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Counting lattices in products of trees

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Abstract. A BMW group of degree (m,n) is a group that acts simply transitively on vertices of the product of two regular trees of degrees m and n. We show that the number of commensurability classes of BMW groups of degree (m,n) is bounded between $(mn)^{\alpha mn}$ and $(mn)^{\beta mn}$ for some $0 < \alpha < \beta$. In fact, we show that the same bounds hold for virtually simple BMW groups. We introduce a random model for BMW groups of degree (m,n) and show that asymptotically almost surely a random BMW group in this model is irreducible and hereditarily just-infinite.

1. Introduction

Given $n \in \mathbb{N}$, let T_n denote the regular tree of valence n. A *BMW group of degree* (m,n) is a subgroup of $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ that acts simply transitively on the vertex set of $T_m \times T_n$. Using these groups, Wise [20, 21] and Burger–Mozes [4] produced the first examples of non-residually finite and virtually simple CAT(0) groups respectively. BMW groups have been extensively studied since and have rich connections to the study of automata groups and commensurators (see Caprace's survey [18]). Particularly relevant here is work of, amongst others, Rattaggi [15–17] and Radu [14].

In this paper, our goal is two-fold: (1) estimate the number of BMW groups and virtually simple BMW groups up to abstract commensurability, and (2) define and study a random model for BMW groups. Although conceptually related, the two parts are independently presented.

Counting BMW groups

Let $\mathrm{BMW}(m,n)$ be the set of all BMW groups of degree (m,n) up to conjugacy in $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$. Let $\doteq$ be the equivalence relation of abstract commensurability, i.e., the groups Γ and Λ satisfy $\Gamma \doteq \Lambda$ if they have isomorphic finite index subgroups. In analogy to counting results for hyperbolic manifolds (see Remark 1.1),

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Caprace [18, Problem 4.26] asks for an estimate on the number of abstract commensurability classes of BMW groups of degree (m,n) as $m,n \to \infty$. Addressing this question, we give the following result:

Theorem A. *There exist* $0 < \alpha < \beta$ *such that, for all sufficiently large m and n,*

$$(mn)^{\alpha mn} \leq |\,\mathrm{BMW}_{vs}(m,n)/\dot{\simeq}| \leq |\,\mathrm{BMW}(m,n)/\dot{\simeq}| \leq (mn)^{\beta mn},$$

where $\mathrm{BMW}_{vs}(m,n)$ *is the collection of BMW groups, up to conjugacy, of degree* (m,n) *that contain an index 4 simple subgroup.*

All BMW groups contain an index 4 normal subgroup, so the index in the above theorem is as small as possible.

Remark 1.1. Compare the above result with the bounds obtained by [2,9]: there exist $0 < \alpha' < \beta'$ such that the number of commensurability classes of hyperbolic manifolds of volume at most v is bounded between $v^{\alpha' v}$ and $v^{\beta' v}$.

A random model for irreducible BMW groups

The random model we define is based on a combinatorial description of BMW groups (more precisely, of involutive BMW groups) given in Section 2.1. We postpone the definition of the model to Section 5, and only highlight its main properties in Theorem B below. This model does not capture all possible BMW groups. It was chosen predominantly for its relative ease of computations on the one hand, and its naturality on the other.

A BMW group is *irreducible* if it does not contain a subgroup of finite index that is isomorphic to the direct product of two free groups. A group is *just-infinite* if it is infinite and has only finite proper quotients. It is *hereditarily just-infinite* if all its finite-index subgroups are just-infinite.

Theorem B. *A random BMW involution group of degree* (m,n)*, with* $n > m^5$*, is hereditarily just-infinite and, in particular, is irreducible with probability at least* $1 - \frac{C}{m}$*, where* C *is a constant that is independent of* m *and* n*.*

Remark 1.2. In fact, when $n > m^5$, we have that:

- An arbitrary BMW group of degree (m,n) is irreducible if and only if it has non-discrete projections to both its factors. Something stronger happens in the random model: asymptotically almost surely, the projection to $\mathrm{Aut}(T_m)$ (resp. to $\mathrm{Aut}(T_n)$) of a random BMW group in the model contains the universal groups $U(A_m)$ (resp. $U(A_n)$).

- By the previous remark and a rigidity theorem for BMW groups [5, Theorem 1.4.1] (see also Theorem 3.8), asymptotically almost surely, two random BMW groups are not isomorphic.

We give the following conjecture regarding this random model.

Conjecture 1.3. In the above range of m and n, a random BMW group is asymptotically almost surely not residually finite and consequently is virtually simple by Theorem B.

More generally, we conjecture the following.

Conjecture 1.4. As $m, n \to \infty$, the proportion of virtually simple BMW groups of degree (m, n), up to conjugacy, tends to 1.

Positive evidence for Conjecture 1.4 has been given by Rattagi [15] and Radu [14] for small values of m and n.

Outline

In Section 2 we discuss involutive BMW groups and a combinatorial description of them. In Section 3, we bound the number of BMWs from above, and in Section 4, we prove the more difficult lower bound, giving Theorem A. In Section 5 we present our random model for BMW groups. Next, in Section 6 and Section 7, we show that the local actions are alternating or symmetric with high probability. Finally, in Section 8 we prove Theorem B. We note that sections Section 3 and Section 4 can be read independently of Section 5, Section 6, Section 7 and Section 8 (and vice versa).

2. Involutive BMW Groups

An *involutive* BMW group Γ of degree (m, n) is a BMW group such that for every edge e of $T_m \times T_n$, there is some $g \in \Gamma$ that interchanges the endpoints of e. Since Γ acts simply transitively on vertices, such an element g must be an involution. We let $\mathrm{BMW}_{\mathrm{inv}}(m, n)$ denote the set of all involutive BMW groups of degree (m, n), up to conjugacy in $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$. A BMW group of degree (m, n) is said to be *irreducible* if and only if the projection of Γ to either (and hence both) $\mathrm{Aut}(T_n)$ or $\mathrm{Aut}(T_m)$ is not discrete. We remark that Γ is irreducible if and only if it does not virtually split as a direct product of two non-trivial free groups [4, Proposition 1.2].

Any tree T is bipartite; let $\mathrm{Aut}^+(T)$ be the index 2 subgroup of $\mathrm{Aut}(T)$ preserving the bi-partition of T. Given a BMW group Γ of degree (m, n), denote by

$$\Gamma^+ = \Gamma \cap (\mathrm{Aut}^+(T_m) \times \mathrm{Aut}^+(T_n))$$

the index 4 subgroup of Γ preserving the bipartitions of T_m and T_n. This subgroup is always torsion-free [14, Lemma 3.1].

2.1. Structure sets

In this section we describe structure sets which encode presentations for BMW groups. For the rest of this article, we fix countable indexing sets $a_1, a_2, \dots$ and $b_1, b_2, \dots$, and for each $k \in \mathbb{N}$, we set $A_k := \{a_1, \dots, a_k\}$ and $B_k := \{b_1, \dots, b_k\}$.

Definition 2.1 (Structure set). An (m,n)-*structure set* S is a collection of subsets of $A_m \sqcup B_n$ such that:

(1) each element of S is of the form $\{a_i, b_k, a_j, b_l\}$, where $a_i, a_j \in A_m$ and $b_k, b_l \in B_n$, and

(2) for every $a \in A_m$ and $b \in B_n$, $\{a, b\}$ is a subset of exactly one set in S.

Let $\mathcal{S}_{m,n}$ denote the set of all (m,n)-structure sets.

For a structure set S, denote by R_S the set of *words in* $A_m \sqcup B_n$ defined as

$$R_S = \{a_i b_k a_j b_l \mid \{a_i, b_k, a_j, b_l\} \in S\}.$$

Remark 2.2. In the definition of a structure set, the elements a_i and a_j (and similarly b_k, b_l) of a set $\{a_i, b_k, a_j, b_l\} \in S$ are not assumed to be distinct, so some $\{a_i, b_k, a_j, b_l\} \in S$ may have fewer than 4 elements. We often still write repeating elements in these subsets, e.g., $\{a_i, b_k, a_i, b_l\}$ instead of $\{a_i, b_k, b_l\}$. In this example, the word $a_i b_k a_i b_l$ is one of the words in R_S corresponding to $\{a_i, b_k, a_i, b_l\} = \{a_i, b_k, b_l\}$.

Remark 2.3. A useful point of view on structure sets is given by partitions of the complete bi-partite graph: Let $K_{m,n}$ be the complete bi-partite graph on $A_m \sqcup B_n$. Given $\{a_i, b_k, a_j, b_l\} \in S$, one can assign to it the closed (possibly degenerate) path of length 4 in $K_{m,n}$ connecting the vertices a_i, a_j to the vertices b_k, b_l. In this way, one can think of an (m,n)-structure set as a partition of the edges of the complete bi-partite graph on the vertices $A_m \sqcup B_n$ into closed paths of length 4 such that each edge belongs to exactly one such path.

A (combinatorial) *square complex* is a 2-complex in which 2-cells (squares) are attached along combinatorial paths of length four. A *VH-complex* is a square complex in which the set of 1-cells (edges) is partitioned into vertical and horizontal edges such that the attaching map of each square alternates between them. We regard the product $T_m \times T_n$ of trees as a VH-complex where an edge is horizontal if it lies in $T_m \times \{v\}$ for some $v \in T_n$, and vertical if it lies in $\{v\} \times T_n$ for some $v \in T_m$.

Definition 2.4 (Marking). A *marking* $\mathcal{M}$ on $T_m \times T_n$ is a choice of a base vertex $o \in T_m \times T_n$ and an identification of the horizontal (resp. vertical) edges incident to o with A_m (resp. B_n). An element $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ is said to *fix* $\mathcal{M}$ if g fixes o and fixes all edges adjacent to o.

Fix a marking $\mathcal{M}$ on $T_m \times T_n$ with base vertex o. Let $\mathrm{BMW}_{\mathcal{M}}(m,n)$ be the set of all involutive BMW groups of degree (m,n), up to an automorphism fixing $\mathcal{M}$. In other words, two BMW groups Γ and Γ' of degree (m,n) are equal in $\mathrm{BMW}_{\mathcal{M}}(m,n)$ if and only if $\Gamma = g\Gamma' g^{-1}$, where g is an element of $\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ that fixes $\mathcal{M}$. Let $\mathcal{S}_{m,n}$ be the set of all (m,n)-structure sets. We now describe how to obtain a bijection:

$$\Phi\colon \mathrm{BMW}_{\mathcal{M}}(m,n) \to \mathcal{S}_{m,n}.$$

Let Γ be an involutive BMW group of degree (m,n). As Γ is involutive, each edge of $T_m \times T_n$ is stabilized by a unique element of Γ. By a slight abuse of notation, we let a_i (resp. b_i) denote the element of Γ that stabilizes the edge that is adjacent to o with label a_i (resp. b_i). As Γ acts freely and transitively on the vertices of $T_m \times T_n$, its action induces a well-defined, Γ-invariant labeling of the edges of $T_m \times T_n$ which we generally call the Γ*-induced labeling*. Moreover, it is readily checked that the 1-skeleton of $T_m \times T_n$ with this labeling is the Cayley graph for Γ with generating set $\{a_1, \dots, a_m, b_1, \dots, b_n\}$ (where bigons in this Cayley graph are collapsed to edges).

We now describe how to form an (m,n)-structure set S associated to Γ. Let S be the collection of subsets $\{a_i, b_k, a_j, b_l\}$ such that there exists a square in $T_m \times T_n$ whose edges are labeled a_i, b_k, a_j, b_l with respect to the Γ-induced labeling. Note that since Γ acts simply transitively on the vertices of $T_m \times T_n$ and preserves the Γ-induced labeling, it suffices to only consider the squares containing o.

To show that S is a structure set, let $a \in A_m$ and $b \in B_n$. There exists a unique square s which contains both edges incident to o labeled by a and b. Since Γ acts simply transitively on vertices, any other square which contains two edges labeled by a and b is in the orbit of s and consequently its edges have the same labels as s. Thus, there is a unique $\{a_i, b_k, a_j, b_l\} \in S$ containing a and b. So, S is indeed an (m,n)-structure set. We say that S is the structure set *associated with* Γ, and we define $\Phi([\Gamma]) = S$, where $[\Gamma]$ is the equivalence class in $\mathrm{BMW}_{\mathcal{M}}(m,n)$ containing Γ.

