Ann. Inst. H. Poincaré D
Comb. Phys. Interact. (Online first)
DOI 10.4171/AIHPD/174

Fluctuations of dimer heights on contracting square-hexagon lattices

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Abstract. We study perfect matchings on the square-hexagon lattice with $1 \times n$ periodic edge weights and with one of the following boundary conditions: (1) each remaining vertex on the bottom boundary is followed by $(m-1)$ removed vertices; (2) the bottom boundary can be divided into finitely many alternating line segments, each of which has a fixed positive length in the scaling limit, such that all the vertices along each line segment are either removed or retained. In case (1), we show that under certain homeomorphism from the liquid region to the upper half-plane, the height fluctuations converge to the Gaussian free field in the upper half-plane. In case (2), when the edge weights $x_1, \dots, x_n$ in one period satisfy the condition that $x_{i+1} = O(\frac{x_i}{e^{N\alpha}})$, where $\alpha > 0$ is a constant independent of N, we show that the height fluctuations converge to a sum of independent Gaussian free fields.

1. Introduction

A perfect matching, or a dimer configuration, on a graph is a subset of edges such that each vertex is incident to exactly one edge in the subset. Dimer configurations appear naturally in statistical physics to model the structure of matter, for example, the perfect matchings on the hexagon lattice is a mathematical model for the molecule structure of graphite. Through explicit combinatorial correspondence, the dimer model is also closely related to other lattice models in statistical mechanics, including the Ising model [25, 27], the 1-2 model [15, 16, 26, 28] and a general polygon model [14]. By developing the technique of Kasteleyn, Temperley and Fisher [18, 40], the partition function (weighted sum of configurations) of dimer configurations on a finite plane graph can be expressed explicitly as the determinant or Pfaffian of a weighted adjacency matrix; the local statistics can be computed [19]. By studying the spectral curve of the periodic dimer model using techniques from algebraic geometry, the sharp phase transition result can be established [22, 24]. The asymptotics of the rescaled dimer height function on a graph approximating a simply connected domain

2020 Mathematics Subject Classification. Primary 82B20; Secondary 82B23, 82B27.
Keywords. Dimer, Gaussian free field, Schur polynomial.

can be studied by a variational principle [10,23] and also by the asymptotics of certain symmetric functions (see [1, 2, 5, 6, 9, 13, 32, 36–38]).

The Gaussian free field (GFF) is a Gaussian process indexed by high-dimensional parameters. The main aim of this paper is to investigate the connection between the height fluctuations of the dimer model on a contracting square-hexagon lattice and the Gaussian free field. It was first shown in [20, 21] that the (non-rescaled) height function for the dimer model with uniform underlying measure on a simply connected square grid with Temperley boundary condition converges to a GFF in distribution. The result was later proved for the whole-plane isoradial graph [11] and the simply connected isoradial graph with Temperley boundary condition [29]. For boundary conditions other than the Temperley boundary condition, the convergence of height fluctuation for the dimer model with uniform underlying measure on a contracting hexagon lattice to GFF was proved in [38]; the corresponding result on an Aztec diamond (contracting square grid) with uniform underlying measure was proved in [9] by analyzing the asymptotics of the Schur function in a neighborhood of $(1, 1, \ldots, 1)$ [7].

A related model is the dimer model on a contracting square-hexagon lattice whose underlying measure depends on periodically assigned edge weights with period $1 \times n$. In [5, 32], we studied this model by establishing an identity of the partition function of the dimer model on such a graph and the value of the Schur function depending on edge weights and then analyzed the asymptotics of the Schur function in a neighborhood of a generic point $(x_1, \ldots, x_N)$. The law of large numbers for the rescaled height function was proved for two specific boundary conditions on the bottom boundary: (1) each remaining vertex on the boundary is followed by $(m - 1)$ removed vertices, where $m \geq 1$ is a positive integer; (2) the bottom boundary is divided to alternating line segments with either all vertices removed or all the vertices preserved in each segment. We shall call the first boundary condition the *staircase boundary condition* and the second boundary condition the *piecewise boundary condition*. In this paper, we study the non-rescaled height fluctuations for the dimer model on the contracting square-hexagon lattice with the above two boundary conditions and show that they converge to GFF in a certain sense, building on the analysis of the Schur function at a generic point (see [5, 32]), and the techniques to relate fluctuations of particle systems determined by Schur generating functions and GFF developed in [7].

The paper is organized as follows. In Section 2, we introduce the contracting square-hexagon lattice and the main technical tools used in this paper; we also state the main results proved in this paper. In Section 3, we introduce the staircase boundary conditions and review the limit shape result for the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the staircase boundary conditions. In Section 4, we prove that certain statistics constructed from the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the staircase boundary conditions converge to Gaussian distribution in the scaling limit. In Section 5, we introduce the piecewise boundary conditions and review the limit shape result for the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the piecewise boundary conditions. In Section 6, we prove that certain statistics constructed from the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the piecewise boundary conditions converge to a sum of finitely many independent Gaussian random variables in the scaling limit, whose number depends on the size of the period n. In Section 7, we show that the statistics constructed from the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the staircase boundary conditions converge to a GFF in the upper half-plane, under a homeomorphism from the liquid region to the upper half-plane. In Section 8, we show that the statistics constructed from the dimer model on a contracting square-hexagon lattice with $1 \times n$ periodic edge weights and the piecewise boundary conditions convergence to a sum of n independent GFFs in the upper half-plane.

2. Background

In this section, we define a general class of graphs (the contracting square-hexagon lattices) on which the height fluctuations of dimer configurations are studied in this paper. Dimer models on such graphs have been studied in [3–5, 30–34], and the limit shape results were explicitly established. We also review the main technical tools used in this paper, including the Schur function, the Young diagram, etc.

2.1. Square-hexagon lattices

Consider a doubly-infinite binary sequence

$$\check{c} = (\dots, c_{-2}, c_{-1}, c_0, c_1, c_2, \dots) \in \{0, 1\}^{\mathbb{Z}}$$

indexed by integers

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The *whole-plane square-hexagon lattice* associated with the sequence $\check{c}$ is a bipartite plane graph $\mathrm{SH}(\check{c})$ defined as follows. Its vertex set is a subset of $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. Each vertex of $\mathrm{SH}(\check{c})$ is either black or white, and we identify the vertices with points on the plane. For $m \in \mathbb{Z}$, the black vertices have y-coordinate m, while the white vertices have y-coordinate $m - \frac{1}{2}$. We will label all the vertices with coordinate m as vertices in the $(2m)$th row, and all the vertices with coordinate $m - \frac{1}{2}$ as vertices in the $(2m-1)$th row.

We further assume that

- each black vertex in the $(2m)$th row is adjacent to two white vertices in the $(2m+1)$th row;
- if $c_m = 1$, each white vertex on the $(2m-1)$th row is adjacent to exactly one black vertex in the $(2m)$th row; if $c_m = 0$, each white vertex on the $(2m-1)$th row is adjacent to two black vertices in the $(2m)$th row.

See Figure 1.

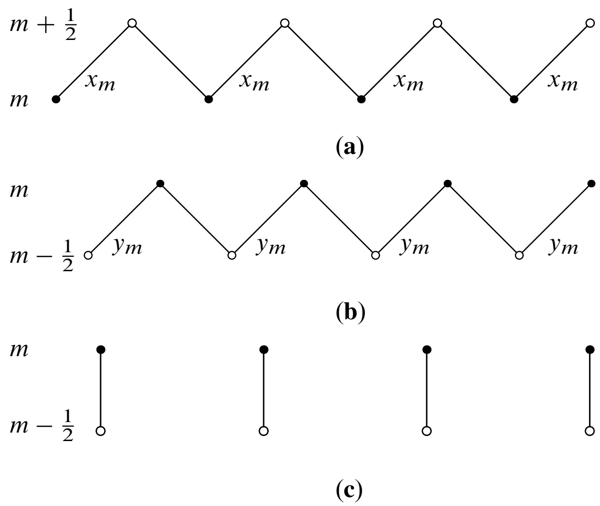

Figure 1. Graph structures of the square-hexagon lattice $\mathrm{SH}(\check{c})$ on the $(2m-1)$th, $(2m)$th, and $(2m+1)$th rows, which depend on the values of (c_m): (a) between the $(2m)$th and the $(2m+1)$th rows; (b) between the $(2m-1)$th and the $(2m)$th rows when $c_m = 0$; (c) between the $(2m-1)$th and the $(2m)$th rows when $c_m = 1$. Black vertices are along the $(2m)$th row, while white vertices are along the $(2m-1)$th and $(2m+1)$th rows.

Note that for any $\check{c} \in \{0,1\}^{\mathbb{Z}}$, each face of $\mathrm{SH}(\check{c})$ is either a square or a hexagon. If $c_i = 0$ for all $i \in \mathbb{Z}$, $\mathrm{SH}(\check{c})$ is a square grid, while if $c_i = 1$ for all i, $\mathrm{SH}(\check{c})$ is a hexagonal lattice.

We shall assign edge weights to the whole-plane square-hexagon lattice $\mathrm{SH}(\check{c})$ satisfying the following assumption; see Figure 1.

Assumption 2.1. *For $m \geq 1$, we assign weight $x_m > 0$ to each NE-SW edge joining the $(2m)$th row to the $(2m+1)$th row of $\mathrm{SH}(\check{c})$. We assign weight $y_m > 0$ to each NE-SW edge joining the $(2m-1)$th row to the $(2m)$th row of $\mathrm{SH}(\check{c})$, if such an edge exists. We assign weight 1 to all the other edges.*

A *contracting square-hexagon lattice* is built from a whole-plane square-hexagon lattice as follows.

Definition 2.2. Let $N \in \mathbb{N}$. Let $\Omega = (\Omega_1, \dots, \Omega_N)$ be an N-tuple of positive integers such that $1 = \Omega_1 < \Omega_2 < \dots < \Omega_N$. Set $m = \Omega_N - N$.

The contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$ is a subgraph of $\mathrm{SH}(\check{c})$ built of $2N$ or $2N + 1$ rows. The rows of $\mathcal{R}(\Omega, \check{c})$ inductively, starting from the bottom, can be enumerated as follows:

- The first row consists of vertices (i, j) with $i = \Omega_1 - \frac{1}{2}, \dots, \Omega_N - \frac{1}{2}$ and $j = \frac{1}{2}$. We call this row the boundary row of $\mathcal{R}(\Omega, \check{c})$.
- When $k = 2s$, for $s = 1, \dots, N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to at least one vertex in the $(2s - 1)$th row of the whole-plane square-hexagon lattice $\mathrm{SH}(\check{c})$ lying between the leftmost vertex and rightmost vertex of the $(2s - 1)$th row of $\mathcal{R}(\Omega, \check{c})$.
- When $k = 2s + 1$, for $s = 1, \dots, N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to two vertices in the $(2s)$th row of $\mathcal{R}(\Omega, \check{c})$.

See Figure 2 for an example of a contracting square-hexagon lattice.

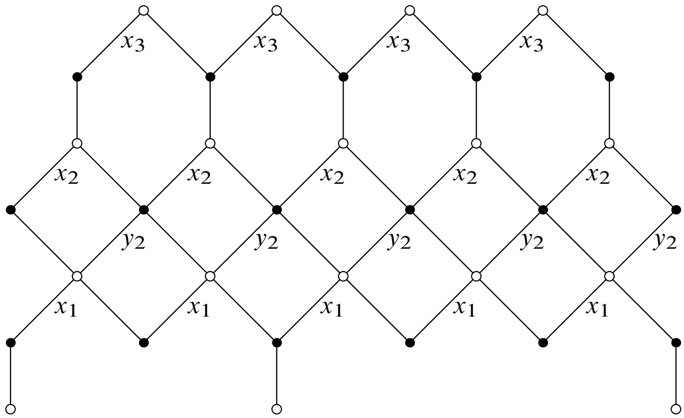

Figure 2. Contracting square-hexagon lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6)$, $(c_1, c_2, c_3) = (1, 0, 1)$.

2.2. Partitions, Young diagrams and Schur functions

We denote by $\mathbb{GT}_N$ the set of N-tuples λ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$, and let $\mathbb{GT}_N^+$ be a subset of $\mathbb{GT}_N$ consisting of all the λ's in $\mathbb{GT}_N$ such that $\lambda_N \geq 0$. For $\lambda \in \mathbb{GT}_N^+$, let

$$|\lambda| := \sum_{i=1}^{N} \lambda_i.$$

A graphic way to represent a non-negative signature μ is through its *Young diagram* Y_λ, a collection of $|\lambda|$ boxes arranged in non-increasing rows aligned on the left:

with λ_1 boxes on the first row, λ_2 boxes on the second row, ..., λ_N boxes on the Nth row. Note that elements in $\mathbb{GT}_N^+$ are in bijection with all the Young diagrams with N rows (rows are allowed to have zero length).

Definition 2.3. Let Y, W be two Young diagrams. We say that $Y \subset W$ *differ by a horizontal strip* if the collection of boxes in $Z = W \setminus Y$ contains at most one box in every column. We say that they *differ by a vertical strip* if Z contains at most one box in every row.

We say that two non-negative signatures λ and μ *interlace* and write $\lambda \prec \mu$ if $Y_\lambda \subset Y_\mu$ differ by a horizontal strip. We say they *cointerlace* and write $\lambda \prec' \mu$ if $Y_\lambda \subset Y_\mu$ differ by a vertical strip.

Definition 2.4. Let $\lambda \in \mathbb{GT}_N^+$ be a partition of length N. We define the counting measure $m(\lambda)$ corresponding to λ as follows:

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\Big(\frac{\lambda_i + N - i}{N}\Big).$$

Moreover, if λ is a random partition with distribution ρ, we use $m(\rho)$ to denote the corresponding random counting measure.

Definition 2.5. Let $\lambda \in \mathbb{GT}_N$. The Schur function is

$$s_\lambda(u_1, \dots, u_N) = \frac{\det_{i,j=1,\dots,N}(u_i^{\lambda_j + N - j})}{\prod_{1 \le i < j \le N}(u_i - u_j)}.$$

2.3. Dimer model

Definition 2.6. A dimer configuration, or a perfect matching, M of a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$ is a set of edges $((i_1, j_1), (i_2, j_2))$ such that each vertex of $\mathcal{R}(\Omega, \check{c})$ belongs to a unique edge in M. The set of perfect matchings of $\mathcal{R}(\Omega, \check{c})$ is denoted by $\mathcal{M}(\Omega, \check{c})$.

Definition 2.7. Let $M \in \mathcal{M}(\Omega, \check{c})$ be a perfect matching of $\mathcal{R}(\Omega, \check{c})$. A V-edge $e = ((i_1, j_1), (i_2, j_2))$ is a present edge in the dimer configuration M such that $\max\{j_1, j_2\} \in \mathbb{N}$ (i.e., its higher extremity is black). A Λ-edge $e = ((i_1, j_1), (i_2, j_2))$ is a present edge in M such that its higher vertex is a white vertex, or equivalently, $\max\{j_1, j_2\} \in \mathbb{N} + \frac{1}{2}$. In other words, the edges in M whose lower vertex is in an odd row are V-edges, and those ones whose lower vertex is in an even row are Λ-edges. If an edge $e = ((i_1, j_1), (i_2, j_2))$ is an V-edge (resp. Λ-edge), then both of its end-points (i_1, j_1) and (i_2, j_2) are called V-vertices (resp. Λ-vertices). Note that whether a vertex is a V-vertex or a Λ-vertex depends on the dimer configuration M.

Definition 2.8. The partition function of the dimer model of a finite graph G with edge weights $(w_e)_{e\in E(G)}$ is given by

$$Z = \sum_{M\in\mathcal{M}} \prod_{e\in M} w_e,$$

where $\mathcal{M}$ is the set of all perfect matchings of G. The Boltzmann dimer probability measure on M induced by the weights w is thus defined by declaring that the probability of a perfect matching is equal to $\frac{1}{Z}\prod_{e\in M} w_e$.

We shall associate to each perfect matching in $\mathcal{M}(\Omega,\check{c})$ a sequence of non-negative signatures, one for each row of the graph.

Construction 2.9. To the boundary row $\Omega = (\Omega_1 < \cdots < \Omega_N)$ of a contracting square-hexagon lattice is naturally associated a non-negative signature ω of length N by

$$\omega = (\Omega_N - N, \dots, \Omega_1 - 1).$$

Let $j \in \{2,\dots,2N+1\}$. Assume that the jth row of $\mathcal{R}(\Omega,\check{c})$ has n_j V-vertices and m_j Λ-vertices. The dimer configuration at the jth row of $\mathcal{R}(\Omega,\check{c})$ corresponds to a signature $\mu \in \mathbb{GT}^+_{n_j}$ such that

- $\mu = (\mu_1,\dots,\mu_{n_j})$.
- We label all the V-vertices on the jth row by the 1st V-vertex, the 2nd V-vertex, …, the n_jth V-vertex, such that the 1st V-vertex is the rightmost V-vertex on the jth row. for $1 \le k \le n_j$, μ_k is the number of Λ-vertices to the left of the kth V-vertex.

Then we have the following assertion.

Theorem 2.10 ([5, Theorem 2.13]). *For given Ω, $\check{c}$, let ω be the signature associated to Ω. Then Construction 2.9 defines a bijection between the set of perfect matchings $\mathcal{M}(\Omega,\check{c})$ and the set $S(\omega,\check{c})$ of sequences of non-negative signatures*

$$\{\mu^{(2N+1)}, \mu^{(2N)}, \dots, \mu^{(2)}, \mu^{(1)}, \mu^{(0)}\},$$

where the signatures satisfy the following properties:

- *All the parts of $\mu^{(0)}$ are equal to 0.*
- *The signature $\mu^{(2N+1)}$ is equal to ω.*
- *For $0 \le i \le 2N+1$, $\mu^{(i)} \in \mathbb{GT}^+_i$.*
- *The signatures satisfy the following (co)interlacement relations:*

$$\mu^{(2N+1)} \prec' \mu^{(2N)} \succ \mu^{(2N-1)} \prec' \cdots \succ \mu^{(2)} \prec' \mu^{(1)} \succ \mu^{(0)}.$$

Moreover, if $c_k = 1$, then $\mu^{(2N+3-2k)} = \nu^{(2N+2-2k)}$.

The following proposition, proved in [5], shows that the partition function of dimer configurations on a contracting square-hexagon lattice can be computed by a Schur function depending on the boundary condition and the edge weights. Therefore, it opens the door for investigating the asymptotics of the periodic dimer model on a contracting square-hexagon lattice by studying the corresponding Schur functions.

Proposition 2.11. *Let $\mathcal{R}(\Omega, \check{c})$ be a contracting square-hexagon lattice built from a whole-plane square-hexagon lattice* $\mathrm{SH}(\check{c})$ *with edge weights $\{x_i, y_i, 1\}_{1\leq i\leq N}$ assigned as in Assumption 2.1. Let*

$$I_2 = \{i \mid i \in \{1, 2, \dots, N\}, c_i = 0\}.$$

Then the partition function for perfect matchings on $\mathcal{R}(\Omega, \check{c})$ is given by

$$Z = \Big[\prod_{i\in I_2} \Gamma_i\Big] s_\omega(x_1, \dots, x_N),$$

where ω is the N-tuple corresponding to the boundary row of $\mathcal{R}(\Omega, \check{c})$, and Γ_i is defined by

$$\Gamma_i = \prod_{t=i+1}^{N} (1 + y_i x_t).$$

Proof. See [5, Proposition 2.18]. ■

2.4. Main results

Now we state the main results proved in the paper.

Let C_0^∞ be the space of smooth real-valued functions with compact support in the upper half-plane $\mathbb{H}$. The *Gaussian free field* Ξ on $\mathbb{H}$ with the zero boundary condition is a collection of Gaussian random variables $\{\xi_f\}_{f\in C_0^\infty}$ indexed by functions in C_0^∞, such that the covariance of two Gaussian random variables ξ_{f_1}, ξ_{f_2} is given by

$$\mathrm{Cov}(\xi_{f_1}, \xi_{f_2}) = \int_{\mathbb{H}}\int_{\mathbb{H}} f_1(z) f_2(w) G_{\mathbb{H}}(z, w)\, dz\, d\overline{z}\, dw\, d\overline{w},$$

where

$$G_{\mathbb{H}}(z, w) := -\frac{1}{2\pi} \ln\left|\frac{z - w}{z - \overline{w}}\right|, \quad z, w \in \mathbb{H},$$

is the Green's function of the Dirichlet Laplacian on $\mathbb{H}$.

The Gaussian free field Ξ can also be considered as a random distribution on C_0^∞ of $\mathbb{H}$ such that for any $f \in C_0^\infty$, we have

$$\Xi(f) = \int_{\mathbb{H}} f(z)\Xi(z)\,dz := \xi_f,$$

where $\Xi(z)$ is the generalized function corresponding to the linear functional Ξ. Note that GFF is conformally invariant in the sense that for any simply connected domain, $\mathcal{S} \subsetneq \mathbb{C}$, and let $\phi\colon \mathcal{S} \to \mathbb{H}$ be a conformal map from $\mathcal{S}$ to $\mathbb{H}$. Then the GFF on $\mathcal{S}$ is

$$\Xi_{\mathcal{S}}(z) := \Xi(\phi(z)).$$

See [39] for more about GFF.

Consider a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$. Let ω be a signature corresponding to the boundary row.

Let $\kappa \in [0, 1)$ and $\lambda \in \mathbb{GT}_{N-\lfloor \kappa N \rfloor}$. Let

$$p_j^\kappa := p_j^{(N-\lfloor \kappa N \rfloor)} = \sum_{i=1}^{N-\lfloor \kappa N \rfloor} (\lambda_i + (N - \lfloor \kappa N \rfloor) - i)^j \quad \text{for } j = 1, 2, \ldots,$$

where λ is the random partition corresponding to the dimer configuration at level κ of the square-hexagon lattice.

The main results proved in this paper (Theorems 2.12 and 2.13) state that in the liquid region, the fluctuations of certain observables of the random perfect matchings on a contracting square-hexagon lattice converge to the pullback of GFF in the upper half-plane. The liquid region, simply speaking, is the region in the simply connected domain approximated by the rescaled square-hexagon lattice where either the density of each type of edges (Λ-edges or V-edges as defined in Definition 2.7) is in the open interval $(0, 1)$ (i.e., neither 0 nor 1) or each type of edges has strictly positive probability to occur. See Definition 3.7 for a rigorous description of the liquid region.

Let

$$M_j^\kappa := \frac{N^{-(j+1)}\sqrt{\pi}}{j+1}(p_{j+1}^\kappa - \mathbf{E}p_{j+1}^\kappa) \tag{2.1}$$

and

$$U(z) = \frac{z}{n} \sum_{i\in\{1,2,\ldots,n\}\cap I_2} \frac{y_i}{1 + y_i z},$$

$$V(z) = \frac{z}{n} \sum_{j=1}^{n} \frac{1}{z - x_j},$$

$$W(z) = \frac{z}{n} \sum_{j=1}^{n} \left(\frac{mz^{m-1}}{z^m - x_j^m} - \frac{1}{z - x_j}\right).$$

Define

$$\chi_{\mathcal{L}}(z) = \frac{(W(\overline{z}) + V(\overline{z}))U(z) - (U(\overline{z}) + W(\overline{z}))V(z) + (V(\overline{z}) - U(\overline{z}))W(z)}{U(z) - U(\overline{z}) - V(z) + V(\overline{z})}, \quad (2.2)$$

$$\kappa_{\mathcal{L}}(z) = \frac{W(\overline{z}) - W(z) + V(\overline{z}) - V(z)}{U(z) - U(\overline{z}) - V(z) + V(\overline{z})}. \quad (2.3)$$

Theorem 2.12 discusses asymptotics of the perfect matchings under the staircase boundary conditions.

Theorem 2.12. *Consider random dimer configurations on the square-hexagon lattice with staircase boundary condition on the bottom boundary with corresponding partition given by*

$$\lambda(N) = ((m-1)(N-1), (m-1)(N-2), \dots, (m-1), 0), \quad (2.4)$$

where $m \geq 1$ is a positive integer. Let

$$\mathbb{S} = \left\{ z \in \mathbb{C} : 0 < \operatorname{Arg} z < \frac{\pi}{m} \right\}. \quad (2.5)$$

Then for $0 < \kappa \leq 1$, $j \in \mathbb{N}$,

$$M_j^\kappa \to \mathbf{M}_j^\kappa \quad \text{as } N \to \infty, \quad \text{and} \quad \mathbf{M}_j^\kappa = \int_{z \in \mathbb{S}; \kappa_{\mathcal{L}}(z) = \kappa} \chi_{\mathcal{L}}(z)^j \, \Xi(z) \, d\chi_{\mathcal{L}}(z).$$

Here $\Xi(z)$ is the Gaussian free field in $\mathbb{S}$, and $(\chi_{\mathcal{L}}, \kappa_{\mathcal{L}})$ is a homeomorphism from $\mathbb{S}$ to the liquid region defined by (2.2) and (2.3).

Theorem 2.13 discusses asymptotics of the perfect matchings under the piecewise boundary conditions.

Theorem 2.13. *Let M_j^κ be defined as in (2.1) but with respect to piecewise boundary conditions satisfying Assumptions 5.1, 5.2 and 5.3. Then the liquid region is a disjoint union of n simply connected components $\mathcal{S}_1, \dots, \mathcal{S}_n$. Then we have*

$$M_j^\kappa \to \mathbf{M}_j^\kappa \quad \text{as } N \to \infty,$$

where

$$\mathbf{M}_j^\kappa = \sum_{i=1}^{n} \int_{z \in \mathbb{H}, \kappa_{\mathcal{S}_i}(z) = \kappa} \chi_{\mathcal{S}_i}(z)^j \, \Xi_i(z) \, d\chi_{\mathcal{S}_i}(z).$$

Here for $1 \leq i \leq n$, $\Xi_i(z)$'s are n independent Gaussian free fields in $\mathbb{H}$, and for each $i \in [n]$, $(\chi_{\mathcal{S}_i}, \kappa_{\mathcal{S}_i})$ is a homeomorphism from the upper half-plane to $\mathcal{S}_i$ defined by (8.2), (8.4) and (8.3), (8.5), respectively.

Here Assumptions 5.1, 5.2 and 5.3 discuss the piecewise boundary condition and conditions of edge weights. The key point is that as the size of the graph N goes to infinity, the ith largest edge weight is exponentially small in N compared to the $(i-1)$th largest edge weight; see Assumptions 5.1, 5.2 and 5.3 for details.

The major new techniques developed in this paper are approaches to deal with boundary conditions. For the staircase boundary conditions, we prove a homeomorphism from the liquid region to $\mathbb{S}$, given by a certain non-real root of an algebraic equation; then the (unrescaled) height fluctuations converge to a pullback of GFF in $\mathbb{S}$. For the piecewise boundary conditions, the results for convergence of height fluctuations to GFF is built upon results in [32], where it is proved that the liquid region splits to disconnected components. In this paper, we further prove that the height fluctuations in each component of the liquid region is a pullback of an independent GFF under a homeomorphism from the component of the liquid region to the upper half-plane.

3. Staircase boundary conditions

In this section, we introduce the staircase boundary conditions on the bottom boundary of a contracting square-hexagon lattice and review the limit shape result of the dimer model on such a lattice.

Consider a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$ with edge weights assigned as in Assumption 2.1. Suppose that the configuration on the bottom row corresponds to the partition given by (2.4). More precisely, each remaining vertex on the boundary row is followed by $(m-1)$ removed vertices in the boundary row; the leftmost vertex and the rightmost vertex on the boundary row are both preserved in $\mathcal{R}(\Omega, \check{c})$. In [35, Example 1.3.7], the Schur function of such a signature is computed explicitly as follows:

$$s_{\lambda(N)}(x_1, \ldots, x_N) = \prod_{1 \le i < j \le N} \frac{x_i^m - x_j^m}{x_i - x_j}. \tag{3.1}$$

Definition 3.1. Let

$$X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N. \tag{3.2}$$

Let ρ_N be a probability measure on $\mathbb{GT}_N$. The *Schur generating function* with respect to ρ_N, X is given by

$$\mathcal{S}_{\rho_N, X}(u_1, \ldots, u_N) = \sum_{\lambda \in \mathbb{GT}_N} \rho_N(\lambda) \frac{s_\lambda(u_1, \ldots, u_N)}{s_\lambda(x_1, \ldots, x_N)}.$$

For a positive integer s, let $\overline{s} = [s \bmod n]$. We make the following assumption on edge weights.

Assumption 3.2. *Assume that the edge weights x_i $(1 \le i \le N)$, y_j $(j \in I_2)$ are periodic in i and j with period n, i.e.,*

$$x_{\overline{i}} = x_i, \tag{3.3}$$
$$y_{\overline{j}} = y_j \tag{3.4}$$

for $1 \le i \le N$, $j \in I_2$.

Lemma 3.3. *Let $x_i > 0$ $(1 \le i \le N)$, y_j $(j \in I_2)$ be edge weights of a contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$ satisfying Assumptions 2.1 and 3.2. Let*

$$X^{(N-t)} = (x_{\overline{t+1}}, \dots, x_{\overline{N}}),$$
$$Y^{(t)} = (x_{\overline{1}}, \dots, x_{\overline{t}})$$

for each integer t satisfying $0 \le t \le N-1$, where $x_i > 0$ $(1 \le i \le n)$ are weights of NE-SW edges joining the $(2i)$th row to the $(2i+1)$th row of the contracting square-hexagon lattice, see Figure 2. Let $\lambda(N)$ be the partition corresponding to the configuration on the boundary row, and let ρ^k be the probability measure on $\mathbb{GT}^+_{N-t}$ which is the distribution of partitions corresponding to the dimer configuration on the kth row of vertices of $\mathcal{R}(\Omega, \check{c})$, counting from the bottom. Then we have

$$\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \dots, u_{N-t}) = \frac{s_{\lambda(N)}(u_1, \dots, u_{N-t}, Y^{(t)})}{s_{\lambda(N)}(X^{(N)})} \prod_{i \in \{1,\dots,t\} \cap I_2} \prod_{j=1}^{N-t} \frac{1 + y_{\overline{i}} u_j}{1 + y_{\overline{i}} x_{\overline{t+j}}}$$

for $k = 2t+1$, $t = 0, 1, \dots, N-1$.

Moreover,

$$\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \dots, u_{N-t}) = \frac{s_{\lambda(N)}(u_1, \dots, u_{N-t}, Y^{(t)})}{s_{\lambda(N)}(X^{(N)})} \prod_{i \in \{1,\dots,t+1\} \cap I_2} \prod_{j=1}^{N-t} \frac{1 + y_{\overline{i}} u_j}{1 + y_{\overline{i}} x_{\overline{t+j}}}$$

for $k = 2t+2$, $t = 0, 1, \dots, N-1$.

Proof. See [5, Lemma 3.17]. ∎

Lemma 3.4. *Let $\kappa \in (0,1)$. Let $\mathcal{R}(\Omega, \check{c})$ be a contracting square-hexagon lattice. Let $\{\lambda\lfloor(1-\kappa)N\rfloor\}_{N\in\mathbb{N}}$ be a sequence of partitions corresponding to dimer configurations on the $[2(N - \lfloor(1-\kappa)N\rfloor) + 1]$th row of $\mathcal{R}(\Omega, \check{c})$, counting from the bottom.*

In particular, $\lambda(N)$ is a fixed partition corresponding to the bottom boundary configuration. Let ρ^k be the measure on the configurations of the kth row, counting from the bottom. Let X be an N-tuple of integers given by (3.2), which are also edge weights of $\mathcal{R}(\Omega,\check{c})$ satisfying (3.3). Let $\rho_{\lfloor(1-\kappa)N\rfloor} := \rho^{2(N-\lfloor(1-\kappa)N\rfloor)+1}$ be a probability measure on $\mathbb{GT}^+_{\lfloor(1-\kappa)N\rfloor}$. Note that $\rho_N := \delta_{\lambda(N)}$ is the distribution of partitions corresponding to the dimer configurations on the bottom row, in which $\lambda(N)$ has probability 1 to occur. Let $\mathcal{S}_{\rho_{\lfloor(1-\kappa)N\rfloor},X}(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})$ be the Schur generating function corresponding to $\rho_{\lfloor(1-\kappa)N\rfloor}$ and X. Then we have

(1) *Assume $1\le i\le n$, then*

$$\lim_{N\to\infty}\frac{1}{(1-\kappa)N}\frac{\partial\log\mathcal{S}_{\rho_{(1-\kappa)N},X}(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})}{\partial u_i}\Big|_{\substack{(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})\\=(x_{1+N-\lfloor(1-\kappa)N\rfloor},\dots,x_N)}}$$
$$= H_i(X,Y,\kappa),$$

where

$$H_i(X,Y,\kappa)=\frac{1}{(1-\kappa)n}\left\{\Big[\sum_{\substack{j\in\{1,2,\dots,n\},\\ j\neq i}}\Big(\frac{mx_i^{m-1}}{x_i^m-x_j^m}-\frac{1}{x_i-x_j}\Big)\Big]+\frac{m-1}{2x_i}\right\}$$
$$+\frac{\kappa}{(1-\kappa)n}\sum_{j\in\{1,2,\dots,n\}\cap I_2}\frac{y_j}{1+y_jx_i}. \tag{3.5}$$

(2) *Assume $1\le i,j\le\lfloor(1-\kappa)N\rfloor$ and $i\neq j$. Then*

$$\lim_{N\to\infty}\frac{\partial^2\log\mathcal{S}_{\lfloor\rho_{(1-\kappa)N}\rfloor,X}(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})}{\partial u_i\partial u_j}\Big|_{\substack{(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})\\=(x_{1+N-\lfloor(1-\kappa)N\rfloor},\dots,x_N)}}$$
$$= G(x_i,x_j),$$

where

$$G(x_i,x_j)=\begin{cases}\frac{m^2x_i^{m-1}x_j^{m-1}}{(x_i^m-x_j^m)^2}-\frac{1}{(x_i-x_j)^2} & \text{if } x_i\neq x_j,\\ 0 & \text{otherwise.}\end{cases}$$

(3) *Assume $i,j,k\in\{1,2,\dots,n\}$ are three distinct integers, then*

$$\lim_{N\to\infty}\frac{\partial^3\log\mathcal{S}_{\rho_{\lfloor(1-\kappa)N\rfloor},X}(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})}{\partial u_i\partial u_j\partial u_k}\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}=0.$$

Proof. The results can be obtained by applying Lemma 3.3, (3.1) and Definition 3.1 and making explicit computations. ∎

Remark 3.5. Lemma 3.4 still holds if we define $\rho_{\lfloor(1-\kappa)N\rfloor} := \rho^{2(N-\lfloor(1-\kappa)N\rfloor)+2}$ and $\{\lambda\lfloor(1-\kappa)N\rfloor\}_{N\in\mathbb{N}}$ is a sequence of signatures corresponding to dimer configurations on the $[2(N-\lfloor(1-\kappa)N\rfloor)+2]$th row of $\mathcal{R}(\Omega,\check{c})$.

Proposition 3.6. *Let $\mathcal{R}(\Omega(N),\check{c})$ be a contracting square-hexagon lattice with the configuration at the bottom boundary given by*

$$\Omega(N) = (1, m+1, 2m+1, \ldots, (N-1)m+1).$$

Assume also that the edge weights are assigned as in Assumption 2.1 (see Figure 2 for an example) and satisfy Assumption 3.2. Let

$$F_{\kappa,m}(z) = \frac{\kappa z}{n(1-\kappa)} \sum_{i\in\{1,2,\ldots,n\}\cap I_2} \frac{y_i}{1+y_i z} + \sum_{j=1}^{n} \frac{z}{n(z-x_j)} + \frac{z}{n(1-\kappa)} \sum_{j=1}^{n} \left(\frac{m z^{m-1}}{z^m - x_j^m} - \frac{1}{z-x_j} \right).$$

Let ρ^k be the measure on the configurations of the kth row, and choose $\kappa \in (0,1)$, such that $k = [2\kappa N]$. Then the corresponding random counting measure $m(\rho^k)$, as defined in Definition 2.4, converges to $\mathbf{m}^\kappa$ in probability as $N \to \infty$, and the moments of $\mathbf{m}^\kappa$ are given by

$$\int_{\mathbb{R}} y^p \mathbf{m}^\kappa(dy) = \sum_{i=1}^{n} \frac{1}{2(p+1)\pi\mathbf{i}} \oint_{x_{t+i}} \frac{dz}{z} [F_{\kappa,m}(z)]^{p+1}. \tag{3.6}$$

Proof. See of [5, Section 8]. ∎

We can compute the Stieltjes transform of the limit measure $\mathbf{m}^\kappa$ when x is in a neighborhood of infinity by

$$\mathrm{St}_{\mathbf{m}^\kappa}(x) := \sum_{j=0}^{\infty} \frac{\int_{\mathbb{R}} y^j \mathbf{m}^\kappa(dy)}{x^{j+1}} = -\sum_{i=1}^{n} \frac{1}{2\pi\mathbf{i}} \oint_{x_{t+i}} \frac{dz}{z} \log\left(1 - \frac{F_{\kappa,m}(z)}{x}\right). \tag{3.7}$$

Integrating by parts, we have

$$\sum_{j=0}^{\infty} \frac{\int_{\mathbb{R}} y^j \mathbf{m}^\kappa(dy)}{x^{j+1}} = \sum_{i=1}^{n} \frac{1}{2\pi\mathbf{i}} \oint_{x_{t+i}} \log(z) \frac{\partial_z (1 - \frac{F_{\kappa,m}(z)}{x})}{1 - \frac{F_{\kappa,m}(z)}{x}} \, dz.$$

The integrand has poles at roots of

$$F_{\kappa,m}(z) = x.$$

Definition 3.7. Let $\mathcal{R}$ be the rescaled square-hexagon lattice, i.e., $\mathcal{R} = \frac{1}{N}\mathcal{R}(\Omega, \check{c})$, with coordinates (χ, κ). Let $\mathcal{L}$ be the set of (χ, κ) inside $\mathcal{R}$ such that the density $d\mathbf{m}^{\kappa}(\frac{\chi}{1-\kappa})$ is not equal to 0 or 1. Then $\mathcal{L}$ is called the *liquid region*. Its boundary $\partial\mathcal{L}$ is called the *frozen boundary*.

