Exact Calabi–Yau categories and odd-dimensional Lagrangian spheres

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Abstract. An exact Calabi–Yau structure, originally introduced by Keller, is a special kind of smooth Calabi–Yau structure in the sense of Kontsevich–Vlassopoulos (2021). For a Weinstein manifold M, the existence of an exact Calabi–Yau structure on the wrapped Fukaya category $\mathcal{W}(M)$ imposes strong restrictions on its symplectic topology. Under the cyclic open-closed map constructed by Ganatra (2019), an exact Calabi–Yau structure on $\mathcal{W}(M)$ induces a class $\tilde{b}$ in the degree one equivariant symplectic cohomology $\mathrm{SH}^1_{S^1}(M)$. Any Weinstein manifold admitting a quasi-dilation in the sense of Seidel–Solomon [Geom. Funct. Anal. 22 (2012), 443–477] has an exact Calabi–Yau structure on $\mathcal{W}(M)$. We prove that there are many Weinstein manifolds whose wrapped Fukaya categories are exact Calabi–Yau despite the fact that there is no quasi-dilation in $\mathrm{SH}^1(M)$; a typical example is given by the affine hypersurface $\{x^3 + y^3 + z^3 + w^3 = 1\} \subset \mathbb{C}^4$. As an application, we prove the homological essentiality of Lagrangian spheres in many odd-dimensional smooth affine varieties with exact Calabi–Yau wrapped Fukaya categories.

1. Introduction

Given a closed Lagrangian submanifold L in some symplectic manifold M, one of the most important questions in symplectic topology is to determine whether it represents a non-trivial homology class $[L] \in H_n(M; \mathbb{Z})$. In particular, for Weinstein manifolds, we have the following conjecture (see, for example, [24, Section 5]).

Conjecture 1 (Folklore). *Let M be any Weinstein manifold and $L \subset M$ a closed, oriented, exact Lagrangian submanifold with vanishing Maslov class. Then, its homology class $[L] \in H_n(M; \mathbb{Z})$ is primitive.*

It is, however, already not obvious to see whether the homology class of an odd-dimensional Lagrangian sphere L is necessarily non-trivial, since the topological intersection number $[L] \cdot [L]$ vanishes.

This is one of the motivations for Seidel and Solomon to introduce in [67] a refined version of the usual intersection number, called the *q-intersection number*, between

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exact Lagrangian submanifolds which are unobstructed (e.g., when they are simply connected). More precisely, Seidel–Solomon's theory relies on the existence of a distinguished cohomology class $b \in \mathrm{SH}^1(M)$ in the first-degree symplectic cohomology, which satisfies

$$\Delta(b) = h \in \mathrm{SH}^0(M)^\times \tag{1.1}$$

under the Batalin–Vilkovisky (BV) operator

$$\Delta : \mathrm{SH}^*(M) \to \mathrm{SH}^{*-1}(M),$$

where $\mathrm{SH}^0(M)^\times$ is the set of invertible elements in $\mathrm{SH}^0(M)$. The class b will be called a *quasi-dilation* (it is defined in Lecture 19 of [56] as a class satisfying

$$\Delta(hb) = h,$$

which differs slightly from the convention used here), and in the special case when $h = 1$, it is called a *dilation*. Taking an algebraic viewpoint, a quasi-dilation b can be regarded as a noncommutative vector field (i.e., a degree one Hochschild cocycle) over the Fukaya category $\mathcal{F}(M)$ of compact Lagrangians, so one can deform the objects of $\mathcal{F}(M)$ along b. For b-equivariant objects $L_0, L_1 \subset M$, one then gets an infinitesimal $\mathbb{C}^*$-action on their endomorphism space $\mathrm{CF}^*(L_0, L_1)$, which induces a derivation $\Phi_{\tilde{L}_0, \tilde{L}_1}$ on the Floer cohomology algebra $\mathrm{HF}^*(L_0, L_1)$, whose non-trivialness is ensured by condition (1.1). The q-intersection number between L_0 and L_1 is defined by considering the generalized eigenspaces of $\Phi_{\tilde{L}_0, \tilde{L}_1}$:

$$L_0 \bullet_q L_1 := \mathrm{Str}\big(e^{\log(q)\Phi_{\tilde{L}_0, \tilde{L}_1}}\big), \tag{1.2}$$

where Str is the supertrace.

Note that the q-intersection number, as opposed to the usual topological intersection number, detects odd-dimensional Lagrangian homology spheres as

$$L \bullet_q L = 1 - q.$$

This, together with some other algebraic properties, enabled Seidel to prove that a (finite-type complete) Liouville manifold M cannot contain infinitely many disjoint Lagrangian spheres if it admits a dilation [63, Theorem 1.4]. Moreover, in the cases where we have a good understanding of the geometry of the dilation, it is possible to make the bound on the number of pairwise disjoint Lagrangian spheres explicit by relating it to the ordinary topology of M; in this way, the non-triviality of the homology classes of Lagrangian spheres in certain Milnor fibers is proved, which provides evidences for Conjecture 1. However, as is already observed in [63], the dilation condition $\Delta(b) = 1$ imposes very strong restrictions on the Liouville manifold M. For

example, the only known examples of simply connected 3-dimensional smooth affine varieties admitting dilations are Milnor fibers associated to A_m singularities.

It is the purpose of this paper to generalize Seidel's results on the homological essentiality of odd-dimensional Lagrangian spheres to a more general class of Weinstein manifolds M. More precisely, the condition we need is the existence of a cohomology class $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ in the degree one S^1-equivariant symplectic cohomology, satisfying

$$\mathbf{B}(\tilde{b}) = h \in \mathrm{SH}^0(M)^\times, \tag{1.3}$$

where the map

$$\mathbf{B} : \mathrm{SH}^*_{S^1}(M) \to \mathrm{SH}^{*-1}(M)$$

is the connecting map in Gysin's long exact sequence (2.2) relating the ordinary and equivariant symplectic cohomologies; see Section 2.3. The class $\tilde{b}$ will be called a *cyclic dilation*. Our main result proves that if the geometry of the class $\tilde{b}$ is sufficiently simple, which means that it appears as a class in the S^1-equivariant Floer cohomology group $\mathrm{HF}^1_{S^1}(\lambda)$ of a Hamiltonian with relatively small slope $\lambda > 0$ at infinity, then the homology class of any Lagrangian sphere in M is non-trivial; see Theorem 13 below for the precise statement. Conjecturally, the class of Weinstein manifolds admitting cyclic dilations contains all the smooth affine varieties with log Kodaira dimension $-\infty$ as a subclass. Some examples and evidences are provided in Sections 2.4 and 6.4.

The notion of a cyclic dilation has an algebraic counterpart, which is known as an *exact Calabi–Yau structure* on a homologically smooth A_∞-category (cf. Definition 2). To understand when does the wrapped Fukaya category $\mathcal{W}(M)$ of a Liouville manifold M admit the structure of an exact Calabi–Yau category is actually the original motivation of this work. As will be explained in Section 4.3, an exact Calabi–Yau structure on $\mathcal{W}(M)$ is related to a cyclic dilation $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ via the cyclic open-closed string map defined by Ganatra [28]. This algebraic interpretation enables us to implement the Koszul duality between A_∞-algebras to deduce the existence of cyclic dilations for many interesting examples of Liouville manifolds, including those without dilations or quasi-dilations; see Section 2.4.

The paper is organized as follows. Section 2 is essentially an overview of the contents of this paper, where the motivations for considering the cyclic dilation condition (1.3) are discussed and many of our results are summarized. Section 3 is a brief sketch of some basic algebraic notions and facts which already exist in the literature, and they are included purely for self-containedness. Section 4 contains our main geometric inputs, where various moduli spaces arising from the parametrized Floer theory are considered. In Section 5, we then apply the Floer theoretical techniques collected in Section 4 to study the Lagrangian submanifolds in Liouville manifolds with cyclic dilations. We generalize Seidel–Solomon's construction of q-intersection numbers in the case of a single Lagrangian sphere, and Theorem 13 is proved there.

Finally, in Section 6, we apply A_∞-Koszul duality and Lefschetz fibration techniques to produce examples of Liouville manifolds which carry cyclic dilations.

2. Background and results

For simplicity, we will work throughout this paper with a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$, whose algebraic closure is denoted by $\bar{\mathbb{K}}$. All the dg or A_∞-categories in this paper will be defined over $\mathbb{K}$, so do the corresponding homotopy or homology theories. When the categories are split-generated by finitely many objects in the sense of [61], it is convenient to use an equivalent language, namely, dg or A_∞-algebras over the semisimple ring

$$\mathbb{k} := \bigoplus_{i \in I} \mathbb{K}e_i,$$

where I is a finite set and $\{e_i\}_{i \in I}$ is a set of idempotents indexed by I. In this way, we are not going to distinguish below between an A_∞-category $\mathcal{A}$ split-generated by finitely many objects $\{S_i\}_{i \in I}$ and its endomorphism algebra of the object $\bigoplus_{i \in I} S_i$ in the formal enlargement $\mathcal{A}^{\mathrm{tw}}$, which is an A_∞-algebra over $\mathbb{k}$.

All the dg or A_∞-algebras in this paper will be $\mathbb{Z}$-graded. The Hochschild chain complex of an A_∞-algebra $\mathcal{A}$ will be denoted by $\mathrm{CH}_*(\mathcal{A})$, and we use $\mathrm{HH}_*(\mathcal{A})$ to denote its homology. The more familiar notation $\mathrm{CC}_*(\mathcal{A})$ will be reserved for the cyclic chain complex $\mathrm{CH}_*(\mathcal{A}) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!]$, which computes the (positive) cyclic homology $\mathrm{HC}_*(\mathcal{A})$. The definitions of these complexes will be briefly recalled in Section 3.2.

2.1. Exact Calabi–Yau structures

Let $\mathcal{A}$ be a homologically smooth A_∞-algebra over some semisimple ring $\mathbb{k}$. It can be regarded as a bimodule over itself, which is known as the diagonal bimodule, and by slight abuse of notation, we will still denote it by $\mathcal{A}$. By our assumption, $\mathcal{A}$ is a perfect bimodule. Its dual bimodule, $\mathcal{A}^\vee$, is defined as

$$\mathcal{A}^\vee := R\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e),$$

where $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. $\mathcal{A}$ is a *weak smooth n-Calabi–Yau algebra* if there exists a non-degenerate Hochschild cycle $\eta \in \mathrm{CH}_{-n}(\mathcal{A})$, i.e., a cocycle which induces an isomorphism

$$\mathcal{A}^\vee[n] \cong \mathcal{A}$$

between $\mathcal{A}$-bimodules. Recall that $\mathrm{CH}_*(\mathcal{A}) \cong \mathcal{A} \otimes_{\mathcal{A}^e}^L \mathcal{A} \cong R\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)$. The reader may refer to [74], Section 8 for basic definitions and properties related to Calabi–Yau algebras.

Associated to $\mathcal{A}$ there is a long exact sequence [51, Theorem 2.2.1]

$$\cdots \to \mathrm{HC}_{-*-1}(\mathcal{A}) \xrightarrow{S} \mathrm{HC}_{-*+1}(\mathcal{A}) \xrightarrow{B} \mathrm{HH}_{-*}(\mathcal{A}) \xrightarrow{I} \mathrm{HC}_{-*}(\mathcal{A}) \to \cdots, \quad (2.1)$$

relating Hochschild and cyclic homologies of $\mathcal{A}$, which is known as *Connes's long exact sequence* [51]. The following definition is our main subject of study in this paper.

Definition 2 ([20, Definition 2.3.6]). A weak smooth n-Calabi–Yau structure on $\mathcal{A}$ is said to be exact if the Hochschild homology class $[\eta]$ lies in the image of Connes's map $B : \mathrm{HC}_{-n+1}(\mathcal{A}) \to \mathrm{HH}_{-n}(\mathcal{A})$.

Notice that the notion of an exact Calabi–Yau structure is strictly more restrictive than a *smooth Calabi–Yau structure* in the sense of Kontsevich–Vlassopoulos [18,45], which is defined as a negative cyclic cycle $\widetilde{\eta} \in \mathrm{CC}_{-n}^{-}(\mathcal{A})$ whose induced Hochschild cycle in $\mathrm{CH}_{-n}(\mathcal{A})$ under the inclusion map of homotopy fixed points $\iota : \mathrm{CC}_{*}^{-}(\mathcal{A}) \to \mathrm{CH}_{*}(\mathcal{A})$ defines a weak smooth n-Calabi–Yau structure on $\mathcal{A}$. This can be easily seen from the following commutative diagram [51, Proposition 5.1.5]:

$$\begin{array}{ccc} \mathrm{HC}_{-n+1}(\mathcal{A}) & \longrightarrow & \mathrm{HC}_{-n}^{-}(\mathcal{A}) \\ & \searrow{\scriptstyle B} & \downarrow{\scriptstyle [\iota]} \\ & & \mathrm{HH}_{-n}(\mathcal{A}) \end{array}$$

Just as a Calabi–Yau structure $[\eta] \in \mathrm{HH}_{-n}(\mathcal{A})$ is the noncommutative analog of a holomorphic volume form Ω, the existence of a lift $[\widetilde{\eta}]$ in $\mathrm{HC}_{-n}^{-}(\mathcal{A})$ corresponds to the (trivial) fact that Ω is necessarily closed. Since Connes's differential B is the noncommutative analog of the de Rham differential, the exact Calabi–Yau condition imposed on $\mathcal{A}$ is analogous to the exactness of Ω as a differential form. This explains the terminology.

We remark that an important class of examples of exact Calabi–Yau A_∞-algebras is the so-called *superpotential algebras* introduced by Ginzburg [33], which is roughly a dg algebra whose underlying associative algebra is modeled on some localization of the path algebra and whose differential is specified by a superpotential lying in the commutator quotient $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$; see Section 3.1 for details. As a special case, we have the Ginzburg dg algebra $\mathcal{G}(Q, w)$ associated to a quiver with potential (Q, w); see [33].

2.2. Wrapped Fukaya categories

Let M be a $2n$-dimensional Liouville manifold with $c_1(M) = 0$, which is obtained by completing a Liouville domain $\overline{M}$ with the cylindrical end $[1, \infty) \times \partial \overline{M}$. Associated to M is a $\mathbb{Z}$-graded A_∞-category $\mathcal{W}(M)$, well defined up to quasi-isomorphism,

known as the *wrapped Fukaya category* [6]. The objects of $\mathcal{W}(M)$ are closed, exact, oriented, Spin Lagrangian submanifolds with vanishing Maslov class, together with certain non-compact exact Lagrangian submanifolds which are modeled at infinity as cones over Legendrian submanifolds in the contact boundary $\partial \bar{M}$.

In the case when M is Weinstein, for any handlebody decomposition of M, there is a set of distinguished objects in $\mathcal{W}(M)$, namely, the Lagrangian cocores $L_1, \ldots, L_k$ of the n-handles. It is proved in [13, 29] that $\mathcal{W}(M)$ is generated by these cocores. In particular, denote by $\mathrm{CW}^*(L_i, L_j)$ the wrapped Floer cochain complex of two cocores $L_i, L_j \subset M$, we have an equivalence

$$D^{\mathrm{perf}}(\mathcal{W}(M)) \cong D^{\mathrm{perf}}(\mathcal{W}_M)$$

between the derived wrapped Fukaya category and the derived category of perfect modules over the wrapped Fukaya A_∞-algebra

$$\mathcal{W}_M := \bigoplus_{1 \le i, j \le k} \mathrm{CW}^*(L_i, L_j),$$

which can be regarded as an A_∞-algebra over the semisimple ring $\Bbbk = \bigoplus_{1 \le i \le k} \mathbb{K} e_i$.