Additionally, we conclude that Γ has the presentation

$$\langle A_m \sqcup B_n \mid \{a^2 \mid a \in A_m\} \cup \{b^2 \mid b \in B_n\} \cup R_S \rangle. \tag{1}$$

This follows since we can take the 1-skeleton of $T_m \times T_n$, label it with the Γ-induced labeling (i.e., form the Cayley graph for Γ) and attach 2-cells corresponding to the relations R_S. The resulting complex can also be obtained from the Cayley complex for Γ by collapsing each bigon corresponding to the relations a^2 and b^2 to an edge. In fact, it is just $T_m \times T_n$ with the Γ-induced edge labeling.

We now need to check that Φ is well defined. Suppose that Γ' is an involutive BMW group of degree (m,n) that is conjugate to Γ by some $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ that fixes $\mathcal{M}$. Then, as g fixes $\mathcal{M}$, the Γ-induced labeling and Γ'-induced labeling of $T_m \times T_n$ agree on all squares that contain o. It follows by construction that the structure sets associated to Γ and Γ' are equal. Consequently, Φ is well defined.

We now check that Φ is injective. Suppose that Γ and Γ' are involutive BMW groups of degree (m,n) and that

$$\Phi([\Gamma]) = \Phi([\Gamma']).$$

Consequently, the Γ-induced labeling and the Γ'-induced labeling of $T_m \times T_n$ agree on all squares which contain o. It now readily follows that Γ' is conjugate to Γ by some element $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ that fixes the marking $\mathcal{M}$. Thus, Γ and Γ' are equal in $\mathrm{BMW}_{\mathcal{M}}(m,n)$.

Finally, we check that Φ is surjective. Let S be an (m,n)-structure set. Then the group Γ with presentation as in (1) is an involutive BMW group (see, e.g., [18]). Moreover, $\Phi([\Gamma]) = S$. We have thus shown the following result.

Proposition 2.5. *Let $\mathcal{M}$ be a marking of $T_m \times T_n$. There is a bijection*

$$\Phi\colon \mathrm{BMW}_{\mathcal{M}}(m,n) \to \mathcal{S}_{m,n}.$$

Moreover, for each $S \in \mathcal{S}_{m,n}$, each representative of $\Phi^{-1}(S)$ has the presentation

$$\langle A_m \sqcup B_n \mid \{a^2 \mid a \in A_m\} \cup \{b^2 \mid b \in B_n\} \cup R_S \rangle.$$

Let $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ be an automorphism. We describe how g acts on markings. Let $\mathcal{M}$ be a marking of $T_m \times T_n$ with base vertex o. Then g induces a new marking $\mathcal{M}' = g\mathcal{M}$ whose base vertex is $o' = go$ and such that the label of an edge e adjacent to o' is equal to the label of $g^{-1}e$ under the marking $\mathcal{M}$. This action of g also induces a bijection

$$\Psi_g\colon \mathrm{BMW}_{\mathcal{M}}(m,n) \to \mathrm{BMW}_{\mathcal{M}'}(m,n)$$

by sending $[\Gamma] \in \mathrm{BMW}_{\mathcal{M}}(m,n)$ to $[g\Gamma g^{-1}] \in \mathrm{BMW}_{\mathcal{M}'}(m,n)$.

Let S be an (m,n) structure set, and let $\mu \in \mathrm{Sym}(A_m)$ and $\nu \in \mathrm{Sym}(B_m)$ be permutations. We can form a new (m,n)-structure set S' by applying the permutation $(\mu,\nu) \in \mathrm{Sym}(A_m) \times \mathrm{Sym}(B_n) \leq \mathrm{Sym}(A_m \sqcup B_n)$ to the subsets in S. We say that S' is a *relabeling of P induced by μ and ν*.

Let $\Gamma' = g\Gamma g^{-1}$ for some $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$. Since Γ acts vertex transitively, we may assume, without loss of generality, that g fixes o. Thus, g induces permutations $\mu \in \mathrm{Sym}(A_m)$ and $\nu \in \mathrm{Sym}(B_m)$ on the labels (in the marking $\mathcal{M}$) of the edges incident to o. It readily follows that the structure set of S' of Γ' is obtained from the structure set S of Γ by relabeling.

Thus, if Γ and Γ' are conjugate BMW groups, then their associated structure sets are the same up to a relabeling, regardless of a choice of marking. Conversely, suppose that S' is an (m,n)-structure set that is a relabeling of a structure set S induced by $\mu \in \mathrm{Sym}(A_m)$ and $\nu \in \mathrm{Sym}(B_n)$. Then we can choose a marking $\mathcal{M}$ on $T_m \times T_n$ and a BMW group Γ whose associated structure set is P. Additionally, we can choose a $g \in \mathrm{Aut}_{\mathcal{M}}(T_m \times T_n)$ so that the induced action of g on the labels of the horizontal and vertical edges around o is given by μ and ν respectively. From this, we have that $g\Gamma g^{-1}$ is a BMW group conjugate to Γ whose structure set is S'. We have thus shown the following proposition.

Proposition 2.6. *There is a bijection*

$$\Psi\colon \mathrm{BMW}(m,n) \to \mathcal{S}_{m,n}/\mathit{relabeling}.$$

2.2. Local actions

Let X be a locally finite graph. For every vertex $v \in V(X)$, let $E(v)$ be the set of edges of X incident to v. If a group Γ acts on X, the *local action* of Γ on X at the vertex v is the induced action of $\mathrm{Stab}_\Gamma(v)$ on the set $E(v)$. By abuse of terminology, we will also refer to the image of the action $\mathrm{Stab}_\Gamma(v) \to \mathrm{Sym}(E(v)) \cong \mathrm{Sym}(n)$ as the local action, where $n = |E(v)|$. The *local actions of* $\Gamma \leq \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ are the local actions of Γ on T_m and T_n. More specifically, we call the local action of Γ on T_m the *A-tree local action*, and the local action of Γ on T_n the *B-tree local action*.

We show how to read off the local action of an involutive BMW group of degree (m,n) from the corresponding structure set. First note that since a BMW group of degree (m,n) acts transitively on the vertices of $T_m \times T_n$, its local actions on T_m (resp. T_n) at different vertices are conjugate actions. We can thus refer to *the* local action on T_m (resp. on T_n) as the conjugacy class of the local action at some vertex of T_m (resp. T_n).

Let us focus on the local action on T_n. Let $\mathcal{M}$ be a marking of $T_m \times T_n$ with base vertex o. Let π_A and π_B be the projections of $T_m \times T_n$ to the first and second factors, respectively. Let $o_B = \pi_B(o) \in T_n$. Edges incident to o_B in T_n are labeled by elements of B_n as follows: the label of an edge e incident to o_B is the label of the unique edge e' of $T_m \times T_n$ incident to o such that $\pi_B(e') = e$.

Let Γ be an involutive BMW group of degree (m,n) with structure set S. The local action of Γ at the vertex o_B can be identified with a subgroup of $\mathrm{Sym}(A_m) \simeq \mathrm{Sym}(m)$. Recall that Γ is generated by the elements $a_1, \dots, a_m, b_1, \dots, b_n$. Observe that $\mathrm{Stab}_\Gamma(o_B) = \langle a_1, \dots, a_m \rangle$, and thus the local action on T_n is generated by the

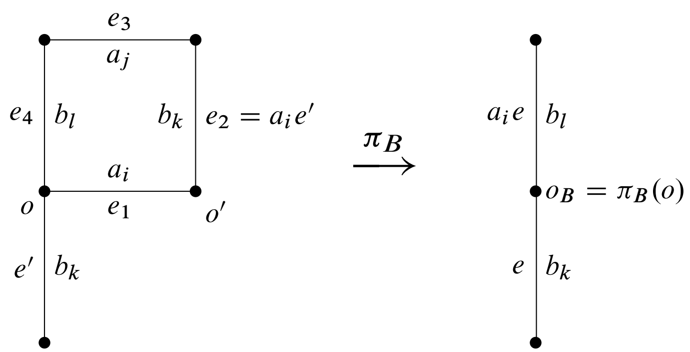

Figure 1. Determining the action of $\alpha_i(k)$.

action of each one of $a_1, \ldots, a_m \in A_m$. Denote by $\alpha_i \in \mathrm{Sym}(n) \simeq \mathrm{Sym}(B_n)$ the action of a_i on B_n.

Lemma 2.7. *Let $1 \leq i \leq m$ and let $1 \leq k, l \leq n$. Then, $\alpha_i(k) = l$ if and only if $\{a_i, b_k, a_j, b_l\} \in S$ for some $a_j \in A_m$.*

Proof. Fix $1 \leq i \leq m$ and $1 \leq k \leq n$. Then, by the definition of a structure set, there exists a unique $\{a_i, b_k, a_j, b_l\} \in S$ containing both a_i and b_k. Thus, to prove the claim, it is enough to show that $\alpha_i(k) = l$. Let e be the edge of T_m that is labeled by b_k and incident to $o_B = \pi_B(o)$. To show that $\alpha_i(k) = l$, we need to show that the label of $a_i e$ is b_l.

There exists a unique edge $e' \in T_m \times T_n$ labeled b_k and incident to o. This edge satisfies $\pi_B(e') = e$. The element a_i acts on $T_m \times T_n$ by mapping o to the endpoint o' of the unique edge e_1 labeled a_i incident to o. Since Γ preserves the Γ-induced labeling, $a_i e'$ is the unique edge e_2 of $T_m \times T_n$ incident to o' labeled b_k. The edges e_1, e_2 are adjacent edges of a (unique) square in $T_m \times T_n$. Let e_1, e_2, e_3, e_4 be the edges of this square as shown in Figure 1. By the definition of S, their respective Γ-induced labels are a_i, b_k, a_j, b_l. We get that $a_i e = \pi_B(a_i e') = \pi_B(e_2) = \pi_B(e_4)$. Since e_4 is incident to o, the label of $a_i e$ is the same as that of e_4, namely b_l. ∎

We call the involutions $\alpha_1, \ldots, \alpha_m \in \mathrm{Sym}(n)$ the *B-tree local involutions*. Similarly, we can define the *A-tree local involutions* $\beta_1, \ldots, \beta_n \in \mathrm{Sym}(m)$ corresponding to the local actions of $b_1, \ldots, b_n$ on the tree T_m.

2.3. Virtual simplicity of BMW groups

The following theorem is a corollary of [3, Propositions 3.3.1, 3.3.2] and [4, Theorem 4.1].

Theorem 2.8 (Burger–Mozes). *Let $m, n \geq 6$, let Γ be an irreducible BMW group of degree (m, n), and assume that the local actions of Γ on T_m and T_n contain the alternating groups* $\mathrm{Alt}(m)$ *and* $\mathrm{Alt}(n)$*, respectively. Then, Γ is hereditarily just-infinite.*

For a group Γ, its *finite residual* $\Gamma^{(\infty)}$ is the intersection of all finite-index subgroups of Γ. The following widely known lemma is a tool to prove virtual simplicity of a group.

Lemma 2.9. *If a group Γ is hereditarily just-infinite and is not residually finite, then $\Gamma^{(\infty)}$ is a finite index, simple subgroup of Γ.*

Proof. Since Γ is not residually finite, the finite residual $\Gamma^{(\infty)}$ is a non-trivial normal subgroup of Γ. By assumption Γ is just-infinite, thus $\Gamma^{(\infty)}$ must have finite index in Γ. As Γ is hereditary just-infinite, $\Gamma^{(\infty)}$ is itself just-infinite, and thus cannot contain any non-trivial infinite-index normal subgroup. On the other hand, by definition and since $\Gamma^{(\infty)}$ has finite index in Γ, $\Gamma^{(\infty)}$ cannot have any non-trivial finite-index normal subgroup. Thus, $\Gamma^{(\infty)}$ is simple. ■

3. Upper bounds on BMW counts

In this section we give upper bounds for the number of conjugacy classes of involutive BMW groups. We then use a result of Burger–Mozes–Zimmer to bound the number of BMW groups that are abstractly commensurable to a given BMW group with primitive local actions and simple type-preserving subgroup.