4. Central limit theorem for staircase boundary conditions

In this section, we construct certain statistics from the (random) dimer configuration on a contracting square-hexagon lattice with staircase boundary conditions and show that they converge in distribution to Gaussian random variables in the scaling limit. The main theorem proved in this section is the following.

Theorem 4.1. *The collection of random variables*

$$\left\{N^{-l}\left[p_l^{((1-\kappa)N)} - \mathbb{E}p_l^{((1-\kappa)N)}\right]\right\}_{l\in\mathbb{N};\,\kappa=a_1,\ldots,a_m} \tag{4.1}$$

converges as $N \to \infty$, in the sense of moments, to the Gaussian vector with zero mean and covariance

$$\begin{aligned}\lim_{N\to\infty} & \frac{\operatorname{cov}\left(p_{l_1}^{\lfloor(1-\kappa_1)N\rfloor}, p_{l_2}^{\lfloor(1-\kappa_2)N\rfloor}\right)}{N^{l_1+l_2}} \\ &= \frac{(1-\kappa_1)^{l_1}(1-\kappa_2)^{l_2}}{(2\pi\mathbf{i})^2}\sum_{i=1}^n\sum_{j=1}^n \oint_{|z-x_i|=\varepsilon}\oint_{|w-x_j|=\varepsilon}\left(\sum_{i=1}^n \frac{z}{n(z-x_i)} + \frac{zH(z)}{1-\kappa_1}\right)^{l_1} \\ &\quad\times\left(\sum_{i=1}^n \frac{w}{n(w-x_j)} + \frac{wH(w)}{1-\kappa_2}\right)^{l_2} Q(z,w)\,dz\,dw,\end{aligned}$$

where

- *$\varepsilon > 0$ is sufficiently small such that the disk centered at x_i with radius ε contains exactly one singularity x_i of the integrand;*
- *the z- and w-contours of integration are counter-clockwise.*

The idea we use to prove Theorem 4.1 is to compute the moments of (4.1) and then show that they satisfy Wick's probability theorem in the $N \to \infty$ limits; this gives the Gaussian distribution of these random variables as well as the explicit form of the covariance. The major new ingredients, compared to [7], are the computations under the staircase boundary conditions.

Let X be given by (3.2), and let $V_N(X)$ be the Vandermonde determinant, i.e.,

$$V_N(X) = \prod_{1\le i<j\le N}(x_j - x_i).$$

Proposition 4.2. *Let ρ_N be a probability measure on $\mathbb{GT}_N$, and let $\lambda \in \mathbb{GT}_N$. Let $U = (u_1, u_2, \ldots, u_N) \in \mathbb{C}^N$. Then*

$$\begin{aligned}\mathbf{E}\sum_{i=1}^{N}(\lambda_i + N - i)^k &:= \sum_{\lambda \in \mathbb{GT}_N} \rho_N(\lambda) \sum_{i=1}^{N}(\lambda_i + N - i)^k \\ &= \frac{1}{V_N(U)} \sum_{i=1}^{N} (u_i \partial_i)^k V_N(U) \mathcal{S}_{\rho_N, X}(U)\Big|_{U=X}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\mathbf{E}\left(\sum_{i=1}^{N}(\lambda_i + N - i)^k \sum_{j=1}^{N}(\lambda_j + N - j)^l\right) \\ &\quad = \frac{1}{V_N(U)} \sum_{i=1}^{N} (u_i \partial_i)^k \sum_{j=1}^{N} (u_j \partial_j)^l V_N(U) \mathcal{S}_{\rho_N, X}(U)\Big|_{U=X}, \end{aligned} \tag{4.3}$$

where ∂_i represents $\frac{\partial}{\partial u_i}$.

Proof. Let $\lambda \in \mathbb{GT}_N$, and let s_λ be the Schur function with respect to λ. By [6, Proposition 4.3], we have

$$\frac{1}{V_N(U)} \sum_{i=1}^{N} (u_i \partial_i)^k V_N(U) s_\lambda(U) = \sum_{i=1}^{N} (\lambda_i + N - 1)^k s_\lambda(U). \tag{4.4}$$

Dividing by $s_\lambda(X)$ both sides of (4.4), then taking expectations for λ with respect to the distribution ρ_N and evaluating at $U = X$, we obtain (4.2). Expression (4.3) can be obtained in a similar way by performing the above process twice. ∎

Let $f(x_1, \ldots, x_r)$ be a function of r variables. Define

$$\operatorname*{Sym}_{x_1,\ldots,x_r} f(x_1, \ldots, x_r) = \frac{1}{r!} \sum_{\sigma \in \Sigma_r} f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(r)}),$$

where Σ_r is the symmetric group of r elements.

For an integer $l > 0$ and $t \in (0, 1]$, let

$$U_t = (u_1, u_2, \ldots, u_{\lfloor tN \rfloor}), \tag{4.5}$$

$$X_t = (x_{N - \lfloor tN \rfloor + 1}, \ldots, x_N), \tag{4.6}$$

$$\mathcal{F}_{(l,t)}(U_t) = \frac{1}{\mathcal{S}_{\rho_{\lfloor tN \rfloor}, X_t}(U_t) V_{\lfloor tN \rfloor}(U_t)} \sum_{i=1}^{\lfloor tN \rfloor} (u_i \partial_i)^l V_{\lfloor tN \rfloor}(U_t) \mathcal{S}_{\rho_{\lfloor tN \rfloor}, X_t}(U_t), \tag{4.7}$$

where $\rho_{\lfloor tN \rfloor}$ is a probability measure on $\mathbb{GT}_{\lfloor tN \rfloor}$.

In order to analyze the asymptotics, we first introduce the following technical lemma.

Lemma 4.3. *Let $f(z)$ be a complex analytic function in a neighborhood of* 1*, and let r be a positive integer. Then*

$$\operatorname*{Sym}_{z_1,\dots,z_{r+1}} \frac{f(z_1)}{(z_1-z_2)\cdots(z_1-z_{r+1})}\Big|_{\substack{(z_1,\dots,z_{r+1})\\=(1,\dots,1)}} = \frac{1}{(r+1)!}\frac{\partial^r f(z)}{\partial z^r}\Big|_{z=1}.$$

Proof. See [6, Lemma 5.5]. ∎

Proposition 4.4. *Assume the assumptions of Lemma 3.4 hold. We use the notation ∂_i to denote $\frac{\partial}{\partial u_i}$. Then we have*

(1) *the degrees of N in the functions $\mathcal{F}_{(l,t)}(U_t)|_{U_t=X_t}$ are at most $l+1$;*

(2) *for $1 \le i \le N$, the degrees of N in functions $\partial_i \mathcal{F}_{(l,t)}(U_t)|_{U_t=X_t}$ are at most l; moreover,*

$$\begin{aligned}\partial_i \mathcal{F}_{(l,t)}(U_t)|_{U_t=X_t} &= \partial_i\Bigg[\sum_{r=0}^{l}\binom{l}{r}(r+1)! \\ &\quad\times \sum_{\substack{\{a_1,\dots,a_{r+1}\}\\ \subset\{1,2,\dots,\lfloor tN\rfloor\}}} \operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{u_{a_1}^l(\partial_{a_1}[\log S_{\rho_{\lfloor tN\rfloor},X_t}])^{l-r}}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{r+1}})}\Bigg]\Bigg|_{U_t=X_t} \\ &\quad + T_{(l,t)}(U_t)|_{U_t=X_t},\end{aligned}$$

where the degree of N in $T_{(l,t)}(X_t)$ is less than l;

(3) *the degrees of N in the functions $\partial_i\partial_j\mathcal{F}_{(l,t)}(U_t)|_{U_t=X_t}$ are at most $l-1$ for any $1 \le i, j \le N$, $i \ne j$.*

Proof. When $t=1$ and $X_1=(1,\dots,1)$, the proposition is proved in [7, Lemma 5.5]. Consider a general $S_{\rho_{\lfloor tN\rfloor},X_t}$ with X_t given by (4.6). Since

$$S_{\rho_{\lfloor tN\rfloor},X_t}(X_t)=1,$$

the function $\log S_{\rho_{\lfloor tN\rfloor},X_t}$ is well defined in a neighborhood of X_t. Note that

$$\frac{\partial_i S_{\rho_{\lfloor tN\rfloor},X_t}}{S_{\rho_{\lfloor tN\rfloor},X_t}} = \partial_i(\log S_{\rho_{\lfloor tN\rfloor},X_t}).$$

This way we can write $\mathcal{F}_{(l,t)}(U_t)$ as a large sum of factors of the form

$$\frac{c_0 u_i^{l-s_0}(\partial_i^{s_1}[\log S_{\rho_{\lfloor tN\rfloor},X_t}])^{d_1}\cdots(\partial_i^{s_t}[\log S_{\rho_{\lfloor tN\rfloor},X_t}])^{d_t}}{(u_i-u_{a_1})\cdots(u_i-u_{a_r})},$$

where $i, a_1, \ldots, a_r$ are distinct indices, $s_j, d_j \in \mathbb{N} \cup \{0\}$ for $j = 1, \ldots, t$, and

$$s_1 < s_2 < \cdots < s_t, \quad r + s_0 + \sum_{j=1}^{t} s_j d_j = l. \tag{4.8}$$

Moreover, c_0 depends on r, s_j, d_j, but is independent of $N, a_1, \ldots, a_r$. By symmetry, we can write

$$\mathcal{F}_{(l,t)}(U_t) = \sum_{r,\{s_j\},\{d_j\}} (r+1)! \sum_{\substack{\{a_1,\ldots,a_{r+1}\} \\ \subset\{1,2,\ldots,\lfloor tN \rfloor\}}} \underset{a_1,\ldots,a_{r+1}}{\mathrm{Sym}} \frac{c_0 u_{a_1}^{l-s_0} (\partial_{a_1}^{s_1}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_1} \cdots (\partial_{a_1}^{s_t}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_t}}{(u_{a_1} - u_{a_2}) \cdots (u_{a_1} - u_{a_{r+1}})}, \tag{4.9}$$

where the first sum are over r, $\{s_j\}$, $\{d_j\}$ satisfying (4.8), and c_0 depends on r, $\{s_j\}$, $\{d_j\}$.

Applying Lemma 3.3, (3.1) and Definition 3.1 and making explicit computations, we can compute $(\partial_{a_1}^{s_w}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])$ and obtain that the degree of N in it is at most 1 for all $s_w \geq 0$. Hence for each $1 \leq w \leq t$, the degree of N in each factor $(\partial_{a_1}^{s_w}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_w}$ is at most d_w. For each given choice of $\{a_1, \ldots, a_{r+1}\}$, we define an equivalence relation on the set $\{a_1, \ldots, a_{r+1}\}$: for $1 \leq i, j \leq r+1$, we say a_i and a_j are equivalent if and only if $[a_i \bmod n] = [a_j \bmod n]$. Let $A_1, \ldots, A_w$ be all the distinct equivalence classes under this equivalence relation, where w is a positive integer satisfying $w \leq r+1$. For $1 \leq i \leq w$, let $C_i = \{a_1, \ldots, a_{r+1}\} \setminus A_i$.

For $1 \leq i \leq r+1$, let $a_1, \ldots, \widehat{a}_i, \ldots, a_{r+1}$ be r distinct integers obtained from $a_1, \ldots, a_{r+1}$ by removing a_i. Then

$$\begin{aligned}
&\underset{a_1,\ldots,a_{r+1}}{\mathrm{Sym}} \frac{c_0 u_{a_1}^{l-s_0} (\partial_{a_1}^{s_1}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_1} \cdots (\partial_{a_1}^{s_t}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_t}}{(u_{a_1} - u_{a_2}) \cdots (u_{a_1} - u_{a_{r+1}})} \\
&\quad = \frac{1}{(r+1)!} \sum_{i=1}^{w} \binom{r}{|C_i|} |A_i|!|C_i|! \\
&\qquad \times \underset{A_i}{\mathrm{Sym}} \Bigg(\underset{C_i}{\mathrm{Sym}} \frac{c_0 u_{a_i}^{l-s_0} (\partial_{a_i}^{s_1}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_1} \cdots (\partial_{a_i}^{s_t}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_t}}{\prod_{j \in C_i} (u_{a_i} - u_{a_j})} \\
&\qquad \times \frac{1}{\prod_{a_j \in A_i} (u_{a_i} - u_{a_j})} \Bigg) \\
&\quad = \sum_{i=1}^{w} \frac{|A_i|}{r+1} \underset{A_i}{\mathrm{Sym}} \Bigg(\frac{c_0 u_{a_i}^{l-s_0} (\partial_{a_i}^{s_1}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_1} \cdots (\partial_{a_i}^{s_t}[\log S_{\rho_{\lfloor tN \rfloor}, X_t}])^{d_t}}{\prod_{j \in C_i} (u_{a_i} - u_{a_j})} \\
&\qquad \times \frac{1}{\prod_{a_j \in A_i} (u_{a_i} - u_{a_j})} \Bigg).
\end{aligned} \tag{4.10}$$

By Lemma 3.4 and (4.8), the degree of N in

$$\frac{c_0 u_{a_i}^{l-s_0}(\partial_{a_i}^{s_1}[\log S_{\rho_{\lfloor tN \rfloor},X_t}])^{d_1}\cdots(\partial_{a_i}^{s_t}[\log S_{\rho_{\lfloor tN \rfloor},X_t}])^{d_t}}{\prod_{j\in C_i}(u_{a_i}-u_{a_j})}$$

is at most $l-r$. By Lemma 4.3, the degree of N in (4.10) is at most $l-r$. Summing over all the choices $\{a_1,\dots,a_{r+1}\}\subset\{1,2,\dots,\lfloor tN\rfloor\}$ (there are $O(N^{r+1})$ such choices), we obtain that the degree of N in $\mathcal{F}_{(l,t)}(U_t)$ is at most $l+1$; then part (1) of the proposition follows. ∎

For positive integers l_1, l_2, we define

$$\mathcal{G}_{l_1,l_2,t}(U_t)=l_1\sum_{r=0}^{l_1-1}\binom{l_1-1}{r}\sum_{\{a_1,\dots,a_{r+1}\}\subset\{1,2,\dots,\lfloor tN\rfloor\}}(r+1)!$$
$$\times\operatorname*{Sym}_{a_1,\dots,a_{r+1}}\frac{u_{a_1}^{l_1}\partial_{a_1}[\mathcal{F}_{(l_2,t)}](\partial_{a_1}[\log S_{\rho_{\lfloor tN \rfloor},X_t}])^{l_1-1-r}}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{r+1}})}.$$

Lemma 4.5. *Assume the assumption of Lemma 3.4 holds. Let l_1, l_2 be arbitrary positive integers, and $t\in(0,1]$, then*

$$\frac{1}{V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN \rfloor},X_t}}\sum_{i_1=1}^{\lfloor tN\rfloor}(u_{i_1}\partial_{i_1})^{l_1}\sum_{i_2=1}^{\lfloor tN\rfloor}(u_{i_2}\partial_{i_2})^{l_2}[V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN \rfloor},X_t}]$$
$$=\mathcal{F}_{(l_1,t)}(U_t)\mathcal{F}_{(l_2,t)}(U_t)+\mathcal{G}_{(l_1,l_2,t)}(U_t)+T(U_t),\tag{4.11}$$

where $\mathcal{G}_{(l_1,l_2,t)}(U_t)|_{U_t=X_t}$ has N-degree at most l_1+l_2 and $T(U_t)|_{U_t=X_t}$ has N degree less than l_1+l_2. Moreover, for any index i, the function $\partial_i\mathcal{G}_{(l_1,l_2,t)}(U_t)|_{U_t=X_t}$ has N-degree less than l_1+l_2.

Proof. The proof follows from arguments similar to ones in the proof for the case $X=1^N$, $t=1$ [7, Lemma 5.7]. We sketch the idea here. Note that the left-hand side of (4.11) is exactly

$$\frac{1}{V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN \rfloor},X_t}}\sum_{i_1=1}^{\lfloor tN\rfloor}(u_{i_1}\partial_{i_1})^{l_1}[V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN \rfloor},X_t}\mathcal{F}_{(l_2)}(U_t)].$$

It can be rewritten as the sum of terms of the form

$$\operatorname*{Sym}_{a_1,\dots,a_{r+1}}\frac{c_0 u_{a_1}^{l_1-s_0}\partial_{a_1}^{s_1}[\mathcal{F}_{(l_2,t)}](\partial_{a_1}^{s_2}[\log S_{\rho_{\lfloor tN \rfloor},X_t}])^{d_2}\cdots(\partial_{a_1}^{s_p}[\log S_{\rho_{\lfloor tN \rfloor},X_t}])^{d_p}}{(u_{a_1}-u_{a_2})(u_{a_1}-u_{a_3})\cdots(u_{a_1}-u_{a_{r+1}})},$$

where $r, s_0, s_1, \dots, s_p, d_2, \dots, d_p$ are non-negative integers and

$$s_2<s_3<\cdots<s_p,\qquad s_0+s_1+s_2d_2+\cdots+s_pd_p+r=l_1.$$

Then $\mathcal{F}_{(l_1,t)}(U_t)\mathcal{F}_{(l_2,t)}(U_t)$ comes from the terms with $s_1=0$; $\mathcal{G}_{(l_1,l_2,t)}(U_t)$ comes from the terms with $s_0=0, s_1=1, s_2=1, d_2=l_1-1-r$. The N-degrees of these terms can be obtained by applying Lemma 3.4. ■

Let s be a positive integer. For a subset $\{j_1,\dots,j_p\}\subset\{1,2,\dots,s\}$, let $\mathcal{P}^s_{j_1,\dots,j_p}$ be the set of all pairings of the set $\{1,2,\dots,s\}\setminus\{j_1,\dots,j_p\}$. The set $\mathcal{P}^s_{j_1,\dots,j_p}$ is non-empty only when $s-p$ is even. For a pairing P, let $\prod_{(a,b)\in P}$ denote the product over all pairs (a,b) from this pairing.

Proposition 4.6. *Assume that the assumptions of Lemma 3.4 hold. Let $s, l_1,\dots,l_s$ be arbitrary positive integers, and let $t\in(0,1]$. Then*

$$\frac{1}{V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}}\sum_{i_1=1}^{\lfloor tN\rfloor}(u_{i_1}\partial_{i_1})^{l_1}\sum_{i_2=1}^{\lfloor tN\rfloor}(u_{i_2}\partial_{i_2})^{l_2}\cdots\sum_{i_s=1}^{\lfloor tN\rfloor}(u_{i_s}\partial_{i_s})^{l_s}[V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}]$$
$$=\sum_{p=0}^{s}\sum_{\{j_1,\dots,j_p\}\subset\{1,2,\dots,s\}}\mathcal{F}_{(l_{j_1},t)}(U_t)\dots\mathcal{F}_{(l_{j_p},t)}(U_t)$$
$$\times\Big(\sum_{P\in\mathcal{P}^s_{j_1,\dots,j_p}}\prod_{(a,b)\in P}\mathcal{G}_{(l_a,l_b,t)}(U_t)+T^{1;s}_{j_1,\dots,j_p}(U_t)\Big),$$

where $T^{1;s}_{j_1,\dots,j_p}(U_t)|_{U_t=X_t}$ has N-degree less than $\sum_{i=1}^s l_i-\sum_{i=1}^p l_{j_i}$.

Proof. The proposition can be proved by induction on s similarly to the proof of [6, Proposition 5.10], where the case $X=1^N$ is proved. ■

Let l be a positive integer, and let $t\in(0,1]$. Let

$$E_{l,t}:=\mathcal{F}_{(l,t)}(X_t)=\frac{1}{V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}}\sum_{i=1}^{\lfloor tN\rfloor}(u_i\partial_i)^l V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}(U_t)\Big|_{U_t=X_t}.$$

Lemma 4.7. *Assume the assumptions of Lemma 3.4 hold. Let $s, l_1,\dots,l_s$ be arbitrary positive integers, and let $t\in(0,1]$. Then*

$$\frac{1}{V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}}\Big(\sum_{i_1=1}^{\lfloor tN\rfloor}(u_{i_1}\partial_{i_1})^{l_1}-E_{l_1,t}\Big)\Big(\sum_{i_2=1}^{\lfloor tN\rfloor}(u_{i_2}\partial_{i_2})^{l_2}-E_{l_2,t}\Big)\cdots$$
$$\times\Big(\sum_{i_s=1}^{\lfloor tN\rfloor}(u_{i_s}\partial_{i_s})^{l_s}-E_{l_s,t}\Big)V_{\lfloor tN\rfloor}S_{\rho_{\lfloor tN\rfloor},X_t}\Big|_{U_t=X_t}$$
$$=\sum_{P\in\mathcal{P}^s_{\emptyset}}\prod_{(a,b)\in P}\mathcal{G}_{(l_a,l_b,t)}(U)\Big|_{U_t=X_t}+T_{\emptyset}(U_t)|_{U_t=X_t},$$

where $T_\emptyset(U_t)|_{U_t=X_t}$ has N-degree less than $\sum_{i=1}^s l_i$.

Assume the distribution of λ is $\rho_{N-\lfloor\kappa N\rfloor}$. Let $\mathbf{E}$ be the expectation under the probability measure $\rho_{N-\lfloor\kappa N\rfloor}$, and let

$$\begin{aligned}&\operatorname{cov}(p_k^{(N-\lfloor\kappa N\rfloor)}, p_l^{(N-\lfloor\kappa N\rfloor)})\\&\quad= \mathbf{E}(p_k^{(N-\lfloor\kappa N\rfloor)} \cdot p_l^{(N-\lfloor\kappa N\rfloor)}) - \mathbf{E}p_k^{(N-\lfloor\kappa N\rfloor)}\mathbf{E}p_l^{(N-\lfloor\kappa N\rfloor)}.\end{aligned}$$

Then by Lemma 4.7 with $s = 2$, we have

$$\lim_{N\to\infty} \frac{\operatorname{cov}(p_k^{(N-\lfloor\kappa N\rfloor)}, p_l^{(N-\lfloor\kappa N\rfloor)})}{N^{k+l}} = \lim_{N\to\infty} \frac{\mathcal{G}_{(k,l,1-\kappa)}(x_{\lfloor\kappa N\rfloor+1}, \ldots, x_N)}{N^{k+l}}. \tag{4.12}$$

We have

$$\begin{aligned}
&\mathcal{G}_{(k,l,1-\kappa)}(x_{\lfloor\kappa N\rfloor+1}, \ldots, x_N)\\
&\quad= k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \binom{k-1}{q}(q+1)!\\
&\qquad\times \operatorname*{Sym}_{a_1,\ldots,a_{q+1}} \frac{u_{a_1}^k \partial_{a_1}[\mathcal{F}_{(l,1-\kappa)}](\partial_{a_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}]^{k-1-q})}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}\Bigg|_{\substack{(u_1,\ldots,\\ u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}\\
&\quad\approx k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \binom{k-1}{q}(q+1)!\\
&\qquad\times \operatorname*{Sym}_{a_1,\ldots,a_{q+1}} \Bigg(\frac{u_{a_1}^k (\partial_{a_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}]^{k-1-q})}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}\\
&\qquad\times \partial_{a_1}\Bigg[\sum_{r=0}^{l} \sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \binom{l}{r}(r+1)!\\
&\qquad\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}} \frac{u_{b_1}^l (\partial_{b_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}])^{l-r}}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg]\Bigg)\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}.
\end{aligned}$$

The approximate equality above contains only leading terms of $\partial_{a_1}[\mathcal{F}_{(l,1-\kappa)}]$; see Proposition 4.4 (2).

We first consider the case that

$$\{a_1, \ldots, a_{q+1}\} \cap \{b_1, \ldots, b_{r+1}\} = \emptyset.$$

By Lemma 3.4, we have

$$\partial_{a_1}\Bigg[\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!$$
$$\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}}\frac{u_{b_1}^l(\partial_{b_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},X_\kappa}])^{l-r}}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg]\Bigg|_{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})=X_{1-\kappa}}$$
$$=\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!(l-r)$$
$$\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}}\frac{u_{b_1}^l(\partial_{b_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},X_\kappa}])^{l-r-1}\partial_{a_1}\partial_{b_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},X_\kappa}]}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg|_{\substack{(u_1,\ldots,\\ u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}$$
$$\approx\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!(l-r)$$
$$\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}}\frac{u_{b_1}^l(H_{b_1}(X,Y,\kappa))^{l-r-1}N^{l-r-1}G(x_{a_1},x_{b_1})}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}.$$

Moreover,

$$\operatorname*{Sym}_{a_1,\ldots,a_{q+1}}\Bigg(\frac{u_{a_1}^k(\partial_{a_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}]^{k-1-q})}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}$$
$$\times\partial_{a_1}\Bigg[\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!$$
$$\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}}\frac{u_{b_1}^l(\partial_{b_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}])^{l-r}}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg]\Bigg)\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}}$$
$$\approx\operatorname*{Sym}_{a_1,\ldots,a_{q+1}}\frac{u_{a_1}^k([H_{a_1}(X,Y,\kappa)]^{k-1-q}N^{k-1-q})}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}$$
$$\times\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!(l-r)$$
$$\times \operatorname*{Sym}_{b_1,\ldots,b_{r+1}}\frac{u_{b_1}^l(H_{b_1}(X,Y,\kappa))^{l-r-1}N^{l-r-1}G(x_{a_1},x_{b_1})}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}.$$

Lemma 4.8. *Let*

$$H(z) = \frac{1}{n} \sum_{j \in \{1,2,\ldots,n\}} \left(\frac{mz^{m-1}}{z^m - x_j^m} - \frac{1}{z - x_j} \right) + \frac{\kappa}{n} \sum_{j \in \{1,2,\ldots,n\} \cap I_2} \frac{y_j}{1 + y_j z}.$$

The contribution of the terms for which $\{a_1, \ldots, a_{q+1}\} \cap \{b_1, \ldots, b_{r+1}\} = \emptyset$ to $\mathcal{G}_{(k,l,1-\kappa)}(X_{1-\kappa})$, as $N \to \infty$, is asymptotically

$$\begin{aligned} &\frac{(1-\kappa)^{k+l} N^{k+l}}{(2\pi \mathbf{i})^2} \sum_{i,j=1}^{n} \oint_{|z-x_i|=\varepsilon} \oint_{|w-x_j|=\varepsilon} \left(\sum_{i=1}^{n} \frac{z}{n(z-x_i)} + \frac{zH(z)}{1-\kappa} \right)^k \\ &\quad \times \left(\sum_{i=1}^{n} \frac{w}{n(w-x_i)} + \frac{wH(w)}{1-\kappa} \right)^l G(z,w)\, dz dw, \end{aligned} \tag{4.13}$$

where $\varepsilon > 0$ is sufficiently small such that for each $1 \le i \le n$, the disk centered at x_i with radius ε contains exactly one singularity x_i of the integrand.

Remark 4.9. Note that $H_i(X, Y, \kappa)$ defined by (3.5) satisfies

$$H_i(X, Y, \kappa) = \frac{1}{1-\kappa} \lim_{z \to x_i} H(z).$$

Proof of Lemma 4.8. Note that

$$\binom{l}{r}(l-r) = l\binom{l-1}{r}.$$

By the computations above, the contribution of the terms when $\{a_1, \ldots, a_{q+1}\} \cap \{b_1, \ldots, b_{r+1}\} = \emptyset$ to $\mathcal{G}_{(k,l,1-\kappa)}(X_{1-\kappa})$, as $N \to \infty$, is asymptotically

$$\begin{aligned} I &:= k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\} \\ \subset \{1,2,\ldots,N-\lfloor \kappa N \rfloor\}}} \binom{k-1}{q} (q+1)! \\ &\quad \times \operatorname*{Sym}_{a_1,\ldots,a_{q+1}} \left[\frac{u_{a_1}^k ([H_{a_1}(X,Y,\kappa)]^{k-1-q} N^{k-1-q})}{(u_{a_1} - u_{a_2}) \cdots (u_{a_1} - u_{a_{q+1}})} \right. \\ &\quad \times l \sum_{r=0}^{l-1} \sum_{\substack{\{b_1,\ldots,b_{r+1}\} \subset \{1,2,\ldots,N-\lfloor \kappa N \rfloor\}, \\ \{b_1,\ldots,b_{r+1}\} \cap \{a_1,\ldots,a_{q+1}\} = \emptyset}} \binom{l-1}{r} (r+1)! \\ &\quad \times \left. \operatorname*{Sym}_{b_1,\ldots,b_{r+1}} \frac{u_{b_1}^l (H_{b_1}(X,Y,\kappa))^{l-r-1} N^{l-r-1} G(x_{a_1}, x_{b_1})}{(u_{b_1} - u_{b_2}) \cdots (u_{b_1} - u_{b_{r+1}})} \right] \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor \kappa N \rfloor}) \\ = X_{1-\kappa}}}. \end{aligned}$$

We consider the equivalence relation on $\{a_1, \ldots, a_{q+1}\}$ (resp. $\{b_1, \ldots, b_{r+1}\}$) such that for $1 \le i \le j \le q+1$, a_i and a_j (resp. for $1 \le i \le j \le r+1$, b_i and b_j) are equivalent if and only if $(a_i \mod n) = (a_j \mod n)$ (resp. $(b_i \mod n) = (b_j \mod n)$). Let $A_1, \ldots, A_h$ (resp. $B_1, \ldots, B_g$) be all the equivalence classes in $\{a_1, \ldots, a_{q+1}\}$ (resp. $\{b_1, \ldots, b_{r+1}\}$) under such an equivalence relation, where h, g are positive integers satisfying $1 \leq h \leq q+1$, $1 \leq g \leq r+1$. For $1 \leq i \leq h$ and $1 \leq j \leq g$, let

$$C_i = \{a_1, \ldots, a_{q+1}\} \setminus A_i, \quad D_j = \{b_1, \ldots, b_{r+1}\} \setminus B_j.$$

Then we have

$$\begin{aligned}
I =& k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\} \\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \binom{k-1}{q} (q+1)! \sum_{i=1}^{g} \frac{|A_i|}{(q+1)} \\
&\times \operatorname*{Sym}_{A_i} \left[\frac{u_{a_i}^k ([H_{a_i}(X,Y,\kappa)]^{k-1-q} N^{k-1-q})}{\prod_{a_s \in C_i} (u_{a_i} - u_{a_s})} \frac{1}{\prod_{a_t \in A_i \setminus \{a_i\}} (u_{a_i} - u_{a_t})} \right. \\
&\times l \sum_{r=0}^{l-1} \sum_{\substack{\{b_1,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}, \\ \{b_1,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset}} \binom{l-1}{r} (r+1)! \sum_{j=1}^{h} \frac{|B_j|}{(r+1)} \\
&\times \operatorname*{Sym}_{B_j} \left(\frac{u_{b_j}^l (H_{b_j}(X,Y,\kappa))^{l-r-1} N^{l-r-1} G(x_{a_i}, x_{b_j})}{\prod_{b_v \in D_j} (u_{b_j} - u_{b_v})} \right. \\
&\left. \left. \times \frac{1}{\prod_{b_w \in B_j \setminus \{b_j\}} (u_{b_j} - u_{b_w})} \right) \right] \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor}) \\ = X_{1-\kappa}}}.
\end{aligned}$$

By Lemma 4.3, we have

$$\begin{aligned}
I =& k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\} \\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \binom{k-1}{q} (q+1)! \sum_{i=1}^{g} \frac{|A_i|}{(q+1)} \\
&\times \frac{1}{|A_i|!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}} \left(\frac{u_{a_i}^k ([H_{a_i}(X,Y,\kappa)]^{k-1-q} N^{k-1-q})}{\prod_{a_s \in C_i} (u_{a_i} - u_{a_s})} \right. \\
&\times l \sum_{r=0}^{l-1} \sum_{\substack{\{b_1,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}, \\ \{b_1,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset}} \binom{l-1}{r} (r+1)! \sum_{j=1}^{h} \frac{|B_j|}{(r+1)} \frac{1}{|B_j|!} \frac{\partial^{|B_j|-1}}{\partial u_{b_j}^{|B_j|-1}} \\
&\left. \times \frac{u_{b_j}^l (H_{b_j}(X,Y,\kappa))^{l-r-1} N^{l-r-1} G(x_{a_i}, x_{b_j})}{\prod_{b_v \in D_j} (u_{b_j} - u_{b_v})} \right) \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor}) \\ = X_{1-\kappa}}} \\
=& \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\} \\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \sum_{i=1}^{g} \frac{k!}{(k-1-q)!} \frac{1}{(|A_i|-1)!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}}
\end{aligned}$$

$$
\times \Bigg(\frac{u_{a_i}^k([H_{a_i}(X,Y,\kappa)]^{k-1-q}N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})}
$$

$$
\times \sum_{r=0}^{l-1} \sum_{\substack{\{b_1,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset}} \frac{l!}{(l-1-r)!} \sum_{j=1}^{h} \frac{1}{(|B_j|-1)!} \frac{\partial^{|B_j|-1}}{\partial u_{b_j}^{|B_j|-1}}
$$

$$
\times \frac{u_{b_j}^l(H_{b_j}(X,Y,\kappa))^{l-r-1}N^{l-r-1}G(x_{a_i},x_{b_j})}{\prod_{b_v\in D_j}(u_{b_j}-u_{b_v})}\Bigg)\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\=X_{1-\kappa}}}.
$$

Using the residue theorem, we obtain

$$
I = \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \sum_{i=1}^{g} \frac{k!}{(k-1-q)!} \operatorname*{Res}_{z=u_{a_i}} \Bigg[\frac{z^k([H(z)]^{k-1-q}N^{k-1-q})}{(z-u_{a_i})^{|A_i|}\prod_{a_s\in C_i}(z-u_{a_s})}
$$

$$
\times \sum_{r=0}^{l-1} \sum_{\substack{\{b_1,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset}} \frac{l!}{(l-1-r)!}
$$

$$
\times \sum_{j=1}^{h} \operatorname*{Res}_{w=u_{b_j}} \left(\frac{w^l(H(w))^{l-r-1}N^{l-r-1}G(z,w)}{(w-u_{b_j})^{|B_j|}\prod_{b_v\in D_j}(w-u_{b_v})} \right) \Bigg] \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\=X_{1-\kappa}}}
$$

$$
\approx N^{k+l} \sum_{s=1}^{n} \operatorname*{Res}_{z=x_s} \Bigg[\Bigg(\frac{1}{N} \sum_{i=\lfloor\kappa N\rfloor+1}^{N} \frac{z}{z-x_i} + zH(z) \Bigg)^k
$$

$$
\times \Bigg[\sum_{j=1}^{n} \operatorname*{Res}_{w=x_j} \Bigg(\Bigg(\frac{1}{N} \sum_{i=\lfloor\kappa N\rfloor+1}^{N} \frac{w}{w-x_i} + wH(w) \Bigg)^l G(z,w) \Bigg) \Bigg] \Bigg]
$$

$$
\approx \frac{N^{k+l}(1-\kappa)^{k+l}}{(2\pi\mathbf{i})^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \oint_{|z-x_i|=\varepsilon} \left(\frac{1}{n} \sum_{p=1}^{n} \frac{z}{z-x_p} + \frac{zH(z)}{1-\kappa} \right)
$$

$$
\times \oint_{|w-x_j|=\varepsilon} \left(\frac{1}{n} \sum_{q=1}^{n} \frac{w}{w-x_q} + \frac{wH(w)}{1-\kappa} \right) G(z,w)\, dw dz.
$$

Then the lemma follows. ∎

Now we consider the case where

$$
|\{a_1,\ldots,a_{q+1}\}\cap\{b_1,\ldots,b_{r+1}\}| = 1. \tag{4.14}
$$

Without loss of generality, we suppose that $a_1 = b_1$, and all the other indices are distinct.

