Combinatorics of the quantum symmetric simple exclusion process, associahedra and free cumulants

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Abstract. The quantum symmetric simple exclusion process (QSSEP) is a model of quantum particles hopping on a finite interval and satisfying the exclusion principle. Recently, Bernard and Jin have studied the fluctuations of the invariant measure for this process, when the number of sites goes to infinity. These fluctuations are encoded into polynomials, for which they have given equations and proved that these equations determine the polynomials completely. In this paper, we give an explicit combinatorial formula for these polynomials, in terms of Schröder trees. We also show that, quite surprisingly, these polynomials can be interpreted as free cumulants of a family of commuting random variables.

1. Introduction

The exclusion process is a model of particles hopping on a one-dimensional lattice, which has been extensively studied in statistical mechanics, probability, and combinatorics, see, e.g., [5] for an overview. Recently, quantum versions of these processes have been introduced and the study of their fluctuations has been undertaken. In particular, Bernard and Jin [1] have studied the invariant measure of this quantum process and shown that the asymptotic behaviour of the fluctuations of this measure, when the number of sites grows, is encoded into a family of polynomials (called "loop polynomials") $Q_\sigma(x_1, \ldots, x_n)$ indexed by cyclic permutations σ of $1, 2, \ldots, n$. The first few values of these polynomials, for permutations of small size, are given by (we write a cyclic permutation of $1, \ldots, n$ as the sequence $1, \sigma(1), \sigma^2(1), \ldots$)

$$Q_1 = x_1,$$
$$Q_{12} = x_1(1 - x_2),$$
$$Q_{123} = x_1(1 - 2x_2)(1 - x_3) = Q_{132},$$
$$Q_{1234} = x_1(1 - 2x_3 - 3x_2 + 5x_2x_3)(1 - x_4) = Q_{1243} = Q_{1432} = Q_{1342},$$
$$Q_{1324} = x_1(1 - x_3 - 4x_2 + 5x_2x_3)(1 - x_4) = Q_{1423}.$$

They point out that the coefficients of these polynomials are integers and seem to be connected with a well-known combinatorial and geometric object, the associahedron. In this paper, we provide an explicit combinatorial expression for these polynomials. More precisely, we prove that these polynomials can be written as sums of monomials indexed by Schröder trees, which are themselves in natural bijection with the faces of the associahedron. Moreover, this combinatorial formula also shows that these polynomials can be interpreted as free cumulants of some family of commuting random variables. Free cumulants have been introduced in the theory of free probability by Speicher, as an analogue of the cumulants appearing in probability theory. They are very efficient tools for making computations, with many applications to the theory of random matrices. See, e.g., the references [6, 7] for an overview of these fields. It is however quite surprising to see these quantities appearing in the analysis of the QSSEP.

This paper is organized as follows: in the next section, we describe the physical model for the quantum symmetric simple exclusion process and how the loop polynomials arise from the analysis of the fluctuations of the two-point correlation functions. In Section 3, we recall the explicit characterization of the loop polynomials obtained in [1], and in Section 4, we introduce some combinatorial notions needed to understand the explicit formula for the loop polynomials. These are non-crossing partitions, free cumulants, Schröder trees, the cluster complex and the associahedra. The connections between Schröder trees and free cumulants have been explored in [4] (see also [2, Section 1.2]), and we use some of their results. Section 5 is devoted to stating and proving the combinatorial formula for the loop polynomials. Finally, in Section 6, we show that the loop polynomials can also be interpreted as free cumulants of a family of commuting random variables.

2. The model

In this section, we describe briefly the physical model at the origin of the construction of the loop polynomials. A more precise description can be found in [1], to which we refer for further information. Our aim here is just to explain the physical context in which the loop polynomials appear. Readers interested only in the combinatorial aspect of the problem can jump directly to the next section, where we give the precise algebraic definition of the loop polynomials.

Let $c_i^\dagger$, c_i, indexed by integers $i = 0, 1, \ldots, N - 1$, be a family of fermionic creation and annihilation operators, satisfying the usual anticommutation relations

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta.$$

These operators are defined on a Hilbert space V of dimension 2^N and describe a system of fermionic particles situated at the points $0, 1, \ldots, N-1$. The quantum symmetric exclusion process has a Hamiltonian H_t, which is a random self-adjoint operator on V, depending on the time $t \geq 0$ and satisfying a stochastic differential equation

$$dH_t = \sum_{j=0}^{N-1} c_{j+1}^\dagger c_j \, dW_t^j + c_j^\dagger c_{j+1} d\bar{W}_t^j,$$

where W_t^j, $j = 0, \ldots, N-1$, are a family of independent complex Brownian motions. The Hamiltonian describes the motions of the quantum particles which, due to Brownian noise, can jump from their site to one of the nearest neighbouring sites. It is supplemented by boundary conditions at the points 0 and $N-1$, where the system is in contact with a reservoir, with which it can exchange particles. From the Hamiltonian, one deduces the equation of motion for the density matrix ρ_t, a positive self-adjoint matrix on V, with $\mathrm{Tr}(\rho_t) = 1$:

$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathscr{L}_{bdry}(\rho_t)dt.$$

Here $\mathscr{L}_{bdry}$ is a linear operator of Linblad form describing the boundary conditions at the points 0 and $N-1$. As $t \to \infty$, the density matrix converges in distribution to a random density matrix ρ which represents the stationary distribution of the process.

