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Asymptotics of multivariate sequences in the presence of a lacuna

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Abstract. We explain a discontinuous drop in the exponential growth rate for certain multivariate generating functions at a critical parameter value in even dimensions $d \geq 4$. This result depends on computations in the homology of the algebraic variety where the generating function has a pole. These computations are similar to, and inspired by, a thread of research in applications of complex algebraic geometry to hyperbolic PDEs, going back to Leray, Petrowski, Atiyah, Bott and Gårding. As a consequence, we give a topological explanation for certain asymptotic phenomena appearing in the combinatorics and number theory literature. Furthermore, we show how to combine topological methods with symbolic algebraic computation to determine explicitly the dominant asymptotics for such multivariate generating functions, giving a significant new tool to attack the so-called connection problem for asymptotics of P-recursive sequences. This in turn enables the rigorous determination of integer coefficients in the Morse–Smale complex, which are difficult to determine using direct geometric methods.

1. Introduction

Let $k \geq 1$ be an integer, and for P and Q coprime polynomials over the complex numbers, let

$$F(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})^k} = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = \sum_{\mathbf{r} \in \mathbb{Z}^d} a_{\mathbf{r}} z_1^{r_1} \cdots z_d^{r_d} \tag{1.1}$$

be a rational Laurent series converging in some open domain $\mathcal{D} \subset \mathbb{C}^d$. The field of analytic combinatorics in several variables (ACSV) describes the asymptotic determination of the coefficients $a_{\mathbf{r}}$ via complex analytic methods. Let $\mathcal{V} = \mathcal{V}_Q$ denote the algebraic set $\{\mathbf{z} : Q(\mathbf{z}) = 0\}$ containing the singularities of $F(\mathbf{z})$. The methods of ACSV, summarized below, vary in complexity depending on the nature of $\mathcal{V}$. When $\mathcal{V}$ is a smooth manifold, for instance when Q and ∇Q do not vanish simultaneously, explicit formulae may be obtained that are universal outside of cases when the curvature of $\mathcal{V}$ vanishes [30, 32]. When $\mathcal{V}$ is the union of transversely intersecting smooth

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surfaces, similar residue formulae hold [4, 31, 32]. The next most difficult case is when $\mathcal{V}$ has an isolated singularity whose tangent cone is quadratic, locally of the form $x_1^2 - \sum_{j=2}^d x_j^2$. These points, satisfying the *cone point hypotheses* [6, Hypotheses 3.1], are called *cone points*; the necessary complex analysis in the neighborhood of a cone point singularity, based on the work in [3], was carried out in [6].

Let $|\mathbf{r}| = |r_1| + \cdots + |r_d|$. In each of these cases, asymptotics may be found of the form

$$a_{\mathbf{r}} \sim C(\hat{\mathbf{r}})|\mathbf{r}|^{\beta}\mathbf{z}_*(\hat{\mathbf{r}})^{-\mathbf{r}}, \tag{1.2}$$

where C and $\mathbf{z}_*$ depend continuously on the direction $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$. A very brief summary of the methodology is as follows. The multivariate Cauchy integral formula gives

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}, \tag{1.3}$$

where $T \subseteq \mathcal{D}$ is a torus in the domain of convergence, and $\mathrm{d}\mathbf{z}/\mathbf{z}$ is the logarithmic holomorphic volume form $z_1^{-1} \cdots z_d^{-1} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_d$. Expand the chain of integration T so that it passes through the variety $\mathcal{V}$, touching it for the first time at a point $\mathbf{z}_*$ where the logarithmic gradient of Q is normal to $\mathcal{V}$, and continuing to at least a multiple $(1+\varepsilon)$ times this polyradius. Let $\mathcal{I}$ be the intersection with $\mathcal{V}$ swept out by the homotopy of the expanding torus. The residue theorem, described in Definition 3.5 below, says that integral (1.3) is equal to the integral over the expanded torus plus the integral of a certain residue form over $\mathcal{I}$. Typically, $\mathbf{z}^{-\mathbf{r}}$ is maximized over $\mathcal{I}$ at $\mathbf{z}_*$, and integrating over $\mathcal{I}$ yields asymptotics of the form (1.2).

In the case of an isolated singularity with quadratic tangent cone, [6, Theorem 3.7] gives such a formula but excludes the case where $d = 2m > 2k + 1$ is an even integer and $d - 1$ is greater than twice the power k in the denominator of (1.1). In that paper, the asymptotic estimate obtained is only $a_{\mathbf{r}} = o(|\mathbf{r}|^{-m}\mathbf{z}_*^{-\mathbf{r}})$ for all m, due to the fact that the generalized Fourier transform of $(x_1^2 - \sum_{j=2}^d x_j^2)^{-k}$ is supported on a conical hypersurface in $\mathbb{R}^d$. In many cases, when the support of the Fourier transform has nonempty interior, the asymptotics of $a_{\mathbf{r}}$ are nothing other than the Fourier transform; see [6, Lemma 6.3]. However, when the Fourier transform has lower-dimensional support, one can say only that on directions within the support, the constant factor in the leading asymptotic term $C|\mathbf{r}|^{\alpha}\mathbf{z}_*^{-\mathbf{r}}$ blows up, with superpolynomial decay in directions interior to where the Fourier transform vanishes, being less than $|\mathbf{r}|^{-m}\mathbf{z}_*^{-\mathbf{r}}$ for any m. This leaves open the question of what the correct asymptotics are, and whether they are smaller by an exponential factor.

In [5], it is shown via diagonal extraction that, for $k = 1$ and a class of polynomials Q with an isolated real hyperbolic quadratic singularity, in fact $a_{n,\ldots,n}$ has strictly smaller exponential order than expected. Diagonal extraction applies only to coefficients of monomials precisely on the diagonal – i.e., where every variable has the same power – leaving open the question of behavior in a neighborhood of the diagonal,[1] and leaving open the question of whether this behavior holds beyond the particular class, for all polynomials with cone point singularities. The purpose of the present paper is to use ACSV methods to show that indeed the behavior is universal for cone points, to prove that it holds in a neighborhood of the diagonal, and to give a topological explanation.

2. Main results and outline

Let F, P, Q and $\{a_{\mathbf{r}}\}$ be as in (1.1), choosing signs so that $Q(\mathbf{0}) > 0$. Throughout the paper, we denote by $\mathbf{L}\colon \mathbb{C}_*^d \to \mathbb{R}^d$ the coordinatewise log-modulus map

$$\mathbf{L}(\mathbf{z}) := \log|\mathbf{z}| = (\log|z_1|, \dots, \log|z_d|).$$

Let $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$, and let $\mathcal{M} := \mathbb{C}_*^d \setminus \mathcal{V}$ be the domain of holomorphy of $\mathbf{z}^{-\mathbf{r}} F(\mathbf{z})$ for sufficiently large $\mathbf{r}$. Let amoeba(Q) denote the amoeba of polynomial Q defined by amoeba$(Q) := \{\mathbf{L}(\mathbf{z}) : \mathbf{z} \in \mathcal{V}\}$. It is known [13] that the components of the complement of the amoeba are convex and correspond to Laurent series expansions for F, each component being a logarithmic domain of convergence for one series expansion. Let B denote the component of the amoeba complement amoeba$(Q)^c$ such that the given series $\sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ converges whenever $\mathbf{z} = \exp(\mathbf{x} + i\mathbf{y})$ with $\mathbf{x} \in B$.

We refer to the torus $T(\mathbf{x}) := \mathbf{L}^{-1}(\mathbf{x})$ as the *torus over* $\mathbf{x}$. For any $\mathbf{r} \in \mathbb{R}^d$, we denote $\hat{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and

$$h_{\hat{\mathbf{r}}}(\mathbf{z}) := -\sum_{j=1}^{d} \hat{r}_j \log|z_j|.$$

For a subset $A \subset \mathbb{C}^d$, when $\hat{\mathbf{r}}$ and $\mathbf{z}_*$ are understood, we use the shorthand

$$A(-\varepsilon) := A \cap \{\mathbf{z} : h_{\hat{\mathbf{r}}}(\mathbf{z}) < h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon\}. \tag{2.1}$$

Assume that $\mathcal{V}$ intersects the torus $\{\exp(\mathbf{x}_* + i\mathbf{y}) : \mathbf{y} \in (\mathbb{R}/(2\pi))^d\}$ at the unique point $\mathbf{z}_* = \exp(\mathbf{x}_*)$. We will be dealing with the situation where $\mathcal{V}$ has a quadratic singularity at $\mathbf{z}_*$. More specifically, we will assume that Q has a real hyperbolic singularity at $\mathbf{z}_*$.

[1] To see why this distinction could matter, consider the function $(x-y)/(1+x+y)$ that generates differences of binomial coefficients $\binom{i+j+1}{i} - \binom{i+j+1}{j}$. The diagonal coefficients where $i = j = n$ are zero but the growth of those nearby approaches a constant times $n^{-1/2}4^n$.

Definition 2.1 (Quadratic singularity). We say that Q has a *real hyperbolic quadratic singularity* at $\mathbf{z}_*$ if $Q(\mathbf{z}_*) = 0$, the gradient $\nabla Q(\mathbf{z}_*) = 0$, and the quadratic part q_2 of $Q(\mathbf{z}) = q_2(\mathbf{z}) + q_3(\mathbf{z}) + \cdots$ at $\mathbf{z}_*$ is a real quadratic form of signature $(1, d-1)$; in other words, there exists a real linear coordinate change so that $q_2(\mathbf{u}) = u_d^2 - \sum_{j=1}^{d-1} u_j^2 + O(|\mathbf{u}|^3)$ for $\mathbf{u}$ a local coordinate centered at $\mathbf{z}_*$.

Example 2.2 (Computing a quadratic singularity: the GRZ (Gillis–Reznick–Zeilberger) function). Let e_j denote the elementary symmetric function of degree j in any number $d \geq j$ of arguments. Let $\mathbf{z} = (z_1, z_2, z_3, z_4)$ and define

$$Q_{4,27}(\mathbf{z}) := 1 - e_1(\mathbf{z}) + 27e_4(\mathbf{z}).$$

We will use the polynomial $Q_{4,27}$ and the generating function $F_{4,27}(\mathbf{z}) := 1/Q_{4,27}(\mathbf{z})$, which appeared as studies of [14] as discussed further below, as a running example. The function $Q_{4,27}$ has a zero at the point $\mathbf{z}_* := (1/3, 1/3, 1/3, 1/3)$ where all coordinates are equal. Recentering at this point via the substitution $\mathbf{z} = \mathbf{z}_* + \mathbf{u}$ yields the polynomial $\widetilde{Q}_{4,27} = 3e_2(\mathbf{u}) + 9e_3(\mathbf{u}) + 27e_4(\mathbf{u})$, with leading term at the origin the quadratic $3e_2(\mathbf{u})$. Writing this quadratic form as $(1/2)\,\mathbf{u}^\intercal M \mathbf{u}$, where M is the matrix with zeros on the diagonal and ones everywhere else, the eigenvalues of M are $3, -1, -1, -1$. This quadratic form thus has signature $(1, 3)$, and $Q_{4,27}$ has a real hyperbolic singularity at $\mathbf{z}_*$.

Returning to a general polynomials Q with a real hyperbolic singularity, denote by $T_{\mathbf{x}_*}(B)$ the open tangent cone in $\mathbb{R}^d$ to the component B of the amoeba complement amoeba$(Q)^c$, consisting of all vectors $\mathbf{v}$ at $\mathbf{x}_* := \mathbf{L}(\mathbf{z}_*)$ such that $\mathbf{x}_* + \varepsilon\mathbf{v} \in B$ for sufficiently small ε. The inequality defining $T_{\mathbf{x}_*}(B)$ is the same as the inequality $\widetilde{Q}(\mathbf{v}) > 0$, where $\widetilde{Q}$ is the leading (homogeneous quadratic) term of $Q(\exp(\mathbf{x}_* + \mathbf{v} + i\mathbf{y}_*))$, along with an inequality specifying $T_{\mathbf{x}_*}(B)$ rather than $-T_{\mathbf{x}_*}(B)$.

Definition 2.3 (Tangent cone; supporting vector). The vector $\mathbf{r}$ is said to be *supporting* at $\mathbf{z}_*$ if $h_{\mathbf{r}}$ attains its maximum on the closure of B at $\mathbf{x}_*$ and if $\{dh_{\mathbf{r}} = 0\}$ intersects the tangent cone $T_{\mathbf{x}_*}(B)$ only at the origin. The set of supporting vectors is a cone over an open set $\widehat{\mathcal{E}}$ of unit vectors, defined by the interior of a *dual cone* related to the amoeba (see [33, Definition 6.26] for further details).

Example 2.4 (GRZ function, continued). When $Q = Q_{4,27}$, the singularity $\mathbf{z}_* = (1/3, 1/3, 1/3, 1/3)$ corresponds to the point $\mathbf{x}_* = (-\log 3, -\log 3, -\log 3, -\log 3)$ on the boundary of the amoeba. The tangent cone $T_{\mathbf{x}}(B)$ is one of the two cones where $\widetilde{Q}_{4,27}(\mathbf{x}) = e_2(\mathbf{x}) > 0$. For an ordinary power series, B and $T_{\mathbf{x}_*}(B)$ point in the negative direction (more precisely, they contain a translate of the negative orthant). The dual cone points into the positive orthant and is always bounded by a component of the *algebraic dual* to $\widetilde{Q}_{4,27}$, which for homogeneous quadratics is also a quadratic form whose matrix is the inverse matrix to the one representing $\widetilde{Q}_{4,27}$. Up to a constant multiple, this is the matrix L with -2 on the diagonal and 1 everywhere else. The cone of supporting vectors is therefore the positive component of the set of vectors $\mathbf{v}$ such that $\mathbf{v}^{\mathsf{T}} L \mathbf{v} > 0$. Equivalently, $\mathbf{r}$ is supporting at $\mathbf{z}_*$ if and only if $(r_1 + r_2 + r_3 + r_4)^2 - 3\sum_{j=1}^{4} r_j^2 > 0$, comprising the circular cone of vectors making an angle of less than $\pi/6$ with the positive diagonal. The set $\widehat{\mathcal{E}}$ is corresponding open disk in the unit simplex.

Theorem 2.5 (Main theorem). *Let P be holomorphic in $\mathbb{C}^d$, Q a Laurent polynomial, k a nonnegative integer, B a component in the complement of the amoeba of Q, and $\sum_{\mathbf{r} \in E} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ the corresponding Laurent series expansion of $F = P/Q^k$.*

Suppose that Q has real hyperbolic quadratic singularity at $\mathbf{z}_* = \exp(\mathbf{x}_*)$ such that $\mathbf{x}_*$ belongs to the boundary of B, and $\mathbf{z}_*$ is the unique intersection of the torus $\mathbf{T}(\mathbf{x}_*)$ with $\mathcal{V}$.

Let $K \subseteq \widehat{\mathcal{E}}$ be a compact set, and suppose that d is even and $2k < d$.

(i) *If $\varepsilon > 0$ is small enough, then for any $\hat{\mathbf{r}} \in K$, there exists a compact cycle $\Gamma(\hat{\mathbf{r}})$, whose volume is bounded as $\hat{\mathbf{r}}$ varies over K, such that the cycle $\Gamma(\hat{\mathbf{r}})$ is supported on $\mathcal{M}(-\varepsilon)$ and*
$$a_{\mathbf{r}} = \int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k} \frac{d\mathbf{z}}{\mathbf{z}}. \tag{2.2}$$

(ii) *If P is a polynomial, then*
$$a_{\mathbf{r}} = \int_{\gamma(\hat{\mathbf{r}})} \mathrm{Res}_{\mathcal{V}} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k} \frac{d\mathbf{z}}{\mathbf{z}} \tag{2.3}$$
for all but finitely many $\mathbf{r} \in E$, where $\gamma(\hat{\mathbf{r}})$ is a compact $(d-1)$ cycle in $\mathcal{V}(-\varepsilon)$, whose volume is bounded as $\hat{\mathbf{r}}$ varies over K, where $\mathrm{Res}_{\mathcal{V}}$ is the residue operator defined below in Section 3.

The heuristic meaning of this result is that, for purposes of computing the Cauchy integral, the chain of integration in (1.3) can be slipped below the height $h_{\mathbf{r}}(\mathbf{z}_*)$ of the singular point $\mathbf{z}_*$.

Motivation and examples

Positivity conjectures for the coefficients of families of generating functions go back at least 90 years. An analysis of a discretized wave equation in two spatial dimensions [12] required nonnegativity of the coefficients of $1/e_2(1-x, 1-y, 1-z)$. A proof was given by Szegő [37], showing that in fact the coefficients of $e_2(1-x, 1-y, 1-z)^{-\beta}$ are nonnegative for all $\beta \geq 1/2$. Nonnegativity conjectures have been made for coefficients of many other rational symmetric functions, including the Askey–Gasper function [2] $1/(1 - e_1(x, y, z) + 4e_3(x, y, z))$, another function $1/e_3(1 - x, 1 - y, 1 - z, 1 - w)$ proposed by Szegő, and the Lewy–Askey function $1/e_2(1 - x, 1 - y, 1 - z, 1 - w)$. Several of these cases were solved by Scott and Sokal [36], who established coefficient positivity for functions of the form $1/Q(1 - x_1, \dots, 1 - x_n)$, where Q is the spanning tree polynomial of any series-parallel graph; see also [21] for related work on positivity.

A necessary condition for nonnegativity of coefficients is asymptotic nonnegativity. Broad theorems that give the asymptotic behavior of coefficients in cases such as these are therefore quite useful. In addition to establishing asymptotic nonnegativity, understanding more precisely the exponential drop in coefficient behavior in the presence of a lacuna allows one to understand what precision and how many initial cases are needed for a brute force proof of coefficient nonnegativity. Another motivation is to push the boundaries of ACSV: behaviors that arise only in sufficiently high dimension may be useful to understand, and may even be common or generic there. For instance, the first case of “nontrivial multiplicity” in the analysis of multivariate generating functions came from positivity studies (see below).

As a running example to accompany definitions and results, we draw on the Gillis–Reznick–Zeilberger (GRZ) family of generating functions [14], which has origins in the work of Askey and Gasper [2]. It is further discussed in [5, Theorems 9–12].