3.1. Upper bound on conjugacy classes of involutive BMWs

Proposition 3.1. *There are at most $(mn)^{mn}$ conjugacy classes of involutive BMW groups of degree (m, n).*

Proof. Recall that $\mathcal{S}_{m,n}$ denotes the set of (m, n)-structure sets. Every $S \in \mathcal{S}_{m,n}$ defines a function $f_S \colon A_m \times B_n \to A_m \times B_n$ by $f_S(a, b) = (a', b')$ if $\{a, b, a', b'\} \in S$. This function is well defined by the definition of a structure set. We can reconstruct S from f_S by

$$S = \{\{a, b, a', b'\} \mid (a', b') = f_S(a, b), a \in A_m, b \in B_n\}.$$

Thus we get an injective map $\mathcal{S}_{m,n} \hookrightarrow (A_m \times B_n)^{A_m \times B_n}$ mapping $S \mapsto f_S$. Consequently,

$$|\,\mathrm{BMW}(m, n)| \leq |\,\mathrm{BMW}_{\mathcal{M}}(m, n)| = |\mathcal{S}_{m,n}| \leq |(A_m \times B_n)^{A_m \times B_n}| = (mn)^{mn},$$

where the equality $|\operatorname{BMW}_{\mathcal{M}}(m,n)| = |\mathcal{S}_{m,n}|$ follows from the bijection in Proposition 2.5. ■

Remark 3.2. For general (not necessarily involutive) BMW groups, using the (m,n)-datum defined in [14] (which is analogous to structure sets defined here), a similar proof gives a number $\beta > 0$ such that the number of conjugacy classes of BMW groups is bounded by $(mn)^{\beta mn}$.

3.2. (m,n)-complexes and type-preserving subgroups

In this subsection we associate to each involutive BMW-group a certain edge-labeled square complex that completely describes the group. These complexes will allow us to deduce a count on the number of involutive BMW groups with the same type-preserving subgroup up to conjugation.

Definition 3.3. An (m,n)-*complex* is a VH-complex Y such that:

(1) Y has exactly 4 vertices: v_{00}, v_{10}, v_{01} and v_{11};

(2) there are exactly m edges between v_{00} and v_{10} (resp. v_{01} and v_{11}), all of which are horizontal and labeled by distinct elements of A_m;

(3) there are exactly n edges between v_{00} and v_{01} (resp. v_{10} and v_{11}), all of which are vertical and labeled by distinct elements of B_n;

(4) for each horizontal edge e_1 and vertical edge e_2 of Y, there is a unique square containing both e_1 and e_2;

(5) there is a label-preserving vertex-transitive action on Y.

Remark 3.4. By (4) above, an (m,n)-complex contains exactly mn squares. We also note that the label preserving automorphism group of an (m,n)-complex is precisely $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 3.5. *Fix a marking on $T_m \times T_n$, and let Γ be an involutive BMW group of degree (m,n). Label the edges of $T_m \times T_n$ by the Γ-induced labeling. Then $\Gamma^+ \backslash (T_m \times T_n)$ is an (m,n)-complex.*

Conversely, let Y be an (m,n)-complex. Then the set S, consisting of the subsets $\{a_i, b_k, a_j, b_l\}$, where a_i, b_k, a_j, b_l are the labels of the edges of squares in Y, is an (m,n)-structure set.

Proof. Let Γ be as in the statement of the lemma. The type-preserving subgroup $\Gamma^+ < \Gamma$ acts freely, so we can consider the quotient complex $Z := \Gamma^+ \backslash (T_m \times T_n)$. As edge labels pass to the quotient and as $\Gamma/\Gamma^+ \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ acts transitively on the vertices of Z, Z is an (m,n)-complex as required. The proof of the converse statement follows from Definitions 2.1 and 3.3. ■

We will need the following lemma counting the number of possible (m,n)-complexes with isomorphic square complexes.

Lemma 3.6. *Given any (m,n)-complex C, there are at most $2(n!m!)^2$ distinct (m,n)-complexes that are isomorphic to C as unlabeled square complexes (i.e., isomorphic via a cellular isomorphism that does not necessarily preserve labels or the VH-structure).*

Proof. Let Y be an the underlying square complex of an (m,n)-complex C. There are at most two ways of giving Y a suitable VH-structure. After choosing a VH-structure, Y has 4 vertices, $2m$ horizontal edges and $2n$ vertical edges, and there are at most $(n!m!)^2$ ways to choose labels to obtain an (m,n)-complex. Thus there are at most $2(n!m!)^2$ possible (m,n)-complexes that are isomorphic to Y as unlabeled square complexes. ∎

The next lemma bounds the number of involutive BMW groups with conjugate type-preserving subgroups.

Lemma 3.7. *For each $\Gamma \in \mathrm{BMW}(m,n)$, there are at most $2(n!m!)^2$ conjugacy classes of $\Lambda \in \mathrm{BMW}(m,n)$ with Γ^+ conjugate to Λ^+.*

Proof. Suppose that $\Gamma^+ = g\Lambda^+g^{-1}$ for some $g \in \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$. We then have that $Y_1 := \Gamma^+ \backslash (T_m \times T_n)$ is isomorphic to $Y_2 := g\Lambda^+g^{-1} \backslash (T_m \times T_n)$ as unlabeled square complexes. By Lemma 3.6, Y_2 can be one of at most $2(n!m!)^2$ possible (m,n)-complexes. By Lemma 3.5 and Proposition 2.5, such an (m,n)-complex completely determines $g\Lambda g^{-1}$ up to a conjugation. The lemma now follows. ∎

Recall that a subgroup $F \leq \mathrm{Sym}(n)$ is *primitive* if no non-trivial partition of $\{1,\dots,n\}$ is stabilized by F. The following theorem is a reformulation of a result of Burger–Mozes–Zimmer, which builds on a superrigidity theorem of Monod and Shalom [11].

Theorem 3.8 ([5, Theorem 1.4.1]). *Let $G = (\mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)) \rtimes R$, where $R \leq \mathbb{Z}/2\mathbb{Z}$ permutes the factors when $m = n$ and is trivial otherwise, and let $\Gamma, \Gamma' \leq \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ be cocompact lattices with primitive local actions. Then any isomorphism $\phi\colon \Gamma \to \Gamma'$ is induced by conjugation in G, i.e., there exist some $g \in G$ such that $ghg^{-1} = \phi(h)$ for all $h \in \Gamma$.*

Recall that two groups are *abstractly commensurable* if they have isomorphic finite index subgroups. The previous theorem implies the following.

Proposition 3.9. *Let Γ be an involutive BMW group of degree (m,n) with primitive local actions and with Γ^+ simple. Then, up to conjugacy, there are at most $2(n!m!)^2$ involutive BMW groups of degree (m,n) that are abstractly commensurable with Γ.*

Proof. Let Λ be an involutive BMW group of degree (m,n) that is abstractly commensurable to Γ. As Γ^+ is simple, Λ contains a finite index subgroup H that is isomorphic to Γ^+. It follows from Theorem 3.8 that H is type-preserving and has four orbits of vertices, so $H = \Lambda^+$. By Theorem 3.8, Γ^+ is conjugate to Λ^+. The result now follows from Lemma 3.7. ∎

4. Counting commensurability classes of BMW groups

In order to count commensurability classes of BMW groups, we first define a partial structure set S_0 (see definition below). We show in Theorem 4.2 that if Γ is an involutive BMW group whose structure set contains S_0, then the index 4 subgroup Γ^+ is simple. We then deduce the lower bound of Theorem A by showing that there are sufficiently many such Γ.

Definition 4.1. A *partial structure set* S_0 is a collection of subsets of $A_m \sqcup B_n$ of the form $\{a_i, b_k, a_j, b_l\}$, for $a_i, a_j \in A_m$ and $b_k, b_l \in B_n$, such that for every $a \in A_m$ and $b \in B_n$ *at most* one subset of S_0 contains both a and b.

Our starting point is the following involutive BMW group Δ (denoted as $\Gamma_{4,5,9}$ by Radu [14]).

Theorem ([14, Theorem 5.5]). *The involutive BMW group Δ of degree $(4,5)$, whose associated $(4,5)$-structure set is*

$$\begin{aligned} S_\Delta := \big\{&\{a_1,b_1,a_1,b_1\},\{a_1,b_2,a_1,b_2\},\{a_2,b_3,a_1,b_3\},\{a_2,b_1,a_2,b_1\}\\ &\{a_3,b_2,a_2,b_2\},\{a_3,b_3,a_3,b_1\},\{a_1,b_4,a_1,b_4\},\{a_4,b_5,a_1,b_5\}\\ &\{a_3,b_5,a_2,b_4\},\{a_4,b_2,a_4,b_1\},\{a_4,b_4,a_4,b_3\}\big\},\end{aligned}$$

satisfies $\Delta^{(\infty)} = \Delta^+$, where $\Delta^{(\infty)}$ is the intersection of all finite index subgroups of Δ.

Let $\alpha_1^\Delta, \dots, \alpha_4^\Delta$ be the corresponding B-tree local involutions of S_Δ, and let $\beta_1^\Delta, \dots, \beta_5^\Delta$ be the A-tree local involutions. The A-tree and B-tree local involutions can be computed explicitly (using Lemma 2.7) to show that

$$\langle \alpha_1^\Delta, \dots, \alpha_4^\Delta \rangle = \mathrm{Sym}(5) \quad \text{and} \quad \langle \beta_1^\Delta, \dots, \beta_5^\Delta \rangle = \mathrm{Sym}(4).$$

Thus, the local actions of Δ on T_4 and T_5 are Sym(4) and Sym(5), respectively. Let $m \geq 13, n \geq 14$ be integers.

For natural numbers $k \leq k'$, denote by $[\![k, k']\!] := \{k, k+1, \dots, k'\}$. Fix the following three involutions in $\mathrm{Sym}([\![6, n]\!])$:

$$\begin{aligned}
\alpha'_1 &= (7 \quad 10)(8 \quad 11)(12 \quad 13)(14 \quad 15)(16 \quad 17)\dots,\\
\alpha'_2 &= (9 \quad 7)(10 \quad 8)(11 \quad 12)(13 \quad 14)(15 \quad 16)\dots,\\
\alpha'_3 &= (6 \quad 9).
\end{aligned}$$

Note that these three involutions generate $\mathrm{Sym}([\![6, n]\!])$. Similarly, fix the following three involutions of $\mathrm{Sym}([\![5, m]\!])$:

$$\begin{aligned}
\beta'_1 &= (6 \quad 9)(7 \quad 10)(11 \quad 12)(13 \quad 14)(15 \quad 16)\dots,\\
\beta'_2 &= (8 \quad 6)(9 \quad 7)(10 \quad 11)(12 \quad 13)(14 \quad 15)\dots,\\
\beta'_3 &= (5 \quad 8).
\end{aligned}$$

These involutions generate $\mathrm{Sym}([\![5, m]\!])$.

Define the following partial structure sets:

$$\begin{aligned}
S_{AR} &= \{\{a_i, b_k, a_i, b_{\alpha'_i(k)}\} \mid 1 \leq i \leq 3,\ 6 \leq k \leq n\},\\
S_{BR} &= \{\{a_i, b_k, a_{\beta'_k(i)}, b_k\} \mid 5 \leq i \leq m,\ 1 \leq k \leq 3\},\\
S_{AC} &= \{\{a_4, b_k, a_4, b_k\} \mid 6 \leq k \leq 8\},\\
S_{BC} &= \{\{a_i, b_k, a_i, b_k\} \mid 5 \leq i \leq 7,\ 4 \leq k \leq 5\},\\
S_{AB} &= \{\{a_i, b_k, a_{\beta'_{k-5}(i)}, b_{\alpha'_{i-4}(k)}\} \mid 5 \leq i \leq 7,\ 6 \leq k \leq 8\},\\
S_A &= \{\{a_i, b_k, a_i, b_{\alpha'_{i-4}(k)}\} \mid 5 \leq i \leq 7,\ 9 \leq k \leq n,\ \alpha'_{i-4}(k) > 8\},\\
S_B &= \{\{a_i, b_k, a_{\beta'_{k-5}(i)}, b_k\} \mid 8 \leq i \leq m,\ 6 \leq k \leq 8,\ \beta'_{k-5}(i) > 7\},\\
S_M &= \{\{a_4, b_n, a_m, b_{n-1}\},\ \{a_{m-2}, b_4, a_{m-1}, b_{n-2}\}\},\\
S_{C1} &= \{\{a_i, b_n, a_i, b_n\},\ \{a_i, b_{n-1}, a_i, b_{n-1}\} \mid 8 \leq i < m\},\\
S_{C2} &= \{\{a_{m-1}, b_k, a_{m-1}, b_k\},\ \{a_{m-2}, b_k, a_{m-2}, b_k\} \mid 9 \leq k \leq n-3\}.
\end{aligned}$$

Let

$$S_0 := S_\Delta \cup S_{AR} \cup S_{BR} \cup S_{AC} \cup S_{BC} \cup S_{AB} \cup S_A \cup S_B \cup S_M \cup S_{C1} \cup S_{C2}. \quad (\dagger)$$

See Figure 2 and Table 1.