The quantities of interest are the fermion two-point functions in the steady measure, namely

$$G_{ij} = \mathrm{Tr}(c_j^\dagger c_i \rho),$$

which are random variables, and their connected correlation functions (known as cumulants in mathematical language) defined for families of integers $i_1, i_2, \ldots, i_p$, $j_1, j_2, \ldots, j_p \in [0, N-1]$ by

$$E[G_{i_1 j_1} G_{i_2 j_2} \ldots G_{i_p j_p}]^c.$$

In the limit of $N \to \infty$, the leading cumulants scale as N^{-p+1} and only the ones for which $j_1, \ldots, j_p$ is a cyclic permutation of $i_1, \ldots, i_p$ have a nonzero limit. More precisely, if $i_1/N, i_2/N, \ldots, i_p/N$ converge to real numbers $u_1, u_2, \ldots, u_p \in [0, 1]$ as $N \to \infty$, then

$$E[G_{i_1 i_p} G_{i_p i_{p-1}} \ldots G_{i_2 i_1}]^c = \frac{1}{N^{p-1}} g_p(u_1, \ldots, u_p) + O\left(\frac{1}{N^p}\right)$$

for some functions g_p. The g_p turn out to be piecewise polynomial functions, being given by a certain polynomial in each sector corresponding to an ordering of the u_i. In order to study the g_p, it is thus convenient to consider their restrictions to these

different sectors. Specifically, introduce functions $Q_\sigma(x_1, \ldots, x_p)$ defined for $0 \leq x_1 \leq x_2 \leq \cdots \leq x_p \leq 1$, indexed by circular permutations σ of $1, \ldots, p$ which satisfy

$$E[G_{i_1 i_{\sigma^{p-1}(1)}} G_{i_{\sigma^{p-1}(1)} i_{\sigma^{p-2}(1)}} \ldots G_{i_{\sigma(1)} i_1}]^c$$
$$= \frac{1}{N^{p-1}} Q_\sigma(x_1, \ldots, x_p) + O\left(\frac{1}{N^p}\right). \tag{2.1}$$

Here we assume that the limits of the i_k/N exist as $N \to \infty$ and correspond, once ordered, to the points $x_1, \ldots, x_p$. The permutation σ encodes the order in which the x_i occur in the left-hand side of (2.1). The Q_σ are polynomials and they are determined by some algebraic conditions which we describe in the next section. While the algebraic definition of the loop polynomials is unambiguous, the fact that they satisfy (2.1) is not proved in full generality and still partly conjectural. It would be nice to have a complete proof of this fact.

3. Characterization of the loop polynomials

In this section, we give the characterization of the loop polynomials Q_σ appearing in formula (2.1) following [1]. Whereas their original definition is valid when the x_i form a set of ordered real numbers in the interval $[0, 1]$, the polynomials Q_σ can be defined purely algebraically as follows.

Definition 3.1. The loop polynomials form a family of polynomials $Q_\sigma(x_1, \ldots, x_n)$, indexed by circular permutations of $1, \ldots, n$, with the following properties:

(1) Q_σ is of degree 1 in each of the variables x_i.

(2) If $n = 1$, then $Q_1(x_1) = x_1$; if $n = 2$, then $Q_{12}(x_1, x_2) = x_1(1 - x_2)$.

(3) For $n \geq 2$, they satisfy the boundary conditions

$$Q_\sigma = x_1 P_\sigma(x_2, \ldots, x_{n-1})(1 - x_n),$$

where P_σ is a polynomial in $n - 2$ variables.

(4) For $i = 1, 2, \ldots, n - 1$, they satisfy a continuity condition

$$Q_\sigma|_{x_i = x_{i+1}} = Q_{s_i \sigma s_i}|_{x_i = x_{i+1}}.$$

(5) For $i = 1, 2, \ldots, n - 1$, they satisfy an exchange relation

$$([x_i](Q_\sigma + Q_{s_i \sigma s_i}))|_{x_i = x_{i+1}} - ([x_{i+1}](Q_\sigma + Q_{s_i \sigma s_i}))|_{x_i = x_{i+1}}$$
$$= 2([x_i]Q_{\sigma^-}(x^-))([x_{i+1}]Q_{\sigma^+}(x^+)).$$

Here s_i is the transposition $(i\ i+1)$ and, if P is a polynomial in several variables, $[x]P$ denotes the coefficient of the monomial x in P. It is a polynomial in the remaining variables. Since σ is a cycle, the permutation $s_i\sigma = \sigma^+\sigma^-$ is the product of two disjoint cycles, denoted by σ^+, which moves $i+1$, and σ^-, which moves i. The variables denoted by x^- are those with the indices moved by σ^- and similarly for x^+.

Using the relations in the definition above, Bernard and Jin prove in [1] that the polynomials Q_σ are uniquely defined, moreover they show that the Q_σ, where $\sigma = (123\ldots n)$, can be computed by induction on n.

Let us explain in more details, following [1], how conditions (3), (4), (5) can be used to compute the loop polynomials. Fix some integer $i \in [1, n-1]$. Since the polynomials have degree one in each variable, it is possible to expand Q_σ as

$$Q_\sigma = A + x_i B + x_{i+1} C + x_i x_{i+1} D,$$

where the polynomials A, B, C, D do not depend on x_i, x_{i+1}. If we do the same for $Q_{s_i\sigma s_i}$, namely

$$Q_{s_i\sigma s_i} = A' + x_i B' + x_{i+1} C' + x_i x_{i+1} D',$$

then (4) implies that

$$A = A', \quad D = D', \quad B + C = B' + C'.$$

Let us consider the factorization $s_i\sigma = \sigma^-\sigma^+$ and the polynomial

$$\Delta := ([x_i]Q_{\sigma^-}(x^-))([x_{i+1}]Q_{\sigma^+}(x^+)).$$

The continuity (4) and exchange (5) conditions imply that

$$B - C' = B' - C = \Delta. \tag{3.1}$$

Since all cyclic permutations are conjugated, these relations allow thus to compute Q_σ for any cycle of length n if we know the value at one particular cycle of length n and the values for cycles of smaller length. Since the value of $Q_{123\ldots n}$ can be computed by induction on n, this shows that the polynomials Q_σ are uniquely defined. It is not clear a priori, from these purely algebraic properties, that the polynomials exist, since there might be too many constraints on them, in particular it is not clear why the boundary conditions (3) will be fulfilled, but our explicit formula below will provide a direct proof that, indeed, polynomials satisfying all these properties exist.

