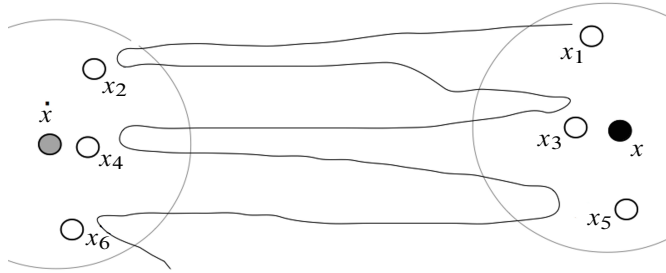


Figure 4. A tail point of a snake is presented by a black point.

Proof. Let x be a point of the sphere, s be a positive integer and $r > 0$. Let us say that x is a *tail point* with parameters (s, r) , if there exists a snake $x_1 <_T \dots <_T x_s$ in X such that all points with odd indexes are at distance at most r from x , and the points with even indexes are at distance at most r from the point \dot{x} antipodal to x . The union of all tail points of parameters (s, r) we denote by $\text{Tail}_{s,r}$. (See Figure 4.)

Fix some positive δ much smaller than ε and define open subsets U_0, \dots, U_d of S^d as follows:

- Let U_0 be the set of all points at distance greater than $\delta/2$ from $\text{Tail}_{2,\varepsilon}$.
- Let U_1 be the set of points at distance smaller than δ from $\text{Tail}_{2,\varepsilon}$ but at distance greater than $\delta/2$ from $\text{Tail}_{3,\varepsilon+\delta}$.
- Let U_2 be the set of points at distance smaller than δ from $\text{Tail}_{3,\varepsilon+\delta}$, but at distance greater than $\delta/2$ from $\text{Tail}_{3,\varepsilon+2\delta}, \dots$
- Let U_d be points at distance less than δ from $\text{Tail}_{d+1,\varepsilon+(d-1)\delta}$ and of distance greater than $\delta/2$ from $\text{Tail}_{d+2,\varepsilon+d\delta}$.

If some of the sets discussed above are empty, we use the convention that a distance to an empty set is ∞ . It is clear that all the subsets U_i are open.

First suppose that U_0, U_1, \dots, U_d do not cover the sphere S^d . Consider a point x not belonging to their union. We observe that the sets

$$\text{Tail}_{2,\varepsilon}, \quad \text{Tail}_{3,\varepsilon+\delta}, \quad \dots, \quad \text{Tail}_{d+2,\varepsilon+d\delta}$$

are at distance no more than $\delta/2$ from this point x . In particular, the set $\text{Tail}_{d+2,\varepsilon+d\delta}$ is not empty. If this happens for arbitrary small δ , then by a compactness argument there exists a point $x \in S^d$ and a snake on $d + 2$ points with all the odd points at distance at most ε from x and all the even points at distance at most ε from \dot{x} . And the claim of the lemma follows.

For the proof of the lemma it is therefore sufficient to prove that (whatever δ we choose) the sets U_0, \dots, U_d cannot cover the sphere. Suppose that it is not the

case: the union of U_i is equal to S^d . We recall that by the Generalized theorem of Lusternik–Schnirelmann (see, e.g., [18]) we know that

If the union of $d + 1$ sets, each of the sets is open or closed, covers S^d , then for one of these sets there exists a point x of S^d such that both x and the antipodal point \dot{x} belong to this set.

We apply this theorem for our open sets U_i . We deduce that if our sets U_0, U_1, \dots, U_d cover the sphere, then there exists k such that U_k contains a pair of antipodal points.

First assume that $k = 0$. Consider x such that both x and \dot{x} belong to U_0 . Since X is an ε -net of the sphere, there exists a point $a \in X$ at distance at most ε from x , and there exists a point $b \in X$ at distance at most ε from \dot{x} . Observe that if $a <_T b$, then $x \in \text{Tail}_{2,\varepsilon}$, and if $a >_T b$, then $\dot{x} \in \text{Tail}_{2,\varepsilon}$. We get a contradiction with the definition of the set U_0 .

