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Erratum to "Ergodic components of partially hyperbolic systems"

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Abstract. This erratum addresses two issues with the proofs in the paper [Comment. Math. Helv. 92, 131–184 (2017)].

1. Introduction

This erratum addresses two issues with the proofs in the paper [2]. The first issue is that [2, Prop. 6.4] as stated is not correct.[1] For instance, the automorphism

$$\mathbb{Z}^2 \to \mathbb{Z}^2, \quad (x, y) \mapsto (5x + 2y, 2x + y)$$

gives a counterexample as it fixes a coset of $\mathbb{Z} \times 2\mathbb{Z}$. The flaw in the proof is that it confuses invertibility in $GL(n, \mathbb{Z})$ with invertibility in $GL(n, \mathbb{R})$ and the notions are not equivalent. In fact, the proposition holds in the following revised version.

Proposition 1. *Let G be a torsion free, finitely-generated, nilpotent group and suppose $\phi \in \operatorname{Aut}(G)$ is such that $\phi(g) \neq g$ for all $g \in G$ other than the identity element $e \in g$. If H is a normal, ϕ-invariant subgroup, then ϕ fixes at most* finitely many *cosets of H.*

We prove this revised version below. The original proposition ([2, Prop. 6.4]) is used in only two places in the proofs of [2, Thm. 4.3 and Lem. 6.5]. For [2, Lem. 6.5], it is straightforward to adapt the proof to use this new proposition in place of [2, Prop. 6.4]. For [2, Thm. 4.3], more work is required and we show how to recover its proof in a section below.

The other issue to address in the original paper comes at the start of Section 8, which deals with AB-systems. That section states that hfh^{-1} is homotopic to f_{AB} and uses this to lift hfh^{-1} to a map on $N \times \mathbb{R}$. In fact, the two functions are not

[1] Note that the numbering of sections in some preprint versions may differ from the published version.

homotopic in general. For instance, for the linear partially hyperbolic maps on the 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ given by the matrices

$$\begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

both have vertical center foliations and the identity map is a leaf conjugacy between the two systems. The two systems are not homotopic to each other and attempting to lift the two systems to AI-system on $\mathbb{T}^2 \times \mathbb{R}$ as in [2, Sec. 8] will not work. To fix this, we amend the definition of an AB-system to add the homotopy as an assumption. That is, a partially hyperbolic diffeomorphism f is an *AB-system* if

(1) it preserves the orientation of the center bundle E^c,

(2) there is a leaf conjugacy h between f and an AB-prototype f_{AB}, and

(3) hfh^{-1} is homotopic to f_{AB}.

This additional assumption can always be achieved by lifting f and f_{AB} to finite covers:

Proposition 2. *If a partially hyperbolic diffeomorphism f satisfies conditions (1) and (2) above, then a lift of f to a finite cover satisfies all of (1), (2), and (3).*

The proof of this is given in the final section of this erratum.

For those readers interested only in the case where the nilmanifold N is a torus $\mathbb{T}^d$, we have structured the proofs below so that most of the details specific to the non-toral case may be skipped over.

Karel Dekimpe suggested an alternative method to establish Proposition 1. Instead of proving the proposition directly, one can instead show the following fact, from which the proposition follows as a corollary:

> *Let G be a finitely generated torsion free nilpotent group and $\varphi \in \mathrm{Aut}(G)$ be fixed point free. Assume that H is a φ invariant subgroup of G such that G/H is torsion free. Then it follows that the induced automorphism on G/H is also fixed point free.*

Here, a "fixed point free" automorphism is one where the identity element is the only fixed point.

2. Proof of Proposition 1

This section gives a proof of Proposition 1. We first prove this in the abelian case and then use induction on the length of the upper central series to handle the non-abelian case.