Example 2.6 (GRZ function at criticality). In four variables, let $Q(\mathbf{z}) = 1 - e_1(\mathbf{z}) + 27e_4(\mathbf{z})$ as in Example 2.2 and let $F_\lambda(\mathbf{z}) := 1/Q(\mathbf{z})$. It is shown in [5] via ACSV results for smooth functions that the diagonal exponential growth rate $|a_{n,n,n,n}|^{1/n}$ of the power series coefficients of F_λ is a function of λ that approaches 81 as $\lambda \to 27$. At the critical value 27, however, the denominator Q of F has a real hyperbolic quadratic singularity at $\mathbf{z}_* := (1/3, 1/3, 1/3, 1/3)$. Recalling the computation of $\widehat{\mathcal{E}}$ in Example 2.4, Theorem 2.5 has the immediate consequence that the exponential growth of $a_\mathbf{r}$ for $\hat{\mathbf{r}}$ making an angle of less than $\pi/6$ with the diagonal $\widehat{\mathbf{1}}$ is strictly less than that of $\mathbf{z}_*^{-\mathbf{r}} = 81^{|\mathbf{r}|}$. In particular, there is a drop in the exponential rate at criticality.

Example 2.7 (KZ function). Another motivating example comes from [19], where it was shown that nonnegativity of the coefficients of the Lewy–Askey function would follow from nonnegativity of the coefficients of a different 4-variable function $F_{\mathrm{KZ}}(\mathbf{z}) = 1/Q_{\mathrm{KZ}}(\mathbf{z}) = 1/(1 - e_1(\mathbf{z}) + 2e_3(\mathbf{z}) + 4e_4(\mathbf{z}))$, which we will call the Kauers–Zeilberger (KZ) function. Computing the first and second partial derivatives of Q_{KZ} at a generic point (u, u, u, u), we find that Q_{KZ} has a real hyperbolic quadratic singularity when every coordinate equals $(\pm\sqrt{3} - 1)/2$, where the point $\mathbf{z}_*$ with coordinates $(\sqrt{3} - 1)/2$ lies on the exponentiated boundary of the amoeba. Computing the matrix of second partial derivatives, we find zeros on the diagonal and a common value off the diagonal. We conclude that the tangent cone and supporting vectors are the same as for the GRZ function. Again, because $d = 4$ and $k = 1$, the hypotheses of Theorem 2.5 are satisfied and there is a lacuna, meaning an exponential drop in coefficients of F_{KZ}, uniformly as $\hat{\mathbf{r}}$ varies over in any compact neighborhood of the diagonal of vectors making an angle less than $\pi/6$ with the diagonal.

With a little further work, understanding of the exponential drop in the case of quadratic singularities in the lacuna regime can be sharpened considerably. In Section 8, we state a result for general functions satisfying the conditions of Theorem 2.5. The result, Theorem 8.1, sharpens Theorem 2.5 by quantifying the exponential drop by pushing the contour Γ down all the way to the next critical value. It is a direct consequence of Theorem 2.5 together with a deformation result of [4]. In the case of the GRZ function $F_{4,27}$ at criticality, the following explicit asymptotics result.

Theorem 2.8. *The diagonal coefficients $a_{n,n,n,n}$ in the power series expansion of $F_{4,27}(\mathbf{z})$ from Example 2.6 have an asymptotic expansion in decreasing powers of n of the form*

$$a_{n,n,n,n} = 3 \cdot \Big(\frac{(4i\sqrt{2}-7)^n}{n^{3/2}} \frac{(5i - \sqrt{2})\sqrt{-2i\sqrt{2}-8}}{24\pi^{3/2}} + \frac{(-4i\sqrt{2}-7)^n}{n^{3/2}} \frac{(-5i - \sqrt{2})\sqrt{2i\sqrt{2}-8}}{24\pi^{3/2}} \Big) + O(9^n n^{-5/2}). \quad (2.4)$$

More generally, as $\mathbf{r} \to \infty$ and $\hat{\mathbf{r}}$ varies over any compact neighborhood of the diagonal on which the angle to the diagonal is less than $\pi/6$, there is a uniform estimate

$$a_{\mathbf{r}} = p_{\hat{\mathbf{r}}}^n n^{-3/2} \cdot C_{\hat{\mathbf{r}}} \cos(n\alpha_{\hat{\mathbf{r}}} + \beta_{\hat{\mathbf{r}}}) + O(p_{\hat{\mathbf{r}}}^n n^{-5/3}),$$

where $p_{\hat{\mathbf{r}}}$, $C_{\hat{\mathbf{r}}}$, $\alpha_{\hat{\mathbf{r}}}$, and $\beta_{\hat{\mathbf{r}}}$ vary continuously with $\hat{\mathbf{r}}$ and specialize to produce (2.4) when $\hat{\mathbf{r}}$ is on the diagonal.

Example 2.9 (Critical GRZ function in higher dimensions). Define a d-variable symmetric rational function generalizing the four-variable GRZ function by

$$F_{\lambda,d}(z_1, \dots, z_d) = \frac{1}{1 - e_1 + \lambda e_d}.$$

It is conjectured that the coefficients are nonnegative when $\lambda \le d!$; if so, this would be sharp because the $(1, \dots, 1)$-coefficient is negative when $\lambda > d!$. This conjecture has been verified [35] up to $d = 17$.

The critical value for λ, however, is at the greater value $\lambda_d := (d-1)^{d-1}$. Although $F_{\lambda_d,d}$ cannot have nonnegative coefficients, it is possible that the coefficients are asymptotically nonnegative within the cone of exponential growth. At the critical value, asymptotics are controlled by the analytic behavior of F near the point $(1/(d-1), \dots, 1/(d-1))$, where the cone point hypotheses of [6] are satisfied. For even values of $d \geq 4$, one is in the lacuna regime. Numeric evidence from testing $d = 4, 6, 8$, and 10 suggests that formula (2.4) generalizes to

$$a_{n,\dots,n} = (d-1)\frac{\alpha\zeta^n + \overline{\alpha}\overline{\zeta}^n}{n^{(d-1)/2}} + O(|\zeta|^n n^{-(d+1)/2}), \tag{2.5}$$

where $\pi^{(d-1)/2}\alpha$ and ζ are algebraic numbers whose minimal polynomials are easy to derive using computer algebra.

Section 8 explains these formulae. Once Theorem 2.5 has guaranteed that the contour of integration can be expanded past the real hyperbolic quadratic singularity, some further hypotheses, which are not hard to satisfy, guarantee that the contour can be expanded further to one or more *critical points* where ACSV gives asymptotic approximations. For the GRZ function, one always obtains a pair of complex conjugate *smooth points*, meaning zeros of the denominator where the gradient of the denominator is nonvanishing. In the case of the four-variable GRZ function, these computations are explicitly carried out in Proposition 8.4. The integral may be reduced to an integer combination of saddle-point integrals near these critical points. Indeed, (2.4) and (2.5) are precisely the leading terms of the sum of expansions at two complex conjugate saddle points, multiplied by $d-1$.

Example 2.10 (KZ function, continued). Asymptotics for the KZ function in directions within the support cone follow a similar but not identical pattern. Contrast the KZ asymptotics to the GRZ asymptotics where the factor of $(d-1)$ in (2.5) proved for $d = 4$ and conjectured for even $d \geq 6$ comes from the fact that the original torus of integration in the Cauchy integral formula is homologous to $(d-1)(\gamma + \overline{\gamma})$, where γ and $\overline{\gamma}$ are homology generators near the critical points. In the case of the KZ function, there are two real hyperbolic critical points and eight smooth critical points: four points given by all permutations of $(3/\sqrt{2}-2, 1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and four points given by all permutations of $(-3/\sqrt{2}-2, -1/\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$. Only the first set of smooth points, whose coordinates are positive, affects dominant asymptotics. The local homology coefficients at these points are shown in Example 8.5 all to be 1.

Studying the real hyperbolic quadratic singularities in the lacuna regime $d > 2k+1$ gives the first ACSV applications where the Cauchy torus of integration is represented by a sum of local homology generators with "multiplicities" not in $\{\pm 1, 0\}$. We derive these multiplicities using a synthesis of algebraic topology and rigorous numerics; for explanation of the numeric portion of the method beyond what can be found in Section 8, see [24]. Finally, this work starts an inquiry into how one might distinguish $Q_{4,27}$ from Q_{KZ} geometrically. Ideally, we would like to understand why homology coefficients greater than 1 appear and find effective means of determining when this happens.

Heuristic argument

Our plan is to expand a torus $\mathbf{T}$ of integration representing series coefficients via the Cauchy integral theorem using a homotopy $\mathbf{H}$ that takes it through the point $\mathbf{z}_*$ and beyond. Let $\mathcal{V}_* = \mathcal{V} \cap \mathbb{C}_*^d$ denote the points of $\mathcal{V}$ with no coordinate vanishing. A classical construction when $\mathcal{V}_*$ is smooth, due to Leray, Thom and others, shows that $\mathbf{T}$ is homologous in $H_d(\mathcal{M})$ to a cycle Γ' which coincides above height $h(\mathbf{z}_*) - \varepsilon$ with a tube around a cycle σ; the height $h_{\mathbf{r}}$ is maximized on σ at the point $\mathbf{z}_*$, and the chain σ is the intersection of $\mathbf{H}$ with $\mathcal{V}_*$. We would like to see that σ is homologous to a class supported on $\mathcal{V}(-\varepsilon)$.

To do this, we compute the intersection σ directly in coordinates suggested by the hypotheses of the theorem. In particular, we use local coordinates where, after taking logarithms, $\mathcal{V}$ is the cone $\{z_1^2 - \sum_{j=2}^d z_j^2 = 0\}$, and select a homotopy $\mathbf{H}$ from $\mathbf{x} + i\mathbb{R}^d$ to $\mathbf{x}' + i\mathbb{R}^d$ with $\mathbf{x} \in B$ so that the line segment $\overline{\mathbf{x}\mathbf{x}'}$ is perpendicularly bisected by the support hyperplane to B at $\mathbf{x}_*$. In these coordinates, the intersection class $\mathcal{I}$ is the cone $\{i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d \text{ and } y_1^2 = \sum_{j=2}^d y_j^2\}$. The residue is singular at the origin (in new coordinates) but converges when $d > 2k + 1$. Inside the variety $\mathcal{V}$, the cone $\mathcal{I}$ may be folded down so as to double cover the cone $\{x + i\mathbf{y} : \mathbf{y} \in \mathbb{R}^d, x > 0, y_1 = 0 \text{ and } x^2 = |\mathbf{y}|^2\}$, as shown in Figure 1.

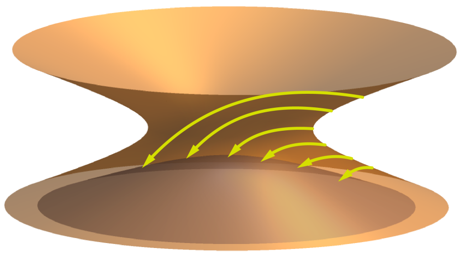

Figure 1. The figure depicts wrapping the one-sheeted hyperboloid around one of the sheets of the two-sheeted hyperboloid. The mapping is rotationally symmetric.

The two covering maps have opposite orientations when d is even. The critical points of $h_{\mathbf{r}}$ restricted to $\mathcal{V}$ are obstructions for deforming the contour of integration downwards, and in this case, the residue integral vanishes and the contour may be further deformed until it encounters the next highest critical point.

Outline of actual proof

The proof cannot precisely follow the heuristic argument because the intersection cycle construction and the residue integral theorem work only where $\mathcal{V}_*$ is smooth. In the case of an isolated quadratic singularity, the intersection cycle σ representing $\mathcal{I}$

can be defined by a homotopy that avoids the singularity, but the intersection class $\mathcal{I}$ may not be folded down as described without passing through the singularity. The effect of this is described in more detail in remarks following the statement of Theorem 5.3. We remark that the same trouble arose in the setting of [6]. There, the authors adopt the method of [3] to reduce the local integration cycle to its projectivized, compact counterpart, the so-called Petrowski or Leray cycles. That path required significant investment into analytic auxiliary results and, more importantly, would not immediately prove that the integration cycle in the presence of lacuna (i.e., when d is even and the denominator degree not too high) allows one to "slide" the integration cycle below the height of the cone point.

Thus we use a different strategy, first perturbing the denominator so that the perturbed varieties on which $Q(\mathbf{z}) = c$ for small c become smooth. This kind of regularization also has the advantage, compared to what was used in [6], that we obtain information about the behavior of coefficients of the generating functions $P/(Q-c)^k$. Next, we study the behavior of the coefficients of the resulting generating functions as $c \to 0$. We denote the zero set of $Q(\mathbf{z}) - c$ by $\mathcal{V}_c$, write $(\mathcal{V}_c)_*$ for the points of $\mathcal{V}_c$ with nonzero coordinates, and denote the restriction of $\mathcal{V}_c$ to its points of height at most $h(\mathbf{z}_*) - \varepsilon$ by $\mathcal{V}_c(\leq -\varepsilon)$. It is easiest to work in the lower-dimensional setting, with σ_c on $(\mathcal{V}_c)_*$ rather than Γ_c on $\mathcal{M}_c$, and to work in relative homology of $(\mathcal{V}_c)_*$ with respect to $\mathcal{V}_c(\leq -\varepsilon)$.

Section 4 lays the theoretical groundwork by computing the explicit intersection cycle in a limiting case of the perturbed variety as $c \downarrow 0$; this is a rescaled limit, and is smooth, in contrast to the variety at $c = 0$. Although the results of Section 4 are subsumed by later arguments, its focus on explicit computation allows for valuable intuition and visualization. Properties of our family of perturbations are given in Section 5. Section 6 uses this approach to complete a relative homology computation in $H_{d-1}((\mathcal{V}_c)_*, \mathcal{V}_c(\leq -\varepsilon))$ for sufficiently small c. This homology group is generated by two cycles, represented by the chain which is an integer combination of a sphere S_c and a hyperboloid. The cycle σ_c is not null-homologous in this relative homology group, but it turns out that in even dimensions, the coefficient of the hyperboloid in σ_c vanishes. The coefficient of S_c does not vanish, but S_c may be shrunk arbitrarily near to $\mathbf{z}_*$ as $c \to 0$. Applying the tube operator shows that the original d-torus of integration T is homologous to $\Gamma_c + \Gamma(\hat{\mathbf{r}})$ in $H_d(\mathcal{M}_c, M_c(\leq -\varepsilon))$, where Γ_c is the tube around S_c and $\Gamma_{\hat{\mathbf{r}}}$ is independent of c. Furthermore, $\Gamma_{\hat{\mathbf{r}}}$ is null-homologous in the relative homology group at $\mathbf{z}_*$ and therefore represented by a cycle lying in $(\mathcal{V}_c)(\leq -\varepsilon)$. The fact that the chain representing σ_c appears to vanish geometrically modulo $\mathcal{V}_c(\leq -\varepsilon)$, in the limit $c \to 0$ as the sphere S_c shrinks to a point, does not indicate that the relative cycle Γ_c vanishes as well: the isomorphism of de Rham and singular cohomology breaks down on singular spaces, and a principled analysis would require introduction of intersection homologies or mixed Hodge structure. We avoid this by desingularization and passing to the limit. For the record, we note that the chain Γ_c remains non-null-homologous as $c \to 0$, although the rank of the homology group $H_d(\mathcal{M}_c, M_c(\le -\varepsilon))$ drops from 2 to 1.

The outline of the proof as it appears in the paper is as follows:

(1) Show that $\sigma_c \cong S_c$ in $H_{d-1}((\mathcal{V}_c)_*, \mathcal{V}_c(\le -\varepsilon))$, provided that d is even. This is accomplished in Section 6.

(2) Pass to a tubular neighborhood to see that T in (1.3) may be replaced by the sum of tubular neighborhoods of S_c and a second chain γ, not depending on c, whose maximum height is at most $h(\mathbf{z}_*) - \varepsilon$. This is accomplished in Section 7.

(3) Dimensional analysis shows that the integral over the tubular neighborhood of S_c goes to zero as $c \downarrow 0$, provided that $2k < d$. This is accomplished in Section 7, proving Theorem 2.5.

(4) Further Morse theoretic analysis shows that the contour γ is an integer times the sum of two standard saddle-point contours. In Section 8, we show that for our motivating example, this integer is in fact 3, proving Theorem 2.8.

3. Preliminaries: Tubes, intersection class, residue form

We recall some topological facts from various sources, most of which are summarized for application to ACSV in [32, pp. 334–338]. Let K be any compact subset of $\mathcal{V}_*$ on which the gradient of the square-free part of Q (the product of its distinct irreducible factors) does not vanish. The well-known tubular neighborhood theorem (for example, [29, Theorem 11.1]) states the following.

Proposition 3.1 (Tubular neighborhood theorem). *The normal bundle over K is trivial, and there is a global product structure of a tubular neighborhood of $\mathcal{V}_*$ in $\mathbb{C}_*^d$.*

This implies the existence of operators $\bullet$ and $\circ$, the product with a small disk and with its boundary, respectively, mapping k-chains in $\mathcal{V}_*$ to $(k+2)$-chains in $\mathcal{M}$ and $(k+1)$-chains in $\mathcal{M}$, respectively, well-defined up to a natural homeomorphism as long as the radius of the disk is sufficiently small. We refer to $\circ\gamma$ as the *tube around* γ and $\bullet\gamma$ as the *tubular neighborhood* of γ. Elementary rules for boundaries of products imply

$$\partial(\circ\gamma) = \circ(\partial\gamma), \quad \partial(\bullet\gamma) = \circ\gamma \cup \bullet(\partial\gamma).$$

Because $\circ$ commutes with ∂, then cycles map to cycles, boundaries map to boundaries, and the map $\circ$ on the singular chain complex of $\mathcal{V}_*$ induces a map on homology $\circ\colon H_*(\mathcal{V}_*) \to H_*(\mathbb{C}_*^d \setminus \mathcal{V})$. This allows one to construct the *intersection class* as in [4, Proposition 2.9].

Definition 3.2 (Intersection class). Suppose Q vanishes on a smooth variety $\mathcal{V}$, and let $\mathbf{T}$ and $\mathbf{T}'$ be two d-cycles in $\mathcal{M}$ that are homologous in $\mathbb{C}_*^d$. Then there exists a unique class $\mathcal{I} = \mathcal{I}(\mathbf{T}, \mathbf{T}') \in H_{d-1}(\mathcal{V}_*)$ such that

$$[\mathbf{T}] - [\mathbf{T}'] = \circ\mathcal{I} \quad \text{in } H_d(\mathcal{M}).$$

The class $\mathcal{I}$ can be represented by the manifold $\mathbf{H} \cap \mathcal{V}$ for any manifold $\mathbf{H}$ with boundary $\mathbf{T} - \mathbf{T}'$ in $\mathbb{C}_*^d$ that intersects $\mathcal{V}$ transversely, with appropriate orientation (or, alternatively, by the image of the fundamental class of $\mathbf{H} \cap \mathcal{V}$ under the natural embedding).

