Lagrangian cobordisms and Lagrangian surgery

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Abstract. Lagrangian k-surgery modifies an immersed Lagrangian submanifold by topological k-surgery while removing a self-intersection. Associated to a k-surgery is a Lagrangian surgery trace cobordism. We prove that every Lagrangian cobordism is exactly homotopic to a concatenation of suspension cobordisms and Lagrangian surgery traces. This exact homotopy can be chosen with as small Hofer norm as desired. Furthermore, we show that each Lagrangian surgery trace bounds a holomorphic teardrop pairing the Morse cochain associated with the handle attachment to the Floer cochain generated by the self-intersection. We give a sample computation for how these decompositions can be used to algorithmically construct bounding cochains for Lagrangian submanifolds. In an appendix, we describe a 2-ended embedded monotone Lagrangian cobordism which is not the suspension of a Hamiltonian isotopy following a suggestion of Abouzaid and Auroux.

## 1. Introduction

1.1. Overview: Lagrangian cobordisms

A Lagrangian cobordism $K: L^+ \rightsquigarrow L^-$ is a Lagrangian submanifold $K \subset X \times \mathbb{C}$ whose projection to $\mathbb{C}$ fibers over the real axis outside of a compact set. This determines a set of "ends" for the Lagrangian cobordism given by Lagrangians $L^+, L^- \subset X$ (see Definition 2.1.1).

These Lagrangian submanifolds were first discussed by Arnold in [3]. Lagrangian cobordisms give an equivalence relation on Lagrangian submanifolds which is coarser than Hamiltonian isotopy (in the sense that if L^+, L^- are Hamiltonian isotopic, they are also Lagrangian cobordant, with K having topology $L^+ \times \mathbb{R}$).

The largest collection of constructions for Lagrangian cobordisms come from [15], which introduced a k-surgery operation on Lagrangian submanifolds and an associated k-trace cobordism for $0 \leq k \leq n - 1$, where $n = \dim(X)/2$. When $k = 0$, this is the Polterovich surgery introduced in [24]. Crucially, Haug provides a geometric
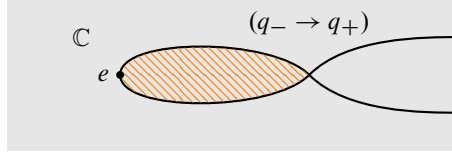
Figure 1. An immersed null cobordism $K^{0,1} \subset \mathbb{C}^0 \times \mathbb{C}$ of the zero-dimensional Whitney sphere $S^0 \subset \mathbb{C}^0$. The patterned region represents a holomorphic teardrop pairing Morse cochain with self-intersection.

meaning to the surgery model described by [4], which writes down a k-trace cobordism for $0 \leq k \leq n$. When $k = n$, the cobordism constructed in [4] is a null-cobordism for the Whitney sphere. For the simplest example, $k = n = 0$, the Whitney sphere is an immersion

$$i^{0,0,+} \colon S^0 = \{q^+, q^-\} \to \mathbb{C}^0 = \{\text{pt}\},$$

and the Polterovich surgery trace/Whitney null-cobordism is the curve drawn in Figure 1.

These surgery traces have some nice properties:

- As manifolds, the ends L^-, L^+ of a Lagrangian surgery trace cobordism

$$K^{k,n-k+1} \colon L^+ \leadsto L^-$$

  differ by topological surgery. The "height" on the Lagrangian trace cobordism given by projection to the real coordinate $\pi_{\mathbb{R}} \colon K^{k,n-k+1} \to \mathbb{R}$ has a single critical point of index $n - k$.

- As an embedding, L^- has one fewer self-intersection than L^+. If $K^{k,n-k+1}$ is graded, one Floer generator associated with this self-intersection lives in degree $n - k - 1$.

1.2. Overview: Floer cohomology and surgery

The Floer complex of an embedded Lagrangian submanifold $CF^\bullet(L)$ is a deformation of the cochain group $CM^\bullet(L)$ by incorporating the counts of holomorphic disks with boundary on L into the differential and product structures of $CM^\bullet(L)$. The underlying cochain group $CM^\bullet(L)$ can be singular cochains, Morse cochains, or

differential forms. The deformation enhances $CF^\bullet(L)$ with a filtered A_∞ structure counting configurations of holomorphic polygons with boundary on L. The algebra structure of $CF^\bullet(L)$ contains an abundance of data about the Lagrangian L. Most importantly, when $CF^\bullet(L)$ is a tautologically unobstructed or weakly unobstructed A_∞ algebra, it has homology groups $HF^\bullet(L)$ which are invariant under Hamiltonian isotopy. Similarly, whenever L^+, L^- embedded monotone Lagrangian cobordant, Biran and Cornea [7] show that $CF^\bullet(L^+)$ and $CF^\bullet(L^-)$ are homotopy equivalent.

For monotone Lagrangian cobordisms with multiple ends, there is a known relation between Polterovich surgery, Lagrangian cobordisms, and exact sequences of Floer cohomology. By using a neck stretching argument to compare holomorphic polygons on the summands of the connect sum to holomorphic polygons with boundary on the surgery, [14, Chapter 10] shows that when Lagrangians $L^+ = L^0 \cup L^1$ and L^- are related by Polterovich surgery that L^- arises as a mapping cone on $L^0 \to L^1$ in the Fukaya category. Because there exists a surgery trace cobordism between L^+ and L^-, we can obtain the same result from [7] without appealing to neck-stretching. Related work [21, 27] shows that a surgery exact triangle in the Fukaya category arises from the Lagrangian Polterovich surgery trace. In the immersed setting, Palmer and Woodward [23] showed that Polterovich surgery leaves Floer cohomology invariant upon incorporating a bounding cochain.

1.3. Overview: Monotone Lagrangian cobordisms

Without placing additional restrictions on the kinds of Lagrangian cobordisms we consider, the equivalence relation given by Lagrangian cobordisms is far too flexible: for example, whenever L^+ and L^- are Lagrangian isotopic, they are embedded Lagrangian cobordant (although the Lagrangian cobordism may be non-orientable). One way to re-impose some rigidity on our Lagrangian cobordisms is to ask that they have well-defined Floer cohomology. In particular,

- if we require K to be exact and embedded with $\dim(K) > 6$, then K has the topology of $L \times \mathbb{R}$ for some $L \subset X$ ([26]); or

- by asking our Lagrangian cobordisms to be monotone and embedded we learn that the ends L^+, L^- are equivalent in the Fukaya category ([7]).

These conditions are so rigid as to make it difficult to find any Lagrangian cobordisms at all! In the first setting, the only known examples of Lagrangian cobordisms are those arising from Hamiltonian isotopy; in the second case, we provide (to our knowledge) the first known example of a 2-ended monotone Lagrangian cobordism which is not a suspension in Appendix A. Previous work of the author ([16]) shows that embedded Lagrangian cobordisms which are unobstructed by bounding cochain

have enough flexibility to make their construction feasible, yet enough rigidity to provide meaningful Floer theoretic results.

1.4. Outline and results

This paper contributes two observations to the theory of Lagrangian cobordisms.

- The first (Section 3) is that Lagrangian cobordisms are flexible enough that they admit a handle decomposition into standard pieces given by k-surgeries. This result only uses standard techniques in symplectic geometry and does not contain any Floer-theoretic computations.

- The second provides rigidity. In Section 4 we relate the geometry of Lagrangian cobordisms to Floer theory. For each standard k-surgery, we exhibit a holomorphic teardrop with boundary on our Lagrangian cobordism, which pairs the surgery handles of the cobordism with its self-intersections (Figure 1). Obstructions arising from the holomorphic teardrop can possibly occur in the local model for 0-surgery, and unobstructedness of this curvature term provides a rigidity criterion for Lagrangian cobordism.

If a 0-surgery is the Lagrangian connect sum at q of two Lagrangians L_0, L_1 which intersect at several points, then the bounding cochain making the surgery trace unobstructed restricts to a bounding cochain on the immersed Lagrangian submanifold $L_0 \cup L_1$. This Lagrangian equipped with bounding cochain is isomorphic to the mapping cone at the intersection point; thus the holomorphic teardrops which appear on the Lagrangian surgery trace give a construction of the [14, Chapter 10] exact surgery triangle. We also provide numerous examples of Lagrangian cobordisms.

- Section 5 contains some examples where we use the Floer cohomology of surgery traces to compute obstructedness/unobstructedness of Lagrangian cobordisms.

- In Appendix A we construct a 2-ended monotone Lagrangian cobordism which is not given by a Hamiltonian isotopy. The construction comes from a proposal due to Abouzaid and Auroux to construct a neither "wide-nor-narrow" monotone Lagrangian submanifold, which we also include.

We now give a more detailed outline of the paper.

In Section 2, we review known constructions of Lagrangian cobordisms and Floer theory of Lagrangian cobordisms. We focus on Biran and Cornea's theorem that "monotone Lagrangian cobordisms provide equivalences in the Fukaya category", and explore the limitations that monotonicity places on this theorem. We give a simple example of an oriented obstructed Lagrangian submanifold whose ends are non-isomorphic in the Fukaya category.

In Section 3.1, we provide some standard tools for decomposing Lagrangian co-
bordisms. In short, when decomposing a Lagrangian cobordism $K \subset X \times \mathbb{C}$, we can
consider decompositions that have boundaries fibering over the X or $\mathbb{C}$ coordinate. In
Proposition 3.1.6 we show that Lagrangian cobordisms are decomposable along the
$\mathbb{C}$-coordinate, and in Proposition 3.1.10 we give a method for decomposing Lagrang-
ian cobordisms along the X-coordinate.

These decompositions are used in Section 3.2 to construct the standard surgery
handle. We review the parameterization of the Whitney sphere and the Lagrangian
null-cobordism for the Whitney sphere in Section 3.2.1. The parameterization is com-
pared to the Lagrangian surgery handle from [4]. In addition to the parameterization
of the surgery handle, Figures 9–13 provide plots of these handles as projections to
the $\mathbb{C}$ coordinate and as multisections of the $T^*\mathbb{R}^n$; we hope that these examples pro-
vide the reader with intuition on the construction and geometry of surgery handles.
We use this particular parameterization of the surgery handle in Section 3.3, where
we prove the main result of this section.

Theorem (Restatement of Theorem 3.3.1). *Let $K : L^+ \rightsquigarrow L^-$ be a Lagrangian cobor-
dism. Then K is exactly homotopic to the concatenation of surgery trace cobordisms
and suspensions of exact homotopies. Furthermore, the Hofer norm of this exact
homotopy can be made as small as desired.*

The proof of Theorem 3.3.1 shows Lagrangian cobordism can be placed into
a good position by an exact homotopy. We additionally comment on the relation
between Lagrangian surgery and anti-surgery. We acknowledge that exact homotopy
is a high price to pay in order to place a Lagrangian cobordism in standard position.
However, it is the strongest equivalence relation we can hope to place as the stan-
dard form must be immersed. We conjecture that unobstructedness of a Lagrangian
submanifold K is preserved by exact isotopies whose Hofer norm is smaller than the
valuation of the bounding cochain on L. In particular, if K is embedded and tau-
tologically unobstructed, the conjecture states that the standard decomposition (with
transverse intersections, given by Theorem 4.2.12) is unobstructed.

Section 4 investigates the relation between self-intersections of Lagrangian sub-
manifolds and topology of the Lagrangian surgery trace. In Section 4.1, we show
that the ends of a surgery trace cobordism can be equipped with a Morse function
so that the critical points of the negative end are in index-preserving bijection with
the self-intersections of the positive end. It follows that whenever $K : L^- \rightsquigarrow L^+$ is
embedded and graded that $\chi(L^-) = \chi(L^+)$. The remainder of the section extends this
relation to Floer cohomology. Given a two-ended *monotone* Lagrangian cobordism
$K : L^+ \rightsquigarrow L^-$ and any other monotone Lagrangian L', the authors of [7] construct a

homotopy equivalence

$$CF^\bullet(L^+, L') \to CF^\bullet(L^-, L'),$$

from which the above equality of Euler characteristics follows (Theorem 2.3.1). As a result, L^+ and L^- are quasi-isomorphic objects in the Fukaya category. We call this isomorphism the "continuation map" associated to a Lagrangian cobordism K. We adopt the name continuation map from the homotopy equivalence on Floer cohomology groups induced by a Hamiltonian isotopy (which is also called the continuation map).

Section 4.2 lays the groundwork by defining Lagrangian cobordisms with double bottlenecks, which allow us to discuss Lagrangian cobordisms whose ends are immersed. We have a version of Theorem 3.3.1 where the pieces in the decomposition are Lagrangian cobordisms with double bottlenecks. When

$$(K, t^-, t^+) \colon (L^+, H^+) \to (L^-, H^-)$$

is a Lagrangian cobordism with double bottlenecks, we prove that for appropriate definition of Floer cochains,

$$\chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+) = \chi^{\mathrm{bot}}(K, t^-, t^+).$$

We then provide a short review of immersed Floer cohomology in Section 4.3. In Section 4.4, we conjecture that the homotopy equivalence from Theorem 2.3.1 can be extended to the non-monotone immersed setting via a pairing of Floer cochains witnessed by a holomorphic teardrop.

Theorem (Restatement of Theorem 4.4.1). *The standard Lagrangian surgery trace bounds a holomorphic teardrop pairing the critical point of the surgery trace with the self-intersection in Floer cohomology.*

Section 5 discusses when we can and cannot find an extension of Theorem 2.3.1 to a (non-monotone) Lagrangian cobordism. In Section 5.1, we use the pairing from Theorem 4.4.1 to justify why the example Lagrangian cobordism given in Figure 5 does not construct a continuation map in the Fukaya category. Section 5.2 applies this conjectural framework to a computation yielding a continuation map associated with a Lagrangian surgery trace cobordism in specific examples (see Figures 2 and 27). In these examples, the holomorphic teardrop contributes to a curvature term $m^0 \colon \Lambda \to CF^\bullet(K^{A,B})$ in Floer cohomology. The existence of a bounding cochain or obstruction of $CF^\bullet(K^{A,B})$ either yields or precludes the construction of a continuation map on Floer cohomology between L_{E_b} and $S_E^1 \cup S_{-E}^1$. In the case of Figure 2, the Lagrangians L_{E_b} and $S_E^1 \cup S_{-E}^1$ are disjoint, so the result is obvious; however
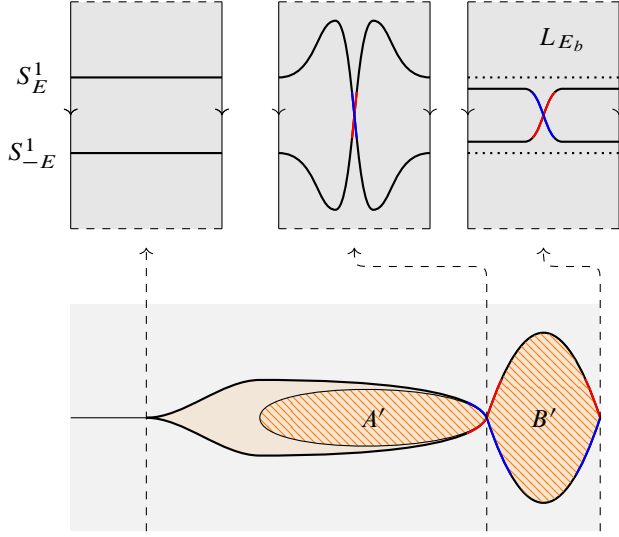
Figure 2. A Lagrangian cobordism corresponding to a Lagrangian surgery which does not yield a continuation map in the Fukaya category. The Lagrangian cobordism is obstructed.

the comparison to the setting of Figure 27 which only differs in terms of the areas A and B is illustrative.

Finally, Appendix A has some constructions of monotone Lagrangian submanifolds. In Section A.1 we complete a proposal of Abouzaid–Auroux to construct a neither narrow-nor-wide Lagrangian submanifold. The ideas of this construction are employed in Section A.2 to construct an embedded oriented 1-ended monotone Lagrangian cobordism. We then use this cobordism, along with a surgery trace, to construct a 2-ended monotone Lagrangian cobordism in Section A.3.

2. Background

Notation

Let X always denote a symplectic manifold of dimension $2n$. There will be many Lagrangian submanifolds of varying dimensions in this article. The dimension of a submanifold will be determined by reverse alphabetical order, so

$$\dim(J) - 2 = \dim(K) - 1 = \dim(L) = n = \dim(M) + 1.$$

In this paper, we will frequently take local coordinates for a Lagrangian submanifold $U \subset L \subset X$, and identify the Weinstein neighborhood with a neighborhood

of $U \subset \mathbb{R}^n \subset \mathbb{C}^n$. We will denote the coordinates near U by $(q_i + {}_J p_i)$. Unless otherwise stated, all Lagrangian submanifold considered are possibly immersed.

2.1. Lagrangian homotopy and cobordism

A homotopy of Lagrangian submanifolds is a smooth map $i_t \colon L \times \mathbb{R} \to X$ with the property that at each $t_0 \in \mathbb{R}$, $i_{t_0} \colon L \to X$ is an immersed Lagrangian submanifold. For each $t_0 \in \mathbb{R}$, a homotopy of Lagrangian submanifolds yields a closed cohomology class $\mathrm{Flux}_{t_0}(i_t) \in H^1(L, \mathbb{R})$, called the flux class of i_t at t_0. The value of the flux class on chains $c \in C_1(L, \mathbb{R})$ is defined by

$$\mathrm{Flux}_{t_0}(i_t)(c) := \int_{i_t : c \times [0, t_0] \to X} \omega.$$

If $\mathrm{Flux}_{t_0}(i_t)$ is exact for all $t_0 \in \mathbb{R}$, we say that this homotopy is an exact homotopy. In the case that i_t is an exact *isotopy*, there exists a time dependent Hamiltonian $H_t \colon X \times \mathbb{R} \to \mathbb{R}$ with the following properties:

- $H_t|_L$ is a primitive for the flux class in the sense that $dH_{t_0}|_L = \mathrm{Flux}_{t_0}(i_t)$.

- The isotopy is generated by the Hamiltonian flow $\phi_t \colon X \times \mathbb{R} \to X$ in the sense that

$$i_t(L) = \phi_t(i_0(L)).$$

Even when i_t is only a *homotopy* we will denote by $H_t \colon L \times \mathbb{R} \to X$ the primitive to the flux class. The *Hofer norm* of such an isotopy is defined to be

$$\int_{\mathbb{R}} \left(\sup_{q \in L} H_t(q) - \inf_{q \in L} H_t(q) \right) dt,$$

which does not depend on the choice of primitive $H_t(q)$. Lagrangian cobordisms are an extension of the equivalence relation of exact homotopy.

Definition 2.1.1 (Arnol'd [3]). Let L^+, L^- be (possibly immersed) Lagrangian submanifolds of X. A *2-ended Lagrangian cobordism* with ends L^+, L^- is a (possibly immersed) Lagrangian submanifold $K \subset (X \times \mathbb{C}, \omega_X + \omega_{\mathbb{C}})$ for which there exists a compact subset $D \subset \mathbb{C}$ so that

$$K \setminus (\pi_{\mathbb{C}}^{-1}(D)) = (L^+ \times \mathbb{R}_{>t^+}) \cup (L^- \times \mathbb{R}_{<t^-}).$$

The sets $\mathbb{R}_{>t^+}$ and $\mathbb{R}_{<t^-}$ are rays pointing along the negative and positive real axis of $\mathbb{C}$, starting at some values $t_- < 0 < t_+$. We denote such a cobordism $K \colon L^+ \rightsquigarrow L^-$.

There is a more general theory of k-ended Lagrangian cobordisms, although for simplicity of notation we only discuss the 2-ended case, and always use "Lagrangian cobordism" to mean "2-ended Lagrangian cobordism". All of the decomposition results of this paper extend to the k-ended setting.
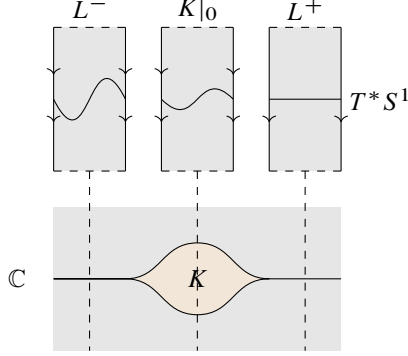
Figure 3. The shadow projection of a Lagrangian cobordism $K \subset T^*S^1 \times \mathbb{C}$ to the $\mathbb{C}$ factor. This Lagrangian cobordism is the suspension of a Hamiltonian isotopy.

The projection to the $\mathbb{C}$ component of a Lagrangian cobordism is called the *shadow projection* of the Lagrangian cobordism; the infimal area of a simply connected region containing the image of the projection is called the *shadow* of a Lagrangian cobordism [10]. We will denote this quantity by $\mathrm{Area}(K)$. See Figure 3 for a diagram of a Lagrangian cobordism. We call the $\mathbb{R}$-component of $X \times \mathbb{C}$ the *cobordism parameter*, and the projection to this coordinate will be denoted by $\pi_{\mathbb{R}} \colon X \times \mathbb{C} \to \mathbb{R}$. Given a Lagrangian cobordism K, we will abuse notation and use $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ to denote the cobordism parameter restricted to K.

Claim 2.1.2. *Let $t_0 \in \mathbb{R}$ be a regular value of the projection $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$. The slice of K at t_0,*

$$K|_{t_0} := \pi_X(\pi_{\mathbb{R}}^{-1}(t_0) \cap K)$$

is a (possibly immersed) Lagrangian submanifold of X.

Lagrangian cobordance extends the equivalence relation of exactly homotopic. We call a Lagrangian cobordism *a suspension* if $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ has no critical points.

Proposition 2.1.3 (Audin–Lalonde–Polterovich [4]). *Let $L^+, L^- \subset X$ be two Lagrangian submanifolds. Then L^+ and L^- are exactly homotopic if and only if there exists a suspension Lagrangian cobordism $K \colon L^+ \rightsquigarrow L^-$ between these two Lagrangians.*

Given an exact homotopy $i_t \colon L \times \mathbb{R} \to X$ whose primitive $H_t \colon L \to \mathbb{R}$ has compact support, there is a *suspension cobordism of i_t* parameterized by

$$L \times \mathbb{R} \to X \times \mathbb{C},$$
$$(q, t) \mapsto (i_t(q), t + {}_J H_t(q)) \in X \times \mathbb{C}.$$

The Hofer norm of i_t is equal to the shadow of the suspension cobordism.

For the purpose of providing some geometric grounding to our discussions, we give an example of a Lagrangian cobordism which is not an exact homotopy. We first note that every compact Lagrangian submanifold $K \subset X \times \mathbb{C}$ gives an example of a Lagrangian cobordism $K \colon \emptyset \rightsquigarrow \emptyset$. While these Lagrangian cobordisms are not very interesting from a Floer-theoretic perspective (as they can be displaced from themselves), they are useful for understanding the kinds of geometry which can appear in a Lagrangian cobordism.

Example 2.1.4 (Sheared product torus). Consider $\mathbb{C} \times \mathbb{C}$ with coordinates given by (q_1, p_1, q_2, p_2). The product Lagrangian torus L_{T^2} is the submanifold parameterized by

$$(\theta_1, \theta_2) \mapsto (\cos(\theta_1) + J \sin(\theta_2), \cos(\theta_1) + J \sin(\theta_2)) \subset \mathbb{C} \times \mathbb{C}.$$

We apply a linear symplectic transformation

$$\phi \colon \mathbb{C}^2 \to \mathbb{C}^2,$$
$$(q_1, p_1, q_2, p_2) \mapsto \left(q_1 + \frac{1}{2} q_2 + J \frac{4}{3} p_1 - \frac{2}{3} p_2, q_2 + \frac{1}{2} q_1 + J \frac{4}{3} p_2 - \frac{2}{3} p_1 \right),$$

so that $K := \phi(L_{T^2})$ is in general position. After taking this shear, $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ is a Morse function with four critical values corresponding to the standard maximum, saddles, and minimum on the torus. Several slices and the shadow of the Lagrangian cobordism are drawn in Figure 4.

2.2. Previous work: Anti-surgery

Lagrangian anti-surgery, introduced by Haug [15], gives a method for embedding the Lagrangian cobordism handle from [4]. Given a Lagrangian $L \subset X$, an *isotropic anti-surgery disk* for L is an embedded isotropic disk $i \colon D^{k+1} \to X$ with the following properties:

- *Clean intersection*. The boundary of D^{k+1} is contained in L. Additionally, the interior of D^{k+1} is disjoint from L, and the outward pointing vector field to D is transverse to L.

- *Trivial normal bundle*. Over D^{k+1}, we can write a splitting

$$(TD^{k+1})^\omega = TD^{k+1} \oplus E.$$

Furthermore, we ask that there is a symplectic trivialization

$$D^{k+1} \times \mathbb{C}^{n-k-1} \to E$$

so that over the boundary, $E|_{\partial D^{k+1}}$ is contained in $TL|_{\partial D^{k+1}}$.
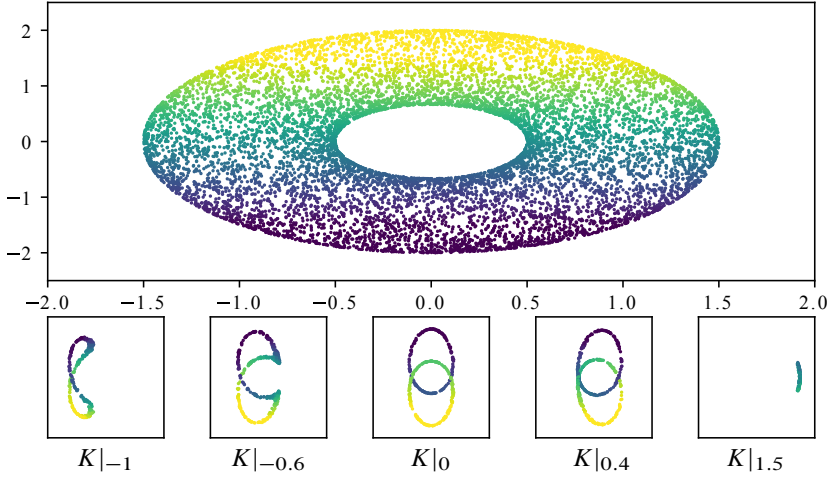
Figure 4. Scatter plots of randomly selected points lying on the sheared product torus $K = \phi(L_{T^2})$. The color corresponds to the value of $\pi_{J\mathbb{R}}: K \to \mathbb{R}$. Top: The shadow projection of the Lagrangian submanifold K. Bottom: Several slices of $K = \phi(L_{T^2})$ at different real values. The critical values of the cobordism parameter $\pi_{\mathbb{R}}: K \to \mathbb{R}$ are at ± 1.5 and $\pm .5$. We use this opportunity to highlight a common misconception: although all slices $K|_t$ are immersed, the Lagrangian K is embedded.

Given an isotropic anti-surgery disk D^{k+1} with boundary on L, [15] produces a Lagrangian $\alpha_{D^{k+1}}(L)$, the anti-surgery of L along D^{k+1}, along with a Lagrangian anti-surgery trace cobordism

$$K_{\alpha_{D^{k+1}}}: L \rightsquigarrow \alpha_{D^{k+1}}(L).$$

As a manifold, $\alpha_{D^{k+1}}(L)$ differs from L by k-surgery along ∂D^{k+1}, and the cobordism parameter $\pi_{\mathbb{R}}: K_{\alpha_{D^{k+1}}} \to \mathbb{R}$ provides a Morse function with a single critical point of index $k + 1$. A k-surgery is a modification of a manifold L which replaces a subset of the form $S^k \times D^{n-k}$ with $D^{k+1} \times S^{n-k-1}$. When compared to L, the anti-surgery $\alpha_{D^{k+1}}(L)$ possesses a single additional self-intersection q_{D^n}. The construction is inspired by an analogous construction for Legendrian submanifolds in [11]. The terminology "anti-surgery" is based on the following observation: given a *Lagrangian* anti-surgery disk D^n for L, the Polterovich surgery [24] of $\alpha_{D^n}(L)$ at the newly created self-intersection point q_{D^n} is Lagrangian isotopic to L. In this sense, anti-surgery and surgery are inverse operations on Lagrangian submanifolds. Accordingly, if L arises from L' by anti-surgery along a disk D^{k+1}, Haug states that L arises from L' by Lagrangian $n - k - 1$ surgery.

These surgeries and anti-surgeries appear in Figure 4, which decompose the product torus into slices related by the creation/deletion of Whitney spheres, surgeries, and anti-surgeries. Some higher-dimensional examples of anti-surgery are in Figures 13 and 12, which draws Lagrangians related by anti-surgery in the cotangent bundle of $\mathbb{R}^2$. In these figures, we have highlighted the isotropic anti-surgery disk corresponding modifications in red; the Lagrangians on the right-hand side all exhibit a single self-intersection at the origin.

2.3. Floer theoretic properties of Lagrangian cobordisms

Our motivation for studying Lagrangian cobordisms comes from their Floer theoretic properties. A fundamental result states that cobordant Lagrangians have homotopic Floer theory.

Theorem 2.3.1 (Biran–Cornea [7]). *Suppose that $K\colon L^+ \rightsquigarrow L^-$ is a monotone embedded Lagrangian submanifold. Let $L' \subset X$ be a monotone test Lagrangian submanifold. Then the chain complexes $CF^\bullet(L^+, L')$ and $CF^\bullet(L^-, L')$ are chain homotopic.*

More generally, Biran and Cornea prove that a Lagrangian cobordism with k-inputs $\{L_i^+\}_{i=1}^k$ and output L^- yields a factorization of L^- into an iterated mapping cone of the L_i^+. In the setting of two-ended monotone Lagrangian cobordisms, applications of Theorem 2.3.1 are limited by lack of examples. In fact, [26] shows that under the stronger condition that K is exact, every embedded exact Lagrangian cobordism $K\colon L^0 \rightsquigarrow L^1$ of $\dim(K) \geq 6$ has the *topology* of $L^0 \times \mathbb{R}$. It is still currently unknown if all such Lagrangian cobordisms are Hamiltonian isotopic to suspensions of Hamiltonian isotopies.

It is expected that Theorem 2.3.1 should extend to more general settings than monotone Lagrangian submanifolds. One of the broader extensions is to the class of *unobstructed immersed Lagrangian cobordisms*. Roughly, unobstructed Lagrangians are those whose counts of holomorphic disks can be made to cancel in cohomology (see the discussion following Definition 4.3.3). The Floer theoretic property of unobstructedness is absolutely necessary to obtain a continuation map in the Fukaya category. We given an example of a Lagrangian cobordism which cannot give a continuation map in the Fukaya category.

Example 2.3.2. Let $S_E^1 \subset T^*S^1$ be the section $\frac{E}{2\pi}\,d\theta$ which bounds an annulus of area E with the zero section. Pick $E_1 \neq E_2$ two positive real numbers. Consider the Lagrangian submanifold which is the disjoint union of two circles

$$S_{E_1}^1 \sqcup S_{-E_1}^1 \subset T^*S^1,$$
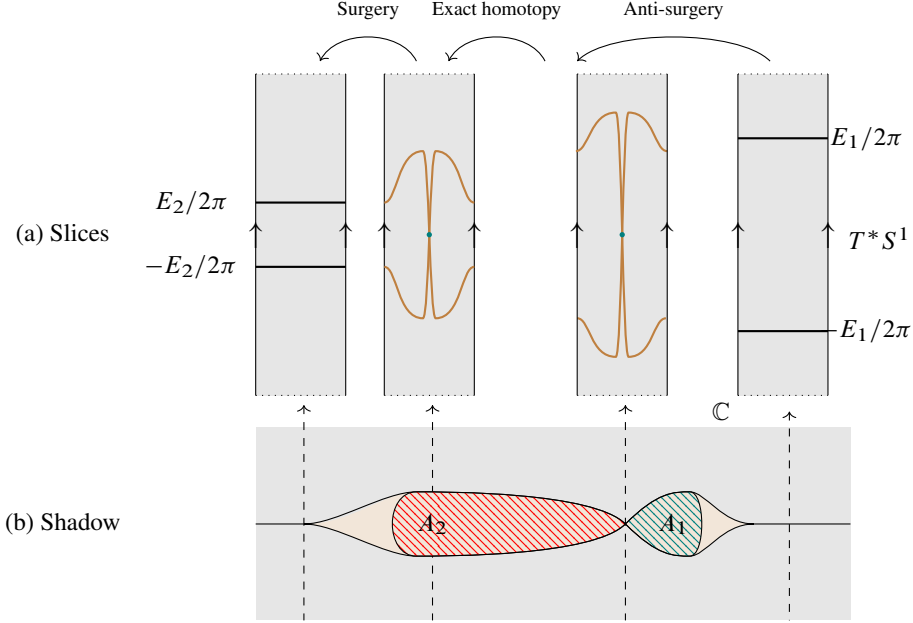
as drawn in Figure 5.