Lemma 3. *Let G be isomorphic to $\mathbb{Z}^d$ and suppose $\phi \in \mathrm{Aut}(G)$ is fixed point free. If H is a normal, ϕ-invariant subgroup, then ϕ fixes at most finitely many cosets of H.*

Proof. Assume $G = \mathbb{Z}^d$ and define a linear map $A\colon \mathbb{Q}^d \to \mathbb{Q}^d$ such that $Az = \phi(z)$ for all $z \in \mathbb{Z}^d$. One can see that $Ax \neq x$ for all $x \in \mathbb{Q}^d$ and so $A - I$ is invertible as a linear automorphism of $\mathbb{Q}^d$, where I denotes the identity map. Let $V \subset \mathbb{Q}^d$ be the span of H; that is, the set of all $\mathbb{Q}$-linear combinations of elements of H. From $(A - I)H \subset H$, it follows that $(A - I)V \subset V$. Since $A - I$ is a finite-dimensional linear automorphism, it must be that $(A - I)V = V$. Note that this is different than saying that $(A - I)H$ and H are equal, which is where the mistake in the original paper occurred.

Now consider the set $(A - I)^{-1}H$. This is an additive subgroup of $\mathbb{Q}^d$ and in general it can have elements in $\mathbb{Q}^d \setminus \mathbb{Z}^d$. Since H is a subgroup of $(A - I)^{-1}H$ and

$$\operatorname{span} H = V = (A - I)^{-1}V = \operatorname{span}(A - I)^{-1}H,$$

it follows that H has finite index as a subgroup of $(A - I)^{-1}H$. Returning to the original problem, consider a fixed coset $\phi(x + H) = x + H$ for some $x \in \mathbb{Z}^d$. Then $(A - I)x \in H$, and so $x \in (A - I)^{-1}H$. This shows that all such fixed cosets lie in $(A - I)^{-1}H$ and so there are only finitely many of them. ■

Lemma 4. *Let $\phi\colon G \to G$ be a group automorphism and let X be a normal ϕ-invariant subgroup. If $\phi|_X$ has at most finitely many fixed points and ϕ fixes at most finitely many cosets of X, then ϕ itself has finitely many fixed points.*

Proof. If $\phi(g) = g$ and $\phi(g') = g'$ are fixed points in the same coset $gX = g'X$, then $\phi(g'g^{-1}) = g'g^{-1}$ is a fixed point in X. Hence, each of the finitely many fixed cosets has finitely many fixed points. ■

Corollary 5. *Suppose ϕ is an automorphism of a group G with center Z, and H is a ϕ-invariant normal subgroup of G. If the induced maps on $Z/(H \cap Z)$ and G/HZ have finitely many fixed points, then the induced map on G/H has finitely many fixed points.*

Proof. Apply the previous lemma to the quotient

$$0 \to Z/(H \cap Z) \to G/H \to G/HZ \to 0.$$

■

Lemma 6. *Suppose G is a finitely generated torsion free nilpotent group and let $\phi: G \to G$ be an automorphism. Let Z denote the center of G. If ϕ is fixed point free, then the only fixed coset $\phi(gZ) = gZ$ is Z itself.*

Proof. As a torsion free abelian group, the center Z is isomorphic to $\mathbb{Z}^d$. By [4, Cor. 2, Sec. 3], G/Z is torsion free. Suppose $\phi(gZ) = gZ \neq Z$ is a fixed coset. Let Y be the subgroup generated by g and Z. Then Y is isomorphic to $\mathbb{Z}^{d+1}$ and within Y, there are infinitely many fixed cosets: $\phi(g^k Z) = g^k Z$ for $k \in \mathbb{Z}$. This contradicts Lemma 3. ∎

Proof of Proposition 1. We prove this by induction on the length of the upper central series of G. The abelian base case is given by Lemma 3. Assume now that G is non-abelian with center Z and that Proposition 1 is already known to hold for the quotient map $\Phi: G/Z \to G/Z$.

Since $\phi|_Z$ has no fixed points other than the identity element, Lemma 3 implies that $\phi|_Z$ fixes at most finitely many cosets of $H \cap Z$. Lemma 6 implies that the only fixed point of Φ is Z. By the inductive hypothesis, Φ fixes at most finitely many cosets of HZ/Z. Then Lemma 5 implies that ϕ (on all of G) fixes at most finitely many cosets of H. ∎

3. Circle bundles over nilmanifolds

Before revising the proof of [2, Thm. 4.3], we first prove the following.