It is straightforward to check that for any $1 \leq i \leq m$ and $1 \leq k \leq n$, $\{a_i, b_k\}$ is a subset of at most one set in S_0, making S_0 a partial structure set.

Theorem 4.2. *If an (m, n)-structure set S contains S_0, then its associated involutive BMW group Γ satisfies that Γ^+ is simple.*

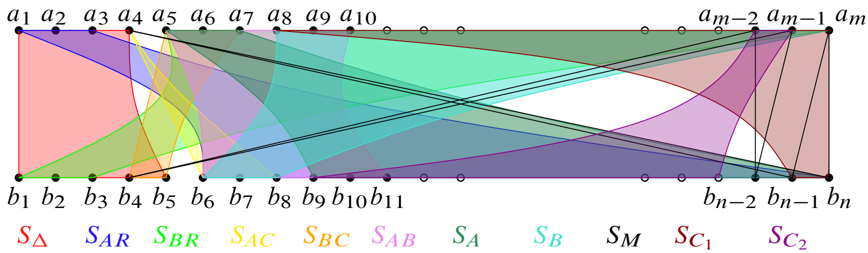

Figure 2. The partial structure set S_0 gives rise to a partial partition of the complete bipartite graph $K_{m,n}$ in the sense of Remark 2.3, each set in S_0 corresponds to a 4-cycle in $K_{m,n}$ contained in the associated highlighted subset in the figure.

	a_1–a_3	a_4	a_5–a_7	a_8–a_{10}	a_{11}–a_{m-3}	a_{m-2}–a_{m-1}	a_m
b_1–b_3	S_Δ	S_Δ	S_{BR}	S_{BR}	S_{BR}	S_{BR}	S_{BR}
b_4	S_Δ	S_Δ	S_{BC}			S_M	
b_5	S_Δ	S_Δ	S_{BC}				
b_6–b_8	S_{AR}	S_{AC}	S_{AB}	S_{AB}, S_B	S_B	S_B	S_B
b_9–b_{11}	S_{AR}		S_{AB}, S_A	S_{AB}		S_{C2}	
b_{12}–b_{n-3}	S_{AR}		S_A			S_{C2}	
b_{n-2}	S_{AR}		S_A			S_M	
b_{n-1}–b_n	S_{AR}	S_M	S_A	S_{C1}	S_{C1}	S_{C1}	S_M

Table 1. Table showing which partial structure sets possibly possess $\{a_i, b_k\}$ as a subset of one its elements. For example, if $\{a_6, b_4\}$ is a subset of some $s \in S_0$, then $s \in S_{BC}$.

To prove the above theorem, we need to show a few lemmas first. Throughout, let S be an (m,n)-structure set containing S_0 and let Γ be its associated involutive BMW group.

Lemma 4.3. *The A-tree and B-tree local actions of Γ are* $\mathrm{Sym}(m)$ *and* $\mathrm{Sym}(n)$*, respectively.*

Proof. We first show the claim for the B-tree local action. Let $\alpha_1, \ldots, \alpha_m$ be the B-tree local involutions of Γ. By Lemma 2.7 and as $S_\Delta, S_{AR} \subset S$, for $1 \le i \le 3$, we get that $\alpha_i = \alpha_i^\Delta \times \alpha_i'$ and that $\alpha_4 = \alpha_4^\Delta \times \gamma_4$, where γ_4 is some unknown permutation in $\mathrm{Sym}([\![6,n]\!])$.

Similarly, as $S_{BR}, S_{BC}, S_{AB}, S_A \subset S$, for $1 \le i \le 3$, we get that $\alpha_{4+i} = \mathrm{id} \times \alpha_i'$, where id is the identity permutation of $\mathrm{Sym}([\![1,5]\!])$. In addition, as $\alpha_1^\Delta, \ldots, \alpha_4^\Delta$ generate $\mathrm{Sym}([\![1,5]\!])$ and $\alpha_1', \ldots, \alpha_3'$ generate $\mathrm{Sym}([\![6,n]\!])$, $\alpha_1, \ldots, \alpha_7$ generate a subgroup of $\mathrm{Sym}(n)$ containing

$$\mathrm{Sym}([\![1,5]\!]) \times \mathrm{Sym}([\![6,n]\!]).$$

Finally, as $S_{BR}, S_B, S_M, S_{C1}, S_{C2} \subset S$, α_{m-1} is the transposition $(4 \quad n-2)$ (S_M is needed to deduce $\alpha_{m-1}(4) = n-2$, and the other relations are needed to ensure everything other than 4 and $n-2$ is fixed). From this we then conclude that $\alpha_1, \dots, \alpha_m$ generate $\mathrm{Sym}(n)$.

The argument for the A-tree local action is similar, so we briefly outline it. Let $\beta_1, \dots, \beta_n \in \mathrm{Sym}([\![1,m]\!])$ be the A-tree local involutions. Using (†), we deduce that $\beta_1, \dots, \beta_8$ generate a subgroup containing $\mathrm{Sym}([\![1,4]\!]) \times \mathrm{Sym}([\![5,m]\!])$. Since β_{n-1} is the transposition $(4 \quad m)$, it follows $\beta_1, \dots, \beta_n$ generate $\mathrm{Sym}(m)$. ∎

Lemma 4.4 (Finite residual). *If a structure set S contains S_0, then its associated involutive BMW group Γ satisfies $\Gamma^{(\infty)} = \Gamma^+$.*

Proof. Clearly $\Gamma^{(\infty)} \le \Gamma^+$, thus it suffices to show that $\Gamma^{(\infty)}$ has index 4 in Γ. More precisely, we show that

$$\Gamma/\Gamma^{(\infty)} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Recall from Proposition 2.5 that Γ has a presentation with generators $A_m \sqcup B_n$, so we may identify A_m and B_n with elements of Γ. For $g \in \Gamma$ denote by $\overline{g}$ its image in $\Gamma/\Gamma^{(\infty)}$. By [18, Proposition 4.2 (vii)] and as $S_\Delta \subset S$, Γ contains the subgroup Δ. By [14, Theorem 5.5] stated above, Δ satisfies

$$\Delta/\Delta^{(\infty)} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2.$$

It follows that $\overline{a}_i = \overline{a}_j$ for all $i, j \le 4$, and $\overline{b}_k = \overline{b}_l$ for all $k, l \le 5$. Denote these elements by $\overline{a}, \overline{b} \in \Gamma/\Gamma^{(\infty)}$, respectively.

In Claim 2 below, we prove that $\overline{b}_i = \overline{b}$ for all $1 \le i \le n$ and $\overline{a}_i = \overline{a}$ for all $1 \le i \le m$. The lemma follows from this as $\Gamma/\Gamma^{(\infty)}$ is generated by $\overline{a}, \overline{b}$ and satisfies the relations $\overline{a}^2 = \overline{b}^2 = [\overline{a}, \overline{b}] = 1$. Hence, it is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

We first prove the following claim, showing that certain generators of Γ are equal in the quotient.

Claim 1. *We claim that $\overline{a}_i = \overline{a}_j$ for all $i, j \ge 5$, and $\overline{b}_k = \overline{b}_l$ for all $k, l \ge 6$.*

Proof of Claim 1. We show that $\overline{b}_k = \overline{b}_l$ for all $k, l \ge 6$. The second claim follows by a similar argument.

Let G be the Schreier graph of the action of $\alpha'_1, \alpha'_2, \alpha'_3$ on $[\![6,n]\!]$, that is, the graph with vertices $[\![6,n]\!]$ and an edge $(k, \alpha'_i(k))$ for every $i \in \{1,2,3\}$ and $k \in [\![6,n]\!]$. The graph G is connected and not bipartite since the natural action of $\mathrm{Sym}([\![6,n]\!]) = \langle \alpha'_1, \alpha'_2, \alpha'_3 \rangle$ on $[\![6,n]\!]$ is primitive.

Let (k, l) be an edge of G. We have that $\alpha'_i(k) = l$ for some $i \in \{1, 2, 3\}$. Consequently, $\{a_i, b_k, a_i, b_l\} \in S_{AR}$ and $a_i b_k a_i b_l \in R_S$ is a relation in Γ. In particular, $b_l = a_i b_k a_i$ (as generators are involutions). Since G is not bipartite and is connected, any two vertices of G are connected by an even length path. It follows that for *any* k and l such that $6 \le k < l \le n$, there is an even number p such that $\overline{b}_l = \overline{a}^p \overline{b}_k \overline{a}^p$. As $\overline{a}^2 = 1$, we deduce that $\overline{b}_k = \overline{b}_l$. The claim follows. ∎

Claim 2. *We claim that $\overline{a} = \overline{a}_i$ for all $1 \le i \le m$, and $\overline{b} = \overline{b}_k$ for all $1 \le k \le n$.*

Proof of Claim 2. By Claim 1, we can define $\overline{a}' := \overline{a}_i$ for all $i \ge 5$ and $\overline{b}' := \overline{b}_k$ for all $k \ge 6$. To prove Claim 2, we need to show that $\overline{a} = \overline{a}'$ and $\overline{b} = \overline{b}'$. We show $\overline{a} = \overline{a}'$. The second statement follows from a similar argument.

First note that since $S_{AR} \subset S$ the word $a_1 b_k a_1 b_{\alpha'_1(k)} \in R_S$ is a relation in Γ for some $6 \le k, \alpha'_1(k) < n - 2$. In the quotient, this relation becomes $\overline{a}\overline{b}'\overline{a}\overline{b}' = 1$ which implies that $\overline{a}$ commutes with $\overline{b}'$. Next, since $S_M \subset S$ the word $a_4 b_n a_m b_{n-1} \in R_S$ is a relation in Γ. In the quotient, this gives $\overline{a}\overline{b}'\overline{a}'\overline{b}' = 1$. As $\overline{b}'$ commutes with $\overline{a}$, we see that $\overline{a} = \overline{a}'$. The claim follows. ∎

This concludes the proof of Lemma 4.4. ∎

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. By Lemma 4.3, the local actions of the group Γ are the full symmetric groups on m and n elements. Moreover, Γ is irreducible and not residually finite as it contains Δ (see [18, Proposition 4.2 vii)]). The theorem then follows from Theorem 2.8, Lemma 2.9 and Lemma 4.4. ∎

Lemma 4.5. *There exists a number $\alpha > 0$ such that, for all integers $m > 0$ and $n > 0$, the number of (m, n)-structure sets is at least $(mn)^{\alpha mn}$.*

Proof. Without loss of generality assume that $m \le n$. Let $\mathcal{I}_n \subseteq \mathrm{Sym}(n)$ be the subset of all involutions. For any m involutions $\alpha_1, \ldots, \alpha_m \in \mathcal{I}_n$, we can define a structure set

$$S = \{\{a_i, b_k, a_i, b_{\alpha_i(k)}\} \mid 1 \le i \le m, 1 \le k \le n\}.$$

Therefore, there are at least $|(\mathcal{I}_n)^m|$ different (m,n)-structure sets. By [7, Theorem 8], the number of involutions in $\mathrm{Sym}(n)$ is

$$|\mathcal{I}_n| \sim \exp\Big(\frac{1}{2}n(\log n - 1) + \sqrt{n}\Big) \ge n^{\frac{1}{4}n}$$

for large n. Thus the number of structure sets of degree (m, n) is at least

$$|(\mathcal{I}_n)^m| \ge n^{\frac{1}{4}mn} \ge (n^2)^{\frac{1}{8}mn} \ge (mn)^{\frac{1}{8}mn},$$

where the last inequality follows from $m \leq n$. By choosing α small enough, we get the claim for all m and n (not just large enough n). ■

Theorem 4.6. *There exists a number $\alpha > 0$ such that, for all sufficiently large natural numbers m and n, there are at least $(mn)^{\alpha mn}$ pairwise non-commensurable, involutive BMW groups Γ of degree (m, n) such that $\Gamma^{(\infty)} = \Gamma^+$ is simple.*

Proof. Note that the partial structure set S_0 defined in (†) has no element containing $\{a_i, b_k\}$ where $i \in [\![11, m-3]\!]$ and $k \in [\![12, n-3]\!]$. Set m', n' to be the number of elements in those integer intervals, respectively – namely $m' = m - 13$ and $n' = n - 14$. We see that we can add to S_0 any partial structure set supported on those elements. Using Lemma 4.5, there is some $\alpha' > 0$ so that there are at least $(m'n')^{\alpha' m'n'}$ different ways of extending S_0 to a partial structure set S'. By further adding the sets $\{a_i, b_j, a_i, b_j\}$ to S' for any i, j such that $\{a_i, b_j\}$ is not contained in an element of S', one obtains a structure set S. Therefore there are at least $(m'n')^{\alpha' m'n'}$ structure sets containing S_0. Additionally, a BMW group Γ associated to such a structure set satisfies that $\Gamma^{(\infty)} = \Gamma^+$ is simple by Theorem 4.2.