Now suppose that for some $k > 0$ the set U_k contains both x and \dot{x} . Since the distance from x to $\text{Tail}_{k+1,\varepsilon+(k-1)\delta}$ is at most δ , we know that $x \in \text{Tail}_{k+1,\varepsilon+k\delta}$. We also know that $\dot{x} \in \text{Tail}_{k+1,\varepsilon+k\delta}$. Therefore, there exist snakes $a_1 <_T \dots <_T a_{k+1}$ and $b_1 <_T \dots <_T b_{k+1}$ in X such that all a_i with even indexes i and all b_j with odd indexes j are at distance $< \varepsilon + k\delta$ from \dot{x} ; and all a_j with odd indexes j and all b_i with even indexes i are at distance $< \varepsilon + k\delta$ from x .

If $a_{k+1} <_T b_{k+1}$, then $a_1 <_T \dots <_T a_{k+1} <_T b_{k+1}$ is a snake and we have $x \in \text{Tail}_{k+2,\varepsilon+k\delta}$. If $a_{k+1} >_T b_{k+1}$, then $\dot{x} \in \text{Tail}_{k+2,\varepsilon+k\delta}$. In both cases we have obtained a contradiction with the definition of U_k , and we have therefore completed the proof of the lemma. ■

A more combinatorial version of Lemma 6.1 is given in Lemma 6.2 below. We recall that the *octahedral triangulation* of a d -dimensional Euclidean sphere (centered at 0) $S^d \subset \mathbb{R}^{d+1}$ is obtained by cutting the sphere by $d + 1$ pairwise distinct coordinate subspaces of dimension d .

Lemma 6.2. *Consider a centrally symmetric triangulation K of the Euclidean sphere S^d , and assume that K is a subdivision of the octahedral triangulation. Let T be an order on vertices of K . Then there exist two antipodal simplices Δ_1 and Δ_2 of this triangulation and a snake on $d + 2$ points, with odd indexed points being vertices of Δ_1 , and even indexed points being vertices of Δ_2 .*

Proof. Consider a triangulation K' , which is the barycentric subdivision of K . By definition, there is a one-to-one correspondence between vertices of K' and simplices of K ; two vertices of K' are adjacent if and only if for the two corresponding simplices of K one is a subset of the other.

Observe that the triangulation K' is also symmetric and also is a subdivision of the octahedral triangulation of S^d . Write in each vertex of K' an integer in the following

way. Let x and x' be two antipodal vertices of K' . Let Δ and Δ' be corresponding (antipodal) simplices of K . Consider maximal s , such that there exists a snake on s points, with all odd indexed vertices in Δ and all even indexed vertices in Δ' , or vice versa. Choose one of these snakes. If its first vertex is in Δ , assign the number $s - 1$ to the point x , and $-(s - 1)$ to the point x' . Analogously, if its first vertex is in Δ' , we assign $-(s - 1)$ to x and $s - 1$ to x' .

Note that any two such snakes of maximal number of points cannot start at opposite simplices. Indeed, otherwise we have snakes $x_1 <_T \dots <_T x_s$ and $y_1 <_T \dots <_T y_s$, the points x_s and y_s are in opposite simplices Δ and Δ' . Since s is the maximal length of snakes, $x_s >_T y_s$ (otherwise we can add y_s to the first snake). In the same way we show that $x_s <_T y_s$ and obtain a contradiction.