We remark that if $\mathcal{V}$ is not smooth but its singularities (where Q and the gradient of its square-free part vanish) have real dimension less than $d-2$, then $\mathbf{H}$ generically avoids the singularities of $\mathcal{V}$, so $\mathcal{I}(\mathbf{T}, \mathbf{T}')$ is well defined. Although the singular set does not always satisfy this dimensional condition, it does so in our applications, where the singular set is zero-dimensional.

For our purposes, the natural cycles to consider are the tori $\mathbf{T}(\mathbf{x})$ for $\mathbf{x}$ in the complement of the amoeba of Q. In this case, there is an especially convenient choice of cobordism between $\mathbf{T}(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}')$, namely the $\mathbf{L}$-preimage of the straight segment ℓ connecting $\mathbf{x}$ and $\mathbf{x}'$ (or its small perturbation). We will be referring to this cobordism as the *standard* one.

Example 3.3. Suppose Q is the linear function $Q(x, y) = 2 - x - y$, whose amoeba is shown in Figure 2. Let $\mathbf{T}$ be the torus $\mathbf{L}(x, y)$ for some (x, y) in the component B_1 of the amoeba complement (shown in Figure 2), and let $\mathbf{T}'$ be the torus $\mathbf{L}^{-1}(x', y')$ for some $(x', y') \in B_2$.

Let T_0 denote the standard torus $S^1 \times S^1$, and let $\mathbf{H}\colon T_0 \times [0, 1] \to \mathbb{C}_*^2$ be the log-linear homotopy defined by

$$\mathbf{H}((\alpha, \beta), t) = (\exp(I\alpha + (1-t)x + tx'), \exp(I\beta + (1-t)y + ty')).$$

The Cauchy integrand T in (1.3) satisfies $[\mathbf{T}] = [T]$. The image of $\mathbf{H}$ under $\mathbf{L}$ is the line segment ℓ between (x, y) and (x', y'). The endpoints have unique $\mathbf{L}$ preimages in $\mathcal{V}$ while each interior point has two preimages in $\mathcal{V}$. Thus, $\mathcal{I} := \mathbf{H} \cap \mathcal{V} = \mathbf{L}^{-1}(\ell)$ is a topological circle in $\mathbb{C}_*^2$, and $[\mathcal{I}]$ is the class in $H_1(\mathcal{V}_*)$ corresponding to a circle γ which, projected to the first coordinate, circles once around the point $(2, 0)$. Taking logarithms, this corresponds to the point $(\log 2, -\infty)$ which is the vertical asymptote at the bottom of the figure. The class $[\mathbf{T}] - [\mathbf{T}'] = \circ\mathcal{I} \in H_2(\mathcal{M})$ is the tube around this circle, which may be written as the set $\{z + z' : z \in \gamma, z' \in \gamma'\}$, where γ' is the image of the unit circle in $\mathbb{C}$ under the diagonal map $z \mapsto (z, z)$. For instance, $\circ\mathcal{I}$ can be taken as the topological torus which is the direct sum of geometric circles

$$\{(2 - e^{i\theta}, e^{i\theta}) + (e^{i\phi}, e^{i\phi}) : 0 \le \theta, \phi < 2\pi\}.$$

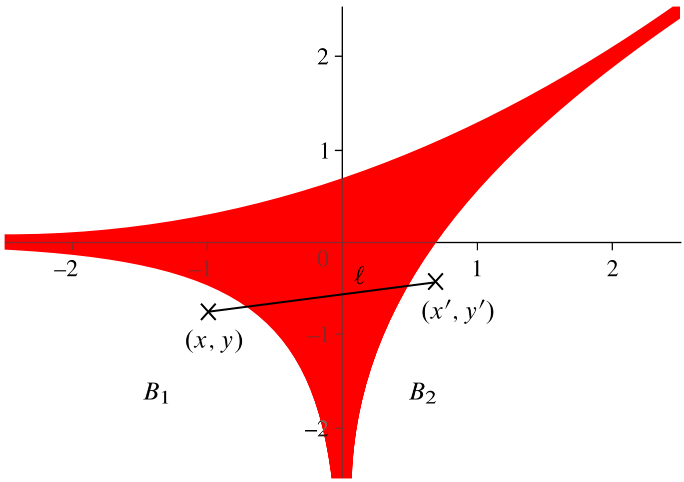

Figure 2. The amoeba of $2 - x - y$.

What are the good choices of $\mathbf{x}'$? We would like to make $F(\mathbf{z})\mathbf{z}^{-\mathbf{r}}\mathrm{d}\mathbf{z}/\mathbf{z}$, the integrand, exponentially small in $|\mathbf{r}|$ when $\mathbf{L}(\mathbf{z}) = \mathbf{x}'$, which happens if we can take $-\mathbf{r} \cdot \mathbf{x}'$ to have arbitrarily small modulus. When Q is a Laurent polynomial, the feasibility of this follows from known facts about cones of hyperbolicity, as we now demonstrate.

First, recall that the Newton polytope of Q is the convex hull of the exponents $\mathbf{m}$ of the monomials of Q,

$$N(Q) = \operatorname{conv}\left(\left\{\mathbf{m} : q_{\mathbf{m}} \neq 0, \, Q(\mathbf{z}) = \sum_{\mathbf{m}} q_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}\right\}\right) \subset \mathbb{R}^d.$$

The Newton polytope has vertices in the integer lattice, and the convex open components of the amoeba complement amoeba$(Q)^c$ map injectively into the integer points in $N(Q)$ (see [11]). Moreover, any vertex of $N(Q)$ has a preimage under this mapping, which is an unbounded component of amoeba$(Q)^c$. The recession cone of the component $B_{\mathbf{m}}$ corresponding to a vertex $\mathbf{m}$ is the interior of the normal cone to $N(Q)$ at $\mathbf{m}$ (i.e., the collection of vectors $\mathbf{d}$ such that $\max_{\mathbf{r} \in N(Q)} \langle \mathbf{d}, \mathbf{r} \rangle$ is uniquely attained at $\mathbf{m}$). Notice that this normal cone is dual to the cone $N(Q)_{\mathbf{m}}$ spanned by $N(Q) - \mathbf{m}$.

Now, let B be the component of amoeba$(Q)^c$ corresponding to the Laurent expansion of F under consideration, and let $\mathbf{m}$ be the corresponding integer vector in $N(Q)$. The vectors $\mathbf{v}$ with $\mathbf{x}_* + \mathbf{v} \in B$ form an open cone contained in $N(Q)_{\mathbf{m}}$. Pick a generic $\mathbf{d}$ in the recession cone of B; then when $t > 0$ is large enough, $\mathbf{x}_* - t\mathbf{d}$ is contained in an unbounded component B' of the complement to amoeba (this follows from the fact that the union of the recession cones of the unbounded components of amoeba$(Q)^c$ are the complement to the set of functionals attaining their maxima on $N(Q)$ at multiple points, a positive codimension fan in $\mathbb{R}^d$). Hence, choosing $\mathbf{x}'$ in this component B' allows one to deform $\mathbf{T}' = \mathbf{T}(\mathbf{x}')$ while avoiding $\mathcal{V}$ so that $h_{\mathbf{r}}$ becomes arbitrarily close to $-\infty$.

Definition 3.4. We call a component B' whose recession cone contains a vector $-\mathbf{d}$ with $\mathbf{d}$ in the recession cone of B *descending* with respect to the component B. Components B' with this property are in general not unique, but any choice of B' works for our argument.

The following result is well known; see, e.g., [4, Proposition 2.14].

Definition 3.5 (Residue form). There is a homomorphism

$$\text{Res}\colon H^d(\mathcal{M}) \to H^{d-1}(\mathcal{V}_*)$$

in de Rham cohomologies such that for any class $\gamma \in H_d(\mathcal{V})$,

$$\int_{\circ\gamma} \omega = \int_{\gamma} \text{Res}(\omega). \tag{3.1}$$

In general, $\text{Res}(\omega)$ can be derived locally from a form representing ω (we also use the notation Res for the corresponding operator on differential forms). When $F = P/Q$ is rational with Q square-free, Res commutes with multiplication by any locally holomorphic function and satisfies

$$Q \wedge \text{Res}(F\,\mathrm{d}\mathbf{z}) = P\,\mathrm{d}\mathbf{z}.$$

More generally, if $F = P/Q^k$, then (see, e.g., [34]) the residue can be expressed in coordinates as

$$\text{Res}\Big[\mathbf{z}^{-\mathbf{r}} F(\mathbf{z})\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}\Big] := \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}c^{k-1}}\Big[\frac{P\mathbf{z}^{-\mathbf{r}}}{\mathbf{z}}\Big]\mathrm{d}\sigma,$$

where σ is the natural area form on $\mathcal{V}$ (characterized by $\mathrm{d}Q \wedge \sigma = \mathrm{d}\mathbf{z}$), and the partial derivatives with respect to c are taken in the coordinates where c is one of the variables.

Putting this together with the definition and construction of the intersection class and Cauchy's integral formula yields the following representation of the coefficients $a_{\mathbf{r}}$.

Proposition 3.6. *Suppose $F = G/Q^k = \sum_{\mathbf{r}\in E} a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$ with G holomorphic and Q a polynomial, the series converging when* $\log|\mathbf{z}|$ *is in the component B of* $\text{amoeba}(Q)^c$. *Let $\mathbf{x} \in B$ and $\mathbf{T}(\mathbf{x}) := \mathbf{L}^{-1}(\mathbf{x})$ be the torus with log-polyradius $\mathbf{x}$. Let $\mathbf{x}'$ be any other point in* $\text{amoeba}(Q)^c$. *Then*

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d}\int_{\mathcal{I}(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}'))} \text{Res}\Big[\mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}\Big] + \frac{1}{(2\pi i)^d}\int_{\mathbf{T}(\mathbf{x}')} \mathbf{z}^{-\mathbf{r}}F(\mathbf{z})\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}. \tag{3.2}$$

Moreover, if $\mathbf{x}'$ *is a descending component* B' *with respect to* B*, and* G *is a polynomial, then for all but finitely many* $\mathbf{r} \in E$*,*

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{I}(\mathbf{T}(\mathbf{x}),\mathbf{T}(\mathbf{x}'))} \mathbf{z}^{-\mathbf{r}} \mathrm{Res}\Big(F(\mathbf{z})\frac{\mathrm{d}\mathbf{z}}{\mathbf{z}}\Big).$$

Proof. The first identity is Cauchy's integral formula, the definition of the intersection class, and (3.1). The second identity follows from the fact that $\sup_{\mathbf{T}(\mathbf{x}')} |G/Q^k|$ and the volume of $\mathbf{T}(\mathbf{x}')$ grow at most polynomially in $|\mathbf{x}'|$ on the torus over $\mathbf{x}'$. For $\mathbf{r} \in N(Q)_{\mathbf{m}}$ large enough in size, the degree of the decay of $|\mathbf{z}^{-\mathbf{r}}|$ overtakes that polynomial growth, so that the last term in (3.2) can be made arbitrarily small. As it is independent of $\mathbf{x}'$ as long as $\mathbf{x}'$ varies in the same component B', it vanishes identically. ∎

4. The limiting quadric

In this section, we focus on the properties of the particular smooth quadratic function

$$q(\mathbf{z}) := -1 + z_1^2 - \sum_{j=2}^{d} z_j^2. \tag{4.1}$$

This section is purely for intuition, in the sense that the work on perturbed varieties in Sections 5–7 draws on the constructions of this section but does not use the results of this section other than recalling the setup of Lemma 4.1 in the proof of Proposition 6.8. We think of this section as "taking place in the log-space" as the linear constructions here correspond to log-linear constructions in $\mathbb{C}_*^d$ in subsequent sections; however, this is a standalone section so this correspondence is also only intuitive. Throughout the section, we work in $\mathbb{C}^d$ with coordinates $z_j = x_j + iy_j$. Note that unlike the local behavior of a function Q near a real hyperbolic quadratic singularity, the quadric q has a nonzero constant term, $q(\mathbf{0}) = -1$. The zero set $\widetilde{\mathcal{V}}$ of q can be viewed as the solution set $\mathcal{V}_c$ to the equation $Q(\mathbf{z}) = c$ near the quadratic singularity of Q, after the variables are scaled by $c^{1/2}$. Our first statement deals with the gradient-like flow on $\widetilde{\mathcal{V}}$ with respect to the function $h := x_1$.

Lemma 4.1. *The function* h *has two critical points* $\mathbf{z}_\pm = (\pm 1, 0, \ldots, 0)$ *on* $\widetilde{\mathcal{V}}$*, both of index* $d-1$*. The stable manifold for* $\mathbf{z}_+$ *is the unit sphere*

$$\mathtt{S} := \Big\{\mathbf{x} + i\mathbf{y} : x_1^2 + \sum_{k=2}^{d} y_k^2 = 0,\ y_1 = x_2 = \cdots = x_d = 0\Big\},$$

and its unstable manifold is the upper lobe of the two-sheeted real hyperboloid

$$\mathtt{H}_+ := \widetilde{\mathcal{V}} \cap \mathbb{R}^d = \Big\{\mathbf{x} + i\mathbf{y} : x_1^2 - \sum_{k=2}^{d} x_k^2 = 0,\ x_1 > 0,\ \mathbf{y} = \mathbf{0}\Big\}.$$

The stable manifold for $\mathbf{z}_-$ is the lower lobe $\mathtt{H}_-$ of this hyperboloid, while the unstable manifold of $\mathbf{z}_-$ is still the sphere $\mathtt{S}$.

Proof. The critical points can be found by a direct computation. Their indices are necessarily $d-1$, as h is the real part of a holomorphic function on a complex manifold [16]. Similarly, direct computation shows that the tangent spaces to $\mathtt{S}$, $\mathtt{H}_\pm$ are the stable/unstable eigenspaces for the Hessian matrices of h restricted to $\widetilde{\mathcal{V}}$ at the critical points. Lastly, as the gradient vector field is invariant with respect to symmetries $\mathbf{y} \mapsto -\mathbf{y}$ and $(x_1, x_2, \dots, x_d) \mapsto (x_1, -x_2, \dots, -x_d)$, leaving $\mathtt{H}_\pm$ and $\mathtt{S}$ invariant, they are the invariant manifolds for the gradient flow. ■

Let $\mathbf{\Phi}\colon \mathbb{R}^d \times [-1,1] \to \mathbb{C}^d$ be the homotopy defined by $\mathbf{\Phi}(\mathbf{y}, t) := t\mathbf{e}_1 + i\mathbf{y}$ (we use a new symbol because $\mathbf{H}$ is in principle only a cobordism). Let $\widetilde{h}$ denote the height function $\widetilde{h}(\mathbf{z}) = -\Re\{z_1\}$.

Theorem 4.2. *The intersection cycle of the homotopy $\mathbf{\Phi}$ with the variety $\widetilde{\mathcal{V}}$ is the union of a hyperboloid $\mathtt{H}$ and a $(d-1)$-sphere $\mathtt{S}$ that intersect in a $(d-2)$-sphere $\mathcal{S}'$. These are given by equations (4.6)–(4.8). The orientation of the intersection cycle is continuous on each of the four smooth pieces, namely the upper and lower half of $\mathtt{H} \setminus \mathcal{S}'$ and the northern and southern hemispheres of $\mathtt{S} \setminus \mathcal{S}'$, but change signs when crossing $\mathcal{S}'$.*

Proof. Writing $\mathbf{z}_j := x_j + iy_j$, the equations for $\mathbf{z}$ such that $\mathbf{z}$ is in the range of $\mathbf{\Phi}$ and $q(\mathbf{z}) = 0$ become

$$|x_1| \le 1, \tag{4.2}$$

$$x_j = 0, \quad 2 \le j \le d, \tag{4.3}$$

$$x_1^2 - y_1^2 = 1 - \sum_{j=2}^{d} y_j^2, \tag{4.4}$$

$$x_1 y_1 = 0. \tag{4.5}$$

The solutions to (4.2)–(4.5) form the union of two sets, one obtained by solving (4.2)–(4.4) when $x_1 = 0$ and the other by solving (4.2)–(4.4) when $y_1 = 0$; these intersect along the solutions to (4.2)–(4.4) when $x_1 = y_1 = 0$. The first of these is the one-sheeted hyperboloid $\mathtt{H} \subseteq i\mathbb{R}^d$ given by

$$-y_1^2 = 1 - \sum_{j=2}^{d} y_j^2. \tag{4.6}$$

The second is the sphere $\mathtt{S} \subseteq \mathbb{R} \times i(\mathbb{R}^{d-1})$ given by

$$x_1^2 + \sum_{j=2}^{d} y_j^2 = 1. \tag{4.7}$$

These intersect at the equator of the sphere $\mathtt{S}$, which is the neck of the hyperboloid $\mathtt{H}$. The intersection set is the sphere $\mathcal{S}'$ in $\{0\} \times i\mathbb{R}^{d-1}$ given by

$$\sum_{j=2}^{d} y_j^2 = 1. \tag{4.8}$$

The intersection class is given by the intersection of $\widetilde{\mathcal{V}}$ with any homotopy intersecting it transversely. While $\mathbf{\Phi}$ does not intersect $\widetilde{\mathcal{V}}$ transversely, it is the limit of the intersections of $\widetilde{\mathcal{V}}$ with arbitrarily small perturbations of $\mathbf{\Phi}$ that do intersect $\widetilde{\mathcal{V}}$ transversely. Let γ_n be a sequence of such transverse intersection cycles converging to $\gamma := \mathtt{H} \cup \mathtt{S}$. Because $\widetilde{\mathcal{V}}$ is smooth, the global product structure on a neighborhood of $\widetilde{\mathcal{V}}$ from the Thom lemma implies that as d-chains,

$$\mathbf{\Phi}(\cdot, -1) - \mathbf{\Phi}(\cdot, 1) = \gamma_n \times S_1 \to \gamma \times S_1,$$

and hence that γ represents the intersection class.

Finally, we determine the orientation via a different perturbation argument. Let us choose a point $p \in \mathcal{S}'$, say for specificity $p = (0, \dots, 0, i)$. The tangent space $T_p(\mathcal{S}')$ is the span of the vectors $i\mathbf{e}_k$ for $2 \le k \le d-1$. The tangent space $T_p(\mathtt{S})$ is obtained by adding the basis vector $\mathbf{e}_1$, while the span of the tangent space $T_p(\mathtt{H})$ is obtained by adding instead the basis vector $i\mathbf{e}_1$. We see that near $\mathcal{S}'$, γ has a product structure $\mathcal{S}' \times \mathbf{W}$, where $\mathbf{W}$ is diffeomorphic to two crossing lines, with tangent cone $xy = 0$ in the plane $\langle \mathbf{e}_1, i\mathbf{e}_1 \rangle$, as in the black lines in Figure 3.