Proposition 7. *Suppose M is a circle bundle with oriented fibers over a nilmanifold N. If M has a compact horizontal submanifold Σ, then M is a trivial bundle.*

Remark. In the case that N is a surface, this proposition follows from the Milnor–Wood inequality since we can use Σ to build a horizontal foliation. For circle bundles in higher dimensions, the bundle is trivial if and only if the Euler class, represented by an element of the cohomology group $H^2(N)$, is equal to zero [5, Sec. 6.2]. This means that it might be possible to prove the proposition using purely algebraic tools. However, we did not see a way to do this and so instead we show directly that the bundle is trivial by constructing a section.

Remark. We consider everything in the C^0 setting here. The circle bundle is defined by a C^0 map $p: M \to N$ and a compact horizontal submanifold Σ is a codimension one C^0 submanifold such that $p|_\Sigma: \Sigma \to N$ is a covering map of finite degree. To show that M is trivial, it enough to find a section; that is, another horizontal submanifold Σ_1 such that $p|_{\Sigma_1}: \Sigma_1 \to N$ is a homeomorphism. To simplify the proof, we assume that the circle fibers are tangent to a C^0 vector field as is the case for the center leaves of a partially hyperbolic skew product.

Proof. Assume Σ intersects each fiber in exactly k points. We may define a metric on each fiber such that the length of every fiber is exactly one and that its points of intersection with Σ are equally spaced; that is, the distance between one point of intersection and the next is exactly $\frac{1}{k}$. We may choose these metrics to vary continuously along M.

Let $\pi\colon \widetilde{M} \to M$ be the universal covering map. We may assume that $\widetilde{M} = \widetilde{N} \times \mathbb{R}$, where $\widetilde{N}$ is the nilpotent Lie group covering N and such that the fibers of M lift to lines of the form $v \times \mathbb{R}$ with $v \in \widetilde{N}$. We further assume that the metric on a fiber of M lifts to a metric on $v \times \mathbb{R}$ which is equal to the standard Euclidean metric given by $\mathbb{R}$. In particular, $\pi^{-1}(\Sigma)$ intersects each fiber $v \times \mathbb{R}$ in a set of points of the form

$$\left\{(v, \sigma(v) + t) : t \in \tfrac{1}{k}\mathbb{Z}\right\}$$

for some $\sigma(v)$ depending on v. We may assume $\sigma\colon \widetilde{N} \to \mathbb{R}$ is continuous. To see this, choose a connected component $\widetilde{\Sigma}$ of $\pi^{-1}(\Sigma)$ and define $\sigma(v)$ to be the unique intersection of $v \times \mathbb{R}$ with $\widetilde{\Sigma}$.

Write $G = \pi_1(M)$, and $H = \pi_1(N)$. The bundle projection $p\colon M \to N$ induces a surjective homomorphism $p_*\colon G \to N$. We now use $\widetilde{\Sigma}$ to define a homomorphism $\tau\colon G \to \frac{1}{k}\mathbb{Z}$. Without loss of generality, assume $\sigma(e) = 0$ where e is the identity element of $\widetilde{N}$. For a deck transformation $g \in G$, let $\tau(g)$ be such that $(e, \tau(g))$ is the unique intersection of $g(\widetilde{\Sigma})$ with $e \times \mathbb{R}$. Similarly to [2, Lem. 7.6], one may show that $\tau\colon G \to \frac{1}{k}\mathbb{Z}$ is a homomorphism. We claim the following.

Claim. There is a (not necessarily unique) homomorphism $\psi\colon H \to \frac{1}{k}\mathbb{Z}$ such that $\psi p_*(g) - \tau(g) \in \mathbb{Z}$ for all $g \in G$.

