Comment. Math. Helv. 98 (2023), 431–508
DOI 10.4171/CMH/552

Pressure at infinity and strong positive recurrence in negative curvature

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(with an appendix by Felipe Riquelme)

Abstract. In the context of geodesic flows of noncompact negatively curved manifolds, we propose three different definitions of entropy and pressure at infinity, through growth of periodic orbits, critical exponents of Poincaré series, and entropy (pressure) of invariant measures. We show that these notions coincide. Thanks to these entropy and pressure at infinity, we investigate thoroughly the notion of strong positive recurrence in this geometric context. A potential is said to be strongly positively recurrent when its pressure at infinity is strictly smaller than the full topological pressure. We show, in particular, that if a potential is strongly positively recurrent, then it admits a finite Gibbs measure. We also provide easy criteria allowing to build such strong positively recurrent potentials and many examples.

1. Introduction

The geodesic flow of a compact connected negatively curved Riemannian manifold M is the typical geometrical example of an *Anosov flow*. Its chaotic behavior reveals itself, in particular, through the existence of infinitely many possible different behaviors of orbits.

A *Gibbs measure* is an ergodic invariant probability measure associated with a given continuous map $F: T^1M \to \mathbb{R}$, with respect to which almost all orbits will spend most of their time in the subsets of T^1M where the potential F is large (see Section 3.3 for the precise definition). In particular, the fact that there exists a Gibbs measure for all Hölder-continuous maps is a quantified way to express that numerous behaviors of orbits are indeed realized as typical trajectories with respect to the Gibbs measures of some Hölder-continuous potentials.

When the manifold M is no longer assumed to be compact, a geometric construction developed in [33] allows to build good candidates for Gibbs measures. However,

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2020 Mathematics Subject Classification. Primary 37D40; Secondary 37A35, 37B40, 37C35, 37D20, 37D35, 37F32.

Keywords. Entropy at infinity, pressure at infinity, strong positive recurrence, noncompact negatively curved manifolds, thermodynamic formalism.

due to noncompactness of M and T^1M, these measures are not necessarily finite, and therefore not always extremely useful.

In [34], Pit and Schapira characterized the finiteness of these measures in terms of the convergence of some geometric series. In [44], in the case of the zero potential $F = 0$, building on [34], Schapira and Tapie proposed a criterion, called *strong positive recurrence*, which implies the finiteness of the associated measure, known as the *Bowen–Margulis–Sullivan measure*. This criterion is as follows: If $\Gamma = \pi_1(M)$, recall that the *critical exponent of* Γ is the exponential growth rate of any orbit of Γ acting on the universal cover $\widetilde{M}$ of M. By a result of Otal and Peigné [31], it also coincides with the topological entropy of the geodesic flow on T^1M. In [44], a *critical exponent at infinity* δ_Γ^∞ is defined, and the authors prove that a critical gap $\delta_\Gamma^\infty < \delta_\Gamma$ implies that the Bowen–Margulis–Sullivan measure is finite. This had been previously shown by Dal'bo, Otal and Peigné in [17] for *geometrically finite manifolds*, for which the critical exponent at infinity is the maximum of the critical exponents among parabolic subgroups. In general, this critical exponent at infinity should be seen as a kind of *entropy at infinity*. Other striking applications of this critical gap have been proved in [16].

The main goal of this paper is to produce a complete study of strong positive recurrence in negative curvature. First, in Sections 4, 5 and 6, we compare this critical exponent at infinity with other, new and old, possible definitions of entropy at infinity and show that they all coincide. At the same time, considering pressures and pressures at infinity instead of entropies, we generalize this study to all Gibbs measures studied in [33,34]. In a second part (Section 7), we give a detailed study of strong positive recurrence in negative curvature. The appendix by F. Riquelme proves important properties of entropy, that are classical in the compact case, but need a careful proof in the noncompact case.

Analogous results have been known for years in the context of symbolic dynamics over a countable alphabet, see [8, 9, 22–24, 39–41].

Let us present our results with more details.

The *topological pressure* of a Hölder-continuous potential $F\colon T^1M \to \mathbb{R}$ is a weighted version of entropy. For a dynamical system on a compact space, there are a lot of different definitions, which all coincide, see for example [49, Chapter 9] or [6]. In the noncompact setting, some of these definitions are meaningless. In [33], following the works of [31, 38] on entropy, three definitions were compared. The *Gurevič pressure* $P_{\mathrm{Gur}}(F)$ is the weighted exponential growth rate of the periodic orbits of the geodesic flow which cross a fixed compact set. The *variational pressure* $P_{\mathrm{var}}(F)$ is the supremum over all invariant probability measures of their measure-theoretic pressures, that is a weighted version of their Kolmogorov–Sinai entropies. The *geometric pressure* $\delta_\Gamma(F)$, a geometric notion specific to geodesic flows also known as *criti-*

cal exponent of (Γ, F), is the weighted exponential growth rate of the orbits of the fundamental group Γ of M acting on its universal cover $\widetilde{M}$.

All the previous discussion applies to the larger setting when $\widetilde{M}$ is still a complete simply connected Riemannian manifold with pinched negative curvature and bounded derivatives of the curvature, and Γ is a discrete group of isometry acting properly on $\widetilde{M}$ possibly with fixed points. In this case, the stabilizer of any point has finite order and $M = \widetilde{M}/\Gamma$ is a *good orbifold*. As considered in [33], the *unit tangent bundle* T^1M is then the set of parametrized bi-infinite geodesics on M with its natural projection from $T^1\widetilde{M}$ and geodesic flow. A Hölder/smooth map on M (respectively, on T^1M) is a map on M (respectively, T^1M) whose lift to $\widetilde{M}$ (respectively, $T^1\widetilde{M}$) is Hölder/smooth. In the sequel, all the results which we present for smooth manifolds can be adapted verbatim to this good orbifold setting. When slight adaptations are required for this generalization, we will specify it in the proof. In the appendix (only), we will restrict to the case where Γ has a subgroup of finite index without torsion. Note that since Γ may not be finitely generated, this is not automatic in our setting.

Standing assumptions in the paper. *We fix once and for all a nonelementary complete connected Riemannian manifold (or good orbifold) M with pinched negative sectional curvature, and bounded first derivative of the sectional curvature.* For all the statements of this section, let us also fix $F\colon T^1M \to \mathbb{R}$ a Hölder-continuous potential.

It has been shown in [31, 38] when $F \equiv 0$ and [33, Theorem 1.1] for general potentials that all these pressures coincide.

Theorem 1.1 (Roblin, Otal–Peigné, Paulin–Pollicott–Schapira). *All notions of pressure coincide:*

$$\delta_\Gamma(F) = P_{\mathrm{var}}(F) = P_{\mathrm{Gur}}(F).$$

We denote this common value by $P_{\mathrm{top}}(F)$, and we call it the topological pressure *of F.*

The terminology differs slightly from [33], where P_{var} was called topological pressure. In retrospect, we consider now that the above terminology is better.

We propose here three notions of pressure at infinity, whose precise definitions will be given in Section 4. The *Gurevič pressure at infinity* $P^\infty_{\mathrm{Gur}}(F)$ measures the weighted exponential growth rate of periodic orbits staying most of the time outside any given compact set. The *variational pressure at infinity* $P^\infty_{\mathrm{var}}(F)$ is the least upper bound of measure-theoretic pressures of invariant probability measures supported mostly outside any given compact set. The *geometric pressure at infinity* $\delta^\infty_\Gamma(F)$ measures the weighted exponential growth rate of those orbits of the fundamental group Γ corresponding to excursions outside any given compact set.

The first main result of this article is the following one.

Theorem 1.2. *All notions of pressures at infinity coincide:*

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F) = P_{\mathrm{Gur}}^\infty(F).$$

We denote this common value by $P_{\mathrm{top}}^\infty(F)$, and we call it the topological pressure at infinity *of F.*

In the special case where F tends to a constant at infinity, the equality $\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F)$ has also been announced in [48] using different methods.

As already implicitly or explicitly noticed for example in [20, 21, 27, 37], this pressure at infinity is deeply related to the phenomenon of loss of mass at infinity. In the vague topology, on a noncompact space, a sequence of probability measures may converge to a finite measure with smaller total mass. As proven by the above authors, if these probability measures have a larger Kolmogorov–Sinai entropy than the entropy at infinity, then they cannot lose the whole mass and converge to the zero measure. In this spirit, as a corollary of Theorem 6.10, we obtain in Corollary 6.11 the following result.

Theorem 1.3. *Let (μ_n) be a sequence of invariant probability measures on T^1M converging in the vague topology to a finite measure μ, with mass $0 \le \|\mu\| \le 1$. Assume that*

$$\int \inf(F, 0)\, \mathrm{d}\mu_n > -\infty$$

for all n. Then their Kolmogorov–Sinai entropies $h_{KS}(\mu_n)$ satisfy the following inequality:

$$\limsup_{n\to\infty}\left(h_{KS}(\mu_n) + \int F\, \mathrm{d}\mu_n\right) \le (1 - \|\mu\|)P_{\mathrm{top}}^\infty(F) + \|\mu\|\, P_{\mathrm{top}}(F).$$

In the geometrically finite case, [27, 37] obtain an improvement of the conclusion of the theorem, with $P_\mu(F)$ instead of $P_{\mathrm{top}}(F)$ on the right, but only for the particular class of potentials F which converge to 0 at infinity, for which $P_{\mathrm{top}}^\infty(F) = P_{\mathrm{top}}^\infty(0)$. An extension of the results of [37] to general manifolds has been announced in [47], cf. also [48, Theorem 1.1]. The strategy used in these papers is different from ours, and does not work yet in general. It would be interesting to obtain their sharper inequality under our weaker assumptions, see [48, Conjecture 5.5].

Once Theorem 1.2 is proven, we can say that a potential F is *strongly positively recurrent* (SPR) when the following *pressure gap* holds:

$$P_{\mathrm{top}}^\infty(F) < P_{\mathrm{top}}(F).$$

We refer the reader to Section 7 for the notions of recurrence, positive recurrence, strong positive recurrence.

An analogous notion of *pressure gap* for potentials on nonpositively curved manifolds, with respect to the set of singular vectors instead of infinity, has been introduced in [12].

As in [44, Theorem 7.1] when $F = 0$, we prove the following extremely useful property of SPR potentials.

Theorem 1.4. *If the potential F is strongly positively recurrent, then it admits a finite Gibbs measure.*

For potentials which vanish at infinity, this has also been announced in [48, Theorem 1.3] using a different strategy. We will show that, on any negatively curved manifold, there exist strongly positively recurrent potentials, see Corollary 4.12. This implies the following new result.

Corollary 1.5. *There exists a Hölder-continuous potential on T^1M which admits a finite Gibbs measure.*

It may be worth pointing out that with their current proofs, all results of [48] quoted above actually rely on the existence of such a potential with finite Gibbs measure (see [48, Lemma 3.9]). Nevertheless, to the best of our knowledge, this fact had not been established beyond geometrically finite manifolds.

We also establish other useful properties. Let m be a finite or infinite Radon measure, invariant under the geodesic flow (g^t). For a given compact subset K in M, and $T \geq T_0$, consider the set $V_{T_0,T}(K)$ of vectors $v \in T^1K$, such that for any $t \in [T_0, T]$, the vector $g^t v$ does not belong to T^1K. These sets $(V_{T_0,T}(K))_{T>T_0}$ decrease when $T \to +\infty$. We say that the flow (g^t) is *exponentially recurrent* with respect to the measure m if there exist a compact set $K \subset M$ whose interior intersects a closed geodesic, and constants $C, \alpha, T_0 > 0$ such that for all $T > T_0$,

$$m(V_{T_0,T}(K)) \leq Ce^{-\alpha T}.$$

In Section 7.4, we establish the following theorem.

Theorem 1.6. *Assume that F has finite topological pressure and finite Gibbs measure m^F. Then F is strongly positively recurrent if and only if the geodesic flow (g^t) is exponentially recurrent with respect to the Gibbs measure m^F.*

Strong positive recurrence says that there exists a compact subset K of M such that the weighted exponential growth rate of the excursions outside K is strictly smaller than the topological pressure. We finish this work with Theorem 7.9, showing that strong positive recurrence does not really depend on the chosen compact set K, in the following sense: We show in Theorem 7.9 that if the potential F is strongly positively recurrent, then for any compact subset K of M, as soon as the interior of K

intersects a closed geodesic, this exponential growth rate of excursions outside K is strictly smaller than the topological pressure.

The first two Sections 2 and 3 contain preliminaries, first on negatively curved geometry and dynamics, and second on thermodynamic formalism, in particular, all different notions of pressures, and the construction of the Gibbs measure m^F.

Sections 4, 5 and 6 on the one hand, and Section 7 on the other hand can be read independently.

Section 4 contains three different definitions of pressures at infinity. In Section 5, we give upper bounds on the growth of certain sets of periodic orbits in terms of entropy and entropy at infinity. We deduce equality of the geometric and Gurevič pressures at infinity $\delta_\Gamma^\infty(F)$ and $P_{\mathrm{Gur}}^\infty(F)$. In Section 6, we show that geometric and variational pressures at infinity $\delta_\Gamma^\infty(F)$ and $P_{\mathrm{var}}^\infty(F)$ coincide. These sections are the technical heart of the paper.

Section 7 is more conceptual. We investigate the notion of strongly positively recurrent potentials in our geometric context, and prove Theorems 1.4 and 1.6.

The appendix by Felipe Riquelme (Theorem A.1) shows that different possible definitions of measure-theoretic entropy, the Kolmogorov–Sinai entropy, the Brin–Katok entropy, and the Katok entropy coincide in our geodesic flow context. This result is well known in the compact case, but not obvious at all without compactness.

2. Negative curvature, geodesic flow

2.1. Geometric preliminaries

Our assumptions and notations are close to those of [33, 34, 44].

Let (M, g) be a smooth complete connected noncompact Riemannian manifold with pinched negative sectional curvature $-b^2 \le K_g \le -a^2$, for some $a, b > 0$, and bounded first derivative of the sectional curvature. Let $\widetilde{M}$ be its universal cover, $\Gamma = \pi_1(M)$ its fundamental group, and $p_\Gamma\colon \widetilde{M} \to M = \widetilde{M}/\Gamma$ the quotient map. We assume that the group Γ is nonelementary, i.e., that the geodesic flow admits at least three different periodic orbits on T^1M. In particular, Γ contains a free group (see for instance [4]). We denote by T^1M and $T^1\widetilde{M}$ the unit tangent bundles of M and $\widetilde{M}$, and by $\pi\colon T^1M \to M$ or $\pi\colon T^1\widetilde{M} \to \widetilde{M}$ the canonical bundle projection. By abuse of notation, we also write $p_\Gamma\colon T^1\widetilde{M} \to T^1M$ for the differential of p_Γ.

Given any two points $x, y \in \widetilde{M}$, the set $[x, y] \subset \widetilde{M}$ will denote the (unique) geodesic segment between x and y.

We fix arbitrarily a point $o \in \widetilde{M}$ that we call *origin*. The boundary at infinity $\partial\widetilde{M}$ is the set of equivalence classes of geodesic rays staying at bounded distance one from another. The *limit set* $\Lambda_\Gamma \subset \partial\widetilde{M}$ is the set of accumulation points $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o$ of the orbit of o. As shown by Eberlein [18], the nonwandering set $\Omega \subset T^1M$ of the geodesic flow is the union of geodesic orbits which admit a lift whose negative and positive endpoints belong to Λ_Γ. The *radial limit set* $\Lambda_\Gamma^{\mathrm{rad}} \subset \Lambda_\Gamma$ is the set of endpoints of geodesics whose images through p_Γ return infinitely often in some compact set:

$$\Lambda_\Gamma^{\mathrm{rad}} := \{\xi \in \Lambda_\Gamma,\ \exists C > 0,\ \exists(\gamma_n) \in \Gamma^{\mathbb{N}},\ \gamma_n o \to \xi,\ d(\gamma_n o, [o\xi)) \leq C\}.$$

We denote by $(g^t)_{t\in\mathbb{R}}$ the geodesic flow acting on T^1M or $T^1\widetilde{M}$. The metric g induces a distance on M and $\widetilde{M}$ that we will simply denote by d. We will also denote by d the distance on T^1M (respectively, on $T^1\widetilde{M}$) defined as follows: for all $v, w \in T^1M$ (respectively, in $T^1\widetilde{M}$), let

$$d(v,w) := \sup_{t\in[-1,1]} d(\pi(g^t v), \pi(g^t w)).$$

This distance is not Riemannian but it is equivalent to the standard Sasaki metric on T^1M (respectively, on $T^1\widetilde{M}$), see [33, Chapter 2] for a discussion on the subject.

The Busemann cocycle is defined for all $\xi \in \partial\widetilde{M}$ and $x, y \in \widetilde{M}$, by

$$\beta_\xi(x,y) = \lim_{z\in[x,\xi),\, z\to\xi} d(x,z) - d(y,z). \tag{1}$$

We will sometimes also write, for all $x, y, z \in \widetilde{M}$,

$$\beta_z(x,y) = d(x,z) - d(y,z).$$

The set of oriented geodesics of $\widetilde{M}$ can be identified with

$$\partial^2\widetilde{M} = (\partial\widetilde{M} \times \partial\widetilde{M}) \setminus \mathrm{Diag}\,.$$

For all $v \in T^1\widetilde{M}$, denote by $v^\pm$ the negative and positive endpoints in $\partial\widetilde{M}$ of the geodesic tangent to v. The unit tangent bundle $T^1\widetilde{M}$ is homeomorphic to $\partial^2\widetilde{M} \times \mathbb{R}$ via the *Hopf parametrization*

$$\mathcal{H}\colon \begin{cases} T^1\widetilde{M} \to \partial^2\widetilde{M} \times \mathbb{R}, \\ \qquad v \mapsto (v^-, v^+, \beta_{v^+}(o, \pi(v))). \end{cases} \tag{2}$$

The geodesic flow acts by translation in these coordinates: for all $v = (v^-, v^+, s)$ and $t \in \mathbb{R}$,

$$g^t(v^-, v^+, s) = (v^-, v^+, t + s).$$

The group Γ acts in these coordinates by

$$\gamma(v^-, v^+, s) = \big(\gamma v^-, \gamma v^+, s + \beta_{v^+}(\gamma^{-1}o, o)\big).$$

In terms of these Hopf coordinates, the nonwandering set Ω is identified with

$$\big((\Lambda^2_\Gamma \setminus \mathrm{Diag}) \times \mathbb{R}\big)/\Gamma.$$

We denote its lift $p_\Gamma^{-1}\Omega$ by $\widetilde{\Omega}$.

Recall that an isometry $\gamma \in \Gamma$ is *hyperbolic* when it admits exactly two fixed points in $\partial\widetilde{M}$. In this case, it acts by translation on the geodesic joining them. The set $\mathscr{P}$ of periodic orbits of the geodesic flow on T^1M is in 1-1 correspondence with the set of conjugacy classes of hyperbolic elements of Γ. Indeed, a periodic orbit p with period $\ell(p)$ can be lifted to a collection $p_\Gamma^{-1}(p)$ of geodesic orbits of $T^1\widetilde{M}$, and each of them, once projected on $\widetilde{M}$, is the oriented translation axis of a unique hyperbolic element γ_p, which acts by translation in the positive direction on the axis, with translation length equal to $\ell(p)$. By construction, all these elements are conjugated one to another.

Not all elements of Γ are hyperbolic. However, the following lemma from [34, Lemma 2.6], variant of the well-known point of view, due to Margulis, of counting elements of Γ inside cones, will allow us to consider only hyperbolic elements.

Lemma 2.1. *Let $\widetilde{K}$ be a compact subset of $\widetilde{M}$ whose interior intersects $\widetilde{\Omega}$. There exist finitely many elements $g_1, \ldots, g_k$ in Γ such that for every $\gamma \in \Gamma$, there exist g_i, g_j among them such that $g_j^{-1}\gamma g_i$ is hyperbolic, and its translation axis intersects $\widetilde{K}$.*

Proof. By [34, Lemma 2.6] applied with $\widetilde{W}$ the interior of $\widetilde{K}$, there exist finite sets $F = \{g_1, \ldots, g_k\}$ and $S = \{s_1, \ldots, s_j\}$ in Γ such that every $\gamma \in \Gamma \setminus S$ satisfies the conclusion of the lemma with respect to F. Consider a hyperbolic element h whose axis intersects $\widetilde{K}$. Then the set $F' = \{g_1, \ldots, g_k, s_1, \ldots, s_j, h\}$ works for every $\gamma \in \Gamma$. Indeed, it works for $\gamma \notin S$ by assumption, and for $\gamma = s_i \in S$ then $s_i^{-1}\gamma h = h$ has a translation axis intersecting $\widetilde{K}$, with $s_i, h \in F'$. ■

The following elementary lemma will be used several times.

Lemma 2.2. *Consider x, y, z three points in a geodesic metric space $\widetilde{M}$, and denote by $[y, z]$ a geodesic between y and z. Then*

$$d(y,x) + d(x,z) - 2d(x,[y,z]) \le d(y,z) \le d(y,x) + d(x,z).$$

We will often need more precise distance estimates, which rely on a negative upper bound of the curvature. The next lemma follows from [33, Lemma 2.5].

Lemma 2.3. *For all $D > 0$ and all $\varepsilon > 0$, there exists $T_0 = T_0(D, \varepsilon) > D$ such that if $x, x', y, y' \in \widetilde{M}$ satisfy $d(x,x') \le D$, $d(y,y') \le D$ and $d(x,y) \ge 2T_0$, then there exists $s_0 \in [-T_0, T_0]$ such that, if v_{xy} (respectively, $v_{x'y'}$) denotes the unit tangent* *vector based at x (respectively, x') tangent to the segment $[x, y]$ (respectively, $[x', y']$), then for all $t \in [T_0, d(x, y) - T_0]$,*

$$d(g^t v_{xy}, g^{t+s_0} v_{x'y'}) \leq \varepsilon.$$

We will also need the following lemma which allows to approximate broken geodesics by axes of hyperbolic elements. If $x \neq y \in \widetilde{M}$, let v_{xy} denote the (oriented and unitary) tangent vector of the geodesic segment $[x, y]$ at x. If $v, w \in T^1_x \widetilde{M}$, set $\measuredangle(v, w) \in [0, \pi]$ for their geometric angle. If $v \in T^1_x \widetilde{M}$ and $w \in T^1_y \widetilde{M}$, denote by $\measuredangle(v, w) \in [0, \pi]$ the geometric angle between v and the image of w through the parallel transport from y to x along $[y, x]$. See Figure 1 for the next lemma.

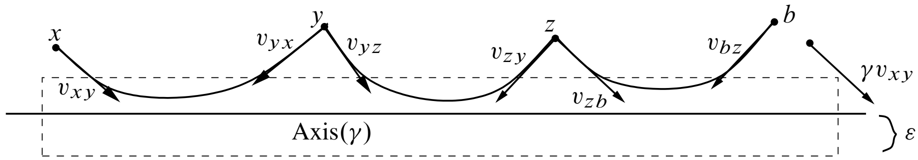

Figure 1. Broken geodesic close to a hyperbolic axis.

Lemma 2.4. *For all $\theta \in (0, \pi)$, and all $\varepsilon > 0$, there exists $C = C(\theta, \varepsilon) > 0$ such that the following holds. Let $x, y, z, b \in \widetilde{M}$ and $\gamma \in \Gamma$ be such that $d(x, y), d(y, z)$ and $d(z, b)$ are at least $2C$, and $d(b, \gamma x) \leq 1/C$. Assume, moreover, that the angles $\measuredangle(v_{yx}, v_{yz})$, $\measuredangle(v_{zy}, v_{zb})$, and $\measuredangle(\gamma v_{xy}, v_{bz})$ are at least θ. Then γ is hyperbolic, the piecewise geodesics $[x, y] \cup [y, z] \cup [z, b]$ is in the ε-neighborhood of its axis except in the C-neighborhood of the points x, y, z and b. Moreover, the period T_γ of γ satisfies*

$$T_\gamma - (6C + 1) \leq d(x, y) + d(y, z) + d(z, b) \leq T_\gamma + 6C + 1.$$

Sketch of proof. By the arguments presented in [33, p. 98], the geodesics from x to b and from x to γx are uniformly close to the union of segments $[x, y] \cup [y, z] \cup [z, b]$, so that v_{xy} and $v_{x,\gamma x}$ on the one hand, and v_{bz} and $v_{\gamma x,x}$ on the other hand, are uniformly close. In particular, adjusting the constants, it implies that the angle between $v_{\gamma x,x}$ and $\gamma v_{x,\gamma x}$ is uniformly bounded from below by, say, $\theta/2$.

When C is large enough, this prevents γ to be parabolic. Indeed, in this case, $v_{\gamma x,x}$ and $\gamma v_{x,\gamma x}$ would be close to the vector from γx to the parabolic fixed point of γ, and therefore very close one from another.

The rest of the proof is an immediate adaptation of arguments of [33, p. 98]. ∎

2.2. Dynamical properties of the geodesic flow

In restriction to its nonwandering set Ω, the geodesic flow satisfies nice dynamical properties. It is *transitive* in the sense that for all nonempty open sets $U, V \subset \Omega$, there exists $T > 0$ such that $g^T U \cap V \neq \emptyset$. And it satisfies a *closing lemma* (see for instance Eberlein in [19, Proposition 4.5.15]): for every compact subset $K \subset \Omega$ and all $\varepsilon > 0$, there exist $\eta > 0$ and $T = T(K, \varepsilon) > 0$, such that for all $v \in K$, and $t > T$ such that $d(g^t v, v) \leq \eta$, there exists a periodic vector p whose period $\ell(p)$ satisfies $|\ell(p) - t| \leq \varepsilon$, and for all $0 \leq s \leq t$, we have $d(g^t p, g^t v) \leq \varepsilon$.

However, we will need similar properties for vectors close to Ω but that may be wandering, and we will also need to make sure that the glued orbit enters an a priori fixed ball. In this direction, we will use several times the following proposition.

Proposition 2.5 (Connecting lemma). *Let K and K' be compact subsets of M whose interiors intersect $\pi(\Omega)$, and $\widetilde{K}$ a compact subset of $\widetilde{M}$ such that $p_\Gamma(\widetilde{K}) = K$. For all $\varepsilon > 0$, there exist $T_0 = T_0(\widetilde{K}, K', \varepsilon) > 0$ and $C_0 = C_0(\widetilde{K}, K', \varepsilon) > 0$ such that the following holds. There exists a construction that associates to any $T \geq 2T_0$ and any $v \in T^1 K$ such that $g^T v \in T^1 K$ a periodic orbit $\wp(v, T)$ that satisfies the following assertions.*

(1) (Shadowing). *The periodic orbit $\wp(v, T)$ has a period belonging to $[T, T + T_0]$, it intersects the interior of $T^1 K'$, and there exists a periodic vector u on this periodic orbit, such that for all $t \in [T_0, T - T_0]$, we have $d(g^t v, g^t u) \leq \varepsilon$.*

(2) (Bounded multiplicity). *For each periodic orbit p with period $T = \ell(p) \geq 2T_0$ going through $T^1 K$, choose arbitrarily a periodic vector $v_p \in T^1 K \cap p$ on p, and denote by $\wp(v_p, \ell(p))$ the corresponding new periodic orbit associated with v_p by our specific construction in* (1). *Then, given any periodic orbit $\wp_0$, the number of periodic orbits p such that $\wp(v_p, \ell(p)) = \wp_0$ is bounded by $C_0 \ell(\wp_0)$.*

Remark 2.6. The first assertion of the above proposition is a standard consequence of transitivity, local product structure, and closing lemma when $v \in \Omega$, but needs a proof otherwise.

The second assertion is more subtle than other similar statements that hold in a compact setting. When M is compact, one usually simply bounds the number of periodic orbits that ε-shadow a fixed orbit $\wp_0$ during most of their period. However, when the manifold (or orbifold) M is not compact, its injectivity radius is not necessarily bounded from below. Therefore, uniformly bounded multiplicity for the number of closed geodesics that stay in a fixed ε-neighborhood of $\wp_0$ is not true in general, notably when $\wp_0$ crosses parts of the manifold where the injectivity radius is much smaller than ε. That is the reason why we consider only those periodic orbits that are constructed through a given procedure, detailed in the proof below, for which we are able to bound the multiplicity.

In the proof and later on, we will need the following notation. As in [34], if $\widetilde{K} \subset \widetilde{M}$ is a compact subset, let us denote by $n_{\widetilde{K}}(\wp)$ the number of lifts $\widetilde{\wp}$ of a given periodic orbit $\wp$ to $T^1\widetilde{M}$ such that $\pi(\widetilde{\wp})$ intersects $\widetilde{K}$.

Proof. The construction of the orbit $\wp(v, T)$ will be explained inside the proof of the first assertion, and the specificities of the construction will be used in the proof of the second assertion.