Now perturb the homotopy as follows. Let $u \colon [-1, 1] \to \mathbb{R}$ be a smooth function that is equal to 1 on $[-1/4, 1/4]$ and vanishes outside of $(-1/2, 1/2)$. Define $\mathbf{\Phi}_\varepsilon(\mathbf{y}, t) := t\mathbf{e}_1 + \varepsilon u(t)\mathbf{e}_d + i\mathbf{y}$, where ε is a real number whose magnitude will

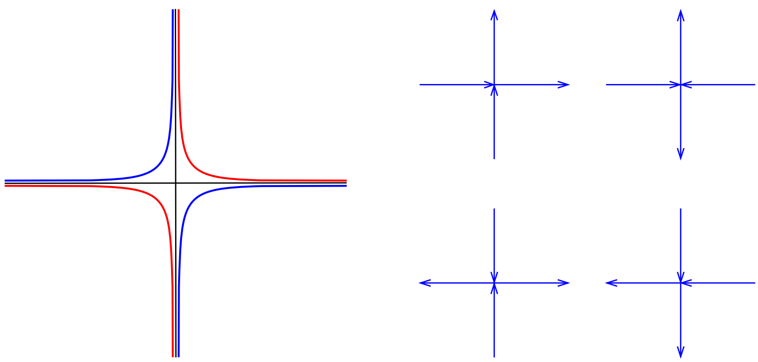

Figure 3. Left: The black line shows $\mathbf{W}$; the blue line shows the projections to the $x_1 y_1$ plane of $\mathbf{W}_\varepsilon$ when $y_d > 0$ and ε is small and positive; the red line shows the projections when ε is small and negative. Right: Orientations of $\mathbf{W}$ consistent with the blue hyperbola.

be chosen sufficiently small and whose sign could be either positive or negative. Because $\mathtt{S}$ and $\mathtt{H}$ intersect only on the subset of $\mathbf{\Phi}$ where $t = 0$, their Hausdorff distance on the set $t \notin (-1/4, 1/4)$ is positive; it follows that for sufficiently small $|\varepsilon|$, the intersection of $\mathbf{\Phi}_\varepsilon$ with $\widetilde{\mathcal{V}}$ is in the subset of $\mathbf{\Phi}_\varepsilon$ where $-1/4 \leq t \leq 1/4$. There $u = 1$ and the equations for the intersection γ_ε are modified from (4.2)–(4.5) as follows:

$$|x_1| \leq 1, \tag{4.2$'$}$$

$$x_j = 0 \quad \text{for } 2 \leq j \leq d-1 \quad \text{and} \quad x_d = \varepsilon, \tag{4.3$'$}$$

$$x_1^2 - y_1^2 = 1 - \varepsilon^2 - \sum_{j=2}^{d} y_j^2, \tag{4.4$'$}$$

$$x_1 y_1 = -\varepsilon y_d. \tag{4.5$'$}$$

Although $\mathbf{\Phi}_\varepsilon$ and $\widetilde{\mathcal{V}}$ still do not intersect transversely, the intersection set $\gamma_\varepsilon := \mathbf{\Phi}_\varepsilon \cap \widetilde{\mathcal{V}}$ is now a manifold. We now fix $y_2, \dots, y_{d-1}$ at a value $\mathbf{y}$ inside the unit ball, setting $x_2 = \varepsilon$ and solving (4.4$'$) and (4.5$'$) for y_1 and y_d as a function of x_1. For $x_1^2 < 1 - |\mathbf{y}|^2$, as $\varepsilon \downarrow 0$, there are two components of the solution, with

$$y_d \to \pm\sqrt{1 - x_1^2 - |\mathbf{y}|^2},$$

respectively. These correspond to different points on the sphere. Fixing one, say with $y_d > 0$, locally γ_ε has a product structure $\mathcal{S}' \times \mathbf{W}_\varepsilon$, where $\mathbf{W}_\varepsilon$ is a hyperbola in quadrants II and IV; see the blue curve in Figure 3. The (oriented) chains $\mathbf{W}_\varepsilon$ converge to $\mathbf{W}$ as $\varepsilon \downarrow 0$, therefore the possible orientations for $\mathbf{W}$ are one of the four shown on the right of Figure 3. The (oriented) chains $\mathbf{W}_\varepsilon$ also converge to $\mathbf{W}$ as $\varepsilon \uparrow 0$, narrowing the choices to the second and third choices in Figure 3, and proving the desired result. ∎

Theorem 4.3. *Let* $\mathtt{n}$ *be the chain given by* $\mathtt{S}$ *with orientation reversed in the southern hemisphere; in other words,* $\mathtt{n}$ *is a sphere, oriented the same as the northern hemisphere of* $\mathtt{S}$*. When* d *is even, the chain* γ *is homotopic to* $\mathtt{n}$ *in* $H_{d-1}(\widetilde{\mathcal{V}})$*.*

Proof. Let $X_1 := \mathbb{R} \times \mathcal{S}'$, and let $\iota_1 : \mathcal{S}' \to X_1$ be the embedding $\mathbf{y} \mapsto (0, \mathbf{y})$. Let $X_2 = [-\pi/2, \pi/2] \times \mathcal{S}'$, and let $\iota_2 : \mathcal{S}' \to X_2$ be the embedding $\mathbf{y} \mapsto (0, \mathbf{y})$. Let X denote the space obtained by gluing X_1 to X_2 modulo the identification of ι_1 and ι_2 (which conveniently identifies identically named points $(0, \mathbf{y})$ in X_1 and X_2). If for $j \in \{1, 2\}$, there are homotopies $T_j : X_j \times [0, 1] \to \widetilde{\mathcal{V}}$ making the maps in Figure 4 commute, then their union modulo the identification is a homotopy $T : X \times [0, 1] \to \widetilde{\mathcal{V}}$.

To prove the lemma, it suffices to construct these in such a way that T_2 is a homotopy from $\mathtt{S}$ to $\mathtt{n}$ and T_1 is a homotopy from $\mathtt{H}$ to a null homologous chain. On X_1, let ρ denote the $\mathbb{R}$ coordinate on X_1 and σ denote the $\mathcal{S}'$ coordinate. Let $\mathbf{z} = (z_2, \dots, z_d)$,

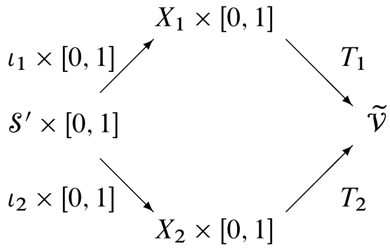

Figure 4. Commuting homotopies define a homotopy on the identification space X.

and let $\mathbf{x}$ and $\mathbf{y}$ denote the real and imaginary parts of $\mathbf{z}$, respectively. Let t denote the $[0,1]$-coordinate of $X_1 \times [0,1]$. We may then define the homotopy T_1 via the equations

$$x_0 = \sin\Big(\frac{\pi}{2}t\Big)\cosh(\rho), \quad y_0 = \cos\Big(\frac{\pi}{2}t\Big)\sinh(\rho),$$
$$x = \sin\Big(\frac{\pi}{2}t\Big)\sigma\sinh\rho, \quad y = \cos\Big(\frac{\pi}{2}t\Big)\sigma\cosh\rho,$$

and check that $T_1((\rho,\sigma),0)$ parametrizes H via

$$y_1 = \sinh(\rho), \quad \mathbf{y} = \cosh(\rho)\sigma.$$

Next we define the map $\tau\colon [-\pi/2,\pi/2] \times [0,1] \to [-\pi/2,\pi/2]$ by $\tau(\rho,t) = (1-t)\rho + t(\min(2\rho,0) - \pi/2)$. This is a linear homotopy from the identity to the map $\rho \mapsto \min(2\rho,0) - \pi/2$, pictured in Figure 5. Define T_2 by the equations

$$x_0 = \sin(\tau(\rho,t)), \quad y = \cos(\tau(\rho,t))\sigma.$$

Again, we verify that $T_2((\rho,\sigma),0)$ parametrizes the chain S via the parametrization $x_0 = \sin(\rho)$ and $\mathbf{y} = \cos(\rho)\sigma$. The parametrization is not one-to-one, mapping the set $\{-\pi/2\} \times \mathcal{S}'$ to the south pole and the set $\{\pi/2\} \times \mathcal{S}'$ to the north pole, however it defines a singular chain homotopy equivalent to a standard parametrization of $\mathcal{S}'$.

Next, we check that the diagram in Figure 4 commutes, mapping $(\mathbf{y}, t)$ in both cases to the point $(\sin(t\pi/2), i\cos(t\pi/2)y_2, \dots, i\cos(t\pi/2)y_d)$. After this, we check that T_2 is a homotopy from S to n. This is clear because the homotopy T_2 leaves the (generalized) longitude component alone while pushing all the southern latitudes to the south pole and stretching the northern latitudes to cover all the latitudes.

Finally, we check that T_1 is a homotopy from H to a null-homologous chain. The map $T_1(\cdot,1)$ maps the imaginary hyperboloid H parametrized by (ρ,σ) into the $\{x_1 > 0\}$ branch of the real two-sheeted hyperboloid $\mathcal{H}'$ defined by $x_1^2 = 1 + \sum_{j=2}^d x_j^2$ and parametrized by cylindrical coordinates (r,σ'). The parametrization is a double covering, with (ρ,σ) and $(-\rho,\sigma)$ getting mapped to the same point. We need to check that the orientations at (ρ,σ) and $(-\rho,-\sigma)$ are opposite. We may

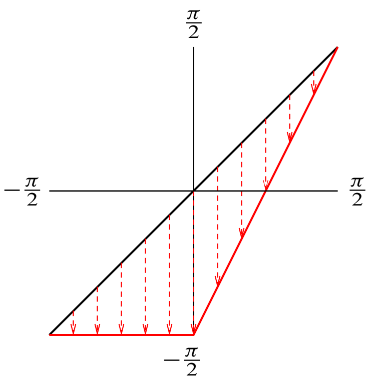

Figure 5. The linear homotopy τ.

parametrize $\mathtt{H}$ by its projection $\mathbf{x}$ onto the last $d-1$ coordinates, then, still preserving orientation, by polar coordinates (r, σ'), where $r > 0$ is the magnitude and $\sigma'(\mathbf{x}) = \mathbf{x}/r$ when $r > 0$ (σ' can be anything when $r = 0$). In these coordinates, the point $(\rho, \sigma) \in \mathtt{H}$ gets mapped to the point

$$(r, \sigma') = \begin{cases} (\sinh(\rho), \sigma), & \rho > 0, \\ (-\sinh(\rho), -\sigma), & \rho < 0. \end{cases}$$

Recalling that the orientation form on $\mathtt{H}$ is given by $\operatorname{sgn}(\rho)\, \mathrm{d}\rho \wedge \mathrm{d}\sigma$, the Jacobian is therefore given by

$$\frac{D(\sigma', r)}{D(\sigma, \rho)} = \begin{cases} \frac{\mathrm{d}\sigma \wedge \cosh(\rho)\, \mathrm{d}\rho}{\mathrm{d}\sigma \wedge \mathrm{d}\rho}, & \rho > 0, \\ \frac{\mathrm{d}(-\sigma) \wedge (-\cosh(\rho))\, \mathrm{d}\rho}{-\mathrm{d}\sigma \wedge \mathrm{d}\rho}, & \rho < 0. \end{cases} \tag{4.9}$$

The central symmetry flips the orientation exactly on even-dimensional spheres, so that (4.9) changes signs with the sign of ρ exactly when $d-2$ is even. This implies that for d even, the two branches locally covering the sheet $\{x_0^2 = |x|^2 + 1,\ x_0 > 0\}$ receive opposite signs and the chain $T_1(\cdot, 1)$ is homologous to zero. ∎

In the next section, we prove perturbed versions of these results leading to identification of certain homology and cohomology classes. To pave the way, we record some further facts about the intersection of the explicit homotopy $\mathbf{\Phi}$ with the quadric.

Proposition 4.4. *There are precisely two critical points for $\widetilde{h}(\mathbf{z}) := -\Re\{z_1\}$ on $\widetilde{\mathcal{V}}$, namely $\pm e_1$. At the higher critical point $-\mathbf{e}_1$, the unstable manifold for the downward gradient flow on $\widetilde{\mathcal{V}}$ is the sphere $\mathtt{S}$, which happens to be a subset of $\mathbf{\Phi}$, with flow lines going longitudinally from the "north pole" $-\mathbf{e}_1$ to the "south pole" $\mathbf{e}_1$. The stable manifold for the downward gradient flow at the north pole is not a subset of $\mathbf{\Phi}$; it is the upper sheet $\mathtt{H}^+$ of the two-sheeted hyperboloid forming the real part of $\mathcal{V}$, namely*

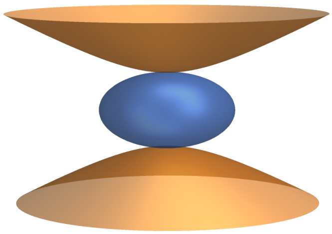

Figure 6. Stable and unstable manifolds at the critical points. A few trajectories of a gradient-like vector field tangent to those manifolds are shown. Note that the vector field is far from Morse–Smale: the stable 3-manifold for one critical point coincides with the unstable 3-manifold of the other, even though both have codimension 3.

the set $\{\mathbf{z} \in \mathbb{R}^d : z_1 > 0 \text{ and } z_1^2 = 1 + \sum_{j=2}^d z_j^2\}$*. At the south pole* $-\mathbf{e}_1$*, these are reversed, with the stable manifold for downward gradient flow equal to* $\mathtt{S}$ *and the unstable manifold being the real surface* $\mathtt{H}^- := \widetilde{\mathcal{V}} \cap \mathbb{R}^d$*; see Figure 6.*

Proof. Once we check that $\mathtt{S}$, $\mathtt{H}^+$ and $\mathtt{H}^-$ are invariant manifolds for the gradient flow on $\widetilde{\mathcal{V}}$, the proposition follows from the dimensions and the fact the ranges of $\widetilde{h}$ on $\mathtt{H}^+$, $\mathtt{S}$ and $\mathtt{H}^-$ are $[1, \infty)$, $[-1, 1]$ and $(-\infty, -1]$, respectively. Invariance of $\mathtt{H}^\pm$ follows from the fact that the gradient is a real map (the gradient at real points is real), and therefore the real subspace, of which $\mathtt{H}^\pm$ are connected components, is preserved by gradient flow. Invariance of $\mathtt{S}$ follows from the same argument after reparametrizing via $(x_1, \ldots, x_d) = (s_1, is_2, \ldots, is_d)$. ∎

5. Perturbation of the variety

Instead of working directly with Q, we consider the small perturbations $Q_c(\mathbf{z}) := Q(\mathbf{z}) - c$. As above, let $\mathcal{V}_c$ denote the zero set of Q_c, let $\omega_c = (P/Q_c^k)\, d\mathbf{z}/\mathbf{z}$ denote the corresponding Cauchy d-form, and let $\mathcal{M}^c = \mathbb{C}_*^d - \mathcal{V}_c$ denote the points where ω_c is analytic. Below we collect several results on the behavior of this deformation.

Proposition 5.1 (Stable behavior). *Under the setup of the previous paragraph,*

(i) *For sufficiently small* $|c| > 0$*, the variety* $\mathcal{V}_c$ *is smooth.*

(ii) *For any index* $\mathbf{r}$*, the coefficient of the power series expansion for* $F_c = P/Q_c^k$ *given by (1.3),*

$$a_{\mathbf{r}}(c) := \left(\frac{1}{2\pi i}\right)^d \int_T \mathbf{z}^{-\mathbf{r}} \frac{P(\mathbf{z})}{Q_c^k} \frac{d\mathbf{z}}{\mathbf{z}},$$

is holomorphic in the disk $|c| < |Q(0)|$*. In particular, any given coefficient is continuous at* $c = 0$ *as a function of* c*.*

Proof. The first statement follows from the Bertini–Sard theorem (the values of c that make $\mathcal{V}_c$ singular is a finite algebraic set). The second follows from the fact that each term in the (converging, under our assumptions) expansion of

$$\frac{P}{(Q-c)^k} = \sum_{l\geq 0} \binom{-k}{l} \frac{P}{Q^{k+l}} c^l$$

is holomorphic and thus integrable over any torus in the domain of holomorphy of F, and the modulus of each term is bounded. ■

We will need to understand the local behavior of $h_{\mathbf{r}}$ on the smooth varieties $\mathcal{V}_c$ near $\mathbf{z}_*$. The following proposition shows that the perturbed varieties have the same geometry as the limiting quadric described in Section 4.

Proposition 5.2 (Local behavior). *Assume that the direction $\hat{\mathbf{r}}$ strictly supports the tangent cone $T_{\mathbf{x}_*}(B)$. Then*

(i) *There is a $\delta > 0$ such that for sufficiently small $|c| \neq 0$, there are precisely two critical points of $h_{\hat{\mathbf{r}}}$ on the variety $\mathcal{V}_c$ in the ball $B_\delta(\mathbf{z}_*)$. These points tend to $\mathbf{z}_*$ as $c \to 0$.*

(ii) *If c is positive and real, these critical points $\mathbf{z}_c^\pm$ are real and can be chosen such that*

$$h_{\hat{\mathbf{r}}}(\mathbf{z}_c^+) > h_{\hat{\mathbf{r}}}(\mathbf{z}_*) > h_{\hat{\mathbf{r}}}(\mathbf{z}_c^-).$$

Proof. By part (i) of Proposition 5.1, $\mathcal{V}_c$ is smooth. The function $h_{\hat{\mathbf{r}}}$ is the real part of the logarithm of the locally holomorphic function $\mathbf{z}^{\hat{\mathbf{r}}}$ near $\mathbf{z}_*$, hence it has a critical point on the smooth complex manifold $\mathcal{V}_c$ if and only if $\mathbf{z}^{\mathbf{r}}$ does, i.e., if $\mathrm{d}\mathbf{z}^{\mathbf{r}}$ is collinear with $\mathrm{d}Q$. This latter condition defines the so-called *log-polar* variety. A local computation implies that under our conditions this is a smooth curve, intersecting $\mathcal{V}$ with multiplicity 2 at $\mathbf{z}_*$ as long as $\mathbf{r}$ is not tangent to the tangent cone $T_{\mathbf{L}(\mathbf{z}_*)}(\log \mathcal{V})$.

Indeed, one can find a real affine-linear coordinate change such that in the new coordinates, centered at $\mathbf{z}_*$,

$$Q = z_1^2 - \sum_{k\geq 2} z_k^2 + O(|\mathbf{z}|^3) \quad \text{and} \quad h = z_1 + \sum_{k\geq 2} a_k z_k + O(|\mathbf{z}|^2),$$

where our conditions on $\mathbf{r}$ imply $\sum_k a_k^2 < 1$. In these coordinates, the log-polar variety is given by the equation $z_k = -a_k z_0 + O(|\mathbf{z}|^2)$. Thus, the log-polar curve intersects $\mathcal{V}_c$ transversely for $|c| \neq 0$ small and consists of 2 geometrically distinct points. A similar computation implies the second statement for real c. ■

The main work in proving Theorem 2.5 will be to prove the following result.