We leave the proof of this to the end and first show that this gives the desired result. By the properties of nilmanifolds [4, Thm. 5], ψ determines a Lie group homomorphism $\psi\colon \widetilde{N} \to \mathbb{R}$ where if we regard H as a discrete subgroup of $\widetilde{N}$ then this is an extension of ψ from H to all of $\widetilde{N}$. Define a submanifold $\widetilde{\Sigma}_1$ as the graph of $\sigma - \psi$; that is, $(v, t) \in \widetilde{\Sigma}_1$ if and only if $t = \sigma(v) - \psi(v)$. By the above claim, for all deck transformations $g \in G$, the intersection of $g(\widetilde{\Sigma}_1)$ with $e \times \mathbb{R}$ lies inside $e \times \mathbb{Z}$. Hence, $\widetilde{\Sigma}_1$ quotients down to a compact horizontal submanifold $\Sigma_1 \subset M$ which intersects each fiber exactly once and therefore shows that the circle bundle is trivial.

It remains to prove the claim.

Proof of the claim. We first consider the abelian case where H is isomorphic to $\mathbb{Z}^d$. Let $\{h_1, \ldots, h_d\}$ be a generating set for H and choose elements $g_i \in G$ such that

$$p_*(g_i) = h_i.$$

Let $z \in G$ be the deck transformation $(v, t) \mapsto (v, t + 1)$ corresponding to going once around a fiber of the circle fibering. Note that $\tau(z) = 1$. As explained in the original proof of [2, Thm. 4.3], $\langle z \rangle$ is the kernel of p_*, and so $\{z, g_1, \dots, g_d\}$ is a generating set for G. Define $\psi: H \to \frac{1}{k}\mathbb{Z}$ by $\psi(h_i) = \tau(g_i)$. Then

$$\psi p_*(z) - \tau(z) = -1 \quad \text{and} \quad \psi p_*(g_i) - \tau(g_i) = 0.$$

As $\psi p_* - \tau$ takes integer values on a generating set for G, it must take integer values on all of G. ■

We now extend this argument to the non-abelian case. Note that both M and N are nilmanifolds. Consider the root set G_1 of the commutator subgroup of G. That is $g \in G_1$ if and only if $g^k \in [G, G]$ for some $k \geq 1$. Such sets are discussed in detail in [1, Chap. 1] (where the notation there is $\sqrt[G]{\gamma_2(G)}$ instead of G_1). In particular, G_1 is a normal subgroup and any homomorphism from G to a torsion free abelian group R is identically zero on G_1 and so factors through $G \to G/G_1 \to R$. We can therefore define a homomorphism $\tau_1: G/G_1 \to \frac{1}{k}\mathbb{Z}$ as the quotient of τ.

Similarly write H_1 for the root set of $[H, H]$. Then H/H_1 is a torsion free abelian group homomorphic to $\mathbb{Z}^d$ for some d (see [1]), and $p_*: G \to H$ descends to a map

$$p_1: G/G_1 \to H/H_1.$$

Adapting the argument above, we may define a map $\psi_1: H/H_1 \to \frac{1}{k}\mathbb{Z}$ such that $\psi_1 p_1 - \tau_1$ takes integer values on all of G/G_1. Then ψ_1 determines a map

$$\psi: H \to \tfrac{1}{k}\mathbb{Z},$$

as desired. ■

4. Revised proof of [2, Theorem 4.3]

This section revises the proof of [2, Thm. 4.3] to use Proposition 1 above in place of the incorrect [2, Prop. 6.4]. By virtue of Proposition 7 above, we need only show that the partially hyperbolic system has a compact us-leaf.

The proof of [2, Thm. 4.3] is unchanged up to the definition of $\hat{\tau}: G \to \mathbb{R}/\mathbb{Z}$ and the first use of [2, Prop. 6.4]. Using instead Proposition 1 above, the most we can say is that $\hat{\tau}$ has a finite image. In other words, there is an integer $k \geq 1$ such that $\tau(G) = \frac{1}{k}\mathbb{Z}$.