Proof of Assertion (1). The reader may follow the proof on Figure 2. We can assume that 2ε is smaller than 1. We fix once for all a vector w in the intersection of Ω and the interior of T^1K'. Up to reducing ε, we can assume that $B(\pi(w), 2\varepsilon) \subset K'$.

By compactness of $\widetilde{K}$, as Λ_Γ is not reduced to a single point, there exists

$$\theta = \theta(\widetilde{K}) > 0$$

such that for all $y \in \widetilde{K}$ and $\widetilde{v} \in T^1_y\widetilde{K}$, there exists $\xi \in \Lambda_\Gamma$ such that $\measuredangle(v_{y\xi}, \widetilde{v}) > \theta$. As the geodesic flow is topologically transitive on Ω, and the action of Γ on Λ_Γ is minimal, we can assume moreover that the positive geodesic orbit on T^1M associated with $(g^t v_{y\xi})_{t\geq 0}$ contains Ω in its closure. Let $C = C(\theta, \varepsilon)$ be the constant provided by Lemma 2.4. Let $\varepsilon' = \min(\varepsilon, 1/(2C)) \leq \varepsilon$. By compactness of $T^1K \cap \Omega$, a uniform property of transitivity holds, in the following sense. There exist

$$T_1 > 2C \quad \text{and} \quad T_2 > T_1 + 6C,$$

that depend only on K, K' and ε', such that the vector $v_{y\xi}$ can be chosen in such a way that the projection on T^1M of $g^{[2C,T_1]}(v_{y\xi})$ intersects $B(w, \varepsilon')$ and the projection on T^1M of $g^{[T_1+6C+1,T_2]}(v_{y\xi})$ intersects once again $B(w, \varepsilon')$.

Let $v \in T^1K$. Set $y_0 = \pi(v) \in M$ and take $y \in \widetilde{K}$ such that $p_\Gamma(y) = y_0$. Let $\widetilde{v} \in T^1_y\widetilde{M}$ be such that $p_\Gamma(\widetilde{v}) = v$. By the above claim, there exists

$$\widetilde{v}' = -v_{y\xi} \in T^1_y\widetilde{M} \quad \text{with } \measuredangle(\widetilde{v}, \widetilde{v}') \leq \pi - \theta$$

such that, with $v' = p_\Gamma(\widetilde{v}')$, the half orbit $(\{g^{-t}v', t \geq 0\})$ is dense in Ω, and at two distinct times $t_1 \in [2C, T_1]$ and $t_2 \in [T_1 + 6C + 1, T_2]$, we have

$$g^{-t_1}v' \in B(w, \varepsilon') \quad \text{and} \quad g^{-t_2}v' \in B(w, \varepsilon').$$

We will see below how it will be important. Set $x = \pi(g^{-t_2}\widetilde{v}')$.

By assumption, $g^T v \in T^1K$ for some $T \geq 0$ large enough (to be made precise later on). Set $z = \pi(g^T\widetilde{v})$. By the same arguments now applied to $-g^T\widetilde{v}$, there exists

$$\widetilde{v}'' \in T^1_z\widetilde{M} \quad \text{with } \measuredangle(g^T\widetilde{v}, \widetilde{v}'') \leq \pi - \theta$$

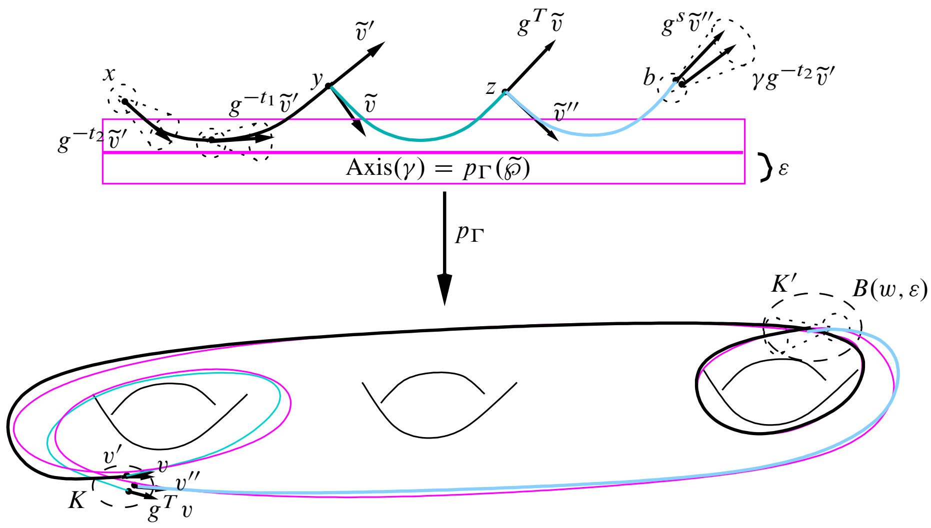

Figure 2. Connecting lemma.

such that, if $v'' = p_\Gamma(\widetilde{v}'')$, the half orbit $(g^t v'')_{t\geq 0}$ is dense in Ω, and for some $s \in [2C, T_1]$, we have $g^s v'' \in B(w, \varepsilon')$. Let $b = \pi(g^s \widetilde{v}'')$ be the base point of $g^s\widetilde{v}''$.

Consider now the broken geodesic path

$$(g^t g^{-t_2}\widetilde{v}')_{0\leq t\leq t_2} \cup (g^t\widetilde{v})_{0\leq t\leq T} \cup (g^t\widetilde{v}'')_{0\leq t\leq s}.$$

It starts from $x = \pi(g^{-t_2}\widetilde{v}')$, has an angle at least θ at $y = \pi(\widetilde{v})$, a second angle at least θ at $z = \pi(g^T\widetilde{v})$, and ends at $b = \pi(g^s\widetilde{v}'')$. Since $p_\Gamma(x)$ and $p_\Gamma(b)$ are both in $\pi(B(w,\varepsilon'))$, there exists $\gamma \in \Gamma$ such that

$$d(\gamma x, b) \leq 2\varepsilon' \leq 1/C.$$

Moreover, if ε is small enough, since $g^{-t_2}v' \in B(w,\varepsilon')$ and $g^s v'' \in B(w,\varepsilon')$ with $\varepsilon' \leq \varepsilon$, up to changing $\gamma \in \Gamma$, the angle $\measuredangle(\gamma g^{-t_2}\widetilde{v}', g^s\widetilde{v}'')$ is at most $\pi - \theta$.

Assume that $T \geq 2C$. Then Lemma 2.4 applies to the sequence of points x, y, z, b. Therefore, γ is hyperbolic. Its translation axis can be written as $\pi(\widetilde{\wp})$, where we choose its lift $\widetilde{\wp}$ to $T^1\widetilde{M}$ oriented so that γ acts by positive translation on it. By Lemma 2.4 again, the broken geodesic path

$$[x,y] \cup [y,z] \cup [z,b]$$

is in the ε-neighborhood of $\pi(\widetilde{\wp})$, except maybe in the C-neighborhood of x, y, z, b. As we chose v' so that $g^{-t_1}v' \in B(w,\varepsilon') \subseteq B(w,\varepsilon)$, with $t_1 \in [2C, d(y,x) - 2C]$, the periodic orbit $\wp = p_\Gamma(\widetilde{\wp})$ intersects $B(w, 2\varepsilon) \subset T^1K'$ near $g^{-t_1}v'$. Moreover, since $T_1 + 6C + 1 \leq d(x, y) \leq T_2$ and $d(y, z) = T$ and $d(z, b) \leq T_1$, it follows from Lemma 2.4 that the translation length $\ell(\gamma)$ of γ satisfies

$$(T_1 + 6C + 1) + T - (6C + 1) \leq \ell(\gamma) \leq T_2 + T + T_1 + 6C + 1.$$

To conclude, let q' be the first point on the geodesic path $[y, z]$ which lies in the ε-neighborhood of the axis of γ, and define a point q as the closest point to q' on the translation axis of γ. Let $\sigma = d(y, q') \geq 0$ so that $q' = \pi(g^\sigma \widetilde{v})$. Let $\widetilde{u}'$ be the tangent vector to the axis of γ at the point q pointing in the same direction as $g^\sigma \widetilde{v}$. By construction, the vector $u := p_\Gamma(g^{-\sigma}\widetilde{u}')$ satisfies, for all $\sigma \leq t \leq T - C$, that

$$d(\pi(g^t u), \pi(g^t v)) \leq \varepsilon.$$

With $\tau = C + 1$, the same kind of estimate holds on $T^1 M$: for all $\tau \leq t \leq T - \tau$, we have

$$d(g^t u, g^t v) \leq \varepsilon.$$

This shows that u satisfies the conclusion of assertion (1), with

$$T_0 = \max(\tau, T_1 + T_2 + 6C + 1).$$

The above procedure of construction of the periodic orbit $(g^t u)$ depends on several arbitrary choices. We define $\wp(v, T)$ as one arbitrary periodic orbit obtained by the above construction.

Proof of Assertion (2). For each periodic orbit p and $v_p \in T^1 K \cap p$ as in the statement, let $\widetilde{v}_p \in T^1 \widetilde{K}$ be the lift of v_p to the universal cover used in the first step in order to define $\wp(v_p, \ell(p))$. Let $\gamma_p \in \Gamma$ be the hyperbolic element whose axis is the lift of p through $y_p = \pi(\widetilde{v}_p)$, oriented in the direction of $\widetilde{v}_p$, and whose translation length is $\ell(p)$.

Assume that $\wp(v_p, \ell(p))$ is equal to a given periodic orbit $\wp_0$. Then, by the construction in the first step, there exist a constant C_1 (depending only on $\widetilde{K}$, K' and ε), a vector $\widetilde{u}_p \in T^1 \widetilde{M}$ and a lift $(\widetilde{\wp}_0)(p)$ of $\wp_0$, that may depend on p, admitting a fundamental domain $((g^t \widetilde{u}_p))_{0 \leq t \leq \ell(\wp_0)}$ whose projection on $\widetilde{M}$ is within Hausdorff distance at most C_1 of $[y_p, \gamma_p y_p]$. In particular, this lift intersects the C_1-neighborhood $\widetilde{K}_{C_1}$ of $\widetilde{K}$ as $\pi(\widetilde{u}_p) \in \widetilde{K}_{C_1}$. Conversely, given a lift $\widetilde{\wp}_0$ of $\wp_0$ intersecting $T^1 \widetilde{K}_{C_1}$, let us show that the number of p with $(\widetilde{\wp}_0)(p) = \widetilde{\wp}_0$ is uniformly bounded. The point $\pi(\widetilde{u}_p)$ can only belong to a compact part of $\widetilde{\wp}_0$ (of length at most $\operatorname{diam} \widetilde{K} + 2C_1$), hence $\pi(g^{\ell(\wp_0)}\widetilde{u}_p)$ is also restricted to a subset of diameter $\operatorname{diam} \widetilde{K} + 2C_1$, and therefore $\gamma_p y_p$ is also restricted to a subset of diameter $\operatorname{diam} \widetilde{K} + 3C_1$. Moreover y_p belongs to the compact set $\widetilde{K}$. For any $R > 0$, there exists a constant $A(R)$ such that, for any $x \in \widetilde{M}$, the number of elements γ of Γ with $\gamma\widetilde{K} \cap B(x, R) \neq \emptyset$ is bounded by $A(R)$: if this number is nonzero, one can pull back by one of these elements to bring $B(x,R)$ to a fixed size neighborhood of $\widetilde{K}$, where the result is obvious by compactness. It follows that the number of possible γ_p is uniformly bounded by $A(\operatorname{diam}\widetilde{K} + 3C_1)$, as claimed.

We have proved that there exists a uniform constant A depending only on $\widetilde{K}$, K' and ε such that the number of periodic orbits p with $\wp(v_p, \ell(p)) = \wp_0$ is bounded from above by A times the number $n_{\widetilde{K}_{C_1}}(\wp_0)$ of lifts $\widetilde{\wp}_0$ of $\wp_0$ that intersect $T^1\widetilde{K}_{C_1}$.

It remains to bound the number $n_{\widetilde{K}_{C_1}}(\wp_0)$ of such lifts $\widetilde{\wp}_0$ of $\wp_0$. Assertion (2) follows from the fact that there exists a constant $B = B(\widetilde{K}, C_1) > 0$ such that for every periodic orbit $\wp_0 \subset T^1M$,

$$n_{\widetilde{K}_{C_1}}(\wp_0) \leq B \times \ell(\wp_0). \tag{3}$$

Let us prove this bound. As $\widetilde{K}_{C_1+1}$ is compact, there exists a constant B such that any point in M has at most B preimages under p_Γ in $\widetilde{K}_{C_1+1}$. Each lift $\widetilde{\wp}_0$ of $\wp_0$ intersecting $T^1\widetilde{K}_{C_1}$ spends a time at least 1 in $\widetilde{K}_{C_1+1}$. Therefore,

$$n_{\widetilde{K}_{C_1}}(\wp_0) \leq \operatorname{Leb}(p_\Gamma^{-1}(\wp_0) \cap \widetilde{K}_{C_1+1}).$$

By the choice of B, this is bounded by $B \operatorname{Leb}(\wp_0) = B\ell(\wp_0)$, proving (3). ■

3. Thermodynamical formalism

Entropy is a well-known measure of the exponential rate of complexity of a dynamical system, and the measure of maximal entropy is an important tool in the ergodic study of hyperbolic dynamical systems.

Pressure is a weighted version of entropy, which is particularly useful for the study of perturbations of hyperbolic systems. The notion of *equilibrium state* is the weighted analogue of the measure of maximal entropy.

In this section, for the geodesic flow of noncompact negatively curved manifolds, we recall some well-known notions and facts from [33] and [34] on the pressure and the construction of the *equilibrium state* or *Gibbs measure* associated with a Hölder-continuous map $F: T^1M \to \mathbb{R}$. This construction has a long story, initiated by the works of Patterson [32] and Sullivan [45] when $F = 0$, by Hamenstädt [25] and Ledrappier [29]. We refer to [33] for detailed historical background and proofs of the assertions in this paragraph. We follow here mainly [33, Chapter 3] and [43], and [34].

3.1. Pressures of Hölder-continuous potentials

Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map in the following sense: there exist $0 < \beta \le 1$ and $C > 0$ such that for all $v, w \in T^1M$ with $d(v,w) \le 1$, we have

$$|F(v) - F(w)| \le C d(v,w)^\beta.$$

Such a map F will be said (β, C)-Hölder-continuous. Let $\widetilde{F} = F \circ p_\Gamma$ be the Γ-invariant lift of F to $T^1\widetilde{M}$.

For $x \neq y \in \widetilde{M}$, recall the notation

$$\int_x^y \widetilde{F} := \int_0^{d(x,y)} \widetilde{F}(g^t v_{x,y})\,\mathrm{d}t.$$

The following statement is easily implied by [33, Lemma 3.2 and Remark (ii) on p. 34].

Lemma 3.1. *Let $F\colon T^1M \to \mathbb{R}$ be a (β, C_F)-Hölder-continuous map on T^1M, and $\widetilde{F}$ its Γ-invariant lift. There exists a constant $c_1 > 0$ depending only on the upper bound of the curvature and the Hölder constants β, C_F, with the following property. Let $D \ge 1$, and consider points $x, y, x', y' \in \widetilde{M}$ with $d(x,y) \le D$ and $d(x', y') \le D$. Then*

$$\left|\int_x^{x'} \widetilde{F} - \int_y^{y'} \widetilde{F}\right| \le c_1 e^D + D\big(|\widetilde{F}(v_{xx'})| + |\widetilde{F}(g^{d(x,x')} v_{xx'})|\big),$$

where $v_{xx'}$ is the tangent vector at x to the geodesic segment from x to x'.

This bound applies, in particular, when x and y are picked in a compact subset $\widetilde{K}$ of T^1M with diameter at most D, and x' and y' are picked in $\gamma\widetilde{K}$ for some $\gamma \in \Gamma$. In this situation, one gets an upper bound $c_1 e^D + 2D \max_{v \in T^1\widetilde{K}} |F(v)|$ which only depends on $\widetilde{K}$.

Proof. By [33, Lemma 3.2 and Remark (ii) on p. 34], we have

$$\left|\int_x^{x'} \widetilde{F} - \int_y^{y'} \widetilde{F}\right| \le c_1 e^D + D \max_{\pi^{-1}(B(x,D))} |\widetilde{F}| + D \max_{\pi^{-1}(B(x',D))} |\widetilde{F}| \tag{4}$$

for some constant c_1. Moreover, on the ball $\pi^{-1}(B(x,D))$ one has the inequality

$$|\widetilde{F}(v) - \widetilde{F}(v_{xx'})| \le C(\widetilde{F}) D$$

as $\widetilde{F}$ is Hölder-continuous and therefore Lipschitz on large scales. One can therefore bound $D \max_{\pi^{-1}(B(x,D))} |\widetilde{F}|$ with $D|\widetilde{F}(v_{xx'})| + C(\widetilde{F}) D^2$, and then bound the second term with $C' e^D$. The last term in (4) is handled similarly. ∎

There are several natural definitions of pressure, that all coincide, as proven in [33, Theorems 4.7 and 6.1], see Theorem 1.1. We recall here these three definitions.

3.1.1. Geometric pressure as a critical exponent. Recall that some point $o \in \widetilde{M}$ has been chosen once and for all. The Poincaré series associated with (Γ, F) is defined by

$$P_{\Gamma,o,F}(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)+\int_o^{\gamma o} \tilde{F}}.$$

The following lemma is elementary, see for instance [33, pp. 34-35].

Lemma 3.2 (Geometric pressure). *The above series admits a* critical exponent $\delta_\Gamma(F) \in \mathbb{R} \cup \{+\infty\}$ *defined by the fact that for all $s > \delta_\Gamma(F)$ (respectively, $s < \delta_\Gamma(F)$), the series $P_{\Gamma,o,F}(s)$ converges (respectively, diverges). Moreover, $\delta_\Gamma(F)$ does not depend on the choice of o and, for any $c > 0$, satisfies*

$$\delta_\Gamma(F) = \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\gamma\in\Gamma, T-c \le d(o,\gamma o) \le T} e^{\int_o^{\gamma o} \tilde{F}}.$$

We call $\delta_\Gamma(F)$ the critical exponent *of (Γ, F) or the* geometric pressure *of F.*

As Γ is nonelementary, one can show (see [33, Lemma 3.3]) that $\delta_\Gamma(F) > -\infty$. Moreover, observe that $\delta_\Gamma(F)$ is finite as soon as F is bounded from above. In [33, Theorem 4.3], it has been shown that the above limsup is in fact a true limit if c is large enough. In what follows, we will never require F to be bounded from above, but we will sometimes assume that $\delta_\Gamma(F)$ is finite.

3.1.2. Variational pressure. Let $\mathcal{M}_1$ be the set of Borel probability measures on T^1M invariant under the geodesic flow, and $\mathcal{M}_{1,\mathrm{erg}}$ the subset of ergodic probability measures. For a given Hölder-continuous potential $F\colon T^1M \to \mathbb{R}$, consider their subsets $\mathcal{M}_1^F$ and $\mathcal{M}_{1,\mathrm{erg}}^F$ of probability measures with $\int F^-\,\mathrm{d}\mu < \infty$, where $F^- = -\inf(F, 0)$ is the negative part of F. Given $\mu \in \mathcal{M}_1$, we denote by $h_{KS}(\mu) = h_{KS}(g^1, \mu)$ its *Kolmogorov–Sinai entropy*, or *measure-theoretic entropy* with respect to g^1 (see the appendix for the definition).

Definition 3.3. The *variational pressure* of F is defined by

$$P_{\mathrm{var}}(F) = \sup_{\mu\in\mathcal{M}_1^F} \left(h_{KS}(\mu) + \int F\,\mathrm{d}\mu\right) = \sup_{\mu\in\mathcal{M}_{1,\mathrm{erg}}^F} \left(h_{KS}(\mu) + \int F\,\mathrm{d}\mu\right).$$

3.1.3. Growth of periodic geodesics and Gurevič pressure. We denote by $\mathcal{P}$ (respectively, $\mathcal{P}'$) the set of periodic (respectively, primitive periodic) orbits of the geodesic flow. Let now K be a compact subset of M whose interior intersects at least a closed geodesic, and $c > 0$ be fixed. Let us denote by $\mathcal{P}_K$ (respectively, $\mathcal{P}_K(t)$, $\mathcal{P}_K(t-c, t)$) the set of periodic orbits $p \in \mathcal{P}$ of the geodesic flow whose projection $\pi(p)$ on M intersects K (respectively, such that $\ell(p) \le t$, $\ell(p) \in (t-c, t]$). The subsets $\mathcal{P}'_K$, $\mathcal{P}'_K(t)$, $\mathcal{P}'_K(t-c, t)$ of $\mathcal{P}'$ are defined similarly.

Denote by $\int_p F$ the integral of F over $(g^t v_p)_{0\leq t\leq \ell(p)}$ for any v_p on p. By [33, Theorem 4.7], the definition below makes sense.

Definition 3.4 (Gurevič pressure). For any compact subset K of M whose interior intersects a closed geodesic and any $c>0$, the *Gurevič pressure* of F is defined by

$$P_{\mathrm{Gur}}(F)=\limsup_{T\to+\infty}\frac{1}{T}\log\sum_{p\in\mathcal{P}_K(T-c,T)} e^{\int_p F}.$$

It does not depend on K nor c. Moreover, when $P_{\mathrm{Gur}}(F)>0$, then

$$P_{\mathrm{Gur}}(F)=\limsup_{T\to+\infty}\frac{1}{T}\log\sum_{p\in\mathcal{P}_K(T)} e^{\int_p F}.$$

Gurevič was the first to introduce this definition (for the potential $F=0$) in the context of symbolic dynamics, see [22]. The equality $P_{\mathrm{Gur}}(F)=P_{\mathrm{var}}(F)$ has been proven in [5] for compact manifolds and $F=0$, in [7] for compact manifolds and Hölder-continuous potentials. The equality $\delta_\Gamma(F)=P_{\mathrm{Gur}}(F)$ is due to Ledrappier [29] in the compact case.

In the noncompact case, when $F\equiv 0$, Sullivan [46] and Otal–Peigné [31] proved that $\delta_\Gamma=P_{\mathrm{var}}$, and Roblin [38] proved that $P_{\mathrm{Gur}}=\delta_\Gamma$. The equality between the three notions of pressures for general Hölder-continuous potentials on noncompact manifolds is done in [33, Theorems 4.7 and 6.1].

3.2. Patterson–Sullivan–Gibbs construction

Let $F\colon T^1M\to\mathbb{R}$ be a Hölder-continuous potential with finite topological pressure. As will be seen in Paragraph 3.3, the construction of a good invariant measure associated with F will use the product structure $\Omega\simeq((\Lambda_\Gamma^2\setminus\mathrm{Diag})\times\mathbb{R})/\Gamma$. The main step is the definition of a good measure ν^F on Λ_Γ, that we will call a *Patterson–Sullivan–Gibbs measure*. We recall it below with more care than usually done, because we will need in Section 7.3 to deal with technical points of the construction.

As stated in Lemma 3.2, the Poincaré series $P_{\Gamma,o,F}(s)$ converges when $s>\delta_\Gamma(F)$ and diverges when $s<\delta_\Gamma(F)$. We say that (Γ,F) is *divergent* if this series diverges at $s=\delta_\Gamma(F)$, and *convergent* if the series converges.

Following the famous Patterson trick, see [32], when (Γ,F) is convergent, we choose a positive nondecreasing map $h\colon\mathbb{R}^+\to\mathbb{R}^+$ with subexponential growth such that for all $\eta>0$, there exists $C_\eta\geq 1$ such that

$$\forall r\geq 0,\ \forall t\geq 0,\quad h(t+r)\leq C_\eta e^{\eta t}h(r), \tag{5}$$

and the series

$$\widetilde{P}_{\Gamma,F}(o,s)=\sum_{\gamma\in\Gamma} h(d(o,\gamma o))\,e^{-sd(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}$$

has the same critical exponent $\delta_\Gamma(F)$, but diverges at the critical exponent $\delta_\Gamma(F)$. The article [32] provides the construction of such a nondecreasing function h, except that (5) is replaced by the following property: for all $\eta > 0$, there exists $r_\eta > 0$ such that

$$\forall r \geq r_\eta,\ \forall t \geq 0, \quad h(t+r) \leq e^{\eta t} h(r).$$

We claim that this property implies (5). Indeed, the result is obvious for $r \geq r_\eta$, while for $r \leq r_\eta$ one may write

$$h(t+r) \leq h(t+r_\eta) \leq e^{\eta t} h(r_\eta) = \frac{h(r_\eta)}{h(0)} e^{\eta t} h(0) \leq \frac{h(r_\eta)}{h(0)} e^{\eta t} h(r).$$

Hence, (5) follows with $C_\eta = h(r_\eta)/h(0)$.

Define now for all $s > \delta_\Gamma(F)$ a probability measure on $\widetilde{M} \cup \partial\widetilde{M}$ by

$$\nu^{F,s} = \frac{1}{\widetilde{P}_{\Gamma,F}(o,s)} \sum_{\gamma\in\Gamma} h(d(o,\gamma o)) e^{-sd(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}} \mathcal{D}_{\gamma o},$$

where $\mathcal{D}_x$ stands for the Dirac mass at x.

By compactness of $\widetilde{M} \cup \partial\widetilde{M}$, we can choose a decreasing sequence $s_k \to \delta_\Gamma(F)$ such that ν^{F,s_k} converges to a probability measure ν^F. As $\widetilde{P}_{\Gamma,o,F}$ diverges at $s = \delta_{\Gamma,F}$, we deduce that ν^F is supported on $\Lambda_\Gamma \subset \partial\widetilde{M}$.

For all $x, y \in \widetilde{M}$ and $\xi \in \partial\widetilde{M}$, recall the following notation from [43, Section 2.2.1] (with an opposite sign convention compared to [33])

$$\rho_\xi^F(x,y) = \lim_{z\in[x,\xi),\, z\to\xi} \int_x^z \widetilde{F} - \int_y^z \widetilde{F}.$$

Observe that $\rho_\xi^0 = 0$ and more generally, when $F \equiv c$ is constant, $\rho^c = c \times \beta$, where β is the usual Busemann cocycle defined in equation (1).

The measure ν^F satisfies the following crucial property. For all $\gamma \in \Gamma$, and ν^F-almost all $\xi \in \partial\widetilde{M}$,

$$\frac{d\gamma_*\nu^F}{d\nu^F}(\xi) = e^{\delta_\Gamma(F)\beta_\xi(o,\gamma o)-\rho_\xi^F(o,\gamma o)}. \tag{6}$$

As a consequence of (6), one gets the following key property, proved in [30]. Recall that for a given set $A \subset \widetilde{M}$, the *shadow* $\mathcal{O}_x(A)$ of A viewed from a point $x \in \widetilde{M}$ is by definition the set of points $y \in \widetilde{M} \cup \partial\widetilde{M}$ such that the geodesic interval $[x,y]$ intersects the set A.

Proposition 3.5 (Shadow lemma). *There exists $R_0 > 0$ such that for every given $R \geq R_0$, there exists a constant $C > 0$ such that for all $\gamma \in \Gamma$,*

$$\frac{1}{C} e^{-\delta_\Gamma(F)d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}} \leq \nu^F(\mathcal{O}_o(B(\gamma o,R))) \leq C e^{-\delta_\Gamma(F)d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}.$$

Observe that the probability measure ν^F constructed above is not unique a priori, but it will be unique in all interesting cases, see Section 7.1 for details.

In fact, we will need a shadow lemma for the family of measures $\nu^{F,s}$, for $s > \delta_\Gamma(F)$. As the uniformity of the constants in the statements with respect to $s > \delta_\Gamma(F)$ will be crucial, we provide a detailed proof.

For $A, B \subset \widetilde{M}$ two sets, we will use the *enlarged shadow* $\mathcal{O}_B(A) = \bigcup_{x \in B} \mathcal{O}_x(A)$, i.e., the set of points $y \in \widetilde{M} \cup \partial\widetilde{M}$ such that there exists some $x \in B$ such that the geodesic interval $[x, y]$ intersects A.

Lemma 3.6 (Orbital Shadow lemma). *For every compact subset $\widetilde{K}$ of $\widetilde{M}$, there exist $r > 0$ and $\tau > 0$ with the following property:*

(1) Upper bound: *For every $\eta > 0$, there exists $c_\eta > 0$ such that for all $\delta_\Gamma(F) < s \leq \delta_\Gamma(F) + \tau$ and $\gamma \in \Gamma$ with $d(o, \gamma o) \geq r$, we have*

$$\nu^{F,s}(\mathcal{O}_{\widetilde{K}}(\gamma \widetilde{K})) \leq c_\eta e^{-(s-\eta)d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}}.$$

(2) Lower bound: *Assume additionally that $\widetilde{K}$ contains $B(o, R_1)$, where $R_1 = R_1(F)$ is a fixed large constant. Then there exists C such that for all $\delta_\Gamma(F) < s \leq \delta_\Gamma(F) + \tau$ and $\gamma \in \Gamma$ with $d(o, \gamma o) \geq r$, we have*

$$\frac{1}{C} e^{-sd(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}} \leq \nu^{F,s}(\mathcal{O}_o(\gamma \widetilde{K})).$$

Proof. By convexity of the distance in nonpositive curvature, if $D = \operatorname{diam}(\widetilde{K}) + d(o, \widetilde{K})$ and $\widetilde{L}$ is the D-neighborhood of $\widetilde{K}$, then for all $\gamma \in \Gamma$, we have

$$\mathcal{O}_{\widetilde{K}}(\gamma \widetilde{K}) \subset \mathcal{O}_o(\gamma \widetilde{L}).$$

Therefore, upon replacing $\widetilde{K}$ with $\widetilde{L}$ in the upper bound, it suffices to prove it for the shadow $\mathcal{O}_o(\gamma \widetilde{K})$. This also shows that without loss of generality, we can assume that $o \in \widetilde{K}$.