As an example, let $\sigma = (1234)$ and $i = 2$ so that $s_i = s_2 = (23)$ and $s_2(1234)s_2 = (1324)$. One has

$$Q_\sigma = x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4)$$

so that

$$A = x_1, \quad B = -3x_1(1 - x_4), \quad C = -2x_1(1 - x_4), \quad D = 5x_1(1 - x_4),$$

moreover

$$s_2\sigma = (134)$$

so that $\sigma^- = (2)$, $x^- = \{x_2\}$ and $\sigma^+ = (134)$, $x^+ = \{x_1, x_3, x_4\}$. Thus

$$Q_{\sigma^-} = x_2, \qquad\qquad [x_2]Q_{\sigma^-} = 1,$$
$$Q_{\sigma^+} = x_1(1 - 2x_3)(1 - x_4), \quad [x_3]Q_{\sigma^+} = -2x_1(1 - x_4).$$

It follows that $\Delta = -2x_1(1 - x_4)$, therefore, by equation (3.1), one has

$$B' = C + \Delta = -4x_1(1 - x_4) \quad \text{and} \quad C' = B - \Delta = -x_1(1 - x_4).$$

Thus, from the knowledge of Q_{1234}, we find the polynomial

$$Q_{1324} = x_1(1 - 4x_2 - x_3 + 5x_2x_3)(1 - x_4).$$

Here are a few more values, for cyclic permutations of length 5, when the polynomials can take four different values:

$$Q_{12345} = x_1(1 - 4x_2 - 3x_3 - 2x_4 + 9x_2x_3 + 7x_2x_4 + 5x_3x_4 - 14x_2x_3x_4)(1 - x_5),$$
$$Q_{13245} = x_1(1 - 6x_2 - x_3 - 2x_4 + 9x_2x_3 + 10x_2x_4 + 2x_3x_4 - 14x_2x_3x_4)(1 - x_5),$$
$$Q_{12435} = x_1(1 - 4x_2 - 4x_3 - x_4 + 12x_2x_3 + 4x_2x_4 + 5x_3x_4 - 14x_2x_3x_4)(1 - x_5),$$
$$Q_{14235} = x_1(1 - 6x_2 - 2x_3 - x_4 + 12x_2x_3 + 7x_2x_4 + 2x_3x_4 - 14x_2x_3x_4)(1 - x_5).$$

Indeed, one can check that

$$Q_{12345} = Q_{13452} = Q_{14523} = Q_{15234} = Q_{15432} = Q_{12543}$$
$$= Q_{13254} = Q_{14325},$$
$$Q_{13245} = Q_{13254} = Q_{15423} = Q_{14523},$$
$$Q_{12435} = Q_{14352} = Q_{15342} = Q_{12534},$$
$$Q_{14235} = Q_{13524} = Q_{15234} = Q_{15324} = Q_{14253} = Q_{13425}$$
$$= Q_{15243} = Q_{14325}.$$

All these examples suggest a combinatorial significance of the loop polynomials. In particular, as pointed out in [1], the sums of the absolute values of the coefficients are Schröder numbers (whose definition is recalled in the next section). We will give a combinatorial formula for the loop polynomials in Section 5, after introducing the objects necessary to describe it in the next section.

4. Some combinatorial objects

This section is devoted to a description of the combinatorial objects which will be used for expressing the loop polynomials.

4.1. Cumulants

First we recall briefly the classical theory of cumulants. In probability theory, the *cumulants* are multilinear expressions $C_n(a_1, \ldots, a_n)$ in random variables which arise in the expansion of the free energy as

$$\log E[e^{\sum_{i=1}^{N} \lambda_i a_i}] = \sum_{n=1}^{\infty} \sum_{i_1 + \cdots + i_N = n} \frac{\lambda_1^{i_1}}{i_1!} \cdots \frac{\lambda_1^{i_N}}{i_N!} C_n(a_I),$$

where E denotes the expectation and $a_I = (a_1, a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_N, \ldots, a_N)$ with i_k occurrences of a_k.

There is a combinatorial way to define the cumulants using the set-partitions of $\{1, \ldots, n\}$, which form a lattice $\mathscr{P}_n$, via the implicit formula

$$E[a_1 \ldots a_n] = \sum_{\pi \in \mathscr{P}_n} C_\pi(a_1, \ldots, a_n), \tag{4.1}$$

where

$$C_\pi(a_1, \ldots, a_n) = \prod_{p \in \pi} C_{|p|}(a_{i_1}, \ldots, a_{i_{|p|}})$$

the product being over the parts p of π with $p = \{i_1, \ldots, i_{|p|}\}$ and $i_1 < i_2 < \cdots < i_{|p|}$. This formula can be inverted to express the cumulants in terms of the "moments", i.e., E evaluated on products of the a_i:

$$C_n(a_1, \ldots, a_n) = \sum_{\pi \in \mathscr{P}_n} E_\pi(a_1, \ldots, a_n)(-1)^{|\pi|-1}(|\pi| - 1)!.$$