Figure 5. An oriented immersed Lagrangian cobordism. If $A_1 \neq A_2$, then K is an obstructed Lagrangian submanifold. The slashed regions correspond to the images of holomorphic teardrops with boundary on K under the projection $\pi_{\mathbb{C}}$. The boundary of these teardrops obstruct the solution to the Maurer–Cartan equation for K.

By applying anti-surgery along the interval $\{0\} \times [-E_1/2\pi, E_1/2\pi]$, we obtain a Lagrangian cobordism to a Lagrangian double section of T^*S^1 intersecting the zero section at $(0, 0)$. We subsequently apply Lagrangian surgery at this self-intersection to obtain $S^1_{E_2} \sqcup S^1_{-E_2}$. Consider the Lagrangian cobordism built from concatenating the anti-surgery and surgery trace cobordism. This is an immersed Lagrangian cobordism which can be perturbed by Hamiltonian isotopy to make the self-intersections transverse. Let $K \subset T^*S^1 \times \mathbb{C}$ be this Lagrangian cobordism, which has a single self-intersection.

The statement of Theorem 2.3.1 cannot be extended to a class of Lagrangian cobordism which contains K. In this setting, $E_1 \neq E_2$, and therefore the Lagrangians $S^1_{E_2} \sqcup S^1_{-E_2}$ and $S^1_{E_1} \sqcup S^1_{-E_1}$ are disjoint. Since $S^1_{E_2} \sqcup S^1_{-E_2}$ and $S^1_{E_1} \sqcup S^1_{-E_1}$ are non-isomorphic objects of the Fukaya category, the Lagrangian cobordism K cannot hope to yield a continuation map.

We propose that the proof of Theorem 2.3.1 fails because the Lagrangian K is an obstructed Lagrangian submanifold, that is, the Floer differential on K does not square to zero due to the possibility of disk/teardrop bubbling (see discussion around

Definition 4.3.3). The Lagrangian cobordism K bounds two holomorphic teardrops of area A_1 and A_2, whose projections under $\pi_{\mathbb{C}}$ are drawn in Figure 5. When these teardrops have differing area, they collectively contribute a non-trivial m^0 curvature term to the Floer cohomology $CF^\bullet(K)$, which cannot be canceled by a bounding cochain.

This example demonstrates that understanding when Lagrangians are unobstructed is essential for building meaningful continuation maps from Lagrangian cobordisms. We will return to Example 2.3.2 in Section 5.1, where we examine the setting where $A_1 = A_2$ and the Lagrangian cobordism K is unobstructed. In previous work, the author [16] showed that bounding cochains for Lagrangian cobordisms could be used to compute wall-crossing transformations for Lagrangian mutations.

3. Lagrangian cobordisms are Lagrangian surgeries

In this section we prove that every Lagrangian cobordism can be decomposed into a composition of Lagrangian surgery traces and exact homotopy suspensions. Section 3.1 gives some constructions for decomposing Lagrangian cobordisms. In Section 3.2 we describe the standard Lagrangian surgery handle. Finally, in Section 3.3 we show that a Lagrangian cobordism can be exactly homotoped to good position, and subsequently decomposed into surgery traces.

3.1. Decompositions of Lagrangian cobordisms

We consider two types of decompositions for Lagrangian cobordisms $K \subset X \times \mathbb{C}$: across the cobordism parameter $\mathbb{C}$ in Section 3.1.4 and across the X-coordinate in Section 3.1.5.

3.1.1. Gluing across the $\mathbb{C}$ parameter: Concatenation. Given Lagrangian cobordisms

$$K^{+0}\colon L^+ \rightsquigarrow L^0, \quad K^{0-}\colon L^0 \rightsquigarrow L^-$$

there exists a concatenation cobordism $K^{0-} \circ K^{+0}\colon L^+ \rightsquigarrow L^-$. The exact homotopy class of the concatenation does not depend on the length of cylindrical component connecting the negative end of K^{+0} to the positive end of K^{0-}. The concatenation operation shows that Lagrangian cobordance is an equivalence relation on the set of Lagrangian submanifolds. In the setting of differentiable manifolds, concatenation can be used to provide a decomposition of any cobordism into a sequence of standard surgery handles.

3.1.2. Gluing across the X parameter: Lagrangian cobordisms with cylindrical boundary.

When describing surgery and trace cobordisms, it is also important to consider Lagrangian cobordisms with cylindrical boundary. We call this decomposition along the X-coordinate.

Definition 3.1.1. Let $L^+ \subset X$ be a Lagrangian submanifold with boundary M. Let $M \times [0, \varepsilon) \subset X$ be a collared neighborhood of the boundary. A Lagrangian cobordism with cylindrical boundary $K: L^+ \rightsquigarrow L^-$ is a Lagrangian submanifold $K \subset X \times \mathbb{C}$ whose boundary has a collared neighborhood of the form $(M \times [0, \varepsilon)) \times \mathbb{R} \subset X \times \mathbb{C}$.

We will use Lagrangians cobordisms with cylindrical boundary to describe local modifications to Lagrangian submanifolds. Let $L^+ = L_\downarrow^+ \cup_M L_\uparrow^+$ be a decomposition of a Lagrangian submanifolds along a surface $M = \partial L_\uparrow^+$. Given $K_\uparrow: L_\uparrow^+ \to L_\uparrow^-$ a Lagrangian cobordism with cylindrical boundary M, we can obtain a Lagrangian cobordism

$$K_\uparrow \cup_{M \times \mathbb{R}} (L_\downarrow^+ \times \mathbb{R}): L^+ \rightsquigarrow (L_\uparrow^- \cup_M L_\downarrow^+).$$

In this case, we say that the Lagrangian $L^- := L_\uparrow^- \cup_M L_\downarrow^+$ arises from modification of L^+ at the set $L_\uparrow^+$.

Definition 3.1.2. We say that $K: L^+ \rightsquigarrow L^-$ decomposes across the X-coordinate along $M \subset L^+$ if there exist Lagrangian cobordisms with cylindrical boundary M,

$$K_\uparrow: L_\uparrow^+ \rightsquigarrow L_\uparrow^-, \quad K_\downarrow: L_\downarrow^+ \rightsquigarrow L_\downarrow^-,$$

so that $L^\pm = L_\uparrow^\pm \cup_M L_\downarrow^\pm \subset X$ is a Lagrangian submanifold and

$$K = K_\uparrow \cup_{M \times \mathbb{R}} K_\downarrow.$$

Proposition 3.1.3. *Suppose that K decomposes across the X-coordinate so we may write $K = K_\uparrow \cup_{M \times \mathbb{R}} K_\downarrow$. Then K is exactly homotopic to either of the following compositions of Lagrangian cobordisms:*

$$(K_\uparrow \cup_{M \times \mathbb{R}} (L_\downarrow^- \times \mathbb{R})) \circ ((L_\uparrow^+ \times \mathbb{R}) \cup_{M \times \mathbb{R}} K_\downarrow),$$

$$((L_\uparrow^- \times \mathbb{R}) \cup_{M \times \mathbb{R}} K_\downarrow) \circ (K_\uparrow \cup_{M \times \mathbb{R}} (L_\downarrow^+ \times \mathbb{R})).$$

Proof. Observe that the Hamiltonian $\pi_{J\mathbb{R}}: X \times \mathbb{C} \to \mathbb{R}$ restricts to the constant zero function on the collar boundary $(M \times [0, \varepsilon)) \times \mathbb{R}$ of $K_\uparrow$. The flow of the associated Hamiltonian vector field on $X \times \mathbb{C}$ is leftward translation of the cobordism parameter. Similarly, the Hamiltonian $-\pi_{J\mathbb{R}}: X \times \mathbb{C} \to \mathbb{R}$ restricts to the constant zero function on the collar boundary of $K_\downarrow$; its associated Hamiltonian flow is rightward translation. Let $\rho: (-\varepsilon, \varepsilon) \to (-1, 1)$ be a smooth increasing function which is constantly ± 1 in

a neighborhood of $\pm\varepsilon$. Let $j\colon K \to X \times \mathbb{C}$ parameterize our Lagrangian cobordism. The Lagrangian isotopy

$$j_t\colon K \times \mathbb{R} \to X \times \mathbb{C},$$

$$q \mapsto \begin{cases} (\pi_X \circ j(q), \pi_\mathbb{C} \circ j(q) + t) & \text{if } q \in K_\uparrow \setminus (M \times [0, \varepsilon)) \times \mathbb{R}, \\ (\pi_X \circ j(q), \pi_\mathbb{C} \circ j(q) - t) & \text{if } q \in K_\downarrow \setminus (M \times (\varepsilon, 0]) \times \mathbb{R}, \\ (\pi_X \circ j(q), \pi_\mathbb{C} \circ j(q) + \rho \cdot t) & \text{if } q \in (M \times (-\varepsilon, \varepsilon)) \times \mathbb{R}, \end{cases}$$

is an exact homotopy with primitive $\pi_{J\mathbb{R}}$ on $K_\uparrow \setminus ((M \times [0, \varepsilon)) \times \mathbb{R})$ and $-\pi_{J\mathbb{R}}$ on $K_\downarrow \setminus (\times \mathbb{R})$. The homotopy fixes the image of $(M \times \varepsilon) \times \mathbb{R}$. ∎

3.1.3. Some general tools for Lagrangian cobordisms. As we will work with immersed Lagrangian cobordisms, we need the following replacement of Weinstein neighborhoods.

Definition 3.1.4. Let X and Y be symplectic manifolds. A *local symplectic embedding* is a map $\phi\colon Y \to X$ so that around every $y \in Y$ there exists a neighborhood U on which $\phi|_U\colon U \to X$ is a symplectic embedding. Let $i\colon L \to X$ be an immersed Lagrangian submanifold. A *local Weinstein neighborhood* is a map from a neighborhood of the zero section in the cotangent bundle of L to X

$$\phi\colon B_\varepsilon^* L \to X,$$

which is a local symplectic embedding, and whose restriction to the zero section makes the diagram

$$\begin{array}{ccc} L & \xrightarrow{\ i\ } & X \\ {\scriptstyle 0}\downarrow & \nearrow {\scriptstyle \phi} & \\ B_\varepsilon^* L & & \end{array}$$

commute.

For both decomposition along the X coordinate and cobordism parameter, we will use the following generalization of Proposition 2.1.3.

Lemma 3.1.5 (Generalized suspension). *Let I be a finite indexing set. Suppose that we are given for each $\alpha \in I$ an exact Lagrangian homotopy $i_t^\alpha\colon L^\alpha \times \mathbb{R} \to X^\alpha$, whose flux primitive is $H_t^\alpha\colon L^\alpha \times \mathbb{R} \to \mathbb{R}$. Let Y be another manifold, and pick functions $\rho^\alpha\colon Y \to \mathbb{R}$. For fixed $q^\alpha \in L^\alpha$, let $(y, d\rho^\alpha(y))\colon Y \to T^*Y$ be the parameterization of the exact Lagrangian section whose primitive is*

$$Y \to \mathbb{R},$$
$$y \mapsto H_{\rho^\alpha(y)}^\alpha(q^\alpha).$$

Then

$$j_{\rho^I}:\left(\prod_{\alpha\in I} L^\alpha\right)\times Y \to \left(\prod_{\alpha\in I} X^\alpha\right)\times T^*Y,$$

$$(q^\alpha, y) \mapsto \left(i^\alpha_{\rho^\alpha(y)}(q^\alpha), y, \sum_{\alpha\in I} H^\alpha_{\rho^\alpha(y)}(q^\alpha)d\rho^\alpha(y)\right)$$

parameterizes a Lagrangian submanifold.

We note that the usual suspension construction is recovered by taking $I = \{1\}$, $Y = \mathbb{R}$, and $\rho(y) = t$.

Proof. For convenience, write j for j_{ρ^I}. Pick local coordinates $y_1 \ldots y_n$ for Y, so that we may locally identify T^*Y with $\mathbb{C}^n$ and write sections of the cotangent bundle as

$$Y \to T^*Y,$$

$$y_i \mapsto \left(y_i + J\sum_{\alpha\in I} H^\alpha_{\rho^\alpha(y)}(q^\alpha)\frac{\partial\rho^\alpha}{\partial y_i}\right).$$

Let $\frac{d}{dt}i^\alpha_t \in TX^\alpha$ be the vector field along the image of i^α_t associated to the isotopy i^α_t. Let (q^α_j) denote local coordinate on the L^α. Since i^α_t is a exact Lagrangian homotopy with flux primitive given by H^α_t, we have that

$$\iota_{\frac{d}{dt}i^\alpha_t}\omega_{X^\alpha}|_{L^\alpha} = \sum_i \partial_{q_j} H^\alpha_t dq^\alpha_j.$$

Let $\partial_{v^\alpha} \in TL^\alpha$, and $\partial_{y_i} \in TY$ be vectors. We compute $j_*(\partial_{y_j})$ and $j_*(\sum_{\alpha\in I} c_\alpha\partial_{v^\alpha})$:

$$j_*(\partial_{y_j}) = \left(\frac{\partial\rho^\alpha}{\partial y_j}\frac{d}{dt}i^\alpha_{\rho^\alpha(y)}, \delta_{ij} + J\sum_{\alpha\in I}\left(\frac{\partial^2\rho^\alpha}{\partial y_i\partial y_j}H_{\rho^\alpha(y)} + \frac{\partial\rho^\alpha}{\partial y_i}\frac{\partial\rho^\alpha}{\partial y_j}\frac{d}{dt}H^\alpha_{\rho^\alpha(y)}\right)\right),$$

$$j_*\left(\sum_{\alpha\in I} c_\alpha\partial_{v^\alpha}\right) = \left(\sum_{\alpha\in I} c_\alpha(i_{\rho(s)})_*\partial_{v^\alpha}, J\sum_{\alpha\in I} c_\alpha\frac{\partial\rho^\alpha}{\partial y_i}\partial^\alpha_v H^\alpha_{\rho^\alpha(y)}\right).$$

We compute the vanishing of the symplectic form on pairs of vectors

$$\sum_{\alpha\in I} b_\alpha\partial_{v^\alpha}, \sum_{\beta\in I} c_\beta\partial_{w^\beta} \in \prod_{\alpha\in I} TL^\alpha.$$

This term vanishes as TL^α is a Lagrangian subspace of ω^α_X:

$$\omega\left(j_*\left(\sum_{\alpha\in I} b_\alpha\partial_{v^\alpha}\right), j_*\left(\sum_{\beta\in I} c_\beta\partial_{w^\beta}\right)\right) = \sum_{\alpha=\beta\in I} b_\alpha c_\beta\omega_{X^\alpha}\left((i_{\rho(s)})_*\partial_{v^\alpha}, (i_{\rho(s)})_*\partial_{w^\beta}\right).$$

The vanishing of the symplectic form on pairs of vectors $j_*(\partial_{y_i})$, $j_*(\sum_{\alpha \in I} c_\alpha \partial_{v^\alpha})$ comes from the assumption that the Lagrangian homotopies L_t^α have flux primitive given by H_t^α:

$$\omega\left(j_*(\partial_{y_i}), j_*\left(\sum_{\alpha \in I} c_\alpha \partial_{v^\alpha}\right)\right) = \sum_{\alpha \in I} c_\alpha\left(\omega_{X^\alpha}\left(\frac{\partial\rho^\alpha}{\partial y_j}\frac{d}{dt}i^\alpha_{\rho^\alpha(y)}, (i_{\rho(s)})_*\partial_{v^\alpha}\right)\right)$$

$$+ \sum_{\alpha \in I} c_\alpha\,\omega_{T^*Y}\left(\delta_{ij}, \frac{\partial\rho^\alpha}{\partial y_j}\partial_v^\alpha H^\alpha_{\rho^\alpha(y)}\right)$$

$$= \sum_{\alpha \in I} c_\alpha\left(\frac{\partial\rho^\alpha}{\partial y_i}\left(\iota_{\frac{d}{dt}i^\alpha_{\rho^\alpha(y)}}\omega_{X^\alpha} + dH^\alpha_{\rho^\alpha(y)}\right)\partial_{v^\alpha}\right) = 0.$$

The vanishing of the symplectic form on pairs of vectors $j_*(\partial_{y_i})$, $j_*(\partial_{y_j})$ corresponds to the fact closed sections of the cotangent bundle are Lagrangian sections.

$$\omega\left(j_*(\partial_{y_i}), j_*(\partial_{y_j})\right) = \sum_{\alpha \in I}\omega_{X^\alpha}\left(\frac{\partial\rho^\alpha}{\partial y_i}\frac{d}{dt}i^\alpha_{\rho^\alpha(y)}, \frac{\partial\rho^\alpha}{\partial y_j}\frac{d}{dt}i^\alpha_{\rho^\alpha(y)}\right)$$

$$+ \omega_Y\left(\delta_{ik} + J\sum_{\alpha \in I}\left(\frac{\partial^2\rho}{\partial y_i\,\partial y_k}H_{\rho(y)} + \frac{\partial\rho^\alpha}{\partial y_i}\frac{\partial\rho^\alpha}{\partial y_k}\frac{d}{dt}H^\alpha_{\rho^\alpha(y)}\right),\right.$$

$$\left.\delta_{jk} + J\sum_{\alpha \in I}\left(\frac{\partial^2\rho}{\partial y_j\,\partial y_k}H_{\rho(y)} + \frac{\partial\rho^\alpha}{\partial y_j}\frac{\partial\rho^\alpha}{\partial y_k}\frac{d}{dt}H^\alpha_{\rho^\alpha(y)}\right)\right)$$

$$= 0. \qquad\qquad\blacksquare$$

3.1.4. Decomposition across the cobordism parameter. Cobordisms can be decomposed into smaller cobordisms along any regular level set of Morse function. We show an analogous decomposition for Lagrangian cobordisms.

Proposition 3.1.6. *Let* $K\colon L^+ \rightsquigarrow L^-$ *be a Lagrangian cobordism with cylindrical boundary* M. *Suppose that* $\pi_{\mathbb{R}}\colon K \to \mathbb{R}$ *has isolated critical values, and that* $0 \in \mathbb{R}$ *is a regular value of the projection* $\pi_{\mathbb{R}}\colon K \to \mathbb{C}$, *so that* $L^0 := K|_0 \subset X$ *is a Lagrangian submanifold. Then there exists Lagrangian cobordisms with cylindrical boundary* M

$$K\|_{(-\infty,0]}\colon L^0 \rightsquigarrow L^-, \quad K\|_{[0,\infty)}\colon L^+ \rightsquigarrow L^0,$$

so that K *and* $K\|_{(-\infty,0]} \circ K\|_{[0,\infty)}$ *are exactly homotopic Lagrangian cobordisms. Furthermore, the construction can be done in such a way that*

- *the Hofer norm of this exact homotopy is as small as desired and,*
- *if* K *is embedded, and the slice* L^0 *is embedded, then* $K\|_{(-\infty,0]} \circ K\|_{[0,\infty)}$ *is embedded as well.*

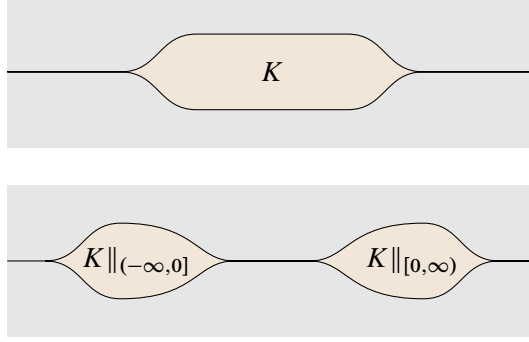A picture of this decomposition is given in Figure 6.

Figure 6. Cobordisms can be decomposed across the cobordism parameter at non-critical values of $\pi_{\mathbb{R}} \circ j \colon K \to \mathbb{R}$.

Proof. We construct a suspension Lagrangian cobordism $J \subset (X \times \mathbb{C}) \times \mathbb{C}$ with ends

$$J \colon K\|_{(-\infty,0]} \circ K\|_{[0,\infty)} \rightsquigarrow K,$$

which will be the suspension of our exact homotopy. Consider the decomposition as sets $K = K|_{t \leq -\varepsilon} \cup_{t=-\varepsilon} K|_{[-\varepsilon,\varepsilon]} \cup_{t=\varepsilon} K|_{[\varepsilon,\infty)}$, where

$$K|_{t \leq -\varepsilon} = \pi_{\mathbb{R}}^{-1}((-\infty,-\varepsilon]), \quad K|_{[-\varepsilon,\varepsilon]} = \pi_{\mathbb{R}}^{-1}([-\varepsilon,\varepsilon]), \quad K|_{[\varepsilon,\infty)} = \pi_{\mathbb{R}}^{-1}([\varepsilon,\infty)).$$

The piece $K|_{[-\varepsilon,\varepsilon]}$ is a suspension, so by Proposition 2.1.3, there exists a primitive $H_t \colon L^0 \times [-\varepsilon,\varepsilon] \to \mathbb{R}$, so that we can parameterize $K|_{[-\varepsilon,\varepsilon]}$ as

$$L^0 \times [-\varepsilon,\varepsilon] \to X \times \mathbb{C},$$
$$(q,t) \mapsto (i_t(q), t + H_t(q)).$$

We choose a truncation profile $\rho(t,s) \colon [-\varepsilon,\varepsilon] \times \mathbb{R} \to [-\varepsilon,\varepsilon]$ which satisfies the following conditions (as indicated in Figure 7):

$$\rho(t,s)|_{s<0} = t, \quad \rho(t,s)|_{|t|>2\varepsilon/3} = t,$$
$$\rho(t,1)|_{|t|<\varepsilon/3} = 0, \qquad \left.\frac{\partial \rho}{\partial s}\right|_{s>1} = 0.$$

Consider the Lagrangian submanifold $J|_{[-\varepsilon,\varepsilon] \times \mathbb{R}}$ given by the generalized suspension from Lemma 3.1.5,

$$j_\rho \colon L^0 \times [-\varepsilon,\varepsilon] \times \mathbb{R} \to X \times T^*[-\varepsilon,\varepsilon] \times T^*\mathbb{R}.$$

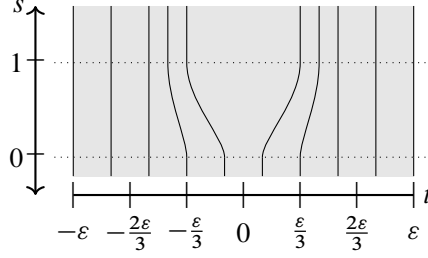This is a Lagrangian cobordism over the s parameter with collared boundaries in both the t and s directions:

Figure 7. Contour plot of the truncation profile $\rho(t, s)$.

- In the s direction, $K|_{|t|=\varepsilon}$ is a boundary for $K|_{[-\varepsilon,\varepsilon]} := J|_{[-\varepsilon,\varepsilon]\times\{s=0\}}$. This extends to a collared boundary $K|_{|t|>2\varepsilon/3}$, and $K|_{|t|>2\varepsilon/3} \times \mathbb{R}_s$ is a collared boundary for $J|_{[-\varepsilon,\varepsilon]\times\mathbb{R}_s}$. Therefore, $J|_{[-\varepsilon,\varepsilon]\times\mathbb{R}_s}$, as a cobordism in the s direction, is a Lagrangian cobordism with cylindrical boundary.

- At each value of s, the collar $(M \times [0, \varepsilon')) \times [-\varepsilon, \varepsilon]_t \subset K|_{[-\varepsilon,\varepsilon]}$ is a collared boundary of the slice $J|_{[-\varepsilon,\varepsilon]\times\{s\}}$.

We are interested in the positive end of this Lagrangian cobordism, which we call $K^+|_{[-\varepsilon,\varepsilon]}$, so that
$$J|_{[-\varepsilon,\varepsilon]\times\mathbb{R}} : K^+|_{[-\varepsilon,\varepsilon]} \rightsquigarrow K|_{[-\varepsilon,\varepsilon]}$$
is a suspension Lagrangian cobordism with collared boundary.

The Lagrangian $K^+|_{[-\varepsilon,\varepsilon]}$ is a suspension, and parameterized by

$$(q, t) \mapsto \left(i_{\rho(t,1)}(q), t + {}_J\frac{d\rho(t, 1)}{dt} H_{\rho(t,1)} \right).$$

We can therefore form a Lagrangian cobordism $J : K^+ \rightsquigarrow K$, where

$$K^+ = K|_{(-\infty,\varepsilon]} \cup_{t=-\varepsilon} K^+|_{[-\varepsilon,\varepsilon]} \cup_{t=\varepsilon} K|_{[\varepsilon,\infty)}.$$

By construction, $\pi_{J\mathbb{R}}(K^+|_{[-\varepsilon/3,\varepsilon/3]}) = 0$, so this portion looks like $L^0 \times [-\varepsilon/3, \varepsilon/3]$. We therefore may assemble Lagrangian cobordisms:

$$K\|_{(-\infty,0]} := K^+|_{(\infty,0]} \cup_{L^0} (L^0 \times [0, \infty)) : L^0 \rightsquigarrow L^-,$$
$$K\|_{[0,\infty)} := K^+|_{[0,\infty)} \cup_{L^0} (L^0 \times (-\infty, 0]) : L^+ \rightsquigarrow L^0.$$

Clearly, $K^+ = K\|_{(-\infty,0]} \circ K\|_{[0,\infty)}$. Additionally, since K^+ fixes the boundary M, both $K\|_{(-\infty,0]}$ and $K\|_{[0,\infty)}$ are Lagrangian submanifolds which fix the boundary M.

It remains to show that this construction can be completed so that the Hofer norm is as small as desired. The norm is

$$\int_{\mathbb{R}} \left(\sup_{(q,t)\in L\times[-\varepsilon,\varepsilon]} \left(\frac{\partial\rho}{\partial s} H_{\rho(s,t)}(q) \right) - \inf_{(q,t)\in L\times[-\varepsilon,\varepsilon]} \left(\frac{\partial\rho}{\partial s} H_{\rho(s,t)}(q) \right) \right)$$

$$\leq \left(\sup_{(q,t)\in L\times[-\varepsilon,\varepsilon]} H_t(q) - \inf_{(q,t)\in L\times[-\varepsilon,\varepsilon]} H_t(q) \right) \int \left(\sup_{t\in[-\varepsilon,\varepsilon]} \frac{\partial\rho}{\partial s} - \inf_{t\in[-\varepsilon,\varepsilon]} \frac{\partial\rho}{\partial s} \right) ds$$

$$\leq 2\varepsilon \left(\sup_{(q,t)\in L\times[-\varepsilon,\varepsilon]} H_t(q) - \inf_{(q,t)\in L\times[-\varepsilon,\varepsilon]} H_t(q) \right),$$

which can be made as small as desired. ∎

Since this construction occurs away from the critical locus, we additionally have a matching of Morse critical points

$$\mathrm{Crit}(\pi_{\mathbb{R}}\colon K \to \mathbb{R}) = \mathrm{Crit}(\pi_{\mathbb{R}}\colon K\|_{(-\infty,0]} \circ K\|_{[0,\infty)} \to \mathbb{R})$$
$$= \mathrm{Crit}(\pi_{\mathbb{R}}\colon K\|_{[0,\infty)} \to \mathbb{R}) \cup \mathrm{Crit}(\pi_{\mathbb{R}}\colon K\|_{(-\infty,0]} \to \mathbb{R}).$$

Recall that the shadow of a Lagrangian cobordism $\mathrm{Area}(K)$ is the infimum of areas of simply connected regions containing the image of $\pi_{\mathbb{C}}\colon K \to \mathbb{C}$, which generalized the Hofer norm.

Claim 3.1.7. *Let K and $K^+ := K\|_{(-\infty,0]} \circ K\|_{[0,\infty)}$ be the Lagrangians from Proposition* 3.1.6*. Then,* $\mathrm{Area}(K) = \mathrm{Area}(K^+)$.

Proof. If K is a suspension arising from exact homotopy H_t, then the shadow can be computed via the Hofer norm:

$$\mathrm{Area}(K) = \int_{\mathbb{R}} \left(\sup_{q\in L^+} H_t(q) - \inf_{q\in L^+} H_t(q) \right) dt.$$

We note that $K|_{[-\varepsilon,\varepsilon]}$ and $K^+|_{[-\varepsilon,\varepsilon]}$ are suspensions. Integrating by change of variables yields

$$\mathrm{Area}(K^+|_{[-\varepsilon,\varepsilon]})$$
$$= \int_{[-\varepsilon,\varepsilon]} \left(\sup_{q\in L^+} \left(\frac{d\rho(t,1)}{dt} H_{\rho(t,1)}(q) \right) - \inf_{q\in L^+} \left(\frac{d\rho(t,1)}{dt} H_{\rho(t,1)}(q) \right) \right) dt$$
$$= \int_{[-\varepsilon,\varepsilon]} \left(\sup_{q\in L^+} H_t(q) - \inf_{q\in L^+} H_t(q) \right) dt = \mathrm{Area}(K|_{[-\varepsilon,\varepsilon]}). \quad ∎$$

The same method allows us to construct Lagrangian cobordisms from Lagrangian submanifolds $K \subset X \times \mathbb{C}$.

Definition 3.1.8. Let $K \subset X \times \mathbb{C}$ be a Lagrangian submanifold (not necessarily a Lagrangian cobordism) with cylindrical boundary. Suppose that t_-, t_+ are regular values of the projection $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$, and that the critical values of $\pi_{\mathbb{R}}$ are isolated. Then by applying Proposition 3.1.6 at t_-, t_+, we can define the *truncation* of K to $[t_-, t_+]$ which is a Lagrangian cobordism with cylindrical boundary

$$K\|_{[t_-, t_+]} \colon K|_{t_+} \rightsquigarrow K|_{t_-}.$$

3.1.5. Decomposition across the X-coordinate. We now look at how to "isolate" a portion of a Lagrangian cobordism across the X-coordinate so that it can be decomposed in the sense of Definition 3.1.2.

Definition 3.1.9. Let $K \colon L^+ \rightsquigarrow L^-$ be an embedded Lagrangian cobordism with $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ having isolated critical points. A dividing hypersurface for K is an embedded hypersurface $M \subset L^0$ with the following properties:

- M divides L^0 in the sense that $L^0 = L_{\downarrow}^0 \cup_M L_{\uparrow}^0$, where $L_{\uparrow/\downarrow}^0$ are the components of $L^0 \setminus M$.

- $M \subset L^0 \subset K$ contains no critical points of $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$.

A dividing hypersurface allows for the following decomposition of our Lagrangian cobordism.

Proposition 3.1.10. *Let $M \subset L^0$ be a dividing hypersurface for $K \colon L^+ \rightsquigarrow L^-$. Then there exists a decomposition of Lagrangian cobordisms up to exact homotopy:*

$$K \sim \widetilde{K}^- \circ \underline{\underline{K}}_M \circ \widetilde{K}^+,$$

so that $\underline{\underline{K}}_M$ is a Lagrangian cobordism which decomposes along M, and $\underline{\underline{K}}_M|_0 = L^0$. Furthermore, this construction can be performed in such a way that

- *the exact homotopy has as small Hofer norm as desired; and*

- *if K is embedded and the slice L^0 is embedded, then $\widetilde{K}^- \circ \underline{\underline{K}}_M \circ \widetilde{K}^+$ is embedded as well.*

Proof. Let $i \colon K \to X \times \mathbb{C}$ be the parameterization of our Lagrangian cobordism. Consider a small collared neighborhood $M \times I_s \subset L^0$. We take $B_\varepsilon^*(M \times I_s)$ a small neighborhood of the zero section inside the cotangent bundle of $(M \times I_s)$. There exists a map $\phi \colon B_\varepsilon^*(M \times I_s) \to X$, which is locally a symplectic embedding, and sends the zero section to $i(M \times I_s)$.