Then we have

$$\underset{a_1,\ldots,a_{q+1}}{\mathrm{Sym}}\left(\frac{u_{a_1}^k(\partial_{a_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}]^{k-1-q})}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}\right.$$
$$\times\partial_{a_1}\Bigg[\sum_{r=0}^{l}\sum_{\substack{\{b_2,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!$$
$$\left.\times\underset{a_1,b_2,\ldots,b_{r+1}}{\mathrm{Sym}}\frac{u_{a_1}^l(\partial_{a_1}[\log S_{\rho_{N-\lfloor\kappa N\rfloor},(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}])^{l-r}}{(u_{a_1}-u_{b_2})\cdots(u_{a_1}-u_{b_{r+1}})}\Bigg]\right)\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\=(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}}$$
$$\approx\underset{a_1,\ldots,a_{q+1}}{\mathrm{Sym}}\left(\frac{u_{a_1}^k N^{k-1-q}H(u_{a_1})^{k-1-q}}{(1-\kappa)^{k-1-q}(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}\right.$$
$$\times\partial_{a_1}\Bigg[\sum_{r=0}^{l}\sum_{\substack{\{b_2,\ldots,b_{r+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{l}{r}(r+1)!$$
$$\left.\times\underset{a_1,b_2,\ldots,b_{r+1}}{\mathrm{Sym}}\frac{u_{a_1}^l N^{l-r}[H(u_{a_1})]^{l-r}}{(1-\kappa)^{l-r}(u_{a_1}-u_{b_2})\cdots(u_{a_1}-u_{b_{q+1}})}\Bigg]\right)\Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\=(x_{\lfloor\kappa N\rfloor+1},\ldots,x_N)}}.$$

The summation over indices when (4.14) holds gives terms of order N^{r+q+1}. We can see that the contribution of the corresponding terms to $\mathcal{G}_{(l,k,1-\kappa)}(X_{1-\kappa})$ as $N\to\infty$ is $\widetilde{I}:=\widetilde{I}_1+\widetilde{I}_2$, where $\widetilde{I}_1$ corresponds to Sym over the elements in the equivalent class including $\{a_1,\ldots,a_{q+1}\}\cap\{b_1,\ldots,b_{r+1}\}$, and $\widetilde{I}_2$ corresponds to the sum of Sym functions over the elements in each equivalent class without $\{a_1,\ldots,a_{q+1}\}\cap\{b_1,\ldots,b_{r+1}\}$. More precisely,

$$\widetilde{I}_1=k\sum_{q=0}^{k-1}\sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}}\binom{k-1}{q}(q+1)!\sum_{i=1}^{g}\frac{|A_i|}{(q+1)}$$
$$\times\underset{A_i}{\mathrm{Sym}}\left\{\frac{u_{a_i}^k([H_{a_i}(X,Y,\kappa)]^{k-1-q}N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})}\frac{1}{\prod_{a_t\in A_i\setminus\{a_i\}}(u_{a_i}-u_{a_t})}\right.$$
$$\times\frac{\partial}{\partial u_{a_i}}\Bigg[\sum_{r=0}^{l}\sum_{\substack{\{b_1,\ldots,\widehat{b}_s,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\ldots,\widehat{b}_s,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset,\\ b_s=a_i}}\sum_{j=1}^{h}\frac{|B_j|}{(r+1)}\mathbf{1}_{b_s\in B_j}\binom{l}{r}(r+1)!$$
$$\left.\times\underset{B_j}{\mathrm{Sym}}\left(\frac{u_{b_j}^l(H_{b_j}(X,Y,\kappa))^{l-r}N^{l-r}}{\prod_{b_v\in D_j}(u_{b_j}-u_{b_v})}\frac{1}{\prod_{b_w\in B_j\setminus\{b_j\}}(u_{b_j}-u_{b_w})}\right)\Bigg]\right\}\Bigg|_{\substack{(u_1,\ldots,\\u_{N-\lfloor\kappa N\rfloor})\\=X_{1-\kappa}}},$$

and

$$\widetilde{I}_2 = k \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\dots,a_{q+1}\} \\ \subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}}} \binom{k-1}{q}(q+1)! \sum_{i=1}^{g} \frac{|A_i|}{(q+1)}$$

$$\times \operatorname*{Sym}_{A_i} \left\{ \frac{u_{a_i}^k ([H_{a_i}(X,Y,\kappa)]^{k-1-q} N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})} \frac{1}{\prod_{a_t\in A_i\setminus\{a_i\}}(u_{a_i}-u_{a_t})} \right.$$

$$\times \frac{\partial}{\partial u_{a_i}} \Bigg[\sum_{r=0}^{l} \sum_{\substack{\{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}, \\ \{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\cap\{a_1,\dots,a_{q+1}\}=\emptyset, \\ b_s=a_i}} \sum_{j=1}^{h} \frac{|B_j|}{(r+1)} \mathbf{1}_{b_s\notin B_j} \binom{l}{r}(r+1)!$$

$$\times \operatorname*{Sym}_{B_j} \left(\frac{u_{b_j}^l (H_{b_j}(X,Y,\kappa))^{l-r} N^{l-r}}{\prod_{b_v\in D_j}(u_{b_j}-u_{b_v})} \frac{1}{\prod_{b_w\in B_j\setminus\{b_j\}}(u_{b_j}-u_{b_w})} \right) \Bigg] \Bigg\} \Bigg|_{\substack{(u_1,\dots, \\ u_{N-\lfloor\kappa N\rfloor}) \\ =X_{1-\kappa}}}.$$

Here we use $\mathbf{1}$ to denote the indicator function. By Lemma 4.3, we infer

$$\widetilde{I}_1 = \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\dots,a_{q+1}\} \\ \subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!}$$

$$\times \sum_{i=1}^{g} |A_i| \frac{1}{|A_i|!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}} \Bigg[\frac{u_{a_i}^k ([H(u_{a_i})]^{k-1-q} N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})}$$

$$\times \frac{\partial}{\partial u_{a_i}} \Bigg(\sum_{r=0}^{l} \sum_{\substack{\{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}, \\ \{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\cap\{a_1,\dots,a_{q+1}\}=\emptyset, \\ b_s=a_i}} \frac{l!}{(l-r)!}$$

$$\times \sum_{j=1}^{h} |B_j| \mathbf{1}_{b_s\in B_j} \frac{1}{|B_j|!} \frac{\partial^{|B_j|-1}}{\partial u_{b_j}^{|B_j|-1}} \frac{u_{b_j}^l (H(u_{b_j}))^{l-r} N^{l-r}}{\prod_{b_v\in D_j}(u_{b_j}-u_{b_v})} \Bigg) \Bigg] \Bigg|_{\substack{(u_1,\dots,u_{N-\lfloor\kappa N\rfloor}) \\ =X_{1-\kappa}}},$$

and

$$\widetilde{I}_2 = \sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\dots,a_{q+1}\} \\ \subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!}$$

$$\times \sum_{i=1}^{g} |A_i| \frac{1}{|A_i|!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}} \Bigg[\frac{u_{a_i}^k ([H(u_{a_i})]^{k-1-q} N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})}$$

$$
\times \frac{\partial}{\partial u_{a_i}}\Bigg(\sum_{r=0}^{l} \sum_{\substack{\{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\dots,\widehat{b}_s,\dots,b_{r+1}\}\cap\{a_1,\dots,a_{q+1}\}=\emptyset,\\ b_s=a_i}} \frac{l!}{(l-r)!}
$$

$$
\times \sum_{j=1}^{h} |B_j| \mathbf{1}_{b_s\notin B_j} \frac{1}{|B_j|!} \frac{\partial^{|B_j|-1}}{\partial u_{b_j}^{|B_j|-1}} \frac{u_{b_j}^{l}(H(u_{b_j}))^{l-r}N^{l-r}}{\prod_{b_v\in D_j}(u_{b_j}-u_{b_v})}\Bigg)\Bigg]\Bigg|_{\substack{(u_1,\dots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}.
$$

By the residue theorem, we have

$$
\begin{aligned}
\widetilde{I}_1 = &\sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\dots,a_{q+1}\}\\ \subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!} \\
&\times \sum_{i=1}^{g} |A_i| \frac{1}{|A_i|!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}} \Bigg[\frac{u_{a_i}^{k}([H(u_{a_i})]^{k-1-q}N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})} \\
&\times \Bigg(\sum_{r=0}^{l} \sum_{\substack{\{b_1,\dots,\widehat{b}_j,\dots,b_{r+1}\}\subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\dots,\widehat{b}_j,\dots,b_{r+1}\}\cap\{a_1,\dots,a_{q+1}\}=\emptyset,\\ b_j=a_i}} \frac{l!}{(l-r)!} \\
&\times \sum_{j=1}^{h} |B_j| \mathbf{1}_{b_j\in B_j} \frac{1}{|B_j|!} \frac{\partial^{|B_j|}}{\partial u_{a_i}^{|B_j|}} \frac{u_{a_i}^{l}(H(u_{a_i}))^{l-r}N^{l-r}}{\prod_{b_v\in D_j}(u_{a_i}-u_{b_v})}\Bigg)\Bigg]\Bigg|_{\substack{(u_1,\dots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}} \\
= &\sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\dots,a_{q+1}\}\\ \subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!} \sum_{i=1}^{g} \operatorname*{Res}_{z=u_{a_i}} \Bigg[\frac{z^{k}([H(z)]^{k-1-q}N^{k-1-q})}{\prod_{a_s\in C_i}(z-u_{a_s})} \\
&\times \frac{1}{(z-u_{a_i})^{|A_i|}} \Bigg(\sum_{r=0}^{l} \sum_{\substack{\{b_1,\dots,\widehat{b}_j,\dots,b_{r+1}\}\subset\{1,2,\dots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\dots,\widehat{b}_j,\dots,b_{r+1}\}\cap\{a_1,\dots,a_{q+1}\}=\emptyset,\\ b_j=a_i}} \sum_{j=1}^{h} |B_j| \mathbf{1}_{b_j\in B_j} \frac{l!}{(l-r)!} \\
&\times \operatorname*{Res}_{w=u_{a_i}} \Bigg(\frac{w^{l}(H(w))^{l-r}N^{l-r}}{\prod_{b_v\in D_j}(w-u_{b_v})} \frac{1}{(w-u_{a_i})^{|B_j|-1}} \frac{1}{(w-z)^2} \Bigg)\Bigg)\Bigg]\Bigg|_{\substack{(u_1,\dots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}} \\
= &\frac{1}{(2\pi\mathbf{i})^2} \sum_{j=1}^{n} \oint_{|z-x_j|=\varepsilon} \Bigg(\sum_{i=1}^{N-\lfloor\kappa N\rfloor} \frac{z}{z-u_i} + zNH(z) \Bigg)^{k} \\
&\times \oint_{|w-x_j|=\varepsilon} \Bigg(\sum_{i=1}^{N-\lfloor\kappa N\rfloor} \frac{w}{w-u_i} + wNH(w) \Bigg)^{l} \frac{1}{(w-z)^2}\, dw dz.
\end{aligned}
$$

We also have

$$\begin{aligned}
\widetilde{I}_2 = &\sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!} \\
&\times \sum_{i=1}^{g} |A_i| \frac{1}{|A_i|!} \frac{\partial^{|A_i|-1}}{\partial u_{a_i}^{|A_i|-1}} \Bigg[\frac{u_{a_i}^k ([H(u_{a_i})]^{k-1-q} N^{k-1-q})}{\prod_{a_s\in C_i}(u_{a_i}-u_{a_s})} \\
&\times \Bigg(\sum_{r=0}^{l} \sum_{\substack{\{b_1,\ldots,\widehat{b}_s,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\},\\ \{b_1,\ldots,\widehat{b}_s,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset,\\ b_s=a_i}} \frac{l!}{(l-r)!} \sum_{j=1}^{h} |B_j| \mathbf{1}_{b_s\notin B_j} \frac{1}{|B_j|!} \\
&\times \frac{\partial^{|B_j|-1}}{\partial u_{b_j}^{|B_j|-1}} \frac{u_{b_j}^l (H(u_{b_j}))^{l-r} N^{l-r}}{\prod_{b_v\in D_j\setminus\{b_s\}}(u_{b_j}-u_{b_v})} \frac{1}{(u_{b_j}-u_{b_s})^2} \Bigg) \Bigg] \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}},
\end{aligned}$$

where $b_j \in B_j$. Using the residue theorem, we can also infer

$$\begin{aligned}
\widetilde{I}_2 = &\sum_{q=0}^{k-1} \sum_{\substack{\{a_1,\ldots,a_{q+1}\}\\ \subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\}}} \frac{k!}{(k-1-q)!} \sum_{i=1}^{g} \operatorname*{Res}_{z=u_{a_i}} \Bigg[\frac{z^k ([H(z)]^{k-1-q} N^{k-1-q})}{\prod_{a_s\in C_i}(z-u_{a_s})} \\
&\times \frac{1}{(z-u_{a_i})^{|A_i|}} \sum_{r=0}^{l} \sum_{\substack{\{b_2,\ldots,b_{r+1}\}\subset\{1,2,\ldots,N-\lfloor\kappa N\rfloor\},\\ \{b_2,\ldots,b_{r+1}\}\cap\{a_1,\ldots,a_{q+1}\}=\emptyset,\\ b_1=a_i}} \frac{l!}{(l-r)!} \sum_{j=1}^{h} \mathbf{1}_{b_1\notin B_j} \\
&\times \operatorname*{Res}_{w=u_{b_j}} \Bigg(\frac{w^l (H(w))^{l-r} N^{l-r}}{\prod_{b_v\in D_j\setminus\{b_1\}}(w-u_{b_v})} \frac{1}{(w-u_{b_j})^{|B_j|}} \frac{1}{(w-z)^2} \Bigg) \Bigg] \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}} \\
= &\frac{1}{(2\pi\mathbf{i})^2} \sum_{j\in\{1,2,\ldots,n\}} \sum_{p\in\{1,2,\ldots,n\},\, p\neq j} \oint_{|z-x_j|=\varepsilon} \Bigg(\sum_{i=1}^{N-\lfloor\kappa N\rfloor} \frac{z}{z-u_i} + zNH(z) \Bigg)^k \\
&\times \oint_{|w-x_p|=\varepsilon} \Bigg(\sum_{i=1}^{N-\lfloor\kappa N\rfloor} \frac{w}{w-u_i} + wNH(w) \Bigg)^l \frac{1}{(w-z)^2}\, dw dz \Bigg|_{\substack{(u_1,\ldots,u_{N-\lfloor\kappa N\rfloor})\\ =X_{1-\kappa}}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{\widetilde{I}}{N^{k+l}} = &\frac{(1-\kappa)^{k+l}}{(2\pi\mathbf{i})^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \oint_{|z-x_i|=\varepsilon} \oint_{|w-x_j|=\varepsilon} \Bigg(\sum_{i=1}^{n} \frac{z}{n(z-x_i)} + \frac{zH(z)}{1-\kappa} \Bigg)^k \\
&\times \Bigg(\sum_{i=1}^{n} \frac{w}{n(w-x_i)} + \frac{wH(w)}{1-\kappa} \Bigg)^l \frac{1}{(z-w)^2}\, dz dw. \qquad (4.15)
\end{aligned}$$

Finally, let us consider the case where

$$|\{a_1, \dots, a_{q+1}\} \cap \{b_1, \dots, b_{r+1}\}| \geq 2.$$

By Lemma 3.4, we can see that the contribution of these terms to $\mathcal{G}_{(l,k)}(X_{1-\kappa})$ has N-degree strictly less than $k + l$.

Therefore, we have the following proposition.

Proposition 4.10. *Assume the assumptions of Lemma 3.4 hold. Then*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{\mathrm{cov}(p_k^{((1-\kappa)N)}, p_l^{((1-\kappa)N)})}{N^{k+l}} \\
&\quad = \frac{(1-\kappa)^{k+l}}{(2\pi\mathbf{i})^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \oint_{|z-x_i|=\varepsilon} \oint_{|w-x_j|=\varepsilon} \left(\sum_{i=1}^{n} \frac{z}{n(z-x_i)} + \frac{zH(z)}{1-\kappa} \right)^k \\
&\quad \times \left(\sum_{j=1}^{n} \frac{z}{n(z-x_j)} + \frac{zH(z)}{1-\kappa} \right)^l Q(z,w)\, dz dw,
\end{aligned}$$

where

$$Q(z,w) = G(z,w) + \frac{1}{(z-w)^2}.$$

Proof. According to formula (4.12), $\lim_{N\to\infty} \frac{\mathrm{cov}(p_k^{((1-\kappa)N)}, p_l^{((1-\kappa)N)})}{N^{k+l}}$ should be the sum of (4.13) and (4.15), divided by N^{k+l}. Then the proposition follows. ■

4.1. Multilevel correlations

Define a mapping $\phi: \{1, \dots, 2N+1\} \to \{\mu^{(i)}, \nu^{(j)}: i, j \in \{1, 2, \dots, N\}\}$ as follows:

$$\phi(n) = \begin{cases} \mu^{(\frac{n-1}{2})} & \text{if } n \text{ is odd,} \\ \nu^{(\frac{n}{2})} & \text{if } n \text{ is even.} \end{cases}$$

For $i \in \{1, 2, \dots, N\}$, define

$$C_i = (x_i, x_{i+1}, \dots, x_N) \in \mathbb{R}^{N-i+1},$$

and for $i \in I_2$, let

$$B_i = y_i C_i = (y_i x_i, y_i x_{i+1}, \dots, y_i x_N) \in \mathbb{R}^{N-i+1}.$$

Let $1 \leq n_1 \leq n_2 \leq \dots \leq n_s = 2N+1$ be positive row numbers of the square-hexagon lattice, counting from the top. For $1 \leq i \leq s$, let ρ_{n_i} be the induced probability measure on dimer configurations of the n_ith row. Then the induced probability measure on the state space

$$\mathbb{GT}_{\lfloor \frac{n_1}{2} \rfloor} \times \mathbb{GT}_{\lfloor \frac{n_2}{2} \rfloor} \times \cdots \times \mathbb{GT}_{\lfloor \frac{n_s}{2} \rfloor}$$

by the measure proportional to the product of weights of present edges of dimer configurations on the square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$ can be expressed as follows:

$$\mathrm{Prob}(\phi(n_s), \ldots, \phi(n_1)) = \rho_{n_s}(\phi(n_s)) \prod_{i=2}^{k} \mathrm{Prob}[\phi(n_{i-1})|\phi(n_i)]. \tag{4.16}$$

Here $\mathrm{Prob}[\phi(n_{i-1})|\phi(n_i)]$ is the probability of $\phi(n_{i-1})$ conditional on $\phi(n_i)$. In particular, we have

$$\mathrm{Prob}[\mu^{(t-1)}|\nu^{(t)}] = \mathrm{pr}_{C_{N-t+1}}(\nu^{(t)} \to \mu^{(t-1)}),$$
$$\mathrm{Prob}[\nu^{(t)}|\mu^{(t)}] = \begin{cases} \mathrm{st}_{B_{N-t+1}}(\mu^{(t)} \to \nu^{(t)}) & \text{if } N-t+1 \in I_2, \\ \mathbf{1}_{\nu^{(t)}=\mu^{(t)}} & \text{otherwise,} \end{cases}$$

where $\mathbf{1}_{\nu^{(t)}=\mu^{(t)}}$ is the indicator of $\nu^{(t)} = \mu^{(t)}$, x_i, y_j are edge weights, and

$$\mathrm{pr}_{C_{N-t+1}}(\nu^{(t)} \to \mu^{(t-1)}) = \begin{cases} x_{N-t+1}^{|\nu^{(t)}|-|\mu^{(t-1)}|} \frac{s_{\mu^{(t-1)}}(x_{N-t+2},\ldots,x_N)}{s_{\nu^{(t)}}(x_{N-t+1},\ldots,x_N)} & \text{if } \mu^{(t-1)} \prec \nu^{(t)}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{st}_{B_{N-t+1}}(\mu^{(t)} \to \nu^{(t)}) = \begin{cases} \frac{y_{N-t+1}^{|\nu^{(t)}|-|\mu^{(t)}|}}{\prod_{j=N-t+1}^{N}(1+y_{N-t+1}x_j)} \frac{s_{\nu^{(t)}}(x_{N-t+1},\ldots,x_N)}{s_{\mu^{(t)}}(x_{N-t+1},\ldots,x_N)} & \text{if } \mu^{(t)} \subset \nu^{(t)}, \\ 0 & \text{otherwise,} \end{cases}$$

see [5, Section 2.4].

Definition 4.11 (Multidimensional Schur generating function). Let $1 \le N_1 \le N_2 \le \cdots \le N_s$ be positive integers. For a probability measure ρ on $\prod_{t=1}^{s} \mathbb{GT}_{N_t}$, we define the s-dimensional Schur generating function with respect to $X = (x_1, \ldots, x_N)$ by

$$\begin{aligned} &\mathcal{S}_{\rho,X}(u_{1,1}, \ldots, u_{N_1,1}; \ldots; u_{1,s}, \ldots, u_{N_s,s}) \\ &\quad = \sum_{\lambda^1 \in \mathbb{GT}_{N_1}, \ldots, \lambda^s \in \mathbb{GT}_{N_s}} \rho(\lambda^1, \ldots, \lambda^s) \prod_{t=1}^{s} \frac{s_{\lambda^t}(u_{1,t}, \ldots, u_{\lfloor \frac{n_t}{2} \rfloor, t})}{s_{\lambda^t}(x_{N-\lfloor \frac{n_s}{2} \rfloor+1}, \ldots, x_N)}. \end{aligned}$$

The multidimensional Schur generating function with respect to $(1, \ldots, 1)$ was defined in [8].

Lemma 4.12. *Let $m_1, \dots, m_k$ be positive integers. Let $1 \leq n_1 \leq n_2 \leq \dots \leq n_k \leq 2N+1$ be positive row numbers of the square-hexagon lattice, counting from the top. Assume that $(\phi(n_s), \dots, \phi(n_1))$ has the distribution ρ defined by (4.16). For $1 \leq s \leq k$, let $\mathcal{D}_l^{(s)}$ be the l-order differential operator defined by*

$$\mathcal{D}_l^{(n_s)} = \frac{1}{V_{\lfloor \frac{n_s}{2} \rfloor}} \left(\sum_{i=1}^{\lfloor \frac{n_s}{2} \rfloor} \left(u_{i,s} \frac{\partial}{\partial u_{i,s}} \right)^l \right) V_{\lfloor \frac{n_s}{2} \rfloor}, \tag{4.17}$$

where $V_{\lfloor \frac{n_s}{2} \rfloor}$ is the Vandermonde determinant on $\lfloor \frac{n_s}{2} \rfloor$ variables $u_{1,s}, \dots, u_{\lfloor \frac{m}{2} \rfloor, s}$. Let

$$\bar{X}_k = (x_{N - \lfloor \frac{n_k}{2} \rfloor + 1}, \dots, x_N).$$

Then

$$\mathcal{D}_{m_1}^{(n_1)} \mathcal{D}_{m_2}^{(n_2)} \cdots \mathcal{D}_{m_k}^{(n_k)} \mathcal{S}_{\rho, X}(u_{1,1}, \dots, u_{\lfloor \frac{n_1}{2} \rfloor, 1}; \dots; u_{1,k}, \dots, u_{\lfloor \frac{n_k}{2} \rfloor, k}) \Big|_{\substack{(u_{1,s}, \dots, u_{\lfloor \frac{n_s}{2} \rfloor, s}) \\ = \bar{X}_s \ \forall 1 \leq s \leq k}}$$
$$= \mathbf{E} \Bigg(\sum_{i=1}^{\lfloor \frac{n_1}{2} \rfloor} \left(\phi(n_1)_{i_1} + \left\lfloor \frac{n_1}{2} \right\rfloor - i_1 \right)^{m_1} \sum_{i_2=1}^{\lfloor \frac{n_2}{2} \rfloor} \left(\phi(n_2)_{i_2} + \left\lfloor \frac{n_2}{2} \right\rfloor - i_2 \right)^{m_2} \cdots$$
$$\times \sum_{i_k=1}^{\lfloor \frac{n_k}{2} \rfloor} \left(\phi(n_k)_{i_k} + \left\lfloor \frac{n_k}{2} \right\rfloor - i_k \right)^{m_k} \Bigg),$$

where $\mathbf{E}$ is the expectation with respect to the probability measure defined by (4.16), and $\mathcal{S}_{\rho,X}$ is the multidimensional Schur generating function as defined in Definition 4.11.

Proof. The theorem follows from explicit computations. ∎

Lemma 4.13. *Suppose the assumptions of Lemma 4.12 hold. For $1 \leq s \leq k$, let*

$$t_s = N - \left\lfloor \frac{n_s}{2} \right\rfloor.$$

For $1 \leq s \leq k-1$, let

$$\widetilde{\mathcal{D}}_{m_s,k}^{(n_s)} := \frac{1}{\widehat{V}_{\lfloor \frac{n_s}{2} \rfloor}} \left(\sum_{i=t_s+1}^{N} \left(u_i \frac{\partial}{\partial u_i} \right)^{m_s} \right) \widehat{V}_{\lfloor \frac{n_s}{2} \rfloor} \prod_{i \in \{t_{s+1}+1, \dots, t_s\} \cap I_2} \prod_{j=t_s+1}^{N} \frac{1 + y_{\bar{i}} u_j}{1 + y_{\bar{i}} x_{\bar{j}}},$$

and let

$$\widetilde{\mathcal{D}}_{m_k,k}^{(n_k)} := \frac{1}{\widehat{V}_{\lfloor \frac{n_k}{2} \rfloor}} \left(\sum_{i=t_k+1}^{N} \left(u_i \frac{\partial}{\partial u_i} \right)^{m_k} \right) \widehat{V}_{\lfloor \frac{n_k}{2} \rfloor},$$

where $\widehat{V}_{\lfloor \frac{n_s}{2} \rfloor}$ *is the Vandermonde determinant on* $\lfloor \frac{n_s}{2} \rfloor$ *variables* $u_{t_s+1}, \ldots, u_N$*. Then*

$$\mathcal{D}_{m_1}^{(n_1)} \mathcal{D}_{m_2}^{(n_2)} \cdots \mathcal{D}_{m_k}^{(n_k)} \mathcal{S}_{\rho,X}(u_{1,1}, \ldots, u_{\lfloor \frac{n_1}{2} \rfloor,1}; \ldots; u_{1,k}, \ldots, u_{\lfloor \frac{n_k}{2} \rfloor,k}) \Big|_{\substack{(u_{1,s},\ldots,u_{\lfloor \frac{n_s}{2} \rfloor,s}) \\ = \bar{X}_s \ \forall 1 \le s \le k}}$$
$$= \widetilde{\mathcal{D}}_{m_1,k}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,k}^{(n_2)} \cdots \widetilde{\mathcal{D}}_{m_k,k}^{(n_k)} \{\mathcal{S}_{\rho_{\lfloor \frac{n_k}{2} \rfloor},X}(u_{N-\lfloor \frac{n_k}{2} \rfloor+1}, \ldots, u_N)\} \Big|_{\substack{(u_1,\ldots,u_N) \\ =(x_1,\ldots,x_N)}},$$

where $\mathcal{S}_{\rho_{\lfloor n_k/2 \rfloor},X}(u_1, \ldots, u_N)$ *is the one-dimensional Schur generating function defined as in Definition 3.1, and* $\rho_{\frac{n_k}{2}}$ *is a probability measure on* $\mathbb{GT}^+_{\lfloor \frac{n_k}{2} \rfloor}$ *defined as in Lemma 3.4.*

Proof. We shall prove the lemma by induction on k. First of all, when $k = 1$, the lemma obviously holds. Assume that the lemma holds when $k = l - 1$, where $l \ge 2$ is a positive integer. Then when $k = l$, we have

$$\mathcal{D}_{m_1}^{(n_1)} \mathcal{D}_{m_2}^{(n_2)} \cdots \mathcal{D}_{m_l}^{(n_l)} \mathcal{S}_{\rho,X}(u_{1,1}, \ldots, u_{\lfloor \frac{n_1}{2} \rfloor,1}; \ldots; u_{1,l}, \ldots, u_{\lfloor \frac{n_l}{2} \rfloor,l}) \Big|_{\substack{(u_{1,s},\ldots,u_{\lfloor \frac{n_s}{2} \rfloor,s}) \\ = \bar{X}_s \ \forall 1 \le s \le l}}$$
$$= \sum_{\lambda^l \in \mathbb{GT}_{\lfloor \frac{n_l}{2} \rfloor}} \rho_{\lfloor \frac{n_l}{2} \rfloor}(\lambda^l) \mathcal{D}_{m_l}^{(n_l)} \left(\frac{s_{\lambda^l}(u_{1,l}, \ldots, u_{\lfloor \frac{n_l}{2} \rfloor,l})}{s_{\lambda^l}(\bar{X}_l)} \right) \widetilde{\mathcal{D}}_{m_1,l-1}^{(n_1)} \cdots \widetilde{\mathcal{D}}_{m_{l-1},l-1}^{(n_{l-1})}$$
$$\mathcal{S}_{\rho(\cdot|\lambda^l),X}(u_{1,1}, \ldots, u_{\lfloor \frac{n_1}{2} \rfloor,1}; \ldots; u_{1,l-1}, \ldots, u_{\lfloor \frac{n_{l-1}}{2} \rfloor,l-1}) \Big|_{\substack{(u_{1,s},\ldots,u_{\lfloor \frac{n_s}{2} \rfloor,s}) \\ = \bar{X}_s \ \forall 1 \le s \le l}},$$

where $\rho(\cdot|\lambda^l)$ is a probability on $\prod_{s=1}^{l-1} \mathbb{GT}_{\lfloor \frac{n_s}{2} \rfloor}$ obtained from ρ by conditional on the configuration λ^l on $\mathbb{GT}_{\lfloor \frac{n_l}{2} \rfloor}$. By the induction hypothesis and Lemma 3.3, we have

$$\mathcal{D}_{m_1}^{(n_1)} \cdots \mathcal{D}_{m_{l-1}}^{(n_{l-1})}$$
$$\mathcal{S}_{\rho(\cdot|\lambda^l),X}(u_{1,1}, \ldots, u_{\lfloor \frac{n_1}{2} \rfloor,1}; \ldots; u_{1,l-1}, \ldots, u_{\lfloor \frac{n_{l-1}}{2} \rfloor,l-1}) \Big|_{\substack{(u_{1,s},\ldots,u_{\lfloor \frac{n_s}{2} \rfloor,s}) \\ = \bar{X}_s \ \forall 1 \le s \le l-1}}$$
$$= \widetilde{\mathcal{D}}_{m_1,l-1}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,l-1}^{(n_2)} \cdots \widetilde{\mathcal{D}}_{m_{l-1},l-1}^{(n_{l-1})}$$
$$\{\mathcal{S}_{\rho_{\lfloor \frac{n_{l-1}}{2} \rfloor}(\cdot|\lambda^l),X}(u_{N-\lfloor \frac{n_{l-1}}{2} \rfloor+1}, \ldots, u_N)\} \Big|_{\substack{(u_1,\ldots,u_N) \\ =(x_1,\ldots,x_N)}}$$
$$= \widetilde{\mathcal{D}}_{m_1,l}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,l}^{(n_2)} \cdots \widetilde{\mathcal{D}}_{m_{l-1},l-1}^{(n_{l-1})} \Bigg\{ \Bigg[\prod_{i \in \{t_l, t_l+1, \ldots, t_{l-1}\} \cap I_2} \prod_{j \in t_{l-1}+1}^{N} \frac{1 + y_{\bar{i}} u_j}{1 + y_{\bar{i}} x_{\bar{j}}} \Bigg]$$
$$\times \frac{s_{\lambda^l}(x_{N-\lfloor \frac{n_l}{2} \rfloor+1}, \ldots, x_{N-\lfloor \frac{n_{l-1}}{2} \rfloor}, u_{N-\lfloor \frac{n_{l-1}}{2} \rfloor+1}, \ldots, u_N)}{s_{\lambda^l}(x_{N-\lfloor \frac{n_l}{2} \rfloor+1}, \ldots, x_N)} \Bigg\} \Bigg|_{\substack{(u_1,\ldots,u_N) \\ =(x_1,\ldots,x_N)}}$$
$$= \widetilde{\mathcal{D}}_{m_1,l}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,l}^{(n_2)} \cdots \widetilde{\mathcal{D}}_{m_{l-1},l}^{(n_{l-1})}$$
$$\left\{ \frac{s_{\lambda^l}(x_{N-\lfloor \frac{n_l}{2} \rfloor+1}, \ldots, x_{N-\lfloor \frac{n_{l-1}}{2} \rfloor}, u_{N-\lfloor \frac{n_{l-1}}{2} \rfloor+1}, \ldots, u_N)}{s_{\lambda^l}(x_{N-\lfloor \frac{n_l}{2} \rfloor+1}, \ldots, x_N)} \right\} \Bigg|_{\substack{(u_1,\ldots,u_N) \\ =(x_1,\ldots,x_N)}}.$$

Note also that

$$\mathcal{D}_{m_l}^{(n_l)}\Big(\frac{s_{\lambda^l}(u_{1,l},\dots,u_{\lfloor\frac{n_l}{2}\rfloor,l})}{s_{\lambda^l}(\bar{X}_l)}\Big)\Big|_{(u_{1,l},\dots,u_{\lfloor\frac{n_l}{2}\rfloor,l})=\bar{X}_l}$$
$$=\sum_{j=1}^{\lfloor\frac{n_l}{2}\rfloor}\Big(\lambda_j^l+\Big\lfloor\frac{n_l}{2}\Big\rfloor-j\Big)^{m_l}$$
$$=\widetilde{\mathcal{D}}_{m_l,l}^{(n_l)}\Big\{\frac{s_{\lambda^l}(u_{N-\lfloor\frac{n_l}{2}\rfloor+1},\dots\dots,u_N)}{s_{\lambda^l}(x_{N-\lfloor\frac{n_l}{2}\rfloor+1},\dots,x_N)}\Big\}\Big|_{\substack{(u_1,\dots,u_N)\\=(x_1,\dots,x_N)}}.$$

Then the lemma follows. ∎

Let $m_1,\dots,m_k$ and $n_1,\dots,n_k$ be as in Lemma 4.12. For $1\le s\le k$, we introduce the notation

$$E_{l,s}=\mathcal{F}_{(l,\frac{n_s}{2N})}(u_{N-\lfloor\frac{n_s}{2}\rfloor}+1,\dots,u_N)\big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)},$$

where $\mathcal{F}$ is defined by (4.7) and can be expressed as in (4.9).

Let $s_1<s_2$ be positive integers between 1 and k. Define

$$G_{s_1,s_2}(u_{N-\lfloor\frac{n_{s_2}}{2}\rfloor+1},\dots,u_N)$$
$$=m_{s_1}\sum_{r=0}^{m_{s_1}-1}\binom{m_{s_1}-1}{r}\sum_{\substack{\{b_1,\dots,b_{r+1}\}\\ \subset\{N-\lfloor\frac{n_{s_1}}{2}\rfloor+1,\dots,N\}}}(r+1)!$$
$$\times\operatorname*{Sym}_{b_1,\dots,b_{r+1}}\frac{u_{b_1}^{m_{s_1}}\frac{\partial}{\partial u_{b_1}}[\mathcal{F}_{(m_{s_2},\frac{n_{s_2}}{2N})}](\frac{\partial}{\partial u_{b_1}}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}])^{m_{s_1}-1-r}}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}.$$

Lemma 4.14. (1) *The degree of N in the expression for $G_{s_1,s_2}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$ is at most $m_{s_1}+m_{s_2}$. Moreover, for any index i the degree of N in the expression for $\frac{\partial}{\partial u_i}G_{s_1,s_2}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$ is less than $m_{s_1}+m_{s_2}$.*

(2) *We have that*

$$\frac{1}{\widehat{V}_{\lfloor\frac{n_{s_1}}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}}m_{s_1}\sum_{i\in\{N-\lfloor\frac{n_{s_1}}{2}\rfloor+1,\dots,N\}}u_i\frac{\partial}{\partial u_i}[\mathcal{F}_{(m_{s_2},\frac{n_{s_2}}{2N})}]$$
$$\times\Big(u_i\frac{\partial}{\partial u_i}\Big)^{m_{s_1}-1}[\widehat{V}_{\lfloor\frac{n_{s_1}}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}]\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$$
$$=G_{s_1,s_2}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}+R,$$

where the degree of N in R is less than $m_{s_1}+m_{s_2}$.

Proof. We first prove part (1). In accordance with Lemma 3.4, the degree of N in $(\frac{\partial}{\partial u_{b_1}}[\log \mathcal{S}_{\rho_{\lfloor n_{s_1}/2\rfloor},X}])^{m_{s_1}-1-r}$ is at most $m_{s_1}-1-r$. In view of Lemma 4.4, the degree of N in the expression $\frac{\partial}{\partial u_{b_1}}[\mathcal{F}_{(m_{s_2},n_{s_2}/(2N))}]$ is at most m_{s_2}. The summation gives $O(N^{r+1})$ terms. Therefore, the degree of N in $G_{s_1,s_2}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$ is at most $m_{s_1}+m_{s_2}$. The fact that the degree of N in $\frac{\partial}{\partial u_i}G_{s_1,s_2}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$ is at most $m_{s_1}+m_{s_2}$ also follows from Lemmas 3.4 and 4.4, and by discussing the cases where $i\in\{b_1,\dots,b_{r+1}\}$ and $i\notin\{b_1,\dots,b_{r+1}\}$ separately.

Now we prove part (2). We have

$$\begin{aligned}
&\frac{1}{\widehat{V}_{\lfloor\frac{n_{s_1}}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}} m_{s_1}\sum_{i\in\{N-\lfloor\frac{n_{s_1}}{2}\rfloor+1,\dots,N\}} u_i\frac{\partial}{\partial u_i}[\mathcal{F}_{(m_{s_2},\frac{n_{s_2}}{2N})}]\\
&\quad\times\left(u_i\frac{\partial}{\partial u_i}\right)^{m_{s_1}-1}[\widehat{V}_{\lfloor\frac{n_{s_1}}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}]\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}\\
&\quad= m_{s_1}\sum_{\substack{t_0+t_1d_1+\dots+t_qd_q+r\\=m_{s_1}-1,\\ t_1<t_2<\dots<t_q}}\binom{m_{s_1}-1}{r}\sum_{\substack{\{b_1,\dots,b_{r+1}\}\\ \subset\{N-\lfloor\frac{n_{s_1}}{2}\rfloor+1,\dots,N\}}}(r+1)!\operatorname*{Sym}_{b_1,\dots,b_{r+1}}\\
&\quad\frac{u_{b_1}^{m_{s_1}-t_0}\frac{\partial}{\partial u_{b_1}}[\mathcal{F}_{(m_{s_2},\frac{n_{s_2}}{2N})}](\frac{\partial^{t_1}}{\partial u_{b_1}}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}])^{d_1}\cdots(\frac{\partial^{t_q}}{\partial u_{b_1}}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_{s_1}}{2}\rfloor},X}])^{d_q}}{(u_{b_1}-u_{b_2})\cdots(u_{b_1}-u_{b_{r+1}})}.
\end{aligned}$$

Recall from the proof of Proposition 4.4 that the degree of N in $\frac{\partial^{t_1}}{\partial u_{b_1}}[\log \mathcal{S}_{\rho_{\lfloor n_{s_1}/2\rfloor},X}]$ is at most 1 for all $t_1\geq 0$. Hence by Lemma 4.4, the degree of N in the expression above is at most

$$d_1+\dots+d_q+m_{s_2}+r+1. \tag{4.18}$$

Given that $t_0+t_1d_1+\dots+t_qd_q+r=m_{s_1}-1$, $t_1<t_2<\dots<t_q$, the maximal of (4.18) is achieved when $t_0=d_2=\dots=d_q=0$, $t_1=1$, $d_1=m_{s_1}-r-1$; with maximal value $m_{s_1}+m_{s_2}$. Then part (2) follows. ∎

Lemma 4.15. *Let $m_1,\dots,m_k$ and $n_1,\dots,n_k$ be as in Lemma 4.12. Then*

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N^{m_1+m_2+\dots+m_k}}(\mathcal{D}^{(n_1)}_{m_1}-E_{m_1,s_1})(\mathcal{D}^{(n_2)}_{m_2}-E_{m_2,s_2})\cdots(\mathcal{D}^{(n_k)}_{m_k}-E_{m_k,s_k})\\
&\quad\mathcal{S}_{\rho,X}(u_{1,1},\dots,u_{\lfloor\frac{n_1}{2}\rfloor,1};\dots;u_{1,k},\dots,u_{\lfloor\frac{n_k}{2}\rfloor,k})\Big|_{\substack{(u_{1,s},\dots,u_{\lfloor\frac{n_s}{2}\rfloor,s})=\bar{X}_s\\ \forall 1\leq s\leq m}}\\
&\quad=\lim_{N\to\infty}\frac{1}{N^{m_1+m_2+\dots+m_k}}\sum_{P\in\mathcal{P}^k_{\emptyset}}\prod_{(s_1,s_2)\in P}G_{s_1,s_2}\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}.
\end{aligned}$$

Proof. The proposition follows from explicit computations and by analyzing the degree of N of each term in the expansion. We sketch the proof here.