### 4.2. Non-crossing partitions and free cumulants

4.2.1. Non-crossing partitions. A set partition of $\{1, 2, \ldots, n\}$ (or any other totally ordered set) is called *non-crossing* if there is no quadruple i, j, k, l such that $i < j < k < l$ while i, k belong to some part of the partition and j, l belong to another part. One can picture a non-crossing partition by putting the points $1, \ldots, n$ in cyclic order on a circle and drawing, for each part of the partition, the convex polygon whose vertices are the elements of the part. The partition is non-crossing if and only if these polygons are disjoint. This construction shows, in particular, that if π is a non-crossing

partition, then its image by the cyclic permutation $(123\ldots n)$ is still non-crossing. For example, here is the non-crossing partition $\pi = \{1, 3, 4\} \cup \{2\} \cup \{5, 6\} \cup \{7\} \cup \{8\}$:
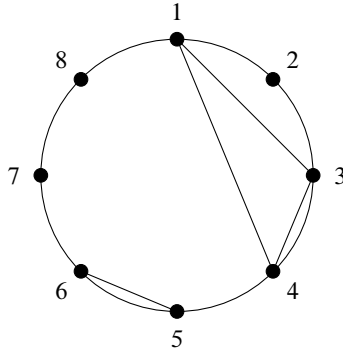
We denote by $\mathrm{NC}(n)$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$. The non-crossing partitions are counted by the Catalan numbers: there are $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ non-crossing partitions of $\{1, 2, \ldots, n\}$. Moreover, the set $\mathrm{NC}(n)$ endowed with the reverse refinement order is a lattice, it is ranked by $\mathrm{rk}(\pi) = n - |\pi|$, where $|\pi|$ is the number of parts of π. We refer to [7] for an in-depth study of the set of non-crossing partitions and for the proofs of all statements about non-crossing partitions.

4.2.2. Kreweras complement and the Möbius function. The *Kreweras complement* of a non-crossing partition is obtained by putting primed points $1', 2', \ldots$ between consecutive points on the circle and forming the largest possible polygons with primed vertices which do not cross the polygons of the original partition. In our example above, we get the partition $K(\pi) = \{1, 5, 7, 8\} \cup \{2, 3\} \cup \{4\} \cup \{6\}$ (where we have removed the primes), see the picture below:
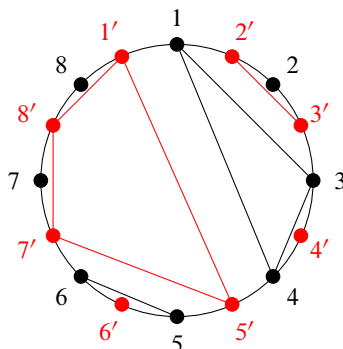
The Kreweras complement is an anti-isomorphism for the order on $\mathrm{NC}(n)$. Note that our definition of the Kreweras complement does not coincide with that of [7], rather, it corresponds to the inverse of the Kreweras complement of [7]. We use this version of the Kreweras map because it is slightly easier to use in our computations.

Recall that for a partially ordered set, its *zeta function* is defined as $\zeta(x, y) = 1$ if $x \le y$ and $\zeta(x, y) = 0$ if not. The *Möbius function* satisfies $\mu(x, y) = 0$ unless $x \le y$ and, for $x \le z$,

$$\sum_{y : x \le y \le z} \mu(x, y)\zeta(x, z) = \delta_{xz}.$$

If we consider ζ as an upper triangular matrix encoding the order relation, then μ is the inverse matrix.

We will consider the Möbius function of $NC(n)$ and put $\mu(\pi) := \mu(\pi, 1_n)$, where 1_n is the partition with one part. The function μ can be expressed, using the Kreweras complement, as

$$\mu(\pi) = \prod_{p \text{ part of } K(\pi)} (-1)^{|p|-1} \text{Cat}_{|p|-1}. \tag{4.2}$$

As recalled in the introduction, free cumulants were introduced by Speicher in the theory of free probability as analogues of cumulants in probability theory. The setting is the following: we consider a unital algebra A over some field k, which is usually taken to be the complex numbers for applications to probability, although the theory can be developed, for a large part, within a purely algebraic framework. This algebra is endowed with a linear form φ such that $\varphi(1) = 1$. The free cumulants form a family $\kappa_n, n = 1, 2, 3, \ldots$, of multilinear forms on A such that κ_n is an n-linear form. Define, for a non-crossing partition π of $[1, n]$, an n-linear form κ_π on A as

$$\kappa_\pi(a_1, \ldots, a_n) = \prod_{p \text{ part of } \pi} \kappa_{|p|}(a_{i_1}, a_{i_2}, \ldots, a_{i_{|p|}}).$$

Here the product is over the parts p of the partition π and, for a part p, we denote by $|p|$ its size and by $i_1, \ldots, i_{|p|}$ its elements listed in increasing order. The following relation defines implicitly the free cumulants as the only sequence κ_n satisfying it for all n and all $a_1, \ldots, a_n \in A$:

$$\varphi(a_1 a_2 \ldots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi(a_1, a_2, \ldots, a_n). \tag{4.3}$$

It is analogous to the combinatorial formula for cumulants (4.1). Using the Möbius function of $NC(n)$, one can invert relation (4.3) and express the free cumulants explicitly as

$$\kappa_n(a_1, a_2, \ldots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \ldots, a_n)\mu(\pi). \tag{4.4}$$

Here φ_π is defined in terms of φ by a formula similar to that for the κ_π:

$$\varphi_\pi(a_1, \ldots, a_n) = \prod_{p \text{ part of } \pi} \varphi(a_{i_1} a_{i_2} \ldots a_{i_{|p|}}).$$