We follow the classical proof of the Shadow lemma, with $\nu^{F,s}$ on $\widetilde{M}$ instead of ν^F on $\partial\widetilde{M}$. By definition, for all $y \in \Gamma o$ and $\alpha \in \Gamma$, we have

$$\frac{d(\alpha_* \nu^{F,s})}{d\nu^{F,s}}(y) = \frac{h(d(\alpha o, y))}{h(d(o, y))} e^{-s(d(\alpha o,y) - d(o,y)) + \int_{\alpha o}^{y} \widetilde{F} - \int_o^y \widetilde{F}}.$$

We deduce that

$$\begin{aligned}
\nu^{F,s}(\mathcal{O}_o(\gamma \widetilde{K})) &= \gamma_*^{-1} \nu^{F,s}(\mathcal{O}_{\gamma^{-1} o}(\widetilde{K})) \\
&= \int_{\mathcal{O}_{\gamma^{-1}o}(\widetilde{K})} \frac{h(d(\gamma^{-1} o, y))}{h(d(o, y))} e^{-s(d(\gamma^{-1}o,y) - d(o,y)) + \int_{\gamma^{-1}o}^{y} \widetilde{F} - \int_o^y \widetilde{F}} \, d\nu^{F,s}.
\end{aligned}$$

The triangular inequality gives

$$d(\gamma^{-1}o, y) \leq d(\gamma^{-1}o, o) + d(o, y).$$

Moreover, since $o \in \widetilde{K}$ and $y \in \mathcal{O}_{\gamma^{-1}o}(\widetilde{K})$, by Lemma 2.2, we have

$$d(\gamma^{-1}o, y) \geq d(\gamma^{-1}o, o) + d(o, y) - 2D.$$

In particular, with $r = 2D$, if $d(\gamma^{-1}o, o) \geq r$, we get $d(\gamma^{-1}o, y) \geq d(o, y)$.

By construction, the map h is nondecreasing and for all $\eta > 0$, there exists $C_\eta > 0$ such that for $\rho \geq 0$, $t \geq 0$, we have $h(t + \rho) \leq C_\eta e^{\eta t} h(\rho)$. Thus, independently of $s > \delta_\Gamma(F)$, we have

$$1 \leq \frac{h(d(\gamma^{-1}o, y))}{h(d(o, y))} \leq \frac{h(d(\gamma^{-1}o, o) + d(o, y))}{h(d(o, y))} \leq C_\eta e^{\eta d(\gamma^{-1}o,o)}.$$

By Lemma 3.1, there exists a positive constant $C(F, \widetilde{K})$, such that uniformly in $y \in \mathcal{O}_{\gamma^{-1}o}(\widetilde{K})$, we have

$$\left| \int_{\gamma^{-1}o}^{y} \widetilde{F} - \int_{o}^{y} \widetilde{F} - \int_{\gamma^{-1}o}^{o} \widetilde{F} \right| \leq C(F, \widetilde{K}).$$

We deduce that, when $s < \delta_\Gamma(F) + 1$,

$$\begin{aligned} \nu^{F,s}(\mathcal{O}_o(\gamma\widetilde{K})) &\leq C_\eta e^{2Ds + C(F,\widetilde{K})} e^{-(s-\eta)d(\gamma^{-1}o,o) + \int_{\gamma^{-1}o}^{o} \widetilde{F}} \times \nu^{F,s}(\mathcal{O}_{\gamma^{-1}o}(\widetilde{K})) \\ &\leq C_\eta e^{2D(\delta_\Gamma(F)+1) + C(F,\widetilde{K})} e^{-(s-\eta)d(\gamma^{-1}o,o) + \int_{\gamma^{-1}o}^{o} \widetilde{F}}. \end{aligned}$$

This concludes the proof of the upper bound.

For the lower bound, we have

$$\nu^{F,s}(\mathcal{O}_o(\gamma\widetilde{K})) \geq e^{-C(F,\widetilde{K})} e^{-sd(\gamma^{-1}o,o) + \int_{\gamma^{-1}o}^{o} \widetilde{F}} \times \nu^{F,s}(\mathcal{O}_{\gamma^{-1}o}(\widetilde{K})).$$

The crucial point is to get a lower bound of the measure on the right hand side. More precisely, as we assume that $\widetilde{K}$ contains a ball centered at o with large radius, we wish to find such a radius $R > 0$ and $\tau > 0$ such that uniformly in $\gamma \in \Gamma$ and $\delta_\Gamma(F) < s < \delta_\Gamma(F) + \tau$, the measure $\nu^{F,s}(\mathcal{O}_{\gamma^{-1}o}(B(o, R)))$ has a positive lower bound. It would follow immediately if we knew that for some $R > 0$, uniformly in $y \in \widetilde{M}$ and $\delta_\Gamma(F) < s < \delta_\Gamma(F) + \tau$, the measure $\nu^{F,s}(\mathcal{O}_y(B(o, R)))$ has a positive lower bound. We follow the usual argument which concludes the proof of the classical Shadow lemma. Imagine by contradiction that there exist

$$s_n \to \delta_\Gamma(F), \quad R_n \to \infty, \quad \text{and} \quad y_n \to y_\infty \in \widetilde{M} \cup \partial\widetilde{M}$$

such that $\nu^{F,s_n}(\mathcal{O}_{y_n}(\widetilde{B}(o, R_n))) \to 0$.

There exists a subsequence s_{n_k} such that $\nu^{F,s_{n_k}}$ converges to some probability measure ν' on the boundary which is supported on the full limit set Λ_Γ. This measure is not a single Dirac mass at y_∞, by nonelementarity. By regularity, we can find an open neighborhood U of y_∞ with $\nu'(U) = 1 - \alpha < 1$. Since $\nu^{F,s_{n_k}}$ converges weakly to ν' and U is open, this entails

$$\nu^{F,s_{n_k}}(U) \leq 1 - \alpha/2$$

for large enough k. For large enough n, the complement of U is then contained in $\mathcal{O}_{y_n}(\widetilde{B}(o, R_n))$ as $y_n \to y_\infty$ and $R_n \to \infty$. This gives

$$\nu^{F,s_{n_k}}(\mathcal{O}_{y_{n_k}}(\widetilde{B}(o, R_{n_k}))) \geq \alpha/2,$$

a contradiction. ∎

3.3. Gibbs measures

Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous potential with finite topological pressure, and let ν^F be a Patterson–Sullivan measure associated with F, as constructed in the previous paragraph.

Denote by $\iota: T^1M \to T^1M$ the involution $v \mapsto -v$, and let $\nu^{F\circ\iota}$ be a Patterson–Sullivan measure associated with $F \circ \iota$. Hopf coordinates allow us to define a Radon measure on $T^1\widetilde{M}$ by the formula

$$\begin{aligned} \mathrm{d}\widetilde{m}^F(v) = \exp\big(&\delta_\Gamma(F)\beta_{v^-}(o,\pi(v)) - \rho_{v^-}^{F\circ\iota}(o,\pi(v)) \\ &+ \delta_\Gamma(F)\beta_{v^+}(o,\pi(v)) - \rho_{v^+}^{F}(o,\pi(v))\big)\,\mathrm{d}\nu^{F\circ\iota}(v_-)\,\mathrm{d}\nu^F(v_+)\,\mathrm{d}t. \end{aligned} \tag{7}$$

By construction, $\widetilde{m}^F$ is invariant under the geodesic flow and it follows from (6) that it is invariant under the action of Γ on $T^1\widetilde{M}$, so that it induces a Radon measure m^F on T^1M.

The following crucial result was shown in [31] for $F = 0$ and in [33, Chapter 6] in general.

Theorem 3.7 ([31]–[33]). *Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous potential with finite topological pressure. Then the following alternative holds. If a measure m^F on T^1M given by the Patterson–Sullivan–Gibbs construction is finite and if, once normalized into a probability measure, it belongs to $\mathcal{M}_1^F$, then it is the unique probability measure realizing the supremum in the variational principle:*

$$P(F) = \sup_{m\in\mathcal{M}_1^F}\Big(h_{KS}(m) + \int_{T^1M} F\,\mathrm{d}m\Big) = h_{KS}\Big(\frac{m^F}{\|m^F\|}\Big) + \int_{T^1M} F\,\frac{\mathrm{d}m^F}{\|m^F\|}.$$

Otherwise, there is no probability measure realizing this supremum.

We will also need the following result, called the *Hopf–Tsuji–Sullivan–Roblin* theorem, see [33, Theorem 5.3] for a more complete statement and a proof.

Theorem 3.8 (Hopf–Tsuji–Sullivan–Roblin theorem, [33]). *Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous potential with finite topological pressure, and let ν^F and m^F be associated with F as above. The following assertions are equivalent.*

(1) *The pair (Γ, F) is divergent, i.e., the Poincaré series $P_{\Gamma,o,F}(s)$ diverges at the critical exponent $\delta_\Gamma(F)$;*

(2) *the measure ν^F gives positive measure to the radial limit set: $\nu^F(\Lambda_\Gamma^{\mathrm{rad}}) > 0$;*

(3) *the measure ν^F gives full measure to the radial limit set: $\nu^F(\Lambda_\Gamma^{\mathrm{rad}}) = 1$;*

(4) *the measure m^F is conservative for the action of the geodesic flow on T^1M;*

(5) *the measure m^F is ergodic and conservative for the action of the geodesic flow on T^1M.*

Together with the above Hopf–Tsuji–Sullivan–Roblin theorem, the Poincaré recurrence theorem implies the following crucial observation:

When the measure m^F is finite, it is ergodic and conservative.

4. Pressures at infinity

In this section, we recall first the notion of *fundamental group outside a compact set* introduced in [34]. Then, to each of the three notions of pressures recalled in Section 3.1, we associate a natural notion of *pressure at infinity*.

4.1. Fundamental group outside a given compact set

For any compact set $\widetilde{K} \subset \widetilde{M}$, as in [16, 34, 44] we define the *fundamental group outside $\widetilde{K}$*, denoted by $\Gamma_{\widetilde{K}}$, as

$$\Gamma_{\widetilde{K}} = \big\{\gamma \in \Gamma,\ \exists x, y \in \widetilde{K},\ [x, \gamma y] \cap \Gamma\widetilde{K} \subset \widetilde{K} \cup \gamma\widetilde{K}\big\}.$$

Considering the last point on such a geodesic segment in $\widetilde{K}$, and the first point in $\gamma\widetilde{K}$, it follows that this set can equivalently be written as

$$\Gamma_{\widetilde{K}} = \big\{\gamma \in \Gamma,\ \exists x, y \in \widetilde{K},\ [x, \gamma y] \cap \Gamma\widetilde{K} = \{x, \gamma y\}\big\}.$$

This subset of Γ corresponds to long excursions of geodesics outside of $K := p_\Gamma(\widetilde{K})$. We stress that this is not a subgroup in general, see examples in [44, Section 7].

Recall from [44, Proposition 7.9] and [44, Proposition 7.7] the following results.

Proposition 4.1. *The following statements hold.*

(1) *Let $\widetilde{K} \subset \widetilde{M}$ be a compact subset, and $\alpha \in \Gamma$. Then $\Gamma_{\alpha\widetilde{K}} = \alpha\Gamma_{\widetilde{K}}\alpha^{-1}$.*

(2) *If $\widetilde{K}_1$ and $\widetilde{K}_2$ are compact subsets of $\widetilde{M}$ such that $\widetilde{K}_1$ is included in the interior of $\widetilde{K}_2$, then there exist finitely many $\alpha_1, \dots, \alpha_k \in \Gamma$ such that*

$$\Gamma_{\widetilde{K}_2} \subset \bigcup_{i,j=1}^{k} \alpha_i \Gamma_{\widetilde{K}_1} \alpha_j^{-1}.$$

In some circumstances, it may be useful to consider different Riemannian structures (M, g_0) and (M, g) on the same orbifold, and compare their fundamental groups outside a given compact set, denoted by $\Gamma^{g_0}_{\widetilde{K}}$ and $\Gamma^{g}_{\widetilde{K}}$ in order to avoid confusions. The following proposition follows from the definition.

Proposition 4.2. *Let $\widetilde{K} \subset \widetilde{M}$ be a compact subset. Let g_0 and g be two complete Riemannian metrics with pinched negative curvature and bounded derivatives of the curvature that coincide outside $p_\Gamma(\widetilde{K})$. Then*

$$\Gamma^{g}_{\widetilde{K}} = \Gamma^{g_0}_{\widetilde{K}}.$$

4.2. Critical exponent at infinity

Consider the associated restricted Poincaré series

$$P_{\Gamma_{\widetilde{K}}}(s, F) = \sum_{\gamma \in \Gamma_{\widetilde{K}}} e^{-sd(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}}.$$

Its critical exponent $\delta_{\Gamma_{\widetilde{K}}}(F) \in [-\infty, +\infty]$, satisfies for all $c > 0$

$$\delta_{\Gamma_{\widetilde{K}}}(F) = \limsup_{t \to +\infty} \frac{1}{t} \log \sum_{\substack{\gamma \in \Gamma_{\widetilde{K}}, \\ t-c \le d(o,\gamma o) \le t}} e^{\int_o^{\gamma o} \widetilde{F}}.$$

We call it the *critical exponent* or *geometric pressure of F outside $\widetilde{K}$*. By construction,

$$\delta_{\Gamma_{\widetilde{K}}}(F) \le \delta_\Gamma(F).$$

Definition 4.3. The *critical exponent at infinity* or *geometric pressure at infinity* of F is defined as

$$\delta^\infty_\Gamma(F) = \inf_{\widetilde{K}} \delta_{\Gamma_{\widetilde{K}}}(F),$$

where the infimum is taken over all compact sets $\widetilde{K} \subset \widetilde{M}$.

An immediate corollary of Proposition 4.1 is the following result.

Corollary 4.4. *Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous potential.*

(1) *Let $\widetilde{K} \subset \widetilde{M}$ be a compact subset, and $\alpha \in \Gamma$. Then $\delta_{\Gamma_{\alpha\widetilde{K}}}(F) = \delta_{\Gamma_{\widetilde{K}}}(F)$.*

(2) *If $\widetilde{K}_1$ and $\widetilde{K}_2$ are compact subsets of $\widetilde{M}$ such that $\widetilde{K}_1$ is included in the interior of $\widetilde{K}_2$, then*

$$\delta_{\Gamma_{\widetilde{K}_2}}(F) \le \delta_{\Gamma_{\widetilde{K}_1}}(F).$$

Corollary 4.4 implies for any Hölder-continuous potential F the very convenient following fact:

$$\delta_\Gamma^\infty(F) = \lim_{R\to+\infty} \delta_{\Gamma_{B(o,R)}}(F).$$

It is worth noting that this critical exponent at infinity can be equal to $-\infty$, in particular, in the trivial situations described in the following lemma, where *all* potentials have critical exponent at infinity equal to $-\infty$.

Lemma 4.5. *Let M be a compact or convex-cocompact Riemannian manifold with pinched negative curvature. For every Hölder-continuous potential $F: T^1M \to \mathbb{R}$, we have*

$$\delta_\Gamma^\infty(F) = -\infty.$$

Proof. By [44, Proposition 7.17], for $\widetilde{K} \subset \widetilde{M}$ large enough, the set $\Gamma_{\widetilde{K}}$ is finite. This immediately implies

$$\delta_\Gamma^\infty(F) \le \delta_{\Gamma_{\widetilde{K}}}(F) = -\infty. \qquad \blacksquare$$

We refer to Corollary 7.7 for more interesting situations where $\delta_\Gamma^\infty(0) \ge 0$ and there exists a Hölder-continuous map $F: T^1M \to \mathbb{R}$ with $\delta_\Gamma^\infty(F) = -\infty$.

4.3. Variational pressure at infinity

Recall that the *vague topology* on the space of Radon measures on T^1M is the weak-* topology on the space of Radon measures viewed as the dual of the space $C_c(T^1M)$ of continuous maps with compact support on T^1M. A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ converges to 0 for the vague topology if and only if for every map $\varphi \in C_c(T^1M)$, it satisfies

$$\lim_{n\to+\infty} \int \varphi \, \mathrm{d}\mu_n = 0.$$

We write this $\mu_n \overset{*}{\rightharpoonup} 0$. This provides the following other natural notion of pressure at infinity.

Definition 4.6. Let F be a Hölder-continuous potential on T^1M. The *variational pressure at infinity* of F is

$$\begin{aligned} P^{\infty}_{\mathrm{var}}(F) &= \sup\Big\{\limsup_{n\to+\infty}\Big(h_{KS}(\mu_n) + \int_{T^1M} F\,\mathrm{d}\mu_n\Big)\ ; \\ &\qquad\qquad (\mu_n)_{n\in\mathbb{N}} \in (\mathcal{M}_1^F)^{\mathbb{N}} \text{ s.t. } \mu_n \overset{*}{\rightharpoonup} 0\Big\} \\ &= \lim_{\varepsilon\to 0}\ \inf_{\substack{K\subset M,\\ K \text{ compact}}}\ \sup\Big\{h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu\ ;\ \mu\in\mathcal{M}_1^F \text{ s.t. } \mu(T^1K)\le\varepsilon\Big\} \\ &= \inf_{\substack{K\subset M,\\ K \text{ compact}}}\ \lim_{\varepsilon\to 0}\ \sup\Big\{h_{KS}(\mu) + \int_{T^1M} F\,\mathrm{d}\mu\ ;\ \mu\in\mathcal{M}_1^F \text{ s.t. } \mu(T^1K)\le\varepsilon\Big\}. \end{aligned}$$

Let us check that these three definitions coincide.

Proof. The limit in ε in the last two lines is a decreasing limit, i.e., an infimum, so it commutes with the infimum over compact subsets K. Hence, it suffices to show that the quantity on the first line, say A, coincides with the quantity on the second line, say B. If a sequence μ_n realizes the supremum in A, then for any $\varepsilon > 0$ and for any compact subset K, one has eventually $\mu_n(T^1K) \le \varepsilon$ by definition of the vague convergence to 0. Therefore, $A \le B$. Conversely, consider sequences ε_n and K_n realizing the infimum in B. Since decreasing ε_n and increasing K_n can only make the infimum smaller, it follows that $\varepsilon'_n = \min(\varepsilon_n, 1/n)$ and $K'_n = K_n \cup B(o,n)$ also realize the infimum in B. We get a sequence of measures $\mu_n \in \mathcal{M}_1^F$ with

$$\mu_n(T^1K'_n) \le \varepsilon'_n \quad \text{and} \quad h_{KS}(\mu_n) + \int_{T^1M} F\,\mathrm{d}\mu_n \to B.$$

Since $T^1K'_n$ increases to cover the whole space and ε'_n tends to 0, we have $\mu_n \overset{*}{\rightharpoonup} 0$. Therefore, $B \le A$. ■

From a dynamical point of view, it would be more natural and apparently more general to consider all compact subsets $\mathcal{K}$ of T^1M, instead of restricting to unit tangent bundles $\mathcal{K} = T^1K$ of compact subsets of M. However, the equality between the three above quantities shows that it would not bring anything to the definition.

In the case $F \equiv 0$, in the context of symbolic dynamics, this definition already appeared in different works, see for example [8, 9, 24, 39].

One can consider a variation around the above definition, requiring additionally that all the measures μ_n are ergodic. We will denote this pressure by $P^{\infty}_{\mathrm{var,erg}}(F)$. We will see in Corollary 6.12 that it coincides with $P^{\infty}_{\mathrm{var}}(F)$, as a byproduct of the proof of Theorem 1.2.

4.4. Gurevič pressure at infinity

To the Gurevič pressure is naturally associated a notion of *Gurevič pressure at infinity*, when considering only periodic orbits that spend an arbitrarily small proportion of their period in a given compact subset. This only makes sense for compact subsets on T^1M whose interior intersects the nonwandering set Ω. As in the preceding sections, we consider only compact subsets K of M, so that we require that the interior of K, denoted by $\overset{\circ}{K}$, intersects the projection $\pi(\Omega)$ of the nonwandering set on M. We recall that $\mathcal{P}_K(T-c,T)$, defined in Section 3.1.3, is the set of periodic orbits intersecting K with length in the interval $[T-c,T]$.

Definition 4.7. Let F be a Hölder-continuous potential on T^1M. For any $c>0$, the *Gurevič pressure at infinity* of F is

$$\begin{aligned} P^\infty_{\mathrm{Gur}}(F) &= \inf_{\substack{K\subset M, K \text{ compact}\\ \overset{\circ}{K}\cap\pi(\Omega)\neq\emptyset}} \lim_{\alpha\to 0} \limsup_{T\to+\infty} \frac{1}{T}\log \sum_{\substack{p\in\mathcal{P}_K(T-c,T);\\ \ell(p\cap T^1K)<\alpha\ell(p)}} e^{\int_p F} \\ &= \lim_{\alpha\to 0} \inf_{\substack{K\subset M, K \text{ compact}\\ \overset{\circ}{K}\cap\pi(\Omega)\neq\emptyset}} \limsup_{T\to+\infty} \frac{1}{T}\log \sum_{\substack{p\in\mathcal{P}_K(T-c,T);\\ \ell(p\cap T^1K)<\alpha\ell(p)}} e^{\int_p F}. \end{aligned}$$

It does not depend on c.

It is not completely obvious from the definition what happens when one increases a compact subset K' to a larger compact subset K. Since one may consider orbits that intersect K but not K', one is allowed more orbits. However, the condition

$$\ell(p\cap T^1K)<\alpha\ell(p)$$

becomes more restrictive for K than for K', allowing less orbits. These two effects pull in different directions. It turns out that the latter effect, allowing less orbits, is stronger. We formulate this statement with a third compact subset K'' as we will need it later on in this form, but for the previous discussion you may take $K'=K''$.

Proposition 4.8. *Consider three compact subsets K'', K', K of M such that the interior of K'' intersects a closed geodesic, and K' is contained in the interior of K. Then, for $\alpha>0$,*

$$\limsup_{T\to+\infty} \frac{1}{T}\log \sum_{\substack{p\in\mathcal{P}_K(T-c,T);\\ \ell(p\cap T^1K)<\alpha\ell(p)}} e^{\int_p F} \leq \limsup_{T\to+\infty} \frac{1}{T}\log \sum_{\substack{p\in\mathcal{P}_{K''}(T-c,T);\\ \ell(p\cap T^1K')<2\alpha\ell(p)}} e^{\int_p F}.$$

Therefore, the infimum in the definition of the Gurevič pressure may be realized by taking an increasing sequence of balls, just like in Corollary 4.4:

$$P^{\infty}_{\mathrm{Gur}}(F) = \lim_{R\to\infty} \lim_{\alpha\to 0} \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\substack{p\in\mathcal{P}_{B(o,R)}(T-c,T);\\ \ell(p\cap T^1B(o,R))<\alpha\ell(p)}} e^{\int_p F}.$$

Proof. Consider a periodic orbit p of length $\ell(p)\in[T-c,T]$ starting from $v\in T^1K$, parametrized by $[0,\ell(p)]$. Fix also $\varepsilon>0$. By the first assertion of Proposition 2.5 there is another periodic orbit p', of length $\ell(p')\in[\ell(p),\ell(p)+T_0]$ for a constant T_0 depending on K and K'' and ε, parametrized by $[0,\ell(p')]$, following p within ε during the interval of time $[T_0,\ell(p)-T_0]$, and intersecting T^1K''. Lemma 3.1 shows that there exists a constant C' such that

$$|\int_p F - \int_{p'} F| \le C'.$$

Moreover, by assertion (2) of Proposition 2.5, there exists C'' such that the multiplicity of the map $p\mapsto p'$ is bounded by $C''T$ if T is large enough.

If ε is such that the ε-neighborhood of K' is included in K, then the times at which p' belongs to T^1K' are of two kind: either they are in $[T_0,\ell(p')-2T_0]$, and then the corresponding point on p belongs to T^1K, or they are not. Hence,

$$\ell(p'\cap T^1K') \le 3T_0 + \ell(p\cap T^1K).$$

Taking into account the multiplicity, we obtain

$$\sum_{\substack{p\in\mathcal{P}_K(T-c,T);\\ \ell(p\cap T^1K)<\alpha\ell(p)}} e^{\int_p F} \le C''T \sum_{\substack{p'\in\mathcal{P}_{K''}(T-c,T+T_0);\\ \ell(p'\cap T^1K')<3T_0+\alpha\ell(p')}} e^{C'+\int_{p'} F}.$$

When T is large enough, we have

$$3T_0 + \alpha\ell(p') < 2\alpha\ell(p').$$

As the interval $[T-c,T+T_0]$ is the union of at most $\frac{T_0}{c}+2$ intervals of length at most c, taking a limsup, we obtain

$$\limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\substack{p\in\mathcal{P}_K(T-c,T);\\ \ell(p\cap T^1K)<\alpha\ell(p)}} e^{\int_p F} \le \limsup_{T\to+\infty} \frac{1}{T} \log \sum_{\substack{p'\in\mathcal{P}_{K''}(T-c,T);\\ \ell(p'\cap T^1K')<2\alpha\ell(p')}} e^{\int_{p'} F}. \qquad \blacksquare$$

4.5. Pressure at infinity is invariant under compact perturbations

In this paragraph, we will show that the critical exponent at infinity is invariant under any compact perturbation of the potential or of the underlying metric.

Proposition 4.9. *Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map, let $A\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map, and let $\widetilde{K} \subset \widetilde{M}$ be a compact subset such that A vanishes outside $p_\Gamma(T^1\widetilde{K})$. Then*

$$\delta_{\Gamma_{\widetilde{K}}}(F + A) = \delta_{\Gamma_{\widetilde{K}}}(F).$$

In particular,

$$\delta_\Gamma^\infty(F + A) = \delta_\Gamma^\infty(F).$$

Proof. By definition, for all $\gamma \in \Gamma_{\widetilde{K}}$, there exist $x, y \in \widetilde{K}$ such that the geodesic segment $[x, \gamma y]$ satisfies $[x, \gamma y] \cap \Gamma\widetilde{K} = \{x, \gamma y\}$. We deduce that

$$\int_x^{\gamma y} (\widetilde{F} + A) = \int_x^{\gamma y} \widetilde{F}.$$

By Lemma 3.1, we deduce that

$$\left| \int_o^{\gamma o} (\widetilde{F} + A) - \int_o^{\gamma o} \widetilde{F} \right| \le 2C(F, \widetilde{K}, A).$$

By definition of $\delta_{\Gamma_{\widetilde{K}}}(F)$ and $\delta_{\Gamma_{\widetilde{K}}}(F + A)$, the result follows immediately. ∎

In the next proposition, we consider two negatively curved Riemannian metrics g_0 and g on M such that there exists $C > 0$ satisfying at every point of M,

$$\frac{1}{C} g_0 \le g \le C g_0 \tag{8}$$

and still denote by g_0 and g their lifts to $\widetilde{M}$. For a given potential $F\colon TM \to \mathbb{R}$, denote by $\delta_{\Gamma_{\widetilde{K}}, g_0}(F)$, $\delta_{\Gamma_{\widetilde{K}}, g}(F)$, $\delta^\infty_{\Gamma, g_0}(F)$, $\delta^\infty_{\Gamma, g}(F)$ the associated critical exponents for the restriction of F to the unit tangent bundles for g and g_0 respectively. It follows from (8) that being Hölder-continuous does not depend on the metric one considers.

Proposition 4.10. *Let (M, g_0) be a Riemannian manifold with pinched negative curvature, and g be another negatively curved metric on M. Let $F\colon TM \to \mathbb{R}$ be a Hölder-continuous potential. Let $\widetilde{K} \subset \widetilde{M}$ be a compact set such that g and g_0 coincide outside of $p_\Gamma(\widetilde{K})$. Then*

$$\delta_{\Gamma_{\widetilde{K}}, g_0}(F) = \delta_{\Gamma_{\widetilde{K}}, g}(F).$$

In particular, $\delta^\infty_{\Gamma, g_0}(F) = \delta^\infty_{\Gamma, g}(F)$.