The lemma obviously holds when $k = 1$, for which both the left-hand side and the right-hand side are 0. When $k = 2$, by Lemma 4.13, we have

$$
\begin{aligned}
\mathcal{E}_2 :=& \mathcal{D}_{m_1}^{(n_1)} \mathcal{D}_{m_2}^{(n_2)} \mathcal{S}_{\rho,X}(u_{1,1},\dots,u_{\lfloor \frac{n_1}{2}\rfloor,1}; u_{1,2},\dots,u_{\lfloor \frac{n_2}{2}\rfloor,2})\Big|_{\substack{(u_{1,s},\dots,u_{\lfloor \frac{n_s}{2}\rfloor,s}) \\ =\bar{X}_s \text{ for } s=1,2}} \\
=& \widetilde{\mathcal{D}}_{m_1,2}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,2}^{(n_2)} \mathcal{S}_{\rho_{\lfloor \frac{n_2}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)} \\
=& \frac{1}{\mathcal{S}_{\rho_{\lfloor \frac{n_1}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)} \widetilde{\mathcal{D}}_{m_1,1}^{(n_1)} \widetilde{\mathcal{D}}_{m_2,2}^{(n_2)} \\
& \Big[\exp(\log[\mathcal{S}_{\rho_{\lfloor \frac{n_2}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)])\Big]\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)} \\
=& \frac{1}{\mathcal{S}_{\rho_{\lfloor \frac{n_1}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N) V_{\lfloor \frac{n_1}{2}\rfloor}(u_{N-\lfloor \frac{n_1}{2}\rfloor+1},\dots,u_N)} \\
& \times \sum_{i\in\{N-\lfloor \frac{n_1}{2}\rfloor+1,\dots,N\}} \left(u_i \frac{\partial}{\partial u_i}\right)^{m_1} \frac{V_{\lfloor \frac{n_1}{2}\rfloor}(u_{N-\lfloor \frac{n_1}{2}\rfloor+1},\dots,u_N)}{V_{\lfloor \frac{n_2}{2}\rfloor}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)} \\
& \times \prod_{l\in\{N-\lfloor \frac{n_2}{2}\rfloor+1,\dots,N-\lfloor \frac{n_1}{2}\rfloor\}\cap I_2} \prod_{s=N-\lfloor \frac{n_1}{2}\rfloor+1}^{N} \frac{1+y_{\bar{l}}u_s}{1+y_{\bar{l}}x_{\bar{s}}} \\
& \times \sum_{j\in\{N-\lfloor \frac{n_2}{2}\rfloor+1,\dots,N\}} \left(u_j \frac{\partial}{\partial u_j}\right)^{m_2} V_{\lfloor \frac{n_2}{2}\rfloor}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N) \\
& \times \Big[\exp(\log[\mathcal{S}_{\rho_{\lfloor \frac{n_2}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)])\Big]\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}.
\end{aligned}
$$

By Lemma 3.3, we have

$$
\begin{aligned}
&\prod_{l\in\{N-\lfloor \frac{n_2}{2}\rfloor+1,\dots,N-\lfloor \frac{n_1}{2}\rfloor\}\cap I_2} \prod_{s=N-\lfloor \frac{n_1}{2}\rfloor+1}^{N} \frac{1+y_{\bar{l}}u_s}{1+y_{\bar{l}}x_{\bar{s}}} \\
&\qquad = \frac{\mathcal{S}_{\rho_{\lfloor \frac{n_1}{2}\rfloor},X}(u_{N-\lfloor \frac{n_1}{2}\rfloor}+1,\dots,u_N)}{\mathcal{S}_{\rho_{\lfloor \frac{n_2}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor}+1,\dots,u_N)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{E}_2 =& \frac{1}{\mathcal{S}_{\rho_{\lfloor \frac{n_1}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N) V_{\lfloor \frac{n_1}{2}\rfloor}(u_{N-\lfloor \frac{n_1}{2}\rfloor+1},\dots,u_N)} \\
& \times \sum_{i\in\{N-\lfloor \frac{n_1}{2}\rfloor+1,\dots,N\}} \left(u_i \frac{\partial}{\partial u_i}\right)^{m_1} V_{\lfloor \frac{n_1}{2}\rfloor}(u_{N-\lfloor \frac{n_1}{2}\rfloor+1},\dots,u_N) \\
& \times \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2}\rfloor},X}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N) \mathcal{F}_{m_2,\lfloor \frac{n_2}{2N}\rfloor}(u_{N-\lfloor \frac{n_2}{2}\rfloor+1},\dots,u_N)\Big|_{\substack{(u_1,\dots,u_N) \\ =(x_1,\dots,x_N)}}.
\end{aligned}
$$

Hence $\mathcal{E}_2$ is a sum of terms of the form

$$\operatorname*{Sym}_{a_1,\ldots,a_{r+1}}\Bigg[c_0 u_{a_1}^{m_1-q_0}\frac{\partial^{q_1}}{\partial u_{a_1}^{q_1}}[\mathcal{F}_{m_2,\frac{n_2}{2N}}]\Big[\frac{\partial^{q_2}}{\partial u_{a_1}^{q_2}}(\log \mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X})\Big]^{d_2}\cdots \\ \times\Big[\frac{\partial^{q_t}}{\partial u_{a_1}^{q_t}}(\log \mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X})\Big]^{d_t}\Big/\big((u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{r+1}})\big)\Bigg]\Bigg|_{\substack{(u_1,\ldots,u_N)\\=(x_1,\ldots,x_N)}}$$

such that

- $r, q_0, q_1, \ldots, q_t, d_2, \ldots, d_t$ are non-negative integers;
- $q_2 < q_3 < \cdots < q_t$;
- we have that

$$q_0 + q_1 + q_2 d_2 + \cdots + q_t d_t + r = m_1; \tag{4.19}$$

- $\{a_1, \ldots, a_{r+1}\} \subset \{N - \lfloor\frac{n_1}{2}\rfloor + 1, \ldots, N\}$.

When $q_1 = 0$, we obtain $E_{m_1,s_1}E_{m_2,s_2}$.

Now we consider the terms corresponding to $q_1 \geq 1$. By Lemma 4.4, the degree of N in $\partial_{a_1}^{q_1}[\mathcal{F}_{m_2,\frac{n_2}{2N}}]$ is at most m_2. By Lemma 3.4, the total degree of N in these terms is at most $m_2 + d_2 + \cdots + d_t + r + 1$. By (4.19) and the assumption that $s_1 \geq 1$, we have

$$m_2 + d_2 + \cdots + d_t + r + 1 \leq m_2 + m_1,$$

and the equality holds when

$$q_0 = d_3 = \cdots = d_t = 0, \quad q_1 = q_2 = 1, \quad d_2 = m_1 - 1 - r.$$

This corresponds to $G_{1,2}$, in which the degree of N is at most $m_1 + m_2$. The degree of N is less than $m_1 + m_2$ in all the other terms. This completes the proof when $k = 2$.

We shall finish the rest of the proof by induction. For $1 \leq l \leq k-1$, let

$$A_l = \prod_{i\in\{t_{l+1}+1,\ldots,t_l\}}\prod_{j=t_l+1}^{N}\frac{1+y_{\bar{i}}u_j}{1+y_{\bar{i}}x_{\bar{j}}}.$$

By Lemma 3.3, we have

$$A_l = \frac{\mathcal{S}_{\rho_{\lfloor\frac{n_l}{2}\rfloor},X}(u_{N-\lfloor\frac{n_l}{2}\rfloor+1},\ldots,u_N)}{\mathcal{S}_{\rho_{\lfloor\frac{n_{l+1}}{2}\rfloor},X}(u_{N-\lfloor\frac{n_{l+1}}{2}\rfloor+1},\ldots,u_N)}.$$

Assume that the lemma holds for $k = r - 1$, where $r \geq 2$ is a positive integer. When $k = r$, by induction hypothesis, we have

$$\frac{1}{\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}} \sum_{i_1 \in \{N-\lfloor \frac{n_1}{2} \rfloor+1,\dots,N\}} \Big(u_{i_1} \frac{\partial}{\partial u_{i_1}}\Big)^{m_1} A_1 \frac{\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor}}{\widehat{V}_{\lfloor \frac{n_2}{2} \rfloor}}$$
$$\times \sum_{i_2 \in \{N-\lfloor \frac{n_2}{2} \rfloor\rfloor+1,\dots,N\}} \Big(u_{i_2} \frac{\partial}{\partial u_{i_2}}\Big)^{m_2} \frac{\widehat{V}_{\lfloor \frac{n_2}{2} \rfloor}}{\widehat{V}_{\lfloor \frac{n_3}{2} \rfloor}} A_2 \cdots$$
$$\times \sum_{i_r \in \{N-\lfloor \frac{n_r}{2} \rfloor+1,\dots,N\}} \Big(u_{i_r} \frac{\partial}{\partial u_{i_r}}\Big)^{m_r} [\widehat{V}_{\lfloor \frac{n_r}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_t}{2} \rfloor},X}]\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$$
$$= \frac{1}{\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}} \sum_{i_1 \in \{N-\lfloor \frac{n_1}{2} \rfloor+1,\dots,N\}} \Big(u_{i_1} \frac{\partial}{\partial u_{i_1}}\Big)^{m_1} [\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}]$$
$$\times \Bigg(\sum_{p=0}^{r-1} \sum_{w_1,\dots,w_p \in \{2,\dots,r\}} \mathcal{F}_{(m_{w_1}, \frac{n w_1}{2N})} \mathcal{F}_{(m_{w_2}, \frac{n w_2}{2N})} \cdots \mathcal{F}_{(m_{w_p}, \frac{n w_p}{2N})}$$
$$\times \Big(\sum_{P \in \mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}\Big)\Bigg)\Bigg|_{\substack{(u_1,\dots,u_N)\\=(x_1,\dots,x_N)}} = S_1 + S_2 + S_3,$$

where by induction hypothesis, the degree of N in $R_{1,w_1,\dots,w_p}$ is less than $\sum_{i=2}^{s} l_i - \sum_{i=1}^{p} l_{w_i}$, and

$$S_1 = \Bigg\{\frac{1}{\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}} \sum_{i_1 \in \{N-\lfloor \frac{n_1}{2} \rfloor+1,\dots,N\}} \Big(u_{i_1} \frac{\partial}{\partial u_{i_1}}\Big)^{l_1} [\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}]\Bigg\}$$
$$\times \Bigg(\sum_{p=0}^{r-1} \sum_{w_1,\dots,w_p \in \{2,\dots,r\}} E_{m_{w_1},w_1} E_{m_{w_2},w_2} \cdots E_{m_{w_p},w_p}$$
$$\times \Big(\sum_{P \in \mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}\Big)\Bigg)\Bigg|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)},$$
$$S_2 = \frac{l_1}{\widehat{V}_{\lfloor \frac{n_1}{2} \rfloor} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}} \sum_{i_1 \in \{N-\lfloor \frac{n_1}{2} \rfloor+1,\dots,N\}} \Bigg\{\Big(u_{i_1} \frac{\partial}{\partial u_{i_1}}\Big)^{l_1-1} [\widehat{V}_{\frac{n_1}{2}} \mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}]\Bigg\}$$
$$\times \Big(u_{i_1} \frac{\partial}{\partial u_{i_1}}\Big) \Bigg(\sum_{p=0}^{r-1} \sum_{w_1,\dots,w_p \in \{2,\dots,t\}} \mathcal{F}_{(m_{w_1}, \frac{n w_1}{2N})} \mathcal{F}_{(m_{w_2}, \frac{n w_2}{2N})} \cdots \mathcal{F}_{(m_{w_p}, \frac{w_p}{2N})}$$
$$\times \Big(\sum_{P \in \mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}\Big)\Bigg)\Bigg|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)},$$

and S_3 consists of all the other terms.

By the definition of $E_{m_i,i}$, we have

$$S_1 = E_{m_1,1}\Big(\sum_{p=0}^{r-1} \sum_{w_1,\dots,w_p\in\{2,\dots,r\}} E_{m_{w_1},w_1}E_{m_{w_2},w_2}\cdots E_{m_{w_p},w_p} \\ \times\Big(\sum_{P\in\mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}\Big)\Big)\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}.$$

By Lemma 4.14, we have

$$S_2 = \Big\{\Big(\sum_{p=0}^{r-1} \sum_{\substack{w_1,\dots,w_p\\ \in\{2,\dots,r\}}} \sum_{x=1}^{p} E_{m_{w_1},w_1}\cdots E_{m_{w_{x-1}},w_{x-1}}E_{m_{w_{x+1}},w_{x+1}}\cdots E_{m_{w_p},w_p} \\ \times\Big[(G_{1,x}+R_{1,x})\Big(\sum_{P\in\mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}\Big)\Big]\Big)\Big\}\Big|_{\substack{(u_1,\dots,u_N)\\=(x_1,\dots,x_N)}},$$

where the degree of N in $R_{1,x}$ is less than $l_1 + l_x$.

Hence we have

$$S_2 = \Big\{\Big[\sum_{p=0}^{r-1} \sum_{\substack{w_1,\dots,w_p\\ \in\{2,\dots,r\}}} \sum_{x=1}^{p} E_{m_{w_1},w_1}\cdots E_{m_{w_{x-1}},w_{x-1}}E_{m_{w_{x+1}},w_{x+1}}\cdots E_{m_{w_p},w_p} \\ \times\Big(G_{1,x}\sum_{P\in\mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{w_1,\dots,\widehat{w}_x,\dots,w_p}\Big)\Big]\Big\}\Big|_{\substack{(u_1,\dots,u_N)\\=(x_1,\dots,x_N)}},$$

where the degree of N in $R_{w_1,\dots,\widehat{w}_x,\dots,w_p}$ is less than $\sum_{i=2}^{t} l_i - \sum_{i=1}^{p} l_{w_i}$.

Note that

$$S_1 + S_2 = \sum_{p=0}^{r} \sum_{w_1,\dots,w_p\in\{1,2,\dots,r\}} E_{m_{w_1},w_1}E_{m_{w_2},w_2}\cdots E_{m_{w_p},w_p} \\ \times\Big(\sum_{P\in\mathcal{P}^r_{w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{w_1,\dots,w_p}\Big)\Big|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)},$$

where the degree of N in $R_{w_1,\dots,w_p}$ is less than $\sum_{i=1}^{t} l_i - \sum_{i=1}^{p} l_{w_i}$.

It remains to show that S_3 does not contribute to the leading terms. Define

$$\mathcal{H}_{j_1,\dots,j_p} = \sum_{P\in\mathcal{P}^r_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} G_{a,b} + R_{1,w_1,\dots,w_p}.$$

By Lemma 4.14, the degree of N in $\mathcal{H}_{j_1,\dots,j_p}|_{U_{N,\kappa}=(1,\dots,1)}$ is at most $\sum_{i=2}^{r} l_i - \sum_{j=1}^{p} l_{w_j}$. Moreover, by Lemma 4.14, for any index i, the degree of N in the expression $\frac{\partial}{\partial u_i}\mathcal{H}_{j_1,\dots,j_p}|_{(u_1,\dots,u_N)=(x_1,\dots,x_N)}$ is less than $\sum_{i=2}^{t} l_i - \sum_{j=1}^{p} l_{w_j}$.

We write

$$\frac{1}{\widehat{V}_{\lfloor\frac{n_1}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X}}\sum_{i_1\in\{N-\lfloor\frac{n_1}{2}\rfloor+1,\ldots,N\}}\left(u_{i_1}\frac{\partial}{\partial u_{i_1}}\right)^{m_1}[\widehat{V}_{\lfloor\frac{n_1}{2}\rfloor}\mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X}]\Bigg(\sum_{p=0}^{r-1}$$
$$\sum_{w_1,\ldots,w_p\in\{2,\ldots,r\}}\mathcal{F}_{(m_{w_1},\frac{nw_1}{2N})}\mathcal{F}_{(m_{w_2},\frac{nw_2}{2N})}\cdots\mathcal{F}_{(m_{w_p},\frac{nw_p}{2N})}\mathcal{H}_{j_1,\ldots,j_p}\Bigg)\Bigg|_{\substack{(u_1,\ldots,u_N)\\=(x_1,\ldots,x_N)}}$$

as a sum of terms of the following form

$$\operatorname*{Sym}_{a_1,\ldots,a_{q+1}}\Big[u_{a_1}^{m_1-s_0}(\partial_{a_1}^{s_1}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X}])^{d_1}\cdots(\partial_{a_t}^{s_t}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X}])^{d_t}\partial_{a_1}^{f_1}\mathcal{F}_{(m_{w_1},\frac{nw_1}{2N})}\cdots$$
$$\times\,\partial_{a_1}^{f_p}\mathcal{F}_{(m_{w_p},\frac{nw_p}{2N})}\partial_{a_1}^{h_0}\mathcal{H}_{j_1,\ldots,j_p}/((u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}}))\Big], \quad (4.20)$$

where

- $\{a_1,\ldots,a_{q+1}\}\subset\{N-\lfloor\frac{n_1}{2}\rfloor+1,\ldots,N\}$;
- $s_1<s_2<\cdots<s_t$ are positive integers;
- $f_1,\ldots,f_p,h_0$ are non-negative integers;
- we have that

$$q+s_0+s_1d_1+\cdots+s_td_t+f_1+\cdots+f_p+h_0=m_1. \quad (4.21)$$

By Lemma 3.4, the degree of N in $([\log\mathcal{S}_{\rho_{\lfloor n_1/2\rfloor},X}])^{d_1}\cdots(\partial_{a_t}^{s_t}[\log\mathcal{S}_{\rho_{\lfloor n_1/2\rfloor},X}])^{d_t}$ is at most $d_1+\cdots+d_t$; therefore, the terms in (4.20) with highest degree of N have the form

$$\operatorname*{Sym}_{a_1,\ldots,a_{q+1}}\frac{u_{a_1}^{m_1}(\partial_{a_1}[\log\mathcal{S}_{\rho_{\lfloor\frac{n_1}{2}\rfloor},X}])^{d_1}\partial_{a_1}^{f_1}\mathcal{F}_{(m_{w_1},\frac{nw_1}{2N})}\cdots\partial_{a_1}^{f_p}\mathcal{F}_{(m_{w_p},\frac{nw_p}{2N})}\partial_{a_1}^{h_0}\mathcal{H}_{j_1,\ldots,j_p}}{(u_{a_1}-u_{a_2})\cdots(u_{a_1}-u_{a_{q+1}})}, \quad (4.22)$$

where

$$s_0=d_2=\cdots=d_t=0,\quad s_1=1.$$

Let $B=\{i\in\{1,2,\ldots,p\}\colon f_i=0\}$. Then

$$(4.22)=\Big[\prod_{i\in B}\mathcal{F}_{(m_{w_i},\frac{nw_i}{2N})}\Big]S(u_1,\ldots,u_N).$$

It suffices to show that the degree of N in S, except for S_1 and S_2, is less than $\sum_{i=1}^{s}l_i-\sum_{i\in B}l_i$. Note that the degree of N in $\partial_{a_1}[\log\mathcal{S}_{\rho_{\lfloor n_1/2\rfloor},X}]^{d_1}$ is at most d_1 by Lemma 3.4.

The summation over

$$\{a_1, \ldots, a_{q+1}\} \subset \left\{N - \left\lfloor \frac{n_1}{2} \right\rfloor + 1, \ldots, N\right\}$$

gives $O(N^{q+1})$ terms. By Lemma 4.4, when $i \notin B$, the degree of N in $\partial_{a_1}^{f_i} \mathcal{F}_{(m_{w_i}, \frac{n w_i}{2N})}$ is at most m_{w_i}. Therefore, the degree of N in $S(u_1, \ldots, u_N)$ is at most

$$\sum_{i=2}^{r} m_i - \sum_{i=1}^{p} m_{w_i} + d_1 + \sum_{i \in \{1,2,\ldots,p\} \setminus B} m_{w_i} + q + 1.$$

By (6.5) and (6.10), if $|B| \le p - 2$, $q + d_1 + 1 \le m_1 - 1$, then the degree of N in $S(u_1, \ldots, u_N)$ is at most

$$\sum_{i=1}^{r} m_i - \sum_{i \in B} m_{w_i} - 1.$$

Therefore, only the terms where at most one f_i is nonzero contribute to the leading order. In these terms, if $h_0 > 0$, then by Lemma 6.8, the degree of N is less than $\sum_{i=1}^{r} m_i - \sum_{i \in B} m_{w_i}$. So only the terms where $h_0 = 0$ and at most one f_i is nonzero contribute to the leading order. These terms are in S_1 and S_2. Then the proof is complete. ■

Proof of Theorem 4.1. The theorem follows from Lemma 4.15 like in the single-level case. The only difference is in the expansion $\{b_1, \ldots, b_{r+1}\} \subset \{1, 2, \ldots, \lfloor (1 - t_1)N \rfloor\}$, while $\{a_1, \ldots, a_{q+1}\} \subset \{1, 2, \ldots, \lfloor (1 - t_2)N \rfloor\}$. The joint Gaussian distribution follows from the fact that the moments satisfy Wick's probability theorem. ■

5. Piecewise boundary conditions

In this section, we introduce the piecewise boundary conditions on the bottom boundary of a contracting square-hexagon lattice and review the limit shape results for perfect matchings on such a graph.

For $N \ge 1$, let $\lambda(N) \in \mathbb{GT}_N^+$. We consider the following special asymptotic case of $\lambda(N)$ as $N \to \infty$. Let

$$\Omega = (\Omega_1 < \Omega_2 < \cdots < \Omega_N) = (\lambda_N(N) + 1, \lambda_{N-1}(N) + 2, \ldots, \lambda_1(N) + N).$$

Indeed, $\Omega_1, \ldots, \Omega_N$ are the locations of the N remaining vertices on the bottom boundary of the contracting square-hexagon lattice. Assume

$$\begin{aligned}\Omega = (&A_1, A_1 + 1, \ldots, B_1 - 1, B_1, A_2, A_2 + 1, \ldots, B_2 - 1, B_2, \ldots, A_s, \\ &A_s + 1, \ldots, B_s - 1, B_s),\end{aligned} \tag{5.1}$$

where $\sum_{i=1}^{s}(B_i - A_i + 1) = N$, and s is a fixed positive integer independent of N. Suppose as $N \to \infty$,

$$A_i(N) = a_i N + o(N), \quad B_i(N) = b_i N + o(N) \quad \text{for } 1 \le i \le s,$$

and $a_1 < b_1 < \cdots < a_s < b_s$ are fixed parameters independent of N and satisfy $\sum_{i=1}^{s}(b_i - a_i) = 1$. Assume the edge weights $\{x_i\}_{i=1}^{N}$ and $\{y_j\}_{j\in I_2\cap\{1,2,\dots,N\}}$ satisfy (3.3) and (3.4), respectively.

Let Σ_N be the permutation group of N elements and let $\sigma \in \Sigma_N$. Let

$$X = (x_1, \dots, x_N).$$

Let $x_1, \dots, x_n$ be all the distinct elements in $\{x_1, \dots, x_N\}$. Let Σ_N^X be the subgroup of Σ_N that preserves the value of X; more precisely,

$$\Sigma_N^X = \{\sigma \in \Sigma_N : x_{\sigma(i)} = x_i \text{ for } 1 \le i \le N\}.$$

Let $[\Sigma_N/\Sigma_N^X]^r$ be the collection of all the right cosets of Σ_N^X in Σ_N. More precisely,

$$[\Sigma_N/\Sigma_N^X]^r = \{\Sigma_N^X\sigma : \sigma \in \Sigma_N\},$$

where for each $\sigma \in \Sigma_N$, $\Sigma_N^X\sigma = \{\xi\sigma : \xi \in \Sigma_N^X\}$ and $\xi\sigma \in \Sigma_N$ is defined by

$$\xi\sigma(k) = \xi(\sigma(k)) \quad \text{for } 1 \le k \le N.$$

For $1 \le j \le N$, let

$$\eta_j^\sigma(N) = |\{k : k > j, x_{\sigma(k)} \ne x_{\sigma(j)}\}|.$$

For $1 \le i \le n$, let

$$\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^\sigma(N) : x_{\sigma(j)} = x_i\},$$

and let $\phi^{(i,\sigma)}(N)$ be the partition obtained by putting all the elements in $\Phi^{(i,\sigma)}(N)$ in decreasing order.

To study the asymptotics under the piecewise boundary conditions, we make the following assumptions.

Assumption 5.1. *Let $(x_1, \dots, x_N)$ be an N-tuple of real numbers at which we evaluate the Schur polynomial such that*

- *N is an integral multiple of n;*
- *$\{x_i\}_{i=1}^{N}$ are periodic with period n, i.e., $x_i = x_j$ for $1 \le i, j \le N$ and $[i \mod n] = [j \mod n]$;*
- $x_1 > x_2 > \cdots > x_n > 0$.

Assumption 5.2. *Assume* $x_{1,N} = x_1 > 0$ *and* $(x_{2,N}, \dots, x_{n,N})$ *changes with* N. *Assume that for each fixed* N, $(x_{1,N}, \dots, x_{n,N})$ *satisfies Assumption* 5.1. *Moreover, assume that*

$$\liminf_{N\to\infty} \frac{\log\bigl(\min_{1\le i<j\le n} \frac{x_{i,N}}{x_{j,N}}\bigr)}{N} \ge \alpha > 0,$$

where α *is a sufficiently large positive constant independent of* N.

Assumption 5.2 requires that as $N \to \infty$, each $x_{i,N}$ is exponentially small in N compared to $x_{(i-1),N}$. One can also see that under Assumption 5.2, except for $x_{1,N}$, all the other weights converge to 0 exponentially fast as $N \to \infty$.

Let $\overline{\sigma}_0 \in [\Sigma_N / \Sigma_N^X]^r$ be the unique element in $[\Sigma_N / \Sigma_N^X]^r$ satisfying the condition that for any representative $\sigma_0 \in \overline{\sigma}_0$, we have

$$x_{\sigma_0(1)} \ge x_{\sigma_0(2)} \ge \cdots \ge x_{\sigma_0(N)}. \tag{5.2}$$

Let $\mathbf{m}_i$ be the limit of the counting measures for $\phi^{(i,\sigma_0)}(N)$ as $N \to \infty$.

Assumption 5.3. *Assume* $x_1, \dots, x_N$ *satisfy Assumption* 5.1. *Let* A_i, B_i *be given as in* (5.1). *For* $1 \le i \le s$, *let*

$$B_i - A_i + 1 = K_i.$$

By (5.1), *we may assume*

$$\begin{aligned}
\lambda_1 = \lambda_2 = \cdots = \lambda_{K_s} &= \mu_1,\\
\lambda_{K_s+1} = \lambda_{K_s+2} = \cdots = \lambda_{K_s+K_{s-1}} &= \mu_2,\\
&\vdots\\
\lambda_{\sum_{t=2}^s K_t} = \lambda_{1+\sum_{t=2}^s K_t} = \cdots = \lambda_{\sum_{t=1}^s K_t} &= \mu_s,
\end{aligned}$$

and note that

$$\mu_1 > \cdots > \mu_s$$

are all the distinct elements in $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$. *Let*

$$J_i = \{t : 1 \le p \le N, 1 \le t \le s, [\sigma_0(p) \bmod n] = i, \lambda_p = \mu_t\}. \tag{5.3}$$

Suppose that all the following conditions hold:

- *If* $1 \le i < j \le n$, $l \in J_i$, *and* $t \in J_j$, *then* $l < t$.
- *For any* p, q *satisfying* $1 \le p \le s$ *and* $1 \le q \le s$, *and* $q > p$,

$$C_1 N \le \mu_p - \mu_q \le C_2 N,$$

where C_1, C_2 *are constants independent of* N.

The key point of Assumption 5.3 is, if we order all the edge weights as in (5.2) and have $x_{\sigma_0(i)}$ correspond to the ith part λ_i of the partition on the bottom, then one cannot have $x_{\sigma_0(i)} = x_{\sigma_0(i)}$ if $\lambda_i \neq \lambda_j$. Assumption 5.3 also supposes that the bottom boundary is divided into finitely many alternating segments of particles and holes; then each segment grows linearly with N.

Let

$$H_{\mathbf{m}_i}(u) = \int_0^{\ln(u)} R_{\mathbf{m}_i}(t)\,dt + \ln\frac{\ln(u)}{u-1},$$

and let $R_{\mathbf{m}_i}$ be the Voiculescu R-transform of $\mathbf{m}_i$ given by

$$R_{\mathbf{m}_i} = \frac{1}{S_{\mathbf{m}_i}^{(-1)}(z)} - \frac{1}{z},$$

where $S_{\mathbf{m}_i}$ is the moment generating function for $\mathbf{m}_i$ given by

$$S_{\mathbf{m}_i}(z) = z + M_1(\mathbf{m}_i)z^2 + M_2(\mathbf{m}_i)z^3 + \cdots,$$

$M_k(\mathbf{m}_i) = \int_{\mathbb{R}} x^k \mathbf{m}_i(dx)$, and $S_{\mathbf{m}_i}^{-1}(z)$ is the inverse series of $S_{\mathbf{m}_i}(z)$. See also [6, Section 2.2] for details.

Proposition 5.4. *Suppose Assumptions 5.2 and 5.3 hold. Let $\kappa \in (0,1)$ be a positive number. Let $\rho_{\lfloor(1-\kappa)N\rfloor}$ be a probability measure on $\mathbb{GT}^+_{\lfloor(1-\kappa)N\rfloor}$ as defined in Lemma 3.4 or Remark 3.5. Let $\mathbf{m}[\rho_{\lfloor(1-\kappa)N\rfloor}]$ be the corresponding random counting measure. Then as $N \to \infty$, $\mathbf{m}[\rho_{\lfloor(1-\kappa)N\rfloor}]$ converge in probability, in the sense of moments to a deterministic measure $\mathbf{m}^\kappa$, whose moments are given by*

$$\int_{\mathbb{R}} x^p \boldsymbol{m}^\kappa(dx) = \frac{1}{2n(p+1)\pi\mathbf{i}} \sum_{i=1}^{n} \oint_{C_1} \frac{dz}{z}\left(zQ'_{i,\kappa}(z) + \frac{n-i}{n} + \frac{z}{n(z-1)}\right)^{p+1},$$

where for $1 \le i \le n$,

$$Q_{i,\kappa}(z) = \begin{cases} \frac{1}{(1-\kappa)n}\Big[H_{\mathbf{m}_i}(z) - (n-i)\log z \\ \qquad + \kappa \sum_{r\in\{1,2,\ldots,n\}\cap I_2} \log\frac{1+y_r z x_1}{1+y_r x_1}\Big] & \text{if } i = 1, \\ \frac{1}{(1-\kappa)n}[H_{\mathbf{m}_i}(z) - (n-i)\log z] & \text{otherwise,} \end{cases}$$

and for $i \ge n+1$,

$$Q_{i,\kappa}(z) = \begin{cases} Q_{(i \bmod n),\kappa}(z) & \text{if } (i \bmod n) \neq 0, \\ Q_{n,\kappa}(z) & \text{otherwise.} \end{cases}$$

Proof. See of [32, Theorem 2.18]. ∎

Lemma 5.5. *Let k be a positive integer such that $1 \le k \le N$. Let*

$$w_i = \begin{cases} u_i & \text{if } 1 \le i \le k, \\ x_i & \text{if } k+1 \le i \le N. \end{cases}$$

Assume $k = qn + r$, where $r < n$, and q, r are positive integers. Let $\lambda \in \mathbb{GT}_N^+$ be an arbitrary partition. Then the Schur function can be computed by the following formula:

$$\begin{aligned} s_\lambda(w_1, \dots, w_N) = & \sum_{\overline{\sigma} \in [\Sigma_N / \Sigma_N^X]^r} \prod_{i=1}^n x_i^{|\phi^{(i,\sigma)}(N)|} \\ & \times \prod_{i=1}^r s_{\phi^{(i,\sigma)}(N)}\Big(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i}, \dots, \frac{u_{qn+i}}{x_i}, 1, \dots, 1\Big) \\ & \times \prod_{i=r+1}^n s_{\phi^{(i,\sigma)}(N)}\Big(\frac{u_i}{x_i}, \frac{u_{n+i}}{x_i}, \dots, \frac{u_{(q-1)n+i}}{x_i}, 1, \dots, 1\Big) \\ & \times \prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}}, \end{aligned}$$

where $\sigma \in \overline{\sigma} \cap \Sigma_N$ is a representative.

Proof. See [32, Proposition 3.4]. ∎

Theorem 5.6. *Under Assumptions 5.2 and 5.3, for each given $\{a_i, b_i\}_{i=1}^n$, when α in Assumption 5.2 is sufficiently large and $k \le n$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \frac{s_{\lambda(N)}(u_1 x_{1,N}, \dots, u_k x_{k,N}, x_{k+1,N}, \dots, x_{N,N})}{s_{\lambda(N)}(x_{1,N}, \dots, x_{N,N})} = \sum_{i=1}^k [Q_i(u_i)], \quad (5.4)$$

where for $1 \le i \le k$,

(1) *if $[i \mod n] \neq 0$,*

$$Q_i(u) = \frac{H_{\mathbf{m}_{i \bmod n}}(u)}{n} - \frac{(n - [i \mod n]) \log(u)}{n},$$

and the convergence of (5.4) is uniform when $u_1, \dots, u_k$ are in an open complex neighborhood of 1;

(2) *if $[i \mod n] = 0$,*

$$Q_i(u) = \frac{H_{\mathbf{m}_n}(u)}{n}.$$

Proof. See [32, Theorem 2.9]. ∎

Lemma 5.7. *Suppose Assumptions 5.2 and 5.3 hold, and let α be given as in Assumption 5.2. For each given $\{a_i, b_i\}_{i=1}^n$, when α is sufficiently large, for any $\sigma \notin \overline{\sigma}_0$, we have*

$$\left|\frac{\prod_{i=1}^n x_{i,N}^{|\phi^{(i,\sigma_0)}(N)|}\prod_{i=1}^n s_{\phi^{(i,\sigma_0)}(N)}(1,\dots,1)}{\prod_{i=1}^n x_{i,N}^{|\phi^{(i,\sigma)}(N)|}\prod_{i=1}^n s_{\phi^{(i,\sigma)}(N)}(1,\dots,1)}\right| \times\left|\frac{\prod_{i<j,x_{\sigma_0(i),N}\neq x_{\sigma_0(j),N}}\frac{1}{x_{\sigma_0(i),N}-x_{\sigma_0(j),N}}}{\prod_{i<j,x_{\sigma(i),N}\neq x_{\sigma(j),N}}\frac{1}{x_{\sigma(i),N}-x_{\sigma(j),N}}}\right| \geq e^{CN^2},$$

where $C > 0$ is a constant independent of N and σ, and increases as α increases. Indeed, we have

$$\lim_{\alpha\to\infty} C = \infty.$$

Proof. See [32, Proposition 4.5]. ■

6. Central limit theorem for piecewise boundary conditions

In this section, we construct certain statistics from the (random) dimer configuration on a contracting square-hexagon lattice with piecewise boundary conditions and show that they converge in distribution to a sum of n independent Gaussian random variables in the scaling limit, where $1 \times n$ is the size of a fundamental domain. The main theorem proved in this section is Theorem 6.1.

Theorem 6.1. *Assume $\kappa_1, \kappa_2 \in (0, 1)$. Then the random variables $\{\frac{1}{N^l}[p_l^{\lfloor(1-\kappa)N\rfloor} - \mathbf{E}p_l^{\lfloor(1-\kappa)N\rfloor}]\}_{l,\kappa}$ converge in distribution to a mean 0 Gaussian vector with covariance given by Proposition 6.16. Moreover, each $\frac{1}{N^l}[p_l^{\lfloor(1-\kappa)N\rfloor} - \mathbf{E}p_l^{\lfloor(1-\kappa)N\rfloor}]$ converges in distribution to the sum of n independent mean 0 Gaussian random variables.*

Again, the idea that we use to prove Theorem 6.1 is to compute the moments of these random variables and then show that they satisfy Wick's probability theorem in the $N \to \infty$ limits; this gives the Gaussian distribution of these random variables as well as the explicit form of the covariance. The major new ingredients, compared to [7], are the application of Lemma 5.5 in computations of moments and the splitting of these variables into a sum of independent Gaussians when the edge weights satisfy Assumptions 5.1, 5.2 and 5.3.

6.1. First order moments

In this section, we compute the expectation of the random variables $\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_i + \lfloor(1-\kappa)N\rfloor - i)^k$.