We now give a lower bound for the number of commensurability classes of involutive BMWs associated to structure sets containing S_0. By Proposition 3.9 the commensurability class of such a BMW group has at most $2(m!n!)^2$ structure sets. Therefore, by the previous paragraph, there are at least $(m'n')^{\alpha' m'n'}/2(m!n!)^2$ commensurability classes of such involutive BMW groups of degree (m, n). By using $m! \leq m^m$ and $n! \leq n^n$ and $m' \geq m/2$ and $n' \geq n/2$ one gets the desired lower bound. ■

5. A random model for BMW groups

For each even $n \in \mathbb{N}$, let $\mathcal{F}_n \subseteq \mathrm{Sym}(n)$ be the subset of fixed-point-free involutions, i.e., involutions which do not fix any element. When we write $\mathcal{F}_n$, it is implied that n is even. Let $(\mathcal{F}_n)^m$ be the set of m-tuples of fixed-point-free involutions. We say that $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$ has *triple matchings* if

$$\alpha_i(k) = \alpha_j(k) = \alpha_p(k)$$

for some $1 \leq k \leq n$ and some distinct $1 \leq i < j < p \leq m$.

Fix $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$ with no triple matchings. We can then show how to canonically define an (m, n)-structure set $S_{\underline{\alpha}}$ with associated B-tree local involutions $\alpha_1, \ldots, \alpha_m$. After fixing a marking $\mathcal{M}$ of $T_m \times T_n$, by Proposition 2.5 this also defines (up to conjugacy) an involutive BMW group $\Gamma_{\underline{\alpha}} \in \mathrm{BMW}_{\mathcal{M}}(m, n)$ with structure set $S_{\underline{\alpha}}$.

For each $1 \leq k < l \leq n$, set $I_{k,l} := \{i \mid \alpha_i(k) = l\}$. Note that since $\underline{\alpha}$ has no triple matchings, $|I_{k,l}| \leq 2$. The structure set $S_{\underline{\alpha}}$ is the collection of subsets $\{a_i, b_k, a_j, b_l\}$

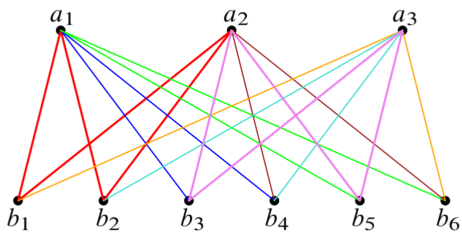

Figure 3. The partition of the edges of the bipartite graph $K_{3,6}$ corresponding to Example 5.1.

such that $1 \leq i \leq j \leq m$ and $1 \leq k < l \leq n$ satisfy $I_{k,l} = \{i, j\}$ (note that i could equal j). It is straightforward to check that $S_{\underline{\alpha}}$ is indeed a structure set and that the B-tree local involutions of $S_{\underline{\alpha}}$ are exactly $\alpha_1, \ldots, \alpha_m$.

Example 5.1. Suppose that $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in (\mathcal{F}_6)^3$ is such that $\alpha_1 = (12)(34)(56)$, $\alpha_2 = (12)(35)(46)$ and $\alpha_3 = (16)(35)(24)$. Note that $\underline{\alpha}$ has no triple matchings. The structure set associated to $\underline{\alpha}$ is then

$$S_{\underline{\alpha}} = \big\{\{a_1, b_1, a_2, b_2\}, \{a_1, b_3, a_1, b_4\}, \{a_1, b_5, a_1, b_6\}, \\ \{a_2, b_3, a_3, b_5\}, \{a_2, b_4, a_2, b_6\}, \{a_3, b_1, a_3, b_6\}, \{a_3, b_2, a_3, b_4\}\big\}.$$

See Figure 3.

A *random element of* $\mathcal{F}_n$ is an element of $\mathcal{F}_n$ chosen uniformly at random. A *random element of* $(\mathcal{F}_n)^m$ is an element of $(\mathcal{F}_n)^m$ chosen uniformly at random, i.e., an m-tuple of m independently chosen, random elements of $\mathcal{F}_n$. We are now ready to define random involutive BMW groups.

Definition 5.2. Suppose a marking $\mathcal{M}$ for $T_m \times T_n$ is fixed. Let $n > 0$ be even and let $\underline{\alpha}$ be a random element of $(\mathcal{F}_n)^m$. If $\underline{\alpha}$ has no triple matchings, we define the corresponding *random involutive BMW group of degree* (m, n) to be $\Gamma_{\underline{\alpha}} \in \mathrm{BMW}_{\mathcal{M}}(m, n)$. On the other hand, if $\underline{\alpha}$ contains a triple matching, then we say that the corresponding random involutive BMW group is not defined.

Let $\mathcal{P}$ be a property of BMW groups. We say that a *random involutive BMW group of degree* (m, n) *satisfies property* $\mathcal{P}$ *with probability* p, if given a random $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$, then with probability p the corresponding random involutive BMW group is defined and satisfies property $\mathcal{P}$.

We will see in Lemma 5.5, if m is a function of n satisfying $m(n) = o(n^{\frac{1}{3}})$, then a random m-tuple $\underline{\alpha} \in (\mathcal{F}_n)^{m(n)}$ has no triple matchings (and consequently defines a random BMW group) with probability tending to 1 as n tend to infinity.

We first prove two elementary lemmas regarding fixed-point-free involutions. Recall that the *double factorial* of an integer n is defined as $n!! := n \cdot (n-2) \cdot (n-4) \cdots 2$ for n even and as $n!! := n \cdot (n-2) \cdot (n-4) \cdots 3 \cdot 1$ for n odd.

Lemma 5.3. *For n even, $|\mathcal{F}_n| = (n-1)!!$.*

Proof. Let $\phi(n) = |\mathcal{F}_n|$, and let $\sigma \in \mathcal{F}_n$. There are $n-1$ options for $\sigma(1)$. After choosing $\sigma(1)$, there are $\phi(n-2)$ ways of completing σ to a fixed-point-free involution. So we get the recursion formula $\phi(n) = (n-1)\phi(n-2)$, with $\phi(2) = 1$. This gives that $\phi(n) = (n-1)!!$. ■

Lemma 5.4. *For n even, let $O_1, \dots, O_k$ be a collection of unordered pairs of distinct elements in $\{1, \dots, n\}$ such that $O_i \cap O_j = \emptyset$ for $i \neq j$. The probability that a random element of $\mathcal{F}_n$ contains the orbit O_i for all $1 \leq i \leq k$ is $\frac{(n-2k-1)!!}{(n-1)!!}$.*

Proof. By Lemma 5.3, there are $(n-1)!!$ fixed-point-free involutions in $\mathrm{Sym}(n)$ of which $(n-2k-1)!!$ have O_i as an orbit for every i. ■

The next lemma shows that when n is sufficiently greater than m, there are no triple matchings with high probability.

Lemma 5.5. *A random $\underline{\alpha} \in (\mathcal{F}_n)^m$ has no triple matchings with probability at least $1 - \frac{4m^3}{n}$.*

Proof. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$. Let p denote the probability that $\underline{\alpha}$ has a triple matching. Let $\Omega \subseteq [\![1, m]\!]^3$ be the set of triples (i, j, k) with $i < j < k$. For each $\omega = (i, j, k) \in \Omega$, let $Z_{\omega,l}$ denote the event where

$$\alpha_i(l) = \alpha_j(l) = \alpha_k(l).$$

Each such event has probability $\frac{1}{(n-1)^2}$ of occurring. Let $Z = \bigcup_{\omega \in \Omega} \bigcup_{l=1}^{n} Z_{\omega,l}$, and note that the probability of Z occurring is p. As $|\Omega| \leq m^3$, we deduce via a union bound that

$$p \leq \frac{m^3 n}{(n-1)^2} \leq \frac{4m^3}{n},$$

where we used that $\frac{1}{n-1} \leq \frac{2}{n}$ for $n \geq 2$. ■

6. A-tree local actions

The aim of this section is to prove the following theorem.

Theorem 6.1. *There is a constant C such the following holds. If Γ is a random BMW involution group of degree (m,n) with $n > m^5$, then the A-tree local action of Γ is* $\mathrm{Sym}(m)$ *with probability at least* $1 - \frac{C}{m}$.

In the next proposition, we give conditions on $\underline{\alpha} \in (\mathcal{F}_n)^m$ that ensure the A-tree local action of the induced BMW group $\Gamma_{\underline{\alpha}}$ is $\mathrm{Sym}(m)$. We then show that all these conditions hold with sufficiently high probability.

We say that $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathcal{F}_n)^m$ has *overlapping matches* if there exist distinct pairs $\{i, j\}, \{i', j'\}$, with $i \neq j$ and $i' \neq j'$, such that $\alpha_i(k) = \alpha_j(k)$ and $\alpha_{i'}(k) = \alpha_{j'}(k)$ for some k.

Proposition 6.2. *Suppose that $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathcal{F}_n)^m$ satisfies:*

(A1) *$\underline{\alpha}$ has no triple matchings,*

(A2) *$\underline{\alpha}$ has no overlapping matches, and*

(A3) *for all i, i', there exists j such that α_i and α_j share a common orbit, and α_j and $\alpha_{i'}$ share a common orbit.*

Then the A-tree local action of $\Gamma_{\underline{\alpha}}$ is $\mathrm{Sym}(m)$.

Proof. As $\underline{\alpha}$ has no triple matchings, we let $\Gamma = \Gamma_{\underline{\alpha}}$ and $S = S_{\underline{\alpha}}$ be respectively the associated BMW group and structure set associated to $\underline{\alpha}$. Let $\beta_1, \dots, \beta_n$ be the A-tree local involutions of Γ.

Suppose that $\alpha_i(k) = \alpha_j(k) = l$ for some $1 \leq i < j \leq m$ and $1 \leq k < l \leq n$. We claim that β_k is the transposition $(i\ j)$. By the definition of S, $\{a_i, b_k, a_j, b_l\} \in S$, and it follows from this that $\beta_k(i) = j$. By (A2), for all distinct $i', j' \neq i, j$, we have that $\beta_{i'}(k) \neq \beta_{j'}(k)$. From this it follows that for all $i' \neq i, j$, $\{a_{i'}, b_k, a_{i'}, b_{\alpha_{i'}(k)}\} \in S$. Thus, $\beta(i') = i'$ for all $i' \neq i, j$. We conclude that β_k is indeed the transposition $(i\ j)$, showing the claim.

Let $1 \leq i < i' \leq m$. By (A3), there exists some $j \neq i, i'$ such that α_i and α_j share a common orbit, and α_j and $\alpha_{i'}$ share a common orbit. By the previous paragraph, this implies there exist $1 \leq l, l' \leq n$ such that β_l is the transposition $(i\ j)$ and $\beta_{l'}$ is the transposition $(j\ i')$. Thus $\beta_l \beta_{l'} \beta_l$ is the transposition $(i\ i')$. Consequently, the subgroup generated by $\{\beta_1, \dots, \beta_n\}$ contains all transpositions and so generates $\mathrm{Sym}(m)$. ∎

The remainder of this section is devoted to showing that a random $\underline{\alpha} \in \mathcal{F}_n^m$ satisfies the three conditions from Proposition 6.2 with sufficiently high probability. Condition (A1) was shown to hold in Lemma 5.5. We thus begin with condition (A2).

Lemma 6.3. *A random $\underline{\alpha} \in (\mathcal{F}_n)^m$ has no overlapping matches with probability at least* $1 - \frac{4m^4}{n}$.