The existence of τ is given by [2, Prop. 6.1 and 6.2]. From the proofs of those results, we can see that then there is a measure μ on $\widetilde{S}$ invariant under the action of G and such that $\tau(g) = \mu[x, g(x))$ for any $x \in \Lambda$ and $g \in G$. Here, Λ is the intersection of the non-open accessibility classes Γ with $\widetilde{S}$. Choose some point $x_0 \in \Lambda$ and for each $t \in \frac{1}{k}\mathbb{Z}$, define a set $X_t \subset \Lambda$ by

$$X_t = \{x \in \Lambda : \mu[x_0, x) = t\}.$$

The sets X_t are disjoint and the action of $g \in G$ on Λ takes X_t to $X_{t+\tau(g)}$. Define $y_t = \sup X_t$ where we are identifying $\widetilde{S}$ with $\mathbb{R}$ in order to define the supremum. Then $\{y_t : t \in \frac{1}{k}\mathbb{Z}\}$ is a discrete subset of Λ which is invariant under the action of G. This implies that for any point y_t, its accessibility class $AC(y_t) \subset \widetilde{M}$ quotients down to a compact us-leaf on M.

5. Proof of Proposition 2

We now prove Proposition 2. We first prove this in a simplified setting where M and M_B are the same manifold (not just homeomorphic manifold) and the leaf conjugacy is the identity. For instance, such assumptions hold for the example on $\mathbb{T}^3$ given at the start of this erratum. After this, we show how the extend this to prove Proposition 2 in the general case.

In both cases, assume $f: M \to M$ is a partially hyperbolic diffeomorphism which preserves the orientation of E^c and $h: M \to M_B$ is a leaf conjugacy to an AB-prototype $f_{AB}: M_B \to M_B$.

Case 1. The homeomorphism h is the identity map on $M = M_B$.

Write G for the simply connected nilpotent Lie group and Γ for the cocompact lattice such that $N = \Gamma \setminus G$ is the nilmanifold associated to the AB-prototype f_{AB}. Quotienting G by $[G, G]$ yields an abelian Lie group isomorphic to $\mathbb{R}^d$ for some d. This defines a projection from G to $\mathbb{R}^d$, and for $x \in G$, we write $\overline{x} \in \mathbb{R}^d$ for its image under the projection. This projection may further be chosen such that Γ is mapped to $\mathbb{Z}^d$. (If the nilmanifold is a torus $N = \mathbb{Z}^d \setminus \mathbb{R}^d$, then the projection $G \to \mathbb{R}^d$ is the identity map and all of the overlines in what follows may be safely ignored.)

Let $A, B: G \to G$ be the commuting Lie group automorphisms defining the AB-prototype. These induce linear automorphisms $\overline{A}$ and $\overline{B}$ on $\mathbb{R}^d$ with the property that

$$\overline{A(x)} = \overline{A}(\overline{x}) \quad \text{and} \quad \overline{B(x)} = \overline{B}(\overline{x}).$$

The universal cover of M_B is $G \times \mathbb{R}$. Define $\beta(x, t) = (B(x), t - 1)$. For $\gamma \in \Gamma$, define $\tau_\gamma(x, t) = (\gamma \cdot x, t)$. Note that $\beta \tau_\gamma = \tau_{B(\gamma)} \beta$ and that every deck transformation may be written in the form $\tau_\gamma \beta^n$ for $\gamma \in \Gamma$ and $n \in \mathbb{Z}$.

Lift f to a diffeomorphism $\tilde{f}: G \times \mathbb{R} \to G \times \mathbb{R}$ such that $\tilde{f}(0 \times \mathbb{R}) = 0 \times \mathbb{R}$, where 0 is the identity element of G. Such a lift exists because of the leaf conjugacy.

This lift then determines an automorphism f_* of the fundamental group $\pi_1(M_B)$ defined by the property $f_*(\tau) \circ \tilde{f} = \tilde{f} \circ \tau$ for all deck transformations τ. Since $0 \times \mathbb{R}$ projects to an f-invariant circle in M_B, one can show that $f_*(\beta) = \beta$. By the leaf conjugacy,

$$\tilde{f}(x \times \mathbb{R}) = A(x) \times \mathbb{R}$$

for all $x \in G$, and so for each $\gamma \in \Gamma$, there is an integer $L(\gamma)$ such that $f_*(\tau_\gamma) = \tau_{A(\gamma)}\beta^{L(\gamma)}$. Using that f_* is a group homomorphism, one can show that

$$L(\gamma_1 \cdot \gamma_2) = L(\gamma_1) + L(\gamma_2) \quad \text{and} \quad A(\gamma_1 \cdot \gamma_2) = A(\gamma_1)B^{L(\gamma_1)}A(\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. This implies that $L\colon \Gamma \to \mathbb{Z}$ is a group homomorphism and that there is $k \geq 0$ such that $L(\Gamma) = k\mathbb{Z}$ and B^k is the identity map on G. If $k = 0$, then f induces the same action on $\pi_1(M_B)$ as the AB-prototype f_{AB} and this would imply the desired result. Therefore, we assume in what follows that $k \geq 1$.