For the piecewise boundary conditions, the proof of Proposition 4.2 still holds. For $\kappa \in (0, 1)$, let

$$\begin{aligned} X_N &= (x_{1,N}, \dots, x_{N,N}), \\ X_{N,\kappa} &= (x_{1+N-\lfloor(1-\kappa)N\rfloor,N}, \dots, x_{N,N}), \\ U_{N,\kappa} &= (u_1, u_2, \dots, u_{\lfloor(1-\kappa)N\rfloor}), \\ U_{N,\kappa,X} &= (u_1 x_{1+N-\lfloor(1-\kappa)N\rfloor,N}, u_2 x_{2+N-\lfloor(1-\kappa)N\rfloor,N}, \dots, u_{\lfloor(1-\kappa)N\rfloor} x_{N,N}). \end{aligned}$$

Let $\lambda \in \mathbb{GT}_{\lfloor(1-\kappa)N\rfloor}$, and let $\rho_{\lfloor(1-\kappa)N\rfloor}$ be a probability measure on $\mathbb{GT}_{\lfloor(1-\kappa)N\rfloor}$ as defined in Proposition 5.4. Then we have

$$\begin{aligned} E_{k,\kappa,N} &:= \mathbf{E} \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor} (\lambda_i + \lfloor(1-\kappa)N\rfloor - i)^k \\ &= \sum_{\lambda \in \mathbb{GT}_{\lfloor(1-\kappa)N\rfloor}} \rho_{\lfloor(1-\kappa)N\rfloor}(\lambda) \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor} (\lambda + \lfloor(1-\kappa)N\rfloor - i)^k \\ &= \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})} \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor} \left(u_i \frac{\partial}{\partial u_i}\right)^k V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X}) \\ &\quad \times \mathcal{S}_{\rho_{\lfloor(1-\kappa)N\rfloor}, X_{N,\kappa}}(U_{N,\kappa,X}) \Big|_{U_{N,\kappa}=(1,\dots,1)}. \end{aligned}$$

For $1 \leq i \leq N$, let

$$v_i = \begin{cases} x_{i,N} & \text{if } 1 \leq i \leq N - \lfloor(1-\kappa)N\rfloor, \\ x_{i,N} u_{i-N+\lfloor(1-\kappa)N\rfloor} & \text{if } N - \lfloor(1-\kappa)N\rfloor + 1 \leq i \leq N. \end{cases}$$

Let $\lambda(N)$ be the partition corresponding to the boundary condition. For $1 \leq i \leq n$, let

$$R(i) = \{1 \leq j \leq \lfloor(1-\kappa)N\rfloor : [(j + N - \lfloor(1-\kappa)N\rfloor) \bmod n] = [i \bmod n]\},$$

and for $1 \leq i \leq \lfloor(1-\kappa)N\rfloor$, let

$$j(i) = \begin{cases} [i + N - \lfloor(1-\kappa)N\rfloor] \bmod n & \text{if } ([i + N - \lfloor(1-\kappa)N\rfloor] \bmod n) \neq 0, \\ n & \text{otherwise.} \end{cases}$$

Assume

$$\lfloor(1-\kappa)N\rfloor = q_{N,\kappa} n + r_{N,\kappa},$$

where $q_{N,\kappa}$ and $r_{N,\kappa}$ are non-negative integers satisfying $r_{N,\kappa} < n$.

By Lemmas 3.3, 5.5 and 5.7, we obtain

$$
\begin{aligned}
E_{k,\kappa,N} =& \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})} \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor} \left(u_i \frac{\partial}{\partial u_i}\right)^k V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X}) \\
& \times \Bigg[\frac{s_{\lambda(N)}(U_{N,\kappa,X}, x_{1,N}, \dots, x_{N-\lfloor(1-\kappa)N\rfloor,N})}{s_{\lambda(N)}(X_N)} \\
& \times \prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \prod_{j=1}^{\lfloor(1-\kappa)N\rfloor} \frac{1+y_{\bar{l}} u_j x_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}}}{1+y_{\bar{l}} x_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}}} \Bigg]\Bigg|_{U_{N,\kappa}=(1,\dots,1)} \\
=& \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})} \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor} \left(u_i \frac{\partial}{\partial u_i}\right)^k V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X}) \frac{T_N}{P_N}\Bigg|_{U_{N,\kappa}=(1,\dots,1)},
\end{aligned}
$$

where

$$
\begin{aligned}
T_N =& \prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \prod_{j=1}^{\lfloor(1-\kappa)N\rfloor} \frac{1+y_{\bar{l}} x_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}} u_j}{1+y_{\bar{l}} x_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}}} \\
& \times \left(\prod_{i=1}^{r_{N,\kappa}} s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \dots, u_{q_{N,\kappa}n+i}, 1, \dots, 1) \right) \\
& \times \left(\prod_{i=r_{N,\kappa}+1}^{n} s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \dots, u_{(q_{N,\kappa}-1)n+i}, 1, \dots, 1) \right) \\
& \times \left(\prod_{i<j, x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{v_{\sigma_0(i)} - v_{\sigma_0(j)}} \right)(1+o(1)) \\
P_N =& \left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma_0)}(N)}(1,\dots,1) \right) \left(\prod_{i<j, x_{\sigma_0(i)}\neq x_{\sigma_0(j)}} \frac{1}{x_{\sigma_0(i)} - x_{\sigma_0(j)}} \right)(1+o(1)).
\end{aligned}
$$

Then we have

$$
E_{k,\kappa,N} = \sum_{j=1}^{n} E_{k,\kappa,N}^{(j)},
$$

where

$$
\begin{aligned}
E_{k,\kappa,N}^{(j)} :=& \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})} \\
=& \sum_{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)} \left(u_i \frac{\partial}{\partial u_i}\right)^k V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X}) \frac{T_N}{P_N}\Bigg|_{U_{N,\kappa}=(1,\dots,1)} \\
=& \sum_{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)} \left(u_i \frac{\partial}{\partial u_i}\right)^k \frac{T_{N,i}}{P_{N,i}}\Bigg|_{U_{N,\kappa}=(1,\dots,1)},
\end{aligned}
$$

in which

$$T_{N,i} = \Big(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \frac{1+y_{\bar{l}}x_{j(i)}u_i}{1+y_{\bar{l}}x_{j(i)}}\Big) s_{\phi^{(j(i),\sigma_0)}(N)}(u_i,1,\dots,1)$$
$$\times\Big(\prod_{j(i)<k\le n}\Big(\frac{1}{u_i x_{j(i)}-x_k}\Big)^{\frac{\kappa N}{n}}\Big)\Big(\prod_{k<j(i)\le n}\Big(\frac{1}{x_k-u_i x_{j(i)}}\Big)^{\frac{\kappa N}{n}}\Big)$$
$$\times\Big(\prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ k \bmod n=j(i),\\ k-N+\lfloor(1-\kappa)N\rfloor\ne i}} [u_i x_{j(i)}-x_k]\Big)e^{No(1)},$$

$$P_{N,i} = s_{\phi^{(j(i),\sigma_0)}(N)}(1,\dots,1)\Big(\prod_{j(i)<k\le n}\Big(\frac{1}{x_{j(i)}-x_k}\Big)^{\frac{\kappa N}{n}}\Big)$$
$$\times\Big(\prod_{k<j(i)\le n}\Big(\frac{1}{x_k-x_{j(i)}}\Big)^{\frac{\kappa N}{n}}\Big)\Big(\prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ k \bmod n=j(i),\\ k-N+\lfloor(1-\kappa)N\rfloor\ne i}} [x_{j(i)}-x_k]\Big)e^{No(1)}.$$

In the expressions above, the $o(1)$ terms converge to 0 uniformly when u_i is in a neighborhood of 1.

If the edge weights satisfy Assumption 5.2, when $j(i)>k$, $u_i x_{j(i)}$ is exponentially small in N compared to x_k, hence we obtain

$$E^{(j)}_{k,\kappa,N} := \lim_{\substack{x_k\to(x_{k \bmod n})\\ \text{for } 1\le k\le N}} \sum_{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)} \Big(u_i\frac{\partial}{\partial u_i}\Big)^k \frac{\widetilde{V}_{N,i}\widetilde{T}_{N,i}}{\widetilde{W}_{N,i}\widetilde{P}_{N,i}}\Big|_{U_{N,\kappa}=(1,\dots,1)},$$

where

$$\widetilde{T}_{N,i} = \Big(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \frac{1+y_{\bar{l}}x_{j(i)}u_i}{1+y_{\bar{l}}x_{j(i)}}\Big) s_{\phi^{(j(i),\sigma_0)}(N)}(u_i,1,\dots,1)$$
$$\times\Big(\prod_{j(i)<k\le n}\Big(\frac{1}{u_i x_{j(i)}-x_k}\Big)^{\frac{\kappa N}{n}}\Big)e^{No(1)},$$

$$\widetilde{P}_{N,i} = s_{\phi^{(j(i),\sigma_0)}(N)}(1,\dots,1)\Big(\prod_{j(i)<k\le n}\Big(\frac{1}{x_{j(i)}-x_k}\Big)^{\frac{\kappa N}{n}}\Big)e^{No(1)},$$

$$\widetilde{V}_{N,i} = \prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ (k \bmod n)=(j(i) \bmod n),\\ k-N+\lfloor(1-\kappa)N\rfloor\ne i}} [u_i x_{j(i)}-x_k],$$

$$\widetilde{W}_{N,i} = \prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ (k \bmod n)=(j(i) \bmod n),\\ k-N+\lfloor(1-\kappa)N\rfloor\ne i}} [x_{j(i)}-x_k].$$

Note that

$$E^{(j)}_{k,\kappa,N} = \sum_{\substack{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j)}} \frac{1}{\widetilde{W}_{N,i}\,\widetilde{P}_{N,i}} \left(u_i\frac{\partial}{\partial u_i}\right)^k \widetilde{V}_{N,i}\exp[\log(\widetilde{T}_{N,i})]\Big|_{\substack{U_{N,\kappa}\\ =(1,\dots,1)}} \tag{6.1}$$

and

$$\frac{\partial \widetilde{T}_{N,i}}{\partial u_i} = \frac{\partial}{\partial u_i}\exp[\log(\widetilde{T}_{N,i})] = \exp[\log(\widetilde{T}_{N,i})]\frac{\partial}{\partial u_i}[\log(\widetilde{T}_{N,i})].$$

Lemma 6.2. *Assume $\kappa \in (0,1)$ and the edge weights satisfy Assumption 5.2, then for $1 \le i \le \lfloor(1-\kappa)N\rfloor$ and $i \in R(j)$,*

$$\lim_{N\to\infty}\frac{1}{N}\frac{\partial[\log\widetilde{T}_{N,i}]}{\partial u_i} = \frac{\kappa}{n}\sum_{l\in I_2\cap\{1,2,\dots,n\}}\frac{y_l x_{j(i)}}{1+y_l x_{j(i)}u_i} - \frac{\kappa}{n}\frac{n-j(i)}{u_i} + \frac{1}{n}H'_{\mathbf{m}_{j(i)}}(u_i),$$

and the convergence is uniform when u_i is in a neighborhood of 1.

Proof. The lemma follows from explicit computations and [9, Theorem 3.6]; see also [6, 13, 17]. ∎

6.2. Second order moments

Let

$$\begin{aligned}
E_{k,l,\kappa,N} &:= \mathbf{E}\left(\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_i+\lfloor(1-\kappa)N\rfloor-i)^k\sum_{j=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_j+\lfloor(1-\kappa)N\rfloor-j)^l\right)\\
&= \sum_{\lambda\in\mathbb{GT}_{\lfloor(1-\kappa)N\rfloor}}\rho_{\lfloor(1-\kappa)N\rfloor}(\lambda)\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_i+\lfloor(1-\kappa)N\rfloor-i)^k\\
&\quad\times\sum_{j=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_j+\lfloor(1-\kappa)N\rfloor-j)^l\\
&= \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})}\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_i\frac{\partial}{\partial u_i}\right)^k\sum_{j=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_j\frac{\partial}{\partial u_j}\right)^l\\
&\quad\times V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})\mathcal{S}_{\rho_{\lfloor(1-\kappa)N\rfloor},X_{N,\kappa}}(U_{N,\kappa,X})\Big|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

Again, by Lemmas 3.3, 5.5 and 5.7, we obtain

$$E_{k,l,\kappa,N} = \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})}\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_i\frac{\partial}{\partial u_i}\right)^k\sum_{j=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_j\frac{\partial}{\partial u_j}\right)^l$$

$$\times V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})\Bigg(\frac{s_{\lambda(N)}(U_{N,\kappa,X},x_{1,N},\dots,x_{N-\lfloor(1-\kappa)N\rfloor,N})}{s_{\lambda(N)}(X_N)}$$

$$\times\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2}\prod_{j=1}^{\lfloor(1-\kappa)N\rfloor}\frac{1+y_{\bar{l}}u_jx_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}}}{1+y_{\bar{l}}x_{\overline{N-\lfloor(1-\kappa)N\rfloor+j}}}\Bigg)\Bigg|_{U_{N,\kappa}=(1,\dots,1)}$$

$$=\frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})}\sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_i\frac{\partial}{\partial u_i}\right)^k$$

$$\times\sum_{j=1}^{\lfloor(1-\kappa)N\rfloor}\left(u_j\frac{\partial}{\partial u_j}\right)^l V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})\frac{T_N}{P_N}\Bigg|_{U_{N,\kappa}=(1,\dots,1)}.$$

Then we have

$$E_{k,l,\kappa,N}=\sum_{s=1}^{n}\sum_{t=1}^{n}E_{k,l,\kappa,N}^{(s,t)},$$

where

$$E_{k,l,\kappa,N}^{(s,t)}:=\frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})}\sum_{\substack{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(s)}}\sum_{\substack{r\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(t)}}\left(u_i\frac{\partial}{\partial u_i}\right)^k$$

$$\times\left(u_r\frac{\partial}{\partial u_r}\right)^l V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})\frac{T_N}{P_N}\Bigg|_{U_{N,\kappa}=(1,\dots,1)}$$

$$=\sum_{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(s)}\left(u_i\frac{\partial}{\partial u_i}\right)^k\sum_{[r\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(t)]}\left(u_r\frac{\partial}{\partial u_r}\right)^l$$

$$\times\frac{V_{N,(i,r)}^{(s,t)}T_{N,(i,r)}^{(s,t)}}{W_{N,(i,r)}^{(s,t)}P_{N,(i,r)}^{(s,t)}}\Bigg|_{U_{N,\kappa}=(1,\dots,1)}\tag{6.2}$$

and where (assume the edge weights satisfy Assumption 5.2)

(1) If $s=t$,

$$T_{N,(i,r)}^{(s,s)}=\Bigg(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2}\frac{1+y_{\bar{l}}x_su_i}{1+y_{\bar{l}}x_s}\Bigg)$$

$$\times\Bigg(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2}\frac{1+y_{\bar{l}}x_su_r}{1+y_{\bar{l}}x_s}\Bigg)s_{\phi^{(s,\sigma_0)}(N)}(u_i,u_j,1,\dots,1)$$

$$\times\Bigg(\prod_{s<k\le n}\left(\frac{1}{u_ix_s-x_k}\right)^{\frac{\kappa N}{n}}\Bigg)\Bigg(\prod_{s<k\le n}\left(\frac{1}{u_rx_s-x_k}\right)^{\frac{\kappa N}{n}}\Bigg)e^{No(1)},$$

$$P_{N,(i,r)}^{(s,s)}=s_{\phi^{(s,\sigma_0)}(N)}(1,\dots,1)\Bigg(\prod_{s<k\le n}\left(\frac{1}{x_s-x_k}\right)^{\frac{2\kappa N}{n}}\Bigg)e^{No(1)},$$

$$V_{N,(i,r)}^{(s,s)} = \Big(\prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ (k \bmod n)=(s \bmod n),\\ k-N+\lfloor(1-\kappa)N\rfloor\notin\{i,r\}}} [u_i x_s - x_k] \Big) \times \Big(\prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ (k \bmod n)=(s \bmod n),\\ k-N+\lfloor(1-\kappa)N\rfloor\notin\{i,r\}}} [u_r x_s - x_k] \Big)(u_i x_s - u_r x_s)$$

$$W_{N,(i,r)}^{(s,s)} = \prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\le k\le N,\\ (k \bmod n)=(s \bmod n),\\ k-N+\lfloor(1-\kappa)N\rfloor\notin\{i,r\}}} [x_s - x_k]^2 (x_{i+N-\lfloor(1-\kappa)N\rfloor} - x_{r+N-\lfloor(1-\kappa)N\rfloor}).$$

In the expressions above, the $o(1)$ terms converge to 0 uniformly when u_i, u_r are in a neighborhood of 1 as $N \to \infty$; moreover, the operator $\frac{\partial^2}{\partial u_r \partial u_s}$ acting on $\log o(1)$ is identically 0 (instead of converging to 0 as $N \to \infty$).

(2) If $s \neq t$,

$$T_{N,(i,r)}^{(s,t)} = \Big(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \frac{1+y_{\bar{l}} x_s u_i}{1+y_{\bar{l}} x_s} \Big) \times \Big(\prod_{l\in\{1,\dots,N-\lfloor(1-\kappa)N\rfloor\}\cap I_2} \frac{1+y_{\bar{l}} x_s u_r}{1+y_{\bar{l}} x_s} \Big) s_{\phi^{(s,\sigma_0)}(N)}(u_i, 1, \dots, 1)$$
$$\times s_{\phi^{(t,\sigma_0)}(N)}(u_r, 1, \dots, 1) \Big(\prod_{s<k\le n} \Big(\frac{1}{u_i x_s - x_k} \Big)^{\frac{\kappa N}{n}} \Big)$$
$$\times \Big(\prod_{t<k\le n} \Big(\frac{1}{u_r x_t - x_k} \Big)^{\frac{\kappa N}{n}} \Big) \Big(\frac{1}{u_i x_s - u_r x_t} \Big) e^{No(1)},$$

$$P_{N,(i,r)}^{(s,t)} = \widetilde{P}_{N,i} \widetilde{P}_{N,r} e^{No(1)}, \quad V_{N,(i,r)}^{(s,t)} = \widetilde{V}_{N,i} \widetilde{V}_{N,r}, \quad W_{N,(i,r)}^{(s,t)} = \widetilde{W}_{N,i} \widetilde{W}_{N,r}.$$

In the expressions above, the $o(1)$ terms converge to 0 uniformly when u_i, u_r are in a neighborhood of 1 as $N \to \infty$; moreover, the operator $\frac{\partial^2}{\partial u_r \partial u_s}$ acting on $\log o(1)$ is identically 0 (instead of converging to 0 as $N \to \infty$).

Lemma 6.3. *Assume $\kappa \in (0,1)$ and the edge weights satisfy Assumption 5.2, then*

(1) *For $1 \le i < j \le \lfloor(1-\kappa)N\rfloor$ and $i, j \in R(s)$,*

$$\lim_{N\to\infty} \frac{\partial^2 [\log T_{N,(i,r)}^{(s,s)}]}{\partial u_i \partial u_r} = \frac{\partial^2}{\partial u_i u_r} \Big[\log \Big(1 - (u_i - 1)(u_r - 1) \frac{u_i H'_{\mathbf{m}_s}(u_i) - u_r H'_{\mathbf{m}_s}(u_r)}{u_i - u_r} \Big) \Big],$$

and the convergence is uniform when u_i is in a neighborhood of 1.

(2) *For* $1 \leq i < j \leq \lfloor (1-\kappa)N \rfloor$ *and* $i \in R(s)$, $j \in R(t)$ *with* $s \neq t$,

$$\lim_{N\to\infty} \frac{\partial^2[\log T^{(s,t)}_{N,(i,r)}]}{\partial u_i \partial u_s} = 0,$$

and the convergence is uniform when u_i *is in a neighborhood of* 1.

Proof. First we consider the case where $i, j \in R(s)$. We have

$$\lim_{N\to\infty} \frac{\partial^2[\log T^{(s,s)}_{N,(i,r)}]}{\partial u_i \partial u_r} = \lim_{N\to\infty} \frac{\partial^2[\log s_{\phi^{(s,\sigma_0)}(N)}(u_i, u_j, 1, \dots, 1)]}{\partial u_i \partial u_r}.$$

Then part (1) of the lemma follows from [9, Theorem 6.8]; see also [7, 13].

Now we consider the case where $i \in R(s)$, $j \in R(t)$ and $s \neq t$. In this case,

$$\begin{aligned}\lim_{N\to\infty} \frac{\partial^2[\log T^{(s,s)}_{N,(i,r)}]}{\partial u_i \partial u_r} &= \lim_{N\to\infty} \frac{\partial^2[-\log(u_i x_{s,N} - u_r x_{t,N})]}{\partial u_i \partial u_r} \\ &= \lim_{N\to\infty} \frac{x_{s,N} x_{t,N}}{(u_i x_{s,N} - u_r x_{t,N})^2},\end{aligned}$$

and the limit is 0 by Assumption 5.2. ■

6.3. Asymptotic analysis

Let

$$\begin{aligned}F^{(s)}_{k,\kappa,N} &= \frac{1}{V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})T_N} \\ &\quad\times \sum_{\substack{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(s)}} \Big(u_i \frac{\partial}{\partial u_i}\Big)^k V_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa,X})T_N. \qquad (6.3)\end{aligned}$$

For simplicity, we use the notation ∂_i to denote $\frac{\partial}{\partial u_i}$. Expanding the right-hand side of (6.1), we can write $E^{(j)}_{k,\kappa,N}$ as a sum of terms of the form

$$\frac{c_0 x^r_{j(i)} (\partial_i^{s_1}[\log \widetilde{T}_{N,i}])^{d_1} \cdots (\partial_i^{s_t}[\log \widetilde{T}_{N,i}])^{d_t}}{(x_{j(i)} - x_{a_1}) \cdots (x_{j(i)} - x_{a_r})}. \qquad (6.4)$$

Similarly, we can write the right-hand side of (6.3) as a large sum of terms of the form

$$\frac{c_0 x^r_{j(i)} u_i^{k-s_0} (\partial_i^{s_1}[\log T_N])^{d_1} \cdots (\partial_i^{s_t}[\log T_N])^{d_t}}{(x_{j(i)}u_i - x_{a_1}u_{a_1-N+\lfloor(1-\kappa)N\rfloor}) \cdots (x_{j(i)}u_i - x_{a_r}u_{a_r-N+\lfloor(1-\kappa)N\rfloor})}$$

such that

- for $1 \le s \le r$, a_s is a positive integer satisfying $N - \lfloor (1-\kappa)N \rfloor + 1 \le a_s \le N$;
- in (6.4), we have $(a_s \bmod n) = (j(i) \bmod n)$ for all $1 \le s \le r$;
- $N - \lfloor (1-\kappa)N \rfloor + i, a_1, \dots, a_r$ are distinct;
- $\{s_j\}_{j=0}^t$ and $\{d_j\}_{j=1}^t$ are non-negative integers satisfying $s_1 < s_2 < \cdots < s_t$;
- we have that

$$r + s_0 + s_1 d_1 + \cdots + s_t d_t = k; \tag{6.5}$$

- c_0 is a constant independent of N and $a_1, \dots, a_r$.

From expression (6.1), we see that any terms obtained by permuting $i, a_1 - N + \lfloor (1-\kappa)N \rfloor, \dots, a_r - N + \lfloor (1-\kappa)N \rfloor$ of (6.4) within $\{1, 2, \dots, \lfloor (1-\kappa)N \rfloor\} \cap R(j)$ are still present in the sum. Let

$$\widetilde{a}_1 = a_1 - N + \lfloor (1-\kappa)N \rfloor.$$

Hence we have

$$E_{k,\kappa,N}^{(j)} = \sum_{\substack{r, \{s_j\}_{j=0}^t, \{d_j\}_{j=1}^t \\ \text{satisfy (6.5)}}} (r+1)! \sum_{\substack{\{\widetilde{a}_1, \dots, \widetilde{a}_{r+1}\} \\ \in \{1,2,\dots,\lfloor (1-\kappa)N \rfloor\} \\ \cap R(j)}} \lim_{\substack{x_{a_1}, \dots, \\ x_{a_{r+1}} \to x_j}}$$
$$\operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\widetilde{a}_1}^{k-s_0} (\partial_{\widetilde{a}_1}^{s_1} [\log \widetilde{T}_{N,\widetilde{a}_1}])^{d_1} \cdots (\partial_{\widetilde{a}_1}^{s_t} [\log \widetilde{T}_{N,\widetilde{a}_1}])^{d_t}}{(x_{a_1} - x_{a_2}) \cdots (x_{a_1} - x_{a_{r+1}})} \Big|_{\substack{U_{N,\kappa} \\ = \{1,\dots,1\}}},$$

where the constant c_0 may depend on r, $\{s_j\}_{j=0}^t$ and $\{d_j\}_{j=1}^t$.

Lemma 6.4. *Let*

$$\widetilde{T}_N = \prod_{l \in \{1,\dots,N-\lfloor (1-\kappa)N \rfloor\} \cap I_2} \prod_{j=1}^{\lfloor (1-\kappa)N \rfloor} \frac{1 + y_{\bar{l}} x_{\overline{N - \lfloor (1-\kappa)N \rfloor + j}} u_j}{1 + y_{\bar{l}} x_{\overline{N - \lfloor (1-\kappa)N \rfloor + j}}}$$
$$\times \left(\prod_{i=1}^r s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \dots, u_{q_{N,\kappa} n + i}, 1, \dots, 1) \right)$$
$$\times \left(\prod_{i=r+1}^n s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \dots, u_{(q_{N,\kappa}-1)n + i}, 1, \dots, 1) \right)$$
$$\times \left(\prod_{\substack{N - \lfloor (1-\kappa)N \rfloor + 1 \le i \le N,\ 1 \le j \le N - \lfloor (1-\kappa)N \rfloor, \\ \widetilde{i} \in R(p),\ \widetilde{j} \in R(q),\ p < q}} \frac{1}{x_i u_{\widetilde{i}} - x_j} \right) (1 + o(1)),$$

$$\widetilde{V}_{\lfloor (1-\kappa)N \rfloor} = \prod_{k=1}^n \prod_{\substack{1 + N - \lfloor (1-\kappa)N \rfloor \le i < j \le N, \\ (i \bmod n) = (j \bmod n) = (k \bmod n)}} (x_i u_{\widetilde{i}} - x_j u_{\widetilde{j}}).$$

Assume $a, b, c \in R(j)$ and are distinct positive integers. Assume that the $o(1)$ in the definition of $\widetilde{T}_N$ converges to 0 uniformly when u_a, u_b, u_c are in a neighborhood of 1. Then

$$\lim_{N\to\infty} \frac{\partial^3 \log[\widetilde{T}_N]}{\partial u_a \partial u_b \partial u_c}\Big|_{U_{N,\kappa}=(1,\dots,1)} = 0.$$

Proof. We have

$$\lim_{N\to\infty} \frac{\partial^3 \log[\widetilde{T}_N]}{\partial u_a \partial u_b \partial u_c}\Big|_{U_{N,\kappa}=(1,\dots,1)} = \lim_{N\to\infty} \frac{\partial^3 \log[s_{\phi^{(j,\sigma_0)}(N)}(u_a, u_b, u_c, 1, \dots, 1)]}{\partial u_a \partial u_b \partial u_c}.$$

To compute the derivative on the right-hand side, note that

$$\lim_{N\to\infty} \mathbf{m}[\phi^{(j,\sigma_0)}(N)] = \mathbf{m}_j.$$

By [9, Theorem 6.8] (see also [7, 13]),

$$\begin{aligned}\lim_{N\to\infty} & \frac{\partial^2 \log[s_{\phi^{(j,\sigma_0)}(N)}(u_a, u_b, u_c, 1, \dots, 1)]}{\partial u_a \partial u_b} \\ &= \frac{\partial^2}{\partial u_a \partial u_b} \log\Big(1 - (u_a - 1)(u_b - 1)\frac{u_a H'_{\mathbf{m}_j}(u_a) - u_b H'_{\mathbf{m}_j}(u_b)}{u_a - u_b}\Big),\end{aligned}$$

and the convergence is uniform when (u_a, u_b, u_c) is in a neighborhood of $(1, 1, 1)$. Therefore, we can take the derivative with respect to u_c on both sides. The right-hand side is independent of u_c, and the derivative with respect to u_c is 0. Then the lemma follows. ∎

Lemma 6.5. *Let $\kappa \in (0, 1)$ and the edge weights satisfy Assumption 5.2. Then*

(1) *The degree of N in $E^{(j)}_{k,\kappa,N}$ is at most $k + 1$.*

(2) *For any integer i satisfying $1 \le i \le \lfloor (1-\kappa)N \rfloor$ and $i \in R(j)$, the degree of N in $\frac{\partial}{\partial u_i} F^{(j)}_{k,\kappa,N}|_{U_{N,\kappa}=(1,\dots,1)}$ is at most k.*

(3) *For any integer i satisfying $1 \le i \le \lfloor (1-\kappa)N \rfloor$ and $i \notin R(j)$, the degree of N in $\frac{\partial}{\partial u_i} F^{(j)}_{k,\kappa,N}|_{U_{N,\kappa}=(1,\dots,1)}$ is less than k.*

(4) *For any integers i_1, i_2 satisfying $1 \le i_1 < i_2 \le \lfloor (1-\kappa)N \rfloor$ and $i_1, i_2 \in R(j)$, the degree of N in $\frac{\partial^2}{\partial u_{i_1} \partial u_{i_2}} F^{(j)}_{k,\kappa,N}|_{U_{N,\kappa}=(1,\dots,1)}$ is less than k.*

Proof. We first consider the asymptotics of

$$\lim_{\substack{x_{a_1},\dots,\\ x_{a_{r+1}}\to x_j}} \operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0} (\partial_{\tilde{a}_1}^{s_1}[\log \widetilde{T}_{N,\tilde{a}_1}])^{d_1} \cdots (\partial_{\tilde{a}_1}^{s_t}[\log \widetilde{T}_{N,\tilde{a}_1}])^{d_t}}{(x_{a_1} - x_{a_2}) \cdots (x_{a_1} - x_{a_{r+1}})}\Big|_{\substack{U_{N,\kappa}\\ =\{1,\dots,1\}}}. \tag{6.6}$$

Following computations similar to the ones in Lemma 6.2, we obtain that the degree of N in each factor $(\partial_{\tilde{a}_1}^{s_l}[\log \widetilde{T}_{N,\tilde{a}_1}])^{d_1}$ is at most d_1. By identity (6.5), the degree of N in (6.6) is at most $k-r$. Summing over the permutations, we obtain that

$$\sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} \lim_{x_{a_1},\dots,x_{a_{r+1}}\to x_j}$$

$$\operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0}(\partial_{\tilde{a}_1}^{s_1}[\log \widetilde{T}_{N,\tilde{a}_1}])^{d_1}\cdots(\partial_{\tilde{a}_1}^{s_t}[\log \widetilde{T}_{N,\tilde{a}_1}])^{d_t}}{(x_{a_1}-x_{a_2})\cdots(x_{a_1}-x_{a_{r+1}})}$$

is the sum of $O(N^{r+1})$ terms, the degree of N in each of which is at most $k-r$. So, the degree of N in $E_{k,\kappa,N}^{(j)}$ is at most $k+1$, and we complete the proof of part (1).

Now we prove parts (2) and (3). We consider the following two cases.

- Assume $i\in R(t)$. Consider

$$\mathcal{D}_i := \frac{\partial F_{k,\kappa,N}^{(j)}}{\partial u_i}\Big|_{U_{N,\kappa}=(1,\dots,1)} = \frac{\partial}{\partial u_i}\Bigg[\sum_{\substack{r\geq 0,\,\{s_i\geq 0\}_{i=0}^t,\{d_i\geq 0\}_{i=1}^t:\\ s_0+s_1d_1+\dots+s_tdt+r=k}}$$

$$\sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} (r+1)! \lim_{\substack{x_{a_w}\to x_j,\\ 1\leq w\leq r+1}}$$

$$\operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0}(\partial_{\tilde{a}_1}^{s_1}[\log T_{N,(\tilde{a}_1,i)}^{(j,t)}])^{d_1}\cdots(\partial_{\tilde{a}_1}^{s_t}[\log T_{N,(\tilde{a}_1,i)}^{(j,t)}])^{d_t}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})}\Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\dots,1)}}$$

with $i\notin\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}$. When $\kappa\in(0,1)$, there are $O(N^{r+1})$ such terms. If $i\in R(t)$, then

$$\begin{aligned}
&\frac{\partial}{\partial u_i}[\partial_{\tilde{a}_1}^{s_q}(\log T_{N,(\tilde{a}_1,i)}^{(j,t)})]^{d_q}\Big|_{U_{N,\kappa}=(1,\dots,1)}\\
&\quad = d_q[\partial_{\tilde{a}_1}^{s_q}(\log T_{N,(\tilde{a}_1,i)}^{(j,t)})]^{d_q-1}\frac{\partial}{\partial u_i}[\partial_{\tilde{a}_1}^{s_q}(\log T_{N,(\tilde{a}_1,i)}^{(j,t)})]\Big|_{U_{N,\kappa}=(1,\dots,1)}\\
&\quad = d_q[\partial_{\tilde{a}_1}^{s_q}(\log \widetilde{T}_{N,\tilde{a}_1}^{(j,t)})]^{d_q-1}\frac{\partial}{\partial u_i}[\partial_{\tilde{a}_1}^{s_q}(\log T_{N,(\tilde{a}_1,i)}^{(j,t)})]\Big|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

By Lemmas 6.2 and 6.3, the degree of N in the expressions above is at most d_q-1 if $r=j$, and is strictly less than d_q-1 when $r\neq j$. Note that $\mathcal{D}_i$ can be written as a sum of terms of the form

$$\lim_{\substack{x_{a_1},\dots,\\ x_{a_{r+1}}\to x_j}} \operatorname*{Sym}_{\substack{a_1,\dots,\\ a_{r+1}}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0}(\partial_i[(\partial_{\tilde{a}_1}^{s_1}[\log T_{N,(\tilde{a}_1,i)}^{(j,t)}])^{d_1}])\cdots(\partial_{\tilde{a}_1}^{s_t}[\log \widetilde{T}_{N,\tilde{a}_1}^{(j,t)}])^{d_t}}{(x_{a_1}-x_{a_2})\cdots(x_{a_1}-x_{a_{r+1}})}\Big|_{\substack{U_{N,\kappa}\\=(1,\dots,1)}}.$$

By Lemmas 6.2 and 6.3, the degree of N in the expressions above is at most $d_1 + \cdots + d_t - 1$ when $r = j$, and is less than $d_1 + \cdots + d_t - 1$ when $r \neq j$. Since $r + s_0 + s_1 d_1 + \cdots + s_t d_t = k$, we have

$$d_1 + \cdots + d_t - 1 \leq k - r - 1,$$

and in the sum over $\{a_1 - N + \lfloor(1-\kappa)N\rfloor, \ldots, a_{r+1} - N + \lfloor(1-\kappa)N\rfloor\} \in \{1, 2, \ldots, \lfloor(1-\kappa)N\rfloor\} \cap R(j)$, it is the sum of $O(N^{r+1})$ of such terms, we obtain that the degree of N in this sum is at most k when $r = j$; and is less than k when $r \neq j$. This completes the proof of part (3).

- Consider the case where

$$\begin{aligned} i \in \{&a_1 - N + \lfloor(1-\kappa)N\rfloor, \ldots, a_{r+1} - N + \lfloor(1-\kappa)N\rfloor\} \\ &\subset \{1, 2, \ldots, \lfloor(1-\kappa)N\rfloor\}. \end{aligned}$$

Note that there are $O(N^r)$ such terms in total since i is fixed. By Lemmas 6.2 and 6.3, the degree of N in

$$\operatorname*{Sym}_{a_1,\ldots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0} (\partial_i [(\partial_{\tilde{a}_1}^{s_1}[\log T_{N,(\tilde{a}_1,i)}^{(j,t)}])^{d_1}]) \cdots (\partial_{\tilde{a}_1}^{s_t}[\log \widetilde{T}_{N,\tilde{a}_1}^{(j,t)}])^{d_t}}{(x_{a_1} - x_{a_2}) \cdots (x_{a_1} - x_{a_{r+1}})} \Big|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}}$$

is at most $l - r$. This completes the proof of part (2).

