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Corrigendum and addendum to Appendix A of "Fractal geometry of the complement of Lagrange spectrum in Markov spectrum"

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Abstract. This erratum corrects the statement and proof of Corollary A.4 of [Comment. Math. Helv. 95, 593–633 (2020)].

1. Introduction

In [1, Section A.1, Appendix A], the description

$$j_1 = \lambda_0(\overline{21}12212332221233^*22212332221233321\overline{12}) = 3.70969985975042\ldots$$

of the right endpoint of the largest interval J containing C contains a typo. In fact, [1, Propositions A.1, A.2, and A.3] are correct, and hence the interval (j_0, j_1) containing C whose left endpoint is $j_0 = \lambda_0(\overline{33^*22212}) = 3.70969985967967\ldots$ and the right endpoint is j_1 is certainly disjoint from L. On the other hand, the claim in [1, Corollary A.4] that (j_0, j_1) is the *largest* interval J containing C with $J \cap L = \emptyset$ is not correct because the quantity j_1 does *not* belong to L (contrary to what is claimed right before [1, Corollary A.4]). Fortunately, this mistake in [1, Corollary A.4] is not hard to fix (see below) and, more importantly, it does not affect the other portions of [1].

In the sequel, we correct the statement and proof of [1, Corollary A.4] in Section 2 below by giving an exact description j_1' of the right endpoint of the largest interval J containing C with $J \cap L = \emptyset$. Also, we take the opportunity to make a complement to [1, Appendix A] (which can be thought as a sort of extra subsection (Section A.4) of [1, Appendix A]) by showing in Section 3 below that the local dimension of L near j_1' is one.

2. Corrections to the statement and proof of [1, Corollary A.4]

Let

$$\begin{aligned} j_1' &= \lambda_0(\overline{21}12212332221233^*22212332221233222122121111\overline{12}) \\ &= 3.70969985975153\ldots \end{aligned}$$

Note that $j_1 < j_1'$ are real numbers whose first 11 decimals coincide.

We affirm that the correct version of [1, Corollary A.4] is as follows.

Corollary 2.1. *The largest interval J containing C, which is disjoint from L, is*

$$J = (j_0, j_1').$$

The proof of this statement starts with the following refinement of [1, Proposition A.2].

Proposition 2.2. *If $m(a) = \lambda_0(a) < 3.70969986$ and a contains*

$$12212332221233^*222123322212,$$

then $m(a) \geqslant j_1'$.

Proof. By [1, Lemma 3.1 (i), (ii)], we have

$$\lambda_0(a) \geqslant \lambda_0(\overline{21}12212332221233^*22212332221233\ldots).$$

By [1, Lemmas 3.11 (xxvi), 3.13 (xxxiii), 4.1 (xxxvii), and 3.1 (i)], we deduce that

$$\lambda_0(a) \geqslant \lambda_0(\overline{21}12212332221233^*2221233222123322212\ldots).$$

Since we are assuming that $m(a) = \lambda_0(a)$ and

$$\begin{aligned} &\lambda_7(\ldots 12212332221233^*2221233222123322212 1\ldots) \geqslant 3.70969985996 \\ &\quad > 3.70969985985 \geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222121\ldots), \end{aligned}$$

we get that

$$\lambda_0(a) \geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222122\ldots).$$

Similarly, since $m(a) = \lambda_0(a)$ and

$$\begin{aligned} &\lambda_7(\ldots 12212332221233^*22212332221233222 1222\ldots) \geqslant 3.70969985979 \\ &\quad > 3.709699859752 \geqslant \lambda_0(\overline{21}12212332221233^*222123322212332\ldots), \end{aligned}$$

we obtain that

$$\lambda_0(a) \geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222 1221\ldots).$$

Again, since $m(a) = \lambda_0(a)$ and

$$\begin{aligned}\lambda_7(\ldots 12212332221233^*222123322212332221 2211\ldots) &\geqslant 3.709699859765\\ > 3.709699859752 &\geqslant \lambda_0(\overline{21}12212332221233^*222123322212332\ldots),\end{aligned}$$

we derive that

$$\lambda_0(a) \geqslant \lambda_0(\overline{21}12212332221233^*2221233222123322212212\ldots).$$

Analogously, since $m(a) = \lambda_0(a)$ and

$$\begin{aligned}\lambda_7(\ldots 12212332221233^*222123322212332221 22122\ldots) &\geqslant 3.709699859753\\ > 3.709699859752 &\geqslant \lambda_0(\overline{21}12212332221233^*222123322212332\ldots),\end{aligned}$$

we get that

$$\begin{aligned}\lambda_0(a) &\geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222122121\ldots)\\ &\geqslant \lambda_0(\overline{21}12212332221233^*2221233222123322212212 11\ldots).\end{aligned}$$

Moreover, since $m(a) = \lambda_0(a)$ and

$$\begin{aligned}\lambda_7(\ldots 12212332221233^*2221233222123322212212112\ldots) &\geqslant 3.70969985975183\\ > 3.70969985975181 &\geqslant \lambda_0(\overline{21}12212332221233^*222123322212332\ldots),\end{aligned}$$

we obtain that

$$\begin{aligned}\lambda_0(a) &\geqslant \lambda_0(\overline{21}12212332221233^*2221233222123322212212111\ldots)\\ &\geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222122121111\ldots).\end{aligned}$$

Finally, since $m(a) = \lambda_0(a)$ and

$$\begin{aligned}&\lambda_7(\ldots 12212332221233^*222123322212332221221211112\ldots)\\ &\geqslant 3.70969985975156 > 3.7096998597515581\\ &\geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222\ldots),\end{aligned}$$

we conclude that

$$\begin{aligned}\lambda_0(a) &\geqslant \lambda_0(\overline{21}12212332221233^*222123322212332221221211111\ldots)\\ &\geqslant \lambda_0(\overline{21}12212332221233^*22212332221233222122121111 1\overline{12}) = j_1',\end{aligned}$$

thanks to [1, Lemma 3.1 (i)]. ■

Once we get the previous statement, we can refine [1, Proposition A.3] as follows.