By the properties of nilmanifolds [4, Thm. 5], L extends to a Lie group homomorphism $L\colon G \to \mathbb{R}$. Since $\mathbb{R}$ is abelian, $L|_{[G,G]} \equiv 0$ and there is a linear map $\bar{L}\colon \mathbb{R}^d \to \mathbb{R}$ such that $\bar{L}(\bar{x}) = L(x)$ for all $x \in G$. Let I denote the identity map on $\mathbb{R}^d$. As $\bar{A}$ is hyperbolic, $\bar{A} - I$ is invertible. Define $\bar{S}\colon \mathbb{R}^d \to \mathbb{R}$ by $\bar{S} = \bar{L}(\bar{A} - I)^{-1}$ and $S\colon G \to \mathbb{R}$ by $S(x) = \bar{S}(\bar{x})$. By Cramer's rule, $\bar{S}(m\mathbb{Z}^d) \subset k\mathbb{Z}$, where $m = \det(\bar{A} - I)$. Using $f_*(\beta\tau_\gamma) = f_*(\tau_{B(\gamma)}\beta)$, one can show $LB(\gamma) = L(\gamma)$ for all $\gamma \in \Gamma$. Hence, $LB = L$ as functions on G and one may use this to show that

$$\bar{L}\bar{B} = \bar{L}, \quad \bar{S}\bar{B} = \bar{S}, \quad SB = S.$$

Define $\Gamma_0 \subset \Gamma$ by $\gamma \in \Gamma_0$ if and only if $\bar{\gamma} \in m\mathbb{Z}^d$. Since $\bar{A}(m\mathbb{Z}^d) = m\mathbb{Z}^d$ and $\bar{B}(m\mathbb{Z}^d) = m\mathbb{Z}^d$, it follows that $A(\Gamma_0) = \Gamma_0$ and $B(\Gamma_0) = \Gamma_0$. Hence, A and B define commuting automorphisms $\hat{A}$ and $\hat{B}$ of a nilmanifold $\hat{N} = \Gamma_0 \setminus G$ that finitely covers N. Using this we define a new AB-prototype $f_{\hat{A}\hat{B}}$ on a new suspension manifold $M_{\hat{B}}$ which finitely covers the original. Further, $\tilde{f}$ quotients to a function

$$\hat{f}\colon M_{\hat{B}} \to M_{\hat{B}},$$

which is a lift of the original f.

Define $\tilde{H}\colon G \times \mathbb{R} \to G \times \mathbb{R}$ by $\tilde{H}(x,t) = (x, t + S(x))$. If $\gamma \in \Gamma_0$, then $S(\gamma) \in k\mathbb{Z}$ and since B^k is the identity, it follows that

$$\beta^{S(\gamma)}(x,t) = (x, t - S(\gamma)),$$

which may be used to show that $\tilde{H}\tau_\gamma = \tau_\gamma\beta^{-S(\gamma)}\tilde{H}$. This implies that $\tilde{H}$ quotients to a diffeomorphism $\hat{H}$ on $M_{\hat{B}}$ and that induced action on $\pi_1(M_{\hat{B}})$ satisfies

$$\hat{H}_*(\beta) = \beta \quad \text{and} \quad \hat{H}_*(\tau_\gamma) = \tau_\gamma\beta^{-S(\gamma)}$$

for all $\gamma \in \Gamma_0$. Further note that $\widehat{H}$ is a leaf conjugacy between $\widehat{f}$ and $f_{\widehat{A}\widehat{B}}$. Since