For $H \colon M \times I_s \to X$ with $|dH| < \varepsilon$ and the support of H contained on an interior subset of $M \times I$, denote by $dH \subset X$ the (possibly immersed) submanifold parameterized by

$$M \times I \xrightarrow{df} B_\varepsilon^*(M \times I_s) \xrightarrow{\phi} X.$$

There exists an open neighborhood $U \subset K$ containing $M \subset L^0 \subset K$ with the property that $(U|_t) \cap B^*_\varepsilon(M \times I_s)$ is a section of the cotangent ball for each t. Next let $H_t(q,s) \colon M \times I_s \to \mathbb{R}$ be the primitive of this section for each t so we can parameterize $i \colon U \to X \times \mathbb{C}$ by

$$M \times I_s \times I_t \to X \times \mathbb{C},$$

$$(q,s,t) \mapsto \left(dH_t, t + J \frac{d}{dt} H_t \right).$$

We now consider a function $\rho(s,t) \colon I_s \times I_t \to \mathbb{R}$ which is constantly 1 in a neighborhood of $\partial(I_s \times I_t)$, and constantly 0 on an interior set $(-\varepsilon', \varepsilon')_s \times (-\varepsilon'', \varepsilon'')_t \subset I_s \times I_t$. Consider the Lagrangian suspension cobordism

$$\widetilde{i} \colon M \times I_s \times I_t \to X \times \mathbb{C},$$

$$(q,s,t) \mapsto \left(d(\rho(s,t)H_t), t + J \frac{d}{dt} \rho(s,t) H_t \right),$$

and by abuse of notation, let $\widetilde{i} \colon K \to X \times \mathbb{C}$ be the Lagrangian cobordism where we have replaced $K|_U$ with the chart parameterized above. The parameterizations $\widetilde{i}$ and i are exactly isotopic.

For $t_0 \in (-\varepsilon'', \varepsilon'')$ and $(q,s) \in M \times (-\varepsilon', \varepsilon')$, we have that $\frac{d}{dt}\widetilde{i}(q,s,t) = 0$. Therefore, $\widetilde{K}\|_{(-\varepsilon'', \varepsilon'')}$ admits a decomposition in the X factor along M. We define

$$\underline{\underline{K}}_M := \widetilde{K}\|_{(-\varepsilon'', \varepsilon'')}, \quad \widetilde{K}^- := \widetilde{K}\|_{<-\varepsilon''}, \quad \widetilde{K}^+ := \widetilde{K}\|_{<\varepsilon''}.$$

Bounding the Hofer norm is similar to the computation for Proposition 3.1.6, and is bounded by the $2(\varepsilon' + \varepsilon'')(\sup_{(q,t,s)} H_t(q,s) - \inf_{(q,t,s)} H_t(q,s))$, which can be made as small as desired. ∎

We write this decomposition as

$$\underline{\underline{K}}_M = \underline{\underline{K}}_\downarrow \cup_{M \times \mathbb{R}} \underline{\underline{K}}_\uparrow. \tag{3.1}$$

By applying Proposition 3.1.3 to our decomposition along M, we further split the Lagrangian cobordism as a composition. There is a sequence of exact homotopies (Figure 8):

$$K \rightsquigarrow \widetilde{K}^- \circ \underline{\underline{K}}_M \circ \widetilde{K}^+ \rightsquigarrow \widetilde{K}^- \circ \left(\underline{\underline{K}}_\downarrow \cup_{M \times \mathbb{R}} \times L^-_\uparrow \times \mathbb{R} \right) \circ \left(\underline{\underline{K}}_\uparrow \cup_{M \times \mathbb{R}} (L^+_\downarrow \times \mathbb{R}) \right) \circ \widetilde{K}^+.$$

This second exact homotopy has Hofer norm bounded by $2\varepsilon'' \operatorname{Area}(K)$.

We note that no part of this construction modifies the height function, so

$$\operatorname{Crit}(\pi_\mathbb{R} \colon K \to \mathbb{R}) = \operatorname{Crit}(\pi_\mathbb{R} \colon \widetilde{K}^- \circ \underline{\underline{K}}_M \circ \widetilde{K}^+ \to \mathbb{R}).$$
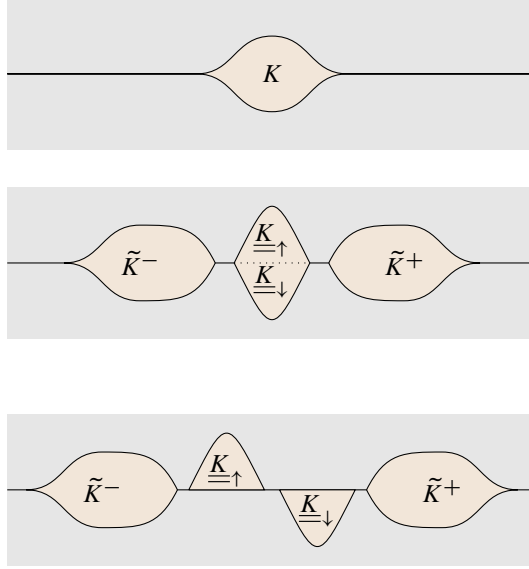
Figure 8. Given a dividing hypersurface M in a Lagrangian cobordism K, one can exactly homotope K to a composition where the middle Lagrangian cobordism decomposes across the X coordinate. This middle component can be further decomposed by applying Proposition 3.1.3.

As in the setting of decomposition along the $\mathbb{C}$ coordinate, we can show that in good cases this decomposition does not modify the Lagrangian shadow. Suppose that $M \subset L^0$ is a dividing hypersurface for $K: L^+ \rightsquigarrow L^-$. Furthermore, suppose that over the chart $M \times I_s \times I_t \subset K$ considered in the proof of Proposition 3.1.10, we have

$$\sup_{q \in K | \pi_{\mathbb{R}}(q) = t} \pi_{J\mathbb{R}} i(q) > \sup_{q \in M \times I_s \times \{t\}} \pi_{J\mathbb{R}} i(q),$$

$$\inf_{q \in K | \pi_{\mathbb{R}}(q) = t} \pi_{J\mathbb{R}} i(q) < \inf_{q \in M \times I_s \times \{t\}} \pi_{J\mathbb{R}} i(q).$$

Then

$$\mathrm{Area}(K) = \mathrm{Area}(\widetilde{K}^- \circ \underline{\underline{K}}_M \circ \widetilde{K}^+).$$

3.2. Standard Lagrangian surgery handle

In this section we give a description of a standard Lagrangian surgery handle. We include many figures in the hope of making the geometry of Lagrangian surgery apparent and start with the simple example of Lagrangian null-cobordism for Whitney spheres.
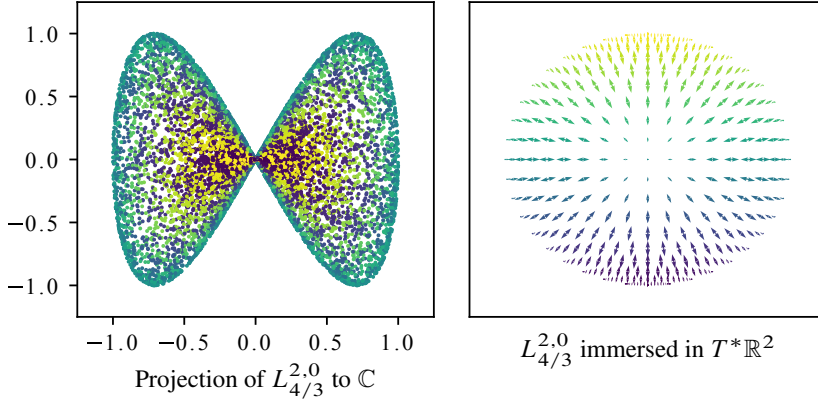
Figure 9. The Whitney sphere $L_{4/3}^{2,0,+} \subset \mathbb{C}^2$. On the left: a projection of randomly sampled points on $L_{4/3}^{2,0,+}$ to the first complex coordinate. The area of the projection is $2 \cdot 4/3$. On the right: $L_{4/3}^{2,0,+}$ drawn as a subset of $T^*\mathbb{R}^2$.

3.2.1. Null-cobordism and the Whitney sphere. We first give a definition of the Whitney sphere in higher dimensions, and show that this is null-cobordant.

Definition 3.2.1. The Whitney sphere of area A is the Lagrangian submanifold

$$L_A^{n,0,+} \subset \mathbb{C}^n,$$

which is parameterized by

$$i_A^{n,0,+} \colon S_r^n \to \mathbb{C}^n,$$
$$(x_0, \ldots, x_n) \mapsto (x_1 + j2x_0x_1, x_2 + j2x_0x_2, \ldots, x_n + j2x_0x_n),$$

where $S_r^n = \{(x_0, \ldots, x_n) \mid \sum_{i=0}^n x_i^2 = r^2\}$, and $r = \sqrt[3]{\frac{4}{3}A}$.

This Lagrangian has a single transverse self-intersection at the pair of points $(\pm r, 0, \ldots, 0) \in S_r^n$. We call these points $q^\pm \in L_A^{n,0,+}$. The quantity A describes the area of the projection of $L_A^{n,0,+}$ to the first complex coordinate:

$$\frac{\text{Area}(\pi_{\mathbb{C}}(L_A^{n,0,+}))}{2} = 2 \cdot \left(\int_0^r 2t\sqrt{r^2 - t^2}\,dt \right) = A.$$

The Lagrangian submanifold $L_A^{1,0,+}$ is the figure eight curve. An example of the Whitney 1-sphere is drawn in the Figure 10 as the slice $K^{1,1}|_1 = L_{4/3}^{1,0}$. In Figure 9 we give a plot of the Whitney 2-sphere, $L_{4/3}^{1,0,+} \subset \mathbb{C}^2$, presented as a set of covectors in the cotangent bundle $T^*\mathbb{R}^2$.
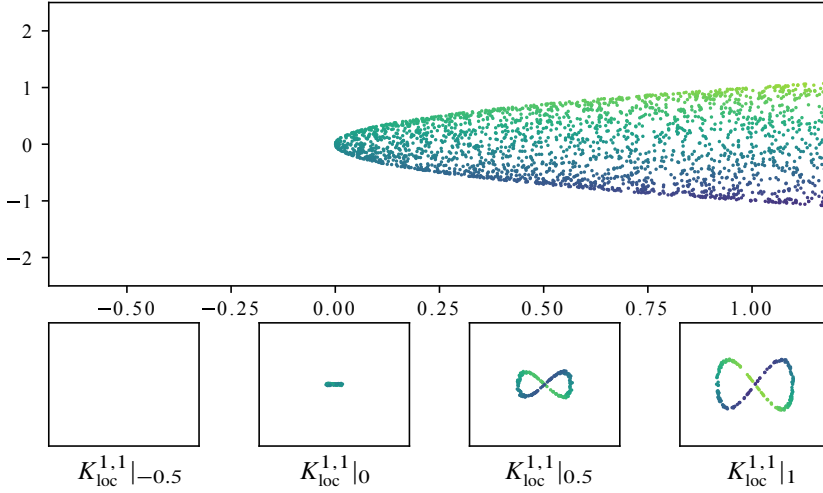
Figure 10. A plot of points on the null-cobordism $K^{1,1} \subset \mathbb{C}^2$ of the Whitney sphere $L_{4/3}^{1,0,+} \subset \mathbb{C}^1$. Points are consistently colored between figures. Top: The shadow projection of this null-cobordism. Bottom: Slices of the null-cobordism at different values of the cobordism parameter.

The Whitney sphere can be extended in one dimension higher to a Lagrangian submanifold parameterized by the disk. Let $r(x_0, \ldots, x_n) = \sum_{i=0}^n x_i^2$. The parameterization

$$j^{n,1} \colon \mathbb{R}^{n+1} \to \mathbb{C}^n \times \mathbb{C},$$
$$(x_0, \ldots, x_n) \mapsto (i_r^{n,0}(x_0, \ldots, x_n), r^2 - J x_0)$$

gives an embedded Lagrangian disk $K^{n,1} \subset \mathbb{C}^n \times \mathbb{C}$ which has the following properties:

- When $r < 0$, the slice $K^{n,1}|_r$ is empty;
- The slice $K^{n,1}|_0$ is not regular;
- When $r > 0$ the slice $K^{n,1}|_r$ is a Whitney sphere of area $A = \frac{4r^3}{3}$.

We will prove that this is a Lagrangian submanifold in Section 3.2.2. In Figure 10 we draw this Lagrangian null-cobordism and its slices, which are Whitney spheres of decreasing radius.

While $K^{n,1}$ is not a Lagrangian cobordism (as it does not fiber over the real line outside of a compact set,) it should be thought of as a model for the null-cobordism of the Whitney sphere. If we desire a Lagrangian cobordism, we may apply Defini-

tion 3.1.8 to truncate this Lagrangian submanifold and obtain a Lagrangian cobordism.

3.2.2. Standard surgery and anti-surgery handle. The standard Lagrangian surgery handle [4] is a Lagrangian $\mathbb{R}^n \times \mathbb{R}^1$ inside $T^*\mathbb{R}^n \times T^*\mathbb{R}^1$, where $T^*\mathbb{R}^1$ is identified with $\mathbb{C}$ by

$$(q, p) \mapsto p + Jq.$$

Let $\sigma_{i,k}$ be $+1$ if $i \leq k$ and -1 otherwise. Consider the function

$$G^{k,n-k+1} \colon \mathbb{R}^{n+1} \to \mathbb{R},$$

$$(x_0, x_1, \ldots, x_n) \mapsto x_0 \left(\sum_{i=1}^{n} \sigma_{i,k} x_i^2 \right) + \frac{1}{3} x_0^3.$$

The graph of $dG^{k,n-k+1}$ parameterizes a Lagrangian submanifold inside the cotangent bundle,

$$\mathbb{R}^{n+1} \to T^*\mathbb{R}^n \times \mathbb{C},$$

$$(x_0, x_1, \ldots, x_n) \mapsto \left(x_1 + J\sigma_{1,k} 2 x_1 x_0, \ldots, x_n + J\sigma_{n,k} 2 x_n x_0, \right.$$

$$\left. x_0 + J \left(x_0^2 + \sum_{i=1}^{n} \sigma_{i,k} x_i^2 \right) \right),$$

whose projection to the $\pi_{J\mathbb{R}}$ coordinate is a Morse function with a single critical point of index $k + 1$. Our convention is that the Morse index of a critical point is dimension of the upward flow space of the point. By multiplying the last coordinate by $-J$, we interchange the real and imaginary parts of the shadow projection.

Definition 3.2.2. For $k \geq 0$, the local Lagrangian $(k, n - k + 1)$ surgery trace is the Lagrangian submanifold $K^{k,n-k+1}_{\mathrm{loc}} \subset (\mathbb{C})^n \times \mathbb{C}$ parameterized by

$$j^{k,n-k+1} \colon \mathbb{R}^{n+1} \to T^*\mathbb{R}^n \times \mathbb{C}$$

$$(x_0, x_1, \ldots, x_n) \mapsto \left(x_1 + J\sigma_{1,k} 2 x_1 x_0, \ldots, x_n + J\sigma_{n,k} 2 x_n x_0, \right.$$

$$\left. x_0^2 + \sum_{i=1}^{n} \sigma_{i,k} x_i^2 - J x_0 \right).$$

The positive and negative slices of this Lagrangian submanifold will be denoted

$$L^{k,n-k,+}_{\mathrm{loc}} := K^{k,n-k+1}_{\mathrm{loc}}|_1, \quad L^{k+1,n-k-1,-}_{\mathrm{loc}} := K^{k,n-k+1}_{\mathrm{loc}}|_{-1}.$$

For $k = -1$, we define $K^{-1,n+2}_{\mathrm{loc}} := (K^{n,1}_{\mathrm{loc}})^{-1}$.
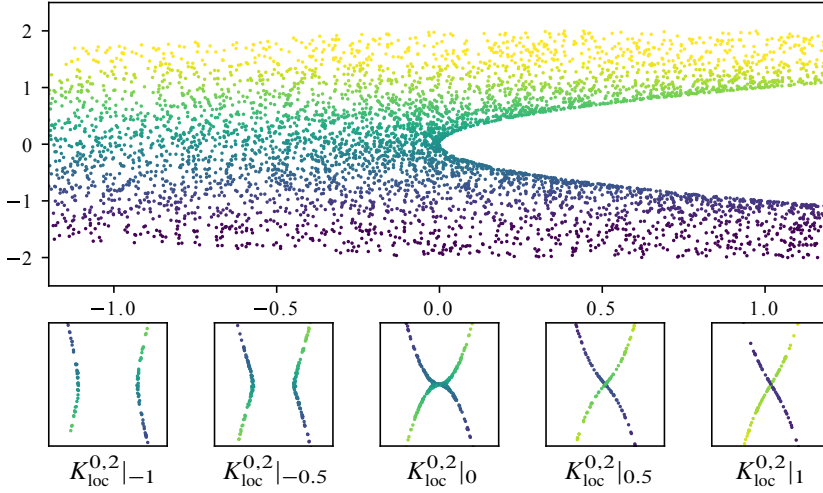
Figure 11. A Lagrangian surgery cobordism $K_{\text{loc}}^{0,2} \subset \mathbb{C}^2$. Top: The shadow projection of the surgery 1-handle $K_{\text{loc}}^{0,2}$. Bottom: Slices of the surgery. The left-most and right-most pictures are models of the surgery and anti-surgery neck.

This will be the local model for Lagrangian surgery trace, which we will construct in Section 3.2.3. Before we proceed with the construction, we state some properties of the surgery trace, and give some examples.

Theorem 3.2.3 (Properties of the standard Lagrangian surgery trace). *The Lagrangian surgery trace*

$$K_A^{k,n-k+1} \colon L_A^{k,n-k,+} \rightsquigarrow L_A^{k+1,n-k-1,-}$$

has the following properties:

- $L_A^{k,n-k,+}$ *is a Lagrangian $S^k \times D^{n-k}$ with a single self-intersection. Its intersection with the first k-coordinates is a Whitney isotropic $L_A^{k,0,+}$;*
- $L_A^{k+1,n-k-1,-}$ *an embedded Lagrangian $D^{k+1} \times S^{n-k-1}$; and*
- $\pi_{\mathbb{R}} \colon K_A^{k,n-k+1} \to \mathbb{R}$ *is Morse, with a single critical point of index $k+1$.*

A particularly relevant example is $K_{\text{loc}}^{0,n+1} \subset \mathbb{C}^{n+1}$, which gives a local model for the Polterovich surgery trace (see Figure 11 for the example $K_{\text{loc}}^{0,2}$).

The slice $L_{\text{loc}}^{k,n-k,+}$ is an immersed Lagrangian submanifold with a single double point,

$$\pi_X \circ j^{k,n-k+1}(\pm 1, 0, \ldots, 0) = (0, \ldots, 0, 0).$$

We denote these points $q_\pm \in L_{\mathrm{loc}}^{k,n-k,+}$. [1] A useful observation is that when the positive end of the surgery trace is restricted to the first k-coordinates,

$$L_{\mathrm{loc}}^{k,n-k,+}|_{\mathbb{C}^k} = L_{\mathrm{loc}}^{k,0,+} \subset \mathbb{C}^k$$

we see an isotropic Whitney sphere. The other end of the Lagrangian surgery trace, $L_{\mathrm{loc}}^{k+1,n-k-1,-}$ is embedded. Furthermore,

$$L_{\mathrm{loc}}^{k+1,n-k-1,-}|_{\mathbb{C}^k} = \emptyset.$$

This allows us to interpret $K_{\mathrm{loc}}^{k,n-k+1}$ as a null-cobordism of a Whitney isotropic in the first k-coordinates. This is a slightly deceptive characterization, as not all Whitney isotropic spheres are null-cobordant; see Remark 4.2.6.

According to our convention (which is that Lagrangian cobordisms go from the positive end to the negative end), the Lagrangian cobordism $K_{\mathrm{loc}}^{k,n-k+1}$ resolves a self-intersection of the input end. For this reason, we say that $K_{\mathrm{loc}}^{k,n-k+1}$ provides a local model of Lagrangian surgery. Given $j: K \to X \times \mathbb{C}$ a parameterization for a Lagrangian cobordism K, the inverse Lagrangian cobordism (denoted by $K^{-1} \subset X \times \mathbb{C}$) is the Lagrangian submanifold parameterized by $(\pi_X \circ j, -\pi_{\mathbb{R}} \circ j + J\pi_{J\mathbb{R}} \circ j)$ (i.e., by reflecting the real parameter of the cobordism). We call the inverse Lagrangian submanifold, $(K_{\mathrm{loc}}^{k,n-k+1})^{-1}$ the local model for Lagrangian anti-surgery.

Example 3.2.4 (Lagrangian Surgery Handle $K_{\mathrm{loc}}^{0,3}$). In Figure 12 we draw slices of the Lagrangian cobordism $K_{\mathrm{loc}}^{0,3}$. In the surgery interpretation, the Lagrangian self-intersection point is an isotropic Whitney sphere $L^{0,0} \subset \mathbb{C}^0$, highlighted in blue.

In the anti-surgery interpretation, the isotropic Lagrangian disk highlighted in red is contracted, collapsing the S^1 boundary to a transverse self-intersection.

Example 3.2.5 (Lagrangian Surgery Handle $K_{\mathrm{loc}}^{1,2}$). In Figure 13 we draw slices of the Lagrangian cobordism $K_{\mathrm{loc}}^{1,2}$. In the surgery interpretation, we resolve the isotropic Whitney S^1 sphere highlighted in blue by replacing it with two copies (an S^0 family) of the null-cobordism $D^{2,0}$.

In the anti-surgery interpretation, the isotropic Lagrangian disk highlighted in red is contracted, collapsing the immersed S^0 boundary and yielding a Lagrangian with a self-intersection.

An important observation is that while the cobordisms $K_{\mathrm{loc}}^{k,n-k+1}$ and $K_{\mathrm{loc}}^{n-k,k+1}$ are topologically inverses, they *are not inverses of each other as Lagrangian cobordisms*. The first is a cobordism between an immersed $S^k \times D^{n-k}$ and an embedded

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[1] The following mnemonics may be useful to the reader: the positive end of the surgery cobordism is immersed and locally looks like the character "+".
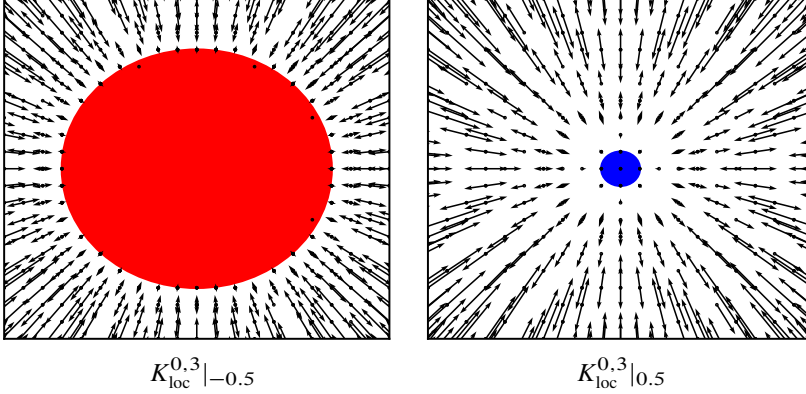
$$K_{\mathrm{loc}}^{0,3}|_{-0.5} \qquad\qquad K_{\mathrm{loc}}^{0,3}|_{0.5}$$

Figure 12. Slices of the surgery 2-handle $K_{\mathrm{loc}}^{0,3} \subset \mathbb{C}^3$ from before and after the critical point. These correspond to models of the surgery and anti-surgery neck. The positive end (right) is an immersed $D^2 \times S^0$, with double point is indicated in blue. The negative end (left) is an embedded $S^1 \times D^1$, whose Lagrangian antisurgery D^2 from [15] is drawn in red.

$D^{k+1} \times S^{n-k-1}$, while the second is between an immersed $S^{n-k-1} \times D^{k+1}$ and an embedded $D^{n-k} \times S^k$ (see Theorem 3.2.3). This can be seen in examples by comparing Figures 12 and 13.

3.2.3. Lagrangian surgery trace. In this section we prove Theorem 3.2.3. First, we apply Section 3.1.5 to build from $K_{\mathrm{loc}}^{k,n-k+1}$ a Lagrangian cobordism with fixed boundary. For this construction, we write $K := K_{\mathrm{loc}}^{k,n-k+1}$. Pick a radius $A \in \mathbb{R}_{>0}$. Let $L^{k,n-k+1,0} = K|_0$; see for instance Figure 10. As a subset of the domain $\mathbb{R}^n \times \mathbb{R}^1$ parameterizing K, the domain parameterizing the Lagrangian $L^{k,n-k+1,0}$ is given by the locus

$$\left\{ (x_0, x_1, \ldots, x_n) \mid x_0^2 + \sum_{i=1}^{n} \sigma_{i,k} x_i^2 = 0 \right\}.$$

We then take hypersurface $M_{k,n-k+1} \subset L^{k,n-k+1,0}$ cut out by $x_0^2 + x_1^2 + \cdots x_n^2 = 1$. As in the proof of Proposition 3.1.10, take an extension $M \times I_s \times I_t \subset K$ which is disjoint from the subset $V = x_0^2 + x_1^2 + \cdots x_n^2 = 1/4$. By using Proposition 3.1.10 to perform a decomposition across the X-coordinate along M, we obtain a Lagrangian cobordism with fixed boundary $\underline{\underline{K}}_\downarrow$. We use the notation from equation (3.1), and we designate the $\downarrow$ component to be the one which contains the origin in $D^n \times D^1$.

Since $(\underline{\underline{K}}_\downarrow) \cap V = (K) \cap V$, we obtain

$$\left(\underline{\underline{K}}_\downarrow \cap V \right)\big|_{1/2} = \left((K)\big|_{1/2} \right) \cap V.$$

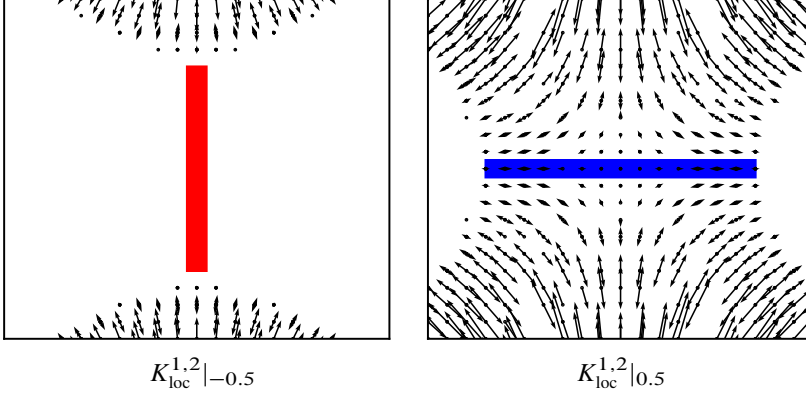$$K_{\text{loc}}^{1,2}|_{-0.5} \qquad\qquad K_{\text{loc}}^{1,2}|_{0.5}$$

Figure 13. Slices of the surgery 2-handle $K_{\text{loc}}^{1,2} \subset \mathbb{C}^2$ from before and after the critical point. These correspond to models of the surgery and anti-surgery neck. The positive end (right) is an immersed $D^1 \times S^1$, whose isotropic Whitney S^1 is highlighted in blue. The negative end (left) is an embedded $S^0 \times D^2$, whose isotropic antisurgery D^1 from [15] is drawn in red.

We define the *standard Lagrangian trace* of area $\frac{1}{6}$ to be

$$K_{1/6}^{k,n-k+1} := \underline{\underline{K}}_{\downarrow} \big\|_{[-1/2,1/2]},$$

where the double vertical bar refers to the truncation from Definition 3.1.8. The standard Lagrangian surgery trace of area A is then defined to be the rescaling (under the map $z \mapsto c \cdot z$ on $\mathbb{C}^n$) of the previously constructed Lagrangian submanifold,

$$K_A^{k,n-k+1} := 6A \cdot K_{1/6}^{k,n-k+1}.$$

The ends of the standard Lagrangian surgery trace of area A will be denoted:

$$K_A^{k,n-k+1} \colon L_A^{k,n-k,+} \rightsquigarrow L_A^{k+1,n-k-1,-}.$$

Remark 3.2.6. Note that in the case of $k = -1, n$, this simply corresponds to truncation

$$K_A^{k,n-k+1} = K_{\text{loc}}^{k,n-k+1}\big\|_{<r}$$

where $A = \frac{4r^3}{3}$.

Remark 3.2.7. When $k = -1$, we have a Lagrangian cobordism $K_A^{-1,n+2} = (K^{n,1})^{-1}$. This case differs slightly from the standard Lagrangian Surgery trace in that the positive end $L_A^{-1,n+1,+}$ is empty, and the negative end $L_A^{0,n,-}$ is a Whitney sphere.
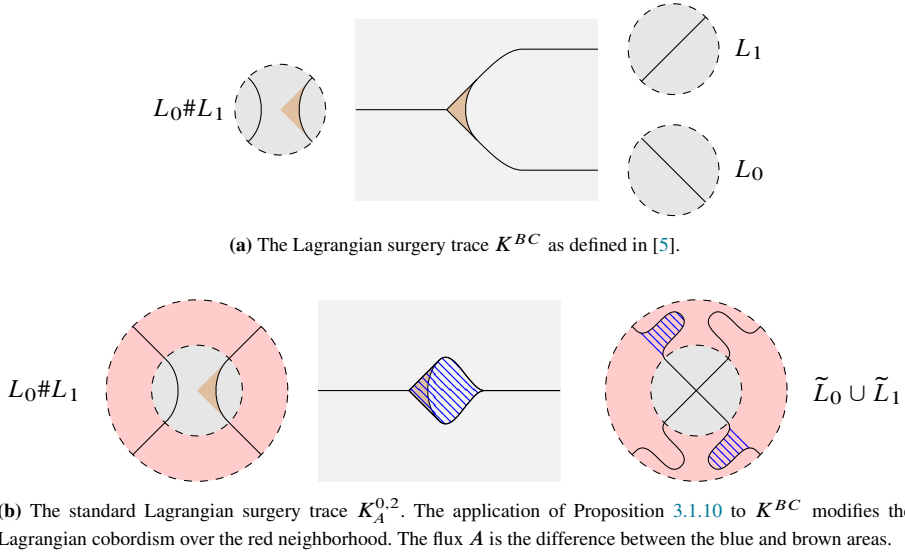
(a) The Lagrangian surgery trace K^{BC} as defined in [5].

(b) The standard Lagrangian surgery trace $K_A^{0,2}$. The application of Proposition 3.1.10 to K^{BC} modifies the Lagrangian cobordism over the red neighborhood. The flux A is the difference between the blue and brown areas.

Figure 14. Comparison between Polterovich surgery and the standard Lagrangian surgery trace. Notice that the fluxes swept over the surgeries differ (both in magnitude and sign).

Example 3.2.8. The ends of the Lagrangian surgery trace $K_A^{0,n} \colon L_A^{0,n,+} \rightsquigarrow L_A^{1,n-1,-}$, which resolves a single transverse intersection of an n-dimensional Lagrangian, do not quite agree with the standard pictures drawn for the Polterovich surgery. In particular, the flux of the surgery (which determines the map $\omega \colon H_2(X, L_A^{1,n-1,-}) \to \mathbb{R}$ in terms of $\omega \colon H_2(X, L_A^{0,n,+}) \to \mathbb{R}$) is surprisingly counterintuitive. We now describe the flux swept out by the local model for the standard Lagrangian surgery trace when $\dim(X) = 2$. This example is based on the computation of flux which appears in [16, Section 4.1] and the discussion surrounding [15, Figure 8].