4.3. Schröder trees and associahedra

4.3.1. Schröder trees. *Schröder trees* are plane, rooted trees such that each internal vertex has at least two descendants. Such trees are counted, in terms of the number of leaves, by the small Schröder numbers $s_n = 1, 1, 3, 11, 45, \ldots$ for $n = 1, 2, 3, \ldots$ with generating series $\frac{1+x-\sqrt{1-6x+x^2}}{4x}$ (the case $n = 1$ is special in that the root is not an internal vertex, it is a leaf). These numbers form the sequence A001003 in [8]. The set of binary trees is a subset, counted again by Catalan numbers: there are Cat_{n-1} binary trees with n leaves. Figure 1 shows the Schröder trees with four leaves. The leaves are the white vertices, the internal vertices are black and the root is denoted by a square. The upper row shows the binary trees, which have the maximal number of vertices (there are $\text{Cat}_3 = 5$ of them in the case of four leaves shown here). For each binary tree, we can contract some of its left internal edges (an internal edge is an edge joining two internal vertices) in order to get a Schröder tree. Below each binary tree, we show the Schröder trees obtained in this way. The first column has four trees, corresponding to the upper left binary tree with two left internal edges and the trees obtained by contracting these edges. The next three columns correspond to binary trees with one left internal edge. Finally, the last binary tree has no left internal edge.
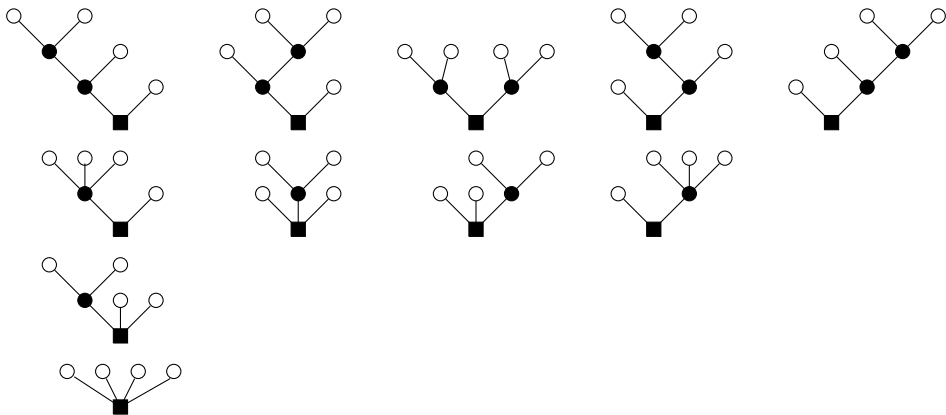

Figure 1. Schröder trees with four leaves.

4.3.2. Prime Schröder trees. A Schröder tree is called *prime* if the rightmost edge of its root is a leaf. The prime Schröder trees are counted by the large Schröder numbers

$$S_n = 2s_{n-1}.$$

Indeed, given a Schröder tree t with $n - 1$ leaves, we can build two prime Schröder trees with n leaves, t_1 and t_2: either one appends a new leaf to the right of the root

of the tree to get t_1, or one builds a new tree by putting the Schröder tree on the left branch of a binary tree with two leaves to get t_2, see the picture below:

It is easy to see that each prime Schröder tree is uniquely obtained in one of these ways from a Schröder tree with one leaf less. We denote by $\mathcal{S}_n$ the set of Schröder trees with n leaves and by $p\mathcal{S}_n \subset \mathcal{S}_n$ that of prime Schröder trees. There are 6 prime Schröder trees with 4 leaves, and they are in the first two columns of Figure 1.

4.3.3. Corners. A *corner* in a Schröder tree is a pair of edges leading to consecutive descendants (from left to right) of an internal vertex. Such a corner is said to belong to the vertex from which the descendants are originated. A Schröder tree with n leaves has exactly $n - 1$ corners. Let t be a Schröder tree with n leaves. Label the corners of the tree from left to right by the numbers $1, 2, \ldots, n - 1$, in a contour exploration of the tree. Here is an example, with a prime Schröder tree having 6 internal vertices and 11 corners:
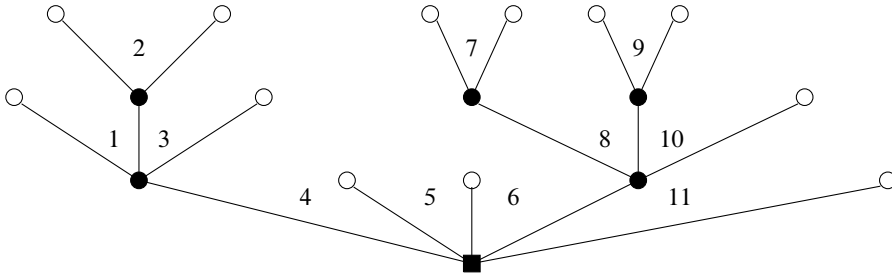

Consider the partition of $\{1, 2, \ldots, n - 1\}$ such that i and j are in the same part if and only if they label corners which belong to the same vertex. It is easy to see that this partition is non-crossing, and we denote it by $\pi(t)$. The number of parts of $\pi(t)$ is equal to the number of internal vertices of t. In our example, the partition is formed by the sets

$$\{1, 3\}, \ \{2\}, \ \{4, 5, 6, 11\}, \ \{7\}, \ \{8, 10\}, \ \{9\}.$$

There is a natural operad built on Schröder trees, which has been used in [4] to recover the Möbius function on $NC(n)$. We will now use some of their constructions. Consider a prime Schröder tree. One obtains a forest from the tree by removing, for each internal vertex, the internal edges pointing out of this vertex, except the leftmost and