Proof. When necessary, denote by $[a,b]^g$ or $[a,b]^{g_0}$ the geodesic segment of the metric g (respectively, g_0) between a and b. By Proposition 4.2, we have

$$\Gamma^{g_0}_{\widetilde{K}} = \Gamma^{g}_{\widetilde{K}}.$$

Let $\gamma \in \Gamma_{\widetilde{K}}$. There exist $x, y \in \widetilde{K}$ such that

$$[x, \gamma y]^{g_0} \cap \Gamma \widetilde{K} = \{x, \gamma y\}.$$

Outside $\Gamma\widetilde{K}$, the metrics g_0 and g coincide, so that (8) is satisfied, the segments $[x,\gamma y]^g$ and $[x,\gamma y]^{g_0}$ are the same, and the integrals of F coincide:

$$\int_{[x,\gamma y]^g} \widetilde{F} = \int_{[x,\gamma y]^{g_0}} \widetilde{F}.$$

Moreover, by compactness, there exists $D > 0$ depending on $\widetilde{K}$, g_0 and g, such that for both metrics,

$$d^{g_0}(x,o) \le D, \quad d^{g}(x,o) \le D, \quad d^{g_0}(y,o) \le D, \quad d^{g}(y,o) \le D.$$

Therefore, using Lemma 3.1, there exists a constant C depending on D and $\sup_{\widetilde{K}}(\widetilde{F})$ such that for both metrics, we have

$$\left| \int_{[o,\gamma o]^g} \widetilde{F} - \int_{[x,\gamma y]^g} \widetilde{F} \right| \le C \quad \text{and} \quad \left| \int_{[o,\gamma o]^{g_0}} \widetilde{F} - \int_{[x,\gamma y]^{g_0}} \widetilde{F} \right| \le C.$$

The result follows by definition of the geometric pressure outside $\widetilde{K}$. ∎

Compact perturbations of a given potential do not change the critical exponent at infinity, but modify the pressure, as shown in the next proposition. This kind of statement is very useful and relatively classical. Similar statements in symbolic dynamics or on geometrically finite manifolds, or for potentials converging to 0 at infinity can be found for example in [27, 37].

Proposition 4.11. *Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous potential, and $A: T^1M \to [0,+\infty)$ a nonnegative Hölder-continuous map with compact support. The map*

$$\lambda \in \mathbb{R} \to \delta_\Gamma(F + \lambda A)$$

is Lipschitz-continuous, convex, nondecreasing, and as soon as the interior of the support of A intersects the nonwandering set Ω, we have

$$\lim_{\lambda \to \infty} \delta_\Gamma(F + \lambda A) = +\infty.$$

Proof. The fact that it is Lipschitz-continuous is an immediate consequence of the definition, and that it is nondecreasing is obvious as $A \geq 0$. Convexity follows from the variational principle (Theorem 1.1) because it is a supremum of affine maps.

Now, if the interior of the support of A intersects Ω, there will be at least an invariant probability measure μ with compact support (supported by a periodic orbit intersecting the interior of the support of A for example) such that $\int A \,\mathrm{d}\mu > 0$. By the variational principle,

$$\delta_\Gamma(F + \lambda A) \geq h_{KS}(\mu) + \int F \,\mathrm{d}\mu + \lambda \int A \,\mathrm{d}\mu,$$

and the latter quantity goes to $+\infty$ when $\lambda \to +\infty$. The result follows. ∎

The combination of Propositions 4.9 and 4.11 provides the following corollary, which will become relevant in Section 7.

Corollary 4.12. *Let F and $A\colon T^1 M \to \mathbb{R}$ be two Hölder-continuous potentials. Assume that F has finite geometric pressure at infinity, and that A is nonnegative, compactly supported, and not everywhere zero on the nonwandering set. Then for $\lambda > 0$ large enough, we have*

$$\delta_\Gamma(F + \lambda A) > \delta_\Gamma^\infty(F + \lambda A).$$

4.6. Infinite pressure

In this paragraph, we prove that if the geometric pressure of a potential is infinite, then its pressure at infinity is also infinite. This is not surprising: everything coming from a compact set is finite, so if the pressure is infinite the major contribution has to come from the complement of compact sets, and therefore the pressure outside any compact set should also be infinite. However, the proof is not completely trivial. It will involve careful splittings of orbits and subadditivity, two themes that will also show up in later proofs. One may think of this proof as a warm-up for the next sections.

Proposition 4.13. *Let $F\colon T^1 M \to \mathbb{R}$ be a Hölder-continuous potential with $\delta_\Gamma(F) = +\infty$. Then $\delta_\Gamma^\infty(F) = +\infty$.*

Proof. We will prove the contrapositive, namely, if there exists a compact set $\widetilde{K}$ of $\widetilde{M}$ with $\delta_{\Gamma_{\widetilde{K}}}(F) < \infty$ then $\delta_\Gamma(F) < \infty$. Adding o to $\widetilde{K}$ if necessary, we can assume $o \in \widetilde{K}$. Fix some $s > \delta_{\Gamma_{\widetilde{K}}}(F)$. Let D be the diameter of $\widetilde{K}$.

Let

$$u_n = \sum_{\gamma \in \Gamma \,:\, d(o,\gamma o) \in (n-1,n]} e^{\int_o^{\gamma o} \widetilde{F}}.$$

We claim that there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$u_n \leq C \sum_{\substack{a,b\in\mathbb{N} \\ 1\leq a,b\leq n-1,\, |a+b-n|\leq C}} u_a u_b + Ce^{sn}. \tag{9}$$

The proof of this inequality is purely geometrical. On the other hand, the proof that this inequality implies the proposition is purely analytical. We postpone the geometrical proof of (9) and explain how to deduce the result assuming this inequality, by a subadditivity argument. Extend u_n by 0 for $n \in (-\infty, -1]$, and define a new sequence

$$v_n = \sum_{n-C}^{n+C} u_i.$$

It satisfies the inequality

$$v_n \leq C_1 \sum_{\substack{1\leq a',b'\leq n-1 \\ a'+b'=n}} v_{a'} v_{b'} + C_1 e^{sn}, \tag{10}$$

for some C_1. To get this inequality, bound each u_i appearing in v_n using (9), and notice that the a, b in the upper bound satisfy $n - 2C \leq a + b \leq n + 2C$ and will therefore appear in one of the products $v_{a'} v_{b'}$ for $a' + b' = n$. We will prove that this sequence v_n grows at most exponentially fast, from which the same result follows for u_n, as desired. For small $z > 0$, define

$$B(z) = \sum_{n\geq 1} C_1 e^{sn} z^n \quad \text{and} \quad V_N(z) = \sum_{n=1}^{N} v_n z^n.$$

The inequality (10) gives

$$V_N(z) \leq B(z) + C_1 V_{N-1}(z)^2. \tag{11}$$

The function B is smooth at 0. Let t be strictly larger than its derivative at 0. Fix z positive and small enough so that $B(z) + C_1(tz)^2 < tz$, which is possible since the function on the left has derivative $< t$. We claim that $V_N(z) \leq tz$ for all N. This is obvious for $N = 0$ as $V_0 = 0$, and the choice of z and the inequality (11) imply that, if it holds at $N - 1$, then it holds at N, concluding the proof by induction. In particular,

$$v_n z^n \leq V_n(z) \leq tz.$$

This proves that v_n grows at most exponentially.

It remains to show (9), using geometry. Let $A > 0$ be large enough ($A > D + 1$ will suffice). Take γ with $d(o, \gamma o) \in (n-1, n]$. We consider two different cases: either $[o, \gamma o] \setminus (B(o, A) \cup B(\gamma o, A))$ does intersect $\Gamma \widetilde{K}$ (we say that γ is recurrent – this terminology is local to this proof), or it does not. The former will give rise to the first term in (9), the latter to the second term.

We start with the nonrecurrent γ's. If $[o, \gamma o] \subset B(o, A) \cup B(\gamma o, A)$, then $d(o, \gamma o)$ is uniformly bounded, so is n, and the formula (9) is obvious for these finitely many n's by taking C large enough. Assume now that n is large. Consider the last point x on $[o, \gamma o] \cap B(o, A) \cap \Gamma \widetilde{K}$, and the first point y on $[o, \gamma o] \cap B(\gamma o, A) \cap \Gamma \widetilde{K}$. Take $\gamma_x \in \Gamma$ such that $x \in \gamma_x \widetilde{K}$, and $\gamma_y \in \Gamma$ such that $y \in \gamma \gamma_y \widetilde{K}$. Note that γ_x and γ_y belong to a finite set $\mathcal{F}_A$ (depending on A), made of these elements of Γ that move o by at most $A + D$. Moreover, $\gamma' = \gamma_x^{-1} \gamma \gamma_y$ belongs to $\Gamma_{\widetilde{K}}$ since $[x, y] \cap \Gamma \widetilde{K} = \{x, y\}$ by construction.

Applying Lemma 3.1 to the compact set $\bigcup_{g \in \mathcal{F}_A} g \widetilde{K}$, we obtain a constant C such that

$$\int_o^{\gamma o} \widetilde{F} \le \int_{\gamma_x o}^{\gamma \gamma_y o} \widetilde{F} + C = \int_o^{\gamma' o} \widetilde{F} + C.$$

Finally, the contribution of the nonrecurrent γ's to u_n is bounded from above by

$$\sum_{\gamma_x, \gamma_y \in \mathcal{F}_A} \sum_{\substack{\gamma' \in \Gamma_{\widetilde{K}} \\ d(o, \gamma' o) \in (n-1-2A-2D, n+2A+2D]}} e^{\int_o^{\gamma' o} \widetilde{F} + C}.$$

The sum over γ_x and γ_y gives a finite multiplicity, and the sum over γ' is bounded by $C(A) e^{ns}$ for some constant $C(A) > 0$ since $s > \delta_{\Gamma_{\widetilde{K}}}(F)$. This is compatible with the second term in the upper bound of (9).

We turn to the contribution to u_n of the recurrent γ's. For such a γ, there is a point x in $[o, \gamma o] \cap \Gamma \widetilde{K} \setminus (B(o, A) \cup B(\gamma o, A))$. Write $x = \gamma' x'$ with $x' \in \widetilde{K}$. Consider the integer a such that $d(o, \gamma' o) \in (a-1, a]$. It satisfies

$$A - D \le a < n - A + D + 1,$$

so if A is large enough one has $1 \le a \le n-1$. Let $\gamma'' = \gamma'^{-1} \gamma$, so that $\gamma = \gamma' \gamma''$. The integer b such that $d(o, \gamma'' o) \in (b-1, b]$ satisfies also

$$0 < A - D \le b < n - A + D + 1 < n.$$

Moreover,

$$\begin{aligned} a + b &= d(o, \gamma' o) + d(o, \gamma'' o) \pm 2 = d(o, \gamma' o) + d(\gamma' o, \gamma o) \pm 2 \\ &= d(o, x) + d(x, \gamma o) \pm (2 + 2D) = d(o, \gamma o) \pm (2 + 2D) \\ &= n \pm (3 + 2D). \end{aligned}$$

This shows that $1 \le a, b \le n-1$ and $|a+b-n| \le 3+2D$. Finally, applying twice Lemma 3.1, we obtain the existence of a constant C' such that

$$\left| \int_o^{\gamma o} \widetilde{F} - \int_o^{\gamma' o} \widetilde{F} - \int_o^{\gamma'' o} \widetilde{F} \right| \le C'.$$

Altogether, this shows that the contribution of recurrent γ's to u_n is bounded by the first term of the right hand side of (9). ■

5. Gurevič and geometric pressure at infinity coincide

In this section, we will study and count the possible excursions of periodic orbits outside large compact sets, and first deduce the inequality

$$P^{\infty}_{\mathrm{Gur}}(F) \le \delta^{\infty}_{\Gamma}(F).$$

The arguments we develop here will also be instrumental in the proof of the inequality $P^{\infty}_{\mathrm{var}}(F) \le \delta^{\infty}_{\Gamma}(F)$ in Section 6.

These inequalities are the heart of Theorem 1.2. The reverse inequalities

$$P^{\infty}_{\mathrm{Gur}}(F) \ge \delta^{\infty}_{\Gamma}(F) \quad \text{and} \quad P^{\infty}_{\mathrm{var}}(F) \ge \delta^{\infty}_{\Gamma}(F)$$

are simpler, and will be proven respectively in Sections 5.2 and 6.1.

Let us explain why the above inequalities are the most surprising and difficult. A major difference between the definition of $\delta^{\infty}_{\Gamma}(F)$ and the two others is that $P^{\infty}_{\mathrm{Gur}}(F)$ and $P^{\infty}_{\mathrm{var}}(F)$ take into account trajectories (respectively periodic / typical) that spend most of the time outside a given large compact set, but can however come back inside this compact set several times, whereas $\delta^{\infty}_{\Gamma}(F)$ considers trajectories that start and finish in a given compact set, but never come back in the meantime. Thus, there are apparently much more trajectories considered in the first two definitions. However, in the next two sections, culminating in Corollaries 5.3 and 6.11, we prove that the above inequalities hold.

The strategy developed below is to cut a given trajectory, which comes back several times inside a given compact set, but spends a small proportion of time inside, into several excursions, and to prove precise upper bounds presented below.

5.1. Excursions of closed geodesics outside compact sets

In this section, we study periodic orbits that intersect (the unit tangent bundle of) a fixed compact subset $K \subset M$, but which spend most of their time away from the R-neighborhood K_R of K.

For all compact subsets $K_1 \subset K_2 \subset M$ and $0 < \alpha \leq 1$, we define

$$\mathcal{P}(K_1, K_2, \alpha) = \left\{p \text{ periodic orbit} ;\ p \cap T^1 K_1 \neq \emptyset,\ \ell(p \cap T^1 K_2) \leq \alpha \ell(p)\right\}$$

and

$$\mathcal{P}(K_1, K_2, \alpha; T, T') = \left\{p \in \mathcal{P}(K_1, K_2, \alpha),\ T \leq \ell(p) \leq T'\right\}. \tag{12}$$

Given a Hölder-continuous potential F, we define for all $T, T' > 0$,

$$\mathcal{N}_F(K_1, K_2, \alpha; T, T') = \sum_{p \in \mathcal{P}(K_1, K_2, \alpha; T, T')} e^{\int_p F}.$$

Theorem 5.1. *Let $K \subset M$ be a compact subset, and $\widetilde{K} \subset \widetilde{M}$ be a compact subset such that $p_\Gamma(\widetilde{K}) = K$. Let $T_0 > 0$. Let $F : T^1 M \to \mathbb{R}$ be a Hölder-continuous potential with $\delta_{\Gamma_{\widetilde{K}}}(F) > -\infty$. Let $\eta > 0$. For all $0 < \alpha \leq 1$ and $R \geq 2$, there exists a positive number $\psi = \psi(\widetilde{K}, F, \eta, \alpha/R)$ such that*

$$\limsup_{T \to +\infty} \frac{1}{T} \log \mathcal{N}_F(K, K_R, \alpha; T, T + T_0) \leq (1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha \delta_\Gamma(F) + \eta + \psi.$$

Moreover, when $\widetilde{K}$, F and η are fixed, $\psi(\widetilde{K}, F, \eta, \alpha/R)$ tends monotonically to 0 when α/R tends to 0.

Remark 5.2. When $\delta_{\Gamma_{\widetilde{K}}}(F) = -\infty$, the statement should be modified, replacing on the right hand side $\delta_{\Gamma_{\widetilde{K}}}(F)$ with an arbitrary real number d, and allowing ψ to depend on d. The same proof applies.

Letting $R \to +\infty$, $\eta \to 0$ and at last K exhaust M and $\alpha \to 0$, we deduce the following corollary.

Corollary 5.3. *Under the same assumptions on M and F as in Theorem 5.1, we have*

$$P^\infty_{\mathrm{Gur}}(F) \leq \delta^\infty_\Gamma(F).$$

Proof. If $\delta_\Gamma(F)$ is infinite, then $\delta^\infty_\Gamma(F)$ is also infinite by Proposition 4.13, and the result is obvious. We can therefore assume $\delta_\Gamma(F) < \infty$. We will also assume $\delta^\infty_\Gamma(F) > -\infty$, as the case $\delta^\infty_\Gamma(F) = -\infty$ can be proved similarly using Remark 5.2.

Let $\eta > 0$. We have to find a compact subset $\widetilde{L}$ of $\widetilde{M}$ whose interior intersects $\pi(\widetilde{\Omega})$, and $\alpha > 0$, such that, with $L = p_\Gamma(\widetilde{L})$, the exponential growth rate of

$$\sum_{\substack{p \in \mathcal{P}_L(T, T+1)\,; \\ \ell(p \cap T^1 L) < \alpha \ell(p)}} e^{\int_p F}$$

is at most $\delta^\infty_\Gamma(F) + 3\eta$. Fix a large compact set $\widetilde{K}$ with $\delta_{\Gamma_{\widetilde{K}}}(F) \leq \delta^\infty_\Gamma(F) + \eta$. We denote respectively by $\widetilde{K}_2$ and $\widetilde{K}_3$ the neighborhoods of size 2 and 3 of $\widetilde{K}$. We wish to apply Theorem 5.1 with $R = 3$, and set $\widetilde{L} = \widetilde{K}_3$, and $L = p_\Gamma(\widetilde{K}_3)$.

There is a difficulty coming from the fact that the definition of the Gurevič pressure involves all periodic orbits going through $\widetilde{L} = \widetilde{K}_3$, while Theorem 5.1 only takes into account those that, additionally, enter $\widetilde{K}$. This difficulty is solved using Proposition 4.8 applied with K instead of K'', K_2 instead of K' and $K_3 = p_\Gamma(\widetilde{K}_3)$ instead of K. This proposition yields that the exponential growth rate of

$$\sum_{\substack{p\in\mathcal{P}_L(T,T+1)\,;\\ \ell(p\cap T^1L)<\alpha\ell(p)}} e^{\int_p F}$$

is bounded by that of

$$\sum_{\substack{p\in\mathcal{P}_K(T,T+1)\,;\\ \ell(p\cap T^1K_2)<2\alpha\ell(p)}} e^{\int_p F}.$$

The latter can be estimated thanks to Theorem 5.1 applied to $R = 2$, $T_0 = 1$ and 2α: this growth rate is bounded by

$$(1-2\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + 2\alpha\delta_\Gamma(F) + \eta + \psi(\alpha),$$

where $\psi(\alpha)$ tends to 0 with α. This quantity converges to $\delta_{\Gamma_{\widetilde{K}}}(F) + \eta \le \delta_\Gamma^\infty(F) + 2\eta$ when α tends to 0, so that for some $\alpha > 0$ it is strictly smaller than $\delta_\Gamma^\infty(F) + 3\eta$. ■

The strategy of the proof of Theorem 5.1 is as follows. A periodic orbit will be cut into two kinds of segments, those which stay in the given compact set K, and the excursions outside this compact set. The weighted growth of the excursions should be controlled by the exponent $\delta_{\Gamma_K}(F)$ multiplied by the proportion of time spent outside K, and the weighted growth of the segments inside K should be controlled by $\delta_\Gamma(F)$ multiplied by the proportion of time spent in K. However, to succeed to get such a control, we need to avoid the situation with several very short excursions in a very close neighborhood of K. For this reason, we need to play with two compact sets, K and its R-neighborhood K_R.

Proof of Theorem 5.1. As in the above proof, we can assume that $\delta_\Gamma(F)$ and $\delta_\Gamma^\infty(F)$ are finite. Let $\widetilde{K} \subset \widetilde{M}$ be a compact set and $\widetilde{K}_R \subset \widetilde{M}$ be its R-neighborhood, and set $K = p_\Gamma(\widetilde{K})$, $K_R = p_\Gamma(\widetilde{K}_R)$. Let D be the diameter of K. Any geodesic segment joining the boundary of $\widetilde{K}$ and the boundary of $\widetilde{K}_R$ has length at least R and at most $D + 2R$. Let also $D' = D'(K, T_0)$ be larger than the diameter of $K \cup \{o\}$, 1 and T_0.

Consider a periodic orbit $p \in \mathcal{P}(K, K_R, \alpha)$ with $\ell(p) \in [T, T + T_0]$. By assumption, $\pi(p) \cap K \neq \emptyset$. We will divide it into long excursions, i.e., those excursions outside both K and K_R, of total length at least $(1-\alpha)\ell(p)$, and periods of time of total length at most $\alpha\ell(p)$ where it stays inside K_R.

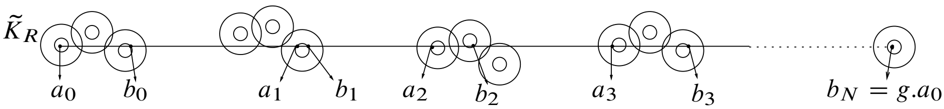

Figure 3. Long excursions outside $\widetilde{K}$ and $\widetilde{K}_R$.

Since p intersects T^1K, we can choose a lift c of p that intersect $T^1\widetilde{K}$. Let g be a hyperbolic isometry whose translation axis is c, and whose translation length is $\ell(p)$, and which translates in the direction given by the orientation of p.

Define inductively points a_i, b_i on c as follows, see Figure 3. Choose first a point a_0 on c inside $\widetilde{K}$. Consider on the geodesic segment $[a_0, g.a_0]$ of c the first points $b_0, a_1 \in \Gamma\partial\widetilde{K}$ such that the open interval (b_0, a_1) does not intersect $\Gamma\widetilde{K}$ and

$$(b_0, a_1) \cap \widetilde{M} \setminus \Gamma\widetilde{K}_R \neq \emptyset.$$

The interval (b_0, a_1) projects through p_Γ into a *long excursion*, i.e., an excursion outside K which also goes outside K_R. Inductively, we define $(b_1, a_2), \dots, (b_{N-1}, a_N)$ by the properties that b_i, a_{i+1} are the first points of $[a_i, g.a_0]$ which lie in $\Gamma\partial\widetilde{K}$ and satisfy

$$(b_i, a_{i+1}) \cap \Gamma\widetilde{K} = \emptyset \quad \text{and} \quad (b_i, a_{i+1}) \cap \widetilde{M} \setminus \Gamma\widetilde{K}_R \neq \emptyset.$$

In other terms, the intervals (b_i, a_{i+1}), $0 \leq i \leq N-1$, are the connected components of $[a_0, g.a_0] \setminus \Gamma\widetilde{K}$ that intersect $\widetilde{M} \setminus \Gamma\widetilde{K}_R$, whereas the segments $[a_i, b_i]$ are included in $\Gamma\widetilde{K}_R$. Finally, set $b_N = g.a_0$.

For all $0 \leq i \leq N$, choose elements $\gamma_i^\pm \in \Gamma$ such that $a_i \in \gamma_i^-\widetilde{K}$ and $b_i \in \gamma_i^+\widetilde{K}$. As $\widetilde{K}$ is compact and the action of Γ is proper, for each i, there are only finitely many choices of such elements $\gamma_i^\pm$. Without loss of generality, set $\gamma_0^- = \mathrm{Id}$ and $\gamma_N^+ = g$.

Choose some $\varepsilon > 0$. The following elementary observations are crucial for the sequel:

(1) As $\bigcup_{0 \leq i \leq N} [a_i, b_i] \subset \Gamma\widetilde{K}_R$, by definition of $\mathcal{P}(K, K_R, \alpha)$ and since $T \leq \ell(p) \leq T + T_0$, we have

$$\ell(p \cap T^1K) \leq \sum_{i=0}^{N} d(a_i, b_i) \leq \alpha(T + T_0) \leq \alpha T + D'.$$

(2) For all $i \in \{0, \dots, N-1\}$, we have $(b_i, a_{i+1}) \subset \widetilde{M} \setminus \Gamma\widetilde{K}$. Moreover, the length of $(b_i, a_{i+1}) \cap \Gamma\widetilde{K}_R$ is at least $2R$ and $\bigcup_i [b_i, a_{i+1}]$ does not intersect the interior of $\Gamma\widetilde{K}$, so that by definition of $\mathcal{P}(K, K_R, \alpha)$,

$$(1-\alpha)T + 2RN \leq \sum_{i=0}^{N-1} d(b_i, a_{i+1}) \leq T + T_0 \leq T + D', \tag{13}$$

and therefore, for $\nu := \frac{1}{2R}(\alpha T + D')$, we have

$$N \leq \nu. \tag{14}$$

For large enough T, we have

$$\nu \leq \frac{\alpha}{R} T. \tag{15}$$

(3) Write $\psi_i = (\gamma_i^-)^{-1}\gamma_i^+ \in \Gamma$ for all $i = 0, \dots, N$. We have

$$|d(o, \psi_i o) - d(a_i, b_i)| \leq 2D',$$

so that

$$\sum_{i=0}^{N} d(o, \psi_i o) \leq \alpha(T + T_0) + 2(N+1)D' \leq \alpha T + 5ND'.$$

Let s_i be the unique integer such that $d(o, \psi_i o) \leq s_i < d(o, \psi_i o) + 1$. Then

$$s_0 + \cdots + s_N \leq \alpha T + 5ND' + N + 1 \leq \alpha T + 7ND'.$$

(4) By definition of $\Gamma_{\widetilde{K}}$, for all $i = 0, \dots, N-1$, since $((\gamma_i^+)^{-1} b_i, \varphi_i (\gamma_{i+1}^-)^{-1} a_{i+1})$ does not intersect $\Gamma \widetilde{K}$, we have $\varphi_i = (\gamma_i^+)^{-1}\gamma_{i+1}^- \in \Gamma_{\widetilde{K}}$. Moreover,

$$|d(o, \varphi_i o) - d(b_i, a_{i+1})| \leq 2D'.$$

Let t_i be the unique integer such that $d(o, \varphi_i o) \leq t_i < d(o, \varphi_i o) + 1$.

(5) As $\sum_{i=0}^{N} d(a_i, b_i) + \sum_{i=0}^{N-1} d(b_i, a_{i+1}) = d(a_0, b_N) = \ell(p) \in [T, T + T_0]$, we get

$$\left| \sum_{i=0}^{N} d(o, \psi_i o) + \sum_{i=0}^{N-1} d(o, \varphi_i o) - T \right| \leq T_0 + (4N + 2)D',$$

and therefore

$$\left| \sum_{i=0}^{N} s_i + \sum_{i=0}^{N-1} t_i - T \right| \leq T_0 + (4N + 2)D' + (2N + 1) \leq 10ND'.$$

(6) By (13), as $d(b_i, a_{i+1}) - 2D' \leq t_i \leq d(b_i, a_{i+1}) + 2D' + 1$, we get

$$(1 - \alpha)T - 2ND' \leq \sum_{i=0}^{N-1} t_i \leq T + 4ND'.$$

(7) Since M has pinched negative sectional curvature and F is (β, C_F)-Hölder-continuous, Lemma 3.1 applied to the compact set $\widetilde{K} \cup \{o\}$ ensures that there exists a constant $C(F, \widetilde{K})$ depending only on the upper bound of the curvature, on $\widetilde{K}$ and the Hölder-continuous constants of F such that for all $i = 0, \dots, N$,

$$\left| \int_{a_i}^{b_i} \widetilde{F} - \int_{o}^{\psi_i o} \widetilde{F} \right| \leq C(F, \widetilde{K}).$$

(8) Similarly, for all $i = 0, \dots, N-1$,

$$\left| \int_{b_i}^{a_{i+1}} \widetilde{F} - \int_o^{\varphi_i o} \widetilde{F} \right| \le C(F, \widetilde{K}).$$

(9) As $\int_p F = \int_{a_0}^{g a_0} \widetilde{F}$, and bounding $2N+1$ from above with 3ν, we deduce

$$\sum_{i=0}^{N} \int_o^{\psi_i o} \widetilde{F} + \sum_{i=0}^{N-1} \int_o^{\varphi_i o} \widetilde{F} - 3C(F,\widetilde{K})\nu \le \int_p F \le \sum_{i=0}^{N} \int_o^{\psi_i o} \widetilde{F} + \sum_{i=0}^{N-1} \int_o^{\varphi_i o} \widetilde{F} + 3C(F,\widetilde{K})\nu. \tag{16}$$

For all $t \in \mathbb{N}$, set

$$\Gamma(t-1,t) = \{\gamma \in \Gamma \,;\, d(o,\gamma o) \in (t-1,t]\} \quad \text{and} \quad \Gamma_{\widetilde{K}}(t-1,t) = \Gamma(t-1,t) \cap \Gamma_{\widetilde{K}}.$$

We also write

$$Q_{F,\Gamma}(t) = \sum_{\gamma \in \Gamma(t-1,t)} e^{\int_o^{\gamma o} \widetilde{F}} \quad \text{and} \quad Q_{F,\Gamma_{\widetilde{K}}}(t) = \sum_{\gamma \in \Gamma_{\widetilde{K}}(t-1,t)} e^{\int_o^{\gamma o} \widetilde{F}}.$$

To each periodic orbit $p \in \mathcal{P}(K, K_R, \alpha)$ with $\ell(p) \in [T, T+T_0]$, we have associated a hyperbolic isometry $g \in \Gamma$ whose axis intersects $\widetilde{K}$ and projects through p_Γ onto $\pi(p)$ and with translation length equal to $\ell(p)$. Then, to each such element g we have associated by the previous construction finite sequences $\varphi_0, \dots, \varphi_{N-1}$ in $\Gamma_{\widetilde{K}}$ and $\psi_0, \dots, \psi_N \in \Gamma$. As one can recover g (and then p, which is the projection of the translation axis of g) from these sequences by the formula $g = \psi_0 \varphi_0 \psi_1 \cdots \varphi_{N-1} \psi_N$, this association is injective.