In [24] the local model for Polterovich surgery of two Lagrangian submanifolds intersecting transversely at a point replaces the Lagrangians $L_0, L_1 \subset \mathbb{R}^2$ (as drawn on the right-hand side of Figure 14 (a)) with the Lagrangian $L_0 \# L_1$ (as drawn on the left-hand side of Figure 14 (a)). Figure 14 (a) also depicts the Lagrangian surgery cobordism

$$K^{BC} \colon (L_0, L_1) \rightsquigarrow (L_0 \# L_1)$$

as defined in [5]. The flux of the surgery – the area highlighted in brown on the left-hand side – is equal to the shadow of K^{BC}. This 3-ended Lagrangian cobordism is not a Lagrangian cobordism with cylindrical boundary (as it has three ends), so it is *not* a local model for the standard Lagrangian trace (as defined in Section 3.2.3).

To obtain a 2-ended Lagrangian cobordism with cylindrical boundary from K^{BC}, one must apply a Lagrangian isotopy which cylindricalizes the boundary (Proposi-

tion 3.1.10). The resulting Lagrangian cobordism

$$K_A^{0,2} \colon (\widetilde{L}_0 \cup \widetilde{L}_1) \rightsquigarrow L_0 \# L_1$$

is drawn in Figure 14 (b). A subtle point is that the positive end of this Lagrangian cobordism is no longer $L_0 \cup L_1$. The construction from Proposition 3.1.10 covers $K_A^{0,2}$ with two charts. The first chart agrees with the Lagrangian cobordism K^{BC} from before. The second chart, contained in the region highlighted in red, is a suspension of a Hamiltonian isotopy of L_i restricted to the red region. The flux of this suspension is the blue hatched region in Figure 14 (b), and equal to Area($K_A^{0,2}$). Observe that

$$\text{Area}(K_A^{0,2}) > \text{Area}(K^{BC}).$$

As a consequence, the area bounded by $\widetilde{L}_0 \cup \widetilde{L}_1$ and $L_0 \# L_1$ has the *opposite sign* of the area between $L_0 \cup L_1$ and $L_0 \# L_1$! The quantity $A = \text{Area}(K_A^{0,2}) - \text{Area}(K^{BC})$ describes the symplectic area bounded by $\widetilde{L}_0 \cup \widetilde{L}_1$ and $L_0 \# L_1$.

The construction of a standard Lagrangian surgery handle allows us to define the standard Lagrangian surgery trace.

Definition 3.2.9. We say that $K \colon L^+ \rightsquigarrow L^-$ is a standard Lagrangian surgery trace if it admits a decomposition across the X coordinate as $K = K_- \cup_{S^{n-k} \times S^n} K_A^{k,n-k+1}$, where K_- is a suspension Lagrangian cobordism with collared boundary.

While the standard Lagrangian surgery trace is a useful cobordism to have, a geometric setup for performing Lagrangian surgery on a given Lagrangian L^+ is desirable. Such a criterion is given in [15] by the anti-surgery disk. In that paper, it was noted that the presence of a Whitney isotropic k sphere was a necessary but not sufficient condition for implanting a Lagrangian surgery handle. We give a sufficient characterization in Remark 4.2.6.

3.3. Cobordisms are iterated surgeries

Having described the Lagrangian surgery operation and trace cobordism, we show that all Lagrangian cobordisms decompose into a concatenation of surgery traces and exact homotopies. This characterization is analogous to the handle body decomposition of cobordisms from the data of a Morse function.

Theorem 3.3.1. *Let $K \colon L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. There is a sequence of Lagrangian cobordisms*

$$K_{H_t^i} \colon L_{i+1}^- \rightsquigarrow L_i^+ \quad \text{for } i \in \{0, \ldots, j\},$$

$$K_i^{k_i, n-k_i+1} \colon L_i^+ \rightsquigarrow L_i^- \quad \text{for } i \in \{1, \ldots, j\},$$

which satisfy the following properties:

- $L^-_{j+1} = L^+$ *and* $L^+_0 = L^-$;
- *each* $K^{k_i, n-k_i+1}$ *is a Lagrangian surgery trace;*
- *each* $K_{H^i_t}$ *is the suspension of an exact homotopy; and*
- *there is an exact homotopy between*

$$K \sim K_{H^j_t} \circ K^{k_j, n-k_j+1}_j \circ K_{H^{j-1}_t} \circ \cdots \circ K_{H^1_t} \circ K^{k_1, n-k_1+1}_1 \circ K_{H^0_t}.$$

Furthermore, this construction can be performed in such a way that the exact homotopy has as small Hofer norm as desired.

The decomposition comes from using the function $\pi_\mathbb{R} \colon K \to \mathbb{R}$ to provide a handle body decomposition of K. We note that unless K is the suspension of an exact isotopy, the decomposition will necessarily be immersed (as the Lagrangian surgery traces are all immersed).

3.3.1. Morse Lagrangian cobordisms.
We first must show that K can be placed into general position by exact homotopy so that $\pi_\mathbb{R}$ is a Morse function (as in Example 2.1.4).

Claim 3.3.2 (Morse lemma for Lagrangian cobordisms). *Let $K \subset X \times \mathbb{C}$ be a Lagrangian cobordism. There exists K', a Lagrangian cobordism exactly homotopic to K, with $\pi_\mathbb{R} \colon K' \to \mathbb{R}$ a Morse function. Furthermore, the construction can be conducted so that*

- *the Hofer norm of the exact homotopy is as small as desired; and*
- *if K is embedded, then K' is embedded as well.*

Proof. By abuse of notation, we will use K to denote the smooth manifold parameterizing the (possibly immersed) Lagrangian cobordism K. Let $\phi \colon B^*_\varepsilon K \to X$ be a local Weinstein neighborhood. At each point $x \in B^*_\varepsilon K$ there is a Darboux neighborhood of $\phi(x)$, which can be chosen to be the product of Darboux neighborhoods of $\pi_X \circ \phi(x)$ and $\pi_\mathbb{C} \circ \phi(x)$. Therefore, there exists around $x \in B^*_\varepsilon K$ Darboux coordinates $(q_0, p_0, \ldots, q_n, p_n)$ so that

$$q_0 := \pi_\mathbb{R} \circ \phi, \quad p_0 := \pi_{J\mathbb{R}} \circ \phi$$

are the pullbacks of the real and imaginary coordinates to the local Weinstein neighborhood. Let $C^\infty_\varepsilon(B^*_\varepsilon K)$ be the smooth functions $B^*_\varepsilon K \to \mathbb{R}$ with compact support disjoint from the boundary. Let $C^\infty_{\mathrm{cob}}(K; \mathbb{R})$ be the functions which agree with $\pi_\mathbb{R} \colon K \to \mathbb{R}$ outside of a compact set. Given $H \in C^\infty_\varepsilon(B^*_\varepsilon K)$, let ψ^t_H be the time t

Hamiltonian flow of H, and let $j_H^t = \phi \circ \psi_H^t$ be the corresponding exactly homotopic immersion of K. We obtain a map

$$\mathscr{P}: C_\varepsilon^\infty(B_\varepsilon^* K) \to C_{\text{cob}}^\infty(K; \mathbb{R}),$$
$$H \mapsto \pi_\mathbb{R} \circ j_H^1,$$

so that $\mathscr{P}(H)$ is the real coordinate of the immersion j_H^t. We will show that this map is a submersion, and in particular open. Let $f: K \to \mathbb{R}$ be a function with compact support, representing a tangent direction of $C_{\text{cob}}^\infty(K; \mathbb{R})$. As $K \subset B_\varepsilon^* K$ is embedded, f can be extended to a compactly supported function $F: B_\varepsilon^* K \to \mathbb{R}$, so that $F|_K = f$, and $F \in C_\varepsilon^\infty(B_\varepsilon^* K)$. The flow of H in the q_0 coordinate is

$$\frac{dq_0}{dt} = \frac{dH}{dp_0}.$$

We define our Hamiltonian $H_f: B_\varepsilon^* K \to \mathbb{R}$ by the integral

$$H_f(x) := \int_{\pi_X(x) \times q_0(x) \times (-\infty, p_0(x))} F \, dp_0.$$

With this choice of Hamiltonian, the Hamiltonian flow at time zero of the real coordinate at a point $x \in K \subset B_\varepsilon^* K$ is given by

$$\frac{d}{dt}(\pi_\mathbb{R} \circ j_{H_f}^t)|_{t=0}(x) = \frac{dq_0}{dt}\Big|_{t=0}(x) = f(x).$$

This shows that $\mathscr{P}$ is a submersion at 0. Since every open set of $C_{\text{cob}}^\infty(K; \mathbb{R})$ contains a Morse function, and the image of $\mathscr{P}$ is open, there is a choice of Hamiltonian H near 0, so that $\mathscr{P}(H) = \pi_\mathbb{R} \circ j_{H_f}^1$ is a Morse function on K.

Because the Hamiltonian can be chosen near zero, we can choose it so that

$$\sup_{x \in K} H(x) - \inf_{x \in K} H(x)$$

is bounded by a constant as small as desired. This shows that the exact homotopy associate to the time 1 flow of H has as small Hofer norm as desired. ∎

A similar argument shows that every K is exactly homotopic to K' with the property that $\text{Crit}(\pi_\mathbb{R}: K' \to \mathbb{R})$ is disjoint from $\mathcal{I}^{\text{si}}(K')$, the set of self-intersections of K'. If $\pi_\mathbb{R}: K \to \mathbb{R}$ is a Morse function whose critical points are disjoint from its self-intersections, we say that the Lagrangian cobordism is a *Morse–Lagrangian cobordism*.
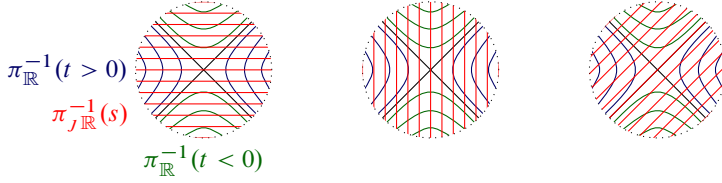
Figure 15. Three cases giving surgery, anti-surgery, and a degenerate Lagrangian cobordism.

3.3.2. Placing cobordisms in good position.

The Lagrangian condition forces a certain amount of independence between the $\pi_{\mathbb{R}} \circ j$ and $\pi_{J\mathbb{R}} \circ j$ projections of the Lagrangian cobordism.

Claim 3.3.3. *Let* $j \colon K \to X \times \mathbb{C}$ *be a Morse–Lagrangian cobordism. Then* $x \in K$ *cannot be a critical point of both* $\pi_{\mathbb{R}} \circ j$ *and* $\pi_{J\mathbb{R}} \circ j$.

Proof. If so, then $j_*(T_p K) \subset TX \subset TX \times T\mathbb{C}$. Since $j_*(T_p K)$ is a Lagrangian subspace, it cannot be contained in any proper symplectic subspace of $T(X \times \mathbb{C})$. ∎

Even when K is a Morse–Lagrangian cobordism, it need not be the case that at a critical point $q \in \mathrm{Crit}(\pi_{\mathbb{R}} \colon K \to \mathbb{R})$ that $K|_{q-\varepsilon}$ is obtained from $K|_{q+\varepsilon}$ by surgery. In the simplest counterexample, $K|_{q-\varepsilon}$ could be obtained from $K|_{q+\varepsilon}$ by anti-surgery; simply knowing the index of a $\pi_{\mathbb{R}} \circ j$ critical point does not determine if it arises from surgery or antisurgery!

Example 3.3.4. We provide some intuition for what additional information is needed to determine if a critical point gives a surgery or an antisurgery. Suppose that q is an index-1 point of a 2-dimensional Lagrangian cobordism K. Then there exist local coordinates around q so that $\pi_{\mathbb{R}} \circ j$ can be written as $q_0^2 - q_1^2$. Since the critical points of $\pi_{J\mathbb{R}} \circ j$ are disjoint from those of $\pi_{\mathbb{R}} \circ j$, the differential of $\pi_{J\mathbb{R}} \circ j$ is non-vanishing at q. We now assume that $\pi_{J\mathbb{R}} \circ j$ is linear in (q_0, q_1) -coordinates (note that this will generally not be the case). We then can write $(\pi_{J\mathbb{R}} \circ j) = a q_0 + b q_1$. The slices of this Lagrangian cobordism are then the level sets of $q_0^2 - q_1^2 = t$, and the primitive describing the exact homotopy between those slices is $a q_0 + b q_1$. We look at three cases (summarized in Figure 15):

(1) If $|a| > |b|$, then $a q_0 + b q_1$ restricted to $q_0^2 - q_1^2 = t$ will have two critical points when $t > 0$. Let q_t, q_t' be these two critical points. Let us make another assumption (which in general does not hold), which is that these critical points are fixed points of the homotopy $K|_t$ (i.e., $\pi_X \circ j(q_{t_0}) = \pi_X \circ j(q_{t_1})$ and $\pi_X \circ j(q_{t_0}') = \pi_X \circ j(q_{t_1}')$ for all $t_0, t_1 > 0$). Since $\lim_{t \searrow 0} q_t = \lim_{t \searrow 0} q_t' = (0,0)$, we obtain that $j(q_t) = j(q_t')$ whenever $t > 0$. We conclude that the positive slices are immersed (making K a surgery).
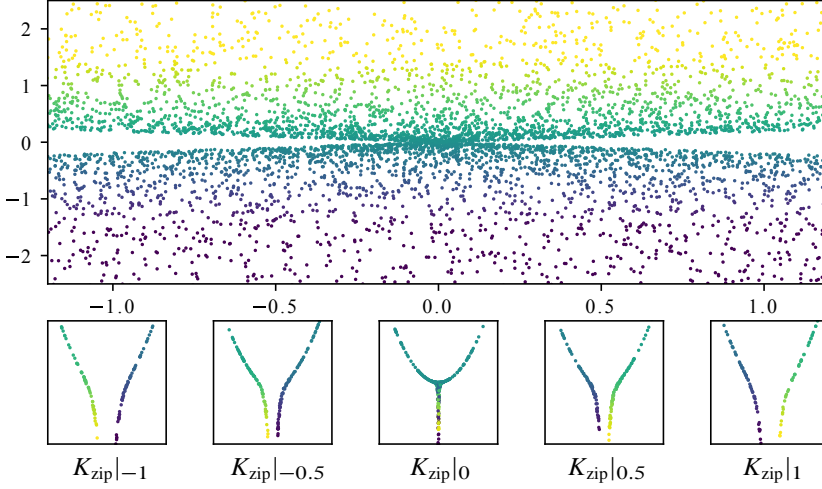
Figure 16. A Morse–Lagrangian cobordism where the critical slice has a codimension-0 set of self-intersection. This Lagrangian cobordism is parameterized by the mapping $(x_0, x_1) \mapsto (x_0 + x_1 - 2Jx_0x_1, 2(x_0^2 - x_1^2) + J(x_0 + x_1))$.

(2) If instead $|b| < |a|$, the same argument holds except that $aq_0 + bq_1$ has critical points on the negative slices of K. K then gives an antisurgery.

(3) The last case is degenerate: when $|a| = |b|$, both the positive and negative slices are embedded, but the critical slice is immersed along a set of codimension 0 (see Figure 16)!

In order for our critical points of $\pi_{\mathbb{R}}$ to correspond to surgeries (Example 3.3.4 (1)), we need to apply another exact homotopy based on an interpolation between Morse functions.

Claim 3.3.5 (Interpolation of Morse functions). *Let $f, g \colon \mathbb{R}^n \to \mathbb{R}$ be Morse functions, each with a single critical point of index k at the origin and $f(0) = g(0) = 0$. Take V any neighborhood which contains the origin. There exists a smooth family of functions $h_c v \colon \mathbb{R}^n \times [0, 1] \to \mathbb{R}$ which satisfies the following properties:*

- *in the complement of V, $h_c|_{\mathbb{R}^n \setminus V} = f|_{\mathbb{R}^n \setminus V}$;*
- *there exists a small neighborhood U of the origin so that $h_1|_U = g|_U$; and*
- *h_1 is Morse with a single critical point;*
- *at time 0, $h_0(x) = f(x)$.*

Proof. Pick coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, so that in a neighborhood of the origin,

$$f = \sum_{i=1}^{n} \sigma_{i,k} x_i^2, \quad g = \sum_{i=1}^{n} \sigma_{i,k} y_i^2.$$

Let $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map so that $\phi_*(\partial_{y_i}) = \partial_{x_i}$. Pick $\phi_c \colon \mathbb{R}^n \times [0, 1/2] \to \mathbb{R}^n$ an isotopy of linear maps smoothly interpolating between $\phi_0 = \mathrm{id}$ and $\phi_{1/2} = \phi$. Take $U_2 \subset V$ a small ball around the origin with the property that for all c, $\phi_c(U_2) \subset V$. Now consider an path of diffeomorphisms $\widetilde{\phi}_c \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying the constraints

$$\widetilde{\phi}_c|_{\mathbb{R}^n \setminus V} = \mathrm{id}, \quad \widetilde{\phi}_0 = \mathrm{id}, \quad \widetilde{\phi}_{1/2}|_{U_2} = \phi|_{U_2}.$$

For $c \in [0, 1/2]$, we define $h_c := f \circ \widetilde{\phi}_c$.

We now define h_c for $c \in [1/2, 1]$. Pick $U_3 \subset U_2$ a neighborhood of the origin with the property that, for every $q \in U_3$ and $\partial_v \in T_q \mathbb{R}^n$, we have

$$|(d(f \circ \phi) - dg)(\partial_v)| \leq \frac{1}{2}|df(\partial v)|.$$

Take an interior subset $U \subset U_3$ which is a neighborhood of the origin. Let ρ be a bump function, which is constantly 1 on U, and 0 outside U_3. Let $\tau \colon [1/2, 1] \to [0, 1]$ be an increasing function smoothly interpolating between $\tau(1/2) = 0$ and $\tau(1) = 1$. For $c \in [1/2, 1]$, let

$$h_c := (1 - \tau(c)\rho) \cdot h_{1/2} + \tau(c)\rho g.$$

It remains to show that h_1 is Morse, with a unique critical point at the origin. For any $q \in \mathbb{R}^n \setminus U_3$, we have that $dh_1 = (\widetilde{\phi}_{1/2})^* \, df$, which is non-vanishing. For any $q \in U$, we have $dh_1 = dg$, which vanishes if and only if $q = 0$. For $q \in U_3 \setminus U$, take $\partial_v \in T_q \mathbb{R}^n$ with the property that $d\rho(\partial_v) = 0$. Then

$$|dh_1(\partial_v)| = |(1 - \rho) \, df(\partial_v) + \rho \, df(\partial_v)| > \frac{1}{2}|df(\partial v)| > 0.$$

This proves that q is not a critical point of h_1. ∎

Proposition 3.3.6. *Let K be a Morse Lagrangian cobordism. Let $q \in K$ be a critical point of the projection $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ of index $k + 1$. There exists*

- *a neighborhood of the origin $U \subset T^* D^n \times \mathbb{C}$, and a symplectic embedding*

$$\phi \colon U \to X \times \mathbb{C},$$

  which respects the splitting so that $\phi(0) = q$; and

- *K' a Morse Lagrangian cobordism exactly homotopic to K,*

so that $K' \cap U = (K^{k,n-k+1}_{\text{loc}}) \cap U$ *under the identification given by* ϕ. *Furthermore,*

- *the critical points of* $\pi_{\mathbb{R}}|_{K'}$ *are in bijection with the critical points of* $\pi_{\mathbb{R}}|_K$;
- *the Hofer norm of the exact homotopy is as small as desired; and*
- *if* K *is embedded, then* K' *is embedded as well.*

Proof. At q take the Lagrangian tangent space $T_q K \subset T_{\pi_X(q)} X \oplus T_{\pi_{\mathbb{C}}(x)} \mathbb{C}$. Since q is a critical point of $\pi_{\mathbb{R}}$, we have that

$$T_q K \subset T_{\pi_X(q)} X \oplus {}_J \mathbb{R}.$$

By dimension counting, any set of vectors $\partial v_0, \partial_{v_1}, \ldots, \partial_{v_n} \in T_q K$ with the property that $(\pi_X)_*(\partial_{v_1}) \neq 0$ for all v_i cannot form a basis of a Lagrangian subspace for $\pi_X(T_q K)$; therefore, there exists a vector $\partial_s \in T_q K$, so that $\pi_X(\partial_s) = 0$, and we may split $T_q K = (\pi_X)_*(T_q K) \oplus {}_J \mathbb{R}$. Choose a Darboux chart

$$U \subset T^* \mathbb{R}^n \times \mathbb{C}, \phi \quad U \to X \times \mathbb{C},$$

which respects the product decomposition, and has $\phi_*(\mathbb{R}^n \times {}_J \mathbb{R}) = T_q K$. Write $\underline{K}$ for $\mathbb{R}^n \times {}_J \mathbb{R}|_U$. Because K and $\underline{K}$ have the same tangent space, we can further restrict to a Weinstein neighborhood of $\underline{K}$ so that K is an exact section of the tangent bundle $B^*_\varepsilon \underline{K}$. By taking a possibly smaller neighborhood, we will identify $\underline{K} = D^n_{r_0} \times [-s_0, s_0]$, where the s-coordinate denotes the ${}_J \mathbb{R}$ direction. Let F be the primitive of this section, so that $K|_{B^* \underline{K}} = dF$.

If we let s be the coordinate on $\underline{K}$ which travels in the ${}_J \mathbb{R}$ direction, then we can compute $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$ at $q \in K$ by

$$f(q) := \pi_{\mathbb{R}}(q) = \partial_s F(q),$$

which is a Morse function on $\underline{K}$ with a single critical point.

We now implement the handle in this neighborhood. Let $G^{k,n-k+1} \colon \underline{K} \to \mathbb{R}$ be the primitive for the handle from Definition 3.2.2, so that $g := \partial_s G^{k,n-k}$ is a Morse function on $\underline{K}$. We will use Claim 3.3.5 to obtain a function h_c interpolating between f and g, and define a preliminary primitive $H^{\text{pre}}_c := \int h_c \, ds$. The section associated to H^{pre}_c will satisfy all the Morse properties we desire; however it does not agree with F outside near the boundary of $\underline{K}$. This is because while f and h_c agree at the boundary of $\underline{K}$, there is no reason for $\int^s_{s_0} h_c(x, s) \, ds$ to match $\int^s_{s_0} f(x, s) \, ds = F(x, s)$ near the boundary of $\underline{K}$. Therefore, we need to add a correcting term to h_c in order to make these integrals agree near the boundary of $\underline{K}$.

We set up this correction using a neighborhood as drawn in Figure 17. Take a set

$$W = D^n_{r_1} \times [s_0/2, 3s_0/4] \subset \underline{K}.$$

Let $\rho \colon W \to \mathbb{R}_{\geq} 0$ be a bump function, vanishing on the boundary, with the property that there exists an $r_2 < r_1$ so that for all $x, y \in D_{r_2}^n$, we have $\rho(x, s) = \rho(y, s)$. Furthermore, assume that

$$\int_{\{x\} \times [s_0/2, 3s_0/4]} \rho(x, s) \, ds = 1.$$

Let $\alpha = \sup_{(x,s) \in \underline{K}} |d\rho|$. Let $\beta = \inf_{(x,s) \in W} |df|$. Since f has no critical points in W, this is greater than 0. To each choice of $V \subset D_{r_2}^n \times [-s_0/2, s_0/2]$ and associated interpolation $h_c \colon \underline{V} \to \mathbb{R}$, we can define a function

$$A_c(x) := \int_{\{x\} \times [-s_0/2, s_0/2]} (h_c(x, s) - f(x, s)) \, ds.$$

We may choose V small enough so that our interpolation satisfies

$$\sup_{x,c} |A_c(x)| < \frac{\beta}{2\alpha}, \quad \sup_{x} |dA_1(x)| < \frac{\beta}{2}.$$

Now consider the function

$$H_c(x, s) = \int_{\{x\} \times [-s_0, s]} h_c(x, s) - A_c(x)\rho(x, s) \, ds.$$

Then $\partial_s H_1(x, s) = h_c(x, s) - A_c(x)\rho(x, s)$. We have that

$$d(\partial_s H_1) = dh_c - d(A_c(x)\rho).$$

By construction $|d(A_c(x)\rho)| < |dh_c|$ inside the region W, and $A_c(x)\rho$ vanishes outside of W. It follows that $\partial_s H_1(x, s)$ has no critical points outside of V. The derivative $\partial_s H_1(x, s)$ is Morse, agrees with g in a neighborhood of the origin, where it has a single critical point. Furthermore, near the boundary of $\underline{K}$, we have

$$H_c(x, s) = F(x, s) \quad \text{for all } (x, s) \in \underline{K} \setminus \big(D_{r_2}^n \times [-s_0/2, 3s_0/4]\big).$$

Consider the Lagrangian section of $T^*\underline{K}$ given by dH_1. This Lagrangian section is exactly isotopic to $K|_{T^*\underline{K}}$, with exact primitive vanishing at the boundary. We therefore have an exactly homotopic family of Lagrangian cobordisms

$$K_c := K \setminus (K|_{T^*\underline{K}}) \cup dH_c,$$

where $\mathrm{Crit}(\pi_{\mathbb{R}} \colon K_0 \to \mathbb{R}) = \mathrm{Crit}(\pi_{\mathbb{R}} \colon K_1 \to \mathbb{R})$ and $K_1|_{T^*\underline{K}} = K_{\mathrm{loc}}^{k,n-k+1}|_{T^*\underline{K}}$.

Finally, for the bound on the Hofer norm: this is given by

$$\int_0^1 \sup_{x,s} (H_c(x, s) - F(x, s)) - \inf_{x,s} (H_c(x, s) - F(x, s)) \, dc.$$
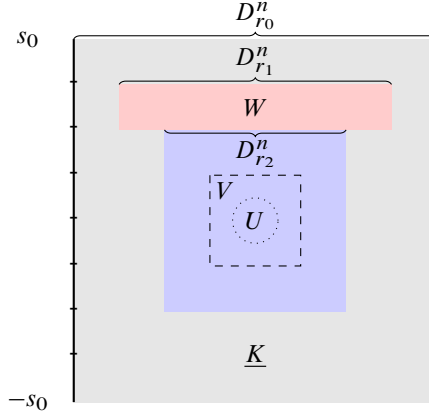
Figure 17. Nested neighborhoods for interpolating primitives. At the end of interpolation: our Lagrangian handle matches the standard surgery handle over U; the region V is used to interpolate between $g = \partial_s G$ and $f = \partial_s F$; the region W is subsequently used to correct h_c so that its integral over the s-coordinate matches F at the boundary of K.

By choosing our initial neighborhood $D_{r_0}^n$ sufficiently small (so both $F(x, s)$ and $G^{k,n-k}$ are nearly zero over the neighborhood), we make

$$\sup_{c,x,s} |(H_c(x, s) - F(x, s)|$$

as small as desired.   ∎

3.3.3. Cobordisms are concatenations of surgeries.

In this section, we prove that every Lagrangian cobordism is exactly homotopic to the concatenation of standard surgery handles.

Proof of Theorem 3.3.1. Let $K: L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. After application of Claim 3.3.2, we obtain K', a Lagrangian cobordism exactly homotopic to K with the property that $\pi_{\mathbb{R}}: K' \to \mathbb{R}$ is Morse with distinct critical values. Enumerate the critical points $\{q_i\}_{i=1}^l = \mathrm{Crit}(\pi_{\mathbb{R}}: K' \to \mathbb{R})$. By Proposition 3.3.6, we may furthermore assume that K' is constructed so that there exists symplectic neighborhoods $U_i \subset T^*D^n \times T^*D^1$, so that

$$K' \cap U_i = K_{\mathrm{loc}}^{k_i, n-k_i+1} \cap U_i.$$

For each q_i take ε_i small enough so that the $(q_i - \varepsilon_i, q_i + \varepsilon_i)$ are disjoint. Take r_i small enough so that the ball of radius r_i centered at the critical point q_i is contained

within the charts U_i. K is exactly homotopic to the composition

$$K\|_{(-\infty, q_l - \varepsilon]} \circ K\|_{[q_l - \varepsilon, q_l + \varepsilon]} \circ K\|_{[q_l + \varepsilon, q_{l-1} + \varepsilon]} \circ \cdots$$
$$\cdots \circ K\|_{[q_2 + \varepsilon, q_1 - \varepsilon]} \circ K\|_{[q_1 - \varepsilon, q_1 + \varepsilon]} \circ K\|_{[q_1 + \varepsilon, \infty)}.$$

By applying Proposition 3.1.10 on $K'\|_{(q_i - \varepsilon_i, q_i + \varepsilon_i)}$ at the dividing hypersurface

$$M_i := \left\{ (x_0, \ldots, x_n) \in U_i \mid \sum_{j=0}^{n} x_j^2 = r_i^2 \right\},$$

we obtain an exact homotopy

$$K'\|_{(q_i - \varepsilon_i, q_i + \varepsilon_i)} \sim K_i^- \circ \underline{\underline{K_i}}_{M_i} \circ K_i^+,$$

where $K_i^{\pm}$ are suspensions, and $K_i^{k_i, n-k_i+1} := \underline{\underline{K_i}}_{M_i}$ are Lagrangian surgery traces.
For each i, let $K_{H_t^i} = K_{i+1}^+ \circ K_i^-$ be the suspension of an exact homotopy. Then

$$K \sim K_{H_t^l} \circ K_l^{k_l, n-k_l+1} \circ K_{H_t^{l-1}} \circ \cdots \circ K_{H_t^1} \circ K_1^{k_1, n-k_1+1} \circ K_{H_t^0}.$$

We finally check the Hofer norm of the exact homotopy above. At each step where we employ an exact homotopy, the operations from Claim 3.3.2 and Propositions 3.3.6, 3.1.6, and 3.1.10 could be conducted in such a way to make the Hofer norm of their associated exact homotopies as small as desired. Since there are a finite number of operations being conducted, the Hofer norm of the exact homotopy between K and a decomposition can be made as small as desired. ∎

Conjecture 3.3.7. *Suppose that L is an unobstructed Lagrangian, whose bounding cochain has valuation c. Let L' be exactly homotopic to L, with the Hofer norm of the exact homotopy less than c. Then L' is unobstructed.*

The conjecture is based on the following observation: for Hamiltonian isotopies, the suspension cobordism K has the property that $CF^\bullet(K)$ is an A_∞ mapping cocylinder between $CF^\bullet(L^+)$ and $CF^\bullet(L^-)$, meaning that there are A_∞ projection maps

$$\pi^{\pm} : CF^\bullet(K) \to CF^\bullet(L^{\pm})$$

and a map (defined on chains, but not a A_∞ homomorphism)

$$\underline{i}^+ : CF^\bullet(L^+) \to CF^\bullet(K),$$

which can be extended to an A_∞ homomorphism
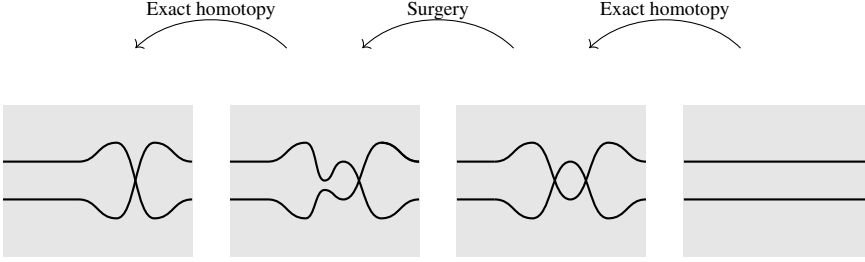
$$i^+ : CF^\bullet(L^+) \to CF^\bullet(K).$$

Figure 18. Rewriting an anti-surgery as exact homotopies and surgeries.