Now we prove part (4). Note that

$$\begin{aligned}
&\frac{\partial^2 F_{k,\kappa,N}^{(j)}}{\partial u_{i_1} \partial u_{i_2}}\Big|_{U_{N,\kappa}=(1,\ldots,1)} \\
&= \frac{\partial^2}{\partial u_{i_1} \partial u_{i_2}} \Bigg[\lim_{\substack{x_k \to (x_{k \bmod n}) \\ \text{for } 1 \leq k \leq N}} \frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa}, X)\widetilde{T}_N} \\
&\quad \times \sum_{\substack{i \in \{1,\ldots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(j)}} \left(u_i \frac{\partial}{\partial u_i}\right)^k \widetilde{V}_{\lfloor(1-\kappa)N\rfloor}(U_{N,\kappa}, X)\widetilde{T}_N \Bigg]\Bigg|_{U_{N,\kappa}=(1,\ldots,1)} \\
&= \frac{\partial^2}{\partial u_{i_1} \partial u_{i_2}} \Bigg[\sum_{\substack{r \geq 0, \{s_i \geq 0\}_{i=0}^t, \{d_i \geq 0\}_{i=1}^t: \\ s_0 + s_1 d_1 + \ldots s_t d_t + r = k}} \sum_{\substack{\{a_1 - N + \lfloor(1-\kappa)N\rfloor, \ldots, a_{r+1} - N + \lfloor(1-\kappa)N\rfloor\} \\ \subset \{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\} \cap R(j)}} (r+1)! \\
&\quad \times \lim_{\substack{x_{a_w} \to x_j, \\ 1 \leq w \leq r+1}} \operatorname*{Sym}_{a_1,\ldots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\tilde{a}_1}^{k-s_0} (\partial_{\tilde{a}_1}^{s_1}[\log \widetilde{T}_N])^{d_1} \cdots (\partial_{\tilde{a}_1}^{s_t}[\log \widetilde{T}_N])^{d_t}}{(x_{a_1} u_{\tilde{a}_1} - x_{a_2} u_{\tilde{a}_2}) \cdots (x_{a_1} u_{\tilde{a}_1} - x_{a_{r+1}} u_{\tilde{a}_{r+1}})} \Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}}.
\end{aligned}$$

The following cases might occur:

- $\{i_1, i_2\} \cap \{\widetilde{a}_1, \ldots, \widetilde{a}_{r+1}\} = \emptyset$, and both ∂_{i_1} and ∂_{i_2} are applied to the same $\log \widetilde{T}_N$. We have

$$\begin{aligned}
&\frac{\partial^2}{\partial u_{i_1} \partial u_{i_2}} \left[\frac{\partial^{s_w} [\log \widetilde{T}_N]}{[\partial u_{\widetilde{a}_1}]^{s_w}} \right]^{d_w} \\
&= d_w(d_w - 1) \left[\frac{\partial^{s_w} [\log \widetilde{T}_N]}{\partial u_{\widetilde{a}_1}^{s_w}} \right]^{d_w - 2} \frac{\partial^{s_w+1} [\log \widetilde{T}_N]}{\partial u_{i_1} \partial u_{\widetilde{a}_1}^{s_w}} \frac{\partial^{s_w+1} [\log \widetilde{T}_N]}{\partial u_{i_2} \partial u_{\widetilde{a}_1}^{s_w}} \\
&\quad + d_w \left[\frac{\partial^{s_w} [\log \widetilde{T}_N]}{\partial u_{\widetilde{a}_1}^{s_w}} \right]^{d_w - 1} \frac{\partial^{s_w+2} [\log \widetilde{T}_N]}{\partial u_{i_1} \partial u_{i_2} \partial u_{\widetilde{a}_1}^{s_w}}.
\end{aligned}$$

  By Lemma 6.4, the degree of N in the expression above is less than $d_w - 1$. Taking into account all the other factors, as well as the sum of $O(N^{r+1})$ terms, in this case the degree of N in $(\partial^2 F_{k,\kappa,N}^{(j)} / \partial u_{i_1} \partial u_{i_2})|_{U_{N,\kappa}=(1,\ldots,1)}$ is less than

$$d_1 + d_2 + \cdots + d_t - 1 + r + 1 \leq k.$$

- $\{i_1, i_2\} \cap \{\widetilde{a}_1, \ldots, \widetilde{a}_{r+1}\} = \emptyset$, and ∂_{i_1} and ∂_{i_2} are applied to different $\log \widetilde{T}_N$. In this case, the degree of N is at most

$$d_1 + d_2 + \cdots + d_t - 2 + r + 1 \leq k - 1.$$

- $i_1 \in \{a_1, \ldots, a_{r+1}\}$ and $i_2 \notin \{a_1, \ldots, a_{r+1}\}$. In this case, we take the sum over $O(N^r)$ terms since one element in $\{a_1, \ldots, a_{r+1}\}$ is fixed to be i_1. Then the degree of N in $(\partial^2 F_{k,\kappa,N}^{(j)} / \partial u_{i_1} \partial u_{i_2})|_{U_{N,\kappa}=(1,\ldots,1)}$ is at most

$$d_1 + d_2 + \cdots + d_t - 1 + r \leq k - 1.$$

- $\{i_1, i_2\} \subset \{a_1, \ldots, a_{r+1}\}$. In this case, we take the sum over $O(N^{r-1})$ terms since two elements in $\{a_1, \ldots, a_{r+1}\}$ are fixed to be i_1 and i_2. Then the degree of N in $(\partial^2 F_{k,\kappa,N}^{(j)} / \partial u_{i_1} \partial u_{i_2})|_{U_{N,\kappa}=(1,\ldots,1)}$ is at most

$$d_1 + d_2 + \cdots + d_t + r - 1 \leq k - 1.$$

Then the lemma follows. ■

6.4. Covariance

Let

$$\begin{aligned}
\widetilde{G}_{\kappa,N,(k,l)}^{(j,s)} = k \sum_{r=0}^{k-1} \binom{k-1}{r} \sum_{\substack{\{a_1 - N + \lfloor (1-\kappa)N \rfloor, \ldots, a_{r+1} - N + \lfloor (1-\kappa)N \rfloor\} \\ \in \{1,2,\ldots,\lfloor (1-\kappa)N \rfloor\} \cap R(j)}} (r+1)! \\
\times \lim_{\substack{x_{a_1}, \ldots, \\ x_{a_{r+1}} \to x_j}} \operatorname{Sym}_{a_1,\ldots,a_{r+1}} \frac{x_{a_1}^r u_{\widetilde{a}_1}^{k-s_0} \partial_{\widetilde{a}_1} [F_{l,\kappa,N}^{(s)}] (\partial_{\widetilde{a}_1} [\log \widetilde{T}_N])^{k-1-r}}{(x_{a_1} u_{\widetilde{a}_1} - x_{a_2} u_{\widetilde{a}_2}) \cdots (x_{a_1} u_{\widetilde{a}_1} - x_{a_{r+1}} u_{\widetilde{a}_{r+1}})}.
\end{aligned}$$

Lemma 6.6. *Let l, k be arbitrary positive integers. Then*

$$E_{k,l,\kappa,N}^{(j,s)} := E_{k,\kappa,N}^{(j)} E_{l,\kappa,N}^{(s)} + \widetilde{G}_{\kappa,N,(l,k)}^{(j,s)}\big|_{U_{N,\kappa}=(1,\dots,1)} + R,$$

where

- *if $j = s$, the degree of N in $\widetilde{G}_{\kappa,N,(l,k)}^{(j,s)}|_{U_{N,\kappa}=(1,\dots,1)}$ is at most $l+k$;*
- *if $j \neq s$, the degree of N in $\widetilde{G}_{\kappa,N,(l,k)}^{(j,s)}|_{U_{N,\kappa}=(1,\dots,1)}$ is less than $l+k$;*
- *the degree of N in R is less than $l+k$.*

Proof. By (6.2) and (6.3), we obtain

$$\begin{aligned}
E_{k,l,\kappa,N}^{(j,s)} &= \lim_{\substack{x_k \to (x_{k \bmod n}) \\ \text{for } 1\le k\le N}} \sum_{\substack{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(j)}} \Big(u_i \frac{\partial}{\partial u_i}\Big)^k \\
&\quad \sum_{\substack{r\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(s)}} \Big(u_r \frac{\partial}{\partial u_r}\Big)^l \frac{V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}}{W_{N,(i,r)}^{(j,s)} P_{N,(i,r)}^{(j,s)}}\Big|_{U_{N,\kappa}=(1,\dots,1)} \\
&= \frac{1}{V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}} \lim_{\substack{x_k \to (x_{k \bmod n}) \\ \text{for } 1\le k\le N}} \sum_{\substack{i\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(j)}} \Big(u_i \frac{\partial}{\partial u_i}\Big)^k V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)} \\
&\quad \times \frac{1}{V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}} \sum_{\substack{r\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(s)}} \Big(u_r \frac{\partial}{\partial u_r}\Big)^l V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}\Big|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

Note that for any integer w, we have

$$\begin{aligned}
&\frac{\partial^w}{\partial u_i^w}\Big[\frac{1}{V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}} \sum_{\substack{r\in\{1,\dots,\lfloor(1-\kappa)N\rfloor\} \\ \cap R(s)}} \Big(u_r \frac{\partial}{\partial u_r}\Big)^l V_{N,(i,r)}^{(j,s)} T_{N,(i,r)}^{(j,s)}\Big]\Big|_{U_{N,\kappa}=(1,\dots,1)} \\
&\quad = \frac{\partial^w}{\partial u_i^w} F_{l,\kappa,N}^{(s)}\Big|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

Then $E_{k,l,\kappa,N}^{(j,s)}$ can be written as a sum of the following terms:

$$\begin{aligned}
&\operatorname*{Sym}_{\substack{a_1,\dots,\\ a_{r+1}}} \frac{c_0 x_{a_1}^r u_{\widetilde{a}_1}^{k-s_0} \partial_{\widetilde{a}_1}^{s_1}[F_{l,\kappa,N}^{(s)}][\partial_{\widetilde{a}_1}^{s_2}(\log T_{N,(\widetilde{a}_1,r)}^{(j,s)})]^{d_2}\cdots[\partial_{\widetilde{a}_1}^{s_t}(\log T_{N,(\widetilde{a}_1,r)}^{(j,s)})]^{d_t}}{(x_{a_1}u_{\widetilde{a}_1}-x_{a_2}u_{\widetilde{a}_2})\cdots(x_{a_1}u_{\widetilde{a}_1}-x_{a_{r+1}}u_{\widetilde{a}_{r+1}})}\Big|_{\substack{U_{N,\kappa}\\ =(1,\dots,1)}} \\
&= \operatorname*{Sym}_{a_1,\dots,a_{r+1}} \frac{c_0 x_{a_1}^r u_{\widetilde{a}_1}^{k-s_0} \partial_{\widetilde{a}_1}^{s_1}[F_{l,\kappa,N}^{(s)}][\partial_{\widetilde{a}_1}^{s_2}(\log \widetilde{T}_N)]^{d_2}\cdots[\partial_{\widetilde{a}_1}^{s_t}(\log \widetilde{T}_N)^{(j,s)}]^{d_t}}{(x_{a_1}u_{\widetilde{a}_1}-x_{a_2}u_{\widetilde{a}_2})\cdots(x_{a_1}u_{\widetilde{a}_1}-x_{a_{r+1}}u_{\widetilde{a}_{r+1}})}\Big|_{\substack{U_{N,\kappa}\\ =(1,\dots,1)}}
\end{aligned}$$

such that

- $r, s_0, s_1, \ldots, s_t, d_2, \ldots, d_t$ are non-negative integers;
- $s_2 < s_3 < \cdots < s_t$;
- we have

$$s_0 + s_1 + s_2 d_2 + \cdots + s_t d_t + r = k; \tag{6.7}$$

- $\{a_1, \ldots, a_{r+1}\} \subset \{N - \lfloor (1-\kappa)N \rfloor + 1, N - \lfloor (1-\kappa)N \rfloor + 2, \ldots, N\} \cap R(j)$.

When $s_1 = 0$, we obtain $F^{(s)}_{l,\kappa,N} F^{(j)}_{k,\kappa,N}|_{U_{N,\kappa}=(1,\ldots,1)}$.

Now we consider the terms corresponding to $s_1 \geq 1$. By Lemma 6.5, the degree of N in $\partial^{s_1}_{\tilde{a}_1}[F^{(s)}_{l,\kappa,N}]$ is at most l when $j = s$ and is less than l when $j \neq s$. Therefore, the total degree of N in these terms is at most $l + d_2 + \cdots + d_t + r + 1$ when $j = s$; and is less than $l + d_2 + \cdots + d_t + r + 1$ when $j \neq s$. By (6.7) and the assumption that $s_1 \geq 1$, we have

$$l + d_2 + \cdots + d_t + r + 1 \leq l + k,$$

and the equality holds when $s_0 = d_3 = \cdots = d_t = 0$, $s_1 = s_2 = 1$, $d_2 = k - 1 - r$; this corresponds to $G^{(j,s)}_{\kappa,N,(k,l)}$, in which the degree of N is $l + k$ when $j = s$, and the degree of N is less than $l + k$ when $j \neq s$. The degree of N is less than $l + k$ in all the other terms. This completes the proof. ∎

Lemma 6.7. *We have*

$$\frac{1}{\widetilde{V}_{\lfloor (1-\kappa)N \rfloor} \widetilde{T}_N} k \sum_{\substack{i \in \{1,2,\ldots,\lfloor (1-\kappa)N \rfloor\} \\ \cap R(j)}} \left(u_i \frac{\partial}{\partial u_i} [F^{(s)}_{l,\kappa,N}] \right) \left(u_i \frac{\partial}{\partial u_i} \right)^{k-1} [\widetilde{V}_{\lfloor (1-\kappa)N \rfloor} \widetilde{T}_N] \Big|_{\substack{U_{N,\kappa} \\ =(1,\ldots,1)}}$$
$$= \widetilde{G}^{(j,s)}_{\kappa,N,(k,l)} \Big|_{U_{N,\kappa}=(1,\ldots,1)} + R,$$

where the degree of N in R is less than $l + k$.

Proof. This follows from the proof of Lemma 6.6. ∎

Lemma 6.8. *Let $i \in \{1, 2, \ldots, \lfloor (1-\kappa)N \rfloor\}$. Then the degree of N in*

$$\frac{\partial}{\partial u_i} \widetilde{G}^{(j,s)}_{\kappa,N,(k,l)} \Big|_{U_{N,\kappa}=(1,\ldots,1)}$$

is less than $k + l$.

Proof. Note that $\widetilde{G}^{(j,s)}_{\kappa,N,(k,l)}$ is the sum of terms

$$\operatorname*{Sym}_{a_1,\ldots,a_{r+1}} \frac{x^r_{a_1} u^{k-s_0}_{\tilde{a}_1} \partial_{\tilde{a}_1}[F^{(s)}_{l,\kappa,N}] (\partial_{\tilde{a}_1}[\log \widetilde{T}_N])^{k-1-r}}{(x_{a_1} u_{\tilde{a}_1} - x_{a_2} u_{\tilde{a}_2}) \cdots (x_{a_1} u_{\tilde{a}_1} - x_{a_{r+1}} u_{\tilde{a}_{r+1}})}.$$

If we take derivatives $\frac{\partial}{\partial u_i}$, the following cases might occur:

(1) $i \in \{a_1, \dots, a_{r+1}\}$. Since one element in $\{a_1, \dots, a_{r+1}\}$ is fixed to be i, we take the sum over $O(N^r)$ terms. By Lemmas 6.4 and 6.5, the degree of N is at most

$$l + (k - 1 - r) + r = l + k - 1;$$

(2) $i \notin \{a_1, \dots, a_{r+1}\}$. In this case, we take the sum over $O(N^{r+1})$ terms. Again by Lemmas 6.4 and 6.5, the degree of N is less than

$$l + k - 1 - r + r + 1 = l + k.$$

Then the lemma follows. ∎

6.5. Products of moments

Recall that $\mathcal{P}^s_{w_1,\dots,w_p}$ is the set of all pairings of the set $\{1, 2, \dots, s\} \setminus \{w_1, \dots, w_p\}$. We have the following lemma concerning the products of moments.

Lemma 6.9. *Let $s, l_1, \dots, l_s$ be positive integers, and let*

$$j_1, \dots, j_s \in \{1, 2, \dots, n\}.$$

Then

$$\begin{aligned}
&\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \left(u_{i_1}\frac{\partial}{\partial u_{i_1}}\right)^{l_1} \sum_{\substack{i_2\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_2)}} \left(u_{i_2}\frac{\partial}{\partial u_{i_2}}\right)^{l_2} \cdots \\
&\quad\times \sum_{\substack{i_s\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_s)}} \left(u_{i_s}\frac{\partial}{\partial u_{i_s}}\right)^{l_s} [\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]\Big|_{U_{N,\kappa}=(1,\dots,1)} \\
&\quad= \sum_{p=0}^{s} \sum_{\substack{w_1<\dots<w_p\\ \in\{1,2,\dots,s\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N} F^{(j_{w_2})}_{l_{w_2},\kappa,N} \cdots F^{(j_{w_p})}_{l_{w_s},\kappa,N} \\
&\quad\quad\times \Bigg(\sum_{P\in\mathcal{P}^s_{w_1,\dots,w_p}} \prod_{(a,b)\in P} \widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R\Bigg)\Bigg|_{U_{N,\kappa}=(1,\dots,1)},
\end{aligned}$$

where the degree of N in R is less than $\sum_{i=1}^{s} l_i - \sum_{i=1}^{p} l_{w_i}$.

Proof. The lemma can be proved by induction on s, similar to the proof of [7, Proposition 5.10]. We shall now sketch the proof. When $s = 1$, the lemma follows from the definition of $F^{(j)}_{l,\kappa,N}$. When $s = 2$, the lemma follows from Lemma 6.6. Assume that the lemma holds for $s = t - 1$, where $t \geq 2$ is a positive integer. When $s = t$, by induction hypothesis, we have

$$\begin{aligned}
&\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)^{l_1} \sum_{\substack{i_2\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_2)}} \Big(u_{i_2}\frac{\partial}{\partial u_{i_2}}\Big)^{l_2}\cdots\\
&\quad\times \sum_{\substack{i_t\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_t)}} \Big(u_{i_t}\frac{\partial}{\partial u_{i_t}}\Big)^{l_t}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]\Big|_{U_{N,\kappa}=(1,\ldots,1)}\\
&\quad= \frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)^{l_1}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]\\
&\quad\times\Bigg(\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\ldots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_p},\kappa,N}\\
&\quad\times\Big(\sum_{P\in\mathcal{P}^t_{1,w_1,\ldots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\ldots,w_p}\Big)\Bigg)\Big|_{U_{N,\kappa}=(1,\ldots,1)}\\
&\quad= S_1 + S_2 + S_3,
\end{aligned}$$

where by induction hypothesis the degree of N in $R_{1,w_1,\ldots,w_p}$ is less than $\sum_{i=2}^{s} l_i - \sum_{i=1}^{p} l_{w_i}$,

$$\begin{aligned}
S_1 &= \Big\{\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)^{l_1}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]\Big\}\\
&\quad\times\Bigg(\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\ldots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_p},\kappa,N}\\
&\quad\times\Big(\sum_{P\in\mathcal{P}^t_{1,w_1,\ldots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\ldots,w_p}\Big)\Bigg)\Bigg|_{U_{N,\kappa}=(1,\ldots,1)},\\
S_2 &= \frac{l_1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \Big\{\Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)^{l_1-1}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]\Big\}\\
&\quad\times\Bigg\{\Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)\Bigg(\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\ldots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_p},\kappa,N}\\
&\quad\times\Big(\sum_{P\in\mathcal{P}^t_{1,w_1,\ldots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\ldots,w_p}\Big)\Bigg)\Bigg\}\Bigg|_{U_{N,\kappa}=(1,\ldots,1)}.
\end{aligned}$$

Indeed, S_1 corresponds to the terms where all the differentiations $\frac{\partial}{\partial u_{i_1}}$ are applied to $\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N$ or u_{i_1}, and S_2 corresponds to the terms where all the differentiations $\frac{\partial}{\partial u_{i_1}}$ except one are applied to $\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N$ or u_{i_1}, and S_3 are all the other terms.

By the definition of $F^{(j)}_{l,\kappa,N}$, we have

$$S_1 = F^{(j_1)}_{l_1,\kappa,N}\Bigg(\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\dots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_s})}_{l_{w_s},\kappa,N} \times\Bigg(\sum_{P\in\mathcal{P}^s_{1,w_1,\dots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\dots,w_p}\Bigg)\Bigg)\Bigg|_{U_{N,\kappa}=(1,\dots,1)}.$$

By Lemma 6.7, we have

$$S_2 = \Bigg\{\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\dots,t\}}}\sum_{x=1}^{p} F^{(j_{w_1})}_{l_{w_1},\kappa,N}\cdots F^{(j_{w_{x-1}})}_{l_{w_{x-1}},\kappa,N}F^{(j_{w_{x+1}})}_{l_{w_{x+1}},\kappa,N}\cdots F^{(j_{w_s})}_{l_{w_s},\kappa,N} \times\Big[(\widetilde{G}^{(j_1,j_x)}_{\kappa,N,(l_1,l_x)} + R_{1,x}) \times\Bigg(\sum_{P\in\mathcal{P}^s_{1,w_1,\dots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\dots,w_p}\Bigg)\Big]\Bigg\}\Bigg|_{U_{N,\kappa}=(1,\dots,1)},$$

where the degree of N in $R_{1,x}$ is less than l_1+l_x.

Hence we have

$$S_2 = \Bigg\{\sum_{p=0}^{t-1}\sum_{\substack{w_1<\cdots<w_p\\ \in\{2,\dots,t\}}}\sum_{x=1}^{p} F^{(j_{w_1})}_{l_{w_1},\kappa,N}\cdots F^{(j_{w_{x-1}})}_{l_{w_{x-1}},\kappa,N}F^{(j_{w_{x+1}})}_{l_{w_{x+1}},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_s},\kappa,N} \times\Bigg(\widetilde{G}^{(j_1,j_x)}_{\kappa,N,(l_1,l_x)}\sum_{P\in\mathcal{P}^t_{1,w_1,\dots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{w_1,\dots,\widehat{w}_x,\dots,w_p}\Bigg)\Bigg\}\Bigg|_{\substack{U_{N,\kappa}\\=(1,\dots,1)}},$$

where the degree of N in $R_{w_1,\dots,\widehat{w}_x,\dots,w_p}$ is less than $\sum_{i=2}^{t} l_i - \sum_{i=1}^{p} l_{w_i}$.

Note that

$$S_1+S_2 = \sum_{p=0}^{t}\sum_{\substack{w_1<\cdots<w_p\\ \in\{1,2,\dots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_p},\kappa,N} \times\Bigg(\sum_{P\in\mathcal{P}^t_{w_1,\dots,w_p}}\prod_{(a,b)\in P}\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{w_1,\dots,w_p}\Bigg)\Bigg|_{U_{N,\kappa}=(1,\dots,1)},$$

where the degree of N in $R_{w_1,\dots,w_p}$ is less than $\sum_{i=1}^{t} l_i - \sum_{i=1}^{p} l_{w_i}$.

It remains to show that S_3 does not contribute to the leading terms. Define

$$\widetilde{H}_{j_1,\dots,j_p} = \sum_{P \in \mathcal{P}^t_{1,w_1,\dots,w_p}} \prod_{(a,b)\in P} \widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R_{1,w_1,\dots,w_p}.$$

In view of Lemma 6.6, the degree of N in $\widetilde{H}_{j_1,\dots,j_p}|_{U_{N,\kappa}=(1,\dots,1)}$ is at most $\sum_{i=2}^t l_i - \sum_{j=1}^p l_{w_j}$. Moreover, according to Lemma 6.8, for any index i, the degree of N in $\frac{\partial}{\partial u_i}\widetilde{H}_{j_1,\dots,j_p}|_{U_{N,\kappa}=(1,\dots,1)}$ is less than $\sum_{i=2}^t l_i - \sum_{j=1}^p l_{w_j}$.

We write

$$\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N} \sum_{\substack{i_1\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \left(u_{i_1}\frac{\partial}{\partial u_{i_1}}\right)^{l_1}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N]$$
$$\times\left(\sum_{p=0}^{t-1}\sum_{\substack{w_1<\dots<w_p\\ \in\{2,\dots,t\}}} F^{(j_{w_1})}_{l_{w_1},\kappa,N}F^{(j_{w_2})}_{l_{w_2},\kappa,N}\cdots F^{(j_{w_p})}_{l_{w_p},\kappa,N}\widetilde{H}_{j_1,\dots,j_p}\right)\Bigg|_{U_{N,\kappa}=(1,\dots,1)}$$

as a sum of terms of the form

$$\lim_{x_{a_1},\dots,x_{a_{r+1}}\to x_{j_1}} \operatorname*{Sym}_{a_1,\dots,a_{r+1}} \tag{6.8}$$
$$\frac{x^r_{a_1}u^{l_1-s_0}_{\tilde{a}_1}(\partial^{s_1}_{\tilde{a}_1}[\log\widetilde{T}_N])^{d_1}\cdots(\partial^{s_t}_{\tilde{a}_t}[\log\widetilde{T}_N])^{d_t}\partial^{f_1}_{\tilde{a}_1}F^{(j_{w_1})}_{l_{w_1},\kappa,N}\cdots\partial^{f_p}_{\tilde{a}_1}F^{(j_{w_p})}_{l_{w_p},\kappa,N}\partial^{h_0}_{\tilde{a}_1}\widetilde{H}_{j_1,\dots,j_p}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})},$$

where

- $\{\tilde{a}_1,\dots,\tilde{a}_{r+1}\}\subset\{1,2,\dots,\lfloor(1-\kappa)N\}\cap R(j_1)$;
- $s_1<s_2<\dots<s_t$ are positive integers;
- $f_1,\dots,f_p,h_0$ are non-negative integers;
- $r+s_0+s_1d_1+\dots+s_td_t+f_1+\dots+f_p+h_0=l_1$.

By Lemma 6.2, the degree of N in $(\partial^{s_1}_{\tilde{a}_1}[\log\widetilde{T}_N])^{d_1}\cdots(\partial^{s_t}_{\tilde{a}_t}[\log\widetilde{T}_N])^{d_t}$ is at most $d_1+\dots+d_t$; therefore, the terms in (6.8) with highest degree of N has the form

$$\operatorname*{Sym}_{a_1,\dots,a_{r+1}}\frac{x^r_{a_1}u^{l_1}_{\tilde{a}_1}(\partial_{\tilde{a}_1}[\log\widetilde{T}_N])^{d_1}\partial^{f_1}_{\tilde{a}_1}F^{(j_{w_1})}_{l_{w_1},\kappa,N}\cdots\partial^{f_p}_{\tilde{a}_1}F^{(j_{w_p})}_{l_{w_p},\kappa,N}\partial^{h_0}_{\tilde{a}_1}\widetilde{H}_{j_1,\dots,j_p}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})}, \tag{6.9}$$

where

$$s_0=d_2=\dots=d_t=0,\quad s_1=1. \tag{6.10}$$

Let

$$B=\{i\in\{1,2,\dots,p\}: f_i=0\}.$$

Then

$$(6.9) = \Big(\prod_{i\in B} F^{(j_{w_i})}_{l_{w_i},\kappa,N}\Big) S(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor}),$$

where $S(u_1,\dots,u_{\lfloor 1-\kappa N\rfloor})$ is a symmetric function. It suffices to show that the degree of N in S, except for S_1 and S_2, is less than $\sum_{i=1}^{s} l_i - \sum_{i\in B} l_i$. Note that the degree of N in $\partial_{\widetilde{a}_1}([\log \widetilde{T}_N])^{d_1}$ is at most d_1 by Lemma 6.2. The summation over $\{\widetilde{a}_1,\dots,\widetilde{a}_{r+1}\} \subset \{1,2,\dots,\lfloor(1-\kappa)N\rfloor\} \cap R(j_1)$ gives $O(N^{r+1})$ terms. By Lemma 6.5, when $i \notin B$, the degree of N in $\partial^{f_i}_{\widetilde{a}_1} F^{(j_{w_i})}_{l_{w_i},\kappa,N}$ is at most l_{w_i}. Therefore, the degree of N in $S(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})$ is at most

$$\sum_{i=2}^{t} l_i - \sum_{i=1}^{p} l_{w_i} + d_1 + \sum_{i\in\{1,2,\dots,p\}\setminus B} l_{w_i} + r + 1.$$

By (6.5) and (6.10), if $|B| \le p-2$, $r_1 + d_1 + 1 \le l_1 - 1$, then the degree of N in $S(u_1,\dots,u_{\lfloor(1-\kappa)N\rfloor})$ is at most

$$\sum_{i=1}^{t} l_i - \sum_{i\in B} l_i - 1.$$

Therefore, only the terms where at most one f_i is nonzero contribute to the leading order. In these terms, if $h_0 > 0$, then by Lemma 6.8, the degree of N is less than $\sum_{i=1}^{s} l_i - \sum_{i\in B} l_i$. So only the terms where $h_0 = 0$ and at most one f_i is nonzero contribute to the leading order. These terms are in S_1 and S_2. Then the proof is complete. ∎

Lemma 6.10. *Let $s, l_1,\dots,l_s$ be positive integers, and let $j_1,\dots,j_s \in \{1,2,\dots,n\}$. Then*

$$\begin{aligned}
&\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N}\Big[\sum_{\substack{i_1\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \Big(u_{i_1}\frac{\partial}{\partial u_{i_1}}\Big)^{l_1} - E^{(j_1)}_{l_1,\kappa,N}\Big]\cdots\\
&\quad\times\Big[\sum_{\substack{i_s\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_s)}} \Big(u_{i_s}\frac{\partial}{\partial u_{i_s}}\Big)^{l_s}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N] - E^{(j_s)}_{l_s,\kappa,N}\Big]\Big|_{U_{N,\kappa}=(1,\dots,1)}\\
&\quad= \sum_{P\in\mathscr{P}^s_{\emptyset}} \prod_{(a,b)\in P} \widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)} + R|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

Proof. The lemma follows from Lemma 6.9 by explicit computations. See also the proof of [7, Lemma 5.11]. ∎

Lemma 6.11. *Let $s, l_1, \dots, l_s$ be positive integers, and let $j_1, \dots, j_s \in \{1, 2, \dots, n\}$. Let $\widetilde{\mathcal{P}}_\emptyset^s \subset \mathcal{P}_\emptyset^s$ consisting of all the pairings of $\{1, 2, \dots, s\}$ such that in each pair (a, b) in the pairing, $j_a = j_b$. Then*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^{l_1+\cdots+l_s}} \frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N}\Bigg[\sum_{\substack{i_1\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}} \left(u_{i_1}\frac{\partial}{\partial u_{i_1}}\right)^{l_1} - E^{(j_1)}_{l_1,\kappa,N}\Bigg]\cdots \\
&\quad\times\Bigg[\sum_{\substack{i_s\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_s)}} \left(u_{i_s}\frac{\partial}{\partial u_{i_s}}\right)^{l_s}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N] - E^{(j_s)}_{l_s,\kappa,N}\Bigg]\Bigg|_{U_{N,\kappa}=(1,\dots,1)} \\
&\quad= \lim_{N\to\infty}\frac{1}{N^{l_1+\cdots+l_s}}\sum_{P\in\widetilde{\mathcal{P}}_\emptyset^s}\prod_{(a,b)\in P}[\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)}|_{U_{N,\kappa}=(1,\dots,1)}\mathbf{1}_{j_a=j_b}],
\end{aligned}$$

where the degree of N in R is less than $\sum_{i=1}^s l_i$.

Proof. The lemma follows from Lemma 6.10 and the fact that the degree of N in R therein is less than $l_1 + \cdots + l_s$, and that the degree of N in $\widetilde{G}^{(j_a,j_b)}_{\kappa,N,(l_a,l_b)}$ is less than $l_a + l_b$ if $j_a \neq j_b$. ∎

6.6. Integral formula for covariance

Assume that $\kappa \in (0, 1)$ and k is a positive integer. Let

$$p_k^{(\lfloor(1-\kappa)N\rfloor)} = \sum_{i=1}^{\lfloor(1-\kappa)N\rfloor}(\lambda_i + \lfloor(1-\kappa)N\rfloor - i)^k,$$

where

$$\lambda = (\lambda_1, \dots, \lambda_{\lfloor(1-\kappa)N\rfloor}) \in \mathbb{GT}_{\lfloor(1-\kappa)N\rfloor}$$

has the distribution $\rho_{\lfloor(1-\kappa)N\rfloor}$ as defined in Lemma 3.4. Explicit computations show that

$$\begin{aligned}
&\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)} - \mathbf{E}p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)} - \mathbf{E}p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)})\cdots \\
&\quad\times(p_{l_s}^{(\lfloor(1-\kappa)N\rfloor)} - \mathbf{E}p_{l_s}^{(\lfloor(1-\kappa)N\rfloor)}) \\
&\quad= \sum_{j_1,\dots,j_s\in\{1,2,\dots,n\}}\frac{1}{\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N}\Bigg[\sum_{\substack{i_1\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_1)}}\left(u_{i_1}\frac{\partial}{\partial u_{i_1}}\right)^{l_1} - E^{(j_1)}_{l_1,\kappa,N}\Bigg]\cdots \\
&\quad\times\Bigg[\sum_{\substack{i_s\in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j_s)}}\left(u_{i_s}\frac{\partial}{\partial u_{i_s}}\right)^{l_s}[\widetilde{V}_{\lfloor(1-\kappa)N\rfloor}\widetilde{T}_N] - E^{(j_s)}_{l_s,\kappa,N}\Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\dots,1)}}.
\end{aligned} \tag{6.11}$$

Lemma 6.12. *We have*

$$\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)})$$
$$=\sum_{j=1}^{n}\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\widetilde{G}_{\kappa,N,(l_1,l_2)}^{(j,j)}\Big|_{U_{N,\kappa}=(1,\dots,1)}.$$

Proof. The lemma follows from (6.11) and Lemmas 6.11 and 6.6. ■

Therefore, in order to obtain an explicit integral formula for the covariance

$$\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)}),$$

it suffices to obtain an explicit integral formula for

$$\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\widetilde{G}_{\kappa,N,(l_1,l_2)}^{(j,j)}\Big|_{U_{N,\kappa}=(1,\dots,1)},$$

where $1\le j\le N$.

We have

$$\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\widetilde{G}_{\kappa,N,(l_1,l_2)}^{(j,j)}\Big|_{U_{N,\kappa}=(1,\dots,1)}$$
$$=\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}l_1\sum_{r=0}^{l_1-1}\binom{l_1-1}{r}\sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}}(r+1)!$$
$$\times\lim_{x_{a_1},\dots,x_{a_{r+1}}\to x_j}\operatorname*{Sym}_{a_1,\dots,a_{r+1}}\frac{x_{a_1}^r u_{\tilde{a}_1}^{l_1}(\partial_{\tilde{a}_1}[\log\widetilde{T}_N])^{l_1-1-r}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})}$$
$$\times\partial_{\tilde{a}_1}\Bigg[\sum_{s=0}^{l_2}\binom{l_2}{s}\sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}}(s+1)!$$
$$\times\lim_{\substack{x_{b_w}\to x_j,\\ 1\le w\le s+1}}\operatorname*{Sym}_{b_1,\dots,b_{s+1}}\frac{c_0x_{b_1}^s u_{\tilde{b}_1}^{l_2}(\partial_{\tilde{b}_1}[\log\widetilde{T}_N])^{l_2-s}}{(x_{b_1}u_{\tilde{b}_1}-x_{b_2}u_{\tilde{b}_2})\cdots(x_{b_1}u_{\tilde{b}_1}-x_{b_{s+1}}u_{\tilde{b}_{s+1}})}\Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\dots,1)}}.$$

We consider the following cases:

- If $\{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}=\emptyset$, we have

$$\partial_{\tilde{a}_1}(\partial_{\tilde{b}_1}[\log\widetilde{T}_N])^{l_2-s}=(l_2-s)(\partial_{\tilde{b}_1}[\log\widetilde{T}_N])^{l_2-s-1}\partial_{\tilde{a}_1}(\partial_{\tilde{b}_1}[\log\widetilde{T}_N]),$$

where the degree of N, when $U_{N,\kappa}=(1,\dots,1)$, is at most l_2-s-1.

Note that by Lemmas 6.2 and 6.3, we have

$$
\begin{aligned}
I_1 := & \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1} \binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,\\ a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} (r+1)! \\
& \times \lim_{x_{a_1},\ldots,x_{a_{r+1}}\to x_j} \mathrm{Sym}_{a_1,\ldots,a_{r+1}} \frac{x_{a_1}^r u_{\tilde{a}_1}^{l_1} (\partial_{\tilde{a}_1}[\log \widetilde{T}_N])^{l_1-1-r}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})} \\
& \times \partial_{\tilde{a}_1}\Bigg[\sum_{s=0}^{l_2} \binom{l_2}{s} \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j),\{a_1,\ldots,a_{r+1}\}\cap\{b_1,\ldots,b_{s+1}\}=\emptyset}} (s+1)! \\
& \times \lim_{\substack{x_{b_w}\to x_j,\\ 1\le w\le s+1}} \mathrm{Sym}_{b_1,\ldots,b_{s+1}} \frac{c_0 x_{b_1}^s u_{\tilde{b}_1}^{l_2} (\partial_{\tilde{b}_1}[\log \widetilde{T}_N])^{l_2-s}}{(x_{b_1}u_{\tilde{b}_1}-x_{b_2}u_{\tilde{b}_2})\cdots(x_{b_1}u_{\tilde{b}_1}-x_{b_{s+1}}u_{\tilde{b}_{s+1}})} \Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}} \\
= & \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1} \binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} (r+1)! \\
& \times \lim_{x_{a_1},\ldots,x_{a_{r+1}}\to x_j} \mathrm{Sym}_{a_1,\ldots,a_{r+1}} \frac{x_{a_1}^r u_{\tilde{a}_1}^{l_1} [A_j(u_{\tilde{a}_1})N]^{l_1-1-r}}{(x_{a_1}u_{\tilde{a}_1}-x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1}-x_{a_{r+1}}u_{\tilde{a}_{r+1}})} \\
& \times \Bigg[\sum_{s=0}^{l_2} \binom{l_2}{s} (l_2-s) \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\\ \cap R(j),\{a_1,\ldots,a_{r+1}\}\cap\{b_1,\ldots,b_{s+1}\}=\emptyset}} (s+1)! \\
& \times \lim_{\substack{x_{b_w}\to x_j,\\ 1\le w\le s+1}} \mathrm{Sym}_{b_1,\ldots,b_{s+1}} \frac{x_{b_1}^s u_{\tilde{b}_1}^{l_2} ([A_j(u_{\tilde{b}_1})N])^{l_2-s-1} B_j(u_{\tilde{a}_1},u_{\tilde{b}_1})}{(x_{b_1}u_{\tilde{b}_1}-x_{b_2}u_{\tilde{b}_2})\cdots(x_{b_1}u_{\tilde{b}_1}-x_{b_{s+1}}u_{\tilde{b}_{s+1}})} \Bigg]\Bigg|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}},
\end{aligned}
$$

where

$$
A_j(z) = \begin{cases} \frac{\kappa}{n}\sum_{l\in I_2\cap\{1,2,\ldots,n\}} \frac{y_l x_1}{1+y_l x_1 z} - \frac{\kappa}{n}\frac{n-1}{z} + \frac{1}{n}H'_{\mathbf{m}_i}(z) & \text{if } j=1, \\ -\frac{\kappa}{n}\frac{n-j}{z} + \frac{1}{n}H'_{\mathbf{m}_j}(z) & \text{if } 2\le j\le n \end{cases} \tag{6.12}
$$

and

$$
B_j(z,w) = \frac{\partial^2}{\partial z\partial w}\left[\log\left(1-(z-1)(w-1)\frac{zH'_{\mathbf{m}_j}(z)-wH'_{\mathbf{m}_j}(w)}{z-w}\right)\right]. \tag{6.13}
$$

By Lemma 4.3, we obtain

$$I_1 \approx \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} \sum_{r=0}^{l_1-1} \frac{l_1!}{(l_1-1-r)!r!} \times \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,\\ a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} \frac{\partial^r}{\partial z^r}[z^{l_1}[NA_j(z)]^{l_1-1-r}]\Big[\sum_{s=0}^{l_2} \frac{l_2!}{(l_2-s-1)!s!} \times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,\\ b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ \{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}=\emptyset}} \frac{\partial^s}{\partial w^s}[w^{l_2}[NA_j(w)]^{l_2-s-1}B_j(z,w)]\Big]\Big|_{(z,w)=(1,1)}.$$

By the residue theorem, we deduce that

$$\begin{aligned} I_1 &\approx \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} \sum_{r=0}^{l_1-1} \frac{l_1!}{(l_1-1-r)!} \times \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,\\ a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} \operatorname*{Res}_{z=1}\Big(\frac{z^{l_1}[NA_j(z)]^{l_1-1-r}}{(z-1)^{r+1}}\Big[\sum_{s=0}^{l_2-1}\frac{l_2!}{(l_2-s-1)!} \\ &\quad \times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,\\ b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ \{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}=\emptyset}} \operatorname*{Res}_{w=1}\Big[\frac{w^{l_2}[NA_j(w)]^{l_2-s-1}B_j(z,w)}{(w-1)^{s+1}}\Big]\Big]\Big) \\ &\approx \frac{1}{(2\pi\mathbf{i})^2}\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\oint_{|z-1|=\varepsilon}\Big(\frac{\lfloor(1-\kappa)N\rfloor}{n}\frac{z}{z-1}+zNA_j(z)\Big)^{l_1} \\ &\quad \times \oint_{|w-1|=\varepsilon}\Big(\frac{\lfloor(1-\kappa)N\rfloor}{n}\frac{w}{w-1}+wNA_j(w)\Big)^{l_2}B_j(z,w)\,dw\,dz \\ &= \frac{1}{(2\pi\mathbf{i})^2}\oint_{|z-1|=\varepsilon}\Big(\frac{1-\kappa}{n}\frac{z}{z-1}+zA_j(z)\Big)^{l_1} \\ &\quad \times \oint_{|w-1|=\varepsilon}\Big(\frac{1-\kappa}{n}\frac{w}{w-1}+wA_j(w)\Big)^{l_2}B_j(z,w)\,dw\,dz. \end{aligned}$$

- If $|\{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}|\geq 2$, then the degree of N in these terms is at most

$$l_2-s+l_1-1-r+r+1+s+1-2 = l_1+l_2-1 < l_1+l_2,$$

therefore the contribution of these terms to the limit is 0.