rightmost ones. The forest thus obtained is a system of non-crossing binary trees. Here is our example with the leaves labelled from left to right:
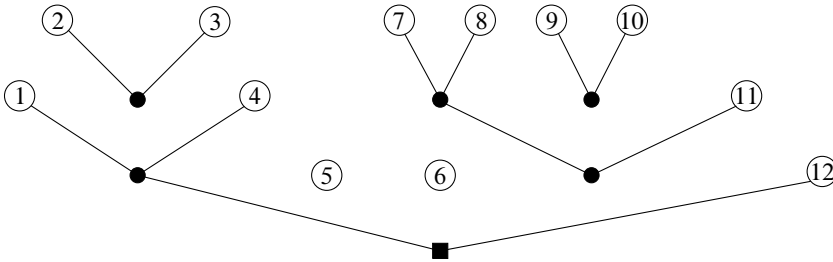
This forest defines a non-crossing partition of the leaves of the tree such that the first and last leaves are in the same component. If we remove this last leaf, then this non-crossing partition is $K(\pi(t))$, the Kreweras complement of $\pi(t)$ (here we use the fact that the Schröder tree is prime). In our example, the non-crossing partition of the leaves is

$$\{1, 4, 12\} \cup \{2, 3\} \cup \{5\} \cup \{6\} \cup \{7, 8, 11\} \cup \{9, 10\},$$

and $K(\pi(t))$ is obtained by removing 12 from the first part of this partition.

Conversely, given a non-crossing partition π of $1, \ldots, n-1$, all prime Schröder trees t such that $\pi(t) = \pi$ are obtained by choosing, for each part of $K(\pi)$, a binary tree whose leaves are numbered by the elements of the part, adding a leaf to the right of the tree containing the leftmost leaf, then joining the binary trees together to form the prime Schröder tree. Since binary trees are counted by Catalan numbers, it follows that the number of prime Schröder trees satisfying $\pi(t) = \pi$ is equal to

$$\prod_{p \text{ part of } K(\pi)} \mathrm{Cat}_{|p|-1}, \tag{4.5}$$

where $|p|$ denotes the number of elements in the part p. Comparing to (4.2), we see that this number is $|\mu(\pi)|$. See [4] for details about this construction.

4.3.4. Cluster complex and associahedra.
Consider a regular polygon drawn in the plane with $n + 1$ vertices enumerated in clockwise order. A pair of vertices which are not adjacent is called a *diagonal*. One can represent the diagonal by a segment joining the two vertices inside the polygon. Two diagonals are *compatible* if the segments they determine do not cross inside the polygon (they may cross however at the vertices). The compatibility relation determines a flag simplicial complex called the *cluster complex*. The maximal subsets of pairwise compatible diagonals determine triangulations of the polygon with vertices on the boundary while, more generally, compatible sets determine *dissections* of the polygon into smaller polygons. Here is

a picture with $n = 7$, where the compatible diagonals are $(1, 6)$, $(3, 6)$ and $(6, 8)$. The resulting dissection has two triangles with vertices $\{1, 6, 8\}$ and $\{6, 7, 8\}$ and two quadrangles with vertices $\{1, 2, 3, 6\}$ and $\{3, 4, 5, 6\}$.
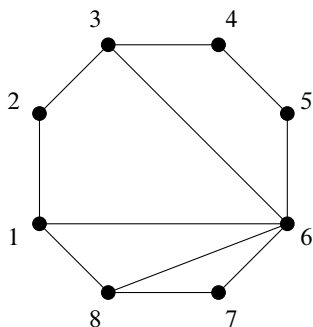
The cluster complex and its dual simplicial complex, the *associahedron*, can be both realized as the boundary of convex polytopes, see, e.g., [3] for more information and relations with Coxeter combinatorics.

There is a simple bijection between the faces of the cluster complex of a polygon with $n + 1$ vertices and Schröder trees. Take a dissection of a polygon with $n + 1$ edges and draw the polygon with the edge $[n, n + 1]$ at the base. Put a vertex inside each polygonal face of the dissection, these vertices correspond to internal vertices of a Schröder tree. The vertex inside the face having $[n, n + 1]$ in its boundary will be the root of the tree. Draw internal edges between the vertices across the diagonals of the dissection and external edges leading to leaves across the segments $[n + 1, 1]$ and $[i, i + 1]$ for $i = 1, 2, \ldots, n - 1$. The resulting tree is a Schröder tree with n leaves, and it is easy to see that this is a bijection between dissections and Schröder trees. Here is the Schröder tree corresponding to the above dissection:
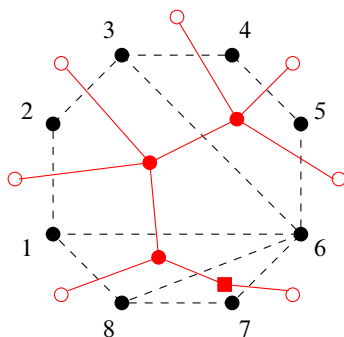

Observe that the corners of the tree are in bijection with the vertices $1, 2, \ldots, n - 1$ of the polygon. The Schröder tree is prime if and only if the vertex n does not belong to one of the diagonals of the dissection. Using this bijection, we could rephrase all

the constructions of the next section in terms of the cluster complex or the associ-
ahedra. However, we think that using Schröder trees makes our constructions easier
to understand.