Theorem 5.3. *Assume the hypotheses of Theorem 2.5. For any compact subset $K \subseteq \widehat{\mathcal{E}}$, one can choose positive numbers ε and c_* as well as a cycle $\Gamma(\hat{\mathbf{r}})$ for every $\hat{\mathbf{r}} \in K$, such that the following hold for every $|c| < c_*$:*

(i) *The cycle $\Gamma(\hat{\mathbf{r}})$ lies in the set $\mathcal{M}_c(-\varepsilon)$; in other words, the cycle $\Gamma(\hat{\mathbf{r}})$ lies below the height level $h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon$ and it avoids $\mathcal{V}_c$ for all c such that $|c| < c_*$.*

(ii) *There is a chain $\Gamma_c \subseteq \mathcal{M}_c$ such that $[\mathbf{T}] \simeq [\Gamma_c] + [\Gamma(\hat{\mathbf{r}})]$ in $H_d(\mathcal{M}_c)$.*

(iii) *The cycle $\Gamma(\hat{\mathbf{r}})$ can be chosen to be $[\circ\gamma(\hat{\mathbf{r}})] + [\mathbf{T}(\mathbf{y})]$, where $\gamma(\hat{\mathbf{r}})$ is a $(d-1)$ cycle in $\mathcal{V}(-\varepsilon)$, and $\mathbf{y}$ is in a descending component B' of the complement of* amoeba(Q) *with respect to B (see Definition 3.4).*

(iv) *For fixed $\mathbf{r}$ as $c \to 0$,*

$$\int_\Gamma \mathbf{z}^{-\mathbf{r}} \omega_c \to \int_\Gamma \mathbf{z}^{-\mathbf{r}} \omega, \tag{5.1}$$

$$\int_{\Gamma_c} \mathbf{z}^{-\mathbf{r}} \omega_c \to 0 \quad \text{if } d > 2k. \tag{5.2}$$

Theorem 5.3 is proven in Section 7. The significance of Theorem 5.3 is that it allows one to deduce the coefficient behavior of generating functions such as the GRZ and KZ functions. Prior to this, it was known how to compute the inverse Fourier transform of $1/\widetilde{Q}$, where $\widetilde{Q}$ is a Lorentzian quadratic, and how to approximate the Fourier transform of $1/Q$ by that of $1/\widetilde{Q}$ when Q is a polynomial whose leading homogeneous part at its singularity of minimal modulus is equal to $\widetilde{Q}$. What is new in Theorem 5.3 is the topological decomposition of the Cauchy torus in a basis of local homology generators. Showing that $[\mathbf{T}]$ can be decomposed into a cycle independent of the perturbation parameter and supported below the cone point, together with a class whose contribution goes to zero as the perturbation shrinks to zero, explains phenomena for which computations were not carried out at the chain level in [3]. Together with Proposition 8.4 and the rigorous numerics at the end of Section 8, the local homology decomposition allows us to deduce the full decomposition of $[\mathbf{T}]$ in local homology generators over which integrals can be computed. The quantitative results in Theorem 2.8 depend on this decomposition and provide the first example of a coefficient greater than 1 in the local homology decomposition of the Cauchy torus.

Proof of Theorem 2.5. The first statement of Theorem 2.5 follows immediately from Theorem 5.3, as

$$\begin{aligned} a_{\mathbf{r}} &= \lim_{c \downarrow 0} a_{\mathbf{r},c} = \lim_{c \downarrow 0} \int_{\mathbf{T}} \mathbf{z}^{-\mathbf{r}} \omega_c = \lim_{c \downarrow 0} \Big[\int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}} \omega_c + \int_{\Gamma_c} \mathbf{z}^{-\mathbf{r}} \omega_c \Big] \\ &= \int_{\Gamma(\hat{\mathbf{r}})} \mathbf{z}^{-\mathbf{r}} \omega \end{aligned}$$

by (5.1), (5.2) and (2.2). The uniform bound in $\hat{\mathbf{r}}$ follows from compactness of K. Indeed, any cycle $\Gamma(\hat{\mathbf{r}})$ satisfies the conclusions for all $\hat{\mathbf{r}}'$ in a small enough open neighborhood of $\hat{\mathbf{r}}$; choosing a finite cover of K by such open vicinities, we prove the claim. To obtain the second statement of Theorem 2.5, we use Proposition 3.6 to see that

$$\int_{\mathbf{T}(\mathbf{x}')} \mathbf{z}^{-\mathbf{r}} \omega_c$$

vanishes for all but finitely many $\mathbf{r}$. Together with (iii) of Theorem 5.3, this implies the conclusion of Theorem 2.5 for polynomial numerators. ■

6. Local homology near a quadratic point

Recall our sign choice for Q, which implies that Q is positive on the real part of the domain of holomorphy for the Laurent expansion under consideration. We are interested in the local topology of the intersections of the singular set $\mathcal{V}_c$ with the height function $h = |\mathbf{z}^\mathbf{r}|$. We start with a result proved in [1, Lemma 1.3], though it dates back at least to [28].

Proposition 6.1. *There exist $\delta, \delta' > 0$ such that if $B = B(\mathbf{z}_*, \delta)$ denotes the ball of radius δ about $\mathbf{z}_*$, then $\mathcal{V}_c \cap B$ is diffeomorphic to the total space of the tangent bundle to the $(d-1)$-dimensional sphere for all $c \in \mathbb{C}$ with $0 < |c| < \delta'$. In particular, the (absolute) homology groups of $\mathcal{V}_c \cap B$ are trivial in dimensions not equal to $d-1$, and $H_{d-1}(\mathcal{V}_c \cap B) \cong \mathbb{Z}$.*

Let $h_* := h(\mathbf{z}_*)$. What we require for our results is a description of the relative homology group $H_{d-1}((\mathcal{V}_c)_* \cap B, \mathcal{V}_c \cap B(h \leq h_* - \varepsilon))$, together with explicit generators. To compute these, we start with the homogeneous situation and then perturb. Denote by q the quadratic part of Q at $\mathbf{z}_*$. This is a real quadratic form, invariant with respect to conjugation, with signature $(1, d-1)$ on the real part of the tangent space at $\mathbf{z}_*$. We denote the two convex cones where $q \geq 0$ by $C_\pm$ and extend Definition 2.3 by considering supporting vectors to C_+ as well as C_-.

Consider the following three surfaces in $\mathbb{C}^d$ of respective codimensions 1, 1 and 2:

(i) the boundary S of the unit ball;

(ii) the hyperplane $H := \{\mathbf{x} + i\mathbf{y} : \mathbf{x} \cdot \hat{\mathbf{r}} = 0\}$ orthogonal to the real vector $\hat{\mathbf{r}}$;

(iii) the complex hypersurface $v := \{q = 0\}$ defined by the quadric.

The transverse intersection of S and H is the equator of S.

Lemma 6.2. *If $\mathbf{r}$ is supporting, then v intersects $S \cap H$ transversely.*

Proof. By the hypothesis that $\mathbf{r}$ is supporting, one can choose h as one local coordinate, changing the rest of the coordinates so that the quadratic form q preserves its Lorentzian form. In these new coordinates, it remains to prove that the functions

$$x_1 = 0, \quad x_1^2 + \sum_2^d y_k^2 - y_1^2 + \sum_2^d x_k^2 = 0, \quad x_1 y_1 - \sum_2^d x_k y_k = 0$$

have independent differentials at their common zeros outside of the origin. This can be checked directly. ∎

Corollary 6.3. *For $\rho > 0$ small enough, there are positive numbers ε_* and c_* such that the manifolds $\{\mathbf{z} : |\mathbf{z} - \mathbf{z}_*| = r\}$, $\{\mathbf{z} : h(\mathbf{z}) = h_*(\mathbf{z}) + \varepsilon\}$ and $\{\mathbf{z} : Q(\mathbf{z}) = c\}$ intersect transversely, provided that $\rho/2 < r < \rho$ while $\varepsilon < \varepsilon_*$ and $|c| < c_*$.*

Proof. For a given $\rho > 0$, introduce new coordinates in which $\mathbf{z}_*$ is the origin and the ρ-ball around $\mathbf{z}_*$ becomes the unit ball in $\mathbb{C}^d$, while rescaling Q by ρ^{-2} and h by ρ^{-1}. The resulting functions become small perturbations (decreasing with ρ) of the quadratic and linear functions in Lemma 6.2, and their zero sets become small deformations Q^ρ and H^ρ of the corresponding varieties.

In particular, the determinants whose nonvanishing witnesses the transversality of the varieties of Q^ρ, H^ρ and S are small deformations of the determinants witnessing the transversality in Lemma 6.2, and therefore are nonvanishing on some open neighborhood U of the set of solutions to $H^\rho = Q^\rho = 0$ intersected with the spherical shell where the distance to the origin is between, say, 1 and $1/2$ for small enough ρ.

For small enough $\varepsilon_*, c_* > 0$, the sets $\{|Q^\rho| \le c_*\} \cap \{|h^\rho| \le \varepsilon_*\} \cap B_1$ are contained in U. Therefore, the varieties $\{Q^\rho = c\}$, $\{H^\rho = \varepsilon\}$ and $\{|\mathbf{z}| = r\}$ are transverse when $|c| \le c_*$, $|\varepsilon| \le \varepsilon_*$ and $1/2 \le r \le 1$. ∎

We will need one more result on the local geometry of $\mathcal{V}$ and $\{h = \text{const}\}$.

Lemma 6.4. *For $\varepsilon \neq 0$, the intersection of the real hyperplane $x_1 = -\varepsilon$ with the quadric*

$$z_1^2 - z_2^2 - \cdots - z_d^2 = c$$

is homotopy equivalent to a $(d-2)$-dimensional sphere for $|c|$ small enough.

Proof. Rescaling, we can assume that $\varepsilon = -1$. Parametrizing $(x_2, \ldots, x_d) = s\xi$ and $(y_2, \ldots, y_d) = t\eta$, where $s, t \ge 0$ and ξ, η are unit vectors in $\mathbb{R}^{d-1}$, we obtain the equations

$$x_1 = 1, \quad 1 - y_1^2 + t^2|\eta|^2 - s^2|\xi|^2 = c, \quad y_1 = st(\eta \cdot \xi). \tag{6.1}$$

Suppose $c = 0$. Then the manifold in question is given by

$$1 + t^2 = s^2 t^2 |\xi \cdot \eta|^2 + s^2.$$

Since $s^2t^2|\xi \cdot \eta|^2 + s^2 \le s^2(1+t^2)$, one can keep ξ, η fixed and retract (s,t) satisfying this equation to $(1,0)$. This retracts the manifold onto the unit $(d-2)$-sphere.

For nonzero c, it can be verified that the manifolds given by (6.1) are transverse, and therefore remain transverse for small c, meaning the intersections are homeomorphic. ∎

Corollary 6.5. *Assume that* $\hat{\mathbf{r}}$ *is supporting. Then, for* $\rho > 0$ *small enough, there are* $\varepsilon, c_* > 0$ *such that*

$$\mathcal{V}_c \cap \{h_{\hat{\mathbf{r}}} = -\varepsilon\} \cap B_\rho(\mathbf{z}_*)$$

is homotopy equivalent to $S^{(d-2)}$ *for* $|c| < c_*$.

Proof. We can choose coordinates in which the quadratic part of Q and $h_{\hat{\mathbf{r}}}$ are given by $z_1^2 - z_2^2 - \cdots - z_d^2$ and x_1, respectively. Then, repeating the argument of Corollary 6.3, we can view rescaled Q and h as small perturbations of the quadratic and linear functions in Lemma 6.4, and apply transversality. ∎

We will be referring to the intersection

$$\mathtt{slab} := \mathtt{slab}_{\rho,\varepsilon} := B_\rho \cap \{|h - h_*| \le \varepsilon\},$$

for ρ, ε satisfying the conditions of Corollary 6.3, as the (ρ,ε)-slab (see Figure 7). We call the intersection of the slab with the boundary ∂B_ρ its *vertical* boundary, and the intersection with $h = h_* - \varepsilon$ its *bottom*.

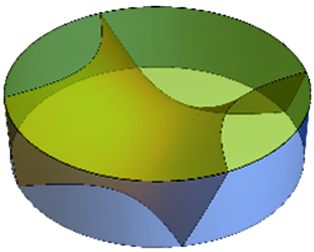

Figure 7. A slab.

Corollary 6.6. *For* ρ, ε_*, c_* *satisfying the conditions of Corollary 6.3, whenever* $|c| < c_*$, *there exists a vector field* $\mathbf{v}$ *on the intersection* $\mathcal{V}_{c,\mathtt{slab}} := \mathcal{V}_c \cap \mathtt{slab}_{\rho,\varepsilon_*}$ *such that the following hold:*

(1) $\mathrm{d}h \cdot \mathbf{v} < 0$ *everywhere outside of the critical points of* h *on* $\mathcal{V}_{c,\mathtt{slab}}$.

(2) *For points on* $\mathcal{V}_{c,\mathtt{slab}}$ *within* $\rho/3$ *from* $\mathbf{z}_*$, *the vector field is the gradient vector field for* $-h$ *on* $\mathcal{V}_c$ *with respect to the standard Hermitian form on* $\mathbb{C}^d$.

(3) *For points at distance between* $\rho/2$ *and* ρ *from* $\mathbf{z}_*$*, the vector field is tangent to the spheres* $\{|\mathbf{z}-\mathbf{z}_*| = \text{const}\}$ *and* $\mathrm{d}h \cdot \mathbf{v} = -1$.

(4) *If* c *is real, the vector field is invariant under conjugation* $\mathbf{v}(\overline{\mathbf{z}}) = \overline{\mathbf{v}(\mathbf{z})}$.

Proof. Let $\mathbf{v}^{(\mathbf{z}_*)}$ denote the gradient vector field for $-h$ on $\mathcal{V}_c$ as in statement (2). For any point $\mathbf{z}$ at distance between ρ and $\rho/2$ from $\mathbf{z}_*$, the transversality conclusions of Corollary 6.3 imply that near $\mathbf{z}$ one can choose coordinates that include the four functions h, $|\mathbf{z}-\mathbf{z}_*|$, $\Re\{Q\}$ and $\Im\{Q\}$. In such coordinates, define $\mathbf{v}^{(\mathbf{z})} := \partial/\partial h$. Since h, Q, distance and the standard Hermitian form are invariant with respect to complex conjugation, we may choose the family $\{\mathbf{v}^{(\mathbf{z})}\}$ to be invariant, in the sense that $\mathbf{v}^{(\overline{\mathbf{z}})}(\overline{\mathbf{w}})$ is the conjugate of $\mathbf{v}^{(\mathbf{z})}(\mathbf{w})$.

Use a partition of unity to glue together the vector fields $\mathbf{v}^{(\mathbf{z})}$, ensuring that the partition gives weight 1 to points in a $\rho/3$ neighborhood of $\mathbf{z}_*$ and zero weight outside the $\rho/2$ neighborhood. This ensures conclusions (1), (2) and (3). If the partition is chosen invariant with respect to conjugation, the last conclusion will be true as well. ∎

Proposition 6.7. *Assume again that* $\mathbf{r}$ *is supporting at* $\mathbf{z}_*$*, where* $\mathbf{z}_*$ *is a quadratic singularity of* Q *with signature* $(1, d-1)$*. Fix* ρ *and* ε *satisfying conditions of Corollary 6.3 and the corresponding* (ρ, ε)*-slab. Letting* $\texttt{bottom}$ *denote* $\mathcal{V}_c \cap \texttt{slab} \cap \{h - h_* = -\varepsilon\}$*, the relative homology group*

$$H_- := H_{d-1}(V_c \cap \texttt{slab}, \texttt{bottom})$$

is free of rank 2 for small enough $|c| \neq 0$*. For small real* $c > 0$*, its generators are given by*

- *an absolute cycle, the image of the generator of* $H_{d-1}(V_c \cap B_r)$ *under the natural homomorphism into* H_-*, and*
- *the relative cycle corresponding to the lobe of the real part of* V_c *located in* $\{h \le h_*\}$.

Proof. The trajectories of the flow along the vector field $\mathbf{v}(\cdot)$ constructed in Corollary 6.6 starting on $\mathcal{V}_{c,\texttt{slab}}$ either converge to the critical points of h on $\mathcal{V}_{c,\texttt{slab}}$, or reach $\texttt{bottom}$. Indeed, the value of h is strictly decreasing outside of the critical points and cannot leave the slab through its side due to conclusion (3) of Corollary 6.6. All trajectories therefore remain in the slab or reach the bottom.

The homology of the pair $(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom})$ is generated by classes represented by the unstable manifolds of the Morse function h at critical points on $\mathcal{V}_c \cap \texttt{slab}$; this is the fundamental theorem of stratified Morse theory, for example, [15, Theorem B]. In our situation, there are exactly two such critical points, $\mathbf{z}_-$ and $\mathbf{z}_+$, both in the real part of $\mathcal{V}_c$ and both of index $d-1$. This proves the statement about the rank of the group.

The long exact sequence of the inclusion of the bottom into $\mathcal{V}_c \cap \texttt{slab}$ gives an exact sequence containing the maps

$$H_{d-1}(\texttt{bottom}) \to H_{d-1}(\mathcal{V}_c \cap \texttt{slab}) \to H_{d-1}(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom}).$$

In accordance with Corollary 6.5, the first of these groups vanishes because $\mathcal{V}_c \cap \texttt{bottom}$ is homotopy equivalent to S_{d-2}. It follows that the absolute cycle generating $H_{d-1}(V_c \cap B_\rho)$ is nonvanishing in $H_{d-1}(\mathcal{V}_c \cap \texttt{slab}, \texttt{bottom})$ and is therefore a generator of H_-.

For $c > 0$, the real part of $\mathcal{V}_c$ located within the lower half of the slab, $\{h < h_*\}$, contains the critical point $\mathbf{z}_-$ (by Proposition 5.2), and the vector field $\mathbf{v}$ is tangent to it (thanks to the reality property mentioned above). Hence it coincides with the unstable manifold of $\mathbf{z}_-$. ■

Of course, the same argument applies to the Morse function $-h$ on V_c, implying that the group

$$H_+ := H_{d-1}(V_c \cap \texttt{slab}, (V_c \cap \texttt{slab}) \cap \{h = h_* + \varepsilon\})$$

also has rank 2 and, for positive real c, is generated by the same absolute cycle together with the analogous relative cycle (the lobe of the real part of $\mathcal{V}_c$ located in $\{h \geq h_*\}$). For small positive c (the situation we will restrict ourselves to from now on), we will denote the generators in H_- by $\texttt{S}_-$ and $\texttt{H}_-$, where $\texttt{S}_-$ is the absolute class represented by the small sphere in V_c and $\texttt{H}_-$ is the relative class represented by the corresponding component of the real part of Q_c. In the same way, we define classes $\texttt{S}_+$ and $\texttt{H}_+$ generating H_+.