If there is a point with assigned number $d + 1$ or larger, the claim of the lemma holds. If not, observe that we can apply Tucker’s lemma to a half of the sphere. We recall that this lemma (see, e.g., [36, Theorem 2.3.1]) claims that if the vertices of a triangulation of the n -ball which is antipodally symmetric on the boundary are labeled with the set $\{\pm 1, \pm 2, \dots, \pm n\}$ such that antipodal vertices on the boundary receive labels which sum to zero then some edge has labels which sum to zero. Applying this lemma to a hemisphere (of S^d), the triangulation K' and the labelling as above, we conclude that there exist two adjacent vertices of K' with two opposite numbers (that is, t and $-t$) written in them. Note that this means that there are snakes on the same number of points with starting points in two antipodal simplices of K . Then at least one of them can be extended for a snake on the larger number of points. This contradiction completes the proof of the lemma. ■

Remark 6.3. A statement similar to Lemma 6.2 can be obtained as a particular case of the Zig-Zag theorem from Simonyi–Tardos [44, Section 3.3]. Given a topologically t -chromatic graph and an ordered coloring of its vertices, this theorem provides a sufficient condition for existence of a full bipartite graph with colors that appear alternating on the two sides of the bipartite subgraph (we do not recall the definition of a topologically t -chromatic graph and refer to [44] for the exact formulation of this result). To study triangulations of spheres one can consider graphs with edges connecting almost antipodal vertices of the triangulation and obtain a version of Lemma 6.2.

Remark 6.4. Similar constructions influence lower bounds in universal ordered protocols (also in other problems), see for example [10, Theorem 9].

In the following definition we consider the l_1 metric on cubes.

Definition 6.5. Let us say that a metric space (M, d_M) *weakly contains arbitrarily large cubes* of dimension d , if there exists a sequence n_i , tending to ∞ and a sequence of mappings of discrete cubes $f_i: [-n_i, n_i]^d \rightarrow M$ (that is, a sequence of mappings

of integer points of $[-n_i, n_i]^d$ to M) such that

$$\lim_{i \rightarrow \infty} \frac{\text{minop}(f_i)}{\text{maxne}(f_i)} = \infty.$$

Here $\text{minop}(f_i)$ is the minimal distance $d_M(f_i(x), f_i(\dot{x}))$, where x, \dot{x} are two antipodal points on the boundary of $[-n_i, n_i]^d$, and $\text{maxne}(f_i)$ is the maximal distance $d_M(f_i(x_1), f_i(x_2))$, where x_1 and x_2 are two neighboring points (points with integer coordinates at l_1 -distance one) in the cube $[-n_i, n_i]^d$.

We also say that a metric space contains *weakly a sequence of arbitrarily large cubes* if for all $d \geq 1$ this space weakly contains arbitrarily large cubes of dimension d .

Now we explain a sufficient condition for $\text{OR}(k) = k$.

Corollary 6.6. *If a metric space M weakly contains arbitrarily large cubes of dimension d , then for any order T on M it holds*

$$\text{OR}_{M,T}(d) = d.$$

In particular, if a metric space M weakly contains a sequence of arbitrarily large cubes, then for any order T on M the order breakpoint of (M, T) is infinite.

Proof. Let T be an order on M . Assume that there exists a sequence n_i , tending to ∞ , and (for each n_i) a mapping f_i of integer points of the boundary of the cube $[-n_i, n_i]^d$ to M , satisfying

$$\frac{\text{minop}(f_i)}{\text{maxne}(f_i)} \rightarrow \infty.$$

We want to prove that (M, T) admits snakes on $d + 1$ points of arbitrarily large elongation.

Take a sufficiently large integer n_i and denote the cube $[-n_i, n_i]^d$ by K_i . The cube K_i contains $(2n_i + 1)^d$ integer points. The boundary S_i of K_i is homeomorphic to a $(d - 1)$ -dimensional sphere, and $(2n_i + 1)^d - (2n_i - 1)^d$ integer points of K_i belong to S_i . Observe that S_i is subdivided into $2d(2n_i)^{d-1}$ unit cubes of dimension $d - 1$, and this is a subdivision of the (image by homeomorphism of the) octahedral partition of S .

Each of the $(d - 1)$ -dimensional unit cubes can be divided into $(d - 1)!$ simplices. Hence, there exists a centrally symmetric triangulation of S_i (consisting of simplices above) such that vertices of this triangulation belong to integer points of K_i and all the simplices have diameters $d - 1$.