Proof. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$. Let p denote the probability that $\underline{\alpha}$ has overlapping matches. Let $\Pi \subseteq [\![1, m]\!]^4$ be the set of quadruples (i, i', j, j') such that $\{i, i'\} \neq \{j, j'\}$, $i \neq i'$ and $j \neq j'$. For each $\pi = (i, i', j, j') \in \Pi$, let $Y_{\pi,k}$ be the event where $\alpha_i(k) = \alpha_{i'}(k)$ and $\alpha_j(k) = \alpha_{j'}(k)$. The probability that each such event occurs is equal to $\frac{1}{(n-1)^2}$. Let $Y = \bigcup_{\pi\in\Pi} \bigcup_{k=1}^{n} Y_{\pi,k}$, and note that the probability that Y occurs is equal to p. Since $|\Pi| \leq m^4$, we deduce via union bound that

$$p \leq \frac{m^4 n}{(n-1)^2} \leq \frac{4m^4}{n}. \qquad \blacksquare$$

Next, we give the probability that two fixed-point-free involutions share a common orbit.

Lemma 6.4. *The probability that two random elements of $\mathcal{F}_n$ share a common orbit is*

$$\sum_{k=1}^{\frac{n}{2}} (-1)^{k+1} \binom{\frac{n}{2}}{k} \frac{(n-2k-1)!!}{(n-1)!!},$$

Moreover, this probability converges to $1 - e^{-\frac{1}{2}}$ as $n \to \infty$.

Proof. Suppose $\alpha, \alpha' \in \mathcal{F}_n$ are chosen uniformly at random. Let $\{O_1, \dots, O_r\}$ be the set of orbits of α, where $r = \frac{n}{2}$. Let $T_i \subseteq \mathcal{F}_n$ be the set of fixed-point-free involutions with orbit O_i, and let Z be the number of fixed-point-free involutions in $\mathcal{F}_n$ with orbit O_i for some $1 \leq i \leq n$. By inclusion-exclusion,

$$Z = \left|\bigcup_{i=1}^{r} T_i\right| = \sum_{k=1}^{r} (-1)^{k+1} \left(\sum_{1\leq n_1 < \cdots < n_k \leq r} |T_{n_1} \cap \cdots \cap T_{n_k}| \right).$$

By Lemma 5.4, $|T_{n_1} \cap \cdots \cap T_{n_k}| = (n-2k-1)!!$ for every $1 \leq n_1 < \cdots < n_k \leq r$. By the above equation, we then have

$$Z = \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} (n-2k-1)!!\,.$$

By Lemma 5.3, we can divide by $(n-1)!!$ to conclude the first claim.

We now prove the convergence claim. Set

$$a_{k,r} = (-1)^{k+1} \binom{r}{k} \frac{(2r-2k-1)!!}{(2r-1)!!} \quad \text{for } k \leq r,$$

and $a_{k,r} = 0$ otherwise. We want to show that $\lim_{r\to\infty} \sum_{k=1}^{r} a_{k,r}$ is equal to $1 - e^{-\frac{1}{2}}$.

We first note that for all $k \le r$,

$$\begin{aligned} a_{k,r} &= (-1)^{k+1}\binom{r}{k}\frac{1}{(2r-1)(2r-3)\cdots(2r-2k+1)} \\ &= \frac{(-1)^{k+1}}{k!}\frac{r(r-1)\cdots(r-k+1)}{(2r-1)(2r-3)\cdots(2r-2k+1)} \\ &= \frac{(-1)^{k+1}}{k!}\prod_{i=0}^{k-1}\frac{r-i}{2r-2i-1}. \end{aligned}$$

Thus $|a_{k,r}| \le \frac{1}{k!}$ for all k and r. Since $\sum_{k=1}^{\infty}\frac{1}{k!} < \infty$, Tannery's theorem implies that

$$\lim_{r\to\infty}\sum_{k=1}^{\infty} a_{k,r}$$

exists and is equal to

$$\sum_{k=1}^{\infty}\big(\lim_{r\to\infty} a_{k,r}\big).$$

Clearly,

$$\lim_{r\to\infty} a_{k,r} = \frac{(-1)^{k+1}}{k!2^k}.$$

Therefore,

$$\lim_{r\to\infty}\sum_{k=1}^{r} a_{k,r} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!2^k} = 1-\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-1}{2}\Big)^k = 1-e^{-\frac{1}{2}}. \qquad \blacksquare$$

Corollary 6.5. *There exists a number N such that whenever $n \ge N$, the probability that two random elements in $\mathcal{F}_n$ share a common orbit is at least $\frac{1}{3}$.*

Remark 6.6. It can be shown using estimates that N in the previous corollary, can be taken to be 2.

Finally, we show that the third property of Proposition 6.2 holds with high probability.

Lemma 6.7. *Let N be as in Corollary 6.5, $n \ge N$ and $\underline{\alpha} \in (\mathcal{F}_n)^m$ be a random element. Then with probability at least $1-2m^2(\frac{8}{9})^m$ the following property holds: for all $1 \le i < i' \le m$ there exists a j such that α_i and α_j share a common orbit, and α_j and $\alpha_{i'}$ share a common orbit.*

Proof. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$. For each $1 \le i < i' \le m$ and $j \ne i, i'$, let $Y_{i,j,i'}$ be the event that both α_i and α_j share a common orbit, and α_j and $\alpha_{i'}$ share a common orbit. Since (up to conjugation) we can treat α_j as fixed, and α_i and $\alpha_{i'}$ as independently randomly chosen, we get that the event that α_i and α_j share an orbit and the event that α_j and $\alpha_{i'}$ share an orbit are independent. By Corollary 6.5 each of them occurs with probability at least $\frac{1}{3}$. Therefore, the probability of the event $Y_{i,j,i'}$ is

$$\mathbb{P}(Y_{i,j,i'}) \geq \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Let $Z_{i,i'}$ be the event that no $Y_{i,j,i'}$ occurs for any $j \notin \{i, i'\}$, and let $Z = \bigcup_{i \neq i'} Z_{i,i'}$. Observe that the property in the lemma's statement holds exactly when the event Z does not occur. Since $Z_{i,i'} = \bigcap_{j \notin \{i,i'\}} Y^c_{i,j,i'}$ and the events $Y_{i,j,i'}$ are independent, we have that

$$\mathbb{P}(Z_{i,i'}) \leq \left(1 - \frac{1}{9}\right)^{m-2} = \left(\frac{8}{9}\right)^{m-2} \leq 2\left(\frac{8}{9}\right)^m,$$

and by a union bound that

$$\mathbb{P}(Z) \leq \sum_{i \neq i'} \mathbb{P}(Z_{i,i'}) \leq 2m^2\left(\frac{8}{9}\right)^m. \qquad \blacksquare$$

Proof of Theorem 6.1. Suppose first that $n \geq N$, where N is as in Corollary 6.5. By Lemmas 5.5, 6.3 and 6.7, and by a union bound, a random $\underline{\alpha} \in \mathcal{F}_n^m$ satisfies the three properties of Proposition 6.2 with probability at least

$$p(m,n) := 1 - \frac{4m^3}{n} - \frac{4m^4}{n} - 2m^2\left(\frac{8}{9}\right)^m.$$

Since $n > m^5$, we can pick some constant C such that $p(m,n) \geq 1 - \frac{C}{m}$ as required. Moreover, by choosing C large enough, we can guarantee that this holds for all n (not just for $n \geq N$). $\blacksquare$

7. *B*-tree local action

In this section we show that the B-tree local action of a random BMW group contains the alternating group $\mathrm{Alt}(n)$ asymptotically almost surely (Corollary 7.2). This follows from a generation result for random fixed-point-free involutions (Theorem 7.1). This theorem should be compared to those of Dixon [8] and Liebeck–Shalev [10] which address generation results for random permutations and random involutions respectively. In fact, our proof closely follows that of Liebeck–Shalev.

Theorem 7.1. *Let $m \geq 3$, $\varepsilon > 0$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$ a random element. Then*

$$\mathbb{P}\big(\mathrm{Alt}(n) \leq \langle \alpha_1, \ldots, \alpha_m \rangle\big) \geq 1 - O(n^{-(m\cdot(1/2-\varepsilon)-1)}).$$

Corollary 7.2. *Let $m = m(n)$ be a function of n satisfying $5 \leq m(n) = O(n^{1/3})$, and let $\underline{\alpha} \in (\mathcal{F}_n)^m$ be a random element. The probability that the B-tree local action of the random BMW group $\Gamma_{\underline{\alpha}}$ contains* $\mathrm{Alt}(n)$ *is* $1 - O(1/n)$.

Proof. By Lemma 5.5, the group $\Gamma_{\underline{\alpha}}$ is well defined with sufficiently high probability. As we saw in Section 5, the B-tree local action is generated by $\alpha_1, \ldots, \alpha_m$. The bound above now follows from the theorem. ∎

To prove Theorem 7.1, we follow the same strategy as that of Liebeck–Shalev [10]. Given a group G, we let $\mathcal{M}_G$ denote the set of maximal, proper subgroups of G. Let $\mathcal{N}_n \subset \mathcal{M}_{\mathrm{Sym}(n)}$ be the set of maximal proper subgroups of $\mathrm{Sym}(n)$ which do not contain $\mathrm{Alt}(n)$. If $\mathrm{Alt}(n) \not\leq \langle \alpha_1, \ldots, \alpha_m \rangle$, then the permutations $\alpha_1, \ldots, \alpha_m$ are contained in some subgroup $M \in \mathcal{N}_n$. It follows by a union bound that

$$\begin{aligned} \mathbb{P}\big(\mathrm{Alt}(n) \not\leq \langle \alpha_1, \ldots, \alpha_m \rangle\big) &\leq \sum_{M \in \mathcal{N}_n} \mathbb{P}(\alpha_1, \ldots, \alpha_m \in M) \\ &= \sum_{M \in \mathcal{N}_n} \mathbb{P}(\alpha \in M)^m, \end{aligned} \tag{2}$$

where $\alpha \in \mathcal{F}_n$ is chosen uniformly at random.

Given a group G, Liebeck–Shalev define the function

$$\zeta_G(s) := \sum_{M \in \mathcal{M}_G} [G : M]^{-s},$$

and show that $\zeta_{\mathrm{Alt}(n)}(s) = O(n^{-(s-1)}) \to 0$ as $n \to \infty$ for all $s > 1$ [10, Theorem 3.1]. A similar proof shows that

$$\sum_{M \in \mathcal{N}_n} [\mathrm{Sym}(n) : M]^{-s} = O(n^{-(s-1)})$$

as $n \to \infty$ for all $s > 1$. By Lemma 7.3 below and (2), we have that

$$\begin{aligned} \mathbb{P}(\mathrm{Alt}(n) \not\leq \langle \alpha_1, \ldots, \alpha_m \rangle) &\leq \sum_{M \in \mathcal{N}_n} \mathbb{P}(\alpha \in M)^m \\ &\leq c^m \sum_{M \in \mathcal{N}_n} [\mathrm{Sym}(n) : M]^{-m(\frac{1}{2} - \varepsilon)} \end{aligned}$$

for some constant c. Theorem 7.1 then follows from the last two equations. Thus, we now turn our attention to proving the following lemma.

Lemma 7.3. *For every $\varepsilon > 0$, there exists a constant c such that given any even integer $n > 0$, any $M \in \mathcal{N}_n$ and a random $\alpha \in \mathcal{F}_n$, then*

$$\mathbb{P}(\alpha \in M) \leq c[\mathrm{Sym}(n) : M]^{-\frac{1}{2} + \varepsilon}.$$

Proof. We follow the same outline as the proof of [10, Theorem 5.1]. We have the following two equations

$$|\operatorname{Sym}(n)| = n! = \exp\Big(n\log n - n + \frac{1}{2}\log(2\pi n) + o(1)\Big),$$
$$|\mathcal{F}_n| = (n-1)!! = \exp\Big(\frac{1}{2}(n\log n - n) + O(1)\Big),$$

each following from Stirling's approximation, where for the second equation we also use the identity

$$|\mathcal{F}_n| = (n-1)!! = \frac{(n)!}{2^{\frac{n}{2}}(n/2)!}.$$

Thus, there exists a constant $c_0 \geq 1$ such that

$$c_0^{-1}|\operatorname{Sym}(n)|^{\frac{1}{2}} \leq |\mathcal{F}_n|\cdot(2\pi n)^{\frac{1}{4}} \leq c_0|\operatorname{Sym}(n)|^{\frac{1}{2}}. \tag{3}$$

Additionally, since $(2\pi n)^{\frac{1}{4}} \ll n! = |\operatorname{Sym}(n)|$, we may also assume that c_0 satisfies

$$c_0^{-1}|\operatorname{Sym}(n)|^{\frac{1}{2}-\frac{\varepsilon}{2}} \leq |\mathcal{F}_n| \leq c_0|\operatorname{Sym}(n)|^{\frac{1}{2}}. \tag{4}$$

By the O'Nan–Scott theorem (cf. [1, Appendix]), the maximal subgroups of $\operatorname{Sym}(n)$ are either primitive, direct products of symmetric groups (not transitive) or wreath products of symmetric groups (transitive and imprimitive).