The homomorphism i^+ is an A_∞ homotopy inverse to π^+. A key point is that the lowest order portion of i^+ comes from the Morse continuation map, so i^+ has valuation zero.

For exact homotopy, we expect that there still exist projection maps $\pi^\pm$. There are several difficulties in the construction of these map (principally, it requires a rigorous definition of the Floer theory in this setting). In contrast to the isotopy case, the map i^+ will be given by at lowest order by counts flow lines (between Morse generators) and holomorphic strips (between generators associated to self-intersections). Thus, while we may be able to define a map (not an A_∞ homomorphism)

$$i^+ : CF^\bullet(L^+) \to CF^\bullet(K),$$

the map may *decrease* valuation. We conjecture that the decrease in valuation is bounded by the shadow of the Lagrangian cobordism. Under these circumstances, there is a version of the A_∞ homotopy transfer theorem [16] which allows i^+ to be extended to an A_∞ homotopy inverse to π^+.

As the exact homotopies we consider for our decomposition of Lagrangian cobordisms have as small Hofer norm as desired, a corollary of the conjecture is that decomposition of Lagrangian cobordism preserves unobstructedness.

Finally, we make a remark about anti-surgery versus surgery. We have shown that every Lagrangian cobordism can be decomposed as a sequence of exact homotopies and Lagrangian surgery traces; in particular, the Lagrangian anti-surgery trace $(K^{k,n-k+1})^{-1}$ can be rewritten as a Lagrangian surgery and exact homotopy. An anti-surgery takes an embedded Lagrangian L and adds a self-intersection; one can equivalently think of this as starting with an embedded Lagrangian, applying an exact homotopy to obtain a pair of self-intersection points, and then surgering away one of the self-intersections. This is drawn in Figure 18.

4. Teardrops on Lagrangian cobordisms

One of the main observations about the decomposition described in Theorem 3.3.1 is that the slices near a critical point of $\pi_{\mathbb{R}}\colon K \to \mathbb{R}$ differ

- topologically by a surgery, and

- as immersed submanifolds of X by the creation or deletion of a self-intersection.

An interpretation of the difference is that surgery trades topological chains of L^- for self-intersections of L^+. When K is an embedded oriented Lagrangian submanifold, one can show that this exchange preserves the $\mathbb{Z}/2\mathbb{Z}$ grading of the topological chains. As a consequence,

$$\chi(L^+) = \chi(K) = \chi(L^-).$$

In this section we extend this statement to the immersed setting, and provide evidence that this equality can be upgraded to an isomorphism of Floer cohomology groups. In Section 4.1 we provide a definition for the Euler characteristic of an immersed Lagrangian submanifold with transverse self-intersections (equation (4.2)). Using Theorem 3.3.1, we then prove in Proposition 4.1.2 that for this definition of Euler-characteristic $\chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+)$.

The computation does not immediately extended to the Euler characteristic of a Lagrangian cobordism $K\colon L^+ \rightsquigarrow L^-$ with immersed ends as K will not have transverse self-intersections. We therefore require a standard form for Lagrangian cobordism with transverse self-intersections. The standard form we choose (Lagrangian cobordisms with double bottlenecks) is adopted from [20] who also studied the Floer cohomology of immersed Lagrangian cobordisms. We subsequently show in Proposition 4.2.13 that for Lagrangian cobordisms with double bottlenecks

$$(K, t^-, t^+)\colon (L^-, H^-) \rightsquigarrow (L^+, H^+),$$

we have

$$\chi^{\mathrm{bot}}(K, t^-, t^+) = \chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+), \tag{4.1}$$

where $\chi^{\mathrm{si}}, \chi^{\mathrm{bot}}$ are the Euler characteristic of an appropriate set of Floer cochains for immersed Lagrangians and Lagrangian cobordisms with double bottlenecks.

In Section 4.3.1 we review the construction of immersed Lagrangian Floer cohomology, and in Section 4.4 we provide evidence that equation (4.1) can be extended to chain homotopy equivalences between the immersed Lagrangian Floer cohomologies of K, L^- and L^+ when K is a standard surgery trace.

4.1. Grading of self-intersections, and an observed pairing on cochains

We recover this equality of Euler characteristic on the chain level for Lagrangian cobordisms with self-intersections (Proposition 4.1.2). Suppose that X has a nowhere

vanishing section Ω of $\Lambda_{\mathbb{C}}^n(T^*M, J_M)$. We say that $i: L \to X$ is graded if there exists a function $\theta: L \to \mathbb{R}$, so that the determinant map

$$\det: L \to S^1,$$
$$q \mapsto \Omega(T_q L)^{\otimes 2}/|\Omega(T_q L)|^2$$

can be expressed as $\det(q) = e^{2J\pi\theta}$. In this setting, we can define a self-intersection corrected Euler characteristic for immersed Lagrangian submanifolds. Let

$$\mathcal{I}^{\mathrm{si}}(L) = \{(p \to q) \mid p, q \in L, i(p) = i(q)\}$$

be the set of ordered self-intersections.[2] Note that each self-intersection of L gives rise to two elements of $\mathcal{I}^{\mathrm{si}}$. The *index* of a self-intersection $(p \to q)$ is defined as

$$\mathrm{ind}(p \to q) := n + \theta(q) - \theta(p) - 2\angle(i_* T_p L, i_* T_q L),$$

where $\angle(V, W)$ is the Kähler angle between two Lagrangian subspaces. We particularly suggest reading the exposition in [2] on computation of this index.

Claim 4.1.1 (Index of handle self-intersections). *Consider the parameterization of one boundary of the standard Lagrangian surgery handle obtained from restricting Definition* 3.2.2 *to the positive slice*

$$j^{k,n-k+1}|_{t=1}: L^{k,n-k,+} \to \mathbb{C}^n.$$

Equip $\mathbb{C}^n$ with the standard holomorphic volume form. The index of the self-intersections are

$$\mathrm{ind}(q_- \to q_+) = n - k - 1, \quad \mathrm{ind}(q_+ \to q_-) = k + 1.$$

Proof. To reduce clutter, we write j for $j^{k,n-k+1}$. The hypersurface

$$x_0^2 + \sum_{i=1}^n \sigma_{i,k} x_i^2 = 1 \quad \text{in } \mathbb{R}^{n+1}$$

describes the slice $L^{k,n-k,+}$ as a hypersurface of the local surgery trace. Its tangent space at a point $\gamma(\theta) = (\sin(\theta), 0, \ldots, 0, \cos(\theta))$ is spanned by the basis

$$\{\cos(\theta)\partial_1 - \sin(\theta)\partial_0, \partial_2, \partial_3, \ldots, \partial_n\}.$$

---

[2]The notation reflects our interpretation of each element $(p \to q)$ as being a short Hamiltonian chord starting at p and ending at q.

Let $e_1, \ldots, e_n$ be the standard basis of $T_{\mathbb{C}} \mathbb{C}^n$. Then

$$i_*(\partial_0) = \sum_i 2_J \sigma_{i,k} x_i e_i,$$

$$j_*(\partial_i) = (1 + 2_J \sigma_{i,k} x_0) e_i \quad \text{for } i = 1, \ldots, n,$$

so that the tangent space at $T_{j(\gamma(\theta))} L_{\mathrm{loc}}^{k,n-k,+}$ is spanned by vectors $\vec{v}_i(\theta)$, where

$$\vec{v}_1(\theta) = \left(\cos(\theta) + 2_J \left(\sum_{i=1}^n \sigma_{i,k} (\cos^2(\theta) - \sin^2(\theta)) \right) \right) e_1,$$

$$\vec{v}_i(\theta) = (1 + 2_J \sigma_{i,k} \cos(\theta)) e_i \quad \text{for } i = 2, \ldots, n.$$

Let $z_i(\theta)$ be coefficients so that $\vec{v}_i(\theta) = z_i(\theta) e_i$. Then $\arg(z_i(\theta))$ is decreasing for $i \leq k$ and increasing for $i > k$. The endpoints of the z_i are given by

$$z_1(0) = 1 + 2_J, \qquad z_1(\pi) = -1 + 2_J,$$

$$z_i(0) = 1 + 2_J \sigma_{i,k}, \quad z_i(\pi) = 1 - 2_J \sigma_{i,k}.$$

To compute the index of $(q_- \to q_+)$, we complete the path $z_i^2(\theta)/|z_i^2(\theta)|$ to a loop by taking the short path, and sum the total argument swept out by each of the z_i. For ease of computation, let $\alpha = \arctan(2)$.

- When $i = 1$, the loop $z_1^2(\theta)/|z_1^2(\theta)|$ sweeps out $-2\pi - 4\alpha$ radians; the short path completion yields a contribution of -2π to the total index of this loop.

- When $1 < i \leq k$, the loop $z_i^2(\theta)/|z_i^2(\theta)|$ sweeps out -4α radians; the short path completion yields a total contribution of 0 from this loop.

- When $k < i \leq n$, the loop $z_i^2(\theta)/|z_i^2(\theta)|$ sweeps out 4α radians; the short path completion yields a total contribution of 2π from this loop.

The total argument swept out is $(n - k - 1) \cdot 2\pi$. The index of the self-intersection is

$$\mathrm{ind}(q_- \to q_+) = n - k - 1.$$

By similar computation (or using duality) we see that

$$\mathrm{ind}(q_+ \to q_-) = k + 1. \qquad \blacksquare$$

Let L be a compact graded Lagrangian submanifold, and $f: L \to \mathbb{R}$ be a Morse function. The set of Floer generators is

$$\mathcal{I}(L) = \mathrm{Crit}(f) \cup \mathcal{I}^{\mathrm{si}}(L). \tag{4.2}$$

For each $x \in \mathrm{Crit}(f)$, let $\mathrm{ind}(x)$ be the Morse index. Define the self-intersection Euler characteristic to be

$$\chi^{\mathrm{si}}(L) := \sum_{x \in \mathcal{I}(L)} (-1)^{\mathrm{ind}(x)}.$$

Proposition 4.1.2. *Let $K: L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. Then*

$$\chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+).$$

As was pointed out to me by Ivan Smith, this also follows in the case that K is embedded by the much simpler argument that the Euler characteristic is the signed self-intersection, and noting that we can choose a Hamiltonian push-off so the intersections of $K \cap \phi(K)$ are in index preserving bijections with intersections $L^\pm \cap \phi(L^\pm)$. Nevertheless, we give proof using decomposition as this will motivate Section 4.4.

Proof. As each exact homotopy preserves $\chi^{\mathrm{si}}(L^-)$, we need only check the case that $K^{k,n-k+1}: L^+ \rightsquigarrow L^-$ is a surgery trace. Choose Morse functions $f^\pm: L^\pm \to \mathbb{R}$ which in the local model of the surgery neck given by Definition 3.2.2 agree with the coordinate x_n. The critical points of $f^\pm$ agree outside of the surgery region. Inside the surgery region, we have

$$\mathrm{Crit}(f^+) \cap L_{\mathrm{loc}}^{k,n-k,+} = \mathrm{Crit}(x_n: L_{\mathrm{loc}}^{k,n-k,+} \to \mathbb{R}),$$
$$\mathrm{Crit}(f^-) \cap L_{\mathrm{loc}}^{k+1,n-k-1,-} = \mathrm{Crit}(x_n: L_{\mathrm{loc}}^{k+1,n-k-1,-} \to \mathbb{R}).$$

Recall that in the coordinates from Definition 3.2.2 these Lagrangian submanifolds are parameterized by domains in $\mathbb{R}^{n+1}$ cut out by the equations

$$L_{\mathrm{loc}}^{k,n-k,+} = \left\{(x_0,\ldots,x_n) \mid x_0^2 + \sum_{i=1}^{n} \sigma_{i,k} x_i^2 = 1\right\},$$

$$L_{\mathrm{loc}}^{k+1,n-k-1,-} = \left\{(x_0,\ldots,x_n) \mid x_0^2 + \sum_{i=1}^{n} \sigma_{i,k} x_i^2 = -1\right\}.$$

We compute the critical points of f restricted to the level sets of $\pi_{\mathbb{R}} \circ j$ using the method of Lagrange multipliers:

$$\nabla f = \langle 2x_0, 2\sigma_{1,k} x_1, \ldots, 2\sigma_{n,k} x_n \rangle,$$
$$\mathrm{grad}\, \pi_{\mathbb{R}} \circ j = \langle 0, 0, \ldots, 0, 1 \rangle.$$

So the only critical points in the surgery region occur when $x_0, \ldots, x_{n-1} = 0$. From this, we obtain the following cases:

- If $k = -1$, then f^+ restricted to $L_{\mathrm{loc}}^{-1,n+1,+}$ has no critical points (as it is empty) and $L^{0,n,-}$ has critical points of index 0 and n, which we call e, x.
- If $-1 < k < n$, then f^+ restricted to $L_{\mathrm{loc}}^{k,n-k,+}$ has no critical points, and f^- restricted to $L_{\mathrm{loc}}^{k+1,n-k-1,-}$ has critical points of index $k+1$ and $n-k-1$. We call these critical points $x_\pm$.

Index	$\mathcal{I}^{\mathrm{si}}(L^+)$	Crit(f^+)	Crit(f^-)
$k+1$	$(q_+ \to q_-)$		x_+
$n-k-1$	$(q_- \to q_+)$		x_-

(a) Floer cochains of $\mathcal{I}(L^+)$ and $\mathcal{I}(L^-)$ for a Lagrangian surgery trace $K^{k,n-k+1} \colon L^+ \rightsquigarrow L^-$, when $-1 < k < n$.

Index	$\mathcal{I}^{\mathrm{si}}(L^+)$	Crit(f^+)
-1	$(q_- \to q_+)$	
0		e
n		x
$n+1$	$(q_+ \to q_-)$	

(b) In the degenerate case $K^{n,1}$, $\mathcal{I}(L^-)$ is empty.

Index	$\mathcal{I}^{\mathrm{si}}(L^-)$	Crit(f^-)
-1	$(q_- \to q_+)$	
0		e
f n		x
$n+1$	$(q_+ \to q_-)$	

(c) In the degenerate case $K^{-1,n+2}$, $\mathcal{I}(L^+)$ is empty.

Table 1. Chain level differences in Floer generators before and after surgery.

- If $k = n$, then f^+ restricted to $L_{\mathrm{loc}}^{n,0,+}$ has critical points of index $0, n$, which we call e, x. The function f^- restricted to $L_{\mathrm{loc}}^{k+1,n-k-1,-}$ has no critical points (as it is empty).

The differences between $\mathcal{I}(L^+)$, $\mathcal{I}(L^-)$ are listed in Table 1. From the values listed in Table 1 it follows that $\chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+)$. ∎

This computation leads to the following question: can we extend Proposition 4.1.2 to an equivalence of Floer theory.

4.2. Doubled bottlenecks

Problematically, the decomposition given by Theorem 3.3.1 is not very useful for understanding Floer cohomology for Lagrangian cobordisms with immersed ends, as such Lagrangian cobordisms will not have transverse self-intersections. This is because the standard definition of Lagrangian cobordisms (Definition 2.1.1) does not allow us to easily work with immersed Lagrangian ends. Rather, [20] gives a definition for a *bottlenecked Lagrangian cobordism* which gives a method for concatenating Lagrangian cobordisms with immersed ends in a way that preserves transversality of self-intersections.

4.2.1. Bottlenecked Lagrangian cobordisms. We adapt [20, Definition 7.4] to our setting.

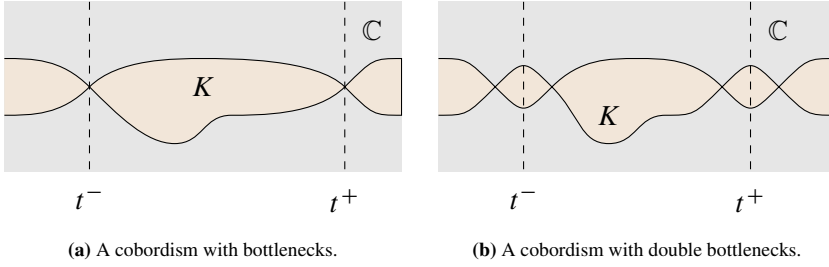(a) A cobordism with bottlenecks.   (b) A cobordism with double bottlenecks.

Figure 19. Lagrangian cobordisms with bottlenecks or double bottlenecks allow us to discuss Lagrangian cobordisms between immersed Lagrangian submanifolds.

Definition 4.2.1. Let $i_0 \colon L_0 \to X$ be an immersed Lagrangian with transverse self-intersections. A bottleneck datum for L is an extension of the immersion to an exact homotopy $i_t \colon L_0 \times I \to X$ with primitive $H_t \colon L_0 \times I \to \mathbb{R}$ satisfying the following conditions:

- *Bottleneck*: $H_0 = 0$ and there exists a bound $C \in \mathbb{R}$ so that $|\frac{dH_t}{dt}| \leq C$ everywhere.

- *Embedded away from* 0: If $i_t(q_0) = i_t(q_1)$, then either $q_0 = q_1$ or $t = 0$.

For simplicity of notation,[3] we will denote the datum of a bottleneck by (L, H_t).

We say that a Lagrangian cobordism $K \subset X \times \mathbb{C}$ is bottlenecked at time t_0 if $K|_{[t_0-\varepsilon, t_0+\varepsilon]}$ is the suspension of a bottleneck datum. The image of the Lagrangian cobordism under $\pi_{\mathbb{C}}$ has a pinched profile (see Figure 19). By application of the open mapping principle, the pinch point prevents pseudoholomorphic disks with boundary on the Lagrangian cobordism from passing from one side of the Lagrangian cobordism to the other (hence the name bottleneck).

Example 4.2.2 (Whitney Sphere). The first interesting example of a bottleneck comes from the Whitney n-sphere $L_A^{n,0} \subset \mathbb{C}^n$. We treat $L_A^{n,0}$ as a Lagrangian cobordism inside of $\mathbb{C}^{n-1} \times \mathbb{C}$. The shadow projection $\pi_{\mathbb{C}} \colon L_A^{n,0} \to \mathbb{C}$ is drawn in Figure 9. The bottleneck on the Lagrangian cobordism occurs at $t = 0$, with bottleneck datum corresponding to an exact homotopy of the Whitney $(n-1)$-sphere $L_A^{n-1,0} \subset \mathbb{C}^{n-1}$. This exact homotopy is parameterized by

$$i_{A(t)}^{n-1,0} \colon S_{r(t)}^{n-1} \to \mathbb{C}^{n-1},$$

where $A(t) = \frac{3}{4}(1 - t^2)^{\frac{3}{2}}$, and $r(t) = \sqrt{1 - t^2}$. The primitive for this exact homotopy is $H_t = 2x_0 t$. Each $L_{A(t)}^{n,0}$ contain a pair of distinguished points, $q_\pm = (\pm r(t), 0, \ldots, 0)$,

---

[3]The primitive of an exact homotopy does not determine the exact homotopy j; however, many properties of the bottleneck are determined by H_t.

which correspond to the self-intersection of the Whitney $(n-1)$ sphere. Note that q_+ is the maximum of H_t on each slice, and q_- is the minimum of H_t on each slice.

Given an immersed Lagrangian $i: L \to X$ with transverse self-intersections, there exists a standard way to produce a bottleneck (which we call a *standard bottleneck*). Pick a local Weinstein neighborhood $\phi: B_r^* L \to X$. Let $h: L \to X$ be a function such that $|dh| < r$, and that $h(p) \neq h(q)$ whenever $(p \to q) \in \mathcal{I}^{\text{si}}(L)$. Let $H_t = t \cdot h$. Then the Lagrangian cobordism submanifold parameterized by

$$L \times (-\varepsilon, \varepsilon) \to X \times T^*(-\varepsilon, \varepsilon),$$

$$(q, t) \mapsto \left(\phi \left(d \left(\frac{t^2}{2} \cdot h \right)_q \right), t +_J H_t(q) \right)$$

is an example of a Lagrangian cobordism with a bottleneck. At each self-intersection $(q_0 \to q_1) \in \mathcal{I}^{\text{si}}(L)$, it will either be the case that $h(q_0) < h(q_1)$ or $h(q_1) < h(q_0)$.

Definition 4.2.3. Let $i_t: L_0 \times I \to X$ with primitive $H_t: L_0 \times I \to \mathbb{R}$ be a bottleneck datum. We say that $(q_0 \to q_1) \in \mathcal{I}^{\text{si}}(L_0)$ has a maximum grading in the base from the bottleneck datum if $\frac{dH_t}{dt}(q_0, 0) > \frac{dH_t}{dt}(q_1, 0)$; otherwise, we say that this generator receives an minimum grading in the base from the bottleneck.

Remark 4.2.4. Our convention for maximal/minimal grading from the base is likely related to the convention of positive/negative perturbations chosen in the paper [8, Remark 3.2.1].

Example 4.2.5 (Whitney sphere, revisited). For the bottleneck $i_{A(t)}^{n-1,0}: S_{r(t)}^{n-1} \to \mathbb{C}^{n-1}$ constructed in Example 4.2.2,

$$\frac{dH_t}{dt}(q_\pm)\Big|_{t=0} = \pm 1$$

so that $(q_- \to q_+)$ has a maximum grading in the base.

We can construct another bottleneck datum so that $(q_- \to q_+)$ has a minimum grading in the base. As $H^1(S^{n-1}) = 0$, all Lagrangian homotopies are exact homotopies. Consider the homotopy $i_{s(t)}^{n-1,0}: L_{B(t)}^{n-1,0} \to \mathbb{C}^{n-1}$, where $B(t) = 1 + t^2$, and $t \in [-1, 1]$. This bottleneck is the exact homotopy of which first decreases the radius of the Whitney sphere, then increases the radius of the Whitney sphere. For this bottleneck, $(q_- \to q_+)$ inherits a minimum grading from the base.

While both $i_{A(t)}^{n-1,0}, i_{B(t)}^{n-1,0}$ provide bottlenecks for the Whitney sphere, they are really quite different as Lagrangians in $\mathbb{C}^{n-1} \times \mathbb{C}$. The first bottleneck can be completed to a null-cobordism by simply adding in two caps (yielding the Whitney n-sphere in $\mathbb{C}^n$); the second bottleneck cannot be closed off without adding in either a handle or another self-intersection.

Remark 4.2.6. Although not relevant to the discussion of bottlenecks, Example 4.2.5 gives us an opportunity to address the discussion at the end of [15, Section 3.4] related to Whitney degenerations and Lagrangian surgery. The question Haug asks is: Does containing a Whitney isotropic sphere suffice for implanting a surgery model? Haug shows that this is not sufficient condition. The specific example considered is the Whitney sphere $L^{2,0,+} \subset \mathbb{C}^2$, which contains a 1-Whitney isotropic

$$L^{1,0,+} \subset L^{2,0,+} \subset \mathbb{C}^2.$$

If there was a Lagrangian surgery trace $K^{1,2} \colon L^{2,0,+} \rightsquigarrow L^-$, which collapsed the 1-Whitney isotropic, then L^- would have the topology of an embedded pair of spheres. Since no such Lagrangian submanifold exists in $\mathbb{C}^2$, we conclude that possessing a 1-Whitney isotropic does not suffice for implanting a surgery handle.

Upon a closer examination, we see that the Whitney 1-isotropic has a small normal neighborhood

$$L^{1,0,+} \times I \subset L^{2,0,+} \subset \mathbb{C} \times \mathbb{C},$$

which gives it the structure of a Lagrangian bottleneck. This is the bottleneck $i_{A(t)}^{n-1,0}$ described above.

Consider instead a Lagrangian submanifold L^+ which contains a Whitney k-isotropic $L^{k,0,+} \times I \subset L$ with a neighborhood giving it the structure of the $i_{B(t)}^{n-1,0}$ bottleneck. Then there exists a Lagrangian surgery trace $K^{k,n-k} \colon L^+ \rightsquigarrow L^-$. This can be immediately observed for instance in Figure 13 – the right-hand side is exactly isotopic to $i_{B(t)}^{1,0}$ (with the cobordism parameter in the vertical direction).

As Example 4.2.5 shows, when one turns an immersed Lagrangian submanifold L_0 into a Lagrangian cobordism with a bottleneck K at $t = 0$, the self-intersections of K with cobordism parameter 0 are in bijection with the self-intersections of L_0. The gradings of the self-intersections of K differ from the gradings of the self-intersections of L_0. When $(q_0 \to q_1) \in \mathcal{I}^{\mathrm{si}}(L^0)$ has a maximum grading in the base, the gradings of self-intersections on K are

$$\mathrm{ind}((q_0, 0) \to (q_1, 0)) = \mathrm{ind}(q_0 \to q_1),$$
$$\mathrm{ind}((q_1, 0) \to (q, 0)) = \mathrm{ind}(q_1 \to q_0) + 1.$$

From a Floer-theoretic viewpoint this is problematic, as the underlying philosophy of Theorem 2.3.1 is that the Floer cohomology of a Lagrangian cobordism should agree with the Floer cohomology of the slice of the cobordism, and this mismatch in index shows that these two groups cannot be the same by Euler characteristic considerations. While bottlenecks give us a way to relate the immersed points of K with the immersed points of the slice at the bottleneck, the self-intersections of K with minimum grading from the base will receive the wrong grading. An additional problem

with self-intersections of K with minimum grading from the base is that one cannot necessarily obtain compactness for moduli spaces of holomorphic teardrops with output on a self-intersection with minimum grading in the base. We therefore use *double bottlenecks* instead.

Definition 4.2.7. Let $i: L \to X$ be an immersion with transverse self-intersections, and $H: L \to \mathbb{R}$, $\phi: B_r^* L \to X$ as in the construction of a standard bottleneck. Furthermore, assume that $dH = 0$ at each self-intersection point. Let $\rho(t) = t(t + \varepsilon)(t - \varepsilon)$, and $\tilde{H}_t(q) := \rho'(t) H(q)$. A standard double bottleneck datum is the exact homotopy

$$i_t: L \times (-\varepsilon, \varepsilon) \to X,$$

$$(q, t) \mapsto \phi(d(\tilde{H}_t)_q)$$

with the property that $\pi_X i_t(q_0) = \pi_X i_t(q_1)$ if and only if $i(q_0) = i(q_1)$ and $t = \pm\varepsilon/\sqrt{3}$ (these are the critical points of ρ).

We say that a Lagrangian cobordism $K \subset X \times \mathbb{C}$ has a double bottleneck at time t_0 if $K|_{[t_0-\varepsilon, t_0+\varepsilon]}$ is the suspension of a standard double bottleneck datum.

If K is a Lagrangian cobordism with standard double bottleneck datum at times t^-, t^+ given by the data $H^\pm: L^\pm \to \mathbb{R}$, $\phi^\pm: B_r^* L^\pm \to X$, then we will write

$$(K, t^+, t^-): (L^+, H^+) \rightsquigarrow (L^-, H^-)$$

and say that (K, t^+, t^-) is a Lagrangian cobordism between L^+, L^- with double bottlenecks determined by H^+, H^-.

Observe that in the setting where $K: L^+ \rightsquigarrow L^-$ is a Lagrangian cobordism, $L^\pm$ are embedded, and $t^\pm$ are chosen as in Definition 2.1.1, then

$$(K, t^+, t^-): (L^+, 0) \rightsquigarrow (L^-, 0)$$

is a Lagrangian cobordism with double bottlenecks.

Each self-intersection $(q_0 \to q_1) \in \mathcal{I}^{\mathrm{si}}(L)$ corresponds to two self-intersections $((q_0, \pm\varepsilon/\sqrt{3}) \to (q_1, \pm\varepsilon/\sqrt{3}))$ in $\mathcal{I}^{\mathrm{si}}(L, H_t)$; if $((q_0, \varepsilon/\sqrt{3}) \to (q_1, \varepsilon/\sqrt{3}))$ has a maximal grading from the base, then $((q_0, -\varepsilon/\sqrt{3}) \to (q_1, -\varepsilon/\sqrt{3}))$ has a minimal grading from the base (and vice versa). To each immersed point $(q_0 \to q_1) \in \mathcal{I}^{\mathrm{si}}(L)$, we can associate a value

$$A_{(q_0 \to q_1)} := \int_{\varepsilon/\sqrt{3}}^{\varepsilon/\sqrt{3}} \left(\tilde{H}_t(q_1) - \tilde{H}_t(q_0) \right) dt.$$

Finally, we observe that for each $(q_0 \to q_1)$, the curves $t + {}_J\tilde{H}_t(q_0)$ and $t + {}_J\tilde{H}_t(q_1)$ bound a strip in $u: \mathbb{R} \times I \to \mathbb{C}$ whose area is $A_{(q_0 \to q_1)}$. Since q_0, q_1 are critical points of h, they are fixed by the homotopy and we obtain a holomorphic strip

$$\{i_0(q_0)\} \times u: \mathbb{R} \times I \to X \times \mathbb{C}$$

with boundary on the double bottleneck.

Given Lagrangian cobordisms with double bottlenecks

$$(K^{+0}, t^+, t^0) \colon (L^+, H^+) \rightsquigarrow (L^0, H^0), \quad (K^{0-}, t^0, t^-) \colon (L^0, H^0) \rightsquigarrow (L^-, H^-),$$

there exists a composition $(K^{0-} \circ K^{+0}, t^+, t^-) \colon (L^+, H^+) \rightsquigarrow (L^-, H^-)$. The composition is covered by the charts $K^{+0}|_{>t^0-\varepsilon}, K^{0-}|_{<t^0+\varepsilon}$ which overlap over the double bottleneck defined by L^0, H^0 and ϕ^0. In the setting where $H^+, H^0, H^- = 0$ (so that L^+, L^0, L^- are embedded) this agrees with the usual definition of composition of Lagrangian cobordisms.

4.2.2. χ^{si} for Lagrangian cobordisms with double bottlenecks. Recall that for immersed compact Lagrangian submanifolds L with transverse self-intersections, the Floer cochains are defined in equation (4.2) to be critical points of an auxiliary Morse function $f \colon L \to \mathbb{R}$ or ordered pairs of points in L whose image in X agree. From this data we defined a self-intersection Euler characteristic. When defining the self-intersection Euler characteristic for Lagrangian cobordisms, we must state which auxiliary Morse functions are admissible, and how to count the self-intersection points at the double bottlenecks. At a minimum, the self-intersection Euler characteristic that we define for Lagrangian cobordisms with double bottlenecks should satisfy the relation

$$\chi^{\mathrm{bot}}((K^{0-} \circ K^{+0}, t^+, t^-)) = \chi^{\mathrm{bot}}(K^{+0}, t^+, t^0) + \chi^{\mathrm{bot}}(K^{0-}, t^0, t^-) - \chi^{\mathrm{si}}(L^0). \quad (4.3)$$

We first handle the issue of the auxiliary Morse function. Pick $f^\pm \colon L^\pm \to \mathbb{R}$ Morse functions. For the Lagrangian cobordism (K, t^+, t^-) we take an admissible Morse function (adapted from [16, Definition 2.1.3]), which is a Morse function $f \colon K \to \mathbb{R}$ satisfying:

- The Morse flow restricted to the fibers above real coordinates t^- and t^+ are determined by $f^\pm$,

$$(\pi_{\mathbb{R}})_* \operatorname{grad} f|_{\pi_{\mathbb{R}}^{-1}(t^-)} = 0, \qquad (\pi_{\mathbb{R}})_* \operatorname{grad} f|_{\pi_{\mathbb{R}}^{-1}(t^{-+})} = 0,$$
$$(\pi_X)_* \operatorname{grad} f|_{\pi_{\mathbb{R}}^{-1}(t^-)} = \operatorname{grad} f^-, \quad (\pi_{\mathbb{R}})_* \operatorname{grad} f|_{\pi_{\mathbb{R}}^{-1}(t^+)} = \operatorname{grad} f^+.$$

- The gradient points outward in the sense that

$$dt(-\operatorname{grad} f)|_{\pi_{\mathbb{R}}(q) < t^-} < 0, \quad dt(-\operatorname{grad} f)|_{t^- < \pi_{\mathbb{R}}(q) < t^- + \varepsilon} > 0,$$
$$dt(-\operatorname{grad} f)|_{\pi_{\mathbb{R}}(q) > t^+} > 0, \quad dt(-\operatorname{grad} f)|_{t^+ - \varepsilon < \pi_{\mathbb{R}}(q) < t^- +} < 0.$$

For such a choice of Morse function, $\mathrm{Crit}(f^+), \mathrm{Crit}(f^-)$ naturally are subsets of $\mathrm{Crit}(f)$, and all critical points of f have cobordism parameter between t^- and t^+ (inclusive). For embedded Lagrangian cobordisms, the Euler characteristic defined using the cochains of an admissible Morse functions satisfies equation (4.3).