5. The formula

In this section, we give an explicit formula for the polynomials Q_σ. For a prime Schrö-
der tree t with $n + 1$ leaves, an integer $k \in [1, n]$ and a circular permutation σ, label the
corners of t from left to right by the numbers $\sigma(k), \sigma^2(k), \ldots, \sigma^{n-1}(k), k$. For each
internal vertex v of t, let $i(v)$ be the smallest label of all corners belonging to v. Define
$x^{t,k,\sigma}$ as the product of $-x_{i(v)}$ over all internal vertices of t. Let us consider again the
Schröder tree from Section 4, and the cycle $\sigma = (2, 4, 8, 5, 9, 1, 6, 11, 10, 3, 7)$, with
$k = 7$:
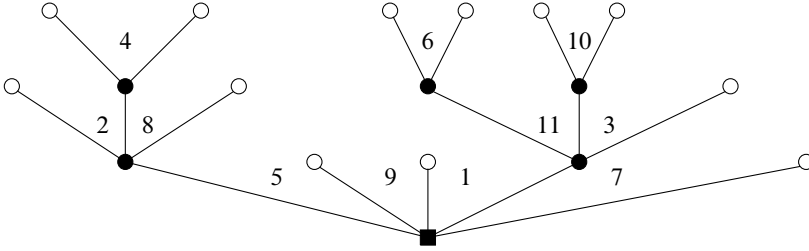
The monomial associated to this tree is then

$$x^{t,7,\sigma} = (-x_2)(-x_4)(-x_1)(-x_6)(-x_3)(-x_{10}).$$

Theorem 5.1. *For each $k \in [1, n]$, one has*

$$Q_\sigma(x_1, \ldots, x_n) = - \sum_{t \in p\mathcal{S}_{n+1}} x^{t,k,\sigma}. \tag{5.1}$$

Proof. Denote by $R_\sigma^k(x_1, \ldots, x_n)$ the polynomial on the right-hand side of (5.1).
First note that, by construction, it is of degree one in each variable. Also, verifying
the formula for $n = 1, 2$ is a trivial exercise.

We now check that R_σ^k depends only on the cyclic permutation σ and not on k.
For this observe that the term $x^{t,k,\sigma}$ can be expressed in terms of the partition $\pi(t)$.
Using (4.5) and the remark following this equation, we can rewrite the definition
of R_σ^k as

$$R_\sigma^k(x_1, \ldots, x_n) = - \sum_{\pi \in \mathrm{NC}(n)} \prod_{p \in \pi} (-x_{i(p)}) |\mu(K(\pi))|. \tag{5.2}$$

Here, for each part p of π, we let $i(p)$ denote the minimum of the numbers $\sigma^{i_1}(k)$,
$\sigma^{i_2}(k), \ldots, \sigma^{i_{|p|}}(k)$, where $p = \{i_1, \ldots, i_{|p|}\}$. We will interpret this formula as a free

cumulant in the next section but for now note that it implies that the polynomial on the right-hand side depends only on the circular permutation σ and not on the chosen k since the factor $|\mu(K(\pi))|$ is invariant by a circular permutation of π. We will therefore denote this polynomial by R_σ.

Let us now check that these polynomials satisfy the boundary conditions. Since 1 is always the smallest corner belonging to its vertex, the variable x_1 is a factor of every term in the sum defining R_σ, therefore x_1 divides R_σ. Since the polynomial R_σ does not depend on k, we can take $k = n$ in the formula. Each prime Schröder tree is either of the form t_1 or t_2, as in the construction of Section 4.2.2 for some Schröder tree t. The corresponding term in the first case is a monomial in $x_1, \ldots, x_{n-1}$ (since n is certainly larger than all the labels of all corners belonging to the root), and in the second case, it is the same monomial multiplied by $-x_n$, therefore the polynomial R_σ has $1 - x_n$ as a factor.

It remains to check the continuity and the exchange conditions (4) and (5). For this we will assume that $k = i + 1$ when computing R_σ and that $k = i$ when computing $R_{s_i \sigma s_i}$. Condition (4) is immediate to check from the definition so we just need to check the exchange condition. For this let $i, i + 1$ be two indices and expand

$$R_\sigma = A + x_i B + x_{i+1} C + x_i x_{i+1} D,$$
$$R_{s_i \sigma s_i} = A' + x_i B' + x_{i+1} C' + x_i x_{i+1} D',$$

where $A, A', B, B', C, C', D, D'$ are polynomials in the other variables $x_1, x_2, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n$. By the continuity condition, one has $A' = A$, $B' + C' = B + C$ and $D' = D$. Consider now the set of all prime Schröder trees t such that the labels i and $i + 1$ (with t labelled by σ) belong to different vertices while i is the smallest label in its vertex and $i + 1$ is not. In the sum from (5.1), these trees contribute to B for R_σ and to C' for $R_{s_i \sigma s_i}$, when the labelling is done according to $s_i \sigma s_i$. Similarly, if we exchange the roles of i and $i + 1$, they contribute to C and to B', respectively. The value $B - C' = B' - C$ is obtained by taking the sum over the remaining trees t, for which i and $i + 1$ label corners which belong to the same vertex and i is the smallest label at this vertex. Such trees contribute both to B and to B'. They have the form
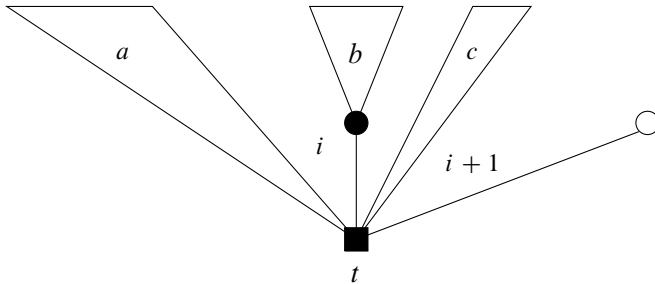
for some trees a, b, c. We have shown the corners labelled i and $i + 1$, which belong to the root. We can cut such a tree into two trees by cutting through the edge at the right of the corner labelled i. Then we exchange the labels $i + 1$ and i to get two trees t^+ and t^- as below:
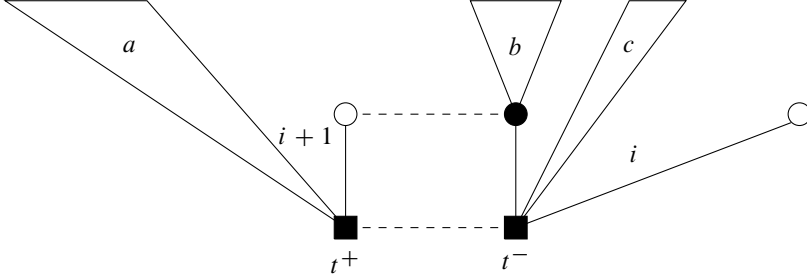
The tree t^+ gives a contribution to formula (5.1) defining $[x_{i+1}]R_{\sigma^+}(x^+)$ while t^- gives a contribution to $[x_i]R_{\sigma^-}(x^-)$, if we compute these polynomials by using the ordering of the cycles σ^- and σ^+ which put i and $i + 1$ at the end. Conversely any pair of such trees can be combined to form a tree t as above. It follows that $B' - C = B - C' = \Delta = ([x_{i+1}]R_{\sigma^+}(x^+))([x_i]R_{\sigma^-}(x^-))$. This proves that the polynomials R_σ satisfy the exchange relations therefore, by uniqueness, one has $Q_\sigma = R_\sigma$, as claimed. ∎