Let us now bound $\mathcal{N}_F(K, K_R, \alpha; T, T+T_0)$. Summing the exponentials of the bounds (16) over all the periodic orbits in $\mathcal{P}(K, K_R, \alpha; T, T+T_0)$, we get the inequality

$$\mathcal{N}_F(K, K_R, \alpha; T, T+T_0) \le e^{3C(F,\widetilde{K})\nu} \sum_{N=0}^{\nu(\alpha,T,T_0,R)} \sum_{\substack{t_0,\dots,t_{N-1},s_0,\dots,s_N \in \mathbb{N} \\ |\sum s_i + \sum t_i - T| \le 10ND' \\ \sum t_i \ge (1-\alpha)T - 2ND'}} Q_{F,\Gamma}(s_0) \cdot Q_{F,\Gamma_{\widetilde{K}}}(t_0) \cdot Q_{F,\Gamma}(s_1) \cdot Q_{F,\Gamma_{\widetilde{K}}}(t_1) \cdots Q_{F,\Gamma_{\widetilde{K}}}(t_{N-1}) \cdot Q_{F,\Gamma}(s_N). \tag{17}$$

The following lemma is a straightforward consequence of the definition of the critical exponents $\delta_\Gamma(F)$ and $\delta_{\Gamma_{\widetilde{K}}}(F)$.

Lemma 5.4. *For all $\eta > 0$, there exists $C_\eta = C_\eta(\widetilde{K}, F, \eta) \geq 1$ such that for all $t > 0$, we have*

$$Q_{F,\Gamma}(t) \leq C_\eta e^{\delta_\Gamma(F)t+\eta t} \quad \textit{and} \quad Q_{F,\Gamma_{\widetilde{K}}}(t) \leq C_\eta e^{\delta_{\Gamma_{\widetilde{K}}}(F)t+\eta t}.$$

We can write the second bound as $Q_{F,\Gamma_{\widetilde{K}}}(t) \leq C_\eta e^{(\delta_{\Gamma_{\widetilde{K}}}(F)-\delta_\Gamma(F))t+\delta_\Gamma(F)t+\eta t}$. Multiplying these bounds, we get

$$\begin{aligned}
&Q_{F,\Gamma}(s_0)\cdot Q_{F,\Gamma_{\widetilde{K}}}(t_0)\cdot Q_{F,\Gamma}(s_1)\cdot Q_{F,\Gamma_{\widetilde{K}}}(t_1)\cdots Q_{F,\Gamma_{\widetilde{K}}}(t_{N-1})\cdot Q_{F,\Gamma}(s_N)\\
&\quad\leq C_\eta^{2N+1}\exp\Big((\delta_\Gamma(F)+\eta)\Big(\sum s_i+\sum t_i\Big)+(\delta_{\Gamma_{\widetilde{K}}}(F)-\delta_\Gamma(F))\Big(\sum t_i\Big)\Big)\\
&\quad\leq C_\eta^{3N}\exp\big((\delta_\Gamma(F)+\eta)T+(|\delta_\Gamma(F)|+\eta)10ND'\\
&\qquad\quad+(\delta_{\Gamma_{\widetilde{K}}}(F)-\delta_\Gamma(F))((1-\alpha)T-2ND')\big)\\
&\quad= C_\eta^{3N}\exp\big((\alpha\delta_\Gamma(F)+(1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F)+\eta)T\\
&\qquad\quad+(|\delta_\Gamma(F)|+\eta+\delta_\Gamma(F)-\delta_{\Gamma_{\widetilde{K}}}(F))10ND'\big).
\end{aligned}$$

Note that this bound does not depend anymore on the choice of the s_i and t_i. In order to bound (17), one should take into account a multiplicity given by the number of possible choices for these integers.

The following combinatorial standard estimate will control the number of possible choices.

Lemma 5.5. *Let $\tau, \kappa \in \mathbb{N}$ be integers with $\kappa < \tau$. The number of ordered integer decompositions of τ of length κ, i.e., the number of $(u_1, \ldots, u_\kappa) \in \mathbb{N}^\kappa$ such that $u_i \geq 0$ and $u_1 + \cdots + u_\kappa \leq \tau$, is equal to*

$$\binom{\tau+\kappa}{\kappa} = \frac{(\tau+\kappa)!}{\kappa!\tau!}.$$

Then $(s_0, t_0, s_1, \ldots, s_N)$ forms an ordered integer decomposition of some integer $\tau \leq T + 10ND'$, with $\kappa = 2N + 1$. Their number is thus bounded by

$$\binom{T+10ND'+2N+1}{2N+1}.$$

Recall that by (14), we have $N \leq \nu$, which is bounded by $T/2$ for large T, so that

$$T+10ND'+2N+1 \leq 8D'T \quad \text{and} \quad 2N+1 \leq 3\nu \leq 8D'\nu.$$

We get

$$\binom{T+10ND'+2N+1}{2N+1} \leq \binom{8D'T}{2N+1} \leq \binom{8D'T}{8D'\nu}$$

thanks to monotonicity properties of binomial coefficients. Summing over all the values of N, we obtain the estimate

$$\begin{aligned}\mathcal{N}_F(K, K_R, \alpha; T, T+T_0) \leq (\nu+1) \cdot \binom{8D'T}{8D'\nu} e^{3C(F,\widetilde{K})\nu} \cdot C_\eta^{3\nu} \\ \times \exp\big((\alpha\delta_\Gamma(F) + (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \eta)T \\ + (|\delta_\Gamma(F)| + \eta + \delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F))10\nu D'\big).\end{aligned}$$

To conclude the proof, we should estimate the exponential growth rate of the various terms in this expression when T tends to infinity. Recall that $\nu \leq \alpha T/R$ by (15). Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, implies that the exponential growth rate of

$$\binom{8D'T}{8D'\nu} \leq \binom{8D'T}{8D'T \cdot \alpha/R}$$

is bounded by $-\rho \log \rho - (1-\rho)\log(1-\rho)$ for $\rho = \alpha/R$. Finally, the exponential growth rate of $\mathcal{N}_F(K, K_R, \alpha; T, T+T_0)$ is bounded by

$$\begin{aligned}\alpha\delta_\Gamma(F) + (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \eta - \rho\log\rho - (1-\rho)\log(1-\rho) \\ + \big(3C(F,\widetilde{K}) + 3\log C_\eta + 10D'(|\delta_\Gamma(F)| + \eta + \delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F))\big)\frac{\alpha}{R}.\end{aligned}$$

This concludes the proof of the theorem. ∎

5.2. Gurevič and geometric pressures at infinity coincide

This paragraph is devoted to the proof of the following part of Theorem 1.2.

Theorem 5.6. *For all Hölder-continuous potentials $F\colon T^1M \to \mathbb{R}$ with finite pressure, we have*

$$P^\infty_{\mathrm{Gur}}(F) = \delta^\infty_\Gamma(F).$$

By Corollary 5.3, it is enough to prove the inequality $P^\infty_{\mathrm{Gur}}(F) \geq \delta^\infty_\Gamma(F)$.

Proof. The set of periodic orbits of the geodesic flow (counted with locally bounded multiplicities in the orbifold case) is in 1-1 correspondence with the set of conjugacy classes of hyperbolic elements of Γ. Let us recall how. Given a periodic orbit $p \subset T^1M$, its preimage $p_\Gamma^{-1}(p) \subset T^1\widetilde{M}$ is a countable union of orbits of the geodesic flow on $T^1\widetilde{M}$. Each of these orbits projects on $\widetilde{M}$ to the translation axis of a hyperbolic element of Γ, which is unique (modulo the pointwise stabilizer of their axis) when requiring that this element translates along the axis with translation length equal to $\ell(p)$, and in the direction given by the direction of $(g^t)_{t>0}$ on this orbit. The number of conjugacy classes of hyperbolic elements (modulo the pointwise stabilizer of their axis) associated with p in this way is equal to the multiplicity of p.

Let $K \subset M$ be a compact subset whose interior intersects a closed geodesic, and containing the projection $p_\Gamma(o)$. Let $\widetilde{K}$ be a compact subset of $\widetilde{M}$ which contains o, such that $p_\Gamma(\widetilde{K}) = K$, and whose interior intersects $\widetilde{\Omega}$. Let N be the maximal multiplicity of p_Γ on $\widetilde{K}$. Let D be the diameter of $\widetilde{K}$.

With the notation of (12), set

$$\mathcal{P}(K,\alpha) := \mathcal{P}(K,K,\alpha) \quad \text{and} \quad \mathcal{P}(K,\alpha;T,T') := \mathcal{P}(K,K,\alpha;T,T').$$

First, by Lemma 2.1, there exists a finite set $\mathcal{G} = \{g_1,\ldots,g_k\} \subset \Gamma$, such that, for all $\gamma \in \Gamma_{\widetilde{K}}$, there exist g_i, g_j (not necessarily unique) such that $g_i^{-1}\gamma g_j$ is hyperbolic with a translation axis which intersects $\widetilde{K}$. Let p_γ be the associated periodic orbit (it depends on the choice of g_i, g_j but this is not a problem). As the axis of $g_i^{-1}\gamma g_j$ intersects $\widetilde{K}$, we deduce that

$$|\ell(p_\gamma) - d(o, g_i^{-1}\gamma g_j o)| \leq 2D.$$

By the triangular inequality, we deduce that

$$|d(o,\gamma o) - \ell(p_\gamma)| \leq 2D + 2 \max_{1\leq i\leq k}(d(o,g_i o)).$$

Similarly, thanks to Lemma 3.1, and using the fact that $\widetilde{F}$ is bounded on the δ-neighborhood of $\Gamma\widetilde{K}$, with $\delta = \max_{1\leq i\leq k}(d(o,g_i o))$, we deduce that there exists a constant $C = C(F,\widetilde{K},g_1,\ldots,g_k)$ such that

$$\left|\int_o^{\gamma o}\widetilde{F} - \int_{p_\gamma} F\right| \leq C.$$

Choose now some $R > 1$, and let $\widetilde{K}_R$ be the R-neighborhood of $\widetilde{K}$. For $\gamma \in \Gamma_{\widetilde{K}_R}$, there exist $a \in \widetilde{K}_R$ and $b \in \gamma\widetilde{K}_R$ such that the geodesic segment $[a,b]$ only meets $\Gamma\widetilde{K}_R$ at its endpoints. Using the above notations, we assume that $g_i^{-1}\gamma g_j$ is hyperbolic with associated periodic orbit p_γ. The point $g_i o$ is at distance at most δ from o, which is at distance at most D from a, and the point $g_j o$ is at distance at most δ from $\gamma g_j o$ which is at distance at most D from b. Therefore, by Lemma 2.3, there exists a constant $T_0 > 0$ depending on δ, D and the bounds on the curvature, such that, when removing segments of length T_0 at the beginning and the end of $[g_i o, \gamma g_j o]$, the middle segment is in a neighborhood of radius less than $1/2$ from the geodesic segment $[a,b]$.

On the other hand, the periodic orbit p_γ associated with $g_i^{-1}\gamma g_j$ admits a translation axis which intersects $\widetilde{K}$. Let $x \in \widetilde{K}$ be a point on this axis and $g_i^{-1}\gamma g_j x \in g_i^{-1}\gamma g_j\widetilde{K}$ its image by $g_i^{-1}\gamma g_j$. By Lemma 2.3, when removing segments of length T_0 at the beginning and the end of the segment $[x, g_i^{-1}\gamma g_j x]$, the middle segment is in a neighborhood of size less than $1/2$ of the geodesic segment $[o, g_i^{-1}\gamma g_j o]$.

The triangular inequality implies that, after removing segments of length $2T_0$ at the beginning and at the end of the geodesic segment $[g_i x, \gamma g_j x]$, this segment is at distance at most $1/2$ of $[g_i o, \gamma g_j o]$, and therefore, at distance at most 1 from $[a, b]$. In particular, as $\gamma \in \Gamma_{\widetilde{K}_R}$, and $R \geq 1$, after removing segments of length $2T_0 + R$ at the beginning and the end of $[g_i x, \gamma g_j x]$, this segment spends the rest of the time outside $\Gamma \widetilde{K}$.

We deduce that the time spent by p_γ inside K is at most $4T_0 + 2R$. In particular, when $\ell(p_\gamma) \geq (4T_0 + 2R)/\alpha$, the periodic orbit p_γ spends a proportion of time at most α inside K. As $|d(o, \gamma o) - \ell(p_\gamma)| \leq 2D + 2\delta$, it implies that as soon as

$$d(o, \gamma o) \geq 2D + 2\delta + \frac{4T_0 + 2R}{\alpha},$$

then p_γ belongs to $\mathcal{P}(K, \alpha)$. In particular, when

$$T > 1 + 2D + 2\delta + \frac{4T_0 + 2R}{\alpha},$$

the above considerations show that for $\gamma \in \Gamma_{\widetilde{K}_R}(T-1, T)$, the associated periodic orbit p_γ belongs to $\mathcal{P}(K, \alpha, T - 1 - 2D - 2\delta, T + 2D + 2\delta)$.

The translation axis of $g_i^{-1} \gamma g_j$ is a lift of p_γ that intersects $\widetilde{K}$. The number of such lifts is at most linear in $\ell(p_\gamma)$, by (3). Therefore, the multiplicity of the above map $\gamma \mapsto p_\gamma$ is at most linear in $\ell(p_\gamma)$.

The above considerations imply that there exist constants C and τ depending only on $K, \widetilde{K}, D, \alpha, F$ such that for $T > 0$ large enough, and all $R > 1$,

$$\sum_{\gamma \in \Gamma_{\widetilde{K}_R},\, T-1 \leq d(o, \gamma o) \leq T} e^{\int_o^{\gamma o} \widetilde{F}} \leq C \times T \times \sum_{p \in \mathcal{P}(K, \alpha, T-1-\tau, T+\tau)} e^{\int_p F}.$$

Taking $\frac{1}{T} \log$ of the above inequality, and letting $T \to +\infty$, and then letting $R \to +\infty$ and $\alpha \to 0$ gives $P_{\mathrm{Gur}}^\infty(F) \geq \delta_\Gamma^\infty(F)$. ■

6. Variational and geometric pressures at infinity coincide

This section is devoted to the proof of the equality between geometric and variational pressures at infinity.

Theorem 6.1. *Let $F \colon T^1 M \to \mathbb{R}$ be a Hölder-continuous potential. Then*

$$\delta_\Gamma^\infty(F) = P_{\mathrm{var}}^\infty(F).$$

The first paragraph contains the proof of the easier inequality $\delta_\Gamma^\infty(F) \leq P_{\mathrm{var}}^\infty(F)$. The harder inequality $P_{\mathrm{var}}^\infty(F) \leq \delta_\Gamma^\infty(F)$ will follow from Section 5, after some reductions. First, in Section 6.2, we introduce a notion of pressure, that we call *Katok* *pressure* in reference to the Katok entropy introduced in [28]. We show that the variational pressure is bounded from above by this new pressure, involving spanning sets. Using the closing lemma, in Section 6.3, we study escape of mass of sequences of probability measures, and relate this new pressure to the Gurevič pressure (which involves weighted growth of periodic orbits), and conclude the proof of the inequality $P_{\mathrm{var}}^{\infty}(F) \le \delta_{\Gamma}^{\infty}(F)$ thanks to Theorem 5.1.

6.1. The first inequality

This paragraph is devoted to the proof of the easier inequality $\delta_{\Gamma}^{\infty}(F) \le P_{\mathrm{var}}^{\infty}(F)$. We deal first with the exceptional situation where $\delta_{\Gamma}(F) = \infty$.

Lemma 6.2. *Under the assumptions of Theorem 6.1, if we assume $\delta_{\Gamma}(F) = \infty$, then for any compact subset K in M and any $C, \varepsilon > 0$, there exists $\mu \in \mathcal{M}_{1,\mathrm{erg}}^{F}$ such that*

$$\mu(T^1 K) < \varepsilon \quad \textit{and} \quad h_{KS}(\mu) + \int F \,\mathrm{d}\mu > C.$$

Proof. The entropy $h_{KS}(\mu)$ of any invariant measure $\mu \in \mathcal{M}_{1,\mathrm{erg}}^{F}$ is nonnegative. Therefore, it suffices to find a measure $\mu \in \mathcal{M}_{1,\mathrm{erg}}^{F}$ with $\mu(T^1 K) < \varepsilon$ and $\int F \,\mathrm{d}\mu > C$. By Theorem 1.1, $P_{\mathrm{var}}(F) = \infty$.

Let $R = R(C, K)$ and $C' = C'(C, K, R)$ be two large enough constants, to be determined later on in the proof. The equality $P_{\mathrm{var}}(F) = \infty$ ensures the existence of a measure $\nu \in \mathcal{M}_1^F$ with $\int F \,\mathrm{d}\nu > C'$, for arbitrarily large $C' > 0$. Taking an ergodic component of ν if necessary, we can assume that ν is ergodic. If $\nu(T^1 K) < \varepsilon$, we are done choosing $\mu = \nu$ and $C' = C$.

Otherwise, consider a ν-typical vector v in $T^1 K$. By the Birkhoff ergodic theorem and Poincaré recurrence theorem, one can find an arbitrarily large $T > 0$ such that

$$\frac{1}{T}\int_0^T F(g^t v)\,\mathrm{d}t > C' \quad \text{and} \quad g^T v \in K.$$

Let K_1 (respectively, K_R) be the neighborhood of size 1 (respectively, R) of K. Consider the open set $\{t \in [0, T],\ g^t v \notin K_1\}$. Inside this set, consider those connected components that contain some t such that $g^t v$ does not belong to K_R. These components have length at least $2(R-1)$. If C' is large enough so that $|F| < C'$ on K_R, we claim that there exists such a component (a, b) such that

$$\int_a^b F(g^t v) > C'(b-a).$$

Indeed, otherwise, one would get

$$\int_0^T F(g^t v) \le C'T$$

by summing the contributions of these big connected components, and integrating the bound $|F| \leq C'$ on the remaining points.

Set $w = g^a v$, and $\tau = b - a$. The piece of orbit $(g^t w)_{0 \leq t \leq \tau}$ has length larger than $2(R-1)$, its projection to M starts and ends in ∂K_1 and remains outside K_1 in between, and it satisfies

$$\int_0^\tau F(g^t w)\,\mathrm{d}t \geq \tau C'.$$

Using the connecting lemma 2.5 in the compact subset K_1, we get a closed orbit $(g^t w')_{0 \leq t \leq \tau + s}$, with $s \leq T_0$, for some T_0 depending only on K_1, which stays at distance at most $1/2$ of the orbit of w for $T_0 \leq t \leq \tau - T_0$. In particular, $g^t w'$ can belong to K only for $t \leq T_0$ or $\tau - T_0 \leq t \leq \tau + s$. Define the measure μ as the uniform probability measure along this periodic orbit. If R has been chosen large enough compared to T_0, we deduce

$$\mu(T^1 K) \leq \frac{3T_0}{2(R-1)} \leq \varepsilon.$$

Let us now check that $\int F\,\mathrm{d}\mu$ is large. First,

$$\left| \int_0^\tau F(g^t w')\,\mathrm{d}t - \int_0^\tau F(g^t w) \right|$$

is bounded by a constant C_0 depending only on K, by Lemma 3.1. Second,

$$\int_\tau^{\tau+s} F(g^t w')$$

is bounded from below by a constant $-C_1$ depending only on K, as s is bounded by T_0 and F is bounded on the $(T_0 + 2)$-neighborhood of K. We get

$$\int_0^{\tau+s} F(g^t w')\,\mathrm{d}t \geq \int_0^\tau F(g^t w)\,\mathrm{d}t - C_0 - C_1 \geq C'\tau - C_0 - C_1.$$

If $C' = C'(K, C, R)$ is large enough, this is at least $C(\tau + s)$, as desired. ■

Proposition 6.3. *Under the assumptions of Theorem 6.1, let F be a Hölder-continuous map. Then $\delta_\Gamma^\infty(F) \leq P_{\mathrm{var}}^\infty(F)$.*

Proof. If $\delta_\Gamma(F) = \infty$, Lemma 6.2 shows that one can find a sequence of measures $\mu_n \in \mathcal{M}_1^F$ tending weakly to 0 such that $h_{KS}(\mu_n) + \int_{T^1 M} F\,\mathrm{d}\mu_n$ tends to infinity. Therefore, $P_{\mathrm{var}}^\infty(F) = \infty$, and the result is obvious. If $\delta_\Gamma^\infty(F) = -\infty$, the result is also obvious.

Assume now that $\delta_\Gamma(F) < \infty$ and $\delta_\Gamma^\infty(F) > -\infty$. Choose for every $R \in \mathbb{N} \setminus \{0\}$ a Hölder-continuous map $0 \leq \chi_R \leq 1$ which approximates $\mathbf{1}_{T^1 p_\Gamma B(o,R)}$ on $T^1 M$: $\chi_R \equiv 1$ on $T^1(p_\Gamma B(o, R-1))$ and $\chi_R \equiv 0$ outside $T^1(p_\Gamma B(o, R))$. Next define

$F_{n,R} = F - n\chi_R$, for all $n \in \mathbb{N}$, and note that $F_{n,R} = F$ outside $T^1 p_\Gamma B(o, R)$ so that $\delta_{\Gamma_{B(o,R)}}(F) = \delta_{\Gamma_{B(o,R)}}(F_{n,R})$ by Proposition 4.9. As a consequence,

$$\delta_\Gamma(F_{n,R}) \geq \delta_{\Gamma_{B(o,R)}}(F_{n,R}) = \delta_{\Gamma_{B(o,R)}}(F) \geq \delta_\Gamma^\infty(F).$$

By the variational principle [33, Theorem 1.1], we can find for all $\varepsilon > 0$ a measure $\mu_{n,R,\varepsilon} \in \mathcal{M}_1^{F_{n,R}}$, such that

$$h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1 M} F_{n,R} d\mu_{n,R,\varepsilon} > \delta_\Gamma(F_{n,R}) - \varepsilon \geq \delta_\Gamma^\infty(F) - \varepsilon.$$

Since $F_{n,R} = F$ outside of a compact subset, $\mu_{n,R,\varepsilon}$ also belongs to $\mathcal{M}_1^F$. Therefore, we have

$$\begin{aligned}
\delta_\Gamma(F) &\geq h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1 M} F \, \mathrm{d}\mu_{n,R,\varepsilon} \\
&\geq n\mu_{n,R,\varepsilon}(T^1 p_\Gamma B(o, R-1)) + h_{KS}(\mu_{n,R,\varepsilon}) + \int_{T^1 M} F_{n,R} \, \mathrm{d}\mu_{n,R,\varepsilon} \\
&\geq n\mu_{n,R,\varepsilon}(T^1 p_\Gamma B(o, R-1)) + \delta_\Gamma^\infty(F) - \varepsilon.
\end{aligned}$$

Choose any sequence $\varepsilon_k \to 0$, $R_k \to \infty$, $n_k \to \infty$, and $\mu_k = \mu_{n_k, R_k, \varepsilon_k}$. As $\delta_\Gamma(F) < \infty$, we get from the above on the one hand that for all $R > 0$,

$$\limsup_{k\to\infty} \mu_k(T^1 p_\Gamma(o, R)) = 0,$$

and on the other hand that

$$\liminf_{k\to\infty} (h_{KS}(\mu_k) + \int F \, \mathrm{d}\mu_k) \geq \delta_\Gamma^\infty(F).$$

This proves that

$$P_{\mathrm{var}}^\infty(F) \geq \delta_\Gamma^\infty(F). \qquad \blacksquare$$

Remark 6.4. Since the proof only needs ergodic measures, it even proves the slightly stronger result

$$\delta_\Gamma^\infty(F) \leq P_{\mathrm{var,erg}}^\infty(F) \leq P_{\mathrm{var}}^\infty(F).$$

6.2. Katok pressure

The proof of Theorem 6.1 will rely on the following notion of pressure, extending to general potentials a notion of entropy introduced by A. Katok in [28] in the case $F = 0$.

For all $v \in T^1\widetilde{M}$ and $\varepsilon, T > 0$, the *dynamical ball* $B(v, \varepsilon; -T, T)$ is defined by

$$B(v, \varepsilon; -T, T) = \{w \in T^1\widetilde{M} \; ; \; \forall t \in [-T, T], \; d(g^t v, g^t w) \leq \varepsilon\}.$$

As in [33], it is more convenient to deal with symmetric dynamical balls. Recall from [33, Lemma 3.14] that for all $0 < \varepsilon \le \varepsilon'$, there exists $T_{\varepsilon,\varepsilon'} \ge 0$, such that for all $v \in T^1\widetilde{M}$ and $T > 0$, we have

$$B(v, \varepsilon'; -T - T_{\varepsilon,\varepsilon'}, T + T_{\varepsilon,\varepsilon'}) \subset B(v, \varepsilon; -T, T) \subset B(v, \varepsilon'; -T, T).$$

As in [44, Remark 3.1], on T^1M, we define two kinds of dynamical balls for $v \in T^1M, \varepsilon, T > 0$: the small dynamical ball

$$B_\Gamma(v, \varepsilon; -T, T) = p_\Gamma(B(\widetilde{v}, \varepsilon; -T, T)),$$

where $\widetilde{v} \in T^1\widetilde{M}$ is a lift of $v \in T^1M$ and the big dynamical ball

$$\begin{aligned} B_{\mathrm{dyn}}(v, \varepsilon; -T, T) &= \{w \in T^1M \ ; \ \forall t \in [-T, T], d(g^t v, g^t w) \le \varepsilon\} \\ &\supset B_\Gamma(v, \varepsilon; -T, T). \end{aligned} \tag{18}$$

Both balls coincide as soon as the injectivity radius of M is bounded from below and ε is small enough. More generally, if along the geodesic $(g^t v)_{-T \le t \le T}$, the injectivity radius at all points $\pi(g^t v)$ is larger than ε, then

$$B_{\mathrm{dyn}}(v, \varepsilon; -T, T) = B_\Gamma(v, \varepsilon; -T, T). \tag{19}$$

We will mainly use the small dynamical balls, that are more convenient in our geometric context, but less natural from the dynamical point of view.

Given a probability measure μ on T^1M, $\delta \in (0,1)$ and $\varepsilon, T > 0$, we will say that a set $V \subset T^1M$ is $(\mu, \delta, \varepsilon; -T, T)$-*spanning*, respectively *dynamically*-$(\mu, \delta, \varepsilon; -T, T)$-*spanning*, if

$$\mu\Big(\bigcup_{v \in V} B_\Gamma(v, \varepsilon; -T, T)\Big) \ge \delta, \quad \text{respectively } \mu\Big(\bigcup_{v \in V} B_{\mathrm{dyn}}(v, \varepsilon; -T, T)\Big) \ge \delta.$$

Of course, a $(\mu, \delta, \varepsilon; -T, T)$-spanning set is also dynamically-$(\mu, \delta, \varepsilon; -T, T)$-spanning.

Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous potential. Let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure on T^1M, invariant under the geodesic flow, such that

$$\int F^- \, \mathrm{d}\mu < \infty.$$

Definition 6.5. Set

$$S_F(\mu, \delta, \varepsilon; -T, T) = \inf \sum_{v \in V} e^{\int_{-T}^{T} F(g^t v)\, \mathrm{d}t},$$

where the infimum is taken over all $V \subset T^1M$ that are $(\mu, \delta, \varepsilon; -T, T)$-spanning. Similarly define $S_F^{\mathrm{dyn}}(\mu, \delta, \varepsilon; -T, T)$ as the infimum of the same quantity over all dynamically- $(\mu, \delta, \varepsilon; -T, T)$-spanning sets.

The *Katok pressure* of F with respect to μ at level δ is defined by

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) = \sup_{\varepsilon>0} \limsup_{T\to+\infty} \frac{1}{2T} \log S_F(\mu, \delta, \varepsilon; -T, T).$$

Similarly, define

$$P^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu, F, \delta) = \sup_{\varepsilon>0} \limsup_{T\to+\infty} \frac{1}{2T} \log S_F^{\mathrm{dyn}}(\mu, \delta, \varepsilon; -T, T).$$

The *Katok pressure* of F with respect to μ, respectively the *dynamical Katok pressure*, is

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F) = \inf_{\delta\in(0,1)} P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta),$$

respectively,

$$P^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu, F) = \inf_{\delta\in(0,1)} P^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu, F, \delta).$$

Comparison between the two kinds of dynamical balls in (18) implies the following inequality:

$$P^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu, F) \le P^{\Gamma}_{\mathrm{Katok}}(\mu, F).$$

The first and main inequality of Proposition 6.6 below was shown in [28]. Compactness was assumed, but his proof [28, (1.4), p. 144] does not use the compactness of the underlying manifold. The second inequality below follows obviously from the above considerations.