This leaves us with handling the self-intersections of the Lagrangian cobordism with double bottlenecks (K, t^+, t^-). From the design of the double bottlenecks, we see that there are inclusions $\mathcal{I}^{\mathrm{si}}(L^\pm) \hookrightarrow \mathcal{I}^{\mathrm{si}}(K)$ sending each intersection to the corresponding intersection in the double bottleneck with maximum grading from the base. This inclusion preserves degree. Unfortunately, the double bottlenecks contain additional intersections not corresponding to elements of $\mathcal{I}^{\mathrm{si}}(L^\pm)$ coming from those intersections with minimal degree in the base. We must judiciously throw out some of these intersections.

Definition 4.2.8. Let $(K, t^+, t^-): (L^+, H^+) \rightsquigarrow (L^-, H^-)$ be a Lagrangian cobordism with double bottlenecks, parameterized by $j: K \to X \times \mathbb{C}$. The bottlenecked Floer generators of (K, t^+, t^-) are

$$
\begin{aligned}
\mathcal{I}^{\mathrm{bot}}(K, t^+, t^-) := {}& \mathrm{Crit}(f) \\
\sqcup\, & \{(q^0 \to q^1) \in \mathcal{I}^{\mathrm{si}} \mid t^- \leq \pi_\mathbb{R} \circ j(q^0) \leq t^+\} \\
\sqcup\, & \{(q^0 \to q^1) \in \mathcal{I}^{\mathrm{si}} \mid \pi_\mathbb{R} \circ j(q^0) < t^-,\ \text{and } (q^0 \to q^1) \text{ has max grad. from base}\} \\
\sqcup\, & \{(q^0 \to q^1) \in \mathcal{I}^{\mathrm{si}} \mid t^+ < \pi_\mathbb{R} \circ j(q^0),\ \text{and } (q^0 \to q^1) \text{ has max grad. from base}\}.
\end{aligned}
$$

For $(K, t^+, t^-): (L^+, H^+) \rightsquigarrow (L^-, H^-)$ a Lagrangian cobordism with double bottlenecks, we define

$$
\chi^{\mathrm{bot}}(K, t^+, t^-) := \sum_{x \in \mathcal{I}^{\mathrm{bot}}(K, t^+, t^-)} (-1)^{\mathrm{ind}(x)}.
$$

Claim 4.2.9. $\chi^{\mathrm{bot}}(K, t^+, t^-)$ *satisfies equation* (4.3).

Proof. From the definition $\chi^{\mathrm{bot}}(K, t^+, t^-)$ we see that

$$
\begin{aligned}
\mathcal{I}^{\mathrm{bot}}(K^{+-}, t^+, t^-) = {}& \left(\mathcal{I}^{\mathrm{bot}}(K^{0-}, t^0, t^-) \sqcup (\mathcal{I}^{\mathrm{bot}}(K^{+0}, t^+, t^0) \setminus \mathrm{Crit}(f^0))\right) \\
\sqcup\, & \{(q^0 \to q^1) \in \mathcal{I}^{\mathrm{si}}(K^{0-}) \mid \pi_\mathbb{R} \circ j(q^0) < t^-, \\
& \qquad\qquad\qquad\quad \text{and } (q^0 \to q^1) \text{ has } min \text{ grad. from base}\} \\
\sqcup\, & \{(q^0 \to q^1) \in \mathcal{I}^{\mathrm{si}}(K^{+0}) \mid t^+ < \pi_\mathbb{R} \circ j(q^0), \\
& \qquad\qquad\qquad\quad \text{and } (q^0 \to q^1) \text{ has } min \text{ grad. from base}\}.
\end{aligned}
$$

There is a bijection between $\mathcal{I}^{\mathrm{si}}(L^0)$ and the union of the latter 2 terms, however this bijection increases the index of the critical points by 1. Equation (4.3) immediately follows. ∎

4.2.3. Decomposition of Lagrangian cobordism via double bottlenecks. Observe that the methods used in Proposition 3.1.6 can similarly be used to give a decomposition of a Lagrangian submanifold $K \subset X$ into bottlenecked Lagrangian submanifolds.
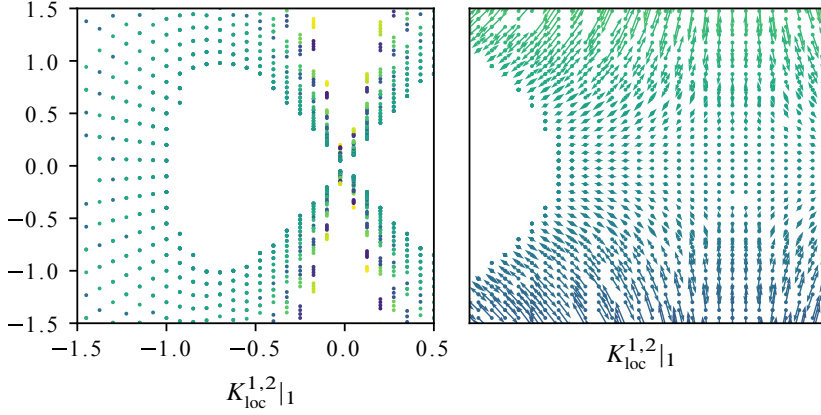
Figure 20. Left: The shadow of $K_{1,2}|_1$ as projected to the first complex coordinate, with bottleneck at the origin. Right: The Lagrangian $K_{1,2}|_1$ drawn as a section of $T^*\mathbb{R}^2$.

Proposition 4.2.10. *Let $K \subset X \times \mathbb{C}$ be a Lagrangian submanifold, and t^-, t^+ regular values of the projection $\pi_{\mathbb{R}} \colon K \to \mathbb{R}$. Suppose $H_t \colon L^{\pm} \times [t^{\pm} - \varepsilon, t^{\pm} + \varepsilon] \to \mathbb{R}$ generates an exact homotopy whose suspension is $K|_{[t^{\pm} - \varepsilon, t^{\pm} + \varepsilon]}$. Furthermore, suppose that over this region, $H_t(q_1) \neq H_t(q_2)$ for each double point $(q_1 \to q_2)$ of the slice $K|_t$. Then there exist choices of Lagrangian bottleneck data $(L^{\pm}, H^{\pm})$ and a Lagrangian cobordism with double bottleneck*

$$(K, t^+, t^-) \colon (L^+, H^+) \rightsquigarrow (L^-, H^-).$$

While the standard surgery handle does not have transverse self-intersections, there is a geometrically pleasing construction of a bottlenecked Lagrangian surgery trace.

Consider $L^{k+1,n-k,+} \subset \mathbb{C}^{n+1}$. We now will now treat $L^{k+1,n-k,+}$ as if it were a Lagrangian cobordism, with first coordinate of $\mathbb{C}^{n+1}$ serving as the cobordism coordinate. We will therefore write

$$L^{k+1,n-k,+} \subset \mathbb{C}^n \times \mathbb{C}_1.$$

From this viewpoint, $L^{k+1,n-k,+} \subset \mathbb{C}^n \times \mathbb{C}_1$ can also be considered as a Lagrangian surgery trace, as $\pi_{\mathbb{R}_1} \colon K^{k+1,n-k+1,+}|_{\pi_{\mathbb{R}_1} < 1/2} \to \mathbb{R}$ has a single critical point. Furthermore, this Lagrangian has a single self-intersection, forming a bottleneck under the $\mathbb{C}_1$ projection: see the shadow projection drawn in Figure 20. By applying Proposition 4.2.10 to $K^{k+1,n-k+1,+}|_{\pi_{\mathbb{R}_1} < 1/2}$, we obtain a double bottlenecked Lagrangian surgery trace.

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Definition 4.2.11. We call the resulting Lagrangian cobordism with double bottlenecks

$$K_{A,B}^{k,n-k+1} \subset \mathbb{C}^{n+1},$$

where the quantity A measures the symplectic area of the class of the teardrop on $K_{A,B}^{k,n-k+1}$, while B is the area of holomorphic strip associated with a double bottleneck.

Note that $K_{A,B}^{k,n-k+1}$ has 2 self-intersections and a single critical point of the height function. By following the method of proof in Theorem 3.3.1, we obtain a doubled-bottleneck decomposition of Lagrangian cobordisms.

Corollary 4.2.12. *Let $(K, t^-, t^+): (L^+, H_t^+) \rightsquigarrow (L^-, H_t^-)$ be a Lagrangian cobordism with double bottlenecks. Then there a sequence of Lagrangian cobordisms with double bottlenecks*

$$(K_{H_t^i}, t_{i+1}^-, t_i^+): (L_{i+1}^-, h_{i+1,t}^-) \rightsquigarrow (L_i^+, h_{i,t}^+) \quad \text{for } i \in \{0, \ldots, j\},$$
$$(K_{A_i,B_i}^{k_i,n-k_i+1}, t_i^+, t_i^-): (L_i^+, h_{i,t}^+) \rightsquigarrow (L_i^-, h_{i,t}^-) \qquad \text{for } i \in \{1, \ldots j\},$$

which satisfy the following properties:

- $L_{j+1}^- = L^+$ and $L_0^+ = L^-$;
- *each $K_{A_i,B_i}^{k_i,n-k_i+1}$ is a Lagrangian surgery trace with double bottlenecks;*
- *each $K_{H_t^i}$ is the suspension of an exact homotopy; and*
- *there is an exact homotopy between*

$$K \sim K_{H_t^j} \circ K_{A_j,B_j}^{k_j,n-k_j+1} \circ K_{H_t^{j-1}} \circ \cdots \circ K_{H_t^1} \circ K_{A_1,B_1}^{k_1,n-k_1+1} \circ K_{H_t^0},$$

whose Hofer norm can be made as small as desired.

We can now upgrade Proposition 4.1.2 to compare the Euler characteristic of a Lagrangian cobordism with immersed ends to the Euler characteristic of the ends.

Proposition 4.2.13. *Let $(K, t^-, t^+): (L^+, H^+) \to (L^-, H^-)$ be a Lagrangian cobordism with double bottlenecks. Then we have a matching of Euler characteristics*

$$\chi^{\mathrm{bot}}(K) = \chi^{\mathrm{si}}(L^-) = \chi^{\mathrm{si}}(L^+).$$

Proof. As in Proposition 4.1.2, it suffices to check for the double bottleneck surgery traces:
$$K_{A,B}^{k,n-k+1}: L_{\mathrm{loc}}^{k,n-k,+} \rightsquigarrow L_{\mathrm{loc}}^{k+1,n-k-1,-}.$$

We discuss the cases $-1 < k < n$. Equip $K_{A,B}^{k,n-k+1}$ with an admissible Morse function f which is given by perturbing the Morse–Bott function $\rho \circ \pi_{\mathbb{R}}$ in Figure 21 by the functions $f^{\pm}$ from Proposition 4.1.2 at the maxima of $\rho \circ \pi_{\mathbb{R}}$. The function $\rho \circ \pi_{\mathbb{R}}$ has a single critical point (the critical point of the Morse surgery handle) with real coordinate between t^+, t^- which lives in degree $n - k$. The remaining critical points of f come from the perturbations $f^{\pm}$, which we enumerated in Table 1. The self-intersections of $K_{A,B}^{k,n-k+1}$ correspond to two copies of the self-intersections of $L_{\mathrm{loc}}^{k,n-k,+}$ whose indexes are shifted depending on whether they receive maximum or minimum grading from the base. The bottleneck cochains $\mathcal{I}^{\mathrm{bot}}(K_{A,B}^{k,n-k+1})$ are listed in Figure 21. From reading the table, we see that

$$\chi^{\mathrm{bot}}(K_{A,B}^{k,n-k+1}) = (-1)^{k+1} + (-1)^{n-k+1},$$

which agrees with $\chi^{\mathrm{si}}(L_{\mathrm{loc}}^{k+1,n-k-1,-})$. ∎

4.3. Overview of immersed Floer cohomology

4.3.1. Immersed Lagrangian Floer cohomology. There are several models based on the work of [1] which produce a filtered A_{∞} algebra associated to an immersed Lagrangian $i: L \to X$. We follow notation from [23] adapted to the Morse cochain setting, although much of our intuition for this filtered A_{∞} algebra comes from [5, 9, 13, 19].

Let $f: L \to \mathbb{R}$ be a Morse function, $i: L \to X$ be a graded Lagrangian immersion with transverse self-intersections, and $\mathcal{I}^{\mathrm{si}}(i) = \{(p \to q) \mid p, q \in L, i(p) = i(q)\}$ be the set of ordered preimages of transverse self-intersections. Assume that

$$\mathrm{Crit}(f) \cap \mathcal{I}(L) = \emptyset.$$

Definition 4.3.1 (Fukaya–Oh–Ohta–Ono [14]). Let R be a commutative ring with unit. The *universal Novikov ring* over R is the set of formal sums

$$\Lambda_{\geq 0} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in R, \ \lambda_i \in \mathbb{R}_{\geq 0}, \ \lim_{i \to \infty} \lambda_i = \infty \right\}.$$

Let k be a field. The *Novikov field* is the set of formal sums

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in k, \lambda_i \in \mathbb{R}, \ \lim_{i \to \infty} \lambda_i = \infty \right\}.$$
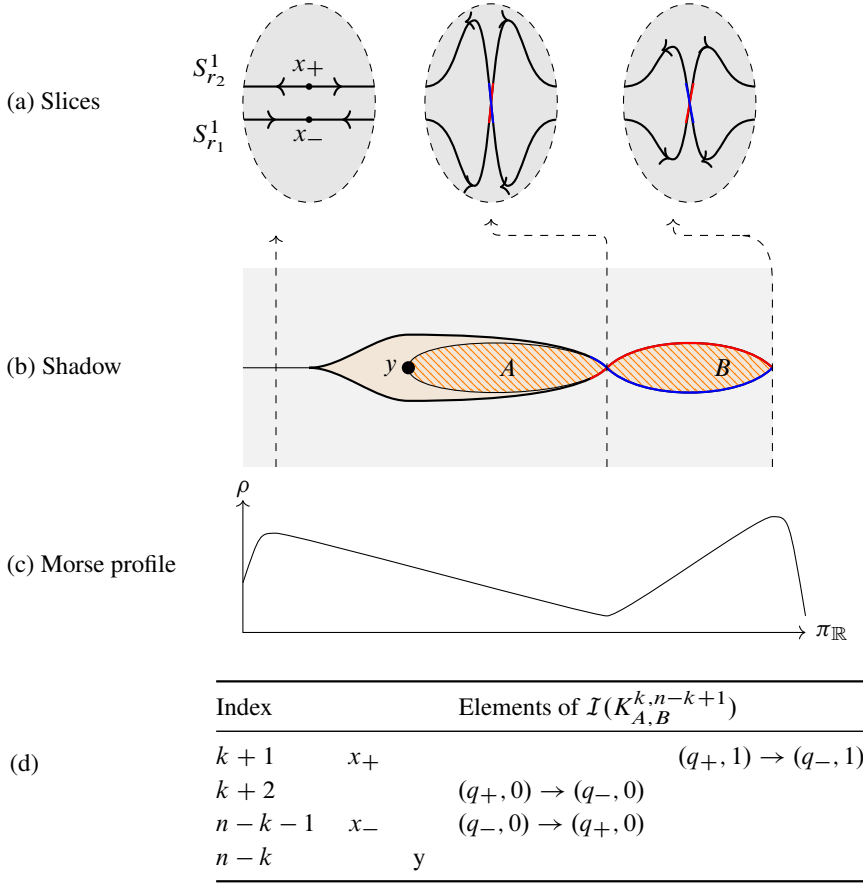
Figure 21. The slices, shadow, and choice of profile ρ used in constructing the Morse function. The drawn example is for $k = 0, n = 2$. (a) Critical points of the function $f^{\pm}$. The gradient flow of $f^{\pm}$ on the slice is indicated by the arrows. (b) The shadow of our Lagrangian cobordism. (c) The profile of a Morse–Bott function on $K_{A,B}^{k,n-k+1}$ which is subsequently perturbed by $f^{\pm}$ at the Morse–Bott maxima and minimum. (d) A tabulation of the indices of elements in $\mathcal{I}^{\text{bot}}(K_{A,b}^{k,n-k+1})$. Observe that cochains listed in rows $k + 1, n - k - 1$ agree with those from Table 1.

An *energy filtration* on a graded Λ-module $A^{\bullet}$ is a filtration $F^{\lambda_i} A^k$ so that

- each A^k is complete with respect to the filtration[4] and has a basis with valuation zero over Λ;
- multiplication by T^{λ} increases the filtration by λ.

---

[4]In the sense that the topology induced by taking a basis of opens to be $F^{\lambda_i} A^k$ is complete.

Associated to an immersed Lagrangian submanifold with transverse self-intersections is a filtered graded Λ-module generated on critical points of f and $(q_0 \to q_1)$:

$$CF^k(L) = \bigoplus_{\substack{x \in \mathrm{Crit}(f), \\ \mathrm{ind}(x)=k}} \Lambda_x \oplus \bigoplus_{\substack{(q_0 \to q_1) \in \mathcal{I}^{\mathrm{si}}(L), \\ \mathrm{ind}(q_0 \to q_1)=k}} \Lambda_{(q_0 \to q_1)}.$$

We will write $\mathcal{I}(L) = \mathrm{Crit}(f) \cup \mathcal{I}^{\mathrm{si}}(L)$ for the set of generators of $CF^\bullet(L)$. The Floer cohomology of L comes with filtered product operations deforming the Morse structure by counts of holomorphic polygons with boundary on L,

$$\langle m^k(x_1, \ldots, x_k), x_0 \rangle = \sum_{\beta \in H_2(X,L)} T^{\omega(\beta)} \# \mathcal{M}_{\mathcal{P}}(L, \beta, \underline{x}_i),$$

where

$$\underline{x} = x_0, x_1, \ldots, x_k \in \mathrm{Crit}(f) \cup \mathcal{I}^{\mathrm{si}}(L),$$

$\mathcal{P}$ is perturbation datum, and $\mathcal{M}_{\mathcal{P}}(L, \beta, \underline{x}_i)$ is the moduli space of $\mathcal{P}$-perturbed pseudoholomorphic treed polygons. We provide a short description of what this data entails. In [9, Section 4.2], a treed disk C is a tree whose vertices v are labeled with disks with internal marked points and $\deg(v)$ boundary marked points and whose edges are labeled with (possibly semi-infinite) intervals. From this data we obtain a space C which is glued together from the disks and intervals labeling the vertices and edges. We write $C = S \cup T$, where S is the *surface* portion of the treed disk, and T are the edges. For a pseudoholomorphic treed polygon, this surface portion is allowed to have strip-like ends. Given a Lagrangian L with Morse function f we examine maps $u \colon C \to X$ which sends $u|_{\partial S}, u|_T \subset L$. Without considering perturbation datum, a pseudoholomorphic treed disk is such a map where $u|_S$ is pseudoholomorphic, and $u|_T$ are flow lines of ∇f. In order to achieve regularity, we need to perturb the J-holomorphic curve and Morse flow line equations by picking perturbations of the almost complex structure and Morse function. A perturbation datum $\mathcal{P}$ is a choice of perturbations for all domains C. These choices must be made coherently, meaning that the perturbations for treed disks which differ by disk bubbling and flow-line breaking have related perturbation datum [9, Definition 4.12].

We reserve $\mathcal{P}^0$ to denote the trivial perturbation datum (so that $\mathcal{M}_{\mathcal{P}^0}(L, \beta, \underline{x}_i)$ consists of configurations of J-holomorphic polygons attached to Morse flow-lines). It is expected that the m^k form a filtered A_∞ algebra (in the sense of [14, Chapter 10]). As we only use these structure coefficients to make some suggestive computations, we do not claim that here.

In good examples, we know that $CF^\bullet(L)$ is an A_∞ algebra. We give such an example below.

$$-1 \qquad (q_- \to q_+)$$

$$\downarrow$$

$$0 \qquad\qquad e$$

$$\vdots$$

$$n \qquad\qquad x$$

$$\downarrow$$

$$n+1 \qquad (q_+ \to q_-)$$

Figure 22. m^1-structure on $CF^\bullet(L_A^{n,0})$ with $n \geq 2$. There are no Morse flow lines contributing the differential on the Floer cohomology. However, the presence of holomorphic teardrops contributes to differential (drawn in red) canceling all homology classes.

4.3.2. Example computation: Whitney sphere. We review a computation from [2] computing the Floer complex for the Whitney sphere $L_A^{n,0} \subset \mathbb{C}^n$. We assume that $n \geq 2$ so that $L_A^{n,0}$ is a graded Lagrangian submanifold. We take a Morse function for the $L_A^{n,0}$ given by the x_1 coordinate. Then $\mathrm{Crit}(f) = \{e, x\}$, where e is the maximum of f in degree 0, and x is a generator in degree n. We take the points $q_\pm = (\pm r(A), 0, \ldots, 0) \in S_{r(A)}^n$ to be the preimages of the self-intersections of this Whitney sphere. The computation from Section 4.1 shows that $(q_+ \to q_-)$ has degree $n + 1$, and $(q_- \to q_+)$ has degree -1.

We now give a description of the moduli space of holomorphic teardrops with boundary on the Whitney sphere. Let $p = (x_1, \ldots, x_n, 0) \in S^n \subset \mathbb{R}^n \times \mathbb{R}$ be a point on the equatorial sphere. Then we can construct a holomorphic teardrop with boundary on the Whitney sphere $L_A^{n,0} \subset \mathbb{C}^n$ which is parameterized by

$$u_p \colon D_\alpha \to \mathbb{C}^n,$$

$$z \mapsto (x_1 z, x_2 z, \ldots, x_n z),$$

where

$$D_\alpha = \{a + \jmath b \mid a \in [0, 1], |b| \leq 2a\sqrt{1 - a^2}\}.$$

Let $\beta \in H_2(X, L)$ be the homology class of this teardrop. The parameter A of the Whitney sphere is constructed so that

$$\omega(\beta) = A.$$

Claim 4.3.2 ([2, Lemma 5.1.2]). *The moduli space of holomorphic teardrops with boundary on S_1^n is regular for the standard almost complex structure and has one component,*

$$\mathcal{M}_J(L_A^{n,0}, \beta, (q_- \to q_+)) = S^{n-1},$$

where β is the class of the teardrop $[u_p]$. The evaluation map

$$\mathrm{ev}\colon \mathcal{M}_J(L_A^{n,0}, \beta, (q_- \to q_+)) \times (D^2 \setminus \{1\}) \to L_A^{n,0}$$

is a homeomorphism onto $L_A^{n,0} \setminus \{q_+, q_-\}$.

Proof. In the version 2 preprint of [2], it is proven that all teardrops with boundary on $L_A^{n,0}$ for the standard almost complex structure are of the form u_p.

It remains to show that these teardrops are regular. In this proof, we will write $L := L_A^{n,0}$. We need only show that the linearized Dolbeault operator

$$D_{u_p}\colon W^{1,k;\varepsilon}(u_p^* T\mathbb{C}^n, \lambda) \to L^{k;\varepsilon}(\Omega^{0,1} \otimes u_p^* T\mathbb{C}^n)$$

surjects. For expositional purposes we will now restrict to $p = (0, 0, \dots, 1)$. Since $T\mathbb{C}^n$ trivializes across the coordinates of $\mathbb{C}^n$ as $\bigoplus_{i=1}^n \mathcal{L}_i$, the pullback $u_p^* T\mathbb{C}^n$ has a trivialization $\bigoplus_{i=1}^n (u_p^* \mathcal{L}_i)$ as well. The proof of Claim 4.1.1 shows that the Lagrangian sub-bundle $(u_p|_{\partial D^2})^* TL$ can also be trivialized. In that proof, the curve $\gamma(\theta) = (\sin(\theta), 0, \dots, 0, \cos(\theta)) \in S^n \subset \mathbb{R}^{n+1}$ has the property that

$$i^{n,0} \circ \gamma(\theta) = (0, \dots, \cos(\theta) + \jmath 2 \sin(\theta) \cdot \cos(\theta))$$

parameterizes the boundary of u_p. The proof of Claim 4.1.1 therefore shows that $(u_p|_{\partial D^2})^* TL$ splits as a sum of real line bundles $\bigoplus_{i=1}^n \lambda_i$. Furthermore, this splitting respects the splitting $T\mathbb{C}^n = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$, in the sense that

$$\lambda_i \subset (u_p|_{\partial D^2})^* \mathcal{L}_i.$$

The map D_{u_p} decomposes across the trivialization

$$\bigoplus_{i=1}^n W^{1,k;\varepsilon}(u_p^* \mathcal{L}_i, \lambda_i) \to \bigoplus_{i=1}^n L^{k;\varepsilon}(\Omega^{0,1} \otimes u_p^* \mathcal{L}_i).$$

Let $D_{u_p}|_{\mathcal{L}_i}\colon W^{1,k;\varepsilon}(u_p^* \mathcal{L}_i, \lambda_i) \to L^{k;\varepsilon}(\Omega^{0,1} \otimes u_p^* \mathcal{L}_i)$ be the restriction of D_{u_p} to each component of the trivialization. We show that each of the $D_{u_p}|_{\mathcal{L}_i}$ surjects.

- For $1 \le i < n$, the argument from Claim 4.1.1 shows that the Maslov index is of the loop determined by λ_i is 0; the Fredholm index of $D_{u_p}|_{\mathcal{L}_i}$ is 1. By automatic regularity in dimension one, this linearized operator surjects.

- For $i = n$, the argument from Claim 4.1.1 shows that the Maslov index is 2; the Fredholm index of $D_{u_p}|_{\mathcal{L}_1}$ is 3. By automatic regularity in dimension one, this linearized operator surjects.

It follows that D_{u_p} is surjective, with Fredholm index

$$\operatorname{ind}(D_{u_p}) = (n-1) + 2 = n + 1.$$

This gives us an $\operatorname{ind}(D_{u_p}) - 2 = n - 1$-dimensional moduli space of teardrops, as expected. ∎

Figure 22 contains a summary of generators and differentials of $CF^\bullet(L_A^{n,0})$ which shows that the Floer cohomology of the Whitney sphere vanishes. We now place this computation in context with the table at the end of Figure 21. In that table, a matching is visible on the chain level between critical points of the surgery handles and the self-intersections of the handles. In the Whitney sphere, the holomorphic teardrops pair the Floer cochains from self-intersections with the Floer cochains from the auxiliary Morse function in the sense that the Floer differential is exact. The goal of Section 4.4 is to show that there are holomorphic teardrops on the Lagrangian surgery trace which similarly pairs critical point of $\rho \circ \pi_{\mathbb{R}} \colon K_{A,B}^{k,n-k+1}$ with a self-intersection in the double bottleneck.

These holomorphic teardrops can be used in examples of immersed Lagrangian cobordism with transverse self-intersection to construct continuation maps; a toy computation is given in Section 5.1.

4.3.3. Deformations and Lagrangian intersection Floer cohomology.
Lagrangian surgery can be interpreted as a geometric deformation of a Lagrangian submanifold. To understand the conjectured equivalent deformation on Floer cohomology, we need to discuss deformations of filtered A_∞ algebras.

Definition 4.3.3. Let A be a filtered A_∞ algebra. Let $\mathfrak{d} \in A$ be an element with $\operatorname{val}(\mathfrak{d}) > 0$. The $\mathfrak{d}$-deformed algebra $(A, \mathfrak{d})$ is the filtered A_∞ algebra whose chains match A, and whose product is given by

$$m_{\mathfrak{d}}^k(a_1 \otimes \cdots \otimes a_k)$$
$$= \sum_{i=0}^{\infty} \sum_{i_0 + \cdots + i_k = i} m^{k+i}(\mathfrak{d}^{\otimes i_0} \otimes a_1 \otimes \mathfrak{d}^{\otimes i_1} \otimes \cdots \otimes \mathfrak{d}^{\otimes i_{k-1}} \otimes a_k \otimes \mathfrak{d}^{\otimes i_k}).$$

We say that $\mathfrak{b}$ is a *bounding cochain* or *is a solution to the Maurer–Cartan equation* if $m_{\mathfrak{b}}^0 = 0$. The space of Maurer–Cartan solutions is denoted by $\mathcal{MC}(A)$. If there exists a bounding cochain for A, we say that A is *unobstructed*. Given N an $A - B$ filtered A_∞ bimodule, and deforming chains $\mathfrak{d}_A \in A$, $\mathfrak{d}_B \in B$, we obtain $(N, \mathfrak{d}_A, \mathfrak{d}_B)$ which

is an $(A, \mathfrak{d}_A) - (B, \mathfrak{d}_B)$ bimodule. The product structure is given by

$$
\begin{aligned}
m^{k_1|1|k_2}_{(A, \mathfrak{d}_A)|N|(B, \mathfrak{d}_B)}&(a_1 \otimes \cdots \otimes a_{k_1} \otimes n \otimes b_1 \otimes \cdots b_{k_2}) \\
&= \sum_{i,j \in \mathbb{N}} \sum_{\substack{i_0 + \cdots + i_{k_1} = i \\ j_0 + \cdots + j_{k_2} = j}} m^{k_1+i|1|k_2+j}_{A|N|B} \big(\mathfrak{d}_A^{\otimes i_0} \otimes a_1 \otimes \mathfrak{d}^{\otimes i_1} \otimes \cdots \\
&\qquad\qquad \cdots \otimes a_k \otimes \mathfrak{d}_A^{\otimes i_k} \otimes n \otimes \mathfrak{d}_B^{\otimes j_0} \otimes b_1 \otimes \mathfrak{d}^{\otimes j_1} \otimes \cdots \\
&\qquad\qquad \cdots \otimes b_k \otimes \mathfrak{d}_B^{\otimes j_k} \big).
\end{aligned}
$$

Observe that when $\mathfrak{b}$ is a bounding cochain for A then the differential on $(A, \mathfrak{b})$ squares to zero. If $\mathfrak{b}_A, \mathfrak{b}_B$ are bounding cochains for A, B, and N is an $A - B$ filtered A_∞ bimodule, then $(N, \mathfrak{b}_A, \mathfrak{b}_B)$ is an $(A, \mathfrak{b}_A) - (B, \mathfrak{b}_B)$ *uncurved* A_∞ bimodule. In particular, the differential on $(N, \mathfrak{b}_A, \mathfrak{b}_B)$ squares to zero.

When discussing the Lagrangian Floer cohomology, we will write $CF^\bullet(L, \mathfrak{d})$ for the Lagrangian Floer cohomology deformed by the element $\mathfrak{d}$. If $CF^\bullet(L)$ is unobstructed, then we say that the Lagrangian L is unobstructed. At lowest order, the Maurer–Cartan equation for a bounding cochain for $CF^\bullet(L)$ states that

$$
m^1_{\text{morse}}(\mathfrak{b}) = m^0 + \text{higher order terms,}
$$

where m^1_{morse} is the Morse differential on L. One interpretation of this is: to first approximation, an unobstructed Lagrangian submanifold is a Lagrangian submanifold where the boundary classes of holomorphic disks with boundary on L cancel out in homology.