A general duality result implies that the relative groups H_- and H_+ are dual to each other, with the coupling given by the intersection index. Briefly, the reason is that the vector field in Corollary 6.3 may be used to deform $\texttt{slab}$ until the boundary of the top flows down to the boundary of the bottom; this makes the space into a manifold with boundary satisfying the hypotheses of [17, Theorem 3.43]. The conclusion of that theorem is an isomorphism between a homology group and a cohomology group, which, combined with Poincaré duality, proves the claim. In fact, we will not use this argument, because we need to compute this coupling explicitly, as follows.

Proposition 6.8. *The intersection pairing between H_- and H_+ is given by*

$$\begin{aligned}
\langle \texttt{H}_+, \texttt{H}_- \rangle &= 0, \\
\langle \texttt{H}_+, \texttt{S}_- \rangle &= (-1)^{d(d-1)/2}, \\
\langle \texttt{S}_+, \texttt{H}_- \rangle &= (-1)^{d(d-1)/2}, \\
\langle \texttt{S}_+, \texttt{S}_- \rangle &= (-1)^{d(d-1)/2} \chi(S^{d-1}) = (-1)^{d(d-1)/2}(1 + (-1)^{d-1}).
\end{aligned}$$

Remark. We pedantically distinguish between $\mathtt{S}_+$ and $\mathtt{S}_-$, although they are the image of the same absolute class, or even chain, in V_c. Also, we note that our orientations of the spheres and their tangent spaces can be in disagreement with the standard orientations induced by the complex structure. By changing the orientation of the chain $\mathtt{S}$, one can suppress the annoying sign factor in the second and third equalities, but not in the last one.

Proof. We can work (after rescaling) in the setup of Lemma 4.1. The cycles representing $\mathtt{H}_c^{\pm}$ are disjoint, explaining the first line. Each intersect $\mathtt{S}$ in precisely one point. Denoting $\partial/\partial x_k$ by ξ_k and $\partial/\partial y_k$ by η_k, the tangent spaces to $\mathtt{S} = \mathtt{S}_\pm$ at $\mathbf{z}_\pm$ are spanned by the vectors

$$\pm\eta_2, \ldots, \pm\eta_d,$$

and the tangent spaces to $\mathtt{H}_\pm$ at $\mathbf{z}_\pm$ are spanned by

$$\pm\xi_2, \ldots, \pm\xi_d,$$

respectively.

In the standard orientation of the complex hypersurface $\widetilde{\mathcal{V}}$, the frame $(\xi_2, \eta_2, \ldots, \xi_d, \eta_d)$ is positive. Hence, the intersection index of $\mathtt{H}_+$ and $\mathtt{S}$ is the parity of the permutation shuffling

$$(\xi_2, \ldots, \xi_d, \eta_2, \ldots, \eta_d)$$

into that standard order, giving the second line. The third line is obtained similarly, taking the signs into account.

The last pairing can be observed by noting that the self-intersection index of a class represented by a manifold of middle dimension in a complex manifold is equal to the Euler characteristics of the conormal bundle of the manifold, under the identification of the collar neighborhood of the manifold with its conormal bundle. This gives $\chi(\mathtt{S}) = (1 + (-1)^{d-1})(-1)^{d(d-1)/2}$, where again the mismatch between the standard orientation of the conormal bundle and the ambient complex variety contributes the factor $(-1)^{d(d-1)/2}$. ■

Importance of the local homology computation lies in the following localization result. Let $\mathbf{u}_* := \mathbf{L}(\mathbf{z}_*) \in \mathbb{R}^d$ be a point on the boundary of amoeba(Q) (recall $\mathbf{L}$ is the logarithmic map $\mathbf{z} \mapsto \log|\mathbf{z}|$).

Theorem 6.9. *Assume that the quadratic critical point $\mathbf{z}_*$ is the only element of $\mathbf{T}(\mathbf{u}_*) \cap \mathcal{V}$, that $\mathbf{z}_*$ lies on the boundary of a component of* amoeba$(Q)^c$ *and that $\mathbf{r}$ is supporting. Then for any $\rho > 0$, there exist $\varepsilon, c_* > 0$ such that for all $c \in (0, c_*)$, the intersection class $\mathcal{I}(\mathbf{T}) \subseteq \mathcal{V}_c$ can be represented by a chain supported on*

$$B_\rho(\mathbf{z}_*) \cup \{h \leq h_* - \varepsilon\}.$$

Proof. Choose ρ small enough so that the conclusions of Corollary 6.3 hold. As the intersection of $\mathcal{V}$ with the torus $\mathbf{T}(\mathbf{u}_*)$ containing $\mathbf{z}_*$ is a single point, standard compactness arguments imply that for sufficiently small positive δ, the intersection of $\mathcal{V}$ with the $\mathbf{L}$-preimage of $B(\mathbf{u}_*, \delta)$ is contained in $B_\rho(\mathbf{z}_*)$. Pick a torus $\mathbf{T}(\mathbf{x})$, where $\mathbf{x}$ is a point in the intersection of B with the component of the complement to the amoeba defining our power series expansion. Choose $\varepsilon > 0$ such that $\{h \le h_* - \varepsilon\}$ intersects $B_\rho(\mathbf{z}_*)$. Let $\mathbf{y}$ be a point in the component B' defined at the end of Section 3, such that $h_{\hat{\mathbf{r}}}(\mathbf{y}) < h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon$. Choose any smooth path $\{\alpha(t) : 0 \le t \le 1\}$ from $\mathbf{x}$ to $\mathbf{y}$ that passes through $B_\rho(\mathbf{z}_*)$ and along which $h_{\hat{\mathbf{r}}}$ decreases. Then the $\mathbf{L}$-preimage of that path is a cobordism between $\mathbf{T}$ and a torus $\mathbf{T}'$ in $\{h \le h(\mathbf{z}_*) - \varepsilon\}$. The transversality conclusion of Corollary 6.3 means that this cobordism, or a small perturbation of it, produces a chain realizing the intersection class $\mathcal{I}(\mathbf{T})$ and satisfying the desired conclusions. ■

We now come to the main result of this section, which completes step (1) of the outline at the end of Section 3.

Theorem 6.10. *For even d, the intersection class $\mathcal{I}(\mathbf{T})$ is equal, up to sign, to $[S_c]$ in $H_{d-1}(\mathcal{V}_c, \mathcal{V}_c(\le -\varepsilon))$.*

Proof. Let e denote the class of $\mathcal{I}(\mathbf{T})$ in the relative homology group H_-. Then, by Lemma 6.7, we have $e = a\mathtt{H}_- + bS_-$ for some integers a and b. We claim that

$$\langle \mathtt{H}_+, e\rangle = \pm 1, \quad \langle S_+, e\rangle = 0. \tag{6.2}$$

The construction of the chain representing the intersection class $\mathcal{I}(\mathbf{T})$ in Theorem 6.9 implies that it meets the chain representing $\mathtt{H}_+$ at precisely one point $\mathbf{z}'_c$. The point $\mathbf{z}'_c$ is not necessarily the point $\mathbf{z}^+_c$, but it is characterized by being the unique point where the homotopy intersects the real variety $\mathcal{V}_{c,\mathbb{R}} \subseteq \mathcal{V}_c$.

The intersection class is represented by a chain that is smooth near $\mathbf{z}'_c$. We need to check that its intersection with the "upper lobe" $\mathtt{H}_+$ is transverse within $\mathcal{V}_c$. Indeed, one can linearly change coordinates centered at $\mathbf{z}'_c$ so that in the new coordinates $\mathbf{z}'$ the homotopy segment runs along the x'_1 axis, and thus the equations defining the cobordism are $x'_2 = \cdots = x'_d = 0$. Then $\mathcal{V}_c$ is given by $z'_1 = R(z'_2, \ldots, z'_d)$ with $\mathrm{d}R|_{\mathbf{z}^+_*} = 0$. By direct computation, the intersection is transversal, and the tangent space to the chain representing the intersection class at $\mathbf{z}^+_*$ is the tangent space to $\mathcal{V}_c$ at $\mathbf{z}^+_*$ multiplied by i.

Because $\mathtt{H}_+$ and e intersect transversely at a single point, the first identity in (6.2) is proved. For the second identity, we again rely on perturbations of the cobordism defining the intersection class. If the path defining the cobordism avoids $\mathbf{z}_*$, for c small enough, the chain realizing $\mathcal{I}(\mathbf{T})$ constructed in Theorem 6.9 will completely avoid the chain representing $\mathtt{S}$, implying that the intersection number of e with $\mathtt{S}_+$ is zero.

To finish, we substitute (6.2) into Proposition 6.8. We compute

$$\pm 1 = \langle \mathtt{H}_+, e \rangle = a \cdot 0 + b \cdot \pm 1,$$

therefore $b = \pm 1$, and

$$0 = \langle \mathtt{S}_+, e \rangle = \pm a \pm b\chi(S^{d-1}).$$

When d is even, the Euler characteristic of the $(d-1)$-dimensional sphere vanishes together with a. ■

7. Proof of the main theorem and Theorem 5.3

We are now ready to prove Theorem 5.3, and thus obtain our main Theorem 2.5. At each stage, it is easiest to prove the result for fixed $\hat{\mathbf{r}}$ and then argue by compactness that the conclusion holds for all $\hat{\mathbf{r}} \in K$. We start with a localization result. Use the notation $\mathcal{I}_c$ to denote intersection class with respect to the perturbed variety $\mathcal{V}_c$.

Lemma 7.1. *Fix $\hat{\mathbf{r}} \in K$. Under the hypotheses of Theorem 6.10, there is an $\varepsilon > 0$ such that the intersection class $\mathcal{I}_c(\mathbf{T}, \mathbf{T}')$ is*

$$\mathcal{I}_c = [S_c] + [\gamma_c],$$

where the cycle $\gamma_c(\hat{\mathbf{r}})$ representing the class $[\gamma_c] \in H_{d-1}(\mathcal{V}_c)$ is supported in $\mathcal{V}_c(<-\varepsilon)$ with respect to $h_{\hat{\mathbf{r}}}$.

Proof. By Theorem 6.10, $\mathcal{I}_c - S_c$ is mapped to zero in the second map of the exact sequence

$$\cdots \to H_{d-1}(\mathcal{V}_c(<-\varepsilon)) \to H_{d-1}(\mathcal{V}_c) \to H_{d-1}(\mathcal{V}_c, \mathcal{V}_c(<-\varepsilon)) \to \cdots .$$

Hence $\mathcal{I}_c - [S_c]$ is represented by a class in $H_{d-1}(\mathcal{V}_c(<-\varepsilon))$. ■

Let Σ denote the singular locus of $\mathcal{V}$, that is, the set $\{\mathbf{z} \in \mathcal{V} : \nabla Q(\mathbf{z}) = \mathbf{0}\}$. The point $\mathbf{z}_*$ is a quadratic singularity, thus isolated, and we may write $\Sigma = \{\mathbf{z}_*\} \cup \Sigma'$, where Σ' is separated from $\mathbf{z}_*$ by some positive distance.

Corollary 7.2. *If the real dimension of Σ is at most $d-2$, then for some $\delta > 0$, the cycles $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta, \hat{\mathbf{r}} \in K\}$ may be chosen so as to be simultaneously supported by some compact Ξ disjoint from Σ.*

Remark. In the case where Σ is the singleton $\{\mathbf{z}_*\}$ or when any additional points $\mathbf{z} \in \Sigma$ satisfy $h(\mathbf{z}) \le h(\mathbf{z}_*) - \varepsilon$ for all $\hat{\mathbf{r}} \in K$, the proof is just one line. This is all our applications presently require, however the greater generality (although most likely not best possible) may be useful in future work.

Proof. The first step is to prove that for fixed $\hat{\mathbf{r}}$, we may choose the cycles $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta\}$ satisfying the conclusion of Lemma 7.1, all supported on a fixed compact set Ξ avoiding Σ. It suffices to avoid Σ' because the condition of being supported on $\mathcal{V}_{-\varepsilon}$ immediately implies separation from $\mathbf{z}_*$. The construction in Theorem 6.9 produces a single homotopy for all c, which is then intersected with each $\mathcal{V}_c$. It follows that the union of the intersection cycles is contained in a compact set. By the dimension assumption, a small generic perturbation avoids Σ' while still being separated from $\mathbf{z}_*$.

Having seen that for fixed $\hat{\mathbf{r}}$, the cycles $\{\gamma_c(\hat{\mathbf{r}}) : 0 < |c| < \delta\}$ may be chosen to satisfy the conclusions of Lemma 7.1 and to be supported on a compact set $\Xi(\hat{\mathbf{r}})$ avoiding Σ, the rest is straightforward. For each $\hat{\mathbf{r}}$, there is a neighborhood $\mathcal{N}(\hat{\mathbf{r}}) \subseteq K$ such that $\hat{\mathbf{s}} \in \mathcal{N}$ and $h_{\hat{\mathbf{r}}}(\mathbf{z}) \leq h_{\hat{\mathbf{r}}}(\mathbf{z}_*) - \varepsilon$ imply $h_{\hat{\mathbf{s}}}(\mathbf{z}) \leq h(\mathbf{z}_*) - \varepsilon/2$. Thus we may choose $\gamma_c(\hat{\mathbf{s}}) = \gamma_c(\hat{\mathbf{r}})$ to be independent of $\hat{\mathbf{s}}$ over $\mathcal{N}(\hat{\mathbf{r}})$. Choosing a finite cover of K by these neighborhoods, the union of the corresponding sets $\Xi(\hat{\mathbf{r}})$ supports the cycles $\gamma_c(\hat{\mathbf{r}})$ for all c and $\hat{\mathbf{r}}$. ■

Theorem 6.9 is a rather standard result about pushing the intersection class below height $h(\mathbf{z}_*)$ except in a small ball about $\mathbf{z}_*$. Our proof of Theorem 6.9 uses an unspecified torus $\mathbf{T}'$ with polyradius in the descending component B' of Definition 3.4, and is therefore not an explicit construction of a chain representing $\mathbf{T}$, but is sufficient to prove Theorem 6.10 and Lemma 7.1 describing the relative homology of the pair $(\mathcal{V}_c, \mathcal{V}_c(\leq -\varepsilon))$.

Equation (7.1) in Lemma 7.3 is all we need to complete the proof of Theorem 5.3. However, in Section 8 we study the asymptotic contributions of lower critical points, these being the dominant contributions in the lacuna setting, when d is even and greater than $2k$. For this purpose, we need a more explicit description of a cycle homologous to $\mathbf{T}$ at height below the critical point: the quadratic approximation of $\mathcal{V}_c$ is only good in a neighborhood of the critical point, however finding a torus disjoint from $\mathcal{V}$ may require traveling further down. The next lemma finds an explicit cycle homologous to $\mathbf{T}$, having height at most $-\varepsilon$ except for an arbitrarily small tube around a piece of $\mathcal{V}_{\leq 0}$, in two ways: one when a torus $\mathbf{T}'$ at height $-\varepsilon$ can be chosen disjoint from $\mathcal{V}$ and a different way when $\mathbf{T}'$ intersects $\mathcal{V}$.

Lemma 7.3. *Choose $\mathbf{x} \in B$, let $\mathbf{y} = -\varepsilon\mathbf{x}$, let $\alpha: [0,1] \to \mathbb{R}^d$ be the line segment from $\mathbf{x}$ to $\mathbf{y}$, and define $\mathbf{T}' = \mathbf{T}(\mathbf{y})$.*

(i) *Suppose that $\mathbf{T}'$ is disjoint from $\mathcal{V}$, as in the proof of Theorem 6.9. Then there exist $\varepsilon, c_*, c' > 0$ such that*

$$[\mathbf{T}] = [\circ S_c] + [\circ\gamma_{c'}] + [\mathbf{T}'] \tag{7.1}$$

for all $|c| < c_*$, where $\gamma_{c'}$ is the cycle from the conclusion of Lemma 7.1 with c replaced by c'.

(ii) *Alternatively, if* $\mathbf{T}'$ *is not disjoint from* $\mathcal{V}$*, then instead of* (7.1) *one has*

$$[\mathbf{T}] = [\circ S_c] + [(\circ\gamma_{c'})_{\geq -\varepsilon} \# \mathbf{T}'], \tag{7.2}$$

where $(\circ\gamma_{c'})_{\geq -\varepsilon} \# \mathbf{T}'$ *is the connected sum of* $(\circ\gamma_{c'})_{\geq -\varepsilon}$ *and* $\mathbf{T}'$ *along their common boundary* $(\circ\gamma_{c'})_{= -\varepsilon} = \partial(\mathbf{T}' \setminus \bullet\gamma_{c'})$.

Proof. For the compact Ξ described in Corollary 7.2, the intersection of $\mathcal{V}$ with Ξ is smooth. By Proposition 3.1, there is a neighborhood of Ξ in $\mathcal{V}_* \setminus \Sigma$ that can be parametrized as a 2-dimensional vector bundle over some compact subset $\Xi' \subseteq \mathcal{V}$. This bundle is naturally coordinatized by the values of Q so that for some small $c'_* > 0$, the tubular neighborhood around $\mathcal{V}_\Xi$ can be identified with $D' \times \mathcal{V}_\Xi$ for $\mathcal{D}' := \{c \in \mathbb{C} : |c| < c'_*\}$. We will denote this neighborhood by $\mathcal{V}_\Xi^{D'}$.

Lemma 7.1 implies that

$$[\mathbf{T}] = [\circ S_c] + [\circ\gamma_c] + [\mathbf{T}']$$

for all small enough $|c|$ (which we may assume from now on to be smaller than $c_* < c'_*$). The class $\circ\gamma_c$ can be represented by a small tube around a cycle $\gamma_c \in \mathcal{V}_c$, which is entirely supported by $\mathcal{V}_\Xi^{D'}$. Using the product structure $\mathcal{V}_\Xi^{D'} \cong D' \times \mathcal{V}_\Xi$, we can identify this tube with a product of a small circle (of radius $\rho(c) > 0$) around $c \in D'$ and γ_*, a cycle in the smooth part of $\mathcal{V}$ obtained by projection of γ_c. When c_* and ρ are sufficiently small, the maximum height of γ_* is $h_* - \varepsilon'$ for some $\varepsilon' > 0$.