Proposition 6.6 (Katok [28]). *Let μ be a g^t-invariant ergodic probability measure. Then for all $\delta > 0$,*

$$h_{KS}(\mu) \le h_{\mathrm{Katok}}(\mu) = P^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu, 0) \le P^{\Gamma}_{\mathrm{Katok}}(\mu, 0).$$

The appendix by F. Riquelme shows that, in the case of geodesic flows on manifolds in negative curvature, these entropies coincide, even in our noncompact setting, cf. Theorem A.1.

In the sequel, we will always work with small dynamical balls and the associated Katok pressure $P^{\Gamma}_{\mathrm{Katok}}(\mu, F)$. Assume that μ is ergodic.

For all $A \subset T^1M$, all $\delta \in (0,1)$ and all $\varepsilon, T > 0$, we define

$$S_{F,A}(\mu, \delta, \varepsilon; -T, T) = \inf_{V\subset A\ (\mu,\delta,\varepsilon;-T,T)\text{-spanning}} \sum_{v\in V} e^{\int_{-T}^{T} F(g^t v)\,dt}$$

and

$$P^{A}_{\mathrm{Katok}}(\mu, F, \delta) = \sup_{\varepsilon>0} \limsup_{T\to+\infty} \frac{1}{2T} \log S_{F,A}(\mu, \delta, \varepsilon; -T, T).$$

The following lemma is elementary but crucial in the sequel.

Lemma 6.7. *Under the assumptions of Theorem 6.1, let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic invariant measure. As soon as $\mu(A) > \delta$, we have*

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \leq P^{A}_{\mathrm{Katok}}(\mu, F, \delta).$$

Moreover, if $\mu(A) \geq 1 - \delta/6$, and F is bounded on A, then

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \geq P^{A}_{\mathrm{Katok}}\Big(\mu, F, \frac{\delta}{2}\Big). \tag{20}$$

Proof. The first inequality is immediate from the definition.

For the second one, let $A' = A \cap g^{-T} A \cap g^T A$. It satisfies $\mu(A') \geq 1 - \delta/2$. Consider V a $(\mu, \delta, \varepsilon; -T, T)$-spanning set. As $\mu(\bigcup_{v \in V} B_\Gamma(v, \varepsilon; -T, T)) \geq \delta$, we get

$$\mu\Big(A' \cap \bigcup_{v \in V} B_\Gamma(v, \varepsilon; -T, T)\Big) \geq \delta/2.$$

For every $v \in V$ such that $\mu(A' \cap B_\Gamma(v, \varepsilon; -T, T)) > 0$, choose an element v' in the intersection $A' \cap B_\Gamma(v, \varepsilon; -T, T)$, and let V' be the set of all such v'. By construction, $V' \subset A$ is a $(\mu, \delta/2, 2\varepsilon; -T, T)$-spanning set.

As F is Hölder-continuous, for $v \in V$ such that $\mu(A' \cap B_\Gamma(v, \varepsilon; -T, T)) > 0$ and $v' \in A' \cap B_\Gamma(v, \varepsilon; -T, T)$, the integrals

$$\int_{-T}^{T} F \circ g^t v \,\mathrm{d}t \quad \text{and} \quad \int_{-T}^{T} F \circ g^t v' \,\mathrm{d}t$$

differ at most by an additive constant depending on the Hölder constants of F, and its L^∞-norm on the ε-neighborhood of A, but not on T. Indeed, as F is bounded on A, and Hölder-continuous, it is also bounded on the ε-neighborhood of A. Moreover, by definition of A', $g^{\pm T} v' \in A$, so that $g^{\pm T} v$ belong to the ε-neighborhood of A. Moreover,

$$d(g^T v, g^T v') \leq \varepsilon \quad \text{and} \quad d(g^{-T} v, g^{-T} v') \leq \varepsilon.$$

Thus, Lemma 3.1 applies and gives the desired bound.

Therefore, up to a multiplicative constant,

$$\sum_{v \in V} e^{\int_{-T}^{T} F(g^t v)\,\mathrm{d}t}$$

is greater than

$$\sum_{v' \in V'} e^{\int_{-T}^{T} F(g^t v')\,\mathrm{d}t}.$$

Up to this multiplicative constant, $S_F(\mu, \delta, \varepsilon; -T, T)$ is greater than $S_{F,A}(\mu, \delta/2, 2\varepsilon; -T, T)$. Taking the limsup of $1/(2T)$ log of these quantities leads to the second inequality. ∎

Since the Katok pressure is defined by taking an infimum over all $(\mu, \delta, \varepsilon; -T, T)$-spanning sets, we deduce the following useful statement.

Lemma 6.8. *Under the assumptions of Theorem 6.1, let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure. Let $\delta > 0$ and $\varepsilon > 0$ be fixed, and for all $T > 0$, let $A_T \subset T^1M$ be a set such that $\mu(A_T) > \delta$. Then*

$$P^\Gamma_{\mathrm{Katok}}(\mu, F) \leq \limsup_{T\to+\infty} \frac{1}{2T} \log S_{F,A_T}(\mu, \delta, \varepsilon; -T, T).$$

We will use the following analogue of Proposition 6.6 for general potentials.

Proposition 6.9. *Under the assumptions of Theorem 6.1, let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map, and $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure on T^1M such that $\int F^- \,\mathrm{d}\mu < \infty$. Then*

$$h_{KS}(\mu) + \int_{T^1M} F \,\mathrm{d}\mu \leq P^\Gamma_{\mathrm{Katok}}(\mu, F).$$

Proof. Let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure and F a Hölder-continuous potential. Let $\delta \in (0, 1)$ be fixed.

For all $\eta > 0$ and $T > 0$, set

$$G_{T,\eta}(F) = \left\{ v \in T^1M \ ; \ \forall t \geq T, \ \left| \frac{1}{2t} \int_{-t}^{t} F(g^s v) \,\mathrm{d}s - \int F \,\mathrm{d}\mu \right| \leq \eta \right\}.$$

Birkhoff ergodic theorem implies that for all $\eta > 0$, we have

$$\lim_{T\to+\infty} \mu(G_{T,\eta}(F)) = 1.$$

Therefore, there exist $T_0 > 0$ and a compact subset $A_{\delta,\eta} \subset G_{T_0,\eta}(F)$ such that

$$\mu(A_{\delta,\eta}) > 1 - \frac{\delta}{6}.$$

Therefore, by (20),

$$\begin{aligned} P^\Gamma_{\mathrm{Katok}}(\mu, F, \delta) &\geq P^{A_{\delta,\eta}}_{\mathrm{Katok}}\left(\mu, F, \frac{\delta}{2}\right) \\ &= \limsup_{T\to+\infty} \frac{1}{2T} \log \inf_{V\subset A_{\delta,\eta}\ (\mu,\delta/2,\varepsilon;-T,T)\text{-spanning}} \sum_{v\in V} e^{\int_{-T}^{T} F(g^t v)\,\mathrm{d}t}. \end{aligned} \tag{21}$$

Let $\mathcal{S}_T \subset A_{\delta,\eta}$ be a finite $(\mu, \delta/2, \varepsilon; -T, T)$-spanning set. As $A_{\delta,\eta} \subset G_{T_0,\eta}(F)$ and thanks to the definition of $G_{T_0,\eta}(F)$, for $T \geq T_0$, we have

$$\sum_{v\in\mathcal{S}_T} e^{\int_{-T}^{T} F(g^t v)\,\mathrm{d}t} \geq e^{2T(\int F\,\mathrm{d}\mu - \eta)} \#\mathcal{S}_T \geq e^{2T(\int F\,\mathrm{d}\mu - \eta)} \inf \#V,$$

the infimum being taken over all $(\mu, \delta/2, \varepsilon, T)$-spanning sets V.

Minimizing over $\mathcal{S}_T$, equation (21) leads to

$$P^{\Gamma}_{\mathrm{Katok}}(\mu, F, \delta) \geq \int F \,\mathrm{d}\mu - \eta + P^{\Gamma}_{\mathrm{Katok}}(\mu, 0, \delta/2).$$

Together with Proposition 6.6, this concludes the proof of Proposition 6.9 since $\delta \in (0, 1)$ and $\eta > 0$ can be arbitrarily small. ■

6.3. Escape of mass and pressure at infinity

This paragraph is devoted to the proof of the following result, of independent interest, which implies Corollary 6.11, a key step in the proof of Theorem 6.1.

Theorem 6.10. *Let $K \subset M$ be a compact set whose interior intersects $\pi(\Omega)$, and let $\widetilde{K} \subset \widetilde{M}$ be a compact subset such that $p_\Gamma(\widetilde{K}) = K$. Let $F\colon T^1 \to \mathbb{R}$ be a Hölder-continuous potential with $\delta_{\Gamma_{\widetilde{K}}}(F) > -\infty$. Let $\eta > 0$. For all $0 < \alpha \leq 1$ and $R \geq 4$, there exists a positive number $\psi = \psi(\widetilde{K}, F, \eta, \alpha/R)$ with the following property. For every $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ with $\mu(T^1 K_R) \leq \alpha$, we have*

$$h_{KS}(\mu) + \int_{T^1 M} F \,\mathrm{d}\mu \leq (1-\alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha\delta_\Gamma(F) + \eta + \psi.$$

Moreover, when $\widetilde{K}$, F and η are fixed, $\psi(\widetilde{K}, F, \eta, \alpha/R)$ tends monotonically to 0 when α/R tends to 0.

Letting K grow to exhaust M, we deduce the following corollary, which provides the second half of Theorem 6.1 (the first inequality $\delta^\infty_\Gamma(F) \leq P^\infty_{\mathrm{var}}(F)$ has been proved in Proposition 6.3).

Corollary 6.11. *Let F be a Hölder-continuous potential on $T^1 M$. Let $(\mu_n)_{n\geq 0} \in (\mathcal{M}^F_1)^{\mathbb{N}}$ be a sequence of probability measures which converges in the vague topology to a measure μ. Then*

$$\limsup_{n\to+\infty} \left(h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n \right) \leq (1 - \|\mu\|)\delta^\infty_\Gamma(F) + \|\mu\|\delta_\Gamma(F).$$

In particular, when $\mu_n \overset{*}{\rightharpoonup} 0$, then

$$\limsup_{n\to+\infty} \left(h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n \right) \leq \delta^\infty_\Gamma(F),$$

so that $P^\infty_{\mathrm{var}}(F) \leq \delta^\infty_\Gamma(F)$.

Proof. When $\delta_\Gamma(F) = \infty$, then $\delta^\infty_\Gamma(F) = \infty$ by Proposition 4.13, and the result is obvious. We can therefore assume that $\delta^\infty_\Gamma(F) < \infty$. We will deal with the case $\delta^\infty_\Gamma(F) > -\infty$, as the case $\delta^\infty_\Gamma(F) = -\infty$ can be treated similarly.

Let $\varepsilon > 0$. Let K be a large compact subset of M, and $\widetilde{K}$ a compact subset of $\widetilde{M}$ satisfying $p_\Gamma(\widetilde{K}) = K$ and

$$\delta_{\Gamma_{\widetilde{K}}}(F) \leq \delta_\Gamma^\infty(F) + \varepsilon \quad \text{and} \quad \|\mu\| \leq \mu(T^1K) + \varepsilon.$$

There are only countably many values of r for which $\mu(\partial T^1K_r)$ has positive measure as these sets are disjoint. Therefore, we can pick r such that $\mu(\partial T^1K_r) = 0$. Replacing K with K_r, we can assume $\mu(\partial T^1K) = 0$.

We apply Theorem 6.10 to $\eta = \varepsilon$, obtaining a function ψ. Let R be large enough so that $\psi(1/R) \leq \varepsilon$. We can also ensure that $\mu(\partial T^1K_R) = 0$. For large enough n, we have

$$\mu_n(T^1K) \geq \mu(T^1K) - \varepsilon \quad \text{and} \quad \mu_n(T^1K_R) \leq \mu(T^1K_R) + \varepsilon \leq \|\mu\| + \varepsilon.$$

In particular,

$$\mu_n(T^1K_R) \geq \mu_n(T^1K) \geq \|\mu\| - 2\varepsilon.$$

Let us estimate $h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n$ for such an n, fixed from now on.

We can write μ_n as an average of ergodic measures:

$$\mu_n = \int_\Omega \nu_\omega \,\mathrm{d}\mathbb{P}(\omega),$$

where all the ν_ω are invariant probability measures for g_t. Since

$$\infty > \int F^- \,\mathrm{d}\mu_n = \int \left(\int F^- \,\mathrm{d}\nu_\omega \right) \mathrm{d}\mathbb{P}(\omega),$$

almost all the measures ν_ω belong to $\mathcal{M}^F_{1,\mathrm{erg}}$. The entropy of a convex combination of probability measures is given by [26, Proposition 4.3.16 (2)]. We can therefore apply Theorem 6.10 to each of the ν_ω (with $\alpha = \nu_\omega(T^1K_R)$) and then average with respect to $\mathbb{P}$, yielding

$$\begin{aligned}
h_{KS}(\mu_n) + \int F \,\mathrm{d}\mu_n &= \int \left(h_{KS}(\nu_\omega) + \int F \,\mathrm{d}\nu_\omega \right) \mathrm{d}\mathbb{P}(\omega) \\
&\leq \int \big((1 - \nu_\omega(T^1K_R))\delta_{\Gamma_{\widetilde{K}}}(F) + \nu_\omega(T^1K_R)\delta_\Gamma(F) + \varepsilon + \psi(1/R) \big) \,\mathrm{d}\mathbb{P}(\omega) \\
&= (1 - \mu_n(T^1K_R))\delta_{\Gamma_{\widetilde{K}}}(F) + \mu_n(T^1K_R)\delta_\Gamma(F) + \varepsilon + \psi(1/R) \\
&\leq (1 - \|\mu\| + 2\varepsilon)(\delta_\Gamma^\infty(F) + \varepsilon) + (\|\mu\| + \varepsilon)\delta_\Gamma(F) + 2\varepsilon.
\end{aligned}$$

As ε is arbitrary, this gives the conclusion. ∎

Let us point out that when $F = 0$ and M is geometrically finite, under the same hypotheses, a stronger version of Corollary 6.11 appears in [37, Theorem 1.1]:

$$\limsup_{n\to+\infty} h_{KS}(\mu_n) \le (1-\|\mu\|)\delta_\Gamma^\infty(0) + \|\mu\| h_{KS}\Big(\frac{\mu}{\|\mu\|}\Big).$$

In [47, 48], Velozo announces an analogous inequality for pressure on general negatively curved manifolds, in the case of potentials going to 0 at infinity. Our approach is valid for all Hölder-continuous potentials, but gives a weaker inequality. However, it provides enough information for our purposes.

Corollary 6.12. *The pressures $P^\infty_{\mathrm{var}}(F)$ and $P^\infty_{\mathrm{var,erg}}(F)$ are equal.*

Proof. We have obviously the inequality

$$P^\infty_{\mathrm{var,erg}}(F) \le P^\infty_{\mathrm{var}}(F).$$

Moreover,

$$P^\infty_{\mathrm{var}}(F) \le \delta_\Gamma^\infty(F)$$

by Corollary 6.11. Finally, Remark 6.4 gives the inequality

$$\delta_\Gamma^\infty(F) \le P^\infty_{\mathrm{var,erg}}(F).$$

Together, these inequalities show that all these quantities coincide. ∎

Proof of Theorem 6.10. As the result is obvious if $\delta_\Gamma(F) = \infty$, we may assume that $\delta_\Gamma(F) < \infty$. Let $K \subset M$ be a compact subset, $R > 0$, and K_R the R-neighborhood of K. Let $\eta > 0$.

Let $\mu \in \mathcal{M}^F_{1,\mathrm{erg}}$ be an ergodic probability measure on T^1M, and $0 < \alpha \le 1$ such that $\mu(T^1K_R) \le \alpha$. Let $\varepsilon > 0$ be small enough (how small exactly will be prescribed at the end of the proof).

Let A be a large compact subset containing K_R, with $\mu(T^1A) > 1-\varepsilon$. Let T_0 be the constant given by assertion (1) of Proposition 2.5 (Connecting lemma) applied with A and K in the role of K and K'. Define

$$A_T = \Big\{ w \in T^1A, \Big| \frac{1}{2T}\int_{-T}^{T} F\circ g^t w \,\mathrm{d}t - \int F\,\mathrm{d}\mu \Big| \le \varepsilon \\ \text{and } \frac{1}{2T}\int_{-T}^{T} \mathbf{1}_{T^1K_R}(g^t w)\,\mathrm{d}t \le \alpha+\varepsilon \Big\}.$$

By the Birkhoff ergodic theorem, there exists $T_1 > 0$ such that for $T \ge T_1$, we have $\mu(A_T) \ge 1-\varepsilon$. Then

$$\mu(A_T \cap g^{T+T_0}T^1A \cap g^{-T-T_0}T^1A) \ge 1-3\varepsilon.$$

The strategy is to bound

$$h_{KS}(\mu) + \int F \, \mathrm{d}\mu$$

from above, in terms of periodic orbits, and use Theorem 5.1 to prove Theorem 6.10.

Consider a maximal $(\varepsilon; -T, T)$-separated subset V of

$$\mathcal{A}_T = A_T \cap g^{T+T_0} T^1 A \cap g^{-T-T_0} T^1 A$$

in the sense that the small dynamical balls $B_\Gamma(v; \varepsilon, -T, T)$, for $v \in V$, are pairwise disjoint. By maximality,

$$\mathcal{A}_T \subseteq \bigcup_{v \in V} B_\Gamma(v, 2\varepsilon; -T, T).$$

Therefore, V is also a $(\mu, \delta, 2\varepsilon; -T, T)$ spanning set for any $\delta \leq 1 - 3\varepsilon$. Proposition 6.9 and Lemma 6.8 ensure that $h_{KS}(\mu) + \int_{T^1 M} F \, \mathrm{d}\mu$ is bounded from above by the exponential growth rate of the sums

$$\sum_{v \in V} e^{\int_{-T}^{T} F(g^t v) \, \mathrm{d}t},$$

over such sets V (that depend implicitly on T).

Now, to each $v \in V$, we will associate a periodic orbit and bound the above sum in terms of $\mathcal{N}_F(K, K_R, \alpha; T - \tau, T + \tau)$ for some constant $\tau > 0$.

Take $v \in V$. As it belongs to $\mathcal{A}_T$, both points $g^{T+T_0} v$ and $g^{-T-T_0} v$ belong to $T^1 A$. By the connecting lemma (Proposition 2.5) applied to the sets A and K, we deduce the existence of a periodic vector v_p, and associated periodic orbit $p(v)$, with $|\ell(p(v)) - 2T| \leq 3T_0$, and $d(g^t v_p, g^t v) \leq \varepsilon/3$ for all $-T \leq t \leq T$. Note that we have used a longer orbit to start with since Proposition 2.5 only gives a good distance control T_0-away from the endpoints of the original geodesic. Since the interior of K intersects $\pi(\Omega)$, it also follows from Proposition 2.5 that we can also require that the orbit $p(v)$ intersects K.

By Lemma 3.1, $\int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t$ is close to $\int_{-T}^{T} F(g^t v) \, \mathrm{d}t$ up to a constant depending only on A and F. Since $v \in \mathcal{A}_T$, the latter integral is close to $2T \int F \, \mathrm{d}\mu$, up to $2T\varepsilon$. Altogether, we get

$$\left| \int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t - \ell(p(v)) \int F \, \mathrm{d}\mu \right| \leq C_0 + \ell(p(v))\varepsilon$$

for some C_0 depending only on A, ε, F. In particular, there exists T_3 such that for $T \geq T_3$, $\ell(p(v))$ is also large, so that this inequality becomes

$$\left| \frac{1}{\ell(p(v))} \int_0^{\ell(p(v))} F(g^t v_p) \, \mathrm{d}t - \int F \, \mathrm{d}\mu \right| \leq 2\varepsilon.$$

Similarly, we obtain, for T large enough,

$$\frac{\ell(p(v) \cap T^1 K_{R/2})}{\ell(p(v))} \leq \alpha + 2\varepsilon,$$

starting from the same properties for the orbit of v due to the definition of $\mathcal{A}_T$, and using the fact that the orbits of v and v_p remain close to each other up to ε, so the orbit of v_p can be in $K_{R/2}$ only at times when the orbit of v is in K_R, except for times in a bounded interval.

Moreover, as the set V is $(\varepsilon; -T, T)$ separated, and the periodic orbit $p(v)$ associated with each $v \in V$ is $\varepsilon/3$-close to it, two parametrized orbits $p(v)$ and $p(v')$ are separated by at least $\varepsilon/3$ when $v \neq v'$. Therefore, the multiplicity of $v \mapsto p(v)$ can only come from different choices of base points on the resulting orbit, separated by at least $\varepsilon/3$, plus orbifold multiplicities bounded by the compactness of A. Hence, this multiplicity is bounded by some multiplicative constant times T.

Therefore, up to a multiplicative constant, $\sum_{v \in V} e^{\int_{-T}^{T} F \circ g^t v \, \mathrm{d}t}$ is bounded by

$$T \mathcal{N}_F(K, K_{R/2}, \alpha + 2\varepsilon, 2T - \tau, 2T + \tau),$$

for some $\tau > 0$ depending on A, F, ε, T_0 but independent of T. Let $\widetilde{K}$ be a compact set such that $p_\Gamma(\widetilde{K}) = K$. Applying Theorem 5.1 with $\eta/2$ and $R/2$, we get that its exponential growth rate is bounded by

$$(1 - \alpha - 2\varepsilon)\delta_{\Gamma_{\widetilde{K}}}(F) + (\alpha + 2\varepsilon)\delta_\Gamma(F) + \eta/2 + \psi\Big(\frac{\alpha + 2\varepsilon}{R/2}\Big),$$

where ψ is a function tending to 0 at 0. If ε is small enough, say $\varepsilon \leq \varepsilon_0$, then the error term $2\varepsilon(\delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F))$ is bounded by $\eta/2$, and we get a bound

$$(1 - \alpha)\delta_{\Gamma_{\widetilde{K}}}(F) + \alpha\delta_\Gamma(F) + \eta + \psi\Big(\frac{\alpha + 2\varepsilon}{R/2}\Big).$$

Finally, we choose $\varepsilon = \alpha\varepsilon_0$, so that $(\alpha + 2\varepsilon)/(R/2)$ is a function of α/R that tends to 0 when α/R tends to 0. This is the desired bound. ■

7. Strong positive recurrence

In symbolic dynamics, the notion of strong positive recurrence appeared in several works, as mentioned in the introduction, see for example [8, 9, 13, 22–24, 39, 40, 42]. In our geometric context, when $F = 0$, the notion appeared in [16, 44] under the terminology of "strongly positively recurrent manifold" or "strongly positively recurrent action". Independently, it appeared (still in the case $F = 0$) among people interested by geometric group theory, see for example [2, 50, 51], under the name of "actions with a growth gap" or later "statistically convex-cocompact manifolds". We follow the ergodic terminology of *strong positive recurrence* below, extending the point of view developed in [44], in the spirit of the works of symbolic dynamics.

7.1. Different notions of recurrence

Recall some definitions which are classical in symbolic dynamics, and were introduced for the geodesic flow in negative curvature in [34, 44]. Recall that $\mathcal{P}$ and $\mathcal{P}'_K$ have been defined in Paragraph 3.1.3, and $n_{\widetilde{K}}$ after Remark 2.6.

Definition 7.1. A Hölder-continuous potential $F\colon T^1M \to \mathbb{R}$ with finite topological pressure is said to be:

(1) *recurrent* if there exists a compact subset $K \subset M$ whose interior intersects the projection $\pi(\Omega)$ of the nonwandering set, with a compact lift $\widetilde{K}$ to $\widetilde{M}$ such that
$$\sum_{p\in\mathcal{P}} n_{\widetilde{K}}(p) e^{\int_p (F-\delta_\Gamma(F))} = +\infty;$$

(2) *positively recurrent* if it is recurrent with respect to some compact subset $K \subset M$ whose interior intersects $\pi(\Omega)$, with a lift $\widetilde{K}$ to $\widetilde{M}$, and for some $N \geq 1$,
$$\sum_{p\in\mathcal{P}'_K,\, n_{\widetilde{K}}(p)\leq N} \ell(p) e^{\int_p (F-\delta_\Gamma(F))} < +\infty;$$

(3) *strongly positively recurrent* if its pressure at infinity satisfies
$$P^{\infty}_{\mathrm{top}}(F) < P_{\mathrm{top}}(F).$$

Let us introduce another quantitative notion of recurrence, which involves an invariant measure.

Let $K \subset M$ be a compact subset, $\widetilde{K} \subset \widetilde{M}$ a compact subset such that $p_\Gamma(\widetilde{K}) = K$. For all $T > 0$ large enough, as in [44], we define[1] $U_T(\widetilde{K}) \subset \widetilde{M}$ as the open set
$$U_T(\widetilde{K}) = \{y \in \widetilde{M} \cup \partial\widetilde{M},\ \exists x \in \widetilde{K},\ d(x,y) > T \text{ and } [x,y)_T \cap \Gamma\widetilde{K} \subset \widetilde{K}\},$$

where $[x,y)_T$ denotes the geodesic segment of length T starting from x on $[x,y)$. When we write this, we implicitly require $d(x,y) > T$. In other words, $y \in U_T(\widetilde{K})$ if

[1] In [16], the definition has been slightly modified to guarantee that it remains open when $\widetilde{M}$ is a Gromov-hyperbolic metric space.

there exists some geodesic $[x, y)$ starting in $\widetilde{K}$ and ending at y, which does not meet $\Gamma\widetilde{K} \setminus \widetilde{K}$ until time T.

For technical reasons, we will need to work with the following slightly larger sets:

$$U_{T_0,T}(\widetilde{K}) = \{y \in \widetilde{M} \cup \partial\widetilde{M},\ \exists x \in \widetilde{K},\ d(x, y) > T \text{ and } [x, y)_{[T_0,T]} \cap \Gamma\widetilde{K} \subset \widetilde{K}\},$$

where $[x, y)_{[T_0,T]}$ denotes the geodesic segment of length $T - T_0$ starting at distance T_0 from x on $[x, y)$ (assuming again $d(x, y) > T \geq T_0$). In other words, $y \in U_{T_0,T}(\widetilde{K})$ if there exists some geodesic $[x, y)$ starting in $\widetilde{K}$ and ending at y, which does not meet $\Gamma\widetilde{K} \setminus \widetilde{K}$ between times T_0 and T.

Let us define $\widetilde{V}_T(\widetilde{K}) \subset T^1\widetilde{K}$ (respectively, $\widetilde{V}_{T_0,T}(\widetilde{K}) \subset T^1\widetilde{K}$) as the set of unit vectors tangent at x to a geodesic segment $[x, y)$, for some $y \in U_T(\widetilde{K})$ (respectively, $U_{T_0,T}(\widetilde{K})$) and x associated with y as above. Finally, let

$$V_T(\widetilde{K}) = p_\Gamma(\widetilde{V}_T(\widetilde{K})) \quad \text{and} \quad V_{T_0,T}(\widetilde{K}) = p_\Gamma(V_{T_0,T})(\widetilde{K})).$$

All these sequences

$$(U_T(\widetilde{K}))_{T>0}, \quad (U_{T_0,T}(\widetilde{K}))_{T>T_0}, \quad (\widetilde{V}_T(\widetilde{K}))_{T>0},$$
$$(\widetilde{V}_{T_0,T}(\widetilde{K}))_{T>T_0}, \quad (V_T(\widetilde{K}))_{T>0}, \quad (V_{T_0,T}(\widetilde{K}))_{T>T_0}$$

are nonincreasing when $T \to \infty$.

Definition 7.2. The geodesic flow (g^t) is said to be *exponentially recurrent* with respect to an invariant (not necessarily finite) measure m if there exist a compact subset $K \subset M$ whose interior intersects $\pi(\Omega)$ and some compact subset $\widetilde{K} \subset \widetilde{M}$ with $p_\Gamma(\widetilde{K}) = K$ such that, for all $T_0 \geq 0$, there exist $C > 0$ and $\alpha > 0$ such that for $T \geq T_0$,

$$m(V_{T_0,T}(\widetilde{K})) \leq C \exp(-\alpha T).$$

In [34, Theorems 1.2, 1.4 and 1.6], the following result, reformulated here thanks to Theorem 3.8, is proven.

Theorem 7.3 (Pit–Schapira). *Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure.*

1. *The potential F is recurrent if and only if (Γ, F) is divergent, if and only if m^F is ergodic and conservative*
2. *The potential F is positively recurrent if and only if m^F is finite.*
3. *The potential F is positively recurrent if and only if it is recurrent and there exists a compact subset $K \subset M$ whose interior intersects at least a closed geodesic, and a compact subset $\widetilde{K} \subset \widetilde{M}$ with $p_\Gamma(\widetilde{K}) = K$, such that*

$$\sum_{\gamma\in\Gamma_{\widetilde{K}}} d(o, \gamma o)e^{-\delta_\Gamma(F)d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}} < +\infty.$$

In Section 7.3, we will prove the following result.