Combining the generation result in [13, 29] with [28, Theorem 3] implies that the wrapped Fukaya category $\mathcal{W}(M)$ of any Weinstein manifold M carries a smooth Calabi–Yau structure. Also, in favorable situations, it is possible to upgrade the smooth Calabi–Yau structure on $\mathcal{W}(M)$ to an exact one. For example, the author studied in [49] the wrapped Fukaya A_∞-algebras of the 6-dimensional Milnor fibers associated to the isolated singularities

$$x^p + y^q + z^r + \lambda xyz + w^2 = 0$$

in $\mathbb{C}^4$, where $\lambda \ne 0, 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1$, and identified them with *Calabi–Yau completions* in the sense of Keller [41, 42] of certain directed quiver algebras, which in particular shows that their wrapped Fukaya categories all admit exact Calabi–Yau structures. Earlier results in this direction include the work of Etgü–Lekili [26], which proves the existence of an exact Calabi–Yau structure in the case when M is a 4-dimensional Milnor fiber of type A_m or D_m (char$(\mathbb{K}) \ne 2$ in the latter case), and Ekholm–Lekili [23], which shows the same to be true when M is a plumbing of T^*S^n's according to any tree, where $n \ge 3$. See also [68], where a relation between the wrapped Fukaya category $\mathcal{W}(Y_\phi; b_0)$ of some quasi-projective 3-folds Y_ϕ arising from meromorphic quadratic differentials ϕ, twisted by some non-trivial bulk term $b_0 \in H^2(Y_\phi; \mathbb{Z}/2)$, and the completed Ginzburg algebras arising from ideal triangulations of marked bordered surfaces is conjectured.

However, it is in general not true that the wrapped Fukaya category of any Weinstein manifold carries an exact Calabi–Yau structure. The first set of such counterexamples is found by Davison [20], who studied the fundamental group algebra

$\mathbb{K}[\pi_1(Q)]$ of a $K(\pi, 1)$ space Q, and showed that when Q is a hyperbolic manifold, $\mathbb{K}[\pi_1(Q)]$ is not exact Calabi–Yau. Note that, for a closed manifold which is topologically $K(\pi, 1)$, we have a quasi-isomorphism

$$\mathcal{W}_{T^*Q} := \mathrm{CW}^*(T_q^*Q, T_q^*Q) \cong \mathbb{K}[\pi_1(Q)]$$

between (formal) A_∞-algebras [3].

2.3. Symplectic cohomologies

There is a closed string counterpart of our discussions in Section 2.2. Recall that, for a Liouville manifold M with $c_1(M) = 0$, one can define, using a Hamiltonian function which is quadratic at infinity, the symplectic cohomology $\mathrm{SH}^*(M)$, which carries the structure of a $\mathbb{Z}$-graded unital algebra over $\mathbb{K}$. There is also an S^1-equivariant theory, denoted as $\mathrm{SH}_{S^1}^*(M)$, whose construction will be recalled in Section 4.1. Analogous to Connes's long exact sequence (2.1), $\mathrm{SH}^*(M)$ and $\mathrm{SH}_{S^1}^*(M)$ fit into the following Gysin-type long exact sequence [11, Theorem 1.3]:

$$\cdots \to \mathrm{SH}^{*-1}(M) \xrightarrow{\mathbf{I}} \mathrm{SH}_{S^1}^{*-1}(M) \xrightarrow{\mathbf{S}} \mathrm{SH}_{S^1}^{*+1}(M) \xrightarrow{\mathbf{B}} \mathrm{SH}^*(M) \to \cdots, \qquad (2.2)$$

where the composition $\mathbf{B} \circ \mathbf{I}$ gives the BV operator Δ.

Remark 3. Note that, in the above, we have used the bold letters $\mathbf{I}$, $\mathbf{B}$, and $\mathbf{S}$ to denote the maps corresponding to I, B, and S in Connes's long exact sequence (2.1). This is to emphasize that we are dealing with closed string invariants. As a convention, we will use the notations $\mathbb{I}$, $\mathbb{B}$, and $\mathbb{S}$ for the open string counterparts of the maps $\mathbf{I}$, $\mathbf{B}$, and $\mathbf{S}$. In particular, there is a long exact sequence

$$\cdots \to \mathrm{HC}_{-*-1}(\mathcal{W}(M)) \xrightarrow{\mathbb{S}} \mathrm{HC}_{-*+1}(\mathcal{W}(M)) \xrightarrow{\mathbb{B}} \mathrm{HH}_{-*}(\mathcal{W}(M))$$
$$\xrightarrow{\mathbb{I}} \mathrm{HC}_{-*}(\mathcal{W}(M)) \to \cdots, \qquad (2.3)$$

which is simply (2.1) applied to the wrapped Fukaya category $\mathcal{W}(M)$.

To relate the two long exact sequences (2.2) and (2.3), we implement the *cyclic open-closed string map* constructed by Ganatra [28], which on the cohomology level descends to a map

$$[\widetilde{\mathrm{OC}}] : \mathrm{HC}_*(\mathcal{W}(M)) \to \mathrm{SH}_{S^1}^{*+n}(M),$$

from which one obtains the following geometric interpretation of an exact Calabi–Yau structure on $\mathcal{W}(M)$.

Proposition 4. *Let M be a non-degenerate Liouville manifold; its wrapped Fukaya category $\mathcal{W}(M)$ carries an exact Calabi–Yau structure if and only if the connecting map $\mathbf{B} : \mathrm{SH}_{S^1}^1(M) \to \mathrm{SH}^0(M)$ in (2.2) hits an invertible element $h \in \mathrm{SH}^0(M)^\times$.*

Proposition 4 will be proved in Corollary 37. In the above, the non-degeneracy condition on a Liouville manifold is introduced by Ganatra in [27], which ensures that the open-closed map $[OC] : HH_*(\mathcal{W}(M)) \to SH^{*+n}(M)$ is an isomorphism. A Liouville manifold M is said to be *non-degenerate* if there is a finite collection of Lagrangians $\{L_i\}$ in M such that OC restricted to the full A_∞-subcategory $\mathcal{L}(M) \subset \mathcal{W}(M)$ formed by $\{L_i\}$ hits the identity $1 \in SH^0(M)$. As we have seen in Section 2.2, any Weinstein manifold is non-degenerate since one can take $\mathcal{L}(M)$ to be the full A_∞-subcategory of cocores.

In other words, an exact Calabi–Yau structure on $\mathcal{W}(M)$ of a non-degenerate Liouville manifold M induces a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$ mentioned in the introduction. Note that if M admits a quasi-dilation in the sense of (1.1), then it also admits a cyclic dilation $\tilde{b}$ which arises as the image of b under the map

$$\mathbf{I} : SH^1(M) \to SH^1_{S^1}(M).$$

It is natural to ask whether the converse is true. We postpone the discussions about whether the cyclic dilation condition (1.3) is strictly weaker than the quasi-dilation condition (1.1) to Section 2.4 and look here at an immediate geometric implication by assuming the existence of a cyclic dilation.

Let $L \subset M$ be a closed exact Lagrangian submanifold, equipped with a rank 1 local system ν so that the isomorphism $SH^*(T^*L) \cong H_{n-*}(\mathcal{L}L; \nu)$ holds [4], where $\mathcal{L}L$ denotes the free loop space of L. There is an S^1-equivariant version of Viterbo functoriality, namely, the (S^1-equivariant lift of) the *Cieliebak–Latschev map* constructed by Cohen–Ganatra [18]

$$[\widetilde{CL}] : SH^*_{S^1}(M) \to H^{S^1}_{n-*}(\mathcal{L}L; \nu), \tag{2.4}$$

which is compatible with the Viterbo functoriality and the Gysin sequence; see Section 5.1. Combined with Proposition 4, we can reinterpret Davison's non-existence result mentioned in Section 2.1 in the following slightly more general form.

Proposition 5. *Let M be a Liouville manifold which admits a cyclic dilation; then it does not contain any closed, orientable, exact Lagrangian submanifold $L \subset M$ which is hyperbolic.*

Proposition 5 will be proved in Section 5.1. In particular, when $M = T^*Q$, it follows from Proposition 4 and the formality result mentioned at the end of Section 2.2 that the fundamental group algebra $\mathbb{K}[\pi_1(Q)]$ cannot be exact Calabi–Yau if Q is hyperbolic, which recovers [20, Corollary 6.2.4].

Analogous to what Seidel and Solomon have done in the case of dilations and quasi-dilations [56, 67], one can use Lefschetz fibrations to produce more examples of Liouville manifolds which admit cyclic dilations starting from the known ones. More precisely, we prove in Section 6.2 the following theorem.

Theorem 6. *Let M be a $2n$-dimensional Liouville manifold with $n \geq 3$. Suppose that $\pi : M \to \mathbb{C}$ is an exact symplectic Lefschetz fibration with smooth fiber F. If F admits a cyclic dilation, then the same is true for the total space M.*

2.4. Trichotomy of affine varieties

The well-known trichotomy of Riemannian manifolds says that positively curved, flat, and negatively curved manifolds have distinct geometric behaviors. In symplectic topology, there is an analogy of this trichotomy for Liouville manifolds. Geometrically, this can be understood by studying the existence and abundance of J-holomorphic maps $u : S \to M^{\mathrm{in}}$ with finite energy in the interior M^{in} of the associated Liouville domain $\bar{M}$, where S is a punctured sphere; see [53].

For simplicity, we restrict our attention to the case when $M \subset \mathbb{C}^N$ is an n-dimensional smooth affine variety, equipped with the restriction of the constant symplectic form on the ambient affine space, in which case the aforementioned trichotomy has a numerical description in terms of the *log Kodaira dimension*

$$\kappa(M) \in \{-\infty\} \cup \{0, \ldots, n\}. \tag{2.5}$$

This is defined by choosing a compactification X of M so that X is a smooth projective variety, and the divisor $D = X \setminus M$ has simple normal crossing. $\kappa(M)$ is defined as the Kodaira–Iitaka dimension of the line bundle $K_X + D$ over X. We will be interested here in the cases when $\kappa(M) = -\infty$, $\kappa(M) = 0$ (in which case M is known as *log Calabi–Yau*), and $\kappa(M) = n$ (in which case M is *log general type*). These should be thought of as analogs of positively curved, flat, and negatively curved Riemannian manifolds, respectively. The existence question of a cyclic dilation (or equivalently, an exact Calabi–Yau structure on $\mathcal{W}(M)$) will be considered separately in these three cases.

First, let M be a smooth affine variety with $\kappa(M) = -\infty$. An important class of such manifolds is given by the Milnor fibers $M_{a_1,\ldots,a_{n+1}} \subset \mathbb{C}^{n+1}$ associated to the Brieskorn singularities

$$z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} = 0,$$

where $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$. In this paper, we will study the simplest non-trivial case, namely, a Fermat affine cubic 3-fold $M_{3,3,3,3} \subset \mathbb{C}^4$.

Theorem 7. *The manifold $M_{3,3,3,3}$ admits a cyclic dilation.*

This will be proved in Section 6.1 using essentially algebraic arguments. Abstractly, one should think of Theorem 7 as a consequence of the Koszul duality between the compact and the wrapped Fukaya categories of $M_{3,3,3,3}$.

Another key point of the proof is to show that, up to quasi-isomorphism, the wrapped Fukaya A_∞-algebra of $M_{3,3,3,3}$ is concentrated in non-positive degrees, which is expected to be true for any $M_{a_1,\ldots,a_{n+1}}$ with $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$, although the verification is more involved in general.

Remark 8. Since our proof of Theorem 7 relies on the results of [23], it is also dependent on the Legendrian surgery description of the wrapped Fukaya category due to Bourgeois–Ekholm–Eliashberg [8]. Details of the proofs of the results sketched in [8, 23] can be found in the recent work [22].

Note that if $a_i \geq 3$ for all i, then $M_{a_1,\ldots,a_{n+1}}$ does not admit a quasi-dilation. This is argued in [63, Example 2.7] for dilations, and the argument there extends trivially to the more general case of quasi-dilations (a sketch is given in the proof of Corollary 66). In particular, Theorem 7 shows that the existence of a cyclic dilation is strictly weaker than having a quasi-dilation.

Combining Theorems 6 and 7, we have the following corollary.

Corollary 9. *Take the affine hypersurface* $\{p(z_1, \ldots, z_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$ *such that*

$$p(z) = z_1^3 + z_2^3 + z_3^3 + z_4^3 + \tilde{p}(z_5, \ldots, z_{n+1}) \tag{2.6}$$

has an isolated singularity at the origin. Let M be the Milnor fiber associated to p; then M admits a cyclic dilation.

Remark 10. More interesting examples of Liouville manifolds admitting cyclic dilations are established in the recent work of Zhou [79] based on the machinery of Diogo–Lisi [21]. In particular, his result implies that $M_{a,\ldots,a}$ admits a cyclic dilation as long as $n \geq a$, therefore generalizing Theorem 7 above. Our method has the advantage that it is applicable to examples beyond complements of smooth divisors in projective varieties. As an example, see Proposition 59.

Second, we consider the case when M is a smooth log Calabi–Yau variety. These manifolds provide important local examples for testing the validity of mirror symmetry and have been studied extensively in the existing literature [5, 31, 32, 34, 54]. As an illustration for the general situation, we consider here the simplest case when

$$\dim_{\mathbb{C}}(M) = 2,$$

and make the following observation.[1] Recall that a log Calabi–Yau surface with *maximal boundary* is the complement $X \setminus D$, where X is a smooth projective surface, and $D \subset X$ is a singular anticanonical divisor with nodal singularities.

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[1]The author thanks Daniel Pomerleano for suggesting this approach to prove Proposition 11, which greatly simplifies the original argument.

Proposition 11. *Let M be an affine log Calabi–Yau surface with maximal boundary; then M admits a cyclic dilation if and only if it admits a quasi-dilation.*

Proof. With our assumptions, one can arrange so that the Conley–Zehnder indices of the periodic orbits are 0, 1, and 2; see for example [31, 54]. In particular, the cochain complex $SC^*(M)$ defining the symplectic cohomology $SH^*(M)$ is supported in these three degrees. Thus, any cyclic dilation can only come from a cocycle in $SC^1(M)$; see our discussions in Section 4.2 for details. ∎

We expect the same to be true in higher dimensions, although no insights can be drawn from the argument above.

Finally, let us take a look at the case when M is a smooth affine variety of log general type. To get some concrete examples, one can take any Milnor fiber $M_{a_1,\dots,a_{n+1}}$ as above, but now with $\sum_{i=1}^{n+1} \frac{1}{a_i} < 1$. In complex dimension 2, the Milnor fibers associated to Arnold's 14 exceptional unimodal singularities are affine surfaces of log general type, since they are complements of ample divisors in K3 surfaces; see [48].

Via the Abel–Jacobi map, we can embed a genus two curve Σ_2 in its Jacobian variety $J(\Sigma_2)$; let M be the complement in $J(\Sigma_2)$ of the image of Σ_2. Clearly, M is log general type. On the other hand, since there is an embedding $D^*T^2 \# D^*T^2 \hookrightarrow M$ from the plumbing of two copies of the disk cotangent bundles over T^2 into M as a Liouville subdomain, Lagrangian surgery produces a genus two exact Lagrangian surface in M. One can therefore use Proposition 5 to conclude that there is no cyclic dilation in $SH^1_{S^1}(M)$. This example can be generalized to the case when M is the complement of a nearly tropical hypersurface in the abelian variety $(\mathbb{C}^*)^n / \Gamma$, where $\Gamma \subset \mathbb{R}^n$ is a lattice; see [5, Section 10].

It seems that genus two exact Lagrangian surfaces can also be established in the 4-dimensional Milnor fibers M_{a_1,a_2,a_3} with $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} < 1$, by imitating the strategy of Keating [38]. However, it is not true that hyperbolic exact Lagrangian submanifolds can always be constructed in varieties of log general type. For instance, this is the case of the complement M of $n + 2$ generic hyperplanes in $\mathbb{CP}^n$, with $n \geq 2$. These manifolds are known as higher-dimensional pair of pants, and are studied extensively in the context of mirror symmetry; see for example [47]. Since M is uniruled by $(n + 2)$-punctured holomorphic spheres, similar argument as in the proof of [25, Theorem 1.7.5] excludes the existence of hyperbolic Lagrangians in M.