Case 1: M is primitive. The main theorem of [13] shows that every primitive subgroup $M \in \mathcal{N}_n$ satisfies $|M| \leq 4^n$. Furthermore, there exists a constant c_1 such that

$$4^n \leq c_1|\operatorname{Sym}(n)|^{\frac{\varepsilon}{2}}.$$

By (4),

$$\begin{aligned}\frac{|M\cap\mathcal{F}_n|}{|\mathcal{F}_n|} \leq \frac{|M|}{|\mathcal{F}_n|} &\leq c_0c_1\frac{|\operatorname{Sym}(n)|^{\frac{\varepsilon}{2}}}{|\operatorname{Sym}(n)|^{\frac{1}{2}-\frac{\varepsilon}{2}}}\\ &\leq c_0c_1|\operatorname{Sym}(n)|^{-\frac{1}{2}+\varepsilon} \leq c_0c_1[\operatorname{Sym}(n):M]^{-\frac{1}{2}+\varepsilon}.\end{aligned}$$

Case 2: M is not transitive. In this case, M can be identified with $\operatorname{Sym}(k)\times\operatorname{Sym}(l)$ for some $k, l < n$ such that $k + l = n$. Furthermore, $M\cap\mathcal{F}_n = \mathcal{F}_k\times\mathcal{F}_l$ if both k and l are even, and $M\cap\mathcal{F}_n = 1$ otherwise. By (3), we get that

$$\begin{aligned}\frac{|M\cap\mathcal{F}_n|}{|\mathcal{F}_n|} \leq \frac{|\mathcal{F}_k||\mathcal{F}_l|}{|\mathcal{F}_n|} &\leq c_0^3\Big(\frac{|\operatorname{Sym}(k)||\operatorname{Sym}(l)|}{|\operatorname{Sym}(n)|}\Big)^{\frac{1}{2}}\cdot\Big(\frac{n}{2\pi kl}\Big)^{\frac{1}{4}}\\ &\leq c_0^3\Big(\frac{|M|}{|\operatorname{Sym}(n)|}\Big)^{\frac{1}{2}} \leq c_0^3[\operatorname{Sym}(n):M]^{-\frac{1}{2}},\end{aligned}$$

where the third inequality follows since $n \leq 2\pi kl$.

Case 3: M is transitive and imprimitive. In this case, M can be identified with $\mathrm{Sym}(k) \wr \mathrm{Sym}(l)$ for some $k, l < n$ such that $kl = n$. In wreath product notation, every permutation $\alpha \in M$ can be written as $\alpha = (\pi_1, \dots, \pi_l) \cdot \tau$, where $\pi_i \in \mathrm{Sym}(k)$ and $\tau \in \mathrm{Sym}(l)$. It readily follows that α is a fixed-point-free involution if and only if the following three conditions hold:

(1) Up to relabeling the l-element set that $\mathrm{Sym}(l)$ acts on, τ has the form

$$\tau = (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$$

for some $m \le \frac{l}{2}$,

(2) $\pi_1 = \pi_2^{-1}, \dots, \pi_{2m-1} = \pi_{2m}^{-1}$, and

(3) $\pi_{2m+1}, \dots, \pi_l$ are fixed-point-free involutions in $\mathrm{Sym}(k)$.

Fixing an involution $\tau \in \mathrm{Sym}(l)$ with m transpositions, the number of fixed-point-free involutions $\alpha \in M \cap \mathcal{F}_n$ which can be written as $\alpha = (\pi_1, \dots, \pi_l) \cdot \tau$ is

$$|\mathrm{Sym}(k)|^m |\mathcal{F}_k|^{l-2m} = (k!)^m ((k-1)!!)^{l-2m} \le c_0^{l-2m} (k!)^{\frac{l}{2}} \le c_0^l (k!)^{\frac{l}{2}},$$

where the first inequality follows from (4). We thus get the bound

$$|M \cap \mathcal{F}_n| \le c_0^l (k!)^{\frac{l}{2}} \cdot |\mathcal{I}_l|,$$

where $\mathcal{I}_l$ is the set of all involutions in $\mathrm{Sym}(l)$. It is shown in [7, Theorem 8] (and refined in [12]) that

$$\begin{aligned}|\mathcal{I}_l| &\le \exp\Big(\frac{1}{2} l \log l - \frac{1}{2} l + \sqrt{l} + O(1)\Big) \\ &\le \exp\Big(\frac{1}{2} l \log l - \frac{1}{2} l + \frac{1}{2}\sqrt{2\pi l} + O(1)\Big).\end{aligned}$$

Therefore, by Stirling's approximation, there exists c_2 such that $|\mathcal{I}_l| \le c_2 (l!)^{\frac{1}{2}}$ for all l. From the last two equations we deduce that

$$|M \cap \mathcal{F}_n| \le c_2 c_0^l (k!)^{\frac{l}{2}} (l!)^{\frac{1}{2}} = c_2 c_0^l ((k!)^l l!)^{\frac{1}{2}}.$$

By (3) and as $|M| = (k!)^l l!$, we get

$$\frac{|M \cap \mathcal{F}_n|}{|\mathcal{F}_n|} \le c_0 c_2 c_0^l (2\pi n)^{\frac{1}{4}} \Big(\frac{(k!)^l l!}{n!}\Big)^{\frac{1}{2}} \le c_2 c_0^{l+1} (2\pi n)^{\frac{1}{4}} [\mathrm{Sym}(n) : M]^{-\frac{1}{2}}. \tag{5}$$

Recall that $M = \mathrm{Sym}(k) \wr \mathrm{Sym}(l)$ is a maximal subgroup of $\mathrm{Sym}(n)$ that preserves a partition of n elements into l subsets of size k. Without loss of generality, let the n-element set be $\{1, \dots, k\} \times \{1, \dots, l\}$ and suppose that M preserves the partition $\bigsqcup_{j=1}^{l} \{1, \dots, k\} \times \{j\}$.

Consider the subgroup H of $\mathrm{Sym}(n)$ that stabilizes the set $\{i\} \times \{1, \dots, l\}$ for each $1 \leq i \leq k$, and which fixes pointwise the set $\{k\} \times \{1, \dots, l\}$. Then H is a copy of $(\mathrm{Sym}(l))^{k-1}$ which satisfies $H \cap M = 1$. Thus,

$$[\mathrm{Sym}(n) : M] \geq |H| = (l!)^{k-1}.$$

Since $l!$ is super-exponential, there exists c_3 so that $c_0^l \leq c_3 \cdot (l!)^{\frac{\varepsilon}{2}}$. We get that

$$c_0^l \leq c_3 \cdot (l!)^{\frac{\varepsilon}{2}} \leq c_3[\mathrm{Sym}(n) : M]^{\frac{\varepsilon}{2}}, \tag{6}$$

where the second inequality follows as $[\mathrm{Sym}(n) : M] \geq (l!)^{k-1} \geq l!$. As $l! \geq \frac{2^l}{2}$, as $n = kl$ and as $l \leq \frac{n}{2}$, we also get that

$$[\mathrm{Sym}(n) : M] \geq (l!)^{k-1} \geq \frac{2^{l(k-1)}}{2} \geq \frac{2^{\frac{n}{2}}}{2}.$$

Moreover, there exists a constant c_4 such that $(2\pi n)^{\frac{1}{4}} \leq \frac{c_4}{2}(2^{\frac{n}{2}})^{\frac{\varepsilon}{2}}$. We then get that

$$(2\pi n)^{\frac{1}{4}} \leq \frac{c_4}{2}(2^{\frac{n}{2}})^{\frac{\varepsilon}{2}} \leq c_4[\mathrm{Sym}(n) : M]^{\frac{\varepsilon}{2}}. \tag{7}$$

By (5), (6) and (7), it follows that

$$\frac{|M \cap \mathcal{F}_n|}{|\mathcal{F}_n|} \leq c_0c_2c_3c_4[\mathrm{Sym}(n) : M]^{-\frac{1}{2}+\varepsilon}.$$

Setting $c = \max\{c_0c_1, c_0^3, c_0c_2c_3c_4\}$ completes the proof. ∎

8. Irreducibility of random BMWs

The aim of this section is to complete the proof of Theorem B. More precisely, we prove the following theorem.

Theorem 8.1. *There is a constant C such that the following holds. If Γ is a random BMW involution group of degree (m, n) with $n > m^5$, then all of the following hold with probability at least $1 - \frac{C}{m}$:*

(1) *the A-tree local action of Γ is* $\mathrm{Sym}(m)$*;*

(2) *the B-tree local action of Γ is either* $\mathrm{Sym}(n)$ *or* $\mathrm{Alt}(n)$*;*

(3) *Γ is irreducible;*

(4) *Γ is hereditarily just-infinite.*

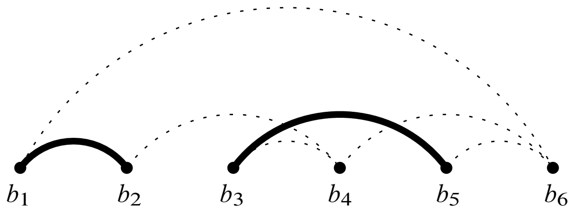

Figure 4. The graph $\mathcal{G}_{\underline{\alpha}}$ for the permutations in Example 5.1. Bold edges represent 'black' edges and dotted edges represent 'white' edges.

Conclusions (1) and (2) follow directly from Theorem 6.1 and Corollary 7.2 respectively. Moreover, conclusion (4) follows from conclusions (1)–(3) and Theorem 2.8. Thus, we are left to prove conclusion (3) regarding the irreducibility of Γ. By a theorem of Caprace [6, Theorem 1.2 (vi)], conclusion (3) is implied by conclusions (1) and (2) as long as

$$n \notin \left\{\frac{m!}{2}-1,\ \frac{m!}{2},\ m!-1,\ \frac{m!(m-1)!}{4}-1,\ \frac{m!(m-1)!}{4},\ \frac{m!(m-1)!}{2}-1,\ \frac{m!(m-1)!}{2},\ m!(m-1)!-1\right\}.$$

Thus, in order to finish the proof Theorem 8.1, it is enough to show that (3) holds whenever n is one of the values above. To do so, we actually show conclusion (3) holds whenever $n > m^8$ (covering the above finite cases) by using the theorem of Trofimov–Weiss stated below.

Suppose that Γ is a group acting vertex-transitively on a locally finite connected graph X. We do not assume the action is faithful. Given a vertex $v \in X$, let Γ_v be its stabilizer, and let $\Gamma_v^{[i]}$ be the pointwise stabilizer of the set of all vertices distance i or less from v. Recall that the *local action* of Γ is the subgroup of $\mathrm{Sym}(m)$ induced by the action of Γ_v on the edges adjacent to v. The following is a consequence of a theorem of Trofimov–Weiss, as reformulated by Caprace [18, Section 4.5].

Theorem 8.2 ([19, Theorem 1.4]). *Suppose that a group Γ acts vertex transitively on a connected locally finite graph X with 2-transitive local action. If $\Gamma_v^{[6]} \not\leq \Gamma_v^{[7]}$ for some $v \in V(X)$, then the image of the action $\Gamma \to \mathrm{Aut}(X)$ is not discrete.*

Recall that a BMW group $\Gamma \leq \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ acts on T_m by projecting to the first factor. We prove Theorem 8.1 (3) by showing that the hypotheses of Theorem 8.2 are satisfied with sufficiently high probability for a random BMW group. We do this by investigating the following graph.

Definition 8.3. Given $\underline{\alpha} \in (\mathcal{F}_n)^m$, define the following simplicial graph $\mathcal{G}_{\underline{\alpha}}$ whose edges are colored black and white such that:

- the vertex set of $\mathcal{G}_{\underline{\alpha}}$ is $B_n = \{b_1, \ldots, b_n\}$;
- vertices b_i and b_j are joined by an edge if there is some $1 \leq k \leq m$ such that $\alpha_k(i) = j$. This edge is black if there exist distinct $k \neq k'$ such that $\alpha_k(i) = \alpha_{k'}(i) = j$, and is white otherwise.