- If $|\{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}|=1$, then

$$\begin{aligned}
I_2 :=& \lim_{N\to\infty}\frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1}\binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} (r+1)! \\
&\times \lim_{x_{a_1},\dots,x_{a_{r+1}}\to x_j} \operatorname{Sym}_{a_1,\dots,a_{r+1}} \frac{x_{a_1}^r u_{\tilde a_1}^{l_1}(\partial_{\tilde a_1}[\log \widetilde T_N])^{l_1-1-r}}{(x_{a_1}u_{\tilde a_1}-x_{a_2}u_{\tilde a_2})\cdots(x_{a_1}u_{\tilde a_1}-x_{a_{r+1}}u_{\tilde a_{r+1}})} \\
&\times \partial_{\tilde a_1}\Bigg[\sum_{s=0}^{l_2}\binom{l_2}{s} \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ |\{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}|=1}} (s+1)! \\
&\times \lim_{\substack{x_{b_w}\to x_j,\\ 1\le w\le s+1}} \operatorname{Sym}_{b_1,\dots,b_{s+1}} \frac{c_0 x_{b_1}^s u_{\tilde b_1}^{l_2}(\partial_{\tilde b_1}[\log \widetilde T_N])^{l_2-s}}{(x_{b_1}u_{\tilde b_1}-x_{b_2}u_{\tilde b_2})\cdots(x_{b_1}u_{\tilde b_1}-x_{b_{s+1}}u_{\tilde b_{s+1}})}\Bigg]\Bigg|_{U_{N,\kappa}=(1,\dots,1)} \\
=& \lim_{N\to\infty}\frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1}\binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} (r+1)! \\
&\times \lim_{x_{a_1},\dots,x_{a_{r+1}}\to x_j} \operatorname{Sym}_{a_1,\dots,a_{r+1}} \frac{x_{a_1}^r u_{\tilde a_1}^{l_1}[A_j(u_{\tilde a_1})N]^{l_1-1-r}}{(x_{a_1}u_{\tilde a_1}-x_{a_2}u_{\tilde a_2})\cdots(x_{a_1}u_{\tilde a_1}-x_{a_{r+1}}u_{\tilde a_{r+1}})} \\
&\times \partial_{\tilde a_1}\Bigg[\sum_{s=0}^{l_2}\binom{l_2}{s} \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ |\{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}|=1}} (s+1)! \\
&\times \lim_{\substack{x_{b_w}\to x_j,\\ 1\le w\le s+1}} \operatorname{Sym}_{b_1,\dots,b_{s+1}} \frac{x_{b_1}^s u_{\tilde b_1}^{l_2}([A_j(u_{\tilde b_1})N])^{l_2-s}}{(x_{b_1}u_{\tilde b_1}-x_{b_2}u_{\tilde b_2})\cdots(x_{b_1}u_{\tilde b_1}-x_{b_{s+1}}u_{\tilde b_{s+1}})}\Bigg]\Bigg|_{U_{N,\kappa}=(1,\dots,1)}.
\end{aligned}$$

By Lemma 4.3, we deduce $I_2 := I_3 + I_4$, where

$$\begin{aligned}
I_3 :=& \lim_{N\to\infty}\frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1}\binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\dots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}} \\
&\frac{\partial^r}{\partial u_{\tilde a_1}^r}[u_{\tilde a_1}^{l_1}[A_j(u_{\tilde a_1})N]^{l_1-1-r}]\partial_{\tilde a_1}\Bigg[\sum_{s=0}^{l_2}\binom{l_2}{s} \\
&\times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\dots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\dots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ \{a_1,\dots,a_{r+1}\}\cap\{b_1,\dots,b_{s+1}\}=\{b_1\}}} \frac{\partial^s}{\partial u_{\tilde b_1}^s}(u_{\tilde b_1}^{l_2}[A_j(u_{\tilde b_1})N]^{l_2-s})\Bigg]\Bigg|_{U_{N,\kappa}=(1,\dots,1)}
\end{aligned}$$

$$= \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1} \binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}}$$

$$\frac{\partial^r}{\partial u_{\widetilde{a}_1}^r}[u_{\widetilde{a}_1}^{l_1}[A_j(u_{\widetilde{a}_1})N]^{l_1-1-r}]\Big[\sum_{s=0}^{l_2}\binom{l_2}{s}$$

$$\times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ \{a_1,\ldots,a_{r+1}\}\cap\{b_1,\ldots,b_{s+1}\}=\{b_1\}}} \frac{\partial^{s+1}}{\partial u_{\widetilde{b}_1}^{s+1}}(u_{\widetilde{b}_1}^{l_2}[A_j(u_{\widetilde{b}_1})N]^{l_2-s})\Big]\Big|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}},$$

and

$$I_4 = \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} l_1 \sum_{r=0}^{l_1-1} \binom{l_1-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}}$$

$$\frac{\partial^r}{\partial u_{\widetilde{a}_1}^r}[u_{\widetilde{a}_1}^{l_1}[A_j(u_{\widetilde{a}_1})N]^{l_1-1-r}]\partial_{\widetilde{a}_1}\Big[\sum_{s=0}^{l_2}\binom{l_2}{s}$$

$$\times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ \{a_1,\ldots,a_{r+1}\}\cap\{b_1,\ldots,b_{s+1}\}=\{b_j\},j\neq 1}} \frac{\partial^{s}}{\partial u_{\widetilde{b}_1}^{s}}(u_{\widetilde{b}_1}^{l_2}[A_j(u_{\widetilde{b}_1})N]^{l_2-s})\Big]\Big|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}}$$

$$= 0.$$

By the residue theorem, we infer

$$I_2 = \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} \sum_{r=0}^{l_1-1} \frac{l_1!}{(l_1-1-r)!} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,a_{r+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \in\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j)}}$$

$$\operatorname*{Res}_{z=u_{\widetilde{a}_1}}\Big[\frac{z^{l_1}[A_j(z)N]^{l_1-1-r}}{(z-1)^{r+1}}\Big]\Big[\sum_{s=0}^{l_2}\frac{l_2!}{(l_2-s)!}$$

$$\times \sum_{\substack{\{b_1-N+\lfloor(1-\kappa)N\rfloor,\ldots,\\ b_{s+1}-N+\lfloor(1-\kappa)N\rfloor\}\\ \subset\{1,2,\ldots,\lfloor(1-\kappa)N\rfloor\}\cap R(j),\\ |\{a_1,\ldots,a_{r+1}\}\cap\{b_1,\ldots,b_{s+1}\}|=1}} (s+1)\operatorname*{Res}_{w=u_{\widetilde{b}_1}}\Big(\frac{w^{l_2}[A_j(w)N]^{l_2-s}}{(w-1)^{s+2}}\Big)\Big]\Big|_{\substack{U_{N,\kappa}\\=(1,\ldots,1)}}$$

$$= \lim_{N\to\infty} \frac{1}{N^{l_1+l_2}} \frac{1}{(2\pi\mathbf{i})^2} \oint_{|z-1|=\varepsilon}\Big(\frac{\lfloor(1-\kappa)N\rfloor}{n}\frac{z}{z-1}+NzA_j(z)\Big)^{l_1}$$

$$\times \oint_{|w-1|=\varepsilon}\Big(\frac{\lfloor(1-\kappa)N\rfloor}{n}\frac{w}{w-1}+NzA_j(w)\Big)^{l_2}\frac{1}{(z-w)^2}\,dwdz$$

$$= \frac{1}{(2\pi\mathbf{i})^2}\oint_{|z-1|=\varepsilon}\left(\frac{(1-\kappa)}{n}\frac{z}{z-1}+zA_j(z)\right)^{l_1}$$
$$\times\oint_{|w-1|=\varepsilon}\left(\frac{(1-\kappa)}{n}\frac{w}{w-1}+wA_j(w)\right)^{l_2}\frac{1}{(z-w)^2}\,dwdz.$$

Then we have the following proposition.

Proposition 6.13. *We have*

$$\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)}-\mathbf{E}p_{l_2}^{(\lfloor(1-\kappa)N\rfloor)})$$
$$=\sum_{j=1}^{n}\frac{1}{(2\pi\mathbf{i})^2}\oint_{|z-1|=\varepsilon}\left(\frac{(1-\kappa)}{n}\frac{z}{z-1}+zA_j(z)\right)^{l_1}$$
$$\times\oint_{|w-1|=\varepsilon}\left(\frac{(1-\kappa)}{n}\frac{w}{w-1}+wA_j(w)\right)^{l_2}\left[B_j(z,w)+\frac{1}{(z-w)^2}\right]dwdz,$$

where for $1\leq j\leq n$, $A_j(z)$ and $B_j(z,w)$ are given by (6.12) *and* (6.13).

6.7. Central limit theorem in multiple levels

Let

$$1\geq\kappa_1\geq\kappa_2\geq\cdots\geq\kappa_k>0,\quad 1\leq n_1\leq n_2\leq\cdots\leq n_k\leq 2N+1,$$

such that for $1\leq i\leq k$,

$$\left\lfloor\frac{n_i}{2}\right\rfloor=\lfloor(1-\kappa_i)N\rfloor$$

and

$$U_{N,\kappa_1,\ldots,\kappa_k,X}$$
$$=(u_{1,1}x_{1+N-\lfloor(1-\kappa_1)N\rfloor,N},u_{2,1}x_{2+N-\lfloor(1-\kappa_1)N\rfloor,N},\ldots,u_{\lfloor(1-\kappa_1)N\rfloor,1}x_{N,N};$$
$$u_{1,2}x_{1+N-\lfloor(1-\kappa_2)N\rfloor,N},u_{2,2}x_{2+N-\lfloor(1-\kappa_2)N\rfloor,N},\ldots,u_{\lfloor(1-\kappa_2)N\rfloor,2}x_{N,N};\ldots;$$
$$u_{1,k}x_{1+N-\lfloor(1-\kappa_k)N\rfloor,N},u_{2,k}x_{2+N-\lfloor(1-\kappa_k)N\rfloor,N},\ldots,u_{\lfloor(1-\kappa_k)N\rfloor,k}x_{N,N}).$$

Let $\mathcal{S}_{\rho,X}(U_{N,\kappa_1,\ldots,\kappa_s,X})$ be the multidimensional Schur generating function as defined in Definition 4.11, where ρ is the joint distribution of partitions on the n_1th, n_2th, ..., n_kth row of the square-hexagon lattice, counting from the top. Then explicit computations show that

$$\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa_1)N\rfloor)}p_{l_2}^{(\lfloor(1-\kappa_2)N\rfloor)}\cdots p_{l_k}^{(\lfloor(1-\kappa_k)N\rfloor)}$$
$$=\mathcal{D}_{l_1}^{(n_1)}\mathcal{D}_{l_2}^{(n_2)}\cdots\mathcal{D}_{l_k}^{(n_k)}\mathcal{S}_{\rho,X}(U_{N,\kappa_1,\ldots,\kappa_k,X})\Big|_{(u_{1,s},\ldots,u_{\lfloor\frac{n_s}{2}\rfloor,s})=(1,\ldots,1)\ \forall 1\leq s\leq k},$$

where $\mathcal{D}_{l_i}^{(n_i)}$ is defined in (4.17).

Lemma 6.14. *Suppose the assumptions of Lemma 4.12 hold. For $1 \leq s \leq k$, let $t_s = N - \lfloor \frac{n_s}{2} \rfloor$. Let*

$$\mathbf{D}_{l_s}^{(n_s)} := \frac{1}{\widehat{V}_{\lfloor \frac{n_s}{2} \rfloor}} \left(\sum_{i=t_s+1}^{N} \left(u_i \frac{\partial}{\partial u_i} \right)^{l_s} \right) \widehat{V}_{\lfloor \frac{n_s}{2} \rfloor},$$

where $\widehat{V}_{\lfloor \frac{n_s}{2} \rfloor}$ is the Vandermonde determinant on $\lfloor \frac{n_s}{2} \rfloor$ variables $u_1 x_{t_s+1}, u_2 x_{t_s+2}, \ldots, u_{\lfloor \frac{n_s}{2} \rfloor} x_N$. Then

$$\mathcal{D}_{l_1}^{(n_1)} \mathcal{D}_{l_2}^{(n_2)} \cdots \mathcal{D}_{l_k}^{(n_k)} \mathcal{S}_{\rho,X}(U_{N,\kappa_1,\ldots,\kappa_k,X})\Big|_{(u_{1,s},\ldots,u_{\lfloor \frac{n_s}{2} \rfloor,s})=(1,\ldots,1),\ \forall 1\leq s\leq k}$$
$$= \frac{1}{\mathcal{S}\rho_{\lfloor \frac{n_1}{2} \rfloor}, X(U_{N,\kappa_1,X})} \mathbf{D}_{l_1}^{(n_1)} \frac{\mathcal{S}_{\rho_{\lfloor \frac{n_1}{2} \rfloor},X}(U_{N,\kappa_1,X})}{\mathcal{S}_{\rho_{\lfloor \frac{n_2}{2} \rfloor},X}(U_{N,\kappa_2,X})} \cdots$$
$$\times \mathbf{D}_{l_{k-1}}^{(n_{k-1})} \frac{\mathcal{S}_{\rho_{\lfloor \frac{n_{k-1}}{2} \rfloor},X}(U_{N,\kappa_{k-1},X})}{\mathcal{S}_{\rho_{\lfloor \frac{n_k}{2} \rfloor},X}(U_{N,\kappa_k,X})} \mathbf{D}_{l_k}^{(n_k)} \{\mathcal{S}_{\rho_{\lfloor \frac{n_k}{2} \rfloor},X}(U_{N,\kappa_k,X})\}\Big|_{\substack{(u_1,\ldots,u_N)\\=(1,\ldots,1)}},$$

where $\mathcal{S}_{\rho_{\lfloor n_k/2 \rfloor},X}(U_{N,\kappa_k,X})$ is the one-dimensional Schur generating function defined as in Definition 3.1, and $\rho_{\frac{n_k}{2}}$ is a probability measure on $\mathbb{GT}^+_{\lfloor \frac{n_k}{2} \rfloor}$ defined as in Lemma 3.4.

Proof. The lemma follows from arguments similar to those used in the proof of Lemma 4.13. ∎

For $1 \leq s \leq N$, let $F_{k,\kappa,N}^{(s)}$ be defined as in (6.3). Let

$$\widetilde{G}_{\kappa_1,\kappa_2,N,(k,l)}^{(j,s)} = k \sum_{r=0}^{k-1} \binom{k-1}{r} \sum_{\substack{\{a_1-N+\lfloor(1-\kappa_1)N\rfloor,\ldots,a_{r+1}-N+\lfloor(1-\kappa_1)N\rfloor\} \\ \in\{1,2,\ldots,\lfloor(1-\kappa_1)N\rfloor\}\cap R(j)}} (r+1)!$$
$$\times \lim_{x_{a_1},\ldots,x_{a_{r+1}}\to x_j} \operatorname{Sym}_{a_1,\ldots,a_{r+1}} \frac{x_{a_1}^r u_{\tilde{a}_1}^{k-s_0} \partial_{\tilde{a}_1}[F_{l,\kappa_2,N}^{(s)}](\partial_{\tilde{a}_1}[\log \widetilde{T}_{N,\kappa_1}])^{k-1-r}}{(x_{a_1}u_{\tilde{a}_1} - x_{a_2}u_{\tilde{a}_2})\cdots(x_{a_1}u_{\tilde{a}_1} - x_{a_{r+1}}u_{\tilde{a}_{r+1}})},$$

where

$$\widetilde{T}_N = \prod_{l\in\{1,\ldots,N-\lfloor(1-\kappa_1)N\rfloor\}\cap I_2} \prod_{j=1}^{\lfloor(1-\kappa_1)N\rfloor} \frac{1+y_{\bar{l}}\overline{x_{N-\lfloor(1-\kappa)N\rfloor+j}}u_j}{1+y_{\bar{l}}\overline{x_{N-\lfloor(1-\kappa)N\rfloor+j}}}$$
$$\times \left(\prod_{i=1}^{r} s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \ldots, u_{q_{N,\kappa}n+i}, 1, \ldots, 1) \right)$$
$$\times \left(\prod_{i=r+1}^{n} s_{\phi^{(j(i),\sigma_0)}(N)}(u_i, u_{n+i}, \ldots, u_{(q_{N,\kappa}-1)n+i}, 1, \ldots, 1) \right)$$
$$\times \left(\prod_{\substack{N-\lfloor(1-\kappa)N\rfloor+1\leq i\leq N, \\ 1\leq j\leq N-\lfloor(1-\kappa)N\rfloor, \tilde{i}\in R(p), \tilde{j}\in R(q), p<q}} \frac{1}{x_i u_{\tilde{i}} - x_j} \right)(1+o(1)).$$

Lemma 6.15. *We have*

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^{l_1+\cdots+l_s}}\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa_1)N\rfloor)}-\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa_1)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa_2)N\rfloor)}-\mathbf{E}p_{l_2}^{(\lfloor(1-\kappa_2)N\rfloor)})\cdots\\&\quad\times(p_{l_s}^{(\lfloor(1-\kappa_k)N\rfloor)}-\mathbf{E}p_{l_s}^{(\lfloor(1-\kappa_k)N\rfloor)})\\&\quad=\lim_{N\to\infty}\frac{1}{N^{l_1+\cdots+l_s}}\sum_{\substack{j_1,\ldots,j_s\\ \in\{1,2,\ldots,n\}}}\sum_{P\in\widetilde{\mathcal{P}}_{\emptyset}^{s}}\prod_{(a,b)\in P}[\widetilde{G}^{(j_a,j_b)}_{\kappa_a,\kappa_b,N,(l_a,l_b)}|_{U_{N,\kappa}=(1,\ldots,1)}\mathbf{1}_{j_a=j_b}].\end{aligned}$$

Proof. The lemma follows from arguments similar to those used in the proof of Lemma 6.11. ■

Proposition 6.16. *Assume* $\kappa_1,\kappa_2\in(0,1)$. *Then*

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^{l_1+l_2}}\mathbf{E}(p_{l_1}^{(\lfloor(1-\kappa_1)N\rfloor)}-\mathbf{E}p_{l_1}^{(\lfloor(1-\kappa_1)N\rfloor)})(p_{l_2}^{(\lfloor(1-\kappa_2)N\rfloor)}-\mathbf{E}p_{l_2}^{(\lfloor(1-\kappa_2)N\rfloor)})\\&\quad=\sum_{j=1}^{n}\frac{1}{(2\pi\mathbf{i})^2}\oint_{|z-1|=\varepsilon}\Big(\frac{(1-\kappa_1)}{n}\frac{z}{z-1}+zA_j(z)\Big)^{l_1}\\&\qquad\times\oint_{|w-1|=\varepsilon}\Big(\frac{(1-\kappa_2)}{n}\frac{w}{w-1}+zA_j(w)\Big)^{l_2}\Big[B_j(z,w)+\frac{1}{(z-w)^2}\Big]dwdz,\end{aligned}$$

where for $1\le j\le n$, $A_j(z)$ *and* $B_j(z,w)$ *are given by* (6.12) *and* (6.13).

Proof. The proposition follows from arguments similar to those used in the proof of Proposition 6.13. ■

Then Theorem 6.1 follows. More precisely, the Gaussian distribution follows from Lemma 6.15 and Wick's probability theorem.

7. Gaussian free field in staircase boundary condition

The goal of this section is to prove Theorem 2.12; we review the results on the limit shapes of complex Burgers equation in [23] and discuss their consequences in the limit shapes of square-octagon lattices. The results proved in this section will be used to show that there exists a homeomorphism from the liquid region to the region $\mathbb{S}$ (defined by (2.5)), and then to show that the fluctuation of certain statistics is the pullback of GFF under such a homeomorphism.

7.1. Height function

Suppose that $\mathcal{R}(\Omega(N),\check{c})$ is a contracting square-hexagon lattice and that the edge weights of SH($\check{c}$) satisfy Assumption 2.1.

The planar dual graph $\mathrm{SH}^*(\check{c})$ of $\mathrm{SH}(\check{c})$ is obtained by placing a vertex of $\mathrm{SH}^*(\check{c})$ inside each face of $\mathrm{SH}(\check{c})$; two vertices of $\mathrm{SH}^*(\check{c})$ are adjacent, or joined by an edge in $\mathrm{SH}^*(\check{c})$, if and only if the two corresponding faces of $\mathrm{SH}(\check{c})$ share an edge of $\mathrm{SH}(\check{c})$.

We place a vertex of $\mathrm{SH}^*(\check{c})$ at the center of each face of $\mathrm{SH}(\check{c})$ and obtain an embedding of $\mathrm{SH}^*(\check{c})$ into the plane. Each face of $\mathrm{SH}^*(\check{c})$ is either a triangle or a square, depending on whether the corresponding vertex of $\mathrm{SH}(\check{c})$ inside the dual face in $\mathrm{SH}^*(\check{c})$ is degree-3 or degree-4.

For a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{c})$, let $\mathcal{R}^*(\Omega,\check{c})$ be a finite triangle-square lattice such that

- $\mathcal{R}^*(\Omega,\check{c})$ is a finite subgraph of $\mathrm{SH}^*(\check{c})$ as constructed above;
- $\mathcal{R}(\Omega,\check{c})$ is the interior dual graph of $\mathcal{R}^*(\Omega,\check{c})$.

In other words, $\mathcal{R}^*(\Omega,\check{c})$ is the subgraph of $\mathrm{SH}^*(\check{c})$ consisting of all the faces of $\mathrm{SH}^*(\check{c})$ corresponding to vertices of $\mathcal{R}(\Omega,\check{c})$; see Figure 3.

Definition 7.1. Let $M \in \mathcal{M}(\Omega,\check{c})$ be a perfect matching of a contracting square-hexagon lattice $\mathcal{R}(\Omega,\check{c})$. We color the vertices of $\mathcal{R}(\Omega,\check{c})$ by black and white as in Section 2.1. A height function h_M is an integer-valued function on vertices of $\mathcal{R}^*(\Omega,\check{c})$ that satisfies the following property.

Let f_1, f_2 be a pair of adjacent vertices of $\mathcal{R}^*(\Omega,\check{c})$. Let (f_1, f_2) denote the non-oriented edge of $\mathcal{R}^*(\Omega,\check{c})$ with endpoints f_1 and f_2, and let $[f_1, f_2\rangle$ (resp. $[f_2, f_1\rangle$) denote the oriented edge starting from f_1 (resp. f_2) and ending in f_2 (resp. f_1).

- When (f_1, f_2) is a dual edge crossing a NW-SE edge or a NE-SW edge of $\mathrm{SH}(\check{c})$:
  - If an oriented dual edge $[f_1, f_2\rangle$ crosses an absent edge e of $\mathrm{SH}(\check{c})$ in M, then $h_M(f_2) = h_M(f_1) + 1$ if $[f_1, f_2\rangle$ has the white vertex or e on the left, and $h_M(f_2) = h_M(f_1) - 1$ otherwise.
  - If an oriented dual edge $[f_1, f_2\rangle$ crosses a present edge e of $\mathrm{SH}(\check{c})$ in M, then $h_M(f_2) = h_M(f_1) - 3$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) + 3$ otherwise.
- When (f_1, f_2) is a dual edge crossing a vertical edge of $\mathrm{SH}(\check{a})$:
  - If an oriented dual edge $[f_1, f_2\rangle$ crosses an absent edge e of $\mathrm{SH}(\check{c})$ in M, then $h_M(f_2) = h_M(f_1) + 2$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) - 2$ otherwise.
  - If an oriented dual edge $[f_1, f_2\rangle$ crosses a present edge e of $\mathrm{SH}(\check{c})$ in M, then $h_M(f_2) = h_M(f_1) - 2$ if $[f_1, f_2\rangle$ has the white vertex of e on the left, and $h_M(f_2) = h_M(f_1) + 2$ otherwise.
- $h_M(f_0) = 0$, where f_0 is the lexicographic smallest vertex of $\mathcal{R}^*(\Omega,\check{c})$.

It is straightforward to verify that the height function above is well defined, by checking that around either a degree-3 vertex or a degree-4 vertex, the total height change is 0. Moreover, since none of the boundary edges of $\mathcal{R}(\Omega, \check{c})$ (by boundary edges we mean edges of $\mathrm{SH}(\check{c})$ joining exactly one vertex of $\mathcal{R}(\Omega, \check{c})$ and one vertex outside $\mathcal{R}(\Omega, \check{c})$) are present in any perfect matching of $\mathcal{R}(\Omega, \check{c})$, the height function restricted on the boundary vertices of $\mathcal{R}^*(\Omega, \check{c})$ is fixed and independent of the random perfect matching; see Figure 3.

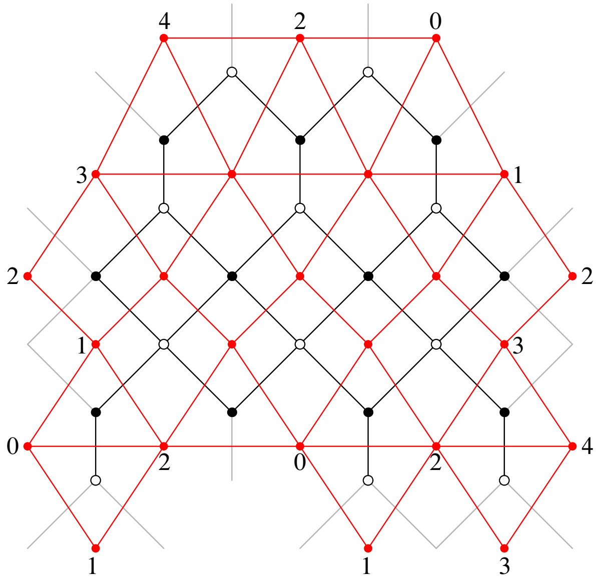

Figure 3. Contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{c})$, dual graph $\mathcal{R}^*(\Omega, \check{c})$ and height function on the boundary. The black lines represent the graph $\mathcal{R}(\Omega, \check{c})$, the gray lines represent boundary edges of $\mathcal{R}(\Omega, \check{c})$, the red lines represent the dual graph $\mathcal{R}^*(\Omega, \check{c})$, and the height function is defined on vertices of the dual graph. The values of the height function on the boundary vertices of $\mathcal{R}^*(\Omega, \check{c})$ are also shown in the figure.

Theorem 7.2 (Law of large numbers for the height function). *Assume that the assumptions of Proposition 3.6 hold.*

Let ρ_N^k be the measure on the configurations of the kth row, and let $\kappa \in (0, 1)$ such that $k = [2\kappa N]$. Let $\mathbf{m}^\kappa$ be the limit of the counting measures $m(\rho_N^k)$ in probability as $N \to \infty$ with moments given by (3.6). Define

$$\mathbf{h}(\chi, \kappa) := 4(1-\kappa) \int_0^{\frac{\chi - \frac{\kappa r}{2n}}{1-\kappa}} d\mathbf{m}^\kappa - 2\chi + 2\kappa. \tag{7.1}$$

Then the random height function h_M associated to a random perfect matching M, as defined by Definition 7.1, has the following law of large numbers:

$$\frac{h_M([\chi N],[\kappa N])}{N} \to \mathbf{h}(\chi,\kappa) \quad \textit{when } N \to \infty,$$

where χ, κ are new continuous parameters of the domain.

Proof. We use the same arguments as in the proof of [5, Theorem 3.7]; more precisely, a combinatorial relation between the height function and counting measure, as well as the convergence results of counting measures as given by Propositions 3.6 and 5.4. ∎

7.2. Variational principle and complex Burger's equation

Let $\mathrm{SH}(\check{c})$ be the whole plane square-hexagon lattice with edge weights assigned as in Assumption 2.1 and periodically with period n such that (3.3) and (3.4) hold. Then $\mathbb{Z}^2$ acts on $\mathrm{SH}(\check{c})$ by translations, which are vertex-color-preserving and edge-weight-preserving isomorphisms of $\mathrm{SH}(\check{c})$. Let $\mathrm{SH}_1(\check{c})$ be the quotient graph of $\mathrm{SH}(\check{c})$ under the action of $\mathbb{Z}^2$. The graph $\mathrm{SH}_1(\check{c})$ is called a *fundamental domain* of $\mathrm{SH}(\check{c})$, which is a finite graph that can be embedded into a torus.

Let γ_x and γ_y be two directed simple cycles winding once around the two homology generators of the torus where $\mathrm{SH}_1(\check{c})$ is embedded. Assume the edge weights of the square-hexagon lattice satisfy Assumption 2.1. We shall modify the edge weights of the graph and construct a modified weighted adjacency matrix (Kasteleyn matrix) for $\mathrm{SH}_1(\check{c})$, which plays an essential role in the analysis of periodic dimer models, see [18, 21, 40].

- Multiply all the edge weights x_i by -1. This way, around each face of degree 4, there are an odd number of "–" signs multiplied by edge weights, while around each face of degree 6, there are an even number of "–" signs multiplied by edge weights.
- Multiply the weight of each edge crossed by γ_x with w (resp. w^{-1}) if the black vertex of the edge is on the left (resp. right) of the path; then multiply the weight of each edge crossed by γ_y with z (resp. z^{-1}) if the black vertex of the edge is on the left (resp. right) of the path.

Let $K(z,w)$ be the weighted adjacency matrix of $\mathrm{SH}_1(\check{c})$ with respect to the modified edge weights after the multiplication above. More precisely, the rows of $K(z,w)$ are labeled by white vertices of $\mathrm{SH}_1(\check{c})$, while the columns of $K(z,w)$ are labeled by black vertices of $\mathrm{SH}_1(\check{c})$. For a black vertex B and a white vertex W of G_1, the entry $K_{BW}(z,w)$ is 0 if B and W are not adjacent; if B and W are joined by an edge e_{BW} in $\mathrm{SH}_1(\check{c})$, then the entry $K_{BW}(z,w)$ is the modified weight of the edge e_{BW}. Let $P(z,w) = \det K(z,w)$ be the *characteristic polynomial*. See [24] for more results

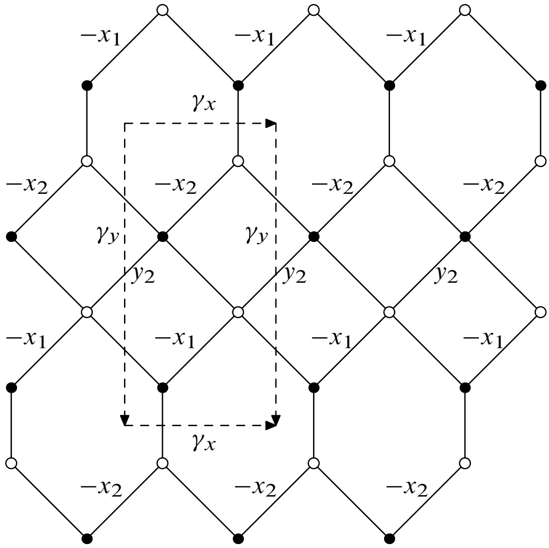

Figure 4. A fundamental domain in a periodic square-hexagon lattice. The subgraph bounded by the dashed lines is a fundamental domain.

about the characteristic polynomial and the phase transitions of the dimer model on a bipartite, periodic graph.

Example 7.3. Consider a fundamental domain of a square-hexagon lattice as illustrated in Figure 4. We have

$$K(z,w) = \begin{pmatrix} z - x_2 & w \\ 1 + y_2 z & z - x_1 \end{pmatrix}$$

and

$$P(z,w) = \det K(z,w) = (z - x_1)(z - x_2) - w(1 + y_2 z).$$

Proposition 7.4. *Let* $\mathbf{h}$ *be the limit height function as given by* (7.1). *In the liquid region, we have*

$$\left(\frac{1}{4}\frac{\partial \mathbf{h}}{\partial x} + \frac{1}{2}, \frac{1}{4}\frac{\partial \mathbf{h}}{\partial y} + \frac{n}{2}\right) = \frac{1}{\pi}(\arg z, -\arg w), \tag{7.2}$$

where the functions z *and* w *solve the differential equation*

$$\frac{z_y}{z} + \frac{w_x}{w} = 0 \tag{7.3}$$

and the algebraic equation $P(z,w) = 0$.

Proof. Use the same arguments as in the proof of [23, Theorem 1]; see also [10]. ■

When the edge weights satisfy Assumptions 2.1 and 3.2, we can choose a fundamental domain such that $P(z,w)$ is linear in w. More precisely, when the period of the graph is $1\times n$, each row of the weighted adjacency matrix $K(z,w)$ has exactly two non-vanishing entries. Choose a fundamental domain consisting of $2n$ rows and 2 columns such that the topmost row is a row of white vertices, and the rightmost column is a column of white vertices. Assume γ_x is oriented from the left to the right and γ_y is oriented from the top to the bottom. Let $\mathrm{SH}_1(\check{c})$ be the toroidal graph constructed from the fundamental domain above by identifying the left and right boundary as well as the top and bottom boundary. Let v_b be a white vertex of $\mathrm{SH}_1(\check{c})$. The following cases might occur.

- When v_b has degree 3:
  - If the vertex is not incident to an edge crossed by γ_x, the two non-vanishing entries of $K(z,w)$ on the row corresponding to v_b are $z-x_i$ and 1.
  - If the vertex is incident to an edge crossed by γ_x, the two non-vanishing entries of $K(z,w)$ on the row corresponding to v_b are $z-x_n$ and w.
- When v_b has degree 4:
  - If the vertex is not incident to an edge crossed by γ_x, the two non-vanishing entries of $K(z,w)$ on the row corresponding to v_b are $z-x_i$ and $1+y_{i+1}z$.
  - If the vertex is incident to an edge crossed by γ_x, the two non-vanishing entries of $K(z,w)$ on the row corresponding to v_b are $z-x_{n-1}$ and $w(1+y_nz)$.

In the toroidal graph $\mathrm{SH}_1(\check{c})$, each vertex is adjacent to exactly two vertices, with possible multiple edges joining two adjacent vertices. We may consider $\mathrm{SH}_1(\check{c})$ with vertices located on a circle, then $P(z,w)=\det K(z,w)$ counts the (signed) partition function of dimer configurations on the circle.

Solving the equation $P(z,w)=0$ for w, we have

$$w=R(z),$$

where $R(z)$ is a rational function of z (quotient of two polynomials in z). By the explanations above, $R(z)$ can be written down explicitly as

$$R(z)=\frac{\prod_{i=1}^{n}(z-x_i)}{\prod_{j=\{1,2,\dots,n\}\cap I_2}(1+y_jz)}.$$

Therefore, given $P(z,w)=0$, (7.3) becomes

$$z_x+\frac{R(z)}{zR'(z)}z_y=0. \tag{7.4}$$

Lemma 7.5. *Let z be a solution of the equation*

$$F_{\kappa,m}(z) = \frac{\chi}{1-\kappa}, \tag{7.5}$$

in the upper half-plane, then z also satisfies the differential equation (7.4).