Proposition 2.3. *The open interval $J = (j_0, j_1')$ containing C is disjoint from L.*

Proof. The argument follows the same lines as the (short) proof in the original article [1] after replacing [1, Proposition A.2] by Proposition 2.2 above. ∎

At this point, we can deduce Corollary 2.1 above from the previous proposition by noticing that $j_1' \in L$ because, for any $\beta, \widetilde{\beta} \in \{1,2,3\}^{\mathbb{N}}$, we have

$$\max_{j\in\{-7,7,14\}} \lambda_j(\beta^T 12212332221233^* 22212332221233222122121111 1\widetilde{\beta}) < 3.709699859751525 < j_1',$$

so that j_1' is the limit of Markov values of periodic words with periods of the form

$$\underbrace{21\ldots21}_{n \text{ times}}\, 12212332221233^* 2221233222123322212212111 11\, \underbrace{12\ldots12}_{n \text{ times}}$$

for $n \in \mathbb{N}$.

3. Local dimension of L near j_1'

Corollary 3.1. *We have*

$$\forall \varepsilon > 0, \quad HD(M \cap (j_1', j_1' + \varepsilon)) = HD(L \cap (j_1', j_1' + \varepsilon)) = 1.$$

Proof. As we said right after Proposition 2.3 above, one has that j_1' is equal to

$$\lim_{n\to\infty} \lambda_0(\underbrace{21\ldots21}_{n \text{ times}}\, 12212332221233^* 2221233222123322212212111 11\, \underbrace{12\ldots12}_{n \text{ times}}).$$

By continuity of the function λ_0, given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$ and $\theta, \widetilde{\theta} \in \{1,2\}^{\mathbb{N}}$, one has

$$\lambda_0(\theta^T (21)^n 12212332221233^* 2221233222123322212212111 11 (12)^n \widetilde{\theta}) \in [j_1', j_1' + \varepsilon).$$

Moreover, as it was also said right after Proposition 2.3 above, we have

$$\max_{j\in\{-7,7,14\}} \lambda_j(\beta^T (21)^n 12212332221233^* 2221233222123322212212111 11 (12)^n \widetilde{\beta}) < 3.709699859751525 < j_1'$$

for any $\beta, \widetilde{\beta} \in \{1,2,3\}^{\mathbb{N}}$.

Therefore,

$$\begin{aligned}&\lambda_0(\theta^T(21)^n 12212332221233^* 222123322212332221221211111(12)^n\widetilde{\theta})\\&\quad = m(\theta^T(21)^n 12212332221233^* 222123322212332221221211111(12)^n\widetilde{\theta})\\&\qquad \in L \subset M,\end{aligned}$$

since, if $\theta = (a_1, a_2, \ldots)$ and $\widetilde{\theta} = (b_1, b_2, \ldots)$, then

$$\lambda_0(\theta^T(21)^n 12212332221233^* 222123322212332221221211111(12)^n\widetilde{\theta})$$

is the limit (when $m \to \infty$) of Markov values of periodic words with periods of the form

$$\begin{aligned}&a_m a_{m-1}\ldots\\&\ldots a_1(21)^n 12212332221233^* 222123322212332221221211111(12)^n b_1 b_2 \ldots b_m.\end{aligned}$$

Let us define the following sets:

$$\begin{aligned}K_n &= \big\{[3; 2, 2, 2, 1, 2, 3, 3, 2, 2, 2, 1, 2, 3, 3, 2, 2, 2, 1, 2, 2, 1, 2, 1, 1, 1, 1, 1, (12)^n, \widetilde{\theta}],\\&\qquad \widetilde{\theta} \in \{1, 2\}^{\mathbb{N}}\big\},\\ \widetilde{K}_n &= \big\{[0; 3, 2, 1, 2, 2, 2, 3, 3, 2, 1, 2, 2, 1, (12)^n, \theta], \theta \in \{1, 2\}^{\mathbb{N}}\big\}.\end{aligned}$$

The previous discussion implies that

$$K_n + \widetilde{K}_n \subset L \cap (j_1', j_1' + \varepsilon) \subset M \cap (j_1', j_1' + \varepsilon),$$

whenever $n \geqslant n_0$. Note that K_n and $\widetilde{K}_n$ are two regular Cantor sets diffeomorphic to

$$C(2) = \big\{[0; a_1, a_2, \ldots], a_n \in \{1, 2\}, \forall n \in \mathbb{N}\big\},$$

which is a non-essentially affine regular Cantor set of class $\mathcal{C}^2$ (see [2, Proposition 1]) with $HD(C(2)) > 0.5$. In particular,

$$HD(K_n) = HD(\widetilde{K}_n) = HD(C(2)) > 1/2.$$

It follows from the dimension formula from [3] that

$$HD(K_n + \widetilde{K}_n) = \min\big\{1, HD(K_n) + HD(\widetilde{K}_n)\big\} = 1,$$

so that

$$HD(M \cap (j_1', j_1' + \varepsilon)) = HD(L \cap (j_1', j_1' + \varepsilon)) = 1. \qquad \blacksquare$$

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