There exists a homeomorphism of the annulus $D' - D_{\rho(c)}(c)$ fixing its outer boundary and sending the small circle $\partial D_{\rho(c)}(c)$ around c into the circle of radius c_*. Extend this homeomorphism, fiberwise, to all of the tubular neighborhood $\mathcal{V}_\Xi^{D'}$. Furthermore, extend it to the complement of $\mathcal{V}_\Xi^{D'}$ in such a way that it is identity outside of a small vicinity of $\mathcal{V}_\Xi^{D'}$ (and thus near S_c and $\mathbf{T}$, $\mathbf{T}'$). Choose c_* smaller if necessary and take $\circ\gamma$ to be the c_*-tube around γ_* for all c with $|c| < c_*$. Then this cycle avoids $\mathcal{V}_c$ for all c with $|c| < c_*$ and has maximum height less than $h_* - \varepsilon$, where ε is positive once c_* has been chosen sufficiently small with respect to ε'. This completes the proof of case (i).

For case (ii), let $\alpha \colon [0, 1] \to \mathbb{R}^d$ be as in Theorem 6.9 parametrizing the line segment from $\mathbf{x}$ to some $\mathbf{y} \in B'$. Let $\mathbf{T}''$ denote the torus with polyradius $\mathbf{y}$, and let $\mathbf{T}'$ be the torus at height ε' which is the slice of the homotopy in Theorem 6.9 for some $t' \in (0, 1)$. Let $\mathbf{y}'$ be the corresponding basepoint.

The homotopy swept out by tori with polyradii $\alpha(t)$ intersects $\mathcal{V}$ and defines an intersection cycle γ. The homotopy $\{\mathbf{L}^{-1}\alpha(t) : 0 \leq t \leq 1\}$ intersects $\mathcal{V}$, yielding an intersection $(d-1)$-chain $\gamma_{\geq\varepsilon}$, which is not a cycle. Its boundary is the $(d-1)$-cycle $\gamma_{=-\varepsilon}$ (assuming, without loss of generality, transversal intersections). These objects are illustrated in Figure 8. Comparing the known expression for $[\mathbf{T}]$

$$[\mathbf{T}] = [\circ S_c] + [\circ\gamma] + [\mathbf{T}''] \tag{7.3}$$

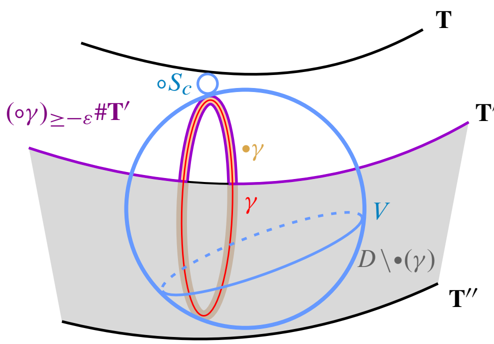

Figure 8. Removing a neighborhood of $\mathcal{V}$ from an expanding torus homotopy creates a region in $\mathcal{M}$ whose boundary is a useful cobordism. The cobordism in (7.3) is given by $(\circ\gamma)_{\geq -\varepsilon}\#\mathbf{T}' + \partial(D \setminus \bullet\gamma) + \circ S_c$.

to the desired expression for $[\mathbf{T}]$

$$[\mathbf{T}] = [\circ S_c] + [(\circ\gamma)_{\geq -\varepsilon}\#\mathbf{T}']$$

using the chain-level identity

$$(\circ\gamma)_{\geq -\varepsilon}\#\mathbf{T}' = (\circ\gamma)_{\geq -\varepsilon} + (\mathbf{T}' \setminus \bullet(\gamma)),$$

we see that the difference is represented by the chain

$$\begin{aligned}(\circ\gamma)_{\geq -\varepsilon} + (\mathbf{T}' \setminus \bullet(\gamma)) - \circ\gamma - \mathbf{T}'' - \mathbf{T}'' &= -(\circ\gamma)_{\leq -\varepsilon} + (\mathbf{T}' \setminus \bullet(\gamma)) - \mathbf{T}'' \\ &= \partial(D \setminus \bullet(\gamma)),\end{aligned}$$

where D is the $(d+1)$-chain given by the $\mathbf{L}$-preimage of $\{\alpha(t) : t' \leq t \leq 1\}$. Since the difference is a boundary, this establishes (7.2). ■

Proof of Theorem 5.3. Let $\gamma(\hat{\mathbf{r}})$ be chosen as in the first conclusion of Lemma 7.3, let

$$\Gamma(\hat{\mathbf{r}}) := \circ\gamma(\hat{\mathbf{r}}) + \mathbf{T}(\mathbf{y}),$$

which must satisfy condition (iii) of Theorem 5.3, and choose $\Gamma_c := \circ S_c$. Conclusion (i) follows from the choice of c_* in the proof of Lemma 7.3. Conclusion (ii) is equation (7.1). As the compact cycle Γ is independent of c, equation (5.1) follows from convergence of ω_c to ω on Γ for each $\mathbf{r}$. It remains only to verify (5.2).

To prove (5.2), choose a local coordinate system in which Q is reduced to its quadratic part, and rescale it by $c^{1/2}$ (either root will work). In this coordinate system $\mathbf{u} = \mathbf{v} + i\mathbf{w}$, we are integrating over the cycle $\circ S_1$, where

$$S_1 = \left\{ v_1^2 + \sum_{k=2}^{d} w_k^2 = 1 : w_1 = v_2 = \cdots = v_d = 0 \right\}.$$

In the new local coordinates $\mathbf{z} = \mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u})$ with ψ holomorphic and $\psi(0) = 1$, the form $\mathbf{z}^{-\mathbf{r}}\omega_c$ becomes

$$\begin{aligned}\mathbf{z}^{-\mathbf{r}}\omega_c &= \mathbf{z}_*^{-\mathbf{r}-\mathbf{1}}\left(1 + c^{1/2}\frac{\mathbf{u}}{\mathbf{z}_*}\right)^{-\mathbf{r}-\mathbf{1}} \frac{P(\mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u}))}{c^k q(\mathbf{u})^k} c^{d/2} \mathrm{d}\mathbf{u} \\ &= c^{d/2-k}\mathbf{z}_*^{-\mathbf{r}-\mathbf{1}} H(\mathbf{u}, c)\mathrm{d}\mathbf{u},\end{aligned}$$

where q is the quadric (4.1) and

$$H(\mathbf{u}, c) := \left(1 + c^{1/2}\frac{\mathbf{u}}{\mathbf{z}_*}\right)^{-\mathbf{r}-\mathbf{1}} \frac{P(\mathbf{z}_* + c^{1/2}\mathbf{u}\psi(\mathbf{u}))}{q(\mathbf{u})^k}.$$

The function $H(\mathbf{u}, c)$ is holomorphic in $\mathbf{u}$ and bounded on $\circ S_1$ uniformly in c. As $c \to 0$, $H(\mathbf{u}, c) \to P(\mathbf{z}_*)/q(\mathbf{u})^k$, and (5.2) follows. ■

8. Application to the Gillis–Reznick–Zeilberger function with critical parameter

Having established the exponential drop, this section extends Theorem 2.5 to obtain more precise asymptotics for $a_{\mathbf{r}}$. Most of what follows concentrates on the GRZ example, however we first state a result holding more generally in the presence of a lacuna. A *critical point at infinity*, formally defined in [4], can be viewed as a sequence of singularities going off to infinity in such a way that the limit of the differential of the height function at the points approaches zero. Here, we note only that there is an effective test for critical points at infinity [4, Algorithm 1] and that our GRZ example does not have any.

Theorem 8.1. *Assume the hypotheses of Theorem 2.5. Fix $\hat{\mathbf{r}}$, and let $c_1 > c_2$ be the heights of the two highest critical points, the highest being the quadratic singularity. Suppose, in addition, that Q has no critical points at infinity in direction $\hat{\mathbf{r}}$ at any height in $[c_2, c_1]$. Then for every $\varepsilon > 0$, there is a neighborhood $\widehat{\mathcal{E}}$ of $\hat{\mathbf{r}}$ such that as $\mathbf{r} \to \infty$ with $\mathbf{r}/|\mathbf{r}| \in \widehat{\mathcal{E}}$,*

$$a_{\mathbf{r}} = O(e^{(c_2+\varepsilon)|\mathbf{r}|}).$$

Theorem 8.1 is an almost immediate consequence of Theorem 2.5 and the following result.

Proposition 8.2 ([4, Theorem 2.4 (ii)]). *Let $[a, b]$ be a real interval, and suppose that $\mathcal{V}_*$ has no finite or infinite critical points $\mathbf{z}$ with $h_{\hat{\mathbf{r}}}(\mathbf{z}) \in (a, b]$. Then for any $\varepsilon > 0$, any chain Γ of maximum height at most b can be homotopically deformed into a chain Γ' whose maximum height is at most $a + \varepsilon$.*

Proof of Theorem 8.1. Apply Proposition 8.2 with $a = c_2$ and $b = c_1$, resulting in the chain Γ'. Applying Theorem 2.5 and the homotopy equivalence of Γ and Γ' in $\mathcal{M}$, we have

$$a_{\mathbf{r}} = \int_{\Gamma'} \mathbf{z}^{-\mathbf{r}} \frac{P}{Q^k} \frac{d\mathbf{z}}{\mathbf{z}} + R,$$

where R decreases super-exponentially and, in the polynomial case, is in fact zero for all but finitely many $\mathbf{r}$. The height condition on Γ' implies that this integral is bounded above by the volume of Γ', multiplied by the maximum value of $|F|$ on Γ', multiplied by $e^{(c_2+\varepsilon)|\mathbf{r}|}$. ∎

In the remainder of this section, as in Example 2.6, we let

$$F(\mathbf{z}) := \frac{1}{1 - z_1 - z_2 - z_3 - z_4 + 27z_1z_2z_3z_4}.$$

Fix $\hat{\mathbf{r}}$ to be the diagonal direction. We will prove Theorem 2.8 by first computing an estimate for $a_{\mathbf{r}}$ up to an unknown integer factor $\mathfrak{m}$. We then use the theory of D-finite functions and rigorous numerical bounds to find the value of $\mathfrak{m}$. Lastly, we indicate how the value of $\mathfrak{m}$ could possibly be determined by topological methods. In order to discuss the sets $\mathcal{V}(\varepsilon)$ relative to different critical heights, we extend the notation in (2.1) via

$$\mathcal{V}_{\leq t} := \mathcal{V} \cap \{\mathbf{z} : h_{\hat{\mathbf{r}}}(\mathbf{z}) < t\}.$$

Proposition 8.3 ([4, Proposition 2.9]). *Let $F = P/Q$, $\mathcal{V}$, the component B and the coefficients $\{a_{\mathbf{r}}\}$ be as in Theorem 2.5. Fix $\hat{\mathbf{r}}$ and suppose the critical values are $c_1 > c_2 > \cdots > c_m$ with c_1 being the height of the quadric singularity $\mathbf{z}_*$. Suppose there are no critical points at infinity of finite height. Then there is a decomposition $\mathcal{C} = \sum_{j=1}^{m} \circ\gamma_i$ in $H_d(\mathcal{V}_*)$ such that for each j, $\gamma_j \in \mathcal{V}_{\leq c_j}$ and is either zero in $H_{d-1}(\mathcal{V}_{\leq c_j})$ or projects to a nonzero element of $H_{d-1}(\mathcal{V}_{\leq c_j}, H_{d-1}(\mathcal{V}_{c_j-\varepsilon}))$. The decomposition is not unique, but the least j for which $\gamma_j \neq 0$ and the projection $\pi\gamma_j$ to $H_{d-1}(\mathcal{V}_{\leq c_j}, H_{d-1}(\mathcal{V}_{c_j-\varepsilon}))$ is well defined.*

These cycles represent classes in integer homology, thus giving a representation of $a_{\mathbf{r}}$ as integer combinations of integrals over homology generators of the respective relative homology groups. Such integrals are generally computable via saddle-point integration. However, determining the integer coefficients appearing in this representation can be extremely difficult, related to the so-called connection problem for solutions of differential equations. Solving the system $Q = 0$ and $\nabla Q = \lambda \nabla h_{\hat{\mathbf{r}}}$, where λ is an additional parameter, gives the set of critical points.

Proposition 8.4. *The critical points of $\mathcal{V}$ are precisely the points $\mathbf{z}_* := (1/3, 1/3, 1/3, 1/3)$, $\mathbf{w} := (\zeta, \zeta, \zeta, \zeta)$ and $\mathbf{w}' := \overline{\mathbf{w}}$, where $\zeta = (-1 + i\sqrt{2})/3$. There are no critical points at infinity. The point $\mathbf{z}_*$ is a quadratic singularity.*

Let $c_1 = h_{\hat{\mathbf{r}}}(\mathbf{z}_*) = \log 81$ and $c_2 = h_{\hat{\mathbf{r}}}(\mathbf{w}) = h_{\hat{\mathbf{r}}}(\mathbf{w}') = \log 9$. Generators for the rank 2 homology group $H_3(\mathcal{V}_{\leq c_2}, \mathcal{V}_{\leq c_2 - \varepsilon})$ are given by the unstable manifold for downward gradient flows at $\mathbf{w}$ and $\mathbf{w}'$, respectively; denote these chains by γ and γ'. The conclusion of Theorem 2.5 in this case is that

$$a_{\mathbf{r}} = \int_{\Gamma} \mathbf{z}^{-\mathbf{r}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}},$$

where $\Gamma \in \mathcal{V}_{<c_1}$. By Proposition 8.3,

$$\Gamma = \mathfrak{m} \circ \gamma + \mathfrak{m}' \circ \gamma'$$

in $H_3(\mathcal{V}_*)$ for some integers $\mathfrak{m}$ and $\mathfrak{m}'$, which must be equal because the coefficients are real. Explicit formulae in [32, Section 9.5] evaluate $\int_\gamma \mathbf{z}^{-\mathbf{r}} F d\mathbf{z}/\mathbf{z}$, which, after adding the complex conjugate, leads to the result in Theorem 2.8 with 3 replaced by $\mathfrak{m}$. To prove Theorem 2.8, it remains to determine the integer $\mathfrak{m}$.

***D*-finite asymptotics and connection coefficients.** A univariate complex function $f(z)$ is called *D-finite* if it satisfies a linear differential equation with polynomial coefficients,

$$p_r(z) f^{(r)}(z) + p_{r-1}(z) f^{(r-1)}(z) + \cdots + p_0(z) f(z) = 0,$$

where $p_r(z) \not\equiv 0$. We call such a linear differential equation with polynomial coefficients a *D-finite* equation. Our approach to determining $\mathfrak{m}$ relies on the fact that the diagonal of a rational function is D-finite [7, 23], and that asymptotics of D-finite function power series coefficients can be determined up to constants which can be rigorously approximated to large accuracy. In general, it is not possible to determine these constants exactly without additional information (in fact, there does not even exist a good characterization of what numbers appear as such constants) but knowing asymptotics of $a_{n,n,n,n}$ up to an *integer* allows us to immediately determine the value of $\mathfrak{m}$.

The process of determining an annihilating D-finite equation of the diagonal of a rational function lies in the domain of *creative telescoping*, a well-developed area of computer algebra. In particular, there are popular packages in MAGMA [22] and MATHEMATICA [20] which take a multivariate rational function and return an annihilating D-finite equation. For the running example of this section, the diagonal

$$f(z) = \sum_{n \geq 0} a_{n,n,n,n} z^n$$

satisfies the linear differential equation

$$z^2(81z^2 + 14z + 1) f^{(3)}(z) + 3z(162z^2 + 21z + 1) f^{(2)}(z) + (21z + 1)(27z + 1) f'(z) + 3(27z + 1) f(z) = 0. \quad (8.1)$$

The following standard results on the analysis of D-finite functions can be found in Flajolet and Sedgewick [10, Section VII.9].

- The solutions of a D-finite equation form a $\mathbb{C}$-vector space, here equal to dimension three.
- A solution of (8.1) can only have a singularity when the leading polynomial coefficient $z^2(81z^2 + 14z + 1)$ vanishes. Here the roots are 0, ζ^4, and its algebraic conjugate $\overline{\zeta}^4$, where ζ is the complex number appearing in the coordinates of the critical point c_2.
- Equation (8.1) is a *Fuchsian* differential equation, meaning its solutions have only regular singular points, and its indicial equation has rational roots. Because of this, at any point $\omega \in \mathbb{C}$, including potentially singularities, any solution of (8.1) has an expansion of the form

$$\left(1-\frac{z}{\omega}\right)^{\alpha} \sum_{j=0}^{d} \left(g_j\left(1-\frac{z}{\omega}\right)\log^j\left(1-\frac{z}{\omega}\right)\right) \tag{8.2}$$

  in a disk centered at ω with a line from ω to the boundary of the disk removed, where α is rational and each g_j is analytic. At any algebraic point $z = \omega$, there are effective algorithms to determine initial terms of the expansion (8.2) for a basis of the vector space of solutions of (8.1).
- If $g(z) = \sum_{n\geq 0} c_n z^n$ is a solution of (8.1) which has no singularity in some disk $|z| < \rho$ except at a point $z = \omega$, and $g(z)$ has an expansion (8.2) in a slit disk near ω (a disk centered at ω minus a ray from the center to account for a branch cut), then asymptotics of c_n are determined by adding asymptotic contributions of the terms in (8.2). In particular, a term of the form $C(1-z/\omega)^{\alpha}\log^r(1-z/\omega)$ with $\alpha \notin \mathbb{N}$ gives an asymptotic contribution of $\omega^{-n}n^{-\alpha-1}\log^r(n)\frac{C}{\Gamma(-\alpha)}$ to c_n. Furthermore, if $g(z)$ has a finite number of singularities in a disk and has an expansion of the form (8.2) at each, then one can simply add the asymptotic contributions coming from each point in the disk to determine asymptotics of c_n.