Consider the pullback T' of the order T with respect to the mapping f_i . By definition T' is an order on vertices of S_i such that $x <_{T'} y \iff f_i(x) <_T f_i(y)$. By

Lemma 6.2 we know that there exists a pair of antipodal simplices Δ_1, Δ_2 and a snake (x_1, \dots, x_{d+1}) on $d + 1$ points oscillating between them.

Consider the image in M (under f_i) of this snake. Observe that

$$d_M(f_i(x_k), f_i(x_l)) \leq (d - 1)\maxne(f_i)$$

for any k, l of the same parity. We assume that i is large enough, so that

$$\minop(f_i) - 2(d - 1)\maxne(f_i) \geq 1.$$

This assumption guarantees that the image of the points of the snake are distinct points in M . Indeed, for the consecutive points $f_i(x_k), f_i(x_{k+1})$ the assumption above implies that $d_M(f_i(x_k), f_i(x_{k+1})) \geq 1$. By definition of the pullback, the fact that $f_i(x_k) \neq f_i(x_{k+1})$ and $x_k <_{T'} x_{k+1}$, we know that $f_i(x_k) <_T f_i(x_{k+1})$. We conclude therefore that

$$f_i(x_1) <_T f_i(x_2) <_T \dots <_T f_i(x_d) <_T f_i(x_{d+1}).$$

Hence, $(f_i(x_1), \dots, f_i(x_{d+1}))$ is indeed a snake, and its width is no more than $(d - 1)\maxne(f_i)$ and its diameter is at least $\minop(f_i) - 2(d - 1)\maxne(f_i)$. Its elongation is at least

$$\frac{\minop(f_i) - 2(d - 1)\maxne(f_i)}{\maxne(f_i)}.$$

From the definition of weak imbeddings of cubes we conclude that this elongation tends to infinity, and this concludes the proof of the corollary. ■

A particular case of Corollary 6.6 is when G is such that for all d there exists a uniform imbedding of \mathbb{Z}^d in G . This condition holds in particular for any group G that contains \mathbb{Z}^∞ as a subgroup. For example, $G = \mathbb{Z} \wr \mathbb{Z}$. Then for any order T on G it holds

$$\text{OR}_{G,T}(k) = k$$

for all k .

We recall that some known examples of groups of infinite Assouad–Nagata dimension do not admit uniform imbeddings of \mathbb{Z}^d . For example, if $G = \mathbb{Z}^2 \wr A$, A is a finite group of cardinality ≥ 2 , then Assouad–Nagata dimension of G is infinite, see [38], who studied also other amenable wreath products; for a general case of any base group of superlinear growth, see [8, Corollary 5.2]. On the other hand, the asymptotic dimension of G is 2, see [38], see also [8, Theorem 4.5] for the upper bounds on dimensional control function. In particular, G cannot contain uniform images of \mathbb{Z}^3 . Wreath product examples mentioned above are discussed in the following subsection.

6.1. Wreath products

Lemma 6.7 (Sequences of arbitrarily large cubes in wreath products). *Let $G = A \wr B$, where A is a finitely generated group of super-linear growth and B is a finitely generated group of cardinality at least two, then G contains weakly a sequence of arbitrarily large cubes.*

The argument we explain below works for all B , but we want to point out that our main interest when B is finite. (If B is infinite and finitely generated, observe that for all $d \geq 1$, G contains uniformly \mathbb{Z}_+^d , and the claim of the lemma follows.) The proof below is reminiscent of an argument in [8, Theorem 4.1].

Proof. Observe that the claim of the lemma does not depend on the generating set in the wreath product. Fix some generating set S_A of A , a set S_B of B and consider a standard generating set S of G , which one can identify with the union of S_A and S_B . Fix $d \geq 1$. We are going to prove that G contains weakly arbitrarily large cubes of dimension d .

First observe that for infinitely many n there exist d disjoint subsets $\Omega_1^n, \Omega_2^n, \dots, \Omega_d^n$ inside the ball $B_{A,S_A}(e, n)$ of radius n such that the following holds. The cardinality of each set Ω_i is n and for each $i, 1 \leq i \leq d$, the length of any path visiting all points of Ω_i^n is $\geq \frac{1}{6}(n - 1)\sqrt{n}$.