Theorem 7.4. *Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure. If $F: T^1M \to \mathbb{R}$ is strongly positively recurrent, then it is positively recurrent.*

This theorem has been proved in [44] for the case $F \equiv 0$, and the proof is almost the same. We provide it here for the sake of completion and the comfort of the reader.

The contrapositive reformulation is extremely useful:

If the measure m^F is infinite, then $\delta_\Gamma^\infty(F) = \delta_\Gamma(F)$.

It has the following corollary.

Corollary 7.5. *Let $F: T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure. Let $p: \bar{M} \to M$ be an infinite Riemannian connected Galois cover of M, and $H = \pi_1(\bar{M}) \lhd \Gamma = \pi_1(M)$. Let $\bar{F} = F \circ dp: T^1\bar{M} \to \mathbb{R}$ be the lift of F to $T^1\bar{M}$. Then*

$$\delta_H^\infty(\bar{F}) = \delta_H(\bar{F}) \leq \delta_\Gamma(F).$$

Proof. The inequality $\delta_H(\bar{F}) \leq \delta_\Gamma(F)$ is immediate since $H \subset \Gamma$. By contradiction, assume that $\delta_H^\infty(\bar{F}) < \delta_H(\bar{F})$. Then the potential $\bar{F}$ would be strongly positively recurrent. By Theorems 7.4 and 3.7, once renormalized into a probability measure, the associated equilibrium measure m^F is finite and unique. By uniqueness, the measure m^F is invariant under the action of the deck group $G = \Gamma/H$. As G is infinite by hypothesis, it is a contradiction with the finiteness of m^F. ∎

Remark 7.6. This corollary does not apply to nonregular cover, even for the zero potential. For example, consider the following construction. Given $\Sigma_\Gamma = \mathbb{H}^2/\Gamma$ a compact genus 2 hyperbolic surface, there exists $H < \Gamma$ a nonnormal subgroup such that $\Sigma_H = \mathbb{H}^2/H$ is a punctured torus with infinite volume. The (nonregular) covering $p: \Sigma_H = \mathbb{H}^2/H \to \Sigma_\Gamma$ does not satisfy the conclusion of the above corollary. Indeed, Σ_H is convex cocompact, nonelementary, with infinite volume. In particular, there exists a large compact subset $\tilde{K} \subset \mathbb{H}^2$ such that $H_{\tilde{K}}$ is finite, so that

$$\delta_H(0) > 0 \quad \text{and} \quad \delta_H^\infty(0) = -\infty.$$

Corollary 7.7. *There exists a complete hyperbolic surface M, with $\delta_\Gamma^\infty(0) > 0$, and a Hölder-continuous potential $F: T^1M \to \mathbb{R}$ such that $\delta_\Gamma^\infty(F) = -\infty$.*

Observe that if $\delta_\Gamma^\infty(0) > -\infty$, then it is nonnegative and every Hölder-continuous potential F which is bounded from below by some constant $-K$ satisfies $\delta_\Gamma^\infty(F) \geq -K$. Therefore, examples satisfying Corollary 7.7 must be unbounded from below.

Proof. Let $M = \mathbb{H}^2/\Gamma$ be a $\mathbb{Z}$-cover of a compact hyperbolic surface. By Corollary 7.5,

$$\delta_\Gamma^\infty(0) = \delta_\Gamma(0) > 0.$$

It is well known that $\delta_\Gamma(0) = 1$ (it follows for instance from [11], see for instance [16] for details on critical exponents of covers). Choose some compact fundamental domain $D \subset M$ with piecewise smooth boundary for the action of the deck group $G = \langle g^n \ ; \ n \in \mathbb{Z}\rangle$. For all $n \in \mathbb{Z}$, set $D_n = g^n D$. Then build a Hölder-continuous map $F : T^1 M \to \mathbb{R}$ such that for all $n \in \mathbb{Z} \setminus \{0\}$ and $v \in T^1 D_n$, we have

$$-|n| \le F(v) \le -(|n| - 1).$$

Considering compact subsets $\widetilde{K}_N$ with $p_\Gamma(\widetilde{K}_N) = \bigcup_{|n|\le N} D_n$, we have

$$\delta_{\Gamma_{\widetilde{K}_N}}(F) \le \delta_{\Gamma_{\widetilde{K}_N}}(0) - N,$$

so that $\delta_\Gamma^\infty(F) = -\infty$. ■

In fact, this construction applies whenever the nonwandering set Ω is noncompact, for the potential $F(v) = -d(\pi(v), p_\Gamma(o))$, for instance.

The next two theorems are proved respectively in Sections 7.4 and 7.5.

Theorem 7.8. *Let $F : T^1 M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure. The potential F is strongly positively recurrent if and only if the geodesic flow is exponentially recurrent with respect to the measure m^F given by the Patterson–Sullivan–Gibbs construction.*

The last result that we shall prove provides very satisfying information on strongly positively recurrent potentials. We will not use it in this paper.

Theorem 7.9. *Let $F : T^1 M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure. If $F : T^1 M \to \mathbb{R}$ is strongly positively recurrent, then for* every *compact set $\widetilde{K} \subset \widetilde{M}$, whose interior intersects $\pi(\Omega)$, we have $\delta_{\Gamma_{\widetilde{K}}}(F) < \delta_\Gamma(F)$.*

In fact, the proof of Theorem 7.8 shows that, for any compact subset K with $\delta_{\Gamma_{\widetilde{K}}}(F) < \delta_\Gamma(F)$, the sets $V_{T_0,T}(\widetilde{K})$ have exponentially small m^F-measure for all $T \ge T_0$. Together with Theorem 7.9, this implies the following corollary.

Corollary 7.10. *Let $F : T^1 M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure and with finite Gibbs measure m^F. Assume that the geodesic flow is exponentially recurrent with respect to m^F. Then, for* any *compact subset $K \subset M$ whose interior intersects $\pi(\Omega)$ and any compact subset $\widetilde{K}$ of $\widetilde{M}$ with $p_\Gamma(\widetilde{K}) = K$, for any $T_0 \ge 0$, there exist $C > 0$ and $\alpha > 0$ such that for all $T \ge T_0$,*

$$m^F(V_{T_0,T}(\widetilde{K})) \le C \exp(-\alpha T).$$

Before proving these results about strong positive recurrence, we provide in the next paragraph ways of constructing strongly positively recurrent potentials.

7.2. Strong positive recurrence through bumps and wells

Adding a bump λA to a potential F, with A a nonnegative compactly supported Hölder-continuous map and $\lambda \to +\infty$, we already proved in Corollary 4.12 the existence of strongly positively recurrent potentials. We restate it below with this terminology.

Corollary 7.11. *On any nonelementary complete connected Riemannian manifold with pinched negative curvature and bounded first derivative of the curvature, there exist Hölder-continuous maps that are strongly positively recurrent.*

It will be convenient to add to a given potential F large bumps of arbitrarily small height. It is what we do in the next proposition.

Proposition 7.12. *Let $F\colon T^1M \to \mathbb{R}$ be a Hölder-continuous map with finite topological pressure. For all $\varepsilon > 0$, there exists a Hölder-continuous map $0 \leq A \leq 1$ compactly supported on T^1M, such that*

$$\delta_\Gamma^\infty(F+\varepsilon A) = \delta_\Gamma^\infty(F) \leq \delta_\Gamma(F) < \delta_\Gamma(F+\varepsilon A).$$

Proof. For a given $\varepsilon > 0$, by the variational principle for $P_{\mathrm{top}}(F)$, there exists a measure $m_\varepsilon \in \mathcal{M}_1^F$, such that

$$P_{\mathrm{top}}(F) = \delta_\Gamma(F) = \sup_{m\in\mathcal{M}_1^F}\left(h_{KS}(m) + \int F\,\mathrm{d}m\right) \leq h_{KS}(m_\varepsilon) + \int F\,\mathrm{d}m_\varepsilon + \frac{\varepsilon}{2}.$$

Choose some compact subset K_ε such that $m_\varepsilon(T^1K_\varepsilon) \geq 1-\varepsilon$. Now, choose some Hölder-continuous map $0 \leq A \leq 1$ with compact support such that $A \equiv 1$ on T^1K_ε. Observe that as soon as $0 < \varepsilon < 1/2$, we have

$$\delta_\Gamma(F+\varepsilon A) \geq h_{KS}(m_\varepsilon) + \int F\,\mathrm{d}m_\varepsilon + \varepsilon m(K_\varepsilon) \geq \delta_\Gamma(F) - \frac{\varepsilon}{2} + \varepsilon(1-\varepsilon) > \delta_\Gamma(F).$$

The result follows. ∎

Adding a bump does not modify the topological pressure at infinity, and increases the topological pressure to produce strongly positively recurrent potentials. On the other hand, subtracting a bump, i.e., adding a well, does not modify the topological pressure at infinity and decreases the topological pressure towards the topological pressure at infinity, as shown in the next statement.

Proposition 7.13. *Let $F : T^1M \to \mathbb{R}$ be a Hölder-continuous map. Then for all $\beta \geq 0$ satisfying $\delta_\Gamma^\infty(F) \leq \delta_\Gamma(F) - \beta$ and all $\eta > 0$, there exists a compactly supported Hölder-continuous function A on T^1M taking values in $[0, \beta]$ such that*

$$\delta_\Gamma(F - A) \leq \delta_\Gamma(F) - \beta + \eta.$$

When $\delta_\Gamma^\infty(F)$ is finite, one may take $\beta = \delta_\Gamma(F) - \delta_\Gamma^\infty(F)$. Then the proposition says that, by perturbing F with a compactly supported potential taking values in $[0, \delta_\Gamma(F) - \delta_\Gamma^\infty(F)]$, one may get a pressure which is arbitrarily close to $\delta_\Gamma^\infty(F)$. The formulation we have given also makes sense when $\delta_\Gamma^\infty(F) = -\infty$, and says in this case that with a compact perturbation one can make the topological pressure arbitrarily negative.

Proof. When $\delta_\Gamma(F) = +\infty$, the conclusion is true for $A = 0$. Therefore, we may assume that F has finite topological pressure. For every $n \in \mathbb{N}$, let A_n be a compactly supported Hölder-continuous map taking values in $[0, \beta]$, equal to β on $p_\Gamma T^1 B(o, n)$. We claim that $\limsup \delta_\Gamma(F - A_n) \leq \delta_\Gamma(F) - \beta$. The proposition follows from this claim by taking $A = A_n$ for large n. Let us prove it.

For every $n \geq 1$, choose an invariant measure μ_n with

$$h_{KS}(\mu_n) + \int (F - A_n)\, \mathrm{d}\mu_n \geq \delta_\Gamma(F - A_n) - \frac{1}{n}.$$

Extracting a subsequence if necessary, we can assume that μ_n converges weakly to an invariant measure μ, with mass $\|\mu\| \in [0, 1]$. Let $\varepsilon > 0$. Choose a large compact subset $K \subset M$ with $\mu(T^1K) > \|\mu\| - \varepsilon$ and $\mu(\partial T^1K) = 0$. For large enough n, we also have $\mu_n(T^1K) > \|\mu\| - \varepsilon$, and therefore

$$\int A_n\, \mathrm{d}\mu_n \geq \beta(\|\mu\| - \varepsilon)$$

as A_n is equal to β on T^1K. We get

$$\begin{aligned} \limsup \delta_\Gamma(F - A_n) &= \limsup \left(h_{KS}(\mu_n) + \int (F - A_n)\, \mathrm{d}\mu_n \right) \\ &\leq \limsup \left(h_{KS}(\mu_n) + \int F\, \mathrm{d}\mu_n \right) - \beta(\|\mu\| - \varepsilon). \end{aligned}$$

Apply now Corollary 6.11, to get an upper bound

$$\limsup \delta_\Gamma(F - A_n) \leq (1 - \|\mu\|)\delta_\Gamma^\infty(F) + \|\mu\|\delta_\Gamma(F) - \beta(\|\mu\| - \varepsilon).$$

With the inequality $\delta_\Gamma^\infty(F) \leq \delta_\Gamma(F) - \beta$, this gives

$$\limsup \delta_\Gamma(F - A_n) \leq \delta_\Gamma(F) - \beta + \beta\varepsilon.$$

As ε is arbitrary, this concludes the proof. ∎

7.3. Strong positive recurrence implies positive recurrence

In this section, we shall prove Theorem 7.4. We follow the proof of [44] in the case $F = 0$.

Assume that F is strongly positively recurrent. By definition, there exists a compact subset $K \subset M$ whose interior intersects at least a closed geodesic, and a compact subset $\widetilde{K} \subset \widetilde{M}$ with $p_\Gamma(\widetilde{K}) = K$, such that

$$\delta_{\Gamma_{\widetilde{K}}}(F) < \delta_\Gamma(F).$$

Without loss of generality, we assume that $o \in \widetilde{K}$.

An elementary computation shows that this strict inequality implies the convergence of the series

$$\sum_{\gamma \in \Gamma_{\widetilde{K}}} d(o, \gamma o) e^{-\delta_\Gamma(F) d(o,\gamma o) + \int_o^{\gamma o} \widetilde{F}}.$$

Therefore, in order to prove that strong positive recurrence implies positive recurrence, by Theorem 7.3 (3), it is enough to show that F is recurrent.

The following inclusion is a variant of an observation of [44]:

$$\Lambda_\Gamma \setminus \Lambda_\Gamma^{\mathrm{rad}} \subset \bigcup_{T_0 > 0} \bigcap_{T > T_0} U_{T_0,T}(\widetilde{K}).$$

Indeed, the set on the right represents points $y \in \partial \widetilde{M}$ such that for some $x \in \widetilde{K}$, the geodesic $[x, y)$ stays a bounded amount of time in $\Gamma.\widetilde{K}$, whereas the set on the left is the set of $y \in \Lambda_\Gamma$ such that the geodesic $[xy)$ eventually leaves every orbit $\Gamma.\widetilde{L}$, for every compact subset $\widetilde{L} \subset \widetilde{M}$.

To prove Theorem 7.4, it suffices to show that for all $T_0 > 0$, we have

$$\nu^F\Big(\bigcap_{T > T_0} U_{T_0,T}(\widetilde{K})\Big) = 0.$$

It then follows by the above inclusion that ν^F gives zero measure to $\Lambda_\Gamma \setminus \Lambda_\Gamma^{\mathrm{rad}}$, and therefore full measure to $\Lambda_\Gamma^{\mathrm{rad}}$. Then Theorem 3.8 implies that F is recurrent.

The following lemma, a variation around [44, equation (29)], is a key step of the proof, and will be useful also in Section 7.4. For $\varepsilon > 0$, let $\widetilde{K}_\varepsilon$ be the ε-neighborhood of $\widetilde{K}$. Let D be the diameter of $\widetilde{K}$. We recall that $\widetilde{K}$ contains o.

Lemma 7.14. *For all $\varepsilon > 0$, there exists $T_0 > 0$ such that for all $T > T_0 + 2D + \varepsilon$, we have*

$$\bigcup_{\substack{\gamma \in \Gamma_{\widetilde{K}_\varepsilon}, \\ d(o,\gamma o) \geq T + D + T_0}} \mathcal{O}_o(\gamma \widetilde{K}) \subset U_{T_0,T}(\widetilde{K}).$$

Moreover, for all $T_1 \geq 0$, there exists a finite set $\{g_1, \ldots, g_N\}$ of elements of Γ such that for all $T > T_1 + 2D$, we have

$$U_{T_1,T}(\widetilde{K}) \cap \Gamma\widetilde{K} \subset \bigcup_{i=1}^{N} \bigcup_{\substack{\gamma\in\Gamma_{\widetilde{K}},\\ d(o,\gamma o)\geq T-2D-T_1}} g_i \cdot \mathcal{O}_{\widetilde{K}}(\gamma\widetilde{K}).$$

Proof. The first inclusion uses the same kind of arguments as for [44, equation (29)]. If $\gamma \in \Gamma_{\widetilde{K}_\varepsilon}$, there exist $x, y \in \widetilde{K}_\varepsilon$ such that the geodesic segment $[x, \gamma y]$ does not intersect $\Gamma\widetilde{K}_\varepsilon$ outside $\{x, \gamma y\}$. Consider $z \in \mathcal{O}_o(\gamma\widetilde{K})$. The geodesic $[o, z]$ intersects $\gamma\widetilde{K}$ at a point γz_0. By Lemma 2.3, there exists $T_0 > 0$ such that the geodesics $[x, \gamma y]$ and $[o, \gamma z_0]$ follow each other up to ε, except in the T_0 neighborhood of the beginning and of the end of these segments. Moreover,

$$d(o, \gamma o) \geq T + D + T_0,$$

so that $d(o, \gamma z_0) \geq T + T_0$. Therefore, $[o, \gamma z_0]$ avoids $\Gamma\widetilde{K}$ along $[T_0, T]$, and so does $[o, z]$.

For the second inclusion, let $T_1 \geq 0$. Introduce a finite family $(g_i)_{1\leq i\leq N}$ of isometries of Γ such that the T_1-neighborhood $\widetilde{K}_{T_1}$ of $\widetilde{K}$ satisfies

$$\widetilde{K}_{T_1} \cap \Gamma\widetilde{K} \subset \bigcup_i g_i\widetilde{K}.$$

Consider a point $y \in U_{T_1,T}(\widetilde{K}) \cap \Gamma\widetilde{K}$. Consider the last copy $g_i\widetilde{K}$ intersected by the segment $[o, y]_{T_1}$, and the first copy $h\widetilde{K}$ intersected by the segment $[o, y]_{T_1+D,T}$. By definition, $\gamma = g_i^{-1}h \in \Gamma_{\widetilde{K}}$. Moreover, it satisfies

$$d(o, \gamma o) \geq T - 2D - T_1,$$

and $y \in g_i \cdot \mathcal{O}_{\widetilde{K}}(\gamma\widetilde{K})$ by construction. The desired inclusion follows. ∎

Lemmas 7.14 and 3.6 have the following corollary, from which Theorem 7.4 follows.

Corollary 7.15. *For all $T_0 \geq 0$ and for all $0 < \eta < \delta_\Gamma(F) - \delta_{\Gamma_{\widetilde{K}}}(F)$, there exist $T_1, C > 0$ such that for $T \geq T_1$, we have*

$$\nu^F(U_{T_0,T}(\widetilde{K})) \leq Ce^{-(\delta_\Gamma(F)-\delta_{\Gamma_{\widetilde{K}}}(F)-\eta)T}. \tag{22}$$

In particular,

$$\nu^F\left(\bigcap_{T>T_0} U_{T_0,T}(\widetilde{K})\right) = 0.$$

Similar statements appeared in [44] and [16], but it appears that some details are welcome on the limit process. We include therefore a detailed (short) argument.

Proof. Let η be as above. By Lemmas 7.14 and 3.6 and conformality of ν^{F,s_n}, for all $s_n > \delta_\Gamma(F)$ close enough to $\delta_\Gamma(F)$, and $T > T_0$ large enough, we have

$$\begin{aligned}\nu^{F,s_n}(U_{T_0,T}(\widetilde{K})) &= \nu^{F,s_n}(\Gamma o \cap U_{T_0,T}(\widetilde{K}))\\ &\leq \sum_{i=1}^{N} \sum_{\substack{\gamma\in\Gamma_{\widetilde{K}},\\ d(o,\gamma o)\geq T-2D-T_0}} \nu^{F,s_n}(g_i \cdot \mathcal{O}_{\widetilde{K}}(\gamma \widetilde{K}))\\ &\leq N \times C \times \sum_{\substack{\gamma\in\Gamma_{\widetilde{K}},\\ d(o,\gamma o)\geq T-2D-T_0}} e^{-(s_n-\eta/2)d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}}.\end{aligned}$$

As the exponential growth rate of

$$\sum_{\substack{\gamma\in\Gamma_{\widetilde{K}},\\ i\leq d(o,\gamma o)\leq i+1}} e^{\int_o^{\gamma o}\widetilde{F}}$$

is $\delta_{\Gamma_{\widetilde{K}}}$ by definition, for large enough i, we get

$$\sum_{\substack{\gamma\in\Gamma_{\widetilde{K}},\\ i\leq d(o,\gamma o)\leq i+1}} e^{\int_o^{\gamma o}\widetilde{F}} \leq e^{(\delta_{\Gamma_{\widetilde{K}}}+\eta/2)i}.$$

Together with the previous equation, this gives for large enough T, for some constant $C > 0$, the inequality

$$\nu^{F,s_n}(U_{T_0,T}(\widetilde{K})) \leq C \times e^{(\delta_{\Gamma_{\widetilde{K}}}(F)+\eta-s_n)T}.$$

Now, ν^F is the weak-$*$ limit $\nu^F = \lim_{n\to\infty}\nu^{F,s_n}$. Therefore, for any open set U, one has $\nu^F(U) \leq \liminf \nu^{F,s_n}(U)$. We obtain

$$\nu^F(U_{T_0,T}(\widetilde{K})) \leq C \times e^{(\delta_{\Gamma_{\widetilde{K}}}(F)+\eta-\delta_\Gamma(F))T}.$$

The result follows. ∎

7.4. Strong positive recurrence and exponential recurrence

Let us prove Theorem 7.8.

Proof. The implication "strong positive recurrence of F implies exponential recurrence of (g^t) with respect to m^F" was essentially shown in the above proof of Theorem 7.4, and in particular equation (22). Indeed, let K and $\widetilde{K}$ be as in the proof of this theorem. On $T^1\widetilde{M}$, the product structure $\widetilde{m}^F \sim \nu^F \times \nu^F \times dt$ (see equation (7)), in the Hopf coordinates (see equation (2)) shows that up to some constant c,

$$m^F(V_{T_0,T}(\widetilde{K})) \leq \widetilde{m}^F(\widetilde{V}_{T_0,T}(\widetilde{K})) \leq c\nu^F(\partial\widetilde{M}) \times \nu^F(U_{T_0,T}(\widetilde{K})).$$

Equation (22) concludes. Note that this proof, combined with Theorem 7.9, implies Corollary 7.10.

Conversely, suppose that (g^t) is exponentially recurrent with respect to m^F, so that for some compact subset $\widetilde{K} \subset \widetilde{M}$ whose interior intersects $\pi(\widetilde{\Omega})$, for every $T_0 > 0$, there exist $\alpha = \alpha(T_0) > 0$ and $C = C(T_0) > 0$ such that, for all $T > T_0$,

$$m^F(V_{T_0,T}(\widetilde{K})) \leq C \exp(-\alpha T). \tag{23}$$

The first step consists in showing that there exists a constant C' such that, for all $T \geq T_0$, we have

$$\nu^F(U_{T_0,T}(\widetilde{K})) \leq C'e^{-\alpha T}. \tag{24}$$

Since the projection p_Γ is finite-to-one on the compact set $T^1\widetilde{K}$, one deduces from (23) that $\widetilde{m}^F(\widetilde{V}_{T_0,T}(\widetilde{K}) \leq C_0 \exp(-\alpha T)$ for some constant C_0. By definition, if $v \in \widetilde{V}_{T_0,T}(\widetilde{K})$, then $v^+ \in U_{T_0,T}(\widetilde{K})$, and $v^- \in \mathcal{O}_{v^+}(\widetilde{K})$. Recall that m^F is supported in Ω. As above, Equations (7) and (2) show that up to some constant c,

$$C_0 \exp(-\alpha T) \geq \widetilde{m}^F(\widetilde{V}_{T_0,T}(\widetilde{K})) \geq \frac{1}{c} \inf_{v\in\widetilde{\Omega}\cap T^1\widetilde{K}} \nu^F(\mathcal{O}_{v^+}(\widetilde{K})) \times \nu^F(U_{T_0,T}(\widetilde{K})).$$

In the above infimum, the vector v varies in the compact set $\widetilde{\Omega} \cap T^1\widetilde{K}$, and ν^F has full support in the limit set, so that this infimum is positive. Therefore, (24) is proven.

In what follows, we will need to consider a compact set $\widetilde{L}$ large enough to satisfy the lower bound in Lemma 3.6. By a standard use of Lemma 2.3, for all $\varepsilon > 0$ there exists $\tau > 0$, such that if $\widetilde{L} \supset \widetilde{K}_\varepsilon \supset \widetilde{K}$ contains an ε-neighborhood of $\widetilde{K}$, uniformly in $T \geq T_0 + 2\tau$, we have

$$\overline{U_{T_0,T}(\widetilde{L})} \subset U_{T_0+\tau,T-\tau}(\widetilde{K}).$$

In particular, it follows from (24) that $\nu^F(\overline{U_{T_0,T}(\widetilde{L})}) \leq C'e^{-\alpha T}$ for some $C' > 0$ and $\alpha > 0$. Until now, T_0 was arbitrary. We choose now T_0 given by the first item in Lemma 7.14.

As $\nu^F = \lim_{s_n\to\delta_\Gamma(F)} \nu^{F,s_n}$, we have

$$\limsup \nu^{F,s_n}(U_{T_0,T}(\widetilde{L})) \leq \nu^F(\overline{U_{T_0,T}(\widetilde{L})}) \leq C'e^{-\alpha T}.$$

Therefore, for all s_n close enough to $\delta_\Gamma(F)$, we have $\nu^{F,s_n}(U_{T_0,T}(\widetilde{L})) \leq C'e^{-\beta T}$ for any $\beta < \alpha$, for instance $\beta = \alpha/2$. Fix some $\varepsilon > 0$. Then Lemma 7.14 gives for some $D = D(\widetilde{L})$,

$$\bigcup_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\geq T+D+T_0}} \mathcal{O}_o(\gamma\widetilde{L}) \subset U_{T_0,T}(\widetilde{L}),$$

so that, as ν^{F,s_n} is supported on Γo,

$$\nu^{F,s_n}\Bigg(\Gamma o \cap \bigcup_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\geq T+D+T_0}} \mathcal{O}_o(\gamma\widetilde{L})\Bigg) \leq C' e^{-\beta T}.$$

In particular, there exists C'' such that, for any large enough k, we have

$$\nu^{F,s_n}\Bigg(\Gamma o \cap \bigcup_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\in[k,k+1]}} \mathcal{O}_o(\gamma\widetilde{L})\Bigg) \leq C'' e^{-\beta k}.$$

As the group Γ acts properly discontinuously on $\widetilde{M}$ and $\widetilde{L}$ is compact, the intersections of shadows in the above union have a bounded multiplicity. Therefore, we deduce that there exists some constant $c > 0$ such that

$$\sum_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\in[k,k+1]}} \nu^{F,s_n}(\mathcal{O}_o(\gamma\widetilde{L})) \leq c e^{-\beta k}.$$

Together with the Orbital Shadow Lemma 3.6, this implies that up to some multiplicative constant, uniformly in s_n, for some $c' > 0$, for all k large enough, we have

$$\sum_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\in[k,k+1]}} e^{-s_n d(o,\gamma o)+\int_o^{\gamma o}\widetilde{F}} \leq c' e^{-\beta k}.$$

Let s_n converge to $\delta_\Gamma(F)$. As $d(o,\gamma o) \in [k, k+1]$, the previous inequality gives, for some $c'' > 0$, for large enough k,

$$\sum_{\substack{\gamma\in\Gamma_{\widetilde{L}_\varepsilon},\\ d(o,\gamma o)\in[k,k+1]}} e^{\int_o^{\gamma o}\widetilde{F}} \leq c'' e^{\delta_\Gamma(F)k-\beta k}.$$

Since $\delta_{\Gamma_{\widetilde{L}_\varepsilon}}(F)$ is the exponential growth rate of the left hand term, we get

$$\delta_{\Gamma_{\widetilde{L}_\varepsilon}}(F) \leq \delta_\Gamma(F) - \beta < \delta_\Gamma(F).$$

In particular, $\delta_\Gamma^\infty(F) < \delta_\Gamma(F)$, proving the strong positive recurrence of F. ∎

7.5. SPR is independent of the compact set

This paragraph is devoted to the proof of Theorem 7.9. Let $F\colon T^1M \to \mathbb{R}$ be a strongly positively recurrent Hölder-continuous potential. Let $K \subset M$ be a compact subset whose interior $\mathring{K}$ intersects $\pi(\Omega)$, and $\widetilde{K} \subset \widetilde{M}$ a compact subset such that $p_\Gamma(\widetilde{K}) = K$. Our proof relies on the following proposition, which provides a convenient upper bound for the growth of $\Gamma_{\widetilde{K}}$.

Proposition 7.16. *Let $A\colon T^1M \to [0,+\infty)$ be a nonnegative Hölder-continuous map whose support is contained in the interior of T^1K. Then*

$$\delta_{\Gamma_{\widetilde{K}}}(F) \le \delta_\Gamma(F - A).$$

Proof. Thanks to Proposition 4.9, we have $\delta_{\Gamma_{\widetilde{K}}}(F) = \delta_{\Gamma_{\widetilde{K}}}(F - A)$. Together with the trivial inequality $\delta_{\Gamma_{\widetilde{K}}}(F - A) \le \delta_\Gamma(F - A)$, this gives the conclusion. ∎

We will also need the following proposition.