6. The polynomials Q_σ as free cumulants

Let us revert to the original interpretation of the polynomials Q_σ, where the x_i form an increasing subset of $[0, 1]$ (see Section 2). Consider the interval $[0, 1]$ equipped with Lebesgue measure and introduce the indicator functions $\Pi_x := 1_{[0,x]}$. We consider them as random variables on the probability space $[0, 1]$. Notice that they satisfy the relations

$$\Pi_x \Pi_y = \Pi_{x \wedge y}.$$

In particular, denoting φ the integration with respect to Lebesgue measure, we have

$$\varphi(\Pi_{u_1} \Pi_{u_2} \ldots \Pi_{u_p}) = \min(u_1, u_2, \ldots, u_p). \tag{6.1}$$

Theorem 6.1. *For each $k \in [1, n]$ and $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$, one has*

$$Q_\sigma(x_1, \ldots, x_n) = \kappa_n(\Pi_{x_{\sigma(k)}}, \ldots, \Pi_{x_{\sigma^{n-1}(k)}}, \Pi_{x_k}). \tag{6.2}$$

Proof. Using formulas (4.4), (5.2) and (6.1), it is enough to check the signs, namely that

$$\prod_{p \text{ part of } K(\pi)} (-1)^{|p|-1} = - \prod_{p \text{ part of } \pi} (-1).$$

This follows from $\sum_{p \text{ part of } K(\pi)} |p| = n$ and the well-known fact that $|K(\pi)| + |\pi| = n + 1$ ($|\pi|$ denotes the number of parts of π), see [7].

Alternatively, we can directly prove that the right-hand side of (6.2) satisfies all the properties of Definition 3.1. Since it is very similar to the proof of Theorem 5.1, we only sketch the argument. Let us call S_σ this right-hand side (it does not depend on k by the cyclic invariance of free cumulants). Property (1) is obvious from the moment-cumulant formula (4.4) and (6.1) while (2) follows from a simple computation. Relations (3) follow from the fact that a free cumulant vanishes if one of its entries is a constant while (4) just states that $\kappa_p(\Pi_{u_1}, \ldots, \Pi_{u_p})$ is a continuous function of the u_j on the cube $[0, 1]^p$. Finally, it remains to check the exchange condition (5). For this let σ be a cyclic permutation and write S_σ and $S_{s_i \sigma s_i}$ as sums over non-crossing partitions, using (4.4) and (6.1). Let π be a non-crossing partition of $[1, n]$ and assume that in the corresponding term of (4.4) the x_i and x_{i+1} are in different parts of π, then it is easy to see that the corresponding terms have the same value in S_σ and $S_{s_i \sigma s_i}$. Suppose now that x_i and x_{i+1} are in the same part p of π, corresponding to the following picture where we show only the part p:
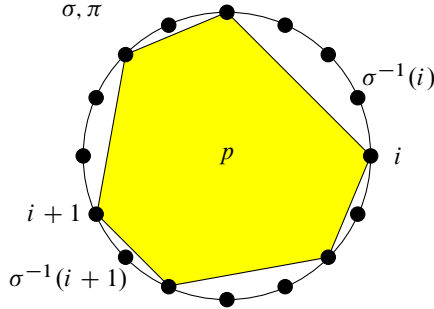
When we multiply σ on the left by s_i, we obtain two cycles σ^+ and σ^- and we can accordingly cut the part p into two parts p^+ and p^-. Each other part of π is either included in σ^+ or in σ^- so that we get two non-crossing partitions, π^+ and π^-, of σ^+ and σ^-:
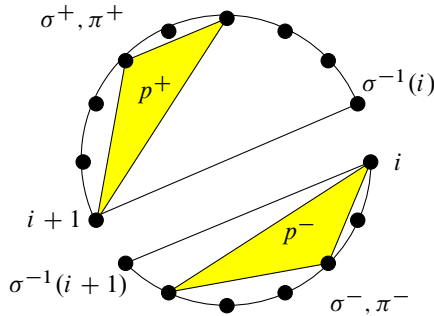
It is then easy to see that, as in the proof of Theorem 5.1, the term in factor of x_i corresponding to the partition π is the product of the terms in factor of x_i and x_{i+1} in S_σ and $S_{s_i \sigma s_i}$, respectively, corresponding to the partitions π^- and π^+. ∎

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