An example of the above graph is shown in Figure 4.

Lemma 8.4. *Suppose that* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$ *satisfies:*

(Irr1) $\underline{\alpha}$ *has no triple matchings;*

(Irr2) *there exists some* $b \in B_n$ *such that all edges in the closed ball* $N_6(b)$ *of* $\mathcal{G}_{\underline{\alpha}}$ *are white;*

(Irr3) $\mathcal{G}_{\underline{\alpha}}$ *is connected;*

(Irr4) $\mathcal{G}_{\underline{\alpha}}$ *contains a black edge;*

(Irr5) *the* A*-tree local action of* $\Gamma_{\underline{\alpha}}$ *is 2-transitive.*

Then, the BMW group $\Gamma_{\underline{\alpha}}$ *is irreducible.*

Proof. Recall from Section 2.1 that the action of the involutive BMW group $\Gamma = \Gamma_{\underline{\alpha}}$ on $T_m \times T_n$ preserves a labeling of edges of $T_m \times T_n$, where horizontal edges are labeled by elements of A_m and vertical edges by B_n. Moreover, recall that the 1-skeleton of $T_m \times T_n$ can be identified with the Cayley graph of Γ. Let $o = (o_A, o_B) \in T_m \times T_n$ be the vertex corresponding to the identity. By the definition of $\Gamma_{\underline{\alpha}}$ and as $\underline{\alpha}$ has no triple matching, if a_i, b_k, a_j, b_l are the labels of the edges of a square in $T_m \times T_n$, then

$$\alpha_i(k) = \alpha_j(k) = l.$$

Let $\pi_A\colon T_m \times T_n \to T_m$ be the projection map.

By a slight abuse of notation, we identify each $b \in B_n$ with the element Γ that interchanges the endpoints of the edge incident to o and labeled by b. We determine the action of b on T_m as follows. Let L be a path in $T_m \times \{o_B\}$ starting at o. Let $a_{i_1}, \ldots, a_{i_r} \in A_m$ be the labels of consecutive edges of L. Now let R be the unique $1 \times r$ rectangle in $T_m \times T_n$ whose bottom left vertex is o, whose left edge is labeled by b and whose top edge is the path bL. Such a rectangle is shown in Figure 5, with $a_{i'_j}$ and b_{k_j} as indicated in Figure 5.

Let L' be the path in $T_m \times \{o_B\}$ corresponding to the bottom of the rectangle R, i.e., L' is the unique edge path starting at o and whose edges have labels $a_{i'_1}, \ldots, a_{i'_r}$. The paths bL and L' have the same projection to T_m. Thus b fixes the projection $\pi_A(L)$ of L onto T_m pointwise if and only if $a_{i_j} = a_{i'_j}$ for all $1 \leq j \leq r$.

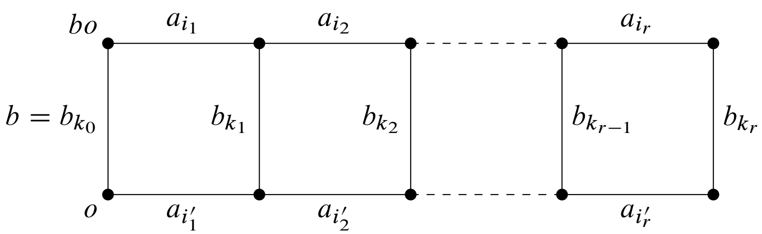

Figure 5. Determining the action of $b \in B_n$ on the A-tree T_m.

By the definition of the graph $\mathcal{G}_{\underline{\alpha}}$, there is a path ℓ in $\mathcal{G}_{\underline{\alpha}}$ traversing in order the vertices $b_{k_0}, \dots, b_{k_r}$. Moreover, the edge joining $b_{k_{j-1}}$ and b_{k_j} is white if and only if $a_{i_j} = a_{i'_j}$. We thus see that b fixes $\pi_A(L)$ if and only if all edges of ℓ are white.

By (Irr2), (Irr3) and (Irr4), there exists some $b \in B_n$ such that all edges in the closed ball $N_6(b)$ are white, but there is a black edge in the closed ball $N_7(b)$. Fix such a $b \in B_n$. It follows that b fixes all paths of length 6 starting at o_A, i.e., $b \in \Gamma_v^{[6]}$. Moreover, there is some path traversing, in order, the vertices $b = b_{k_0}, b_{k_1}, \dots, b_{k_7}$ in $N_6(b)$ such that the edge (b_{k_6}, b_{k_7}) is black. Since, for each j, $(b_{k_{j-1}}, b_{k_j})$ is an edge of $\mathcal{G}_{\underline{\alpha}}$, there is some $a_{i_j} \in A_m$ such that $\alpha_{i_j}(k_{j-1}) = k_j$. Consider the path $\pi_A(L)$ of T_m starting at o_A whose edges are sequentially labeled by $a_{i_1}, \dots, a_{i_7}$. Since the edge (b_{k_6}, b_{k_7}) is black, b cannot fix $\pi_A(L)$, hence $b \notin \Gamma_v^{[7]}$.

Since the A-tree local action is 2-transitive and $\Gamma_v^{[6]} \not\leq \Gamma_v^{[7]}$, it follows from Theorem 8.2 that the projection of $\Gamma_{\underline{\sigma}} \leq \mathrm{Aut}(T_m) \times \mathrm{Aut}(T_n)$ to $\mathrm{Aut}(T_m)$ is not discrete. Therefore, $\Gamma_{\underline{\sigma}}$ is irreducible by [4, Proposition 1.2]. ∎

To prove Theorem 8.1, we need to show that conditions (Irr1)–(Irr5) are satisfied with high probability. This has already been established for (Irr1), (Irr3) and (Irr5) in Lemma 5.5, Corollary 7.2 and Theorem 7.1 respectively. Hence, all that remains is to show that conditions (Irr2) and (Irr4) hold with high probability. To do so, we investigate the following random variable.

Definition 8.5. Let $m, n \in \mathbb{N}$ with n even. Given $\alpha \in \mathcal{F}_n$, define

$$P_\alpha := \{\{i, \alpha(i)\} \mid 1 \leq i \leq n\}.$$

For random $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in (\mathcal{F}_n)^m$, define the random variable

$$M_{n,m}(\underline{\alpha}) := \sum_{i<j} |P_{\alpha_i} \cap P_{\alpha_j}|.$$

Remark 8.6. The number of black edges in $\mathcal{G}_{\underline{\alpha}}$ is at most $M_{n,m}(\underline{\alpha})$, with equality precisely when $\underline{\alpha}$ has no triple matchings. In particular, $\mathcal{G}_{\underline{\alpha}}$ has no black edges if and only if $M_{n,m}(\underline{\alpha}) = 0$.

The following lemma demonstrates that $\mathcal{G}_{\underline{\alpha}}$ has at least one black edge with sufficiently high probability.

Lemma 8.7. *There exists a constant N such that if given any $m, n \in \mathbb{N}$ with n even and $n \geq N$ and a random $\underline{\alpha} \in (\mathcal{F}_n)^m$, then the probability that $\mathcal{G}_{\underline{\alpha}}$ has no black edge is at most $(\frac{2}{3})^{m-1}$.*

Proof. For random $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathcal{F}_n)^m$, define the random variable

$$Y_{n,m}(\underline{\alpha}) = \sum_{i=2}^{m} |P_{\alpha_1} \cap P_{\alpha_i}|.$$

It follows from the definition of $M_{n,m}$ that, for each $\underline{\alpha} \in (\mathcal{F}_n)^m$, we have

$$0 \leq Y_{n,m}(\underline{\alpha}) \leq M_{n,m}(\underline{\alpha}).$$

Therefore, $\mathbb{P}(M_{n,m} = 0) \leq \mathbb{P}(Y_{n,m} = 0)$.

By Corollary 6.5 there is a constant N such that for $n \geq N$, two random elements of $\mathcal{F}_n$ do not have a common orbit with probability at most $2/3$, i.e., $\mathbb{P}(M_{n,2} = 0) \leq \frac{2}{3}$. Since $Y_{n,m}$ is the sum of $m-1$ non-negative independent identically distributed random variables with the same distribution as $M_{n,2}$, we have for $n \geq N$ that

$$\mathbb{P}(M_{n,m} = 0) \leq \mathbb{P}(Y_{n,m} = 0) \leq \left(\frac{2}{3}\right)^{m-1}.$$

The result now follows from Remark 8.6. ■

We now show the following proposition.

Proposition 8.8. *Let $m, n \in \mathbb{N}$ with $n > m^8$ and n even. Then with probability at least $1 - \frac{1}{m}$, for some $b \in B_n$, the closed ball $N_6(b) \subset \mathcal{G}_{\underline{\alpha}}$ only contains white edges.*

We first prove some lemmas that are used in the proof of Proposition 8.8.

Lemma 8.9. *For any even n, we have $\mathbb{E}(M_{n,2}) = \frac{n}{2(n-1)} \leq 1$.*

Proof. Let $(\alpha, \alpha') \in (\mathcal{F}_n)^2$ be a random element. As in Lemma 6.4, we may assume α is fixed and α' is chosen at random. Let $C_1, \ldots, C_r$ be the orbits of α, with $r = \frac{n}{2}$, and let Y_i be the indicator random variable associated to the event that C_i is an orbit of α'. Then $M_{n,2} = \sum_{i=1}^{r} Y_i$, and $\mathbb{P}(Y_i = 1) = \frac{1}{n-1}$ by Lemma 5.4. Therefore, by the linearity of expectation,

$$\mathbb{E}(M_{n,2}) = \frac{n}{2(n-1)} \leq 1,$$

as required. ■

Lemma 8.10. *For any $A > 0$, we have $\mathbb{P}(M_{n,m} \geq A) \leq \frac{m^2}{2A}$.*

Proof. Observe that $M_{n,m}$ is a sum of $\frac{m(m-1)}{2}$ random variables, each having the same probability distribution as $M_{n,2}$. By Lemma 8.9 and linearity of expectation, we see that

$$\mathbb{E}(M_{n,m}) \leq \frac{m(m-1)}{2} \leq \frac{m^2}{2}.$$

The result now follows by applying Markov's inequality. ∎

Proof of Proposition 8.8. By Lemma 8.10, $\mathbb{P}(M_{n,m} \geq \frac{m^3}{2}) \leq \frac{1}{m}$. To prove Proposition 8.8, it thus suffices to show that if $M_{n,m}(\underline{\alpha}) < \frac{m^3}{2}$, then all edges in the closed ball $N_6(b)$ are white for some vertex b. Indeed, if $M_{n,m}(\underline{\alpha}) < \frac{m^3}{2}$, then by Remark 8.6, $\mathcal{G}_{\underline{\alpha}}$ contains fewer than $\frac{m^3}{2}$ black edges. Since vertices of $\mathcal{G}_{\underline{\alpha}}$ have valence at most m, any edge of $\mathcal{G}_{\underline{\alpha}}$ has at most $2m^5$ vertices a distance 5 or less from it. Thus there are at most

$$2m^5 \times \frac{m^3}{2} = m^8$$

vertices that are at a distance of 5 or less from the endpoint of a black edge. Hence, as $m^8 < n$, the closed ball $N_6(b)$ contains no black edge for some vertex b. ∎

Proof of Theorem 8.1. As noted in the paragraph after Theorem 8.1, we may assume $n > m^8$. Lemma 5.5, Proposition 8.8, Corollary 7.2, Lemma 8.7 and Theorem 6.1 each give upper bounds for the probability than one of the conditions (Irr1)–(Irr5) in Lemma 8.4 do not hold. Taking a union bound, we see that there is a constant N such that if $n \geq \max(m^8, N)$, the probability that at least one of (Irr1)–(Irr5) is not satisfied is at most

$$\frac{4m^3}{n} + \frac{1}{m} + \frac{C_1}{m} + \left(\frac{2}{3}\right)^{m-1} + \frac{C_2}{m}$$

for some constants C_1 and C_2. Therefore for some sufficiently large constant C, we deduce that for all even $n > m^8$ conditions (Irr1)–(Irr5) of Lemma 8.4 are satisfied with probability at least $1 - \frac{C}{m}$. Lemma 8.4 ensures that the group $\Gamma_{\underline{\alpha}}$ satisfies conclusion (3) of Theorem 8.1 with probability at least $1 - \frac{C}{m}$. Theorem 8.1 now follows from the discussion immediately after its statement. ∎

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