We have already mentioned an elementary case of the polynomial growth theorem of [29]. The result of this paper states that if A is not virtually cyclic, then the growth function of A satisfies $v_{A,S_A}(n) \geq n(n + 1)/2$. In this case, there exist infinitely many n such that

$$v_{A,S_A}(n + \sqrt{n})/v_{A,S_A}(\sqrt{n}/6) \geq n.$$

Take such an n and put $\varepsilon = \sqrt{n}/2$, choose an $(\varepsilon, \varepsilon)$ -net of the ball $B_{A,S_A}(e, n)$ in A . Place a ball of radius $\varepsilon/3$ around each point of the net. In each ball choose d distinct elements. If n is sufficiently large, then $v_{A,S_A}(\sqrt{n}/6)$ is greater than d , and therefore such choice is possible. Observe also that the number of the points in such a net is at least

$$v_{A,S_A}(n + \sqrt{n})/v_{A,S_A}(\sqrt{n}/6) \geq n.$$

In each ball choose one point and denote the union of the chosen points Ω_1^n . Choose one more point in each ball and denote the union of these points by Ω_2^n and so on. It is clear that the distance between any two points of Ω_i^n is at least $\sqrt{n}/6$, and hence the length of any path visiting all points of Ω_i^n is at least $\frac{1}{6}(n - 1)\sqrt{n}$.

Enumerate the points of $(\varepsilon, \varepsilon)$ -net in an arbitrary order and consider the restriction of this order to Ω_i^n . Fix a non-identity element b in the already fixed generating set of B . Consider a map $\rho: [0, 1, \dots, n]^d \rightarrow A \wr B$. A point with coordinates

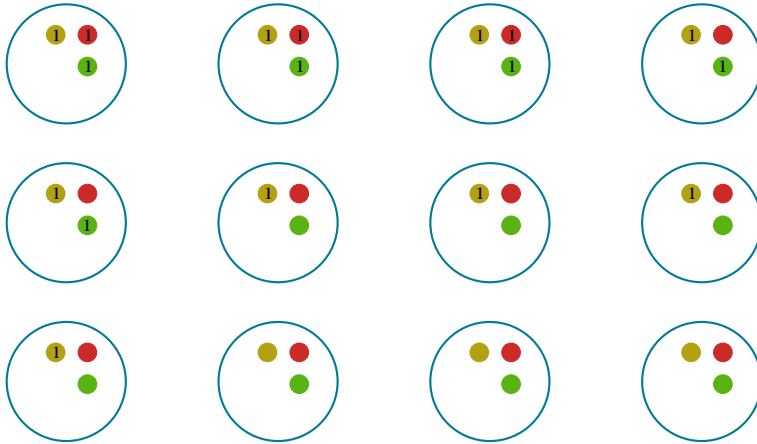


Figure 5. The sets Ω_i^n , here shown for $n = 12$, $d = 3$, $i = 1, 2, 3$. The points of the same color correspond to the same set. The image of $(9, 3, 5)$ under ρ is shown, this is a configuration taking value 1 in the first 9 yellow points, in the first 3 red points and in the first 5 green points.

(z_1, z_2, \dots, z_d) is sent to (e_A, f) where f is the configuration which takes value b in the first z_i elements of Ω_i , $1 \leq i \leq d$, and takes value e_B elsewhere.

Figure 5 shows a possible choice for $A = \mathbb{Z}^2$, $B = \mathbb{Z}/2\mathbb{Z}$. We use an additive notation in B , and a non-identity element b is denoted by 1.

If u, v are points in the cube at distance 1 in l_1 -metric of \mathbb{Z}^d , then the distance between $\rho(u)$ and $\rho(v)$ in the word metric of the wreath product is at most $2n + 1$. Indeed, observe that in this case the values of the configuration of a and that of b differ at one point, which we denote by x . To make this change, it is sufficient to go from identity to x , to make a switch at go back to the identity.