$$\widehat{H}_* \widehat{f}_* \widehat{h}_*^{-1}(\tau_\gamma) = \widehat{H}_* f_* \widehat{h}_*^{-1}(\tau_\gamma) = \tau_{A(\gamma)} \beta^{-SA(\gamma)} \beta^{L(\gamma)} \beta^{S(\gamma)} = \tau_{A(\gamma)},$$

it follows that $\widehat{H} \widehat{f} \widehat{H}^{-1}$ and $f_{\widehat{A}\widehat{B}}$ have the same action on $\pi_1(M_{\widehat{B}})$. Since $M_{\widehat{B}}$ is an Eilenberg–MacLane space of type $K(\pi, 1)$, this implies that the maps $\widehat{H} \widehat{f} \widehat{H}^{-1}$ and $f_{\widehat{A}\widehat{B}}$ are homotopic and completes the proof of Proposition 2 in the case that h is the identity.

Case 2. The homeomorphism $h: M \to M_B$ is not the identity map.

Define a map $g: M_B \to M_B$ by $g = hfh^{-1}$. By construction, g is a homeomorphism which is topologically conjugate to f (a stronger property than leaf conjugacy). For any finite cover $\widehat{M}_B$ of M_B, if g lifts to a homeomorphism $\widehat{g}: \widehat{M}_B \to \widehat{M}_B$, then there is a finite cover $\widehat{M}$ of M and lifts $\widehat{f}: \widehat{M} \to \widehat{M}$ and $\widehat{h}: \widehat{M} \to \widehat{M}_B$ of the maps f and g, respectively, such that

$$\widehat{h}\widehat{f} = \widehat{g}\widehat{h}.$$

In fact, we can just take $\widehat{M}$ to be the same topological space as $\widehat{M}_B$, but with its smooth structure pulled back from M.

We now use the following definition of leaf conjugacy, adapted slightly from the original definition in [3, Chap. 7]. A homeomorphism $f: M \to M$ *preserves a foliation* $\mathcal{F}$ if and only if it sends the leaf through p to the leaf through $f(p)$. If $f: M \to M$ and $f': M' \to M'$ are homeomorphisms which preserve foliations $\mathcal{F}$ and $\mathcal{F}'$, then $(f, \mathcal{F})$ is *leaf conjugate* to $(f', \mathcal{F}')$ if and only if there is a homeomorphism $h: M \to M'$ which carries leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$ and $hf(L) = f'h(L)$ for every leaf L of $\mathcal{F}$. When f or f' is partially hyperbolic, the foliation is understood to be the center foliation.

The proof in Case 1 never directly uses the partial hyperbolicity of f, only the fact that f is a homeomorphism of M_B such that the identity map on M_B is a leaf conjugacy between $(f, \mathcal{F}^c)$ and $(f_{AB}, \mathcal{F}^c)$ where $\mathcal{F}^c$ is the center foliation of f_{AB}. Here in Case 2, g preserves $\mathcal{F}^c$ and the identity map on M_B is a leaf conjugacy between $(g, \mathcal{F}^c)$ and $(f_{AB}, \mathcal{F}^c)$. Following all of the steps of case 1 with g in place of f, we construct an AB-prototype

$$f_{\widehat{A}\widehat{B}}: M_{\widehat{B}} \to M_{\widehat{B}}$$

and a homeomorphism

$$\widehat{H}: M_{\widehat{B}} \to M_{\widehat{B}}$$

such that $\widehat{H}$ is a leaf conjugacy between $f_{\widehat{A}\widehat{B}}$ and a lift $(\widehat{g}, \mathcal{F}^c)$ of $(g, \mathcal{F})$ to a finite cover. As described above, we can then lift f and h to maps

$$\widehat{f}: \widehat{M} \to \widehat{M} \quad \text{and} \quad \widehat{h}: \widehat{M} \to M_{\widehat{B}}$$

so that $\widehat{h}$ is a topological conjugacy between $\widehat{f}$ and $\widehat{g}$. We can then verify that the composition $\widehat{H} \circ \widehat{h}$ is the desired leaf conjugacy between the partially hyperbolic maps $\widehat{f}$ and $f_{\widehat{A}\widehat{B}}$. This completes the proof of Proposition 2.

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