We expect that a smooth affine variety of log general type can never admit a cyclic dilation and will prove the following general statement in Section 6.3.

Theorem 12. *Let M be a smooth affine variety of log general type which contains an exact Lagrangian torus; then it does not admit a cyclic dilation.*

Theorem 12 shows that the higher-dimensional pair of pants mentioned above do not admit cyclic dilations, but it is not helpful in general as there are many contractible

affine varieties of log general type, which conjecturally do not contain any closed exact Lagrangian submanifold. One example is the Ramanujam surface studied in [66].

2.5. Categorical dynamics

We start with a brief overview of the theory of categorical dynamics, which is developed by Seidel in a series of works [56, 62, 63, 65, 67]. Various assumptions will be imposed here to keep the exposition simple enough. Given a quiver with potential (Q, w), one can associate to it two A_∞-algebras. One of them is the (completed) Ginzburg dg algebra $\widehat{\mathcal{G}}(Q, w)$ mentioned at the end of Section 2.1 (see also Section 3.1 below for related backgrounds), while the other one, denoted as $\mathcal{B}(Q, w)$, is introduced by Kontsevich–Soibelman [43]. $\mathcal{B}(Q, w)$ is related to $\widehat{\mathcal{G}}(Q, w)$ via Koszul duality, which means there are quasi-isomorphisms

$$\mathcal{B}(Q, w) \cong R\mathrm{Hom}_{\mathcal{G}(Q,w)}(\Bbbk, \Bbbk), \quad \widehat{\mathcal{G}}(Q, w) \cong R\mathrm{Hom}_{\mathcal{B}(Q,w)}(\Bbbk, \Bbbk),$$

where $\Bbbk := \bigoplus_{i \in Q_0} \mathbb{K} e_i$ is the semisimple ring consisting of copies of $\mathbb{K}$ indexed by the set of vertices Q_0 of Q. Koszul duality between $\mathcal{B}(Q, w)$ and $\mathcal{G}(Q, w)$ induces an isomorphism between their Hochschild cohomologies [40]

$$\mathrm{HH}^*(\mathcal{B}(Q, w)) \cong \mathrm{HH}^*(\mathcal{G}(Q, w)). \tag{2.7}$$

In general, this is only an isomorphism of Gerstenhaber algebras, but since $\mathcal{B}(Q, w)$ is a cyclic A_∞-algebra and $\mathcal{G}(Q, w)$ is exact Calabi–Yau, their Hochschild cohomologies carry naturally induced BV structures. For cyclic A_∞-algebras, this is proved by Tradler [72, Theorem 1], and for smooth Calabi–Yau algebras, this is due to Ginzburg [33, Theorem 3.4.3]. Thus, one expects that (2.7) is a BV algebra isomorphism. See [15, Theorem A], where this is confirmed for Koszul Calabi–Yau algebras.

Let us assume temporarily that the superpotential w is homogeneous and consists only of cubic terms, which in particular implies the formality of the A_∞-structure on $\mathcal{B}(Q, w)$ (since it is a dg algebra with vanishing differential), and works over the ground field $\mathbb{K} = \mathbb{C}$. The graded associative algebra $\mathcal{B}(Q, w)$ then carries a rational $\mathbb{C}^*$-action, which has weight i on the degree i part. According to [62], this $\mathbb{C}^*$-action enables us to define a bigraded refinement $\mathcal{B}(Q, w)^{\mathbf{perf}}$ of the A_∞-category $\mathcal{B}(Q, w)^{\mathrm{perf}}$ of perfect A_∞-modules over $\mathcal{B}(Q, w)$. The $\mathbb{C}^*$-action on $\mathcal{B}(Q, w)$ induces at the infinitesimal level a Hochschild cocycle $\mathrm{eu}_{\mathcal{B}} \in \mathrm{CH}^1(\mathcal{B}(Q, w))$. This particular noncommutative vector field is known as the *Euler vector field*. Under the BV operator, $\mathrm{eu}_{\mathcal{B}}$ goes to a non-zero scalar multiple of the identity. Any object $\mathcal{E}$ of $\mathcal{B}(Q, w)^{\mathrm{perf}}$ which is *rigid* and *simple*, meaning that

$$H^0(\mathrm{hom}_{\mathcal{B}(Q,w)^{\mathrm{perf}}}(\mathcal{E}, \mathcal{E})) \cong \mathbb{C}, \quad H^1(\mathrm{hom}_{\mathcal{B}(Q,w)^{\mathrm{perf}}}(\mathcal{E}, \mathcal{E})) \cong 0,$$

is $\mathbb{C}^*$-equivariant and therefore defines an object in the category $\mathcal{B}(Q, w)^{\mathbf{perf}}$. In particular, it is infinitesimally equivariant with respect to $\mathrm{eu}_{\mathcal{B}}$. For any two such objects $\mathcal{E}_0$ and $\mathcal{E}_1$, the derivation $[\mathrm{eu}_{\mathcal{B}}]$ defines an endomorphism of

$$H^*(\hom_{\mathcal{B}(Q,w)^{\mathrm{perf}}}(\mathcal{E}_0, \mathcal{E}_1)),$$

from which one recovers the weight grading on $\mathcal{B}(Q, w)^{\mathbf{perf}}$.

Geometrically, let M be a $2n$-dimensional Weinstein manifold, and assume that its wrapped Fukaya A_∞-algebra $\mathcal{W}_M$ is quasi-isomorphic to some Ginzburg dg algebra $\mathcal{G}(Q, w)$. In view of Proposition 4, M admits a cyclic dilation. For simplicity, assume further that the Fukaya categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are Koszul dual so that the endomorphism algebra $\mathcal{F}_M$ of a set of split generators in $\mathcal{F}(M)$ can be identified with $\mathcal{B}(Q, w)$. Our assumptions therefore ensure that the closed-open string map

$$[\mathrm{CO}] : \mathrm{SH}^*(M) \to \mathrm{HH}^*(\mathcal{F}(M)) \tag{2.8}$$

is an isomorphism.

Via the quasi-isomorphism $\mathcal{F}_M \cong \mathcal{B}(Q, w)$ and the inverse of (2.8), the Euler vector field $\mathrm{eu}_{\mathcal{B}}$ gives rise to a quasi-dilation $b \in \mathrm{SH}^1(M)$. In other words, the infinitesimal symmetry b integrates to a *dilating* $\mathbb{C}^*$-action on the Fukaya category $\mathcal{F}(M)$ in the sense of [65]. For any two $\mathbb{C}^*$-equivariant objects L_0, L_1 of $\mathcal{F}(M)$, this enables us to define an endomorphism of their Floer cohomology algebra $\mathrm{HF}^*(L_0, L_1)$, which equips $\mathrm{HF}^*(L_0, L_1)$ with an additional $\mathbb{C}$-grading by generalized eigenspaces.

More generally, as mentioned in the introduction, one can start directly with a quasi-dilation $b \in \mathrm{SH}^1(M)$ and consider the infinitesimal deformation of the objects in $\mathcal{F}(M)$ along b. For b-equivariant Lagrangian submanifolds $L_0, L_1 \subset M$, the infinitesimal action of b still defines a derivation $\Phi_{\tilde{L}_0, \tilde{L}_1}$ on the $\mathbb{C}$-algebra $\mathrm{HF}^*(L_0, L_1)$, which appears in the definition of the q-intersection number (1.2). This construction is due to Seidel–Solomon [67]. As an application, Seidel proves in [63] that, for Liouville manifolds with dilations, there is an upper bound on the number of disjoint Lagrangian spheres.

It is natural to ask whether there is a similar theory after removing the assumption that $\mathcal{B}(Q, w)$ (and thus $\mathcal{F}_M$) is formal. Note that, as a consequence of non-formality, the aforementioned $\mathbb{C}^*$-action on $\mathcal{B}(Q, w)$ does not preserve the A_∞-structure.

In general, given any Liouville manifold M with a cyclic dilation $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$, one can try to imitate Seidel–Solomon's construction by making use of the higher-order closed-open string maps introduced by Ganatra [28]; see Section 5.2. Due to the existence of certain obstruction terms (corresponding to some unwanted boundary strata in the relevant moduli spaces), it is in general not possible to obtain endomorphisms on Floer cohomology groups $\mathrm{HF}^*(L_0, L_1)$ of any simply connected Lagrangian submanifolds. However, for an odd-dimensional Lagrangian sphere $L \subset M$

with dimension $n \geq 3$, the obstruction vanishes and we obtain a non-trivial derivation $\Phi_{\widetilde{L},\widetilde{L}}$ on $\mathrm{HF}^*(L, L)$; see Section 5.3. This indicates that one can use cyclic dilations to detect the homological non-triviality of odd-dimensional Lagrangian spheres. In fact, we will prove the following result in Section 5.4.

Theorem 13. *Let M be a $2n$-dimensional Weinstein manifold, where $n \geq 3$ is odd, and $c_1(M) = 0$. Assume that M admits a cyclic dilation $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ and satisfies an additional property $(\widetilde{H})$ (cf. Definition 50). Then, for any Lagrangian sphere $L \subset M$, its homology class $[L] \in H_n(M;\mathbb{K})$ is non-zero.*

In the above, if $\mathbb{K}$ can be taken to be an arbitrary field, then one would be able to arrive at the conclusion that $[L] \in H_n(M;\mathbb{Z})$ is primitive. However, proving Theorem 13 may involve essential difficulties when $\mathrm{char}(\mathbb{K}) = 2$; see Remark 53. Here, property $(\widetilde{H})$ imposes additional restrictions on the cohomology class $\tilde{b}$. More precisely, there is a continuation map

$$\mathrm{HF}^1_{S^1}(\lambda) \to \mathrm{SH}^1_{S^1}(M)$$

from the S^1-equivariant Floer cohomology of a Hamiltonian with slope λ on the cylindrical end $M \setminus \bar{M}$ to the S^1-equivariant symplectic cohomology; see (4.9). Roughly speaking, property $(\widetilde{H})$ says that the class $\tilde{b}$ appears already in $\mathrm{HF}^1_{S^1}(\lambda)$, with $\lambda > 0$ being relatively small with respect to the minimal period of the Reeb orbits on the contact boundary $\partial \bar{M}$. See Definition 50 for details.

This property is expected to be true for many Weinstein manifolds with cyclic dilations, including the Milnor fibers $M_{a_1,\dots,a_{n+1}}$ with $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$. In fact, it follows from the argument in [79, Section 5.2] that it holds for the Milnor fibers $M_{a,\dots,a}$ with $n \geq a$. When the cyclic dilation satisfies $\mathbf{B}(\tilde{b}) = 1$, the assumption of Theorem 13 can be equivalently expressed in terms of the first Gutt–Hutchings capacity of the Weinstein domain $\bar{M}$; see Corollary 51.

In algebraic terms, a related question is the following.

Question 14. Let X be a spherical object in the triangulated A_∞-category $\mathcal{B}(Q,w)^{\mathrm{perf}}$; then is it always true that its class $[X]$ in the Grothendieck group $K_0(\mathcal{B}(Q,w)^{\mathrm{perf}})$ is non-zero?

For Weinstein manifolds whose wrapped Fukaya A_∞-algebra $\mathcal{W}_M$ can be identified with the Ginzburg dg algebra $\mathcal{G}(Q, w)$, the work of Lazarev [46, Theorem 1.8] combined with Koszul duality gives rise to an injective map

$$K_0(\mathcal{F}(M)) \to H_n(M;\mathbb{Z}).$$

Thus, an affirmative answer to Question 14 would imply the homological essentiality of Lagrangian spheres in M. It is in general unknown how to answer Question 14 in odd dimensions, unless Q is a finite oriented tree and w vanishes [56, Remark 15.16].

Theorem 13 will be proved in Section 5.4.

3. Algebraic preliminaries

We summarize in this section some basic algebraic notions and facts that will be used in this paper. The expositions in Sections 3.2 and 3.3 follow essentially from [28, Sections 2 and 3].

3.1. Superpotential algebras

As we have mentioned in Section 2.1, an important class of exact Calabi–Yau algebras is given by superpotential algebras defined by Ginzburg [33], whose definition we will briefly recall here.

Let $\mathcal{A}$ be a finitely generated unital dg algebra over $\Bbbk$; denote by $\Omega^1_{\mathcal{A}}$ the $\mathcal{A}$-bimodule of noncommutative differentials on $\mathcal{A}$, which is the kernel of the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. $\mathcal{A}$ is said to be *quasi-free* if $\Omega^1_{\mathcal{A}}$ is projective. For example, any path algebra of a quiver is quasi-free, so is its localization. Assume from now on that $\mathcal{A}$ is quasi-free; define the space of de Rham differential forms

$$\mathrm{DR}_{\Bbbk}(\mathcal{A}) := T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})/[T_{\mathcal{A}}(\Omega^1_{\mathcal{A}}), T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})],$$

where $T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})$ is the tensor algebra of $\Omega^1_{\mathcal{A}}$ over $\mathcal{A}$, so in particular $\mathrm{DR}_{\Bbbk}(\mathcal{A})$ carries the structure of a dg algebra. Denote by $\mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A})$ the dg vector space of $\Bbbk$-linear (super)derivations on $\mathcal{A}$. For a closed 2-form $\omega \in \mathrm{DR}^2_{\Bbbk}(\mathcal{A})$, we have a map

$$i_{\omega} : \mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A}) \to \mathrm{DR}^1_{\Bbbk}(\mathcal{A}) \qquad (3.1)$$

defined by contracting ω with every derivation in $\mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A})$. We say that ω is *symplectic* if i_{ω} is an isomorphism. Let $\mathbb{D}er_{\Bbbk}(\mathcal{A}) := \mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}) = (\Omega^1_{\mathcal{A}})^{\vee}$ be the bimodule of *double derivations* of $\mathcal{A}$; analogous to (3.1), we have a map

$$\iota_{\omega} : \mathbb{D}er_{\Bbbk}(\mathcal{A}) \to \Omega^1_{\mathcal{A}}.$$

A symplectic form ω is *bisymplectic* if ι_{ω} is an isomorphism.

Given any $a \in \mathcal{A}$ and letting $\omega \in \mathrm{DR}^2_{\Bbbk}(\mathcal{A})$ be bisymplectic, consider the double derivation $H_a \in \mathbb{D}er_{\Bbbk}(\mathcal{A})$ defined by $\iota_{H_a}\omega = Da := a \otimes 1 - 1 \otimes a$. Using H_a, one can define a bracket $\{\cdot, \cdot\}$ on $\mathcal{A}$ by

$$\{a_1, a_2\} = \circ(H_{a_1}(a_2)),$$

where $\circ : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication on $\mathcal{A}$. It can be checked that $\{\cdot, \cdot\}$ descends to a Lie bracket on $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$, and it defines an action of $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$ on $\mathcal{A}$ by derivations. See [73, Appendix A].

On the other hand, given $\theta \in \mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A})$, we can define the Lie derivative L_θ on $T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})$ by

$$L_\theta(a) = \theta(a), \quad L_\theta(Da) = D(\theta(a))$$

for any $a \in \mathcal{A}$ and $Da \in \Omega^1_{\mathcal{A}}$. Clearly, L_θ descends to a map on the quotient $\mathrm{DR}_{\Bbbk}(\mathcal{A})$.