Observe that if we take two antipodal points w, \hat{w} in the boundary of the cube, then the distance between $\rho(w)$ and $\rho(\hat{w})$ is at least $\frac{1}{6}\sqrt{n}(n - 1)$.

Indeed, observe that if we have a pair of antipodal points on the boundary of the cube, then there exists i such that the i -th coordinate of one of them is n and of the other is 0. Therefore, to move from between the images of these antipodal points, we need to visit all points of Ω_i^n , and hence the length of such path is at least $\sqrt{n}(n - 1)/6$.

This completes the proof of the lemma. ■

Corollary 6.8 (Order ratio function for wreath products). *Let $G = A \wr B$ be a wreath product of A and B , where $\#A = \infty, \#B > 1$, A and B are finitely generated. Then either we have a logarithmic upper bound for the order ratio function (and this happens if and only if G has a finite AN-dimension), or the order ratio function is linear (moreover, the order breakpoint is infinite).*

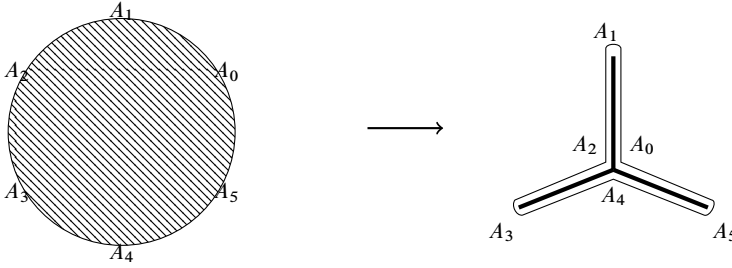


Figure 6. The map from the ball to the tripod.

Proof. Indeed, by a result in [8, Theorem 5.1 and Corollary 5.2] AN-dimension of $A \wr B$ is finite if and only if A is of linear growth and B is finite. The cited theorem deals with the case when B is finite, and it is straightforward that the AN-dimension is infinite when A and B are both infinite, and the dimension is finite when A and B are both finite.

By Theorem 4.4 we know that if AN-dimension is finite, then there is a logarithmic bound for the order ratio function. On the other hand, if A is of super-linear growth or if A and B are infinite, we know by Lemma 6.7 that $A \wr B$ contains weakly arbitrarily large cubes (of arbitrarily large dimension), and hence by Corollary 6.6 the order breakpoint is infinite. ■

6.2. Product of tripods

In Theorem 4.4 we proved that if a metric space M has AN-dimension d , then

$$\text{Br}(M) \leq 2d + 2.$$

Now we will show that this estimate is close to optimal.

Proposition 6.9. *Let M be a Cartesian product of d tripods. Then for any order T on M it holds $\text{OR}_{M,T}(2d) = 2d$; in other words,*

$$\text{Br}(M) \geq 2d + 1.$$

Proof. There exists a continuous map $f: S^{2d-1} \rightarrow M$ such that any two opposite points in S^{2d-1} map to different points.

Indeed, S^{2d-1} is homeomorphic to the boundary of the product D^d of d 2-dimensional unit disks. There is a continuous mapping h from a unit disk to tripod such that h maps any two antipodal points on the boundary of D to points at distance 1.

Denote the mapping $h^d : D^d \rightarrow M$ by f . If x and x' are two opposite points in the boundary of D^d then their projections to one of the disks are opposite points of

boundary of this disk, and projections of $f(x)$ and $f(x')$ to the corresponding tripod are two points at distance 1. (See Figure 6.)

Let T be an order on M and let T' be a pullback of T to S^{2d-1} . By Lemma 6.1, for any ε , there are two opposite points x and x' of S^{2d-1} and a snake

$$x_1 <_{T'} \cdots <_{T'} x_{2d+1}$$

such that points with odd indices are in $B(x, \varepsilon)$ and points with even indices are in $B(x', \varepsilon)$. It is clear that points $f(x_i)$ form a snake of large elongation in (M, T) , so $\text{OR}_{M,T}(2d) = 2d$. ■

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