These results, combined with rigorous algorithms for numerical analytic continuation of D-finite functions, allow us to rigorously determine asymptotics. For our example, the SAGE ORE_ALGEBRA package [18] computes a basis of solutions to equation (8.1) whose expansions at the origin begin with

$$\begin{aligned}
a_1(z) &= \log(z)^2\left(\frac{1}{2} - \frac{3z}{2} + \frac{9z^2}{2} + \cdots\right) + \log(z)(-4z + 18z^2 + \cdots) \\
&\quad + (8z^2 - 48z^3 + \cdots), \\
a_2(z) &= \log(z)(1 - 3z + 9z^2 + \cdots) + (-4z + 18z^2 + \cdots), \\
a_3(z) &= 1 - 3z + 9z^2 + \cdots,
\end{aligned}$$

and a basis of solutions to (8.1) whose expansions at $z = \zeta^4$ begin with

$$b_1(z) = 1 + \Big(\frac{13}{2} + \frac{43\sqrt{2}}{4}i\Big)(z-\zeta^4)^2 + \Big(\frac{8165}{48} + \frac{943\sqrt{2}}{30}i\Big)(z-\zeta^4)^3 + \cdots,$$

$$b_2(z) = \sqrt{z-\zeta^4} + \Big(\frac{13}{3} - \frac{365\sqrt{2}}{96}i\Big)(z-\zeta^4)^{3/2}$$
$$\qquad - \Big(\frac{7071}{1024} - \frac{1041\sqrt{2}}{32}i\Big)(z-\zeta^4)^{5/2} + \cdots,$$

$$b_3(z) = (z-\zeta^4) + \Big(\frac{17}{3} - \frac{31\sqrt{2}}{6}i\Big)(z-\zeta^4)^2 - \Big(\frac{1013}{72} + \frac{1805\sqrt{2}}{36}i\Big)(z-\zeta^4)^3 + \cdots.$$

Since we can compute the power series coefficients of the diagonal generating function $f(z)$ at the origin, we can represent $f(z)$ in the $a_j(z)$ basis. In fact, because $f(z)$ is analytic at the origin, it must be a multiple of $a_3(z)$, and examining constant terms shows that $a_3(z) = f(z)$. Since the coefficients of $f(z)$ grow, it must admit a singularity at $z = \zeta^4$ or $z = \overline{\zeta}^4$ (in fact, we can deduce that it will have a singularity at both because we already know its dominant asymptotic behavior). If we can determine $f(z)$ in terms of the $b_j(z)$ basis, then we will know its expansion in a neighborhood of the origin, and therefore be able to determine asymptotics of its coefficients. Thus, we need to solve a *connection problem*, representing a function given by a basis specified by local information at one point in terms of a basis specified by local information at another point.

To do this, it is sufficient to determine the change of basis matrix converting from the $a_j(z)$ basis into the $b_j(z)$ basis. Using algorithms going back to the Chudnovsky brothers [8, 9] and van der Hoeven [38], and recently improved and implemented by Mezzarobba [25, 26], we can compute this change of basis matrix numerically to any specified precision. The key is to use numeric analytic continuation to evaluate the $a_j(z)$ and $b_j(z)$ to sufficiently high precision near a fixed value of z. Such evaluations can be done using the series expansions around each point (which can be computed efficiently) and rigorous bounds on the error of series truncation [27].

Proof of Theorem 2.8. In particular, computing the change of basis matrix in this example using the SAGE implementation of Mezzarobba gives

$$f(z) = a_3(z) = C_1 b_1(z) + C_2 b_2(z) + C_3 b_3(z),$$

where C_1, C_2, and C_3 are constants which can be rigorously computed to 1000 decimal places in under 10 seconds on a modern laptop. As $b_2(z)$ is the only element of the $b_j(z)$ basis which is singular at $z = c_2$, the dominant singular term in the expansion of $f(z)$ near $z = c_2$ is

$$C_2\sqrt{z-\zeta^4} = -(3.5933098558743233\ldots$$
$$\qquad + i\,0.38132214909311386\ldots)\sqrt{z-\zeta^4}.$$

Thus, $f(z)$ has a singularity at $z = \zeta^4$, and the asymptotic contribution of this singularity to $a_{n,n,n,n}$ is

$$\Psi_1(n) := \frac{(4i\sqrt{2}-7)^n}{n^{3/2}} \frac{0.543449606382202\ldots + i0.259547320313100\ldots}{\sqrt{\pi}} + O(9^n n^{-5/2}).$$

Repeating the same analysis at the point $z = \overline{\zeta}^4$ gives an asymptotic contribution

$$\Psi_2(n) := \frac{(4i\sqrt{2}+7)^n}{n^{3/2}} \frac{0.543449606382202\ldots - i0.259547320313100\ldots}{\sqrt{\pi}} + O(9^n n^{-5/2}),$$

so that $a_{n,n,n,n}$ has the asymptotic expansion $a_{n,n,n,n} = \Psi_1(n) + \Psi_2(n)$.

Comparing this expansion (with numerical coefficients known to 1000 decimal places) with the expansion in (2.4) which has constants that are unknown but restricted to be integers, proves that $\lambda_2 = \lambda_3$ are integers equal to $2.99\ldots$ up to almost 1000 decimal places (almost 1000 decimal places more than needed to make this conclusion), meaning $\mathfrak{m} = 3$. This finishes the proof of Theorem 2.8. ■

Example 8.5. We can repeat this analysis for the KZ function

$$F_{\mathrm{KZ}}(\mathbf{z}) = \frac{1}{1 - e_1(\mathbf{z}) + 2e_3(\mathbf{z}) + 4e_4(\mathbf{z})} = \frac{1}{1-(x+y+z+w)+2(xyz+xyw+xzw+yzw)+4xyzw}.$$

The diagonal of F_{KZ} satisfies the D-finite equation

$$-t^2(16t^2-24t+1)f'''(t) - 3t(32t^2-36t+1)f''(t) - (112t^2-80t+1)f'(t) + 4(1-4t)f(t) = 0$$

with a basis of solutions at the origin consisting of a solution with power series expansion

$$1 + 4t + 40t^2 + 544t^3 + \cdots$$

and two solutions with logarithmic singularities, and a basis of solutions at the singularity $\rho = 3/4 - 1/\sqrt{2}$ determining dominant asymptotics of the diagonal sequence consisting of two analytic solutions at $z = \rho$ and a solution whose expansion at $z = \rho$ is of the form

$$\sqrt{t-\rho} + O((t-\rho)^{3/2}).$$

Numerically computing a change of basis matrix gives an expansion

$$(4.4125\ldots)\sqrt{t-\rho} + O((t-\rho)^{3/2})$$

for the diagonal of F_{KZ}, leading to an asymptotic expansion

$$(0.2577973\ldots)\rho^{-n}n^{-3/2} + O(n^{-5/2})$$

for the diagonal coefficients whose leading coefficient can be computed to any accuracy. Matching this up with the asymptotic contributions of the smooth critical points shows each contributes with a "multiplicity" of one.

9. Concluding remarks

Explaining the multiplicity

We have seen that the integral over $\mathcal{C}(c_2)$ and $\mathcal{C}(c_3)$ appear in the Cauchy integral representation of $a_{n,n,n,n}$ with a multiplicity of 3. Expanding a torus past a smooth critical point leads to a coefficient of 1 when the critical point is a height maximum along the imaginary fiber and zero when it is a height minimum along this fiber. Evidently, when deforming the Cauchy domain of integration past the highest critical point $c_1 = (1/3, 1/3, 1/3, 1/3)$, the resulting chain Γ lying just below this height is not like a simple torus and instead, under gradient flow, has multiplicity 3 in the local homology basis at the diagonal points ζ and $\overline{\zeta}$.

Problem 9.1. Give a direct demonstration of these coefficients being 3.

Our best explanation at present is this. If W is a smooth algebraic hypersurface, Morse theory gives us a basis for $H_{d-1}(W)$ consisting of the unstable manifolds for downward gradient flow at each critical point. The stable manifolds at each critical point are an upper triangular dual to this via the intersection pairing. The original torus of integration is a tube over a torus T_0 in $\mathcal{V}$. If $\mathcal{V}$ were smooth, we would be trying to show that the stable manifold at $\mathbf{w}$ in $\mathcal{V}_*$ intersects T_0 with signed multiplicity ± 3, where $\mathbf{w} = (\zeta, \zeta, \zeta, \zeta)$. This is probably not true in the smooth varieties $\mathcal{V}_c$. However, as $c \to 0$, part of the stable manifold at $\mathbf{w}$ gets drawn toward $\mathbf{z}_* = (1/3, 1/3, 1/3, 1/3)$. Therefore, in the limit, we need to check how many total signed paths in the gradient field ascend from $\mathbf{w}$ to $\mathbf{z}_*$.

By the symmetry, we expect to find these paths along the three partial diagonals: $\{x = y,\ z = w\}$, $\{x = z,\ y = w\}$ and $\{x = w,\ y = z\}$. Solving for gradient ascents on any one of these yields three that go to $\mathbf{z}$ rather than to the coordinate planes. If these all had the same sign, the multiplicity would be 9 rather than 3, therefore, in any one partial diagonal, the three paths are two of one sign and one of the other. It remains to show that the signs are as predicted, that these are the only paths going from $\mathbf{w}$ to $\mathbf{z}_*$, and to rigorize passage from the smooth case to the limit as $c \to 0$.

Computational Morse theory

One of the central problems in ACSV is effective computation of coefficients in integer homology. Specifically, the class $[T] \in H_d(\mathcal{M})$ must be resolved as an integer combination of classes $\circ\sigma$, where $\sigma \in H_{d-1}(\mathcal{V}_*)$ projects to a homology generator for one of the attachment pairs $H_{d-1}(\mathcal{V}_{\leq c}, \mathcal{V}_{\leq c-\varepsilon})$ near a critical point with critical value c. What is known is nonconstructive. There is a highest critical value c where $[T]$ has nonzero homology in the attachment pair. The projection of $[T]$ to $H_{d-1}(\mathcal{V}_{\leq c}, \mathcal{V}_{\leq c-\varepsilon})$ is well defined. If this relative homology element is the projection of an absolute homology element $\sigma \in H_{d-1}(\mathcal{V}_{\leq c} \setminus \mathcal{V}_{\leq c-\varepsilon})$, then there is no *Stokes phenomenon*, meaning one can replace $[T]$ by $[T] - \sigma$ and continue to the next lower attachment pair where $[T] - \sigma$ projects to a nonzero homology element.

The data for this problem is algebraic. Therefore, one might hope for an algebraic solution, which can be found via computer algebra without resorting to numerical methods, rigorous or otherwise. At present, however, we have only heuristic geometric arguments.

Problem 9.2. Given an integer polynomial and rational $\hat{\mathbf{r}}$, algebraically compute the highest critical points $\mathbf{z}$ for which the projection of $[T]$ to the attachment pair is nonzero. Then compute these integer coefficients. Also determine whether T is homologous to a local cycle, and in the case that it is, find a way to continue the computation to the next lower critical point.

Combining computation and topology

One of the main achievements of the present paper is the preceding chain of reasoning that combines topological methods with computer algebra. Computer algebra methods give asymptotic formulae for the diagonal coefficients which includes an unknown constant, computable up to an arbitrarily small (rigorous) error term. These methods say nothing about the behavior of coefficients in a neighborhood of the diagonal. Topological methods show that in a neighborhood of the diagonal, coefficients are given by an asymptotic formula which is the sum of algebraic quantities up to unknown integer factors. This method on its own cannot identify the correct asymptotics without further geometric methods that have, thus far, eluded us. Combining the two analyses determines the integer factors, leading to rigorous asymptotics throughout an open cone containing the diagonal direction.

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References

[1] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals*. Monogr. Math. 83, Birkhäuser, Boston, MA, 1988 Zbl 1297.32001 MR 966191

[2] R. Askey and G. Gasper, Convolution structures for Laguerre polynomials. *J. Anal. Math.* **31** (1977), 48–68 Zbl 0347.33006 MR 486692

[3] M. F. Atiyah, R. Bott, and L. Gårding, Lacunas for hyperbolic differential operators with constant coefficients I. *Acta Math.* **124** (1970), 109–189 Zbl 0191.11203 MR 470499

[4] Y. Baryshnikov, S. Melczer, and R. Pemantle, Stationary points at infinity for analytic combinatorics. *Found. Comput. Math.* **22** (2022), no. 5, 1631–1664 Zbl 1500.05008 MR 4498442

[5] Y. Baryshnikov, S. Melczer, R. Pemantle, and A. Straub, Diagonal asymptotics for symmetric rational functions via ACSV. In *29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*, LIPIcs. Leibniz Int. Proc. Inform. 110, Schloss Dagstuhl Leibniz-Zentrum für Informatik, Wadern, article no. 12, 15 pp., 2018 Zbl 1482.05011 MR 3826131

[6] Y. Baryshnikov and R. Pemantle, Asymptotics of multivariate sequences, part III: Quadratic points. *Adv. Math.* **228** (2011), no. 6, 3127–3206 Zbl 1252.05012 MR 2844940

[7] G. Christol, Diagonales de fractions rationnelles et equations différentielles. In *Study group on ultrametric analysis, 10th year: 1982/83, No. 2*, p. article no. 18, Institut Henri Poincaré, Paris, 1984 MR 747499

[8] D. V. Chudnovsky and G. V. Chudnovsky, On expansion of algebraic functions in power and Puiseux series, I. *J. Complexity* **2** (1986), no. 4, 271–294 Zbl 0629.68038 MR 923022

[9] D. V. Chudnovsky and G. V. Chudnovsky, On expansion of algebraic functions in power and Puiseux series, II. *J. Complexity* **3** (1987), no. 1, 1–25 Zbl 0656.34003 MR 883165

[10] P. Flajolet and R. Sedgewick, *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009 Zbl 1165.05001

[11] M. Forsberg, M. Passare, and A. Tsikh, Laurent determinants and arrangements of hyperplane amoebas. *Adv. Math.* **151** (2000), no. 1, 45–70 Zbl 1002.32018 MR 1752241

[12] K. Friedrichs and H. Lewy, Das Anfangswertproblem einer beliebigen nichtlinearen hyperbolischen Differentialgleichung beliebiger Ordnung in zwei Variablen. Existenz, Eindeutigkeit und Abhängigkeitsbereich der Lösung. *Math. Ann.* **99** (1928), no. 1, 200–221 Zbl 54.0520.01 MR 1512448

[13] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*. Math. Theory Appl., Birkhäuser, Boston, MA, 1994 Zbl 0827.14036 MR 1264417

[14] J. Gillis, B. Reznick, and D. Zeilberger, On elementary methods in positivity theory. *SIAM J. Math. Anal.* **14** (1983), no. 2, 396–398 Zbl 0599.42500 MR 688584

[15] M. Goresky and R. MacPherson, *Stratified Morse theory*. Ergeb. Math. Grenzgeb. (3) 14, Springer, Berlin, 1988 Zbl 0639.14012 MR 932724

[16] P. Griffiths and J. Harris, *Principles of algebraic geometry*. Pure Appl. Math., Wiley-Interscience, New York, 1978 Zbl 0408.14001 MR 507725

[17] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2002 Zbl 1044.55001 MR 1867354

[18] M. Kauers, M. Jaroschek, and F. Johansson, Ore polynomials in Sage. In *Computer algebra and polynomials*, pp. 105–125, Lecture Notes in Comput. Sci. 8942, Springer, Cham, 2015 Zbl 1439.16049 MR 3335570

[19] M. Kauers and D. Zeilberger, Experiments with a positivity-preserving operator. *Exp. Math.* **17** (2008), no. 3, 341–345 Zbl 1157.05003 MR 2455705

[20] C. Koutschan, HolonomicFunctions (User's guide). 2013, Research Institute for Symbolic Computation, University of Linz, Austria, https://www3.risc.jku.at/publications/download/risc_3934/hf.pdf, visited on 22 January 2024

[21] K. Kozhasov, M. Michałek, and B. Sturmfels, Positivity certificates via integral representations. In *Facets of algebraic geometry. Vol. II*, pp. 84–114, London Math. Soc. Lecture Note Ser. 473, Cambridge University Press, Cambridge, 2022 Zbl 1492.14107 MR 4381912

[22] P. Lairez, Computing periods of rational integrals. *Math. Comp.* **85** (2016), no. 300, 1719–1752 Zbl 1337.68301 MR 3471105

[23] L. Lipshitz, The diagonal of a D-finite power series is D-finite. *J. Algebra* **113** (1988), no. 2, 373–378 Zbl 0657.13024 MR 929767

[24] S. Melczer and M. Mezzarobba, Sequence positivity through numeric analytic continuation: uniqueness of the Canham model for biomembranes. *Comb. Theory* **2** (2022), no. 2, article no. 4 Zbl 1498.05027 MR 4449812

[25] M. Mezzarobba, Rigorous multiple-precision evaluation of D-finite functions in SageMath. 2016, arXiv:1607.01967

[26] M. Mezzarobba, Truncation bounds for differentially finite series. *Ann. H. Lebesgue* **2** (2019), 99–148 Zbl 1435.65106 MR 3974489

[27] M. Mezzarobba and B. Salvy, Effective bounds for P-recursive sequences. *J. Symbolic Comput.* **45** (2010), no. 10, 1075–1096 Zbl 1201.65219 MR 2679389

[28] J. Milnor, *Singular points of complex hypersurfaces*. Ann. of Math. Stud. 61, Princeton University Press, Princeton, NJ, 1968 Zbl 0184.48405 MR 239612

[29] J. W. Milnor and J. D. Stasheff, *Characteristic classes*. Ann. of Math. Stud. 76, Princeton University Press, Princeton, NJ, 1974 Zbl 0298.57008 MR 440554

[30] R. Pemantle and M. C. Wilson, Asymptotics of multivariate sequences. I: Smooth points of the singular variety. *J. Combin. Theory Ser. A* **97** (2002), no. 1, 129–161 Zbl 1005.05007 MR 1879131

[31] R. Pemantle and M. C. Wilson, Asymptotics of multivariate sequences. II: Multiple points of the singular variety. *Combin. Probab. Comput.* **13** (2004), no. 4–5, 735–761 Zbl 1065.05010 MR 2095981

[32] R. Pemantle and M. C. Wilson, *Analytic combinatorics in several variables*. Cambridge Stud. Adv. Math 140, Cambridge University Press, Cambridge, 2013 Zbl 1297.05004 MR 3088495

[33] R. Pemantle, M. C. Wilson, and S. Melczer, *Analytic combinatorics in several variables*. 2nd edn., Cambridge Stud. Adv. Math. 212, Cambridge University Press, New York, 2024

[34] F. Pham, *Singularities of integrals. Homology, hyperfunctions and microlocal analysis*. Universitext, Springer, London, 2011 Zbl 1223.32001 MR 2798679

[35] V. Pillwein, On the positivity of the Gillis–Reznick–Zeilberger rational function. *Adv. in Appl. Math.* **104** (2019), 75–84 Zbl 1408.33041 MR 3883174

[36] A. D. Scott and A. D. Sokal, Complete monotonicity for inverse powers of some combinatorially defined polynomials. *Acta Math.* **213** (2014), no. 2, 323–392 Zbl 1304.05074 MR 3286037

[37] G. Szegő, Über gewisse Potenzreihen mit lauter positiven Koeffizienten. *Math. Z.* **37** (1933), no. 1, 674–688 Zbl 0007.34401 MR 1545428

[38] J. van der Hoeven, Fast evaluation of holonomic functions near and in regular singularities. *J. Symbolic Comput.* **31** (2001), no. 6, 717–743 Zbl 0982.65024 MR 1834006

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