Proposition 7.17. *Let $F_1, F_2\colon T^1M \to \mathbb{R}$ be two Hölder-continuous potentials with finite topological pressure that satisfy $F_2 \le F_1$ and $F_2(w) < F_1(w)$ for some $w \in \Omega$. Assume that F_2 admits a finite Gibbs measure m_{F_2}. Then their topological pressures satisfy*

$$P_{\mathrm{top}}(F_2) < P_{\mathrm{top}}(F_1).$$

Proof. For $i = 1, 2$, $P_{\mathrm{top}}(F_i)$ coincides with $P_{\mathrm{var}}(F_i)$, i.e.,

$$\sup\left\{\int F_i \,\mathrm{d}m + h_{KS}(m)\,;\, m \text{ invariant probability measure with } \int F_i^- \,\mathrm{d}\mu_i < +\infty\right\}.$$

As $F_2 \le F_1$, we have

$$\int F_1^- \,\mathrm{d}m \le \int F_2^- \,\mathrm{d}m$$

for any invariant probability measure m. Therefore, when $m = m_{F_2}$,

$$P_{\mathrm{var}}(F_2) = \int F_2 \,\mathrm{d}m_{F_2} + h_{KS}(m_{F_2}) \le \int F_1 \,\mathrm{d}m_{F_2} + h_{KS}(m_{F_2}) \le P_{\mathrm{var}}(F_1).$$

Assume by contradiction that $P_{\mathrm{var}}(F_1) = P_{\mathrm{var}}(F_2)$. Then by the previous inequalities,

$$\int F_1 \,\mathrm{d}m_{F_2} = \int F_2 \,\mathrm{d}m_{F_2}.$$

It implies that $F_1 = F_2$ m_{F_2}-almost surely. As $F_2 \le F_1$ and $F_2 < F_1$ on a neighborhood of w, this contradicts the fact that m_{F_2} has full support in Ω. Therefore, $P_{\mathrm{var}}(F_2) < P_{\mathrm{var}}(F_1)$. ∎

Let us conclude the proof of Theorem 7.9.

Proof of Theorem 7.9. Choose some $w \in \Omega \cap T^1K$ and $\varepsilon > 0$ such that $B(w, 2\varepsilon) \subset T^1K$. Let $A: T^1M \to [0, +\infty)$ be a nonnegative Hölder-continuous map supported in $B(w, \varepsilon)$ with $A(w) > 0$. By Proposition 4.9, for all $\eta > 0$, we have

$$\delta_\Gamma^\infty(F - \eta A) = \delta_\Gamma^\infty(F).$$

Moreover, the map $\eta \mapsto \delta_\Gamma(F - \eta A)$ is Lipschitz-continuous by Proposition 4.11. As F is strongly positively recurrent, for $\eta > 0$ small enough, the map $F - \eta A$ is still strongly positively recurrent. In particular, by Theorem 1.4, it admits a finite Gibbs measure. Therefore, Propositions 7.16 and 7.17 give the inequalities

$$\delta_{\Gamma_{\widetilde{K}}}(F) \leq \delta_\Gamma(F - \eta A) < \delta_\Gamma(F).$$

Theorem 7.9 follows. ∎

A. Entropies for geodesic flows

In this appendix, we prove that three important notions of entropies of an invariant probability measure for the dynamics of the geodesic flow on negatively curved manifolds coincide, namely the Kolmogorov–Sinai, the Katok and the Brin–Katok entropies. These equalities were first proved for dynamical systems defined on compact metric spaces in [28] and [10], and generalized for Lipschitz maps on noncompact manifolds in [36] taking only in consideration ergodic measures. This appendix treats the case of nonergodic measures as well as the equality with Katok and local (Brin–Katok) entropies relative to small dynamical balls. The extension of this appendix to the orbifold setting is open, as discussed in our last paragraph.

A.1. Different notions of entropy

Let $(\widetilde{M}, g)$ be a smooth complete connected Riemannian manifold with pinched negative sectional curvature $-b^2 \leq K_g \leq -a^2$, for some $0 < a \leq b$. Let $M = \widetilde{M}/\Gamma$ be a quotient manifold, with $\Gamma = \pi_1(M)$ a discrete group, and $p_\Gamma: T^1\widetilde{M} \to T^1M$ the differential of the quotient map $\widetilde{M} \to M$. We will denote by (g^t) both geodesic flows on $T^1\widetilde{M}$ and $T^1M = T^1\widetilde{M}/\Gamma$.

For all definitions of entropy, the entropy of the geodesic flow (g^t) with respect to an invariant probability measure μ on T^1M is defined as the entropy of its time-one-map g^1 with respect to μ. If μ is ergodic with respect to the flow, it is not necessarily ergodic with respect to this time-one-map g^1. However, in this case, almost every time $\tau \in \mathbb{R}$ is ergodic, so that the relation $h(g^\tau) = |\tau| h(g^1)$ allows us to assume, without loss of generality, that μ is ergodic with respect to g^1.

A.1.1. The Kolmogorov–Sinai entropy. Let $\mathcal{M}_1$ be the set of g^1 invariant probability measures on T^1M and let $\mu \in \mathcal{M}_1$. In this appendix, the word *partition* always denotes a finite or countable measurable partition of T^1M. Let $\mathcal{P}$ be such a partition. The entropy of $\mathcal{P}$ is defined by

$$H(\mu, \mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

The join $\mathcal{P}^n = \bigvee_{i=0}^n g^{-i}\mathcal{P}$ is the partition whose atoms are the nonempty subsets of the form

$$P_0 \cap g^{-1}P_1 \cap \cdots \cap g^{-n}P_n,$$

where the sets P_i are in $\mathcal{P}$. The entropy of μ with respect to $\mathcal{P}$ is the limit

$$h(\mu, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{P}^n).$$

The Kolmogorov–Sinai entropy of μ is the least upper bound

$$h_{KS}(\mu) := \sup_{\mathcal{P}} h(\mu, \mathcal{P})$$

over all partitions $\mathcal{P}$ with finite entropy.

For any $v \in T^1M$, denote by $\mathcal{P}(v)$ the atom of $\mathcal{P}$ containing v. The Shannon–McMillan–Breiman theorem (see, for instance, [1]) asserts that whenever μ is ergodic, then for μ-a.e. $v \in T^1M$, we have

$$h(\mu, \mathcal{P}) = \lim_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(v)).$$

Moreover, when μ is not ergodic, we have

$$\int_{T^1M} \lim_{n\to\infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(v))\, \mathrm{d}\mu(v) = h(\mu, \mathcal{P}).$$

A.1.2. The Katok entropies. For completeness, let us recall the following definitions. Let d be any metric on $T^1\widetilde{M}$, bi-Lipschitz equivalent to the Sasaki metric. By an abuse of notation, we will denote by d the corresponding induced metric on T^1M and by $B_d(v, r)$ the corresponding metric ball centered at v with radius $r > 0$.

Let $\widetilde{v} \in T^1\widetilde{M}$ and $\varepsilon, T > 0$. The *dynamical ball* $B(\widetilde{v}, \varepsilon; T)$ on the universal cover is defined by

$$B(\widetilde{v}, \varepsilon; T) = \{\widetilde{w} \in T^1\widetilde{M}\ ;\ \forall t \in [0, T],\ d(g^t\widetilde{v}, g^t\widetilde{w}) \le \varepsilon\}.$$

As in [44, Remark 3.1], we consider on T^1M the small dynamical ball $B_\Gamma(v, \varepsilon; T) = p_\Gamma(B(\widetilde{v}, \varepsilon; T))$ and the big dynamical ball

$$B_{\mathrm{dyn}}(v, \varepsilon; T) = \{w \in T^1M\ ;\ \forall t \in [0, T],\ d(g^tv, g^tw) \le \varepsilon\} \supset B_\Gamma(v, \varepsilon; T). \tag{25}$$

Both balls coincide as soon as the injectivity radius of M is bounded from below away from zero uniformly on T^1M and ε is small enough. More generally, if along the orbit $(g^t v)_{0\le t\le T}$, the injectivity radius at the point $\pi(g^t v)$ is larger than ε, then

$$B_{\mathrm{dyn}}(v,\varepsilon;T) = B_\Gamma(v,\varepsilon;T). \tag{26}$$

Given a probability measure μ on T^1M, $\delta\in(0,1)$ and $\varepsilon, T>0$, a set $V\subset T^1M$ is $(\mu,\delta,\varepsilon;T)$-*spanning* (respectively, *dynamically*-$(\mu,\delta,\varepsilon;T)$-*spanning*) if

$$\mu\Big(\bigcup_{v\in V} B_\Gamma(v,\varepsilon;T)\Big)\ge\delta, \quad \text{respectively, } \mu\Big(\bigcup_{v\in V} B_{\mathrm{dyn}}(v,\varepsilon;T)\Big)\ge\delta.$$

Of course, a $(\mu,\delta,\varepsilon;T)$-spanning set is also a dynamically-$(\mu,\delta,\varepsilon;T)$-spanning.

Let $S_\Gamma(\mu,\delta,\varepsilon;T)$ (respectively, $S_{\mathrm{dyn}}(\mu,\delta,\varepsilon;T)$) be the minimal cardinality of a $(\mu,\delta,\varepsilon;T)$-spanning set (respectively, of a dynamically-$(\mu,\delta,\varepsilon;T)$-spanning set).

The Katok entropy of μ with respect to the small (respectively, big) dynamical balls are defined respectively as

$$h^\Gamma_{\mathrm{Katok}}(\mu) = \inf_{\delta>0}\sup_{\varepsilon>0}\limsup_{T\to\infty}\frac{1}{T}\log S_\Gamma(\mu,\delta,\varepsilon;T),$$

and

$$h^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu) = \inf_{\delta>0}\sup_{\varepsilon>0}\limsup_{T\to\infty}\frac{1}{T}\log S_{\mathrm{dyn}}(\mu,\delta,\varepsilon;T).$$

A.1.3. The Brin–Katok entropies. Given a nonempty compact subset $\mathcal{K}\subset T^1M$, we define the local entropies on $\mathcal{K}$ relative respectively to small and big dynamical balls as

$$\overline{h}^\Gamma_{\mathrm{loc}}(\mu,\mathcal{K}) = \sup_{\varepsilon>0}\ \operatorname*{ess\,sup}_{v\in\mathcal{K}}\ \limsup_{T\to\infty,\ g^Tv\in\mathcal{K}}\ -\frac{1}{T}\log\mu(B_\Gamma(v,\varepsilon;T)),$$

and

$$\overline{h}^{\mathrm{dyn}}_{\mathrm{loc}}(\mu,\mathcal{K}) = \sup_{\varepsilon>0}\ \operatorname*{ess\,sup}_{v\in\mathcal{K}}\ \limsup_{T\to\infty,\ g^Tv\in\mathcal{K}}\ -\frac{1}{T}\log\mu(B_{\mathrm{dyn}}(v,\varepsilon;T)).$$

Taking the least upper bound over nonempty compact subsets $\mathcal{K}$ leads to the definition of the upper Brin–Katok local entropies

$$\overline{h}^\Gamma_{BK}(\mu) = \sup_{\mathcal{K}}\ \overline{h}^\Gamma_{\mathrm{loc}}(\mu,\mathcal{K}) \quad \text{and} \quad \overline{h}^{\mathrm{dyn}}_{BK}(\mu) = \sup_{\mathcal{K}}\ \overline{h}^{\mathrm{dyn}}_{\mathrm{loc}}(\mu,\mathcal{K}).$$

A.2. All entropies coincide

The main result of this appendix is stated below. Despite of being expected, its relevance lies in its many potential applications. For example, in [44, Theorem 1.4], a formula relating local entropies of invariant measures through a change of the Riemannian metric has been established, which brings as consequence such a formula for Kolmogorov–Sinai entropies. In particular, it also gives a relationship between topological entropies of geodesic flows coming from perturbations of a given Riemannian metric by the use of measures of maximal entropies on the corresponding dynamics.

Theorem A.1. *Let (M, g) be a complete connected Riemannian manifold with pinched negative curvatures $-b^2 \leq K_g \leq -a^2 < 0$. Let $\mu \in \mathcal{M}_1$ be an ergodic invariant probability measure for the geodesic flow on T^1M. Then*

$$h_{KS}(\mu) = \overline{h}^{\Gamma}_{BK}(\mu) = \overline{h}^{\mathrm{dyn}}_{BK}(\mu) = h^{\Gamma}_{\mathrm{Katok}}(\mu) = h^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu).$$

Proof of Theorem A.1. We will prove Theorem A.1 in two steps. The first step is to prove that the Kolmogorov–Sinai entropy coincides with the local Brin–Katok entropies, and the second one is the analogue with the Katok entropies.

Step 1. The inequality $h_{KS}(\mu) \leq \overline{h}^{\mathrm{dyn}}_{BK}(\mu)$ is due to Brin–Katok [10]. In this reference, equality is proved on a compact manifold, but the proof of this inequality does not use compactness. The inequality $\overline{h}^{\mathrm{dyn}}_{BK}(\mu) \leq \overline{h}^{\Gamma}_{BK}(\mu)$ is immediate from (25). Therefore, we just need to prove that $\overline{h}^{\Gamma}_{BK}(\mu) \leq h_{KS}(\mu)$.

The proof relies on a crucial geometric property: as the curvature is bounded from below, the injectivity radius along a geodesic decays at most exponentially. More precisely, for every compact subset $C \subset M$, there exists a positive constant $c > 0$ such that for all vectors $w \in T^1C$, and all $t \in \mathbb{R}$, we have

$$r_{\mathrm{inj}}(g^t w) \geq c^{-1} e^{-c|t|}. \tag{27}$$

This geometric inequality follows from [14, Theorem 4.7], see also [15, Proposition 4.19].

For the next proposition we do not need the ergodicity of μ. In particular, the corollary stated after its proof is satisfied for any invariant probability measure.

Proposition A.2. *For every compact subset $\mathcal{K} \subset T^1M$ with $\mu(\mathcal{K}) > 0$, and for every $\varepsilon > 0$, there exists a partition $\mathcal{P}_{\mathcal{K}}$ of $\mathcal{K}$ with finite entropy such that, if $\mathcal{P} = \mathcal{P}_{\mathcal{K}} \sqcup (T^1M \setminus \mathcal{K})$, for μ-a.e. $v \in \mathcal{K}$, the sequence $n_k \to \infty$ of return times in $\mathcal{K}$ of $(g^n v)_{n\in\mathbb{N}}$ satisfies*

$$\mathcal{P}^{n_k}(v) \subset B_{\Gamma}(v, \varepsilon; n_k).$$

In particular, for every compact subset $\mathcal{K} \Subset T^1M$*, for* μ*-a.e.* $v \in \mathcal{K}$*,*

$$\limsup_{n\to\infty, g^n v \in \mathcal{K}} -\frac{1}{n}\log\mu(B_\Gamma(v,\varepsilon;n)) \le \limsup_{n\to\infty, g^n v\in\mathcal{K}} -\frac{1}{n}\log\mu(\mathcal{P}^n(v)). \tag{28}$$

Proof. By [35, Proposition 1.34], for every compact subset $\mathcal{K} \subset T^1M$, there exists $c_0 > 0$ such that for all $\delta > 0$, there exists a partition $\mathcal{P}_\delta$ of $\mathcal{K}$ whose atoms $\mathcal{P}_\delta(v)$ for any $v \in T^1M$ all have diameter at most δ, with $\mu(\partial\mathcal{P}_\delta(v)) = 0$, and $\#\mathcal{P}_\delta \le c_0\delta^{-d}$ where d is the dimension of T^1M. As $\mu(\mathcal{K}) > 0$, by the Poincaré recurrence theorem, we know that for μ-a.e. $v \in \mathcal{K}$, infinitely often $g^n v \in \mathcal{K}$. Divide the set $\mathcal{K}$ (up to a measure 0 set) into the return time partition: for all $k \ge 1$, let

$$A_k = \{v \in \mathcal{K},\ g^k v \in \mathcal{K},\ \text{and } g^i v \notin \mathcal{K} \text{ for all } 1 \le i \le k-1\}.$$

For all $k \ge 1$, set $\delta_k = \varepsilon/(Le^c)^k$, where L is the Lipschitz-constant for the time-one-map g^1 of the geodesic flow, and $c > 0$ is the constant associated with the compact subset $C = \pi(\mathcal{K}) \subset M$ from equation (27). For $v \in A_k$, define $\mathcal{P}(v)$ as $\mathcal{P}(v) := \mathcal{P}_{\delta_k}(v) \cap A_k$. For $v \notin \mathcal{K}$, set $\mathcal{P}(v) = T^1M \setminus \mathcal{K}$.

Thanks to the choice of δ_k, an immediate verification shows that for $v \in A_k$, we have $\mathcal{P}(v) \subset B_{\mathrm{dyn}}(v, \varepsilon/e^{ck}; k)$. By equations (26) and (27), in fact, we have in this case

$$\mathcal{P}(v) \subset B_\Gamma\Big(v, \frac{\varepsilon}{e^{ck}}; k\Big) = B_{\mathrm{dyn}}\Big(v, \frac{\varepsilon}{e^{ck}}; k\Big).$$

Recall the notation

$$\mathcal{P}^n(v) = \mathcal{P}(v) \cap g^{-1}\mathcal{P}(gv) \cap \cdots \cap g^{-(n-1)}\mathcal{P}(g^{n-1}v).$$

Now, for almost all $v \in \mathcal{K}$, let $n_k \to \infty$ be the sequence of return times of $(g^n v)_{n\ge 0}$ inside $\mathcal{K}$ (with $n_0 = 0$). Let $\tilde{v}$ be any lift of v into $T^1\widetilde{M}$. By construction of $\mathcal{P}$, and by the above, for almost every $v \in T^1M$, we have

$$\begin{aligned}
\mathcal{P}^{n_k}(v) &\subseteq \mathcal{P}(v) \cap g^{-n_1}\mathcal{P}(g^{n_1}v) \cap \cdots \cap g^{-n_{k-1}}\mathcal{P}(g^{n_{k-1}}v)\\
&\subseteq \bigcap_{i=0}^{k-1} g^{-n_i} B_{\mathrm{dyn}}\Big(g^{n_i}v, \frac{\varepsilon}{e^{c(n_{i+1}-n_i)}}; n_{i+1}-n_i\Big)\\
&= \bigcap_{i=0}^{k-1} g^{-n_i} B_\Gamma\Big(g^{n_i}v, \frac{\varepsilon}{e^{c(n_{i+1}-n_i)}}; n_{i+1}-n_i\Big)\\
&\subseteq \bigcap_{i=0}^{k-1} g^{-n_i} p_\Gamma(B_{\mathrm{dyn}}(g^{n_i}\tilde{v}, \varepsilon; n_{i+1}-n_i))\\
&= \bigcap_{i=0}^{k-1} p_\Gamma(g^{-n_i} B_{\mathrm{dyn}}(g^{n_i}\tilde{v}, \varepsilon; n_{i+1}-n_i)).
\end{aligned}$$

Note that we are strongly using the fact that dynamical balls for the time-one-map coincide with dynamical balls for the flow at integer times. Without loss of generality, we may assume that $\varepsilon \leq c^{-1}$. In particular, the quotient map p_Γ is an isometry restricted to each of the dynamical balls involved in the last intersection, thanks to (27). Hence, we get

$$\begin{aligned}\mathcal{P}^{n_k}(v) &\subseteq p_\Gamma\left(\bigcap_{i=0}^{k-1} g^{-n_i} B_{\mathrm{dyn}}(g^{n_i}\widetilde{v}, \varepsilon; n_{i+1}-n_i)\right)\\ &= p_\Gamma(B_{\mathrm{dyn}}(\widetilde{v}, \varepsilon; n_k))\\ &= B_\Gamma(v, \varepsilon; n_k).\end{aligned}$$

It remains to prove that $\mathcal{P}$ is a partition of finite entropy. By construction recall that

$$\#\{P \in \mathcal{P} : P \subseteq A_k\} \leq c_0 \delta_k^{-d} = c_0 \left(\frac{\varepsilon}{(Le^c)^k}\right)^{-d}.$$

We have

$$\begin{aligned}H_\mu(\mathcal{P}) &= -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P)\\ &= -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\sum_{P\in\mathcal{P}, P\subset A_k}\mu(P)\log\mu(P)\\ &\leq -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\mu(A_k)\log\mu(A_k)\\ &\quad + \sum_{k=1}^{\infty}\mu(A_k)\log\#\{P\in\mathcal{P} : P\subset A_k\}\\ &\leq -\mu(\mathcal{K}^c)\log\mu(\mathcal{K}^c) - \sum_{k=1}^{\infty}\mu(A_k)\log\mu(A_k)\\ &\quad - \left(\sum_{k=1}^{\infty}\mu(A_k)\right)\times\log(c_0\varepsilon^d) + \sum_{k=1}^{\infty}\mu(A_k)\times k\log(Le^c)^d.\end{aligned}$$

The first term is some finite constant. The third term is bounded from above by a constant times $\mu(\mathcal{K})$ and is therefore finite. By the Kac lemma, the last term, up to a constant, is equal to

$$\sum_{k=1}^{\infty} k\mu(A_k) = \mu(T^1M) \leq 1.$$

The second term is finite since [35, Lemma 1.35] together with $\sum_k k\mu(A_k) < \infty$ imply

$$\sum_k \mu(A_k)\log\mu(A_k) < \infty.$$

Therefore, $\mathcal{P}$ has finite entropy. ∎

Integrating (28) over $v \in \mathcal{K}$ on the left, and over $v \in T^1M$ on the right, and using the Shannon–McMillan–Breiman theorem, Proposition A.2 leads to the following corollary.

Corollary A.3. *Under the same assumptions, we have*

$$\begin{aligned}\int_{\mathcal{K}} \limsup_{\substack{n\to\infty,\\ g^n v\in\mathcal{K}}} -\frac{1}{n}\log\mu(B_\Gamma(v,n,\varepsilon))\,\mathrm{d}\mu(v) \\ \leq \int_{T^1M} \limsup_{\substack{n\to\infty,\\ g^n v\in\mathcal{K}}} -\frac{1}{n}\log\mathcal{P}^n(v)\,\mathrm{d}\mu(v) \leq h_{KS}(\mu).\end{aligned}$$

Assume now that $\mathcal{K}$ is large enough so that there exists $v \in \mathcal{K}$ with $\mu(B_d(v,1)) > 0$ and $B_d(v,2) \subset \mathcal{K}$. Let us define

$$\mathcal{K}_{-1} = \{v \in \mathcal{K}\ ;\ d(v,\mathcal{K}^c) \geq 1\} \subset \mathcal{K}.$$

By our assumption, $\mu(\mathcal{K}_{-1}) > 0$. Note that for all $v \in \mathcal{K}_{-1}$, we have

$$\limsup_{\substack{T\to\infty,\\ g^T v\in\mathcal{K}_{-1},\,T\in\mathbb{R}}} -\frac{1}{T}\log\mu(B_\Gamma(v,T,\varepsilon)) \leq \limsup_{\substack{n\to\infty,\\ g^n v\in\mathcal{K},\,n\in\mathbb{N}}} -\frac{1}{n}\log\mu(B_\Gamma(v,n,\varepsilon)).$$

If we consider the essential least upper bound over $v \in \mathcal{K}$ on the left and on the right in (28), using the ergodicity of μ and the Shannon–McMillan–Breiman theorem, we get

$$\overline{h}^{\Gamma}_{\mathrm{loc}}(\mu,\mathcal{K}_{-1}) \leq h(\mu,\mathcal{P}).$$

This already implies $\overline{h}^{\Gamma}_{BK}(\mu) \leq h_{KS}(\mu)$ since the right hand side of the inequality is less than $h_{KS}(\mu)$ and $\mathcal{K} \subset T^1M$ is arbitrary.

Step 2. The goal now is to prove the equality between the Katok entropies and the Kolmogorov–Sinai entropy. The inequality $h_{KS}(\mu) \leq h^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu)$ follows immediately from Katok [28, formula (1.4)], where the author considers coverings instead of spanning sets. In this reference, equality is proved on a compact manifold, but the proof of this inequality does not use compactness. The inequality $h^{\mathrm{dyn}}_{\mathrm{Katok}}(\mu) \leq h^{\Gamma}_{\mathrm{Katok}}(\mu)$ is immediate from (25). Hence by Step 1, we just need to prove that $h^{\Gamma}_{\mathrm{Katok}}(\mu) \leq \overline{h}^{\Gamma}_{BK}(\mu)$.

Let $h := \overline{h}^{\Gamma}_{BK}(\mu)$. By definition of local entropy, for any $\rho > 0$, there exists a compact subset $\mathcal{K} \subset T^1M$ and $\varepsilon > 0$ such that $\mu(\mathcal{K}) > 4/5$ and for μ-a.e. $v \in \mathcal{K}$, we have

$$\limsup_{\substack{T\to\infty,\\ g^T v\in\mathcal{K}}} -\frac{1}{T}\log\mu(B_\Gamma(v,\varepsilon/2;T)) \leq h+\rho.$$

For every $\tau > 0$, set

$$\mathcal{K}_\tau := \{v \in \mathcal{K} : \mu(B_\Gamma(v,\varepsilon/2;T)) \geq \exp(-T(h+2\rho)),\ \forall T \geq \tau,\ g^T v \in \mathcal{K}\}.$$

Then there exists $\tau_0 > 0$ such that $\mu(\mathcal{K}_{\tau_0}) > 3/4$. Note that $\mu(Y_T) > 1/2$ for every $T \geq \tau_0$, where $Y_T = \mathcal{K}_{\tau_0} \cap g^{-T}\mathcal{K}_{\tau_0}$. Let $0 < \delta < 1/2$. Then

$$h^{\Gamma}_{\text{Katok}}(\mu) \leq \limsup_{T\to\infty}\frac{1}{T}\log S_\Gamma(\mu,\delta,\varepsilon;T) \leq \limsup_{T\to\infty}\frac{1}{T}\log S_\Gamma(Y_T,\varepsilon;T),$$

where $S_\Gamma(Y_T,\varepsilon,T)$ is the minimal cardinality of a (ε,T)-spanning set of Y_T.

Choose a maximal $(\varepsilon/2,T)$-separated set $\mathcal{E}$ in Y_T, and denote by $\Sigma_\Gamma(Y_T,\varepsilon/2,T)$ its cardinality. By maximality, $\mathcal{E}$ is also (ε,T)-spanning, so that

$$S_\Gamma(Y_T,\varepsilon,T) \leq \Sigma_\Gamma(Y_T,\varepsilon/2,T).$$

By construction, we have

$$e^{-T(h+\rho)}\Sigma_\Gamma(Y_T,\varepsilon/2,T) \leq \sum_{y\in\mathcal{E}}\mu(B_\Gamma(y,\varepsilon/2;T)) \leq 1.$$

With the above inequalities, we deduce that

$$h^{\Gamma}_{\text{Katok}}(\mu) \leq h+\rho.$$

As ρ is arbitrary, the result follows. ∎

A.3. Comparison between entropies for orbifolds

A Riemannian orbifold is said to be *good* when it is the quotient of a simply connected manifold $\widetilde{M}$ by a discrete group of isometries Γ: it is the setting to which all the results in the article apply except this appendix which assumes, moreover, that the action of Γ is free, i.e., $\widetilde{M}/\Gamma$ is a manifold. A good orbifold $M = \widetilde{M}/\Gamma$ is said to be *very good* when it has a subgroup $\Gamma' < \Gamma$ of *finite index* acting on $\widetilde{M}$ without fixed point, i.e., if M has a finite covering which is a manifold.

Theorem A.1 extends immediately to very good orbifolds since the entropies which we consider are invariant by finite coverings. Unfortunately, it does not extend yet to general good orbifolds for the following reason.

We have crucially used in the proof of Step 1 (which is used for Step 2) the fact that the injectivity radius cannot decrease more than exponentially fast along geodesics. The notion of injectivity radius on orbifold is delicate: note that the length of the shortest geodesic loop based at x goes to 0 as x approaches a singularity. There are however notions of injectivity radius adapted to orbifolds which are automatically positive on compact sets, such as the *cone injectivity radius* considered in [3, Chapter 9]. Nevertheless, it is unknown whether such injectivity radius can decrease faster than exponentially along the geodesics of an orbifolds with bounded sectional curvature. The proof of (27) given for manifolds in [14] is based on the study of the Riemannian heat kernel. Therefore, its adaptation to orbifolds is delicate.

Acknowledgments. The authors thank warmly both Jérôme Buzzi for numerous enlightening discussions about strong positive recurrence, and the referee for his very detailed reading and his comments that improved a lot our text.

Funding. We acknowledge the support of ANR grant CCEM (ANR-17-CE40-0034) and of the Centre Henri Lebesgue ANR-11-LABX-0020-01.

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Received 12 October 2020.

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