Proof. Differentiating (7.5) with respect to χ and κ, we have

$$\frac{\frac{\partial z}{\partial \chi}}{\frac{\partial z}{\partial \kappa}} = \frac{1}{-\frac{z}{n}\sum_{i\in I_2\cap\{1,2,\ldots,n\}}\frac{y_i}{1+y_i z} + \sum_{j=1}^{n}\frac{z}{n(z-x_j)}}. \tag{7.6}$$

Given the different scalings of (x, y) and (χ, κ) (more precisely, in the (x, y)-system, we assume each fundamental domain has height 1 and width 1; while in the (χ, κ) system, we assume each fundamental domain has width 1 and height n), we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \chi} \quad \text{and} \quad \frac{\partial}{\partial y} = n\frac{\partial}{\partial \kappa}.$$

The right-hand side of (7.6) divided by n is exactly $\frac{R(z)}{zR'(z)}$. ∎

Lemma 7.6. *Assume all the edge weights x_j's are distinct. Let $m \geq 1$ be a positive integer. Then*

(1) *For $\kappa = 0$ and any $\chi \in (0, m)$, the equation*

$$F_{0,m}(z) = \chi \tag{7.7}$$

has a unique root satisfying

$$\mathrm{Arg}(z) = \frac{\pi}{m}. \tag{7.8}$$

(2) *For each z satisfying $\mathrm{Arg}(z) = \frac{\pi}{m}$, there exists $\chi \in (0, m)$ such that equation (7.7) holds.*

Proof. We first prove part (1). Equation (7.7) has the form

$$\frac{1}{n}\sum_{j=0}^{n}\frac{mz^m}{z^m - x_j^m} = \chi.$$

Let $\sqrt[m]{a}$ be the non-negative mth root of a non-negative number a. Assume $z = \sqrt[m]{R}e^{\frac{\mathrm{i}\pi}{m}}$, where $R = |z^m| \geq 0$. Then we have

$$\frac{1}{n}\sum_{j=1}^{n}\frac{-mR}{-R - x_j^m} = \chi. \tag{7.9}$$

It suffices to show that (7.9) has a unique solution in $R \in [0, \infty)$. Explicit computations show that (7.9) is equivalent to the equation

$$f(-R) := \sum_{j=1}^{n} (-mR) \prod_{i \in \{1,2,\ldots,n\}, i \neq j} (-R - x_i^m) - \chi n \prod_{i=1}^{n} (-R - x_i^m) = 0.$$

Without loss of generality, assume that

$$0 < x_1 < x_2 < \cdots < x_n.$$

We have

$$\operatorname{sgn}[f(x_i^m)] = (-1)^{n-i}.$$

Moreover, when $\chi > 0$,

$$\operatorname{sgn}[f(0)] = (-1)^{n+1}.$$

Given $\chi < m$, we have

$$\operatorname{sgn}[f(-\infty)] = (-1)^n.$$

Therefore, the equation $f(z) = 0$ has a solution in each of the following intervals:

$$(-\infty, 0), \quad (x_1^m, x_2^m), \quad (x_2^m, x_3^m), \quad \ldots, \quad (x_{n-1}^m, x_n^m).$$

Given $\chi < m$, $f(z)$ is a degree-n polynomial and has at most n distinct roots in $\mathbb{C}$. Therefore, we have $f(z)$ has exactly one root in each of the above intervals. In particular, $f(z) = 0$ has exactly one root in the interval $(-\infty, 0)$, one of whose mth root gives the unique solution of (7.7) satisfying (7.8). Then part (1) of the lemma follows.

Now we prove part (2) of the lemma. Let

$$g(t) = \frac{1}{n} \sum_{j=1}^{n} \frac{mt}{t + x_j^m}.$$

Then

$$g'(t) = \frac{1}{n} \sum_{j=1}^{n} \frac{m x_j^m}{(t + x_j^m)^2} > 0$$

for any $t \in \mathbb{R}$. Moreover,

$$g(0) = 0, \quad \lim_{t \to \infty} g(t) = m.$$

Then part (2) of the lemma follows. ■

Lemma 7.7. *Assume all the edge weights x_j's are distinct. Let $m \geq 1$ be a positive integer. Assume*

$$|I_2 \cap \{1, 2, \dots, n\}| = r.$$

For each $\kappa \in [0, 1]$ and each $\chi \in [0, m]$, the equation

$$F_{\kappa,m}(z) = \frac{\chi}{1-\kappa} \tag{7.10}$$

has a unique root $z_0(\chi, \kappa)$ such that

(1) $\operatorname{Arg} z_0(\chi, 0) = \frac{\pi}{m}$.

(2) $z_0(0, \kappa) = 0$.

(3) $\lim_{\chi \to m + (\frac{r}{n} - 1)\kappa} z_0(\chi, \kappa) = \infty$.

(4) $z_0(\chi, 1) \in [0, +\infty)$.

(5) $z_0(\chi, \kappa)$ *is continuous in* (χ, κ).

Proof. Let

$$g_{m-1,j}(z) = \sum_{k=0}^{m-1} z^k x_j^{m-1-k}.$$

Then when $\kappa = 1$, the equation $(1 - \kappa) F_{\kappa,m} = \chi$ has the form

$$p(z) := \frac{z}{n} \sum_{i \in I_2 \cap \{1,2,\dots,n\}} \frac{y_i}{1 + y_i z} + \frac{z}{n} \sum_{j=1}^{n} \frac{\frac{\partial g_{m-1,j}(z)}{\partial z}}{g_{m-1,j}(z)} - \chi = 0.$$

Note that $p(z)$ is well defined on $(0, +\infty)$ since $g_{m-1,j}(z) > 0$ whenever $z > 0$. When $\chi \in (0, m - 1 + \frac{r}{n})$, we have

$$p(0) < 0 \quad \text{and} \quad p(+\infty) > 0.$$

Moreover, we have the following assertion.

Lemma 7.8. *If $z \in (0, +\infty)$ and $x_i, y_j > 0$, then*

$$p'(z) > 0.$$

Proof. Note that

$$p'(z) = \frac{1}{n} \sum_{i \in I_2 \cap \{1,2,\dots,n\}} \frac{y_i}{(1 + y_i z)^2} + \frac{1}{n} \sum_{j=1}^{n} \frac{(z g'_{m-1,j})' g_{m-1,j} - z (g'_{m-1,j})^2}{g_{m-1,j}^2}.$$

It suffices to show that for $z > 0$, $x_i > 0$, we have

$$T_m(z) := (zg'_{m-1,j})' g_{m-1,j} - z(g'_{m-1,j})^2 \geq 0. \tag{7.11}$$

We prove (7.11) by induction on m. When $m = 1$, we have $g_{m-1,j} = 1$ and $T_m(z) = 0$. Assume (7.11) holds when $m = l - 1$, $l \geq 2$. When $m = l$, we have

$$\begin{aligned} g_{l-1,j} &= zg_{l-2,j} + x_j^{m-1}, \\ T_l(z) &= [zg_{l-2,j} + z^2 g'_{l-2,j}]'(zg_{l-2,j} + x_j^{l-1}) - z(g_{l-2,j} + zg'_{l-2,j})^2 \\ &= z^2 T_{l-1}(z) + x_j^{l-1} g_{l-2,j} + zx_j^{l-1}(3g'_{l-2,j} + g''_{l-2,j}) > 0. \end{aligned}$$

Since $T_{l-1}(z) > 0$ by induction hypothesis, $g_{l-2,j} > 0$, $g'_{l-2,j} > 0$ and $g''_{l-2,j} > 0$. Then the lemma follows. ■

Hence $p(z) = \chi$ has exactly one root in $(0, \infty)$ when $\chi \in (0, m - 1 + \frac{r}{n})$. The root converges to 0 when χ goes to 0, and the root approaches $+\infty$ when χ goes to $m - 1 + \frac{r}{n}$.

The fact that there is a unique root of (7.10) satisfying condition (1) follows from Lemma 7.6. The root when $\kappa = 0$ satisfying condition (1) and the root when $\kappa = 1$ satisfying condition (4) can be considered as boundary conditions for the Burgers equation. More precisely, the slope of height is $\frac{1}{m}$ on the bottom boundary $\kappa = 0$ of the rescaled square-hexagon lattice $\mathcal{R}$, while the slope of height is 0 on the top boundary $\kappa = 1$. Since the surface tension function is strictly convex in the liquid region, the solution of the Burgers equations satisfying the given boundary conditions is unique; see [10,22]. By Lemma 7.5, a root satisfying (1)–(5) is also a solution of the Burgers equation satisfying given boundary conditions, then the lemma follows. ■

Lemma 7.9. *Let $\mathcal{L}$ be the liquid region of the limit shape defined as follows:*

(1) *the region is a subset of*

$$\mathcal{R} := \Big\{(\chi, \kappa) : 0 \leq \kappa < 1; 0 < \chi < m + \Big(\frac{r}{n} - 1\Big)\kappa\Big\};$$

(2) *the solution $z_0(\chi, \kappa)$ of $F_{\kappa,m}(z) = \frac{\chi}{1-\kappa}$ given as in Lemma 7.7 remains non-real in the region;*

(3) *the region includes the bottom boundary $\kappa = 0$, $0 < \chi < m$ when $m \geq 2$.*

Let $\mathcal{L}_o$ be the interior of $\mathcal{L}$. Let $T_{\mathcal{L}}$ be a mapping on $\mathcal{L}$ which maps each point $(\chi, \kappa) \in \mathcal{L}$ to $z_0(\chi, \kappa)$. Then, restricted on $\mathcal{L}_o$, $T_{\mathcal{L}}$ is a homeomorphism from $\mathcal{L}_o$ to $T_{\mathcal{L}}(\mathcal{L}_o)$. Moreover,

$$T_{\mathcal{L}}(\mathcal{L}_o) = \Big\{z \in \mathbb{C} : 0 < \operatorname{Arg} z < \frac{\pi}{m}\Big\},$$

and Arg z *is the principal argument of* z.

Remark 7.10. Note that $\mathcal{R}$ is exactly the scaling limit of the region covered by the contracting square-hexagon lattice with perfect matchings, hence the liquid region is a subset of $\mathcal{R}$.

Proof of Lemma 7.9. First we show that $T_{\mathcal{L}}$ is one-to-one from $\mathcal{L}$ to $T_{\mathcal{L}}(\mathcal{L})$. Then if z is a solution of $F_{\kappa,m}(z) = \frac{\chi}{1-\kappa}$, we have $F_{\kappa,m}(\overline{z}) = \frac{\chi}{1-\kappa}$. Hence we can solve for χ and κ in terms of z as in (2.2) and (2.3).

Therefore, $T_{\mathcal{L}}$ is one-to-one from $\mathcal{L}_o$ to $T_{\mathcal{L}}(\mathcal{L}_o)$. From the expressions for $T_{\mathcal{L}}$, $\chi_{\mathcal{L}}$ and $\kappa_{\mathcal{L}}$, it is straightforward to see that $T_{\mathcal{L}}$, $\chi_{\mathcal{L}}$ and $\kappa_{\mathcal{L}}$ all are continuous.

Then we claim that

$$\mathcal{L}_o = \mathcal{L} \setminus \{\kappa = 0\}. \tag{7.12}$$

Note that $\mathcal{L} \subseteq \mathcal{R}$ and $\{\kappa = 0\}$ is part of the boundary of $\mathcal{R}$. Since $\mathcal{L}_o$ is the interior of $\mathcal{L}$, we must have $\mathcal{L}_o \subseteq \mathcal{L} \setminus \{\kappa = 0\}$. It remains to prove $\mathcal{L}_o \supseteq \mathcal{L} \setminus \{\kappa = 0\}$. To see why that is true, let $(\chi_1, \kappa_1) \in \mathcal{L} \setminus \{\kappa = 0\}$ and $z_1 = T_{\mathcal{L}}(\chi_1, \kappa_1)$. From the definition of $\mathcal{L}$, we have $z_1 \notin \mathbb{R}$. Note that z_1 is the root of $F_{\kappa_1,m}(z) = \frac{\chi_1}{1-\kappa_1}$ as given by Lemma 7.7. Given $0 < \kappa_1 < 1$, we obtain

$$z_1 \notin \{x_j e^{\frac{2t\pi \mathbf{i}}{m}} : 1 \leq j \leq n, 1 \leq t \leq m\}.$$

We shall show that $(\chi_2, \kappa_2) \in \mathcal{L} \setminus \{\kappa = 0\}$ whenever $|\chi_1 - \chi_2|$ and $|\kappa_1 - \kappa_2|$ are sufficiently small. Fix $\varepsilon > 0$ such that

- we have

$$B(z_1, \varepsilon) \cap [\mathbb{R} \cup \{x_j e^{\frac{2t\pi \mathbf{i}}{m}} : 1 \leq j \leq n, 1 \leq t \leq m\}] = \emptyset; \tag{7.13}$$

- $\inf_{z \in \partial B(z_1,\varepsilon)} |F_{\kappa_1,m}(z) - \frac{\chi_1}{1-\kappa_1}| > 0$;
- $F_{\kappa_1,m}(z) = \frac{\chi_1}{1-\kappa_1}$ has a unique zero in $B(z_1, \varepsilon)$.

By (7.13),

$$\left| F_{\kappa_1,m}(z) - \frac{\chi_1}{1-\kappa_1} - F_{\kappa_2,m}(z) + \frac{\chi_2}{1-\kappa_2} \right| < \eta$$

for any $\eta > 0$, whenever $|\chi_1 - \chi_2|$ and $|\kappa_1 - \kappa_2|$ are sufficiently small, and $z \in B(z_1, \varepsilon)$.

Therefore, when $|\chi_1 - \chi_2|$ and $|\kappa_1 - \kappa_2|$ are sufficiently small, we have

$$\left| F_{\kappa_1,m}(z) - \frac{\chi_1}{1-\kappa_1} \right| > \left| F_{\kappa_1,m}(z) - \frac{\chi_1}{1-\kappa_1} - F_{\kappa_2,m}(z) + \frac{\chi_2}{1-\kappa_2} \right|$$

for any $z \in \partial B(z_1, \varepsilon)$. By Rouché's theorem, $F_{\kappa_2,m}(z) - \frac{\chi_2}{1-\kappa_2}$ has a root in $B(z_1, \varepsilon)$, which is as described in Lemma 7.7. Hence $(\chi_2, \kappa_2) \in \mathcal{L} \setminus \{\kappa = 0\}$, and we obtain expression (7.12).

Similar arguments show that for any point $(\chi, 0)$ with $0 < \chi < m$, there is a neighborhood $B_\delta(\chi, 0)$ such that $B_\delta(\chi, 0) \cap \mathbb{H}_+ \in \mathcal{L}$, where $\mathbb{H}_+$ is the upper half-plane.

Now we claim that

$$T_{\mathcal{L}}(\mathcal{L}_o) \subseteq \left\{z \in \mathbb{C}\colon 0 < \operatorname{Arg} z < \frac{\pi}{m}\right\}. \tag{7.14}$$

To see why that is true, assume there exists a point

$$c \in T_{\mathcal{L}}(\mathcal{L}_o) \setminus \left\{z \in \mathbb{C}\colon 0 < \operatorname{Arg} z < \frac{\pi}{m}\right\}.$$

Let $d \in \mathcal{L}_o$ such that

$$T_{\mathcal{L}}(d) = c.$$

By Lemma 7.7, we see along the boundary $\kappa = 1$of $\mathcal{R}$, $z_0 \in [0, \infty)$. By continuity of z_0, we can find $c_1 \in T_{\mathcal{L}}(\partial\mathcal{L}) \cap (0, +\infty)$, $d_1 \in \partial\mathcal{L} \setminus \{(\chi, \kappa)\colon \kappa = 0\}$ such that

$$T_{\mathcal{L}}(d_1) = c_1.$$

In particular, d_1 cannot be along the line $\kappa = 0$ by Lemma 7.7 (1). Since $\mathcal{R}$ is connected, we can find a path p_{dd_1} in $\mathcal{R}$ joining d and d_1 such that $p_{dd_1} \cap \{\kappa = 0\} = \emptyset$. By continuity, $T_{\mathcal{L}}(p_{dd_1})$ is a continuous curve in $\mathbb{C}$ joining the point c satisfying $\operatorname{Arg} c \geq \frac{\pi}{m}$ and the point c_1 satisfying $\operatorname{Arg} c_1 = 0$. (Note some point $(\chi, \kappa) \in p_{dd_1}$ may be outside the liquid region; in that case, $T_{\mathcal{L}}$ maps (χ, κ) to $z_0(\chi, \kappa)$ as in Lemma 7.7 which is real.) By (7.1), $\frac{d\mathbf{h}}{d\chi} \in [-2, 2]$, hence $\frac{1}{4}\frac{d\mathbf{h}}{d\chi} + \frac{1}{2} \in [0, 1]$. By (7.2), we have $\operatorname{Arg}(z_0(\chi, \kappa)) \in [0, \pi]$ for all $(\chi, \kappa) \in \mathcal{R}$. Then there exists $c_2 = T_{\mathcal{L}}(d_2)$ such that $d_2 \in p_{dd_1}$ and $\operatorname{Arg} c_2 = \frac{\pi}{m}$. Since $d_2 = (\chi_{\mathcal{L}(c_2)}, \kappa_{\mathcal{L}(c_2)})$, by (2.2), (2.3), we obtain that

$$\chi_{\mathcal{L}}(c_2) = \frac{1}{n}\sum_{j=1}^{n} \frac{m c_2^m}{c_2^m - x_j^m} \in (0, m) \quad \text{when } \operatorname{Arg}(c_2) = \frac{\pi}{m},$$
$$\kappa_{\mathcal{L}}(c_2) = 0.$$

This contradicts the fact that $p_{dd_1} \cap \{(\chi, \kappa)\colon \chi \in (0, m), \kappa = 0\} = \emptyset$. The contradiction implies (7.14).

We finally show that

$$\left\{z \in \mathbb{C}\colon 0 < \operatorname{Arg} z < \frac{\pi}{m}\right\} \subseteq T_{\mathcal{L}}(\mathcal{L}_o).$$

Assume that there exists $t \in \partial T_{\mathcal{L}}(\mathcal{L}_o)$ such that $t \in \{z \in \mathbb{C}\colon 0 < \operatorname{Arg} z < \frac{\pi}{m}\} \setminus T_{\mathcal{L}}(\mathcal{L}_o)$. Then there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset T_{\mathcal{L}}(\mathcal{L}_o)$ such that

$$\lim_{n \to \infty} t_n = t.$$

By continuity of (2.2) and (2.3), we have

$$\lim_{n\to\infty} \chi(t_n) = \chi(t), \qquad \lim_{n\to\infty} \kappa(t_n) = \kappa(t).$$

Hence $(\chi(t), \kappa(t)) \in \mathcal{L}_o \cup \partial\mathcal{L}_o$ is such that $F_{\kappa(t),m}(z) = \frac{\chi(t)}{1-\kappa(t)}$ has a root t in $\{z \in \mathbb{C} : 0 < \operatorname{Arg} z < \frac{\pi}{m}\}$ as described in Lemma 7.7. Note that $\kappa(t) \neq 1$, because if $\kappa(t) = 1$, then $\operatorname{Arg} t = 0$. Also $\chi(t) \neq 0$, because if $\chi(t) = 0$, then $t = 0$. Moreover, $\chi(t) \neq m + (\frac{r}{n} - 1)\kappa(t)$, because otherwise $t = \infty$. Then $\chi(t), \kappa(t) \in \mathcal{L}$. Since $\operatorname{Arg} t \neq \frac{\pi}{m}$, $\kappa(t) \neq 0$, we have $(\chi(t), \kappa(t)) \in \mathcal{L}_o$ and $t \in T_{\mathcal{L}}(\mathcal{L}_o)$. Then the proof is complete. ■

7.3. Proof of Theorem 2.12

Proof. By Theorem 4.1, we have

$$\lim_{N\to\infty} \frac{\operatorname{cov}(p_{l_1}^{((1-\kappa_1)N)}, p_{l_2}^{((1-\kappa_2)N)})}{N^{l_1+l_2}} = \frac{(1-\kappa_1)^{l_1}(1-\kappa_2)^{l_2}}{(2\pi\mathbf{i})^2} \times \sum_{i=1}^{n}\sum_{j=1}^{n} \oint_{|z-x_i|=\varepsilon} \oint_{|w-x_j|=\varepsilon} [F_{\kappa_1,m}(z)]^{l_1} [F_{\kappa_2,m}(w)]^{l_2} Q(z,w)\, dz dw. \quad (7.15)$$

The poles of $F_{\kappa,m}(z)$ are of 3 types:

(1) $x_1, \dots, x_n$ lying on the positive real axis;

(2) $-\frac{1}{y_j}$ for $j \in I_2 \cap \{1, 2, \dots, n\}$ lying on the negative real axis;

(3) roots of $z^m = x_j^m$ except x_j for $j = 1, \dots, n$, lying on the circle centered at 0 with radius x_j.

We may change the sum of contour integrals in the right-hand side of (7.15) into an integral over a contour enclosing all the poles of $F_{\kappa,m}(z)$ of type (1), yet enclosing no poles of types (2) and (3), with respect to both z and w.

For $\kappa \in (0,1)$, let

$$\mathcal{Z}_1(\kappa) = \Big\{ z : F_{\kappa,m}(z) = \frac{\chi}{1-\kappa}; \chi \in \Big[0, m + \kappa\Big(\frac{r}{n} - 1\Big)\Big]; \ z \text{ is a root as given by Lemma 7.7} \Big\}.$$

Let

$$\mathcal{Z}_2(\kappa) = \{ z : \overline{z} \in \mathcal{Z}_1(\kappa) \} \quad \text{and} \quad \mathcal{Z}(\kappa) = \mathcal{Z}_1(\kappa) \cup \mathcal{Z}_2(\kappa).$$

We claim that for $\kappa \in (0,1)$, $\mathcal{Z}(\kappa)$ is a contour in the complex plane $\mathbb{C}$ enclosing all the poles of $F_{\kappa,m}(z)$ of type (1), yet enclosing no poles of type (2) and (3). By Lemma 7.9, we have

$$\mathcal{Z}_1(\kappa) \subset \{0, +\infty\} \cup \Big\{ z : 0 < \operatorname{Arg} z < \frac{\pi}{m} \Big\}.$$

Note that $z_0(0,\kappa)=0$ and $z_0(m+\kappa(\frac{r}{n}-1),\kappa)=+\infty$ by Lemma (7.7). Since $\mathcal{Z}_2(\kappa)$ is the complex conjugate of $\mathcal{Z}_1(\kappa)$, then the claim follows.

For $\kappa_1,\kappa_2\in(0,1)$, (7.15) becomes

$$\begin{aligned}
\lim_{N\to\infty}&\frac{\operatorname{cov}(p_{l_1}^{((1-\kappa_1)N)},p_{l_2}^{((1-\kappa_2)N)})}{N^{l_1+l_2}}\\
&=\frac{(1-\kappa_1)^{l_1}(1-\kappa_2)^{l_2}}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}(\kappa_1)}\oint_{w\in\mathcal{Z}(\kappa_2)}[F_{\kappa_1,m}(z)]^{l_1}[F_{\kappa_2,m}(w)]^{l_2}\\
&\quad\times Q(z,w)\,dzdw\\
&=\frac{1}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}_1(\kappa_1)}\oint_{w\in\mathcal{Z}_1(\kappa_2)}[\chi_{\mathcal{L}}(z)]^{l_1}[\chi_{\mathcal{L}}(w)]^{l_2}Q(z,w)\,dzdw\\
&\quad-\frac{1}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}_1(\kappa_1)}\oint_{w\in\mathcal{Z}_2(\kappa_2)}[\chi_{\mathcal{L}}(z)]^{l_1}[\chi_{\mathcal{L}}(\overline{w})]^{l_2}Q(z,w)\,dzdw\\
&\quad-\frac{1}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}_2(\kappa_1)}\oint_{w\in\mathcal{Z}_1(\kappa_2)}[\chi_{\mathcal{L}}(\overline{z})]^{l_1}[\chi_{\mathcal{L}}(w)]^{l_2}Q(z,w)\,dzdw\\
&\quad+\frac{1}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}_2(\kappa_1)}\oint_{w\in\mathcal{Z}_2(\kappa_2)}[\chi_{\mathcal{L}}(\overline{z})]^{l_1}[\chi_{\mathcal{L}}(\overline{w})]^{l_2}Q(z,w)\,dzdw\\
&=\frac{1}{(2\pi\mathbf{i})^2}\oint_{z\in\mathcal{Z}_1(\kappa_1)}\oint_{w\in\mathcal{Z}_1(\kappa_2)}[\chi_{\mathcal{L}}(z)]^{l_1}[\chi_{\mathcal{L}}(w)]^{l_2}\\
&\quad\times[Q(z,w)dzdw-Q(z,\overline{w})dzd\overline{w}-Q(\overline{z},w)d\overline{z}dw+Q(\overline{z},\overline{w})d\overline{z}d\overline{w}],
\end{aligned}$$

where

$$Q(z,w)=\frac{m^2z^{m-1}w^{m-1}}{(z^m-w^m)^2}.$$

Note that

$$\begin{aligned}
&Q(z,w)dzdw+Q(z,\overline{w})dzd\overline{w}+Q(\overline{z},w)d\overline{z}dw+Q(\overline{z},\overline{w})d\overline{z}d\overline{w}\\
&\quad=2d\ln\left|\frac{z^m-w^m}{z^m-\overline{w}^m}\right|.
\end{aligned}$$

Hence the random variables $\{M_j^{\kappa}\}_{\kappa\in(0,1),j\in\mathbb{N}}$, defined by (2.1), converge to the Gaussian distribution with mean 0 and limit covariance

$$\begin{aligned}
\lim_{N\to\infty}\operatorname{cov}(M_{j_1}^{\kappa_1},M_{j_2}^{\kappa_2})=\frac{-1}{2\pi(j_1+1)(j_2+1)}\oint_{z\in\mathcal{Z}_1(\kappa_1)}\oint_{w\in\mathcal{Z}_1(\kappa_2)}\chi_{\mathcal{L}}(z)^{j_1+1}\\
\times\chi_{\mathcal{L}}(w)^{j_2+1}d\ln\left|\frac{z^m-w^m}{z^m-\overline{w}^m}\right|.
\end{aligned}\tag{7.16}$$

Integrating by parts, we obtain that the random variables $\{\mathcal{M}_j^{\kappa}\}_{\kappa\in(0,1),j\in\mathbb{N}}$ are Gaussian with mean 0 and covariance

$$\operatorname{cov}(\mathcal{M}_{j_1}^{\kappa_1}, \mathcal{M}_{j_2}^{\kappa_2}) = \oint_{z\in\mathbb{S}:\kappa_{\mathfrak{L}}(z)=\kappa_1} \oint_{w\in\mathbb{S}:\kappa_{\mathfrak{L}}(w)=\kappa_2} \chi_{\mathfrak{L}}(z)^{j_1} \times \chi_{\mathfrak{L}}(w)^{j_2} \frac{d\chi_{\mathfrak{L}}(z)}{dz}\frac{d\chi_{\mathfrak{L}}(w)}{dw}\mathcal{G}_{\mathbb{S}}(z,w)\,dzdw, \tag{7.17}$$

where $\mathcal{G}_{\mathbb{S}}(z,w)$ is Green's function on $\mathbb{S}$ given by

$$\frac{1}{2\pi}\ln\left|\frac{z^m - w^m}{z^m - \overline{w}^m}\right|.$$

Then integration by parts shows that the right-hand sides of (7.16) and (7.17) are equal. ∎

8. Gaussian free field in piecewise boundary condition

In this section, we prove Theorem 2.13.

For $1 \le i \le n$, let

$$F_{i,\kappa}(z) = \frac{1}{n}\frac{z}{z-1} + \frac{1}{1-\kappa} z A_i(z),$$

where $A_i(z)$ is defined by (6.12).

Lemma 8.1. *For any $\chi > 0$, $\kappa \in (0,1)$ and $1 \le i \le n$, the equation*

$$F_{i,\kappa}(z) = \frac{\chi}{1-\kappa} \tag{8.1}$$

has at most one pair of complex conjugate roots.

Proof. See [32, Proposition 7.2]. ∎

Let J_i be defined as in (5.3). Under Assumption 5.2, we may assume that

$$J_i = \begin{cases} \{d_i, d_i+1, \dots, d_{i+1}-1\} & \text{if } 1 \le i \le n-1, \\ \{d_n, d_n+1, \dots, s\} & \text{if } i = n, \end{cases}$$

where for $1 \le i \le n$,

$$1 = d_1 < d_2 < \cdots < d_n \le s.$$

Lemma 8.2. *Let $1 \leq i \leq n$. Let $\mathcal{S}_i$ consist of all the (χ, κ) in $\mathcal{R}$ (the rescaled square-hexagon lattice $\frac{1}{N}\mathcal{R}(\Omega, \check{c})$ in the limit as $N \to \infty$) such that equation (8.1) has exactly one pair of complex conjugate roots. Let $\mathbb{H}$ be the upper half-plane defined by*

$$\mathbb{H} = \{z : \operatorname{Im} z > 0\}.$$

The mapping

$$T_{\mathcal{S}_i} : \mathcal{S}_i \to \mathbb{H}$$

maps $(\chi, \kappa) \in \mathcal{S}_i$ to $t := \mathrm{St}_{\mathbf{m}_i}^{(-1)}(\log z)$, where z is the unique root of (8.1) in $\mathbb{H}$, and $\mathrm{St}_{\mathbf{m}_i}$ is the Stieltjes transform of the measure $\mathbf{m}_i$ defined by (3.7). Let

$$p(t) = \sum_{l \in I_2 \cap \{1,2,\ldots,n\}} \frac{y_l x_1 \exp[\mathrm{St}_{\mathbf{m}_i}(t)]}{1 + y_l x_1 \exp[\mathrm{St}_{\mathbf{m}_i}(t)]},$$

$$q(t) = \frac{\exp(\mathrm{St}_{\mathbf{m}_i}(t))}{\exp(\mathrm{St}_{\mathbf{m}_i}(t)) - 1}.$$

Then $T_{\mathcal{S}_1}$ is a homeomorphism with inverse $t \to (\chi_{\mathcal{S}_1}(t), \kappa_{\mathcal{S}_1}(t))$ for all $t \in \mathbb{H}$ given by

$$\chi_{\mathcal{S}_1}(t) = \frac{\bar{t}(p(t) - q(t)) - t(p(\bar{t}) - q(\bar{t})) - (n-1)(\bar{t} - t)}{n[p(t) - p(\bar{t}) - q(t) + q(\bar{t})]}, \tag{8.2}$$

$$\kappa_{\mathcal{S}_1}(t) = \frac{\bar{t} - t}{p(t) - p(\bar{t}) - q(t) + q(\bar{t})}, \tag{8.3}$$

and for $2 \leq i \leq n$,

$$\chi_{\mathcal{S}_i}(t) = \frac{\bar{t}q(t) - tq(\bar{t}) + (\bar{t} - t)(n - i)}{n[q(t) - q(\bar{t})]}, \tag{8.4}$$

$$\kappa_{\mathcal{S}_i}(t) = \frac{\bar{t} - t}{-q(t) + q(\bar{t})}. \tag{8.5}$$

Proof. The proof of the lemma is an adaptation of [9, Proposition 6.2]; see also [12, Theorem 2.1]. ■

Then Theorem 2.13 follows from arguments similar to those used in the proof of [9, Theorem 6.3].

Acknowledgments. Zhongyang Li thanks Vadim Gorin for helpful discussions. The author thanks anonymous referees for carefully reading the paper.

Funding. The research is supported by National Science Foundation grant DMS 1608896.

References

[1] A. Borodin, Periodic Schur process and cylindric partitions. *Duke Math. J.* **140** (2007), no. 3, 391–468 Zbl 1131.22003 MR 2362241

[2] A. Borodin, Schur dynamics of the Schur processes. *Adv. Math.* **228** (2011), no. 4, 2268–2291 Zbl 1234.60012 MR 2836121

[3] A. Borodin and P. L. Ferrari, Random tilings and Markov chains for interlacing particles. *Markov Process. Related Fields* **24** (2018), no. 3, 419–451 Zbl 1401.05061 MR 3821250

[4] C. Boutillier, J. Bouttier, G. Chapuy, S. Corteel, and S. Ramassamy, Dimers on rail yard graphs. *Ann. Inst. Henri Poincaré D* **4** (2017), no. 4, 479–539 Zbl 1391.82008 MR 3734415

[5] C. Boutillier and Z. Li, Limit shape and height fluctuations of random perfect matchings on square-hexagon lattices. *Ann. Inst. Fourier (Grenoble)* **71** (2021), no. 6, 2305–2386 Zbl 1492.82009 MR 4448539

[6] A. Bufetov and V. Gorin, Representations of classical Lie groups and quantized free convolution. *Geom. Funct. Anal.* **25** (2015), no. 3, 763–814 Zbl 1326.22012 MR 3361772

[7] A. Bufetov and V. Gorin, Fluctuations of particle systems determined by Schur generating functions. *Adv. Math.* **338** (2018), 702–781 Zbl 1400.82064 MR 3861715

[8] A. Bufetov and V. Gorin, Fourier transform on high-dimensional unitary groups with applications to random tilings. *Duke Math. J.* **168** (2019), no. 13, 2559–2649 Zbl 1436.60016 MR 4007600

[9] A. Bufetov and A. Knizel, Asymptotics of random domino tilings of rectangular Aztec diamonds. *Ann. Inst. Henri Poincaré Probab. Stat.* **54** (2018), no. 3, 1250–1290 Zbl 1401.60174 MR 3825881

[10] H. Cohn, R. Kenyon, and J. Propp, A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** (2001), no. 2, 297–346 Zbl 1037.82016 MR 1815214

[11] B. de Tilière, Scaling limit of isoradial dimer models and the case of triangular quadri-tilings. *Ann. Inst. H. Poincaré Probab. Statist.* **43** (2007), no. 6, 729–750 Zbl 1176.82008 MR 3252428

[12] E. Duse and A. Metcalfe, Asymptotic geometry of discrete interlaced patterns: Part I. *Internat. J. Math.* **26** (2015), no. 11, article no. 1550093 Zbl 1359.60021

[13] V. Gorin and G. Panova, Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory. *Ann. Probab.* **43** (2015), no. 6, 3052–3132 Zbl 1390.05240 MR 3433577

[14] G. R. Grimmett and Z. Li, Critical surface of the hexagonal polygon model. *J. Stat. Phys.* **163** (2016), no. 4, 733–753 Zbl 1342.82032 MR 3488571

[15] G. R. Grimmett and Z. Li, The 1-2 model. In *In the tradition of Ahlfors–Bers. VII*, pp. 139–152, Contemp. Math. 696, American Mathematical Society, Providence, RI, 2017 Zbl 1390.82013 MR 3715445

[16] G. R. Grimmett and Z. Li, Critical surface of the 1-2 model. *Int. Math. Res. Not. IMRN* **2018** (2018), no. 21, 6617–6672 Zbl 1407.82020 MR 3873540

[17] A. Guionnet and M. Maïda, A Fourier view on the R-transform and related asymptotics of spherical integrals. *J. Funct. Anal.* **222** (2005), no. 2, 435–490 Zbl 1065.60023 MR 2132396

[18] P. W. Kasteleyn, The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice. *Physica* **27** (1961), no. 12, 1209–1225 Zbl 1244.82014

[19] R. Kenyon, Local statistics of lattice dimers. *Ann. Inst. H. Poincaré Probab. Statist.* **33** (1997), no. 5, 591–618 Zbl 0893.60047 MR 1473567

[20] R. Kenyon, Conformal invariance of domino tiling. *Ann. Probab.* **28** (2000), no. 2, 759–795 Zbl 1043.52014 MR 1782431

[21] R. Kenyon, Dominos and the Gaussian free field. *Ann. Probab.* **29** (2001), no. 3, 1128–1137 Zbl 1034.82021 MR 1872739

[22] R. Kenyon and A. Okounkov, Planar dimers and Harnack curves. *Duke Math. J.* **131** (2006), no. 3, 499–524 Zbl 1100.14047 MR 2219249

[23] R. Kenyon and A. Okounkov, Limit shapes and the complex Burgers equation. *Acta Math.* **199** (2007), no. 2, 263–302 Zbl 1156.14029 MR 2358053

[24] R. Kenyon, A. Okounkov, and S. Sheffield, Dimers and amoebae. *Ann. of Math. (2)* **163** (2006), no. 3, 1019–1056 Zbl 1154.82007 MR 2215138

[25] Z. Li, Critical temperature of periodic Ising models. *Comm. Math. Phys.* **315** (2012), no. 2, 337–381 Zbl 1259.82024 MR 2971729

[26] Z. Li, 1-2 model, dimers, and clusters. *Electron. J. Probab.* **19** (2014), article no. 48 Zbl 1314.60036 MR 3217336

[27] Z. Li, Spectral curves of periodic Fisher graphs. *J. Math. Phys.* **55** (2014), no. 12, article no. 123301 Zbl 1316.82010 MR 3390555

[28] Z. Li, Uniqueness of the infinite homogeneous cluster in the 1-2 model. *Electron. Commun. Probab.* **19** (2014), article no. 23 Zbl 1296.82014 MR 3197119

[29] Z. Li, Conformal invariance of dimer heights on isoradial double graphs. *Ann. Inst. Henri Poincaré D* **4** (2017), no. 3, 273–307 Zbl 1377.82019 MR 3713018

[30] Z. Li, Asymptotics of Schur functions on almost staircase partitions. *Electron. Commun. Probab.* **25** (2020), article no. 51 Zbl 1453.82013 MR 4129675

[31] Z. Li, Limit shape of perfect matchings on contracting bipartite graphs. *Int. Math. Res. Not. IMRN* **2022** (2022), no. 12, 9173–9208 Zbl 07542588 MR 4436205

[32] Z. Li, Schur function at general points and limit shape of perfect matchings on contracting square hexagon lattices with piecewise boundary conditions. *Int. Math. Res. Not. IMRN* **2022** (2022), no. 22, 17591–17661 Zbl 1505.05133 MR 4514451

[33] Z. Li, Limit shape of perfect matchings on rail-yard graphs. *Int. Math. Res. Not. IMRN* **2023** (2023), no. 14, 12231–12297

[34] Z. Li and M. Vuletić, Asymptotics of pure dimer coverings on rail-yard graphs. 2022, arXiv:2110.11393

[35] I. G. Macdonald, *Symmetric functions and Hall polynomials*. Oxford Math. Monogr., The Clarendon Press, Oxford University Press, Oxford, 1998 Zbl 0899.05068

[36] A. Okounkov and N. Reshetikhin, Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram. *J. Amer. Math. Soc.* **16** (2003), no. 3, 581–603 Zbl 1009.05134 MR 1969205

[37] L. Petrov, Asymptotics of random lozenge tilings via Gelfand–Tsetlin schemes. *Probab. Theory Related Fields* **160** (2014), no. 3–4, 429–487 Zbl 1315.60013 MR 3278913

[38] L. Petrov, Asymptotics of uniformly random lozenge tilings of polygons. Gaussian free field. *Ann. Probab.* **43** (2015), no. 1, 1–43 Zbl 1315.60062 MR 3298467

[39] S. Sheffield, Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** (2007), no. 3–4, 521–541 Zbl 1132.60072 MR 2322706

[40] H. N. V. Temperley and M. E. Fisher, Dimer problem in statistical mechanics – an exact result. *Philos. Mag. (8)* **6** (1961), 1061–1063 Zbl 0126.25102 MR 136398

Communicated by Richard W. Kenyon

Received 6 February 2022; revised 23 August 2022.

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