Given two Lagrangian submanifolds L_0, L_1 which intersect transversely, we will write

$$
CF^\bullet(L_0, L_1) := \bigoplus_{x \in L_0 \cap L_1} \Lambda_x
$$

for the Lagrangian intersection Floer cochains of L_0 and L_1. It is expected that is an $CF^\bullet(L_0) - CF^\bullet(L_1)$ filtered A_∞ bimodule, whose bimodule structure comes from counting treed-pseudoholomorphic polygons. Given $\mathfrak{d}_i \in CF^\bullet(L_i)$ deforming cochains, we write $CF^\bullet((L_0, \mathfrak{d}_0), (L_1, \mathfrak{d}_1))$ for the $CF^\bullet(L_0, \mathfrak{d}_0) - CF^\bullet(L_1, \mathfrak{d}_1)$ filtered A_∞ bimodule. Again, as we are primarily interested in making some motivating computations, we will not use the algebraic structures of this complex and simply use the language of A_∞ algebras to make some remarks about the areas of various polygons which naturally occur in Lagrangian submanifolds associated to surgeries.

4.3.4. Running example: Multisection of $T^\ast S^1$. We return to the Lagrangian submanifold $L_E \subset T^\ast S^1$ first defined in Example 2.3.2 , parameterized by

$$
\begin{aligned}
S^1 &\to T^\ast S^1, \\
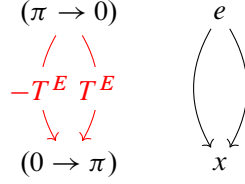\theta &\mapsto \left(2\theta, \frac{E}{8} \sin(\theta) \right).
\end{aligned}
$$

Figure 23. The chain complex $CF^\bullet(L_E)$. The black lines represent the Morse flow lines, while the red edges represent contributions to the differential coming from holomorphic strips.

This is a Lagrangian submanifold with 1-self-intersection where $\theta = 0$ and $\theta = \pi$. With respect to the standard holomorphic form on $T^*S^1 = \mathbb{C}^*$, the generator $(\pi \to 0)$ has degree 0 and $(0 \to \pi)$ has degree 1. There are two holomorphic strips of area E from the $(\pi \to 0)$ generator to the $(0 \to \pi)$ generator.

In this case there are no holomorphic disks or teardrops with boundary on L_E, so we may compute $HF^\bullet(L_E)$, which is isomorphic as a vector space to $H^\bullet(S^1) \oplus H^\bullet(S^1)$. We note that we do *not* expect $CF^\bullet(L_E)$ to be homotopic as a differential graded algebra or filtered A_∞ algebra to $C^\bullet(S^1 \cup S^1)$.

A computation which will be useful later is the Lagrangian intersection Floer cohomology between L_E and a section $S^1_{E'}$ of the cotangent bundle. Here, $S^1_{E'}$ is the Lagrangian section parameterized by

$$\theta \mapsto \left(\theta, \frac{E'}{2\pi}\right),$$

as drawn in Figure 24. Let $\mathfrak{b} := \sum_i a_i T^{D_i} (0 \to \pi) \in CF^\bullet(L_E)$ be a deforming cochain for L_E, whose lowest order term is $a_0 T^{D_0}$. We compute $CF^\bullet((L_E, \mathfrak{b}), S^1_{E'})$. The intersection between these two Lagrangian submanifolds consists of two points,

$$L_E \cap S^1_{E'} = \{p, q\}.$$

There are two holomorphic strips with boundary on L_E and $S^1_{E'}$ which have area A and B. Additionally, there exists a holomorphic triangle with ends limiting to q, p and $(0 \to \pi)$ of area C. The constants A, B, C, E, E' satisfy the relation:

$$B + E' = \frac{E}{2} + C.$$

The differential on $CF^\bullet((L_E, \mathfrak{b}), S^1_{E'})$ is given by

$$m^1(p) = T^A - T^B + a_0 T^{D_0} \cdot T^C + \mathcal{O}(\min(A, D_0 + C)).$$

For this differential to vanish, we obtain the constraint at lowest order that
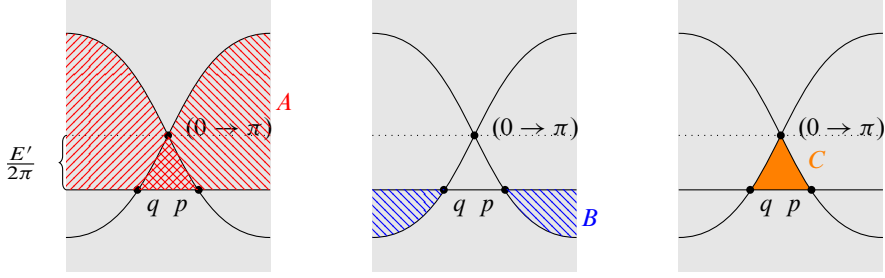
$$D_0 = \frac{E}{2} - E'.$$

Figure 24. Areas of various holomorphic strips and triangles appearing in the computation of $CF^\bullet((L_E, \flat), S_{E'}^1)$.

Furthermore, if this condition is met, there exists some extension of $T^{D_0}(0 \to \pi)$ to a bounding cochain $\flat$ so that $HF^\bullet((L_E, \flat), S_{E'}^1) = H^\bullet(S^1)$.

We will recover this bounding cochain from the Lagrangian surgery trace cobordism in Section 5.2.

4.4. Existence of holomorphic teardrops

4.4.1. Teardrops on surgery traces.
We now examine the Floer cohomology of the bottlenecked surgery handle $K_{A,B}^{k,n-k+1}$. The Floer cochains are given in Figure 21. The Floer products, in addition to counting treed pseudoholomorphic disks, counts treed pseudoholomorphic polygons ([2, 23]). The strip-like ends of these polygons limit to the self-intersections of our Lagrangian submanifold. When a treed pseudo-holomorphic polygon has only 1 strip like end (either input or output), we call it a treed pseudoholomorphic teardrop.

Theorem 4.4.1. *Let $K_{A,B}^{k,n-k+1}$ be the local model of the double bottleneck Lagrangian surgery trace. Take the standard Morse function $\pi_{\mathbb{R}}\colon K_{A,B}^{k,n-k+1} \to \mathbb{R}$, and standard choice of almost complex structure on $\mathbb{C}^{n+1}$. Let β_A be the class of teardrop with boundary on $K_{A,B}^{k,n-k+1}$ which has boundary on the Whitney isotropic contained in $K_{A,B}^{k,n-k+1}|_0$. The moduli space $\mathcal{M}_J(K_{A,B}^{k,n-k+1}, \beta_A, ((q_+, 0) \to (q_-, 0))), x_+)$ is comprised of a single regular treed holomorphic polygon, which is a teardrop.*

Proof. The space of holomorphic teardrops with input on $((q_+, 0) \to (q_-, 0)))$ can be described by the space of holomorphic teardrops with boundary on the isotropic k-sphere contained in $K_{A,B}^{k,n-k+1}$. By arguments similar to Claim 4.3.2, the moduli space of such teardrops is regular and has an evaluation map which sweeps out the homology class of y. ∎
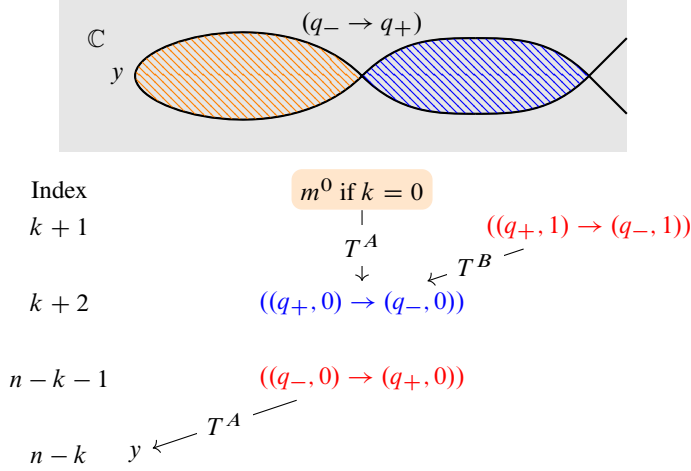
Figure 25. Floer cohomology of the bottlenecked surgery trace $K_{A,B}^{k,n-k+1}$. The holomorphic teardrop is marked in orange and has area A, the holomorphic strip from the bottleneck is marked in blue and has area B.

This leads to the appearance of a term in the differential of the Floer complex of $K_{A,B}^{k,n-k+1}$; see Figure 25.

The most interesting example of this occurs when we take the trace of the standard Polterovich surgery, when $k = 0$. In this setting, the Whitney isotropic is a 1-dimensional sphere, and so the space of holomorphic teardrops is zero-dimensional. This gives us an additional contribution which we can count.

Corollary 4.4.2. *For the standard choice of J, $\mathcal{M}_J(K_{A,B}^{0,n+1}, \beta_A, ((q_-, 0) \to (q_+, 0)))$ consists of a single holomorphic teardrop.*

The presence of this contribution will turn on a non-trivial curvature term

$$T^A((q_+, 0) \to (q_-, 0( ))_-$$

in the A_∞ algebra. For $CF^\bullet(K_{A,B}^{0,n+1})$ to be unobstructed, there must be a Floer cochain whose coboundary is $T^A((q_+, 0) \to (q_-, 0))$. A candidate cochain would be $T^{B-A}((q_+, 1) \to (q_-, 1))$. As deforming cochains are required to have positive valuation, we see that $K_{A,B}^{0,n+1}$ is unobstructed if and only if $B < A$.

5. Sample computation

The previous section suggests the following algorithm for computing a continuation map associated to an unobstructed Lagrangian cobordism. First, decompose the

Lagrangian cobordism K using Theorem 3.3.1 into pieces K_i which are either a suspension of an exact homotopy or surgery trace. If we can show that unobstructed of K implies unobstructedness of each of the K_i (Conjecture 3.3.7), then Theorem 4.4.1 suggests that each K_i provides a continuation map between its ends. The continuation map associated to K would then be given by composing all of these smaller continuation maps together.

More precisely, we conjecture that there exists a version of Floer cohomology for a Lagrangian cobordism with double bottlenecks, $CF^\bullet{}_{\text{bot}}(K)$, which has filtered A_∞ projections

$$CF^\bullet{}_{\text{bot}}(K)$$

$$\overset{\pi^+}{\swarrow} \qquad \overset{\pi^-}{\searrow}$$

$$CF^\bullet(L^+) \qquad\qquad\qquad CF^\bullet(L^-).$$

As a vector space, $CF^\bullet{}_{\text{bot}}(K)$ is generated on $\mathcal{I}^{\text{bot}}(K)$. We observe from Figure 21 that

$$\mathcal{I}^{\text{bot}}(K) = \mathcal{I}(L^-) \cup \mathcal{I}(L^+) \cup (\mathcal{I}(L^+)[1] \setminus \{(q_- \to q_+)\}) \cup \{y\}.$$

Since $\text{ind}(y) = \text{ind}(q_- \to q_+) + 1$, we identify $CF^\bullet{}_{\text{bot}}(K)$ as a vector space with

$$CF^\bullet(L^-) \oplus CF^\bullet(L^+)[1] \oplus CF^\bullet(L^+).$$

Under this identification the projections onto the first and last component agree with the maps $\pi^\pm$ above, and $CF^\bullet(L^+)[1]$ is a filtered A_∞ ideal of $CF^\bullet{}_{\text{bot}}(K)$. Then Theorem 4.4.1 suggests that the maps induced by the Floer differential on $CF^\bullet(K)$,

$$CF^\bullet(L^+) \to CF^\bullet(L^+)[1], \quad CF^\bullet(L^-) \to CF^\bullet(L^+)[1]$$

are invertible (although the inverse will have negative Novikov valuation). A similar approach was used in [16, Claim 4.4.7].

From this map we can construct weakly filtered[5] homotopy equivalences from $CF^\bullet(K)$ to either $CF^\bullet(L^+)$ or $CF^\bullet(L^-)$. A similar story holds for Lagrangian cobordisms which are the suspension of an exact homotopy.

Problematically, weakly filtered maps *do not* preserve the property of unobstructedness. If a weakly filtered map decreases the filtration by at most λ, it only defines a pushforward map on bounding cochains whose valuations are at least λ. In particular, if $CF^\bullet(L^+), CF^\bullet(L^-)$ are weakly filtered homotopy equivalent, and $CF^\bullet(L^+)$ is unobstructed, then $CF^\bullet(L^-)$ is unobstructed provided that the bounding cochain for L^+ has sufficiently large valuation. Therefore, each Lagrangian cobordism coming from a Lagrangian surgery trace $K: L^+ \rightsquigarrow L^-$ only identifies a subset of the

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[5]A weakly filtered map is allowed to decrease the filtration, see [16, Definition A.1.3].

Maurer–Cartan elements of L^+ with those of L^-. To see which bounding cochains are identified: every bounding cochain $\flat$ on K induces bounding cochains on $\pi_*^{\pm}\flat$ on $L^{\pm}$, which sets up a correspondence between subsets of the Maurer Cartan spaces of L^+ and L^-. The objects $(L^{\pm}, \flat^{\pm})$ are expected to be isomorphic in the Fukaya category, with the homotopy equivalences from $CF^{\bullet}_{\text{bot}}(K, \flat)$ to $CF^{\bullet}(L^{\pm}, \flat^{\pm})$ providing a "continuation map". Note that not every bounding cochain on $L^{\pm}$ can be achieved in this manner. This observation was employed in [16] to recover wall-crossing formula for Lagrangian tori via Lagrangian cobordisms related to the mutation operation.

More generally, given a Lagrangian cobordism $K \colon L^+ \rightsquigarrow L^-$, we expect that the decomposition from Theorem 4.2.12 will provide us with an algorithm which

• determines if a bounding cochain exists for K. For simplicity, write

$$K = K_n \circ \cdots \circ K_1,$$

where each $K_i \colon L_i \rightsquigarrow L_{i-1}$ is either a Lagrangian surgery trace or suspension of an exact homotopy. Let $\mathcal{MC}(L)$ denote the set of bounding cochains on L. Then the discussion above produces for each i a subset $B_i \subset \mathcal{MC}(L_i) \times \mathcal{MC}(L_{i-1})$ of bounding cochains restricted from K_i. The fiber product

$$B_1 \times_{\mathcal{MC}(L_1)} B_2 \times_{\mathcal{MC}(L_2)} \times \cdots \times_{\mathcal{MC}(L_{n-1})} B_n$$

then describes the bounding cochains on K.

• computes the homotopy equivalence between $CF^{\bullet}(L^+)$ and $CF^{\bullet}(L^-)$ if they are unobstructed. This homotopy equivalence is the composition of the homotopy equivalences induced by each suspension or surgery trace cobordism.

The construction of such an algorithm is beyond the scope of this paper. We therefore conclude with two sample computations demonstrating how the proposed algorithm might work in practice. In Section 5.1, we look at which choices of surgery parameters make the Lagrangian cobordism constructed in Example 2.3.2 unobstructed. In Section 5.2, we look at a single Lagrangian surgery trace $K_{A,B}^{0,2}$. When $A > B$, we show that this Lagrangian cobordism is unobstructed and gives a continuation map between its ends (when equipped with appropriate bounding cochains). When $A < B$, we show that the Lagrangian cobordism is obstructed (and furthermore that the ends are not isomorphic objects in the Fukaya category for any choice of bounding cochain).

5.1. A return to Example 2.3.2

We now return to a variation of Example 2.3.2. In particular, we discuss the conditions for when antisurgery followed by surgery give an unobstructed Lagrangian

cobordism, and why when unobstructed these Lagrangian cobordisms are mapping cocylinders.

Consider the Lagrangian cobordism K_{A_+, A_-}, which is obtained concatenating the following Lagrangian submanifolds:

- a Lagrangian anti-surgery trace $(K_{A_+}^{1,1})^{-1} \colon S^1_{E_+} \sqcup S^1_{-E_+} \rightsquigarrow L_E$, where L_E is an immersed Lagrangian submanifold in $T^* S^1$;

- a Lagrangian surgery trace $K_{A_-}^{1,1} \colon L_E \rightsquigarrow S^1_{E_-} \sqcup S^1_{-E_-}$.

The slices and shadow of the Lagrangian cobordism K_{A_+, A_-} are drawn in Figure 26 (a) and (b). The relations between E_+, E_-, A_+, A_- and E are

$$E - 2E_+ = A_+, \quad E - 2E_- = A_-.$$

The immersed Floer cohomology of K_{A_+, A_-} is a deformation of the Morse cohomology of K_{A_+, A_-}. We take the standard Morse function for the ends of the cobordism, which is generated on critical points:

$$CM^\bullet(S^1_{E_\pm} \sqcup S^1_{-E_\pm}) = \Lambda \langle e_0^\pm, e_1^\pm, x_0^\pm, x_1^\pm \rangle.$$

Topologically, K_{A_+, A_-} is two pairs of pants sharing a common neck; we take the our Morse function to be a perturbation of the Morse–Bott function which has maximums along the ends, and a minimally graded S^1 along the common neck. The generators and zero-dimensional flow lines contributing to

$$m^1 \colon CM^\bullet(K_{A_+, A_-}) \to CM^\bullet(K_{A_+, A_-})$$

are drawn in Figure 26 (c). Pairs of canceling flow lines are indicated with dotted lines.

In addition to these Morse flow trees, the A_∞ product structure on $CF^\bullet(K_{A_+, A_-})$ counts configurations of holomorphic teardrops. By Theorem 4.4.1 both the surgery and anti-surgery trace contribute holomorphic teardrops with output on $(q_+ \to q_-)$. The lowest order contributions to m^0 and m^1 are listed in Figure 26 (d). The most important term is the curvature term,

$$m^0 = (T^{A_+} - T^{A_-})(q^+ \to q^-) + \mathcal{O}(\min(A_+, A_-)).$$

We break into two cases.

5.1.1. Case 1: $A_+ \neq A_-$. In the event where $A_+ \neq A_-$, the $m^0 \colon \Lambda \to CF^\bullet(K_{A_+, A_-})$ term is non-zero. Furthermore, since $\min(A_+, A_-)$ is the smallest area of a holomorphic teardrop,

$$\mathrm{val}\big(\langle m^k(x_1, \ldots, x_k), (q_+ \to q_-) \rangle\big) > \min(A_+, A_-)$$
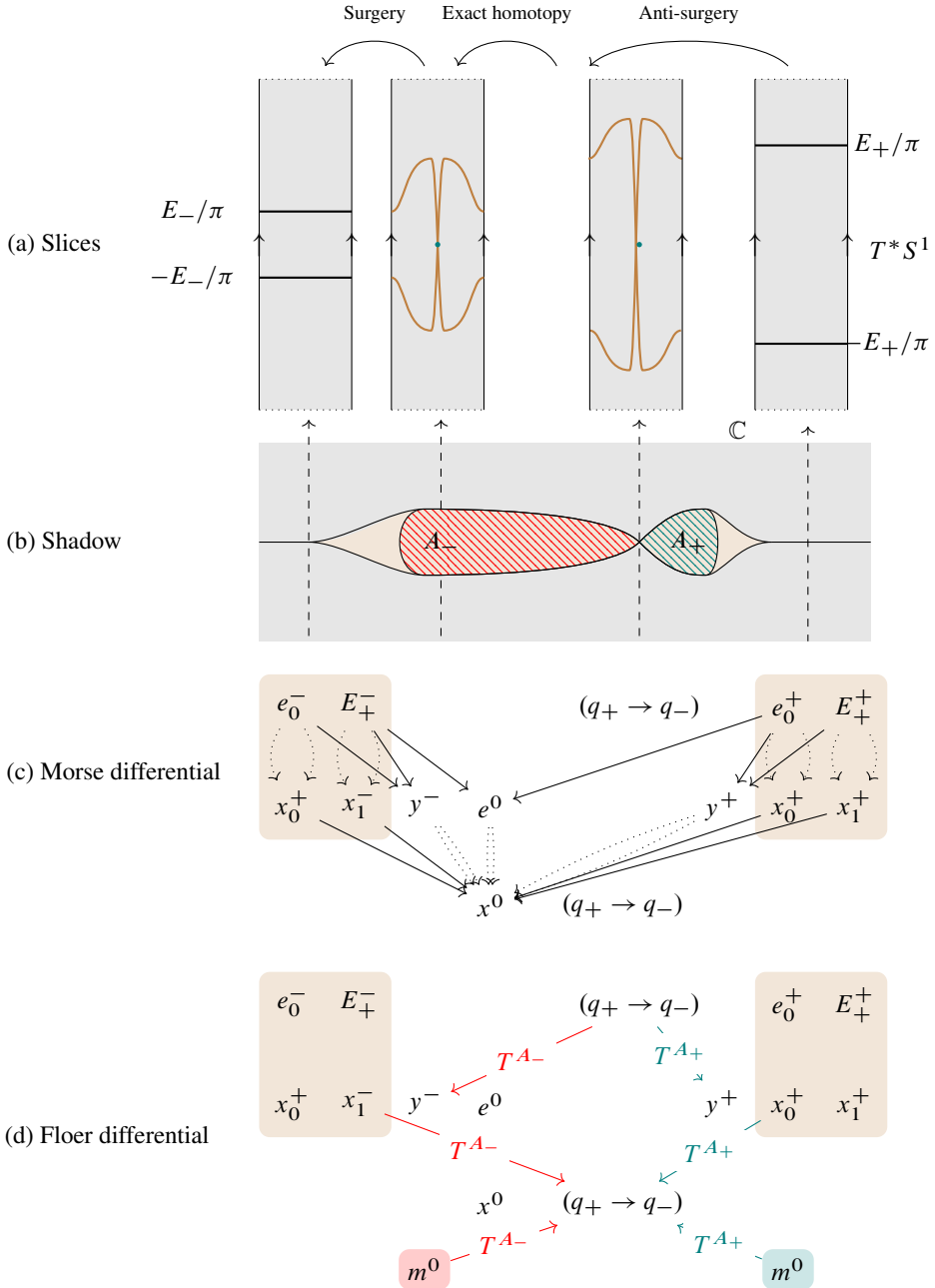
Figure 26. Anti-surgery followed by surgery.

for all x_i satisfying $\text{val}(x_i) > 0$. It follows that there is no solution to the Maurer–Cartan equation, and $CF^\bullet(K_{A_+,A_-})$ is obstructed. Furthermore, clearly $S^1_{E_+} \sqcup S^1_{-E_+}$ is not isomorphic to $S^1_{E_-} \sqcup S^1_{-E_-}$ as an object of the Fukaya category.

5.1.2. Case 2: $A_+ = A_-$. The more interesting example is to consider is when $A_+ = A_-$. In this setting, the curvature term vanishes, and $CF^\bullet(K_{A_+,A_-})$ is tautologically unobstructed. A computation shows that the chain complex $CF^\bullet(K_{A_+,A_-})$ is a mapping cylinder; by using homotopy transfer theorem one can construct a A_∞ morphism

$$\widehat{i^-}: CF^\bullet(S^1_{E_-} \sqcup S^1_{-E_-}) \to CF^\bullet(K_{A_+,A_-}).$$

The composition

$$\Phi = \pi^+ \circ \widehat{i^-}: CF^\bullet(S^1_{E_-} \sqcup S^1_{-E_-}) \to CF^\bullet(S^1_{E_+} \sqcup S^1_{-E_+})$$

is a continuation map, reflecting that K_{A_+,A_-} should provide an equivalence of objects in the Fukaya category.

5.2. Continuations, obstructions, and mapping cones

We now employ the algorithm proposed at the beginning of this section to compute a continuation map for an immersed Lagrangian cobordism. Consider the double-section Lagrangian $L_{E_a} \subset T^*S^1$ discussed in Section 4.3.4. Let $S_{\pm E}$ be the pair of sections $\pm\frac{E}{2\pi} d\theta$ of T^*S^1. When $0 < 2E < E_a$ there is a Lagrangian surgery trace cobordism $L_{E_a} \rightsquigarrow S^1_E \sqcup S^1_{-E}$.

Remark 5.2.1. The direction of the inequality $0 < 2E < E_a$ is a bit subtle. At first glance, the inequality $0 < E_a < 2E$ appears natural because it is the one obtained by using the local surgery model associated with the Polterovich surgery of two Lagrangian submanifolds intersecting transversely at a single point. Because we apply Proposition 3.1.10 to construct the Lagrangian surgery trace, the ends of the Lagrangian surgery trace (see Section 3.2.3) differ by a non-trivial amount of flux from the Polterovich surgery model. The flux swept out is opposite and greater than the area bounded by the surgery neck, leading to the inequality $0 < 2E < E_a$. We give a more detailed account in Example 3.2.8.

One constructs the double bottleneck surgery trace (Definition 4.2.11)

$$K^{0,2}_{A,B}: L_{E_b} \rightsquigarrow S^1_E \sqcup S^1_{-E}$$

by subsequently applying a small exact homotopy, where the flux swept out on the positive end gives the relation $0 < E_b < E_a$. We will drop the superscript and from here on write $K_{A,B}$ for this Lagrangian cobordism. Here, A is the area of the small

teardrop of the surgery, and B is the area of the strip associated with the bottleneck; the slice above the second bottleneck is denoted L_{E_a}. The relations between A, B, E_a, E_b and E are

$$E_a - E_b = 2B, \quad E_a - 2E = 2A.$$

See Figure 27 (a), (b) for slice and shadow of the Lagrangian cobordism. We equip $K_{A,B}$ with an admissible Morse function; this has critical points e_0, e_1, x_0, x_1 near the negative end, e_a, e_b, x_a, x_b corresponding to the Morse critical points of the slices L_{E_a} and L_{E_b}, and an additional critical point y from the surgery handle. The Lagrangian $K_{A,B}$ has two self-intersections; we call 3 generators of the Floer cohomology $(\pi \to 0)_a, (0 \to \pi)_a$, and $(0 \to \pi)_b$, and discard the fourth generator.

5.2.1. Computation of low order products. As it is difficult to compute the full product structure for Floer cochains of $K_{A,B}$, we restrict ourselves to computing smallest-order contributions to the m^0 and m^1 terms on $CF^\bullet{}_{bot}(K_{A,B})$. These lowest order computations are sufficient to prove unobstructedness of Floer groups by standard arguments using the filtration of $CF^\bullet{}_{bot}(K_{A,B})$.

- The differential on the Morse complex of $CM^\bullet(K_{A,B})$ can be fully computed. The Morse flow-lines are drawn in Figure 27 (c). The differential is represented by black dotted and solid arrows in Figure 27 (d). The dotted arrows have canceling contributions.

- By Corollary 4.4.2, there exists a holomorphic teardrop which can either be considered as having input on $(0 \to \pi)_a$ or output on $(\pi \to 0)_a$ with area A:

  - The teardrop with output on $(0 \to \pi)_a$ appears in the m^0 term in the A_∞ structure

    $$m^0 = T^A \cdot (0 \to \pi)_a + (O_A).$$

    This is represented with the highlighted red term in Figure 27 (d). Additionally, this teardrop contributes to the differential,

    $$\langle m^1(x_1), (0 \to \pi)_a \rangle = T^A + \mathcal{O}(A).$$

  - The teardrop with input on $(\pi \to 0)_a$ gives a contribution to the differential

    $$\langle m^1(\pi \to 0)_a, y \rangle = T^A + \mathcal{O}(A).$$

- The holomorphic strips contributing to the differential on $CF^\bullet{}_{bot}(K_{A,B})$ arise from the strips which appear on a double bottleneck

  $$\langle m^1(\pi \to 0)_a, (0 \to \pi)_b \rangle = T^{B+E_b} - T^{B+E_b}.$$

  The double bottleneck also has a holomorphic strip pairing the generator which is "doubled" with maximal and minimal grading from the base

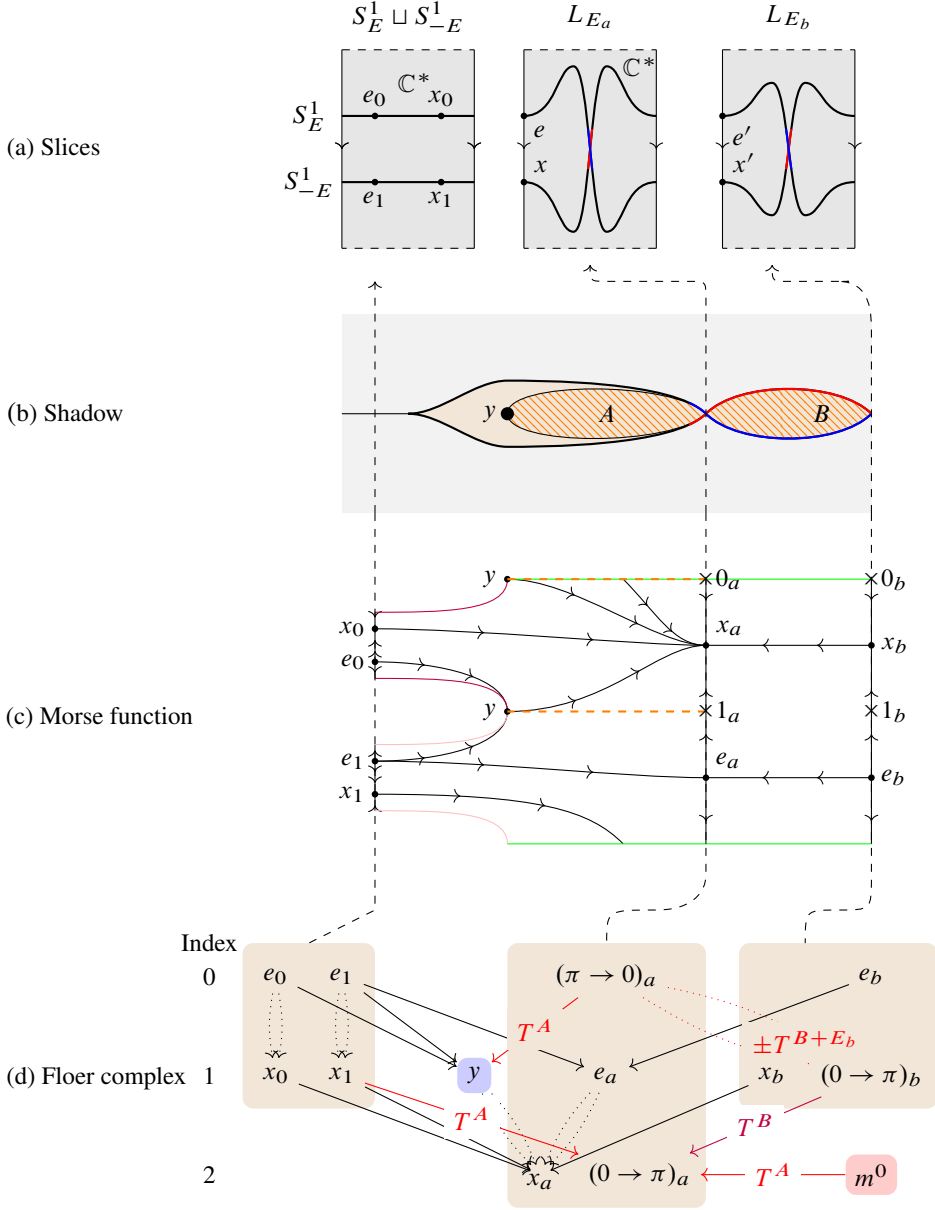 $$\langle m^1(0 \to \pi)_b, (0 \to \pi)_a \rangle = T^B + \mathcal{O}(B).$$

Figure 27. Slices, shadow, Morse function and Floer complex of the surgery trace cobordism.

These contributions are represented by red arrows in Figure 27 (d). We obtain two cases which we consider separately: when $A > B$ and when $A < B$. We discuss the Floer theoretic and geometric implications below.