Consider the triple $(\mathcal{A}, \omega, \theta)$, where $\mathcal{A} = T_{\mathcal{B}}(\mathcal{M})$ is connected [20, Definition 4.3.2] and non-positively graded, with $\mathcal{B}$ being a quasi-free associative algebra concentrated in degree 0 and $\mathcal{M}$ a $\mathcal{B}$-bimodule. The 2-form $\omega \in \mathrm{DR}^2_{\Bbbk}(\mathcal{A})$ is a bisymplectic form which is homogeneous with respect to the grading induced from $\mathcal{A}$, and $\theta \in \mathrm{Der}_{\Bbbk}(\mathcal{A}, \mathcal{A})$ has cohomological degree 1, which satisfies $L_\theta\omega = 0$ and $\theta^2 = 0$. One can associate to this data a dg algebra $\mathcal{G}(\omega, \theta)$, called *Ginzburg dg algebra*, whose underlying graded algebra is given by the free product $\mathcal{A} * \Bbbk[t]$, where

$$|t| = |\omega| - 1.$$

The differential d on $\mathcal{G}(\omega, \theta)$ is defined by considering the noncommutative moment map

$$\mu_{\mathrm{nc}} : \mathrm{DR}^2_{\Bbbk}(\mathcal{A})_{\mathrm{cl}} \to \overline{\mathcal{A}} := \mathcal{A}/\Bbbk$$

on the space of closed cyclic 2-forms $\mathrm{DR}^2_{\Bbbk}(\mathcal{A})_{\mathrm{cl}} \subset \mathrm{DR}^2_{\Bbbk}(\mathcal{A})$, which satisfies

$$D(\mu_{\mathrm{nc}}(\omega)) = \iota_\Delta\omega,$$

where $\Delta \in \mathbb{D}er(\mathcal{A})$ is the double derivation

$$\Delta(a) = a \otimes 1 - 1 \otimes a.$$

Under the assumption that $\mathcal{A}$ is connected, μ_{nc} lifts to a map

$$\widetilde{\mu}_{\mathrm{nc}} : \mathrm{DR}^2_{\Bbbk}(\mathcal{A})_{\mathrm{cl}} \to [\mathcal{A}, \mathcal{A}].$$

With the above notations, we define

$$da = \theta(a), dt = \widetilde{\mu}_{\mathrm{nc}}(\omega).$$

Note that our assumptions on the derivation θ ensures that $d^2 = 0$, so the dg algebra $\mathcal{G}(\omega, \theta)$ is well defined.

Definition 15. The Ginzburg dg algebra $\mathcal{G}(\omega, \theta)$ defined above is a superpotential algebra if $\theta = \{w, \cdot\}$ for some $w \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]$. w is called the superpotential.

If one takes $\mathcal{A}$ above to be the path algebra $\mathbb{K}\widetilde{Q}$ of the double $\widetilde{Q}$ of some quiver $Q = (Q_0, Q_1)$ obtained by adding a reverse a^* to all the arrows $a \in Q_1$ (if a is a cycle of odd degree, then $a^* = a$), the above construction recovers the Ginzburg dg algebra (or the *dg preprojective algebra*, in the terminology of [74]) $\mathcal{G}(Q, w)$ associated to the

Figure 1. The quiver $Q_{1,0,0}$.

quiver with potential (Q, w), which can be defined in a more concrete and elementary way; see [74, Section 9.3].

However, there are also many superpotential algebras which are not of the form $\mathcal{G}(Q, w)$; a typical example is the fundamental group algebra $\mathbb{K}[\pi_1(T^3)]$; see [20, Example 4.3.4]. Another counterexample is the following.

Example 16. Consider the associative algebra $\mathbb{K}[x, y][(xy - 1)^{-1}]$, regarded as a trivially graded dg algebra with vanishing differential. One can show that this algebra has a superpotential description. In fact, consider the path algebra $\mathbb{K}\widetilde{Q}_{1,0,0}$ of the double of the quiver $Q_{1,0,0}$ which consists of a single vertex and a unique cycle x; see Figure 1. Let

$$\mathcal{A} = \mathbb{K}\langle x, y \rangle[(xy - 1)^{-1}]$$

be the localization of $\mathbb{K}\widetilde{Q}_{1,0,0}$ at $xy - 1$, where the generator y represents the reverse of the cycle x in the quiver $\widetilde{Q}_{1,0,0}$. Since $\mathbb{K}\widetilde{Q}_{1,0,0}$ is quasi-free, so is $\mathcal{A}$. In this way, we have identified $\mathbb{K}[x, y][(xy - 1)^{-1}]$ with the dg algebra $(\mathcal{A} * \mathbb{K}[t], d)$ so that $dx = dy = 0$ and $dt = xy - yx$. The superpotential vanishes for dimension reasons.

Theorem 17 ([20, Theorem 4.3.8]). *Let $\mathcal{G}(\omega, \theta)$ be a Ginzburg dg algebra so that ω has cohomological degree $-n + 2$; then $\mathcal{G}(\omega, \theta)$ has a smooth n-Calabi–Yau structure which is exact.*

3.2. Complexes with S^1-actions

This section follows closely [28, Section 2]. Let $C_{-*}(S^1)$ be the dg algebra of chains on S^1; there is a quasi-equivalence of dg algebras

$$C_{-*}(S^1) \cong \mathbb{K}[t]/(t^2), \quad |t| = -1.$$

Note that the algebra structure on $C_{-*}(S^1)$ comes from the group structure of S^1. The following definition is introduced in [11, 28].

Definition 18 ([28, Definitions 1 and 2]). An S^1-complex, or a chain complex with an A_∞ S^1-action, is a strictly unital A_∞-module $\mathcal{P}$ over $\mathbb{K}[t]/(t^2)$. Equivalently, it is a graded $\mathbb{K}$-vector space equipped with operations $\delta_j^{\mathcal{P}} : \mathcal{P} \to \mathcal{P}[1 - 2j]$, with

$\delta_0^{\mathcal{P}} := d^{\mathcal{P}}$ being the differential on $\mathcal{P}$, and $\delta_j^{\mathcal{P}} := \mu_{\mathcal{P}}^{j|1}(t, \ldots, t, \cdot)$, $j > 0$ being the structure maps such that, for each $k \geq 0$, the equation

$$\sum_{j=0}^{k} \delta_j^{\mathcal{P}} \delta_{k-j}^{\mathcal{P}} = 0$$

holds. If the A_∞-module $\mathcal{P}$ is a dg module; i.e., $\delta_j^{\mathcal{P}}$ for $j \geq 2$, it is called a strict S^1-complex.

We will abbreviate $\delta_j^{\mathcal{P}}$ to δ_j as long as there is no confusion.

S^1-complexes form a dg category $C_{-*}(S^1)^{\mathrm{umod}}$, whose morphisms we will now recall. Let $A := C_{-*}(S^1)$ be the quadratic algebra with a degree -1 generator; denote by $\varepsilon : A \to \mathbb{K}$ the trivial augmentation and by $\bar{A} := \ker(\varepsilon)$ the augmentation ideal. Let $\mathcal{P}$ and $\mathcal{Q}$ be strictly unital A_∞-modules over A. A *unital pre-morphism* of degree k from $\mathcal{P}$ to $\mathcal{Q}$ is a collection of maps

$$F^{d|1} : \bar{A}^{\otimes d} \otimes \mathcal{P} \to \mathcal{Q}[k - d]$$

for each $d \geq 0$. Or equivalently, it can be expressed as a set of maps $\{F^d\}$, with

$$F^d := F^{d|1}(t, \ldots, t, \cdot) : \mathcal{P} \to \mathcal{Q}[k - 2d].$$

The space of pre-morphisms in each degree forms the graded vector space of morphisms between the objects $\mathcal{P}$ and $\mathcal{Q}$ in the dg category A^{umod}, which will be denoted by $R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q})$. There is a differential ∂ on $R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q})$, which is defined by

$$(\partial F)^s := \sum_{i=0}^{s} F^i \circ \delta_{s-i}^{\mathcal{P}} - (-1)^{\deg(F)} \sum_{j=0}^{s} \delta_{s-j}^{\mathcal{Q}} \circ F^j.$$

An S^1-*complex homomorphism* is a pre-morphism which is closed under ∂. A homomorphism $F : \mathcal{P} \to \mathcal{Q}$ between S^1-complexes is a *quasi-isomorphism* if the induced map

$$[F^0] : H^*(\mathcal{P}) \to H^{*+\deg(F)}(\mathcal{Q})$$

on cohomologies is an isomorphism.

Let $\mathcal{P}$ and $\mathcal{Q}$ be S^1-complexes; one can define their (derived) tensor product, which is another S^1-complex $\mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}$. To do this, note that we can view $\mathcal{Q}$ as a right A_∞-module over A since A is commutative. The chain complex $\mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}$ has underlying vector space

$$\bigoplus_{d \geq 0} \mathcal{Q} \otimes \bar{A}[1]^{\otimes d} \otimes \mathcal{P},$$

and the differential acts as

$$\partial(q \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes p)$$

$$= \sum_{i=0}^{d} \left((-1)^{|m|} \delta_i^{\mathcal{Q}}(q) \otimes \underbrace{t \otimes \cdots \otimes t}_{d-i} \otimes p + q \otimes \underbrace{t \otimes \cdots \otimes t}_{d-i} \otimes \delta_i^{\mathcal{P}}(p) \right).$$

The tensor product $\mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}$ is functorial in the sense that if $F = \{F^d\} : \mathcal{P}_0 \to \mathcal{P}_1$ is a pre-morphism of S^1-complexes, then there are induced maps

$$F_\# : \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}_0 \to \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}_1, \quad {}_\#F : \mathcal{P}_0 \otimes^{\mathbb{L}}_{S^1} \mathcal{Q} \to \mathcal{P}_1 \otimes^{\mathbb{L}}_{S^1} \mathcal{Q}$$

given by

$$F_\#(q \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes p) = \sum_{j=0}^{d} q \otimes \underbrace{t \otimes \cdots \otimes t}_{d-j} \otimes F^j(p)$$

and

$${}_\#F(p \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes q) = \sum_{j=0}^{d} (-1)^{\deg(F) \cdot |q|} F^j(p) \otimes \underbrace{t \otimes \cdots \otimes t}_{d-j} \otimes q,$$

which are chain maps if F is closed.

Proposition 19 ([28, Proposition 1]). *Let $F : \mathcal{P} \to \mathcal{P}'$ be a quasi-isomorphism of S^1-complexes; then we have induced quasi-isomorphisms between* hom *spaces and tensor products; i.e., we have quasi-isomorphisms*

$$F \circ : R\mathrm{Hom}_{S^1}(\mathcal{P}', \mathcal{Q}) \cong R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q}),$$

$$\circ F : R\mathrm{Hom}_{S^1}(\mathcal{Q}, \mathcal{P}) \cong R\mathrm{Hom}_{S^1}(\mathcal{Q}, \mathcal{P}'),$$

$${}_\#F : \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P} \cong \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}',$$

$$F_\# : \mathcal{P} \otimes^{\mathbb{L}}_{S^1} \mathcal{Q} \cong \mathcal{P}' \otimes^{\mathbb{L}}_{S^1} \mathcal{Q}.$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be two S^1-complexes and $f : \mathcal{P} \to \mathcal{Q}$ a chain map. We say that a homomorphism $F = \{F^d\}$ from $\mathcal{P}$ to $\mathcal{Q}$ is an S^1-*equivariant enhancement* of f if $[F^0] = [f]$. In particular, F has degree $\deg(f)$.

Finally, we notice that if $\mathcal{P}$ and $\mathcal{Q}$ are S^1-complexes, then their (linear) tensor product $\mathcal{P} \otimes \mathcal{Q}$ can also be equipped with an S^1-complex structure with

$$\delta_k^{\mathcal{P} \otimes \mathcal{Q}}(p \otimes q) := (-1)^{|q|} \delta_k^{\mathcal{P}}(p) \otimes q + p \otimes \delta_k^{\mathcal{Q}}(q). \tag{3.2}$$

We call this S^1-action on $\mathcal{P} \otimes \mathcal{Q}$ the *diagonal S^1-action*.

Definition 20 ([28, Definitions 4 and 5]). The homotopy orbit complex of $\mathcal{P}$ is the derived tensor product

$$\mathcal{P}_{hS^1} := \mathbb{K} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}.$$

The homotopy fixed point complex of $\mathcal{P}$ is the chain complex of morphisms

$$\mathcal{P}^{hS^1} := R\mathrm{Hom}_{S^1}(\mathbb{K}, \mathcal{P}).$$

Since $R\mathrm{Hom}_{S^1}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}[u]$, $\mathcal{P}^{hS^1}$ carries the structure of a $\mathbb{K}[u]$-module. The Tate complex of $\mathcal{P}$ is defined as the localization of $\mathcal{P}^{hS^1}$ at u:

$$\mathcal{P}^{\mathrm{Tate}} := \mathcal{P}^{hS^1} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}],$$

where u is a formal variable with $|u| = 2$.

Let $F : \mathcal{P} \to \mathcal{Q}$ be a homomorphism of S^1-complexes; it induces chain maps

$$F^{hS^1} : \mathcal{P}^{hS^1} \to \mathcal{Q}^{hS^1}, \quad F_{hS^1} : \mathcal{P}_{hS^1} \to \mathcal{Q}_{hS^1}, \quad F^{\mathrm{Tate}} : \mathcal{P}^{\mathrm{Tate}} \to \mathcal{Q}^{\mathrm{Tate}}. \quad (3.3)$$

When F is a quasi-isomorphism (of S^1-complexes), F^{hS^1}, F_{hS^1}, and F^{Tate} are quasi-isomorphisms (of chain complexes).

Proposition 21 ([28, Proposition 2]). *If $F : \mathcal{P} \to \mathcal{Q}$ is a homomorphism between S^1-complexes, then the various induced maps in (3.3) intertwine all of the long exact sequences for equivariant homology groups of $\mathcal{P}$ with those for $\mathcal{Q}$.*

For instance, we have the Gysin exact triangle [28, Remark 23]

$$\mathcal{P} \to \mathcal{P}_{hS^1} \to \mathcal{P}_{hS^1}[2] \xrightarrow{[1]} .$$

Taking $\mathcal{P} = \mathrm{CH}_*(\mathcal{A})$ to be the Hochschild chain complex of some strictly unital A_∞-algebra $\mathcal{A}$ recovers Connes's long exact sequence (2.1). If $F : \mathcal{P} \to \mathcal{Q}$ is an S^1-complex homomorphism, there is a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{P} & \longrightarrow & \mathcal{P}_{hS^1} & \longrightarrow & \mathcal{P}_{hS^1}[2] \xrightarrow{[1]} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Q} & \longrightarrow & \mathcal{Q}_{hS^1} & \longrightarrow & \mathcal{Q}_{hS^1}[2] \xrightarrow{[1]}
\end{array}
$$

The structure of an S^1-complex $(\mathcal{P}, \{\delta_j\}_{j \geq 0})$ admits an alternative description by implementing the u-linear model. Let u be a formal variable of degree 2; consider the u-adically completed tensor product

$$\mathcal{P}[\![u]\!] := \mathcal{P} \hat{\otimes}_{\mathbb{K}} \mathbb{K}[\![u]\!] \quad (3.4)$$

in the category of graded vector spaces. An S^1-complex can be equivalently formulated as a graded $\mathbb{K}$-vector space $\mathcal{P}$ equipped with a map $\delta_{eq} : \mathcal{P} \to \mathcal{P}[\![u]\!]$ of total degree 1 defined by

$$\delta_{eq} := \sum_{j=0}^{\infty} \delta_i u^i, \tag{3.5}$$

which satisfies $\delta_{eq}^2 = 0$. The map δ_{eq} is called an *equivariant differential*; note that it extends u-linearly to a map $\mathcal{P}[\![u]\!] \to \mathcal{P}[\![u]\!]$, which we will still denote by δ_{eq}.

The *(positive) S^1-equivariant homology*, *negative S^1-equivariant homology*, and *periodic S^1-equivariant homology* are defined, respectively, as homologies of the following complexes:

$$\mathcal{P}_{hS^1} := (\mathcal{P}((u))/u\mathcal{P}[\![u]\!], \delta_{eq}), \tag{3.6}$$

$$\mathcal{P}^{hS^1} := (\mathcal{P}[\![u]\!], \delta_{eq}), \tag{3.7}$$

$$\mathcal{P}^{\text{Tate}} := (\mathcal{P}((u)), \delta_{eq}). \tag{3.8}$$

3.3. Non-unital Hochschild chain complex

Our exposition here follows [28, Section 3.1]. Let $\mathcal{A}$ be an A_∞-algebra over the semisimple ring $\mathbb{k} := \bigoplus_{i=1}^{r} \mathbb{K}e_i$. For the usual Hochschild chain complex $\text{CH}_*(\mathcal{A})$ to be a strict S^1-complex in the sense of Definition 18, one needs to assume that $\mathcal{A}$ is strictly unital. However, in the geometric context, the Fukaya category is in general only cohomologically unital; we therefore need a replacement of the usual Hochschild chain complex so that it possesses the structure of a strict S^1-complex and is quasi-isomorphic to the usual Hochschild complex $\text{CH}_*(\mathcal{A})$. This construction, known as the *non-unital Hochschild chain complex*, will be recalled below.

Let $\mathcal{A}$ be a cohomologically unital A_∞-algebra over $\mathbb{k}$. As a graded vector space, the non-unital Hochschild chain complex $\text{CH}_*^{\text{nu}}(\mathcal{A})$ consists of two copies of the ordinary Hochschild chain complex, with the grading of the second copy shifted down by 1, i.e.,

$$\text{CH}_*^{\text{nu}}(\mathcal{A}) := \text{CH}_*(\mathcal{A}) \oplus \text{CH}_*(\mathcal{A})[1]. \tag{3.9}$$

With respect to the decomposition (3.9), elements in the complex $\text{CH}_*^{\text{nu}}(\mathcal{A})$ can be written as $\check{\alpha} + \hat{\beta}$, where $\check{\alpha} \in \text{CH}_*(\mathcal{A})$ and $\hat{\beta} \in \text{CH}_*(\mathcal{A})[1]$. As a convention, we will refer to the left factor in $\text{CH}_*^{\text{nu}}(\mathcal{A})$ as the *check factor* and the right factor in $\text{CH}_*^{\text{nu}}(\mathcal{A})$ as the *hat factor*.

The differential b^{nu} on the complex $\text{CH}_*^{\text{nu}}(\mathcal{A})$ can therefore be expressed as a block matrix

$$b^{\text{nu}} := \begin{bmatrix} b & d_{\wedge\vee} \\ 0 & b' \end{bmatrix},$$

where

$$b(\breve{\alpha}) = \sum (-1)^{\maltese_1^k \cdot (1+\maltese_{k+1}^d) + \maltese_{k+1}^{d-1}+1} \mu^{d-i}(x_k \otimes \cdots \otimes x_1 \otimes x_d$$
$$\otimes x_{d-1} \otimes \cdots \otimes x_{k+i+1}) \otimes x_{k+i} \otimes \cdots \otimes x_{k+1}$$
$$+ \sigma(-1)^{\maltese_1^s} x_d \otimes \cdots \otimes \mu^j(x_{s+j+1} \otimes \cdots \otimes x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1$$

is the usual Hochschild differential on the check factor $\mathrm{CH}_*(\mathcal{A})$, with

$$\breve{\alpha} = x_d \otimes \cdots \otimes x_1,$$

b' is the *bar differential* on the hat factor defined by

$$b'(\hat{\beta}) = \sum (-1)^{\maltese_1^s} x_d \otimes \cdots \otimes x_{s+j+1} \otimes \mu_{\mathcal{A}}^j(x_{s+j} \otimes \cdots x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1$$
$$+ \sum (-1)^{\maltese_1^{d-j}} \mu_{\mathcal{A}}^j(x_d \otimes \cdots \otimes x_{d-j+1}) \otimes x_{d-j} \otimes \cdots \otimes x_1,$$

where $\hat{\beta} = x_d \otimes \cdots \otimes x_1$, and $d_{\wedge\vee} : \mathrm{CH}_*(\mathcal{A})[1] \to \mathrm{CH}_*(\mathcal{A})[1]$ is defined by

$$d_{\wedge\vee}(\hat{\beta}) := (-1)^{\maltese_2^d + \|x_1\| \cdot \maltese_2^d + 1} x_1 \otimes x_d \otimes \cdots \otimes x_2 + (-1)^{\maltese_1^{d-1}} x_d \otimes \cdots \otimes x_1.$$

In the above, we have followed the convention of [61]; in particular, the symbol

$$\maltese_i^j := \sum_{k=i}^j \|x_k\|$$

is used to abbreviate the signs, where $\|x_k\| := |x_k| - 1$ is the reduced grading.

Remark 22. The idea of non-unital Hochschild complex also appears in the geometric context, for example, in the Legendrian surgery description of the symplectic cohomology [8].

The natural inclusion $\mathrm{CH}_*(\mathcal{A}) \hookrightarrow \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$ is a quasi-isomorphism since the quotient complex is acyclic; see [28, Lemma 2]. Associated to $\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$ there is a corresponding non-unital version of the (chain-level representative of) Connes's operator $B^{\mathrm{nu}} : \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \to \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})[1]$, which is defined explicitly by

$$B^{\mathrm{nu}}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1)$$
$$:= \sum_i (-1)^{\maltese_1^i \maltese_{i+1}^k + \|x_k\| + \maltese_1^k + 1}(0, x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1}). \quad (3.10)$$

Note that we can write $B^{\mathrm{nu}} = s^{\mathrm{nu}} N$, where

$$s^{\mathrm{nu}}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1) := (-1)^{\maltese_1^k + \|x_k\| + 1}(0, x_k \otimes \cdots \otimes x_1),$$

and

$$N(x_k \otimes \cdots \otimes x_1) := (1 + \lambda + \cdots + \lambda^{k-1})(x_k \otimes \cdots \otimes x_1)$$

is the *norm* of the *cyclic permutation operator*

$$\lambda(x_k \otimes \cdots \otimes x_1) := (-1)^{\|x_1\| \cdot \maltese_2^k + \|x_1\| + \|x_k\|} x_1 \otimes x_k \otimes \cdots \otimes x_2. \qquad (3.11)$$

One can verify that $(B^{\mathrm{nu}})^2 = 0$ and $b^{\mathrm{nu}} B^{\mathrm{nu}} + B^{\mathrm{nu}} b^{\mathrm{nu}} = 0$, which is shown in the following lemma.

Lemma 23 ([28, Lemma 3]). $\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$ *is a strict S^1-complex.*

When $\mathcal{A}$ is strictly unital, there is an S^1-equivariant enhancement of the natural inclusion $\mathrm{CH}_*(\mathcal{A}) \hookrightarrow \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$, which is a quasi-isomorphism of S^1-complexes. Again, one can package everything in the u-linear model and define the equivariant differential on $\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A})$ as

$$b_{\mathrm{eq}} := b^{\mathrm{nu}} + u B^{\mathrm{nu}}.$$

The positive, negative, and periodic cyclic homologies of a cohomologically unital A_∞-algebra $\mathcal{A}$ are then defined, respectively, as the homologies of the following complexes:

$$\mathrm{CC}_*(\mathcal{A}) := (\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \otimes_\mathbb{K} \mathbb{K}((u))/u\mathbb{K}[\![u]\!], b_{\mathrm{eq}}),$$
$$\mathrm{CC}_*^-(\mathcal{A}) := (\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \otimes_\mathbb{K} \mathbb{K}[\![u]\!], b_{\mathrm{eq}}),$$
$$\mathrm{CC}_*^{\mathrm{per}} := (\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{A}) \otimes_\mathbb{K} \mathbb{K}((u)), b_{\mathrm{eq}}).$$

4. Parametrized Floer theory

As the general geometric setup of this paper, (M, θ_M) will be a $2n$-dimensional Liouville manifold. In order to have $\mathbb{Z}$-gradings on various Floer cochain complexes, we assume that $c_1(M) = 0$ and will in fact fix the choice of a trivialization of the canonical bundle K_M.

4.1. Equivariant symplectic cohomology

We recall the definition of the S^1-equivariant symplectic cohomology $\mathrm{SH}_{S^1}^*(M)$, whose construction is sketched in [60] and later carried out in detail in [11]. We will actually follow closely the general framework of [28], which has the advantage of being coordinate free. The idea behind the construction will also be used in Appendix A to define higher-order operations involving the pair-of-pants surface.

The construction of $\mathrm{SH}_{S^1}^*(M)$ is more involved than its non-equivariant version $\mathrm{SH}^*(M)$ in the sense that, besides the Floer differential $\delta_0 = d$, there is now

a sequence of higher-order corrections δ_k, $k \geq 1$, whose definitions make use of parametrized moduli spaces. We now recall the definition of the domains of these parametrized maps.

Definition 24 ([28, Definition 7]). A k-point angle-decorated cylinder consists of a cylinder

$$Z = \mathbb{R} \times S^1,$$

together with a collection of auxiliary marked points $p_1, \ldots, p_k \in Z$ such that their $s \in \mathbb{R}$ coordinates $(p_i)_s$, $1 \leq i \leq k$, satisfy

$$(p_1)_s \leq \cdots \leq (p_k)_s. \tag{4.1}$$

We call these coordinates the heights of the marked points and denote them by

$$h_i := (p_i)_s, \quad i = 1, \ldots, k.$$

Similarly, the $t \in S^1$ coordinates $(p_i)_t$, $1 \leq i \leq k$ of the marked points are called angles, and we introduce the notations

$$\theta_i := (p_i)_t, \quad i = 1, \ldots, k.$$

Denote by $\mathcal{M}_k$ the moduli space of k-point angle-decorated cylinders, modulo translation in the s-direction. Given a marked cylinder $(Z, p_1, \ldots, p_k)$ representing an element of $\mathcal{M}_k$, for a fixed constant $\xi > 0$, define positive and negative cylindrical ends

$$\varepsilon^+ : [0, \infty) \times S^1 \to Z \quad \text{and} \quad \varepsilon^- : (-\infty, 0] \times S^1 \to Z \tag{4.2}$$

by

$$\varepsilon^+(s, t) = (s + h_k + \xi, t) \quad \text{and} \quad \varepsilon^-(s, t) = (s - h_1 + \xi, t + \theta_1), \tag{4.3}$$

respectively. Note that the negative cylindrical end involves a twist by θ_1. If we regard Z as a sphere S^2 with two punctures ζ_{in}, which is the input, and ζ_{out}, which is the output, the parametrizations of ε^+ and ε^- in the t-component can be equivalently described by putting *asymptotic markers* ℓ_{in} and ℓ_{out} at ζ_{in} and ζ_{out}, respectively. These are half-lines in the real projectivized tangent spaces $\mathbb{P} T_{\zeta_{\text{in}}} S^2$ and $\mathbb{P} T_{\zeta_{\text{out}}} S^2$. In the present situation, ℓ_{in} is pointing along the arc $\{\varepsilon^+(s, 0)\}$, while ℓ_{out} is pointing toward p_1, the closest auxiliary marked point to ζ_{out}. Note that there is a non-canonical isomorphism $\mathcal{M}_k \cong (S^1)^k \times [0, 1)^{k-1}$.

$\mathcal{M}_k$ can be compactified to a manifold with corners $\overline{\mathcal{M}}_k$ by adding broken k-point angle-decorated cylinders, by which we mean

$$\bigsqcup_s \bigsqcup_{j_1, \ldots, j_s; j_i > 0, \sum j_i = k} \mathcal{M}_{j_1} \times \cdots \times \mathcal{M}_{j_s}.$$

In particular, the codimension 1 boundary of $\bar{\mathcal{M}}_k$ is covered by the images of the natural inclusions

$$\bar{\mathcal{M}}_{k-j} \times \bar{\mathcal{M}}_j \hookrightarrow \partial\bar{\mathcal{M}}_k, \quad 0 < j < k, \tag{4.4}$$

$$\bar{\mathcal{M}}_k^{i,i+1} \hookrightarrow \partial\bar{\mathcal{M}}_k, \tag{4.5}$$

where $\bar{\mathcal{M}}_k^{i,i+1}$ is the compactification of the locus where $h_i = h_{i+1}$. On $\mathcal{M}_k^{i,i+1}$, there is a forgetful map

$$\pi_i : \mathcal{M}_k^{i,i+1} \to \mathcal{M}_{k-1} \tag{4.6}$$

which remembers only the first one of the angles of the interior marked points with coincident heights. Since π_i is compatible with the choices of cylindrical ends $\varepsilon^\pm$ specified by (4.3), it extends to a map $\bar{\pi}_i : \bar{\mathcal{M}}_k^{i,i+1} \to \bar{\mathcal{M}}_{k-1}$ defined on the compactifications.

In order to write down the Floer equations for the parametrized maps, we need to introduce Floer data on the domain cylinders. We start by specifying the sets of Hamiltonian functions and almost complex structures to work with. We say that a time-dependent Hamiltonian $H_t : S^1 \times M \to \mathbb{R}$ is *admissible* if $H_t = H + F_t$ is the sum of an autonomous Hamiltonian $H : M \to \mathbb{R}$ which is quadratic at infinity, namely,

$$H(r, y) = r^2$$

on the cylindrical end $[r_0, \infty) \times \partial\bar{M}$, where $r \in (1, \infty)$ is the radial coordinate and $r_0 \gg 1$, and a time-dependent perturbation $F_t : S^1 \times M \to \mathbb{R}$. We require that, on $M \setminus \bar{M}$, we have that, for any $r_1 \gg 1$, there exists an $r > r_1$ such that F_t vanishes in a neighborhood of the hypersurface $\{r\} \times \partial\bar{M} \subset M$. For instance, one can take F_t to be a function supported near non-constant orbits of the Hamiltonian vector field X_H, where it is modeled on a Morse function on S^1.

Denote by $\mathcal{H}(M)$ the set of admissible Hamiltonians H_t on M such that all 1-periodic orbits of the Hamiltonian vector field X_{H_t} are non-degenerate, and write $\mathcal{O}_M$ for the set of 1-periodic orbits of X_{H_t}. For an orbit $y \in \mathcal{O}_M$, we define its degree to be $\deg(y) := n - \mathrm{CZ}(y)$, where $\mathrm{CZ}(y)$ is the Conley–Zehnder index of y. With these data, one can define a $\mathbb{Z}$-graded Floer cochain complex of H_t, which is called the *symplectic cochain complex*. It will be denoted by $\mathrm{SC}^*(M)$, with its degree i piece given by

$$\mathrm{SC}^i(M) := \bigoplus_{y \in \mathcal{O}_M, \deg(x)=i} |o_y|_{\mathbb{K}},$$

where $|o_y|_{\mathbb{K}}$ is the $\mathbb{K}$-normalization of the orientation line o_y defined via index theory.

Remark 25. For convenience, we will often choose generators of the orientation lines o_y associated to each Hamiltonian orbit and denote them by y in a slight abuse of

notation. The same convention applies to orientation lines associated to Hamiltonian chords when dealing with open-string invariants.

Let J_t be a time-dependent almost complex structure on M; we say that it is *of weak contact type* on the conical end if there is a sequence $\{r_i\}$ with $\lim_{i \to \infty} r_i = \infty$ so that near $\{r_i\} \times \partial \bar{M}$ we have $dr \circ J_t = -\theta_M$. Denote by $\mathcal{J}(M)$ the set of $d\theta_M$-compatible almost complex structures on M which are of weak contact type on the conical end. Recall that the usual Floer differential $d : SC^*(M) \to SC^{*+1}(M)$ is defined by counting rigid J_t-holomorphic cylinders $u : Z \to M$ with asymptotics at some Hamiltonian orbits $y^+, y^- \in \mathcal{O}_M$. This can be regarded as the special case $k = 0$ of the operations δ_k on the complex $SC^*(M)$ defined below.

Definition 26 ([28]). A Floer datum for a k-point angle-decorated cylinder $(Z, p_1, \ldots, p_k)$ consists of the following:

- choices of positive and negative cylindrical ends for Z, as in (4.2),

- a 1-form $\alpha_Z = dt$ on Z,

- a surface-dependent Hamiltonian function $H_Z : Z \to \mathcal{H}(M)$ which satisfies

$$(\varepsilon^\pm)^* H_Z = \lambda_\pm H_t$$

  for some $\lambda_- > \lambda_+ > 0$, where $H_t \in \mathcal{H}(M)$ is some fixed choice of an admissible Hamiltonian;

- a surface-dependent almost complex structure $J_Z : Z \to \mathcal{J}(M)$ such that

$$(\varepsilon^\pm)^* J_Z = J_t$$

  for some fixed choice of $J_t \in \mathcal{J}(M)$.

Remark 27. In the definition of Floer datum, we require the domain-dependent Hamiltonian H_Z to be conformally equivalent, instead of being equivalent on the two cylindrical ends. This is due to the fact that, in order for the maximum principle to hold in the definition of the equivariant symplectic cohomology, we need an extra condition $\partial_s H_Z \le 0$. This fact was not taken into account in [2, Appendix B], as the energy computations there leaves out the derivatives of the Hamiltonian with respect to the domain.

Universal and consistent choices of Floer data over the moduli spaces $\bar{\mathcal{M}}_k$ for all $k \ge 1$ can be constructed in an inductive way. In our specific situation, this means the following:

- At the boundary strata (4.4), the choices of Floer data should coincide with the product of the Floer data chosen on lower-dimensional moduli spaces. Moreover, the choices vary smoothly with respect to the gluing charts.

- At a boundary stratum of the form (4.5), the Floer datum chosen for a representative $(Z, p_1, \ldots, p_k)$ of an element of $\bar{M}_k^{i,i+1}$ coincides with the one pulled back from the corresponding element of $\bar{M}_{k-1}$ via the forgetful map $\bar{\pi}_i$.

Fix a universal and consistent choice of Floer data, whose existence is ensured by an inductive argument. For any pair of orbits $y^+, y^- \in \mathcal{O}_M$ and any integer $k \geq 1$, we introduce the moduli space $\mathcal{M}_k(y^+; y^-)$ of pairs

$$((Z, p_1, \ldots, p_k), u),$$

where $(Z, p_1, \ldots, p_k) \in \mathcal{M}_k$ and $u : Z \to M$ is a map which satisfies Floer equation

$$(du - X_{H_Z} \otimes dt)^{0,1} = 0,$$

where the $(0, 1)$-part is taken with respect to the domain-dependent almost complex structure J_Z chosen above as part of our Floer data. Additionally, u is required to satisfy the asymptotic conditions

$$\lim_{s \to \pm\infty} (\varepsilon^\pm)^* u(s, \cdot) = y^\pm.$$

The boundary of the Gromov compactification $\bar{\mathcal{M}}_k(y^+; y^-)$ is covered by the images of the natural inclusions

$$\bar{\mathcal{M}}_j(y; y^-) \times \bar{\mathcal{M}}_{k-j}(y^+; y) \hookrightarrow \partial\bar{\mathcal{M}}_k(y^+; y^-),$$
$$\bar{\mathcal{M}}_k^{i,i+1}(y^+; y^-) \hookrightarrow \partial\bar{\mathcal{M}}_k(y^+; y^-),$$

which come from the boundary strata of $\bar{\mathcal{M}}_k$, along with the strata coming from the usual semi-stable strip breaking

$$\bar{\mathcal{M}}_k(y; y^-) \times \bar{\mathcal{M}}(y^+; y) \hookrightarrow \partial\bar{\mathcal{M}}_k(y^+; y^-),$$
$$\bar{\mathcal{M}}(y; y^-) \times \bar{\mathcal{M}}_k(y^+; y) \hookrightarrow \partial\bar{\mathcal{M}}_k(y^+; y^-).$$

For generic choices of Floer data, the moduli spaces $\bar{\mathcal{M}}_k(y^+; y^-)$ are compact manifolds with corners of dimension

$$\deg(y^+) - \deg(y^-) + 2k - 1.$$

Let $((Z, p_1, \ldots, p_k), u)$ be a rigid element of $\bar{\mathcal{M}}_k(y^+; y^-)$, so we have

$$\deg(y^+) = \deg(y^-) - 2k + 1.$$

In this case, there are natural isomorphisms

$$\mu_u : o_{y^-} \to o_{y^+}$$

between the orientation lines defined via index theory. One defines the operation

$$\delta_k : \mathrm{SC}^*(M) \to \mathrm{SC}^{*-2k+1}(M)$$

by a signed count of rigid elements of the moduli spaces $\bar{\mathcal{M}}_k(y^+; y^-)$ for varying asymptotics y^+ and y^-. Our choices of Floer data ensures that the elements of $\bar{\mathcal{M}}_k^{i,i+1}(y^+, y^-)$ are never rigid, from which the identity

$$\sum_{i=0}^{k} \delta_i \delta_{k-i} = 0$$

follows; see [28, Lemma 9] for details. This shows that $(\mathrm{SC}^*(M), \{\delta_k\}_{k\geq 0})$ is an S^1-complex (usually not strict) in the sense of Definition 18.

As in (3.4), the S^1-*equivariant symplectic cohomology* $\mathrm{SH}^*_{S^1}(M)$ is defined to be the cohomology of the complex

$$(\mathrm{SC}^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!], \delta_{\mathrm{eq}}),$$

where $\delta_{\mathrm{eq}} := \sum_{k=0}^{\infty} \delta_k u^k$ and u is a formal variable of degree 2. Note that this is the $\mathcal{P}_{hS^1}$ defined in (3.6).

One can replace the autonomous Hamiltonian H in the above construction (which is quadratic on the conical end) with a Hamiltonian which is linear at infinity, i.e.,

$$H_\lambda(r, y) = \lambda r + C \tag{4.7}$$

on $[r_0, \infty) \times \partial \bar{M}$, where $\lambda \in \mathbb{R}$ and C is some constant, and consider its time-dependent perturbation $H_{\lambda,t} = H_\lambda + F_t$ as above so that all the time-1 orbits of $X_{H_{\lambda,t}}$ are non-degenerate. The space of such Hamiltonians $H_{\lambda,t} : S^1 \times M \to \mathbb{R}$ with varying λ will be denoted by $\mathcal{H}_\ell(M)$, and we have

$$\mathcal{H}_\ell(M) = \bigcup_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda(M),$$

where $\mathcal{H}_\lambda(M)$ is the space of Hamiltonians of the form $H_{\lambda,t}$ with fixed slope λ at infinity. We also require that $\lambda \notin \mathcal{P}_M$, where $\mathcal{P}_M \subset \mathbb{R}$ is the collection of those λ such that the set of 1-periodic orbits $\mathcal{O}_{M,\lambda}$ of $X_{H_{\lambda,t}}$ is not contained in any compact subset of M. The same construction as above gives rise to the S^1-equivariant Floer cochain complex

$$\mathrm{CF}^*_{S^1}(M, \lambda) := (\mathrm{CF}^*(M, \lambda) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!], \delta_{\mathrm{eq}}), \tag{4.8}$$

where $\mathrm{CF}^*(M, \lambda)$ is the Floer cochain complex of $H_{\lambda,t}$. When it is clear from the context which manifold we are working with, we will simply write these Floer complexes as $\mathrm{CF}^*_{S^1}(\lambda)$ and $\mathrm{CF}^*(\lambda)$. The cohomology of (4.8) will be denoted by $\mathrm{HF}^*_{S^1}(\lambda)$. Just

as in the non-equivariant case, for $\lambda_1 < \lambda_2$, one can build equivariant continuation maps

$$\kappa_{\text{eq}}^{\lambda_1,\lambda_2} := \sum_{k=0}^{\infty} \kappa_k^{\lambda_1,\lambda_2} u^k : \text{CF}_{S^1}^*(\lambda_1) \to \text{CF}_{S^1}^*(\lambda_2) \tag{4.9}$$

via counting 1-parameter families of k-point angle decorated cylinders equipped with monotone homotopies connecting elements of $\mathcal{H}_{\lambda_1}(M)$ and $\mathcal{H}_{\lambda_2}(M)$. The maps $\{\kappa_k^{\lambda_1,\lambda_2}\}$ constitute an S^1-complex homomorphism between $\text{CF}^*(\lambda_1)$ and $\text{CF}^*(\lambda_2)$. Passing to direct limit yields an alternative definition of the equivariant symplectic cohomology

$$\text{SH}_{S^1}^*(M) := \varinjlim_\lambda \text{HF}_{S^1}^*(\lambda).$$

This approach of defining $\text{SH}_{S^1}^*(M)$ is carried out in detail in [11]. Note that, for the purpose of defining $\text{SH}_{S^1}^*(M)$, one can assume that $\lambda > 0$. However, the flexibility of allowing negative λ will be needed for our later purposes; see Section 5.4.

Returning to our original setup, one can arrange so that $H_t \in \mathcal{H}(M)$ is a C^2-small and time-independent Morse function in the interior M^{in} of $\bar{M}$; this produces a copy of the Morse complex $\text{CM}^*(H_t)$, which sits in $\text{SC}^*(M)$ as a subcomplex. When passing to cohomology, the inclusion

$$\text{CM}^*(H_t) \hookrightarrow \text{SC}^*(M)$$

induces the classical PSS map

$$\text{PSS} : H^*(M; \mathbb{K}) \to \text{SH}^*(M). \tag{4.10}$$

Since the homotopy S^1-action on the symplectic cochain complex $\text{SC}^*(M)$ comes from reparametrizing the Hamiltonian orbits, one may expect that it becomes trivial when restricted to the subcomplex $\text{CM}^*(H_t)$; i.e., all the operations $\delta_k, k \geq 1$ should vanish on it. This is indeed the case with appropriate choices of Floer data over the moduli spaces $\bar{\mathcal{M}}_k$. As a consequence, the natural inclusion $\text{CM}^*(H_t) \hookrightarrow \text{SC}^*(M)$ can be regarded as an S^1-complex homomorphism, which induces the S^1-*equivariant PSS map*

$$\widetilde{\text{PSS}} : H^*(M; \mathbb{K}((u))/u\mathbb{K}[\![u]\!]) \to \text{SH}_{S^1}^*(M) \tag{4.11}$$

after passing to cohomologies.

As a variant, one can also use a $H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ which is C^2-small and Morse in M^{in}, which will then give rise to maps

$$\text{PSS}_\lambda : H^*(M; \mathbb{K}) \to \text{HF}^*(\lambda), \tag{4.12}$$

$$\widetilde{\text{PSS}}_\lambda : H^*(M; \mathbb{K}((u))/u\mathbb{K}[\![u]\!]) \to \text{HF}_{S^1}^*(\lambda). \tag{4.13}$$

The maps (4.10) and (4.11) can then be thought of as (4.12) and (4.13) composed with the continuation maps and equivariant continuation maps, respectively.

4.2. Cyclic dilations

We consider in this subsection the geometric counterpart of an exact Calabi–Yau structure on a homologically smooth A_∞-category. The precise relationship between these two notions will be established in the next subsection, using Ganatra's construction of the cyclic open-closed string map.

Take any $\lambda \in [0, \infty]$ with $\lambda \notin \mathcal{P}_M$ if $\lambda < \infty$. Recall that the S^1-equivariant Hamiltonian Floer cohomology $\mathrm{HF}^*_{S^1}(\lambda)$ fits into the following Gysin-type long exact sequence:

$$\cdots \to \mathrm{HF}^{*-1}(\lambda) \xrightarrow{\mathbf{I}} \mathrm{HF}^{*-1}_{S^1}(\lambda) \xrightarrow{\mathbf{S}} \mathrm{HF}^{*+1}_{S^1}(\lambda) \xrightarrow{\mathbf{B}} \mathrm{HF}^*(\lambda) \to \cdots ; \qquad (4.14)$$

see [9–11] for a detailed discussion. The BV operator

$$\Delta = [\delta_1] : \mathrm{HF}^*(\lambda) \to \mathrm{HF}^{*-1}(\lambda)$$

coincides on the cohomology level with the composition $\mathbf{B} \circ \mathbf{I}$ (in the non-trivial order). When $M = T^*Q$ is the cotangent bundle of a compact Spin manifold Q and $\lambda = \infty$, the above Gysin exact sequence (4.14) reduces to the well-known long exact sequence in string topology [14]

$$\cdots \to H_{*+1}(\mathcal{L}Q; \mathbb{K}) \xrightarrow{\mathbf{I}} H^{S^1}_{*+1}(\mathcal{L}Q; \mathbb{K}) \xrightarrow{\mathbf{S}} H^{S^1}_{*-1}(\mathcal{L}Q; \mathbb{K}) \xrightarrow{\mathbf{B}} H_*(\mathcal{L}Q; \mathbb{K}) \to \cdots , \tag{4.15}$$

where the maps $\mathbf{I}$ and $\mathbf{B}$ are known as the *erasing map* and the *marking map*, respectively. Because of this, the same terminology will be used to refer to $\mathbf{I}$ and $\mathbf{B}$ in the general case.

Recall that, in string topology, the map $\mathbf{B}$ is defined as follows: by definition, an equivariant homology class $\gamma \in H^{S^1}_{*-1}(\mathcal{L}Q; \mathbb{K})$ can be equivalently realized as a homology class of $H_{*-1}(S^\infty \times_{S^1} \mathcal{L}Q; \mathbb{K})$, whose lift in $H_*(S^\infty \times \mathcal{L}Q; \mathbb{K})$ will be denoted by $\widetilde{\gamma}$. Let Π be the trivial projection $S^\infty \times \mathcal{L}Q \to \mathcal{L}Q$; then $\mathbf{B}(\gamma) = \Pi_*(\widetilde{\gamma})$.

Since the natural inclusion $\mathrm{CM}^*(H_t) \hookrightarrow \mathrm{SC}^*(M)$ is an S^1-complex homomorphism, by Proposition 21 the long exact sequence (4.14) is compatible with the ordinary and equivariant PSS maps (cf. (4.12) and (4.13)), which leads to the following commutative diagram:

$$
\begin{array}{ccccccccc}
\cdots & \to & H^{*-1}(M; \mathbb{K}) & \to & H^{*-1}_{S^1}(M; \mathbb{K}) & \to & H^{*+1}_{S^1}(M; \mathbb{K}) & \to & H^*(M; \mathbb{K}) & \to & \cdots \\
& & \downarrow{\scriptstyle \mathrm{PSS}_\lambda} & & \downarrow{\scriptstyle \widetilde{\mathrm{PSS}}_\lambda} & & \downarrow{\scriptstyle \widetilde{\mathrm{PSS}}_\lambda} & & \downarrow{\scriptstyle \mathrm{PSS}_\lambda} & & \\
\cdots & \longrightarrow & \mathrm{HF}^{*-1}(\lambda) & \xrightarrow{\mathbf{I}} & \mathrm{HF}^{*-1}_{S^1}(\lambda) & \xrightarrow{\mathbf{S}} & \mathrm{HF}^{*+1}_{S^1}(\lambda) & \xrightarrow{\mathbf{B}} & \mathrm{HF}^*(\lambda) & \longrightarrow & \cdots
\end{array}
$$

$$\tag{4.16}$$

where the upper row is the usual Gysin long exact sequence for the trivial S^1-action on M, so in particular, there is an isomorphism

$$H^*_{S^1}(M; \mathbb{K}) \cong H^*(M; \mathbb{K}((u))/u\mathbb{K}[\![u]\!]).$$

We proceed to give a chain-level interpretation of the condition (1.3) that appeared in the introduction. To do this, we need to find the chain-level expression of the coboundary map $\mathbf{B} : \mathrm{HF}^1_{S^1}(\lambda) \to \mathrm{HF}^0(\lambda)$ in the Gysin sequence (4.14). Write a degree 1 cocycle $\widetilde{\beta}$ in the S^1-equivariant Hamiltonian Floer cochain complex $\mathrm{CF}^*(\lambda) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!]$ as $\sum_{k=0}^{\infty} \beta_k \otimes u^{-k}$, where $\beta_k \in \mathrm{CF}^{2k+1}(\lambda)$ and only finitely many terms in the infinite sum are non-zero. If we use $\mathbf{B}_c$ to denote the underlying chain-level map of the marking map $\mathbf{B}$, standard diagram chasing argument enables us to find that (see [11, Proposition 2.9])

$$\mathbf{B}_c\left(\sum_{k=0}^{\infty} \beta_k \otimes u^{-k}\right) = \sum_{k=0}^{\infty} \delta_{k+1}(\beta_k).$$

Definition 28. A cyclic dilation is a cohomology class $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ whose representing cocycle $\widetilde{\beta} \in \mathrm{SC}^1_{S^1}(M)$ consists of a sequence of odd-degree Floer cochains $\{\beta_k\}_{k \geq 0}$ with $\beta_k \in \mathrm{SC}^{2k+1}(M)$, and $\beta_k \neq 0$ for only finitely many k so that the cocycle $\sum_{k=0}^{\infty} \delta_{k+1}(\beta_k)$, after passing to cohomology, defines an invertible element $h \in \mathrm{SH}^0(M)^{\times}$.

By definition, there exists a $\lambda \in \mathbb{R}_{>0} \setminus \mathcal{P}_M$ so that $\widetilde{\beta}$ lies in the image of the equivariant continuation map (cf. (4.9))

$$\kappa^{\lambda,\infty}_{\mathrm{eq}} : \mathrm{CF}^1_{S^1}(\lambda) \to \mathrm{CF}^1_{S^1}(\infty) := \mathrm{SC}^1_{S^1}(M).$$

By slight abuse of notation, the preimage $(\kappa^{\lambda,\infty}_{\mathrm{eq}})^{-1}(\widetilde{\beta}) \in \mathrm{CF}^1_{S^1}(\lambda)$ will still be denoted by $\widetilde{\beta}$, and we will often refer to its cohomology class $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$ as a cyclic dilation.

Let $\lambda \gg 0$ be sufficiently large. We consider an important special case of the above definition, namely, when $h = 1$ is the identity of $\mathrm{SH}^0(M)$. It follows from the exactness of (4.14) that there is a cohomology class $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$ satisfying $\mathbf{B}(\tilde{b}) = 1$ if and only if $1 \in \mathrm{HF}^0(\lambda)$ vanishes under the erasing map $\mathbf{I} : \mathrm{HF}^0(\lambda) \to \mathrm{HF}^0_{S^1}(\lambda)$.

In view of the commutative diagram (4.16), this is precisely the case when the image of the (locally finite) fundamental class $1 \in H^0(M; \mathbb{K})$ vanishes under the composition

$$H^*(M; \mathbb{K}) \hookrightarrow H^*(M; \mathbb{K}((u))/u\mathbb{K}[\![u]\!]) \xrightarrow{\widetilde{\mathrm{PSS}_\lambda}} \mathrm{HF}^*_{S^1}(\lambda). \qquad (4.17)$$

On the other hand, it follows from [77, Lemma 4.2.4] that if M admits a dilation in $\mathrm{HF}^1(\lambda)$, i.e., a class $b \in \mathrm{HF}^1(\lambda)$ which becomes a dilation in $\mathrm{SH}^1(M)$ under the continuation map

$$\kappa^{\lambda,\infty} : \mathrm{CF}^1(\lambda) \to \mathrm{CF}^1(\infty) := \mathrm{SC}^1(M),$$

then $1 \in H^0(M; \mathbb{K})$ lies in the kernel of (4.17). This observation enables us to relate cyclic dilations to the following notion introduced by Zhao [77].

Definition 29 ([77, Definition 4.2.1]). We say that a Liouville manifold M admits a higher dilation if the identity $1 \in H^*(M; \mathbb{K})$ lies in the kernel of the localized equivariant PSS map

$$\widehat{\mathrm{PSS}} : H^*(M; \mathbb{K}((u))) \to \widehat{\mathrm{PSH}}^*(M),$$

where $\widehat{\mathrm{PSH}}^*(M)$ is the completed periodic symplectic cohomology, which is the cohomology of the Tate complex (cf. (3.5))

$$(\mathrm{SC}^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u)), \delta_{\mathrm{eq}}).$$

By [77, Lemma 4.2.5], a higher dilation can be equivalently interpreted using the equivariant PSS map (4.11). Precisely, M admits a higher dilation if and only if $1 \otimes u^{-k} \in H^*(M; \mathbb{K}((u))/u\mathbb{K}[\![u]\!])$ lies in the kernel of the map $\widetilde{\mathrm{PSS}}$ (cf. (4.11)) for every $k \geq 0$. In view of the above discussions, we get the following proposition.

Proposition 30. *If M admits a higher dilation, then it admits a cyclic dilation with $h = 1$.*

Remark 31. In fact, it is an observation made in [67, Remark 6.5] that the existence of a dilation in $\mathrm{HF}^1(\lambda)$ is equivalent to the existence of a cocycle $\beta_0 \in \mathrm{CF}^1(\lambda)$ and a cochain $\beta_{-1} \in \mathrm{CF}^{-1}(\lambda)$ so that $\delta_{\mathrm{eq}}(\beta_{-1} + \beta_0 \otimes u^{-1}) = e$, where $e \in \mathrm{CF}^0(\lambda)$ is the chain-level representative of the identity. It is therefore natural to consider a sequence of cochains $\{\beta_j\}_{j \geq -1}$ with arbitrary length, where $\beta_j \in \mathrm{CF}^{2j+1}(\lambda)$, so that

$$\delta_{\mathrm{eq}}\left(\sum_{j=-1}^{\infty} \beta_j \otimes u^{-j-1} \right) = e, \tag{4.18}$$

where only finitely many β_j can be non-zero. Our discussions above show that (4.18) holds for $\lambda \gg 0$ if and only if M admits a cyclic dilation with $h = 1$. On the other hand, the notion of a higher dilation imposes the additional restriction that $e \otimes u^{-k}$ need to be coboundaries in the complexes $\mathrm{CF}_{S^1}^{-2k}(\lambda)$.

This observation enables us to get some first examples of Liouville manifolds with cyclic dilations. For any closed manifold Q, one can consider the classifying map $f : Q \to B\pi_1(Q)$ for its universal cover. Q is called *rationally inessential* if the fundamental class $[Q] \in H_n(Q; \mathbb{Q})$ vanishes under the pushforward

$$f_* : H_n(Q; \mathbb{Q}) \to H_n(B\pi_1(Q); \mathbb{Q}).$$

In particular, every simply connected closed manifold is rationally inessential. It follows from [77, Corollary 1.1.6] and Proposition 30 stated above that, for any rationally inessential manifold Q, $M = T^*Q$ admits a cyclic dilation over $\mathbb{Q}$. It is, however,

not clear whether such a cotangent bundle admits a quasi-dilation. In fact, it is even unknown whether T^*Q admits a dilation over $\mathbb{Q}$ for any simply connected *formal* manifold Q; see [56, Lecture 18]. More interesting examples of Liouville manifolds which admit cyclic dilations will be established in Section 6.

Remark 32. Related notions are introduced in [79], where the author considers the spectral sequence associated to the u-adic filtration on the equivariant Floer cochain complex $CF^*_{S^1}(\lambda)$, and M is said to admit a k-dilation if, for sufficiently large $\lambda \notin \mathcal{P}_M$, the identity $e \in CF^0(\lambda)$ is killed in the $(k+1)$-th page of the spectral sequence. In particular, any flexible Weinstein manifold admits a 0-dilation, and a dilation in the sense of Seidel–Solomon [67] is a 1-dilation. In general, having a k-dilation for $k \geq 1$ is equivalent to requiring that $\sum_{j=0}^{k-1} \beta_j \otimes u^{-j} \in CF^1_{S^1}(\lambda)$ defines a cyclic dilation with $h = 1$.

4.3. Cyclic open-closed map

We briefly summarize the construction of the cyclic open-closed string map due to Ganatra [28]. Details can be found in [28, Section 5]. See also [30] for applications of the cyclic open-closed string map in the study of mirror symmetry of closed symplectic manifolds.

Roughly speaking, the cyclic open-closed map is a parametrized version of the usual open-closed map

$$OC : CH_*(\mathcal{W}(M)) \to SC^{*+n}(M) \tag{4.19}$$

considered in [2, 58], which keeps track of the S^1-complex structures on both sides.

However, as we have already noticed in Section 3.3, in order to keep track of the S^1-action on the open-string side, one needs to consider the non-unital Hochschild complex $CH^{nu}_*(\mathcal{W}(M))$ instead of the usual Hochschild complex $CH_*(\mathcal{W}(M))$. Thus, the first step toward the construction of an "S^1-equivariant enhancement" of the usual open-closed map OC would be to replace OC by a map

$$OC^{nu} : CH^{nu}_*(\mathcal{W}(M)) \to SC^{*+n}(M)$$

defined on the non-unital Hochschild complex. Following Ganatra, we will call OC^{nu} the *non-unital open-closed string map*. In view of the definition of $CH^{nu}_*(\mathcal{W}(M))$ recalled in Section 3.3, the map OC^{nu} should consist of the check component

$$\widecheck{OC} : CH_*(\mathcal{W}(M)) \to SC^{*+n}(M)$$

and the hat component

$$\widehat{OC} : CH_*(\mathcal{W}(M))[1] \to SC^{*+n}(M),$$

which act, respectively, on the check and hat factors of the non-unital Hochschild complex, and

$$\mathrm{OC}^{\mathrm{nu}}(\check{\alpha}, \widehat{\beta}) = \widetilde{\mathrm{OC}}(\check{\alpha}) + \widehat{\mathrm{OC}}(\widehat{\beta}),$$

where $\check{\alpha} \in \mathrm{CH}_*(\mathcal{W}(M))$ and $\widehat{\beta} \in \mathrm{CH}_*(\mathcal{W}(M))[1]$.

The map $\widetilde{\mathrm{OC}}$ is defined on the ordinary Hochschild chain complex, and its definition in fact coincides with the ordinary open-closed string map (4.19). Recall that OC is defined by considering closed disks $\bar{S}$ equipped with boundary marked points $\zeta_1, \ldots, \zeta_d \in \partial\bar{S}$, which serve as inputs, and an interior marked point ζ_{out}, which is an output. There is also an asymptotic marker ℓ_{out} at ζ_{out} pointing toward ζ_d. One can assign Floer data to such disks $(\bar{S}; \zeta_1, \ldots, \zeta_d; \zeta_{\mathrm{out}}, \ell_{\mathrm{out}})$ in the usual way, and when forming the moduli space of Floer trajectories, the boundary components of $\partial\bar{S} \setminus \{\zeta_1, \ldots, \zeta_d\}$ will be labeled with Lagrangian submanifolds $L_1, \ldots, L_d$ which are objects of the wrapped Fukaya category $\mathcal{W}(M)$ so that L_i is the label of the arc along the boundary between ζ_i and $\zeta_{i+1 \bmod d}$, and the marked points $\zeta_1, \ldots, \zeta_d$ are associated with asymptotics $x_1, \ldots, x_d$, which are time-1 chords of the Hamiltonian vector field X_{H_t} from L_{i-1} to $L_{i \bmod d}$ for some $H_t \in \mathcal{H}(M)$. For any $y_{\mathrm{out}} \in \mathcal{O}_M$, the coefficient before $|o_{y_{\mathrm{out}}}|_{\mathbb{K}}$ in $\widetilde{\mathrm{OC}}(|o_{x_d}|_{\mathbb{K}}, \ldots, |o_{x_1}|_{\mathbb{K}})$ is determined by a signed count of rigid Floer trajectories $u : S \to M$ which satisfy the relevant Floer equation, with boundary conditions determined by the Lagrangian labelings $(L_1, \ldots, L_d)$ and asymptotic conditions specified by $(\vec{x} := (x_d, \ldots, x_1); y_{\mathrm{out}})$.

The definition of the map $\widehat{\mathrm{OC}}$ differs from $\widetilde{\mathrm{OC}}$ in the sense that one now considers the same closed disks $(\bar{S}; \zeta_1, \ldots, \zeta_d; \zeta_{\mathrm{out}}, \ell_{\mathrm{out}})$ with d boundary marked points and an interior marked point as the domains, but the asymptotic marker ℓ_{out} is required to point between ζ_1 and ζ_d. The collection $(\vec{x}; y_{\mathrm{out}})$ of Hamiltonian chords and orbits as above still determines the asymptotic conditions for the corresponding Floer equation. Since the direction of ℓ_{out} remembers the position of ζ_f, its freedom to vary increases the degree of the map by 1, which explains why $\widehat{\mathrm{OC}}$ is a map defined on the shifted Hochschild chain complex $\mathrm{CH}_*(\mathcal{W}(M))[1]$.

Proposition 33 ([28, Lemma 11]). *The non-unital open-closed map*

$$\mathrm{OC}^{\mathrm{nu}} = \widetilde{\mathrm{OC}} \oplus \widehat{\mathrm{OC}}$$

is a chain map.

Consider the natural inclusion

$$\iota : \mathrm{CH}_*(\mathcal{W}(M)) \hookrightarrow \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{W}(M)),$$

whose composition with the non-unital open-closed map gives rise to a chain map

$$\mathrm{OC}^{\mathrm{nu}} \circ \iota : \mathrm{CH}_*(\mathcal{W}(M)) \to \mathrm{SC}^{*+n}(M),$$

which coincides on the chain-level with the usual open-closed string map OC. Since we have learned from Section 3.3 that ι is a quasi-isomorphism, it follows that, as homology level maps,

$$[\mathrm{OC}^{\mathrm{nu}}] = [\mathrm{OC}].$$

The cyclic open-closed string map $\widetilde{\mathrm{OC}}$ will be defined as an S^1-equivariant enhancement of $\mathrm{OC}^{\mathrm{nu}}$, by including higher cyclic chain homotopies. More precisely, it consists of a sequence of maps

$$\mathrm{OC}^k = \widetilde{\mathrm{OC}}^k \oplus \widehat{\mathrm{OC}}^k : \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{W}(M)) \to \mathrm{SC}^{*+n-2k}(M)$$

for each $k \geq 0$ such that

$$\widetilde{\mathrm{OC}}^0 = \widetilde{\mathrm{OC}}, \quad \widehat{\mathrm{OC}}^0 = \widehat{\mathrm{OC}},$$

and for any $k \geq 1$, we have

$$(-1)^n \sum_{i=0}^{k} \delta_i \circ \widetilde{\mathrm{OC}}^{k-i} = \widehat{\mathrm{OC}}^{k-1} \circ \mathbb{B}^{\mathrm{nu}} + \widetilde{\mathrm{OC}}^k \circ b, \tag{4.20}$$

$$(-1)^n \sum_{i=0}^{k} \delta_i \circ \widehat{\mathrm{OC}}^{k-i} = \widehat{\mathrm{OC}}^k \circ b' + \widetilde{\mathrm{OC}}^k \circ (1 - \lambda), \tag{4.21}$$

where $\mathbb{B}^{\mathrm{nu}} : \mathrm{CH}_*^{\mathrm{nu}}(\mathcal{W}(M)) \to \mathrm{CH}_{*-1}^{\mathrm{nu}}(\mathcal{W}(M))$ is the map (3.10) applied to the wrapped Fukaya category, and λ is the cyclic permutation operator defined in (3.11). Roughly speaking, the maps $\widetilde{\mathrm{OC}}^k$ and $\widehat{\mathrm{OC}}^k$ are defined in the same way as $\widetilde{\mathrm{OC}}$ and $\widehat{\mathrm{OC}}$, but with additional interior marked points $p_1, \ldots, p_k$ included in the respective domains, which are located near ζ_{out} and are *strictly radially ordered* in the sense that

$$0 < |p_1| < \cdots < |p_k| < \frac{1}{2} \tag{4.22}$$

with respect to the standard complex coordinate near ζ_{out}. Note that now the cylinders in the moduli spaces $\mathcal{M}_i$, where $0 \leq i \leq k$, break "below" the surface S at ζ_{out} instead of "above" it. As before, ℓ_{out} is required to point toward p_1 if $k > 0$.

Define

$$\widetilde{\mathrm{OC}} := \sum_{k=0}^{\infty} (\widetilde{\mathrm{OC}}^k \oplus \widehat{\mathrm{OC}}^k) u^k;$$

it follows from (4.20) and (4.21) the following.

Theorem 34 ([28, Theorem 1]). *The non-unital open-closed map* $\mathrm{OC}^{\mathrm{nu}}$ *admits a geometrically defined* S^1*-equivariant enhancement*

$$\widetilde{\mathrm{OC}} \in R\mathrm{Hom}_{S^1}(\mathrm{CH}_*^{\mathrm{nu}}(\mathcal{W}(M))[n], \mathrm{SC}^*(M)).$$

Combining [27, Theorem 1.1] with [28, Corollary 1], we have the following theorem.

Theorem 35 (Ganatra). *Let M be a Liouville manifold which is non-degenerate; then the homology level maps*

$$[OC] : HH_*(\mathcal{W}(M)) \to SH^{*+n}(M), \quad [\widetilde{OC}] : HC_*(\mathcal{W}(M)) \to SH^{*+n}_{S^1}(M)$$

are isomorphisms.

We can now fulfill our promise at the beginning of Section 4.2, namely, to explain the relationship between exact Calabi–Yau structures on $\mathcal{W}(M)$ and cyclic dilations.

Proposition 36. *For any Liouville manifold M, there is a commutative diagram*

$$
\begin{array}{ccc}
HC_{*+1}(\mathcal{W}(M)) & \xrightarrow{\;\mathbb{B}\;} & HH_*(\mathcal{W}(M)) \\
\scriptstyle[\widetilde{OC}]\downarrow & & \downarrow\scriptstyle[OC] \\
SH^{*+n+1}_{S^1}(M) & \xrightarrow{\;\mathbf{B}\;} & SH^{*+n}(M)
\end{array}
\qquad (4.23)
$$

where $\mathbb{B}$ is the cohomology level map associated to $\mathbb{B}^{\mathrm{nu}}$.

Proof. This is a direct consequence of Theorem 34 and Proposition 21. ∎

As a corollary, we get the following proof of Proposition 4.

Corollary 37. *Let M be a non-degenerate Liouville manifold; its wrapped Fukaya category $\mathcal{W}(M)$ is exact Calabi–Yau if and only if there exists a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$.*

Proof. Since M is non-degenerate, it follows from Theorem 35 that both of the maps $[OC]$ and $[\widetilde{OC}]$ in the commutative diagram (4.23) are isomorphisms. The corollary would then follow from the fact that $[\eta] \in HH_{-n}(\mathcal{W}(M))$ is non-degenerate if and only if its image under the open-closed map $[OC]$ is an invertible element $h \in SH^0(M)^\times$.

The proof of [28, Theorem 3] shows that the geometrically defined smooth Calabi–Yau structure

$$[\widetilde{\eta}_{\mathrm{std}}] \in HC^-_{-n}(\mathcal{W}(M))$$

induces a non-degenerate class $[\eta_{\mathrm{std}}] \in HH_{-n}(\mathcal{W}(M))$, which is mapped under $[OC]$ to the identity $1 \in SH^0(M)$. According to the definition of Calabi–Yau structures, any two such structures differ by an automorphism of the diagonal bimodule. In our case, any non-degenerate class $[\eta] \in HH_{-n}(\mathcal{W}(M))$ differs from $[\eta_{\mathrm{std}}]$ by an automorphism

of the diagonal bimodule, which is an invertible element of $\mathrm{HH}^0(\mathcal{W}(M))$. By [27, Theorem 1.1], its image under the inverse of the closed-open map

$$[\mathrm{CO}]^{-1} : \mathrm{HH}^*(\mathcal{W}(M)) \to \mathrm{SH}^*(M)$$

is an element $h \in \mathrm{SH}^0(M)^\times$, so is $[\mathrm{OC}]([\eta])$.

On the other hand, if $[\eta] \in \mathrm{HH}_{-n}(\mathcal{W}(M))$ is mapped to a class $h \in \mathrm{SH}^0(M)^\times$. By applying the closed-open map, we get a class $[\mathrm{CO}](h) \in \mathrm{HH}^0(\mathcal{W}(M))^\times$, which induces an automorphism of the diagonal bimodule. Composing this with the isomorphism $\mathcal{W}(M)^\vee[n] \cong \mathcal{W}(M)$ between the diagonal bimodule and its shifted dual induced by $[\eta_{\mathrm{std}}]$, we get another isomorphism $\mathcal{W}(M)^\vee[n] \xrightarrow{\cong} \mathcal{W}(M)$, which is induced by $[\eta]$. This shows that $[\eta]$ is non-degenerate. ∎

5. Lagrangian submanifolds

Let M be a Liouville manifold with $c_1(M) = 0$, and fix a trivialization of its canonical bundle K_M. We consider in this chapter the open-string implications of the existence of a cyclic dilation. To be precise, we will consider Lagrangian submanifolds in M which are objects of the compact Fukaya category $\mathcal{F}(M)$; namely, they satisfy the following assumption.

Assumption 38. $L \subset M$ *is closed, connected, exact, graded, and* Spin.

We will actually fix the choice of a grading on L so that the Lagrangian Floer cohomology $\mathrm{HF}^*(L_0, L_1)$ of two Lagrangian submanifolds $L_0, L_1 \subset M$ is well defined as a $\mathbb{Z}$-graded algebra over $\mathbb{K}$.

5.1. The Cieliebak–Latschev map

Let $L \subset M$ be an exact Lagrangian submanifold satisfying Assumption 38. As a consequence of the Viterbo functoriality [4, 76], we have a map

$$\mathrm{SH}^*(M) \to \mathrm{SH}^*(T^*L) \cong H_{n-*}(\mathscr{L}L; \nu), \tag{5.1}$$

where the latter isomorphism is established in [1] in the case when L is Spin and in [4] in the general case. Since we have required in Assumption 38 that L is Spin, the local system $\nu : \pi_1(\mathscr{L}L) \to \mathbb{K}$ can be dropped out from our notations.

There is an S^1-equivariant analog of (5.1) constructed by Cohen–Ganatra [18] (see also [77, Section 4.4.1] for a detailed exposition), which is an infinite sum

$$\widetilde{\mathrm{CL}} := \sum_{k=0}^\infty \mathrm{CL}_k u^k : \mathrm{SC}^*(M) \otimes_\mathbb{K} \mathbb{K}((u))/u\mathbb{K}[\![u]\!] \to C_{n-*}^\diamond\big(\mathscr{L}L; \mathbb{K}((u))/u\mathbb{K}[\![u]\!]\big)$$

whose degree 0 piece arises from relevant considerations by Cieliebak–Latschev in [16]. In the above, $C_{n-*}^{\Diamond}(\mathscr{L}L; \mathbb{K})$ is a quotient of the dg algebra $C_{n-*}(\mathscr{L}L; \mathbb{K})$ constructed by Cohen–Ganatra in [18, Appendix A.1]. See also [77, Appendix B]. It has the property that the projection

$$C_{n-*}(\mathscr{L}L; \mathbb{K}) \to C_{n-*}^{\Diamond}(\mathscr{L}L; \mathbb{K})$$

is a quasi-isomorphism, and $C_{n-*}^{\Diamond}(\mathscr{L}L; \mathbb{K})$ carries the structure of a strict S^1-complex. $\widetilde{\mathrm{CL}}$ defines an S^1-complex morphism, so it descends to the map (2.4) on the cohomology level. We will give a brief account of Cohen–Ganatra's construction in this section and explain its implications for Lagrangian submanifolds in Liouville manifolds with cyclic dilations.

The construction of the maps $\mathrm{CL}_k : \mathrm{SC}^*(M) \to C_{n-*+2k}^{\Diamond}(\mathscr{L}L; \mathbb{K})$ is in some sense parallel to the construction of the maps $\{\delta_k\}$ in Section 4.1, but we now consider half-cylinders instead of cylinders as our domains. A k-*point angle decorated half-cylinder* is a (positive) half-cylinder $Z^+ \subset Z$ together with a collection of auxiliary interior marked points $p_1, \ldots, p_k \in Z^+$ satisfying (4.1). Denote by $\mathcal{M}_{k,+}$ the moduli space of such half-cylinders. Every element of $\mathcal{M}_{k,+}$ is equipped with a positive cylindrical end

$$\varepsilon^+ : [0, \infty) \times S^1 \to Z^+, \quad (s, t) \mapsto (s + (p_k)_s + \xi, t),$$

for some fixed $\xi > 0$. Note that, unlike the case of $\mathcal{M}_k$, there is no free $\mathbb{R}$-action on the moduli space $\mathcal{M}_{k,+}$.

$\mathcal{M}_{k,+}$ can be compactified to a manifold with corners $\overline{\mathcal{M}}_{k,+}$ by including broken trajectories in the moduli space. The codimension 1 boundary strata of $\overline{\mathcal{M}}_{k,+}$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_{j,+} \times \overline{\mathcal{M}}_{k-j} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, \quad 0 \leq j \leq k, \tag{5.2}$$

$$\overline{\mathcal{M}}_{k,+}^{i,i+1} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, \quad 1 \leq i < k, \tag{5.3}$$

$$\overline{\mathcal{M}}_{k-1,+}^{S^1} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, \tag{5.4}$$

where $\mathcal{M}_{k,+}^{i,i+1}$ is the locus where the i-th and $(i+1)$-th height coordinates coincide, and $\mathcal{M}_{k-1,+}^{S^1}$ is the locus where $h_1 = 0$. There exist forgetful maps

$$\pi_i : \mathcal{M}_{k,+}^{i,i+1} \to \mathcal{M}_{k-1,+}, \quad 1 \leq i \leq k-1,$$

$$\pi_{S^1} : \mathcal{M}_{k-1,+}^{S^1} \to \mathcal{M}_{k-1,+},$$

where the first map has been considered in (4.6), which forgets the point p_{i+1}, while the second map forgets p_1. Note that the maps π_i for $i \geq 1$ extend as maps

$$\overline{\pi}_i : \overline{\mathcal{M}}_{k,+}^{i,i+1} \to \overline{\mathcal{M}}_{k-1,+}$$

on the compactifications. The same holds for π_{S^1}, and we denote its extension by

$$\bar{\pi}_{S^1} : \bar{\mathcal{M}}^{S^1}_{k-1,+} \to \bar{\mathcal{M}}_{k-1,+}.$$

The definition of a Floer datum for a k-point angle-decorated half-cylinder $(Z^+, p_1, \ldots, p_k)$ is completely analogous to that of Definition 26 and will therefore be omitted. Inductively, there exist universal and consistent choices of Floer data for each $k \geq 0$ and each k-point angle decorated half-cylinder in the following sense.

- In a sufficiently small neighborhood of $L \subset M$, the Hamiltonian $H_{Z+} = 0$ near $s = 0$.

- Near the boundary stratum (5.2), the Floer datum coincides with the product of the Floer data chosen on lower-dimensional strata up to conformal equivalence. The Floer data vary smoothly with respect to the gluing charts for the product Floer data.

- Near the boundary strata (5.3), the Floer data are conformally equivalent to the ones obtained by pulling back from $\bar{\mathcal{M}}_{k-1,+}$ via the forgetful maps $\bar{\pi}_i$ for $i = 1, \ldots, k - 1$.

Fixing a universal and consistent choice of Floer data, for each $y \in \mathcal{O}_M$, define $\mathcal{M}_{k,+}(y, L)$ to be the moduli space of pairs $((Z^+, p_1, \ldots, p_k), u)$, where

$$(Z^+, p_1, \ldots, p_k) \in \mathcal{M}_{k,+},$$

and $u : Z^+ \to M$ is a map satisfying the parametrized Floer equation

$$(du - X_{H_{Z+}} \otimes dt)^{0,1} = 0,$$

where the $(0, 1)$-part is taken with respect to J_{Z+}, together with asymptotic and boundary conditions

$$\lim_{s \to \infty} (\varepsilon^+)^* u(s, \cdot) = y,$$
$$u(0, t) = \gamma \quad \text{for some } \gamma \in \mathscr{L}L.$$

For generic choices of H_{Z+}, $\mathcal{M}_{k,+}(y, L)$ is a smooth manifold of dimension

$$n - \deg(y) + 2k,$$

which admits a well-defined Gromov bordification $\bar{\mathcal{M}}_{k,+}(y, L)$, whose codimension 1 boundary is covered by the inclusions

$$\bar{\mathcal{M}}_{k-j,+}(y', L) \times \bar{\mathcal{M}}_j(y, y') \hookrightarrow \partial \bar{\mathcal{M}}_{k,+}(y, L), \tag{5.5}$$
$$\bar{\mathcal{M}}^{i,i+1}_{k,+}(y, L) \hookrightarrow \partial \bar{\mathcal{M}}_{k,+}(y, L), \tag{5.6}$$
$$\bar{\mathcal{M}}^{S^1}_{k-1,+}(y, L) \hookrightarrow \partial \bar{\mathcal{M}}_{k,+}(y, L). \tag{5.7}$$

Choose some Riemannian metric g on L. There is an evaluation map

$$\mathrm{ev} : \mathcal{M}_{k,+}(y, L) \to \mathscr{L}L$$

which is defined by restricting $u \in \mathcal{M}_{k,+}(y, L)$ to $\{0\} \times S^1$ and taking the arc length parametrization of the boundary of u with respect to g and a base point determined by the position of p_1. The map ev admits an extension $\overline{\mathrm{ev}} : \bar{\mathcal{M}}_{k,+}(y, L) \to \mathscr{L}L$ to the boundary strata, and the k-th order Cieliebak–Latschev map

$$\mathrm{CL}_k : \mathrm{SC}^*(M) \to C^{\Diamond}_{n-*+2k}(\mathscr{L}L; \mathbb{K})$$

is defined as

$$\mathrm{CL}_k(|o_y|_{\mathbb{K}}) = (-1)^{\deg(y)} \overline{\mathrm{ev}}_*([\bar{\mathcal{M}}_{k,+}(y, L)]),$$

where $[\bar{\mathcal{M}}_{k,+}(y, L)]$ denotes the fundamental chain.

Proposition 39 ([77, Proposition 4.4.12]). $\widetilde{\mathrm{CL}} = \sum_{k=0}^{\infty} \mathrm{CL}_k u^k$ *defines a morphism of S^1-complexes, and therefore it is an S^1-equivariant enhancement of* CL_0.

The proof follows from an analysis of the boundary strata of $\bar{\mathcal{M}}_{k,+}(y, L)$. In particular, our choice of Floer data ensures that the elements in the strata $\bar{\mathcal{M}}^{i,i+1}_{k,+}(y, L)$ will never contribute, since they are not rigid. For details, see the proof of [77, Proposition 4.4.12]. We will encounter a similar situation in the proof of Proposition 78. On the other hand, the contribution from the stratum $\bar{\mathcal{M}}^{S^1}_{k-1,+}(y, L)$ is non-trivial and can actually be identified with $\delta_1^{\mathrm{top}} \circ \mathrm{CL}_{k-1}$, where δ_1^{top} denotes the chain-level BV operator on $C_{n-*}(\mathscr{L}L; \mathbb{K})$ defined by rotating the loops, which descends to a BV operator on the quotient dg algebra $C^{\Diamond}_{n-*}(