5.2.2. Continuation maps and Unobstructedness: $A > B$. We first note that whenever $A > B$, the curvature term m^0 can be canceled out at lowest order by the differential, as

$$m^1(T^{A-B}(0 \to \pi)_b) = T^A(0 \to \pi)_a + \mathcal{O}(A).$$

We can build a bounding cochain order by order using the filtration on $CF^\bullet_{\text{bot}}(K_{A,B})$, whose first term is

$$\flat_{A,B} := T^{A-B}(0 \to \pi)_b + \mathcal{O}(A - B) \in \mathcal{MC}(K_{A,B}).$$

This gives us a sufficient condition for this Lagrangian cobordism to be unobstructed: the flux swept out by the exact homotopy as defined by the double bottleneck (measured by the area B of the holomorphic strip) must not be greater than the area of the surgery neck taken (as measured by the holomorphic teardrop A). Since

$$\pi^+ \colon CF^\bullet_{\text{bot}}(K_{A,B}) \to CF^\bullet(L_{E_b})$$

is an A_∞ homomorphism, we obtain a bounding cochain $\pi^+_* \flat_{A,B}$ for $CF^\bullet(L_{E_b})$. We can verify this computation by checking the Lagrangian intersection Floer cohomology against a test Lagrangian $S^1_{E'}$. Recall from Section 4.3.4, the complex

$$CF^\bullet((L_{E_b}, \pi^+_* \flat_{A,B}), S^1_{E'})$$

is acyclic whenever the lowest order term of $\pi^+_*(\flat_{A,B})$ is not $T^{\frac{E_b}{2} - E'}$. In this case, the lowest order term of $\flat$ is

$$T^{A-B} = T^{\frac{Ea - 2E}{2} - \frac{Ea - E_b}{2}} = T^{\frac{E_b}{2} - E}.$$

So at lowest order, $CF^\bullet((L_{E_b}, \pi^+_* \flat_{A,B}), S^1_{E'})$ has non-trivial homology if and only if $E = E'$.

The computation shows that the ends of $K_{A,B}$ cannot be distinguished at lowest order by testing them against objects $S^1_{E'}$ in the Fukaya category. We now show that the ends have the same immersed Lagrangian Floer cohomology by showing that $(K_{A,B}, \flat)$ is a mapping cocylinder, providing a homotopy equivalence between the Floer cohomology of its ends. This will construct a continuation map between the ends of the cobordism. First, we note that the projection

$$\pi^+ \colon CF^\bullet_{\text{bot}}(K_{A,B}, \flat) \to CF^\bullet(L_{E_b})$$

is a homotopy equivalence. The homotopy can be described on generators as follows:
Let

$$\mathcal{I}^- = \{e_0, x_0, e_1, x_1\},$$
$$\mathcal{I}^0 = \{y, e_a, x_a, (0 \to \pi)_a\},$$
$$\mathcal{I}^+ = \{(\pi \to 0)_a, e_b, x_b, (0 \to \pi)_b\}.$$

Let $CF^\bullet{}_{\text{bot}}(K_{A,B})|_{\mathcal{I}^*}$ be the subspace spanned by the appropriate set of generators. This is generally not an A_∞ ideal, however

$$CF^\bullet{}_{\text{bot}}(K_{A,B})|_{\mathcal{I}^0} \oplus CF^\bullet{}_{\text{bot}}(K_{A,B})|_{\mathcal{I}^+},$$

is an A_∞ ideal. We additionally consider the map

$$H : CF^\bullet{}_{\text{bot}}(K_{A,B})|_{\mathcal{I}^+} \to CF^\bullet{}_{\text{bot}}(K_{A,B})|_{\mathcal{I}^0},$$

which on the given basis sends

$$H(e_a) = e_1 - e_0, \qquad H(y) = e_0,$$
$$H((0 \to \pi)_a) = T^{-A}(x_1 - x_0), \quad H(x_a) = x_0.$$

Let $i^+ : CF^\bullet(L|_{E_b}) \to CF^\bullet{}_{\text{bot}}(K_{A,B})$ be the chain-level inclusion

$$i^+(e_b) = e_b + e_1 - e_0, \quad i^+(x_b) = x_b + x_0,$$
$$i^+((\pi \to 0)_b) = T^{-B}((\pi \to 0)_a + T^A e_0),$$
$$i^+(0 \to \pi)_b = (0 \to \pi)_b + T^{B-A}(x_1 - x_0).$$

At lowest order, the map i^+ is a homotopy inverse to π^+ with homotopy given by H.

By using the homotopy transfer theorem[6] we may extend i^+ to an A_∞ homomorphism

$$\widehat{i}^+ : CF^\bullet(L_{E_b}, (\pi^+)_* \mathfrak{b}_{A,B}) \to CF^\bullet{}_{\text{bot}}(K_{A,B}, \mathfrak{b}_{A,B}).$$

We combine the inclusion map with the projection

$$\pi^- : CF^\bullet{}_{\text{bot}}(K_{A,B}, \mathfrak{b}_{A,B}) \to CF^\bullet(L^-, (\pi^-)_* \mathfrak{b}_{A,B})$$

to obtain a continuation map

$$\Phi := \pi^- \circ \widehat{i}^+ : CF^\bullet(L_{E_b}, (\pi^+)_* \mathfrak{b}_{A,B}) \to CF^\bullet(S^1_E \sqcup S^1_{-E}, (\pi^-)_* \mathfrak{b}_{A,B}).$$

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[6]There may be some concern that the valuation of H is negative. This is in general not a problem as long as $H \circ m^0$ has positive valuation. Since we have canceled out m^0 by incorporation of a bounding cochain, we can use the homotopy transfer theorem here.

One can furthermore use the continuation map to study how modifications of the bounding cochain on $(L_{E_b}, \mathfrak{b})$ relate to local systems on the Lagrangian $S_E^1, \sqcup S_{-E}^1$. As Φ is a weakly filtered A_∞ homomorphism of energy loss $B - A$, there exists a pushforward map on degree 1 Maurer–Cartan solutions with valuation greater than $A - B$:

$$\Phi_*\colon \{\mathfrak{d} \in \mathcal{MC}^1(L_{E_b}, \mathfrak{b}), \mathrm{val}(\langle \mathfrak{d}, (0 \to \pi)_b \rangle) > A - B\} \to \mathcal{MC}^1(S_E^1 \sqcup S^{-E}).$$

Given an element

$$\mathfrak{d} = c_1(T^{A-B+\varepsilon} + \mathcal{O}(A - B + \varepsilon))((0 \to \pi)_b) + 2c_0(T^\varepsilon + \mathcal{O}(\varepsilon))x_b,$$

we can compute the pushforward to first order:

$$\Phi_*(\mathfrak{d}) = c_0(T^\varepsilon + \mathcal{O}(\varepsilon))x_0 + c_1(T^\varepsilon + \mathcal{O}(\varepsilon))x_1,$$

showing that Φ_* is a surjection. If we interpret graded deformations in $\mathcal{MC}((L_{E_b}, \mathfrak{b}))$ as a local system, this suggests that all local systems on $S_E^1 \sqcup S_{-E}^1$ are realized by deformations of $(L_{E_b}, \mathfrak{b})$).

5.2.3. Obstructedness: $B \geq A$. The other setting of interest is when $B \geq A$, as drawn in Figure 2. Then the Lagrangian cobordism $K_{A,B}$ is obstructed, in the sense that $\mathcal{MC}(K_{A,B})$ is empty. In this setting, L_{E_b} is never isomorphic to $S_E^1 \sqcup S_{-E}^1$ for any choice of deforming cochain, as $E_b < 2E$. This is easily observed as L_{E_b} can be displaced from $S_E^1 \sqcup S_{-E}^1$ by a Hamiltonian isotopy.

A. Monotone two-ended Lagrangian cobordisms

A Lagrangian submanifold $L \subset (X, \omega)$ is called monotone if there exists a $\lambda > 0$ such that the homomorphism

$$\omega, \mu\colon \pi_2(X, L) \to \mathbb{R}$$

are proportional so that $\omega = \lambda\mu$. Monotone Lagrangian submanifolds provide an interesting subset of examples to work with as they are more general than exact Lagrangian submanifolds, and always have well-defined Floer cohomology. As a result, the Floer cohomology of monotone Lagrangian submanifolds has been studied extensively. The pearly-Floer cohomology of monotone Lagrangian submanifolds $CF^\bullet(L)$ is a deformation of the Morse cohomology [5]; as such, the maximal rank of the Floer cohomology group is bounded by the cohomology of L. We say that L is *wide* if this bound is achieved, i.e.,

$$HF^\bullet(L) \simeq H^\bullet(L; \Lambda).$$

Many constructions of wide monotone Lagrangian submanifolds exist (for example, every exact Lagrangian). In contrast, a Lagrangian submanifold is called narrow if $HF^\bullet(L)$ vanishes. The Chiang Lagrangian in dimension n not a prime power is narrow over every characteristic [25, Theorem 4.10]. A dramatic example of a narrow monotone Lagrangian submanifold comes from [22], which produces examples of displaceable monotone Lagrangian submanifolds. The wide-narrow dichotomy (introduced in [6]) observes that the relative sizes of the Floer cohomology groups and the cohomology groups of a monotone Lagrangian submanifold tends to characterize the "rigidity" of a Lagrangian submanifold. More precisely: quantitative measures of the size of a Lagrangian submanifold L (such as the width, or Gromov width of the complement of L) are oftentimes small for narrow Lagrangians and large for wide Lagrangians. It was conjectured in [6] that all Lagrangian submanifolds are either wide or narrow. We include a construction (the main idea was communicated to us by Abouzaid and Auroux) of a monotone Lagrangian submanifold which is neither wide-nor-narrow.

For the same reason that monotonicity is a good property for studying Lagrangian submanifold, the theory of monotone Lagrangian cobordisms is similarly well developed. For instance, it is known that every pair of monotone Lagrangian cobordant submanifolds are isomorphic in the Fukaya category. To our knowledge, we provide the first example of a 2-ended embedded monotone Lagrangian cobordism not given by suspension of a Hamiltonian isotopy.

A.1. A Neither wide nor narrow monotone Lagrangian submanifold

The goal is to obtain a Lagrangian submanifold which is neither wide nor narrow, that is $HF^\bullet(L) \neq 0, H^\bullet(L)$. The approach is the following: Given Lagrangian submanifolds L_1, L_2 which fit into a monotone Lagrangian cobordism

$$K \colon (L_1, L_2) \rightsquigarrow L_3,$$

the work [7] shows that $CF^\bullet(L_3)$ is a mapping cone of a morphism $CF^\bullet(L_1) \to CF^\bullet(L_2)$. If L_1 is narrow and L_2 is wide, then

$$HF^\bullet(L_3) = HF^\bullet(L_2) = H^\bullet(L_2),$$

which is unlikely to be $H^\bullet(L_3)$.

The difficulty of this construction is that generally the surgery you produce is non-monotone. If L_1 and L_2 intersect transversely at distinct points $L_1 \pitchfork L_2 = \{x_1, \ldots, x_k\}$, then a candidate Lagrangian cobordism is the surgery trace cobordism. However, it is rarely the case that the Lagrangian submanifold obtained by surgering

L_1 and L_2 at their intersection points,

$$L_3 = L_1 \#_{x_1,\ldots,x_k} L_2$$

is a monotone Lagrangian submanifold, as any strip of index 1 with boundary on $L_1 \cup L_2$ will give rise to a disk of Maslov index zero on the Lagrangian surgery. One might hope that L_1 and L_2 intersect at single point; problematically, the Lagrangian intersection Floer cohomology $HF^{\bullet}(L_1, L_2)$ would be non-vanishing, which seemingly contradicts the narrowness of L_1.

The proposal of Abouzaid and Auroux is to perform the above construction with Lagrangian submanifolds L_1 and L_2 which intersect cleanly along a single (non-point) connected component, and perform surgery along the component. The upshot is that (in comparison to the transverse intersection case) one would hope that there are no Maslov index 1 strips with boundary in $L_1 \cup L_2$; intuitively, we should think that all of the Maslov index 1 strips have degenerated to flow-lines on the intersection locus for some choice of Morse function on $L_1 \cap L_2$ corresponding to an infinitesimal Hamiltonian perturbation of L_2. In this setting, one can still define the Lagrangian surgery of L_1 and L_2 at their intersection (either using the ideas of [12], or surgery constructions from [17, 20]); this comes with a surgery trace cobordism allowing us to apply the mapping cone construction of Biran and Cornea.

We then need to find monotone L_1 and L_2 which intersect transversely along a single connected component. Let X be a symplectic manifold, and $L \subset X$ be a narrow monotone Lagrangian submanifold. Inside of $X \times \bar{X}$, consider the Lagrangian submanifold

$$L_1 = L \times L \subset X \times \bar{X}.$$

By application of the Künneth formula, L_1 is a narrow monotone Lagrangian submanifold of $X \times \bar{X}$.

We also consider the diagonal Lagrangian

$$L_2 = \{(x, x) \in X \times \bar{X} \mid x \in X\}.$$

This Lagrangian is monotone, and by identifying the Floer cohomology of the diagonal with the fixed-point Floer cohomology of $X \times \bar{X}$, we see that L_2 is wide.

These two Lagrangian submanifolds intersect cleanly along

$$\Delta_L := \{(x, x) \in X \times \bar{X} \mid x \in L\},$$

so the surgery $L_3 := L_1 \#_{\Delta_L} L_2$ is a good candidate for a wide nor-narrow Lagrangian submanifold. It remains to check if the surgery trace cobordism $K : (L_1, L_2) \rightsquigarrow L_3$ is monotone. A criterion is given by [20, Lemma 6.3] on the fundamental groups of the summands of the Lagrangian connect sum, namely at least one of $\pi_1(L_i)$ is torsion in $\pi_1(X \times \bar{X})$. This will clearly depend on the choice of L and X, which we now fix.

As suggested to the author by Cheuk Yu Mak, we take the example

$$L = L_{0,m}^p \subset \mathbb{CP}^{m+1} = X$$

from [22, Theorem 1.4] which is self-displaceable (and therefore narrow) and monotone. The topology of L is $S^1 \times S^m$. Since $X \times \bar{X}$ is simply connected, the image of $\pi_1(L_2)$ is torsion in $\pi_1(X \times \bar{X})$. Therefore, the surgery trace cobordism

$$K \colon (L_1, L_2) \rightsquigarrow L_3$$

is monotone.

Finally, it remains to show that L_3 has different homology than L_2. A computation on the fundamental group shows that when $m > 2$,

$$\begin{aligned}
\pi_1(L_3) &= \pi_1(\mathbb{CP}^{m+1} \setminus \Delta_L) *_{\pi_1(\Delta_L)} \pi_1((L \times L) \setminus \Delta_L) \\
&= \mathbb{Z} \neq \pi_1(L_2)
\end{aligned}$$

as $L_2 \cong \mathbb{CP}^{m+1}$.

A.2. A 1-ended oriented embedded monotone Lagrangian cobordism

We expand on ideas in the construction above to produce a 1-ended oriented embedded monotone Lagrangian cobordism. First, a short observation. Let $i \colon L \to X$ be a Lagrangian immersion. Let $\pi_2(X, i \colon L \to X)$ denote the set of disks

$$u \colon (D, \partial D) \to (X, L)$$

with the property that $u|_{\partial D}$ factors through the immersion i. If i is an embedding, this is exactly the same as $\pi_2(X, i(L))$. Suppose that $i^\pm \colon L \to X$ are exactly homotopic Lagrangian immersions. There is a canonical identification

$$\pi_2(X, i^+ \colon L \to X) \cong \pi_2(X, i^- \colon L \to X),$$

and the Maslov index and symplectic area

$$\begin{aligned}
\mu &\colon \pi_2(X, i^\pm \colon L \to X), \\
\omega &\colon \pi_2(X, i^\pm \colon L \to X),
\end{aligned}$$

are unchanged under this identification. This is not the case when we work with $\pi_2(X, i^\pm(L))$, which cannot usually be identified. Even when they are (i.e., there are no birth/deaths of self-intersections), the symplectic area of polygons with corners of the self-intersection on $i^\pm(L)$ can change. However, from this we can conclude the following.

Claim A.2.1. *Suppose that $i^+\colon L \to X$ is an embedding which is exactly homotopic to $i^-\colon L \to X$ (which may not be an embedding). If $i^-(L) \subset X$ is monotone, then $i^+(L)$ is monotone.*

Let $L \subset X$ be a displaceable oriented monotone embedded Lagrangian submanifold, with a given parameterization $i^L_t\colon L \times [0,1]_t \to X$ of the exact isotopy which displaces the Lagrangian submanifold from itself (for example, from [22, Theorem 1.4]). Additionally we suppose that this exact isotopy is picked so that i_t, i_{1-t} are disjoint for all $t \in (0, 1/4)$. Let $H_t\colon L \to \mathbb{R}$ be the primitive of the flux form. Pick C large enough so that

$$|H_t| < C/2. \tag{A.1}$$

Consider now the curve

$$\gamma_0\colon S^1 = \mathbb{R}_t/4\mathbb{Z} \to T^*S^1 = \mathbb{C}/(100C\,\mathbb{Z})$$

parameterized in Figure 28, whose parameterization over $[1,2]$ and $[3,4]$ is given by $t + 2_JC$ and $(t-2) - 2_JC$, respectively. The self-intersection of the curve occurs at $\gamma_0(0.5) = \gamma_0(2.5)$. The parameterization is picked so that those segments have normal neighborhoods of radius C over those portions of the parameterization. The area A highlighted in blue (notably avoiding the normal neighborhoods) is chosen to be at least $10C$.

Since γ_0 parameterizes a Whitney sphere, we can construct a null-cobordism

$$j^\gamma\colon D^2 \to T^*S^1 \times \mathbb{C},$$
$$j^\gamma(D^2)\colon \emptyset \rightsquigarrow \gamma_0(S^1).$$

We will denote the cobordism parameter on $T^*S^1 \times \mathbb{C}$ by s. We parameterize the Lagrangian null-cobordism so that it is the suspension of an exact homotopy when $s \leq 1$, as drawn in the bottom of Figure 31. The negative end of this cobordism has a chart

$$j^\gamma|_{s\leq 1}\colon S^1 \times (-\infty, 1]_s \to T^*S^1 \times \mathbb{C},$$
$$(t,s) \mapsto (\gamma_s(t), s + {}_JH^\gamma_s(t)).$$

Because we picked $A > 10C$, we can ensure that the following criteria are met in the above chart:

(A) During this portion of the Lagrangian null-cobordism, $\gamma_s(t) = \gamma(t)$ for all $t \in [1,2] \cup [3,4]$. This means that $H^\gamma_s(t)$ is constant in the t parameter over $[1,2] \cup [3,4]$.

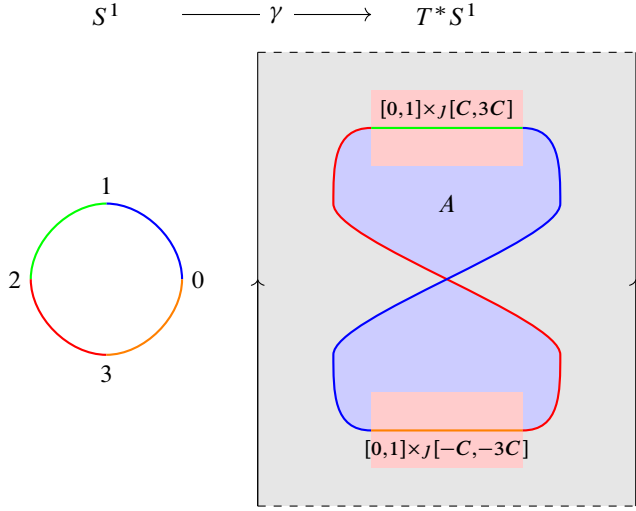(B) If $\gamma_s(t_0) = \gamma_s(t_1)$, with $t_0 \neq t_1$, then $t_0, t_1 \in \{0.5, 2.5\}$.

$$S^1 \qquad \xrightarrow{\hspace{1cm} \gamma \hspace{1cm}} \qquad T^*S^1$$

Figure 28. The path parameterized by γ_0.

(C) For all $s \in [1/8, 1]$, we have $H_s^\gamma(0.5) < -C$, $H_s^\gamma(2.5) > C$.

(D) $\gamma_s|_{t \in (0,1) \cup (2,3)}$ remains disjoint from the normal neighborhoods of radius C drawn in Figure 28.

We now describe the main idea of the construction of the null-cobordism. Consider the Lagrangian null-cobordism

$$j^\gamma \times i_0^L(D^2 \times L): \emptyset \rightsquigarrow \gamma_0(S^1) \times i_0^L(L).$$

Note that this is an immersed monotone Lagrangian null-cobordism with self-intersection along

$$\{0\} \times L \times \mathbb{R}_{<0} \subset \mathbb{C} \times X \times \mathbb{C}.$$

Since L is displaceable from itself, we can hope to make this immersed Lagrangian null-cobordism an embedded Lagrangian null-cobordism by application of an exact homotopy.

We first describe the negative end of this embedded Lagrangian null-cobordism. Consider a cutoff function $\rho_0(t): [0, 1] \to [0, 1]$ with the property that

$$\partial_t \rho < 2, \quad \rho|_{t<1/10} = 0, \quad \text{and} \quad \rho|_{t>9/10} = 1.$$

We look at the Lagrangian submanifold $\overline{i}_0^L: L \times S^1 \to X \times T^*S^1$ drawn in Figure 30, which is obtained by concatenating:

•   the identity cobordism for $i_0: L \to X$ along the blue path,
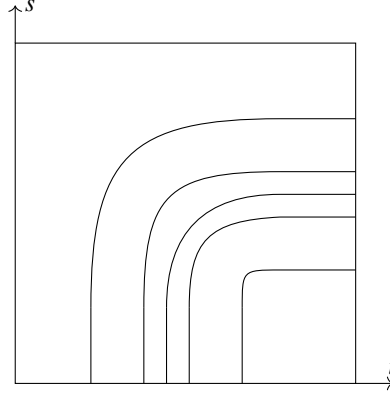
Figure 29. A contour plot of the auxiliary function $\rho_s(t): [0,1]_t \times [0,1]_s \to [0,1]$.

- the suspension of the exact isotopy $i_{\rho_0(t)}: L \times [0,1]_t \to X$ (the green region),

- the identity cobordism for $i_1: L \to X$ along the red path,

- the suspension of the exact isotopy $i_{\rho_0(-t)}: L \times [0,1]_t \to X$ (the orange region).

The blue path and red path are chosen to match the paths of γ_0 drawn in Figure 28, and by our bound on $\partial_t \rho$, and the suspension cobordisms have shadow lying inside the neighborhoods surrounding $\gamma_0|_{t \in [1,2] \cup [3,4]}$. Since L is embedded and each of the four pieces described above are embedded; therefore the only self-intersections of $\bar{i}_0^L: L \times S^1 \to X \times T^*S^1$ occur when the shadows of these four pieces overlap. By item (D), the only such occurrence is when the blue and red path cross. This occurs at $t = 0.5, 1.5$ by item (B). However, the X-component of these pieces are given by

$$i_0: L \to X \quad \text{and} \quad i_1: L \to X,$$

which were chosen to be disjoint. Therefore, the Lagrangian parameterized by $\bar{i}_0^L$ is embedded.

We are now ready to modify the immersed Lagrangian null-cobordism

$$j^\gamma \times i_0^L: D^2 \times L \to (X \times T^*S^1 \times \mathbb{C}),$$

so that the end agrees with $\bar{i}_0^L$. This Lagrangian cobordism is embedded over the portion where the cobordism parameter is greater than 1. We therefore need to replace $j^\gamma \times i_0^L|_{s \leq 1}$ with a suspension of an exact homotopy (defined for $s \leq 1$):

(i)   whose negative end is given by the Lagrangian $\bar{i}_0^L: S^1 \times L \to T^*S^1 \times X$;

(ii) which agrees with $j^\gamma \times i_0^L$ in a neighborhood $s = 1$; and

(iii) is embedded.
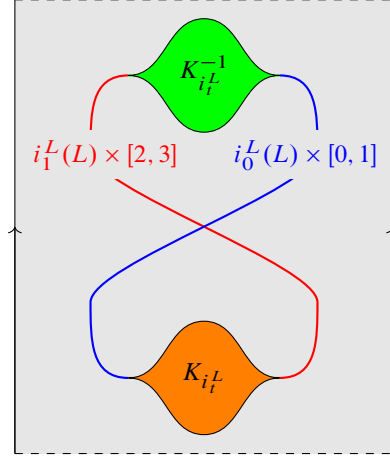
Figure 30. The Lagrangian submanifold $\bar{i}_0^L(q, t) \colon L \times S^1 \to X \times T^*S^1$, projected to the S^1 coordinate.

The slices of the suspension are drawn in Figure 31. We now give an explicit parameterization of this exact homotopy. Let $\rho_s(t) \colon [0, 1]_t \times [0, 1]_s \to [0, 1]$ be a smooth function drawn in Figure 29 with

(a) $\rho_s(0) = 0$, $\rho_0(1) = 1$ and $\rho_1(1) = 0$;

(b) $0 \leq \partial_t \rho_s < 2C$;

(c) when $0 \leq s < 1/4$ and $3/4 \leq s \leq 1$, we have $\partial_s \rho_s = 0$; and

(d) when $0 \leq t < 1/10$ and $9/10 \leq t \leq 1$, we have $\partial_t \rho_s = 0$.

Next, we consider the exact homotopy of Lagrangian submanifolds defined for $s \in [0, 1]$:

$$\bar{i}_s^L \colon L \times S^1 \to X \times T^*S^1,$$

$$(q, t) \mapsto \begin{cases} \left(i_0^L(q), \gamma_s(t)\right) & \text{if } t \in [0, 1), \\ \left(i_{\rho_s(t-1)}^L(q), \gamma_s(t) - J\frac{d\rho_s}{dt}H_{\rho_s(t-1)}(q)\right) & \text{if } t \in [1, 2), \\ \left(i_s^L(q), \gamma_s(t)\right) & \text{if } t \in [2, 3), \\ \left(i_{\rho_s(4-t)}^L, \gamma_s(t) + J\frac{d\rho_s}{dt}H_{\rho_s(4-t)}(q)\right) & \text{if } t \in [3, 4). \end{cases}$$

The map i_s^L is smooth by item (d).

Now consider the Lagrangian suspension cobordism of this exact homotopy

$$j^L|_{(0,1)} \colon L \times S^1 \times (0, 1) \to X \times T^*S^1 \times \mathbb{C},$$
$$(q, t, s) \mapsto \left(\bar{i}_s^L(q, t), s + {}_J\bar{H}_s(q, t)\right),$$
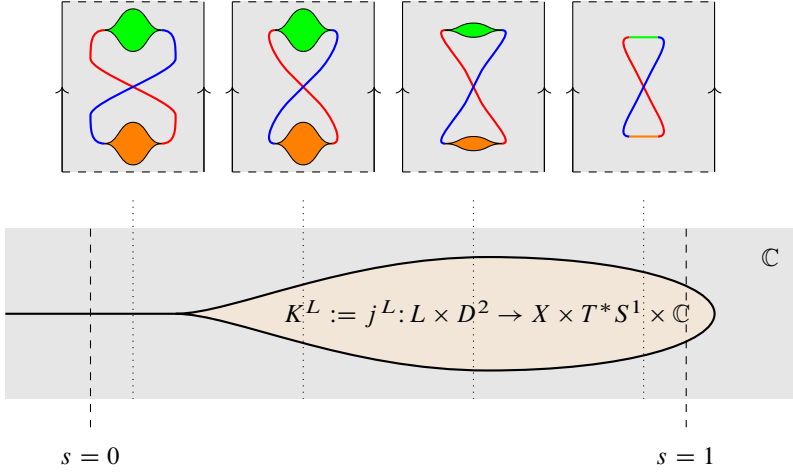
Figure 31. An embedded oriented monotone null-cobordism with connected ends.

where $\bar{H}_s(q, t)$ is the primitive for the exact homotopy $\bar{i}_s^L(q, t)$. By using item (a) of ρ, boundary of the suspension satisfies agrees with $j^\gamma \times i_0^L(D^2 \times L)|_{s=1}$, and by item (d) the domains of the suspension and $j^\gamma \times i_0^L(D^2 \times L)|_{s \geq 1}$ can be glued together to give a smooth Lagrangian cobordism.

We now prove that $j^L|_{(0,1)}$ is an embedding (item (iii)). Observe that the slices of $j^L|_{(0,1)}$ are embedded away from the locus $t \in \{0.5, 2.5\}, s > 1/4$. By employing items (b), (C), and equation (A.1), over the region where $s > 1/4$,

$$\bar{H}_s(q, 0.5) < H_s^\gamma(q, 0.5) + |\partial_t \rho_s||H_t| < C + 2 \cdot \frac{C}{2}$$

$$\leq 0 \leq C - 2 \cdot \frac{C}{2} < H_s^\gamma(q, 1.5) + |\partial_t \rho_s||H_t| < \bar{H}_s(q, 0.5),$$

so $j^L|_{(0,1)}$ is an embedding.

We now describe a Lagrangian null-cobordism of $\bar{i}_0^L(q, t)$.

By items (i) and (ii), we can form an embedded Lagrangian cobordism

$$K^L: \emptyset \to \bar{i}_0^L(L \times S^1),$$

which is parameterized by $j^L|_{(0,1)}$ for slices with $0 < s < 1$, and parameterized by $i_1^L \times j^\gamma$ for slices $s > 1$. We denote this parameterization by

$$j^L: L \times D^2 \to X \times T^* S^1 \times \mathbb{C},$$

which is an embedded Lagrangian null-cobordism.
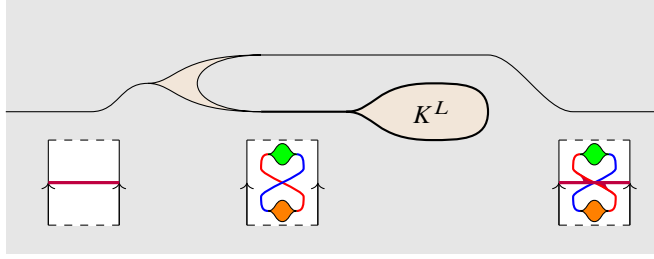
Figure 32. A monotone 2-ended Lagrangian cobordism.

We now prove that $j^L \colon L \times D^2 \to X \times T^*S^1 \times \mathbb{C}$ is monotone. There is an exact homotopy between $\widetilde{j}^L$ and $i_1^L \times j^\gamma$. Since the Maslov index and symplectic area decompose across products, and the symplectic area and Maslov index vanish on

$$\pi_2(X \times T^*S^1 \times \mathbb{C}, j \colon D^2 \to T^*S^1 \times \mathbb{C}), \quad i_1^L \times j^\gamma \colon L \times D^2 \to X \times T^*S^1 \times \mathbb{C},$$

the immersed Lagrangian is monotone with monotonicity constant matching L. By Claim A.2.1, $j^L \colon L \times D^2 \times X \to T^*S^1 \times \mathbb{C}$ is monotone as well.

A.3. A 2-ended monotone embedded Lagrangian cobordism

We now consider the Lagrangian $L_1 = L_{k,m}^p \times L_{k,m}^p \subset X \times \bar{X}$. This can be displaced from itself in a way so that the resulting displacement avoids the diagonal Lagrangian

$$i^{L_2} \colon L_2 \subset X \times \bar{X}.$$

Let $i^{S^1} \colon S^1 \to T^*S^1$ be the zero section which intersects the figure eight exactly at the self-intersection of γ_s. Consider the embedded Lagrangian submanifold

$$\overline{i}_0^{L_1} \colon L_1 \times S^1 \subset (X \times \bar{X}) \times T^*S^1$$

from the previous section. We also take the product Lagrangian

$$i^{L_2} \times i^{S^1} \colon L_2 \times S^1 \to (X \times \bar{X}) \times T^*S^1.$$

These two Lagrangians intersect along $\Delta_L \times \{0\} \subset (X \times \bar{X}) \times T^*S^1$.

We take the Lagrangian surgery cobordism

$$K \colon (\overline{i}_0^{L_1}, i^{L_2} \times i^{S^1}) \rightsquigarrow \overline{i}_0^{L_1} \# (i^{L_2} \times i^{S^1}).$$

By Section A.1, this is a 3-ended monotone Lagrangian cobordism. We concatenate this with the Lagrangian cobordism $K^{L_1} \colon \emptyset \to \overline{i}_0^{L_1}$ constructed in Section A.2 to obtain a 2-ended Lagrangian cobordism

$$K \circ ((K^{L_1})^{-1} \cup L_2 \times \mathbb{R}) \colon i^{L_2} \times i^{S^1} \to \overline{i}_0^{L_1} \# (i^{L_2} \times i^{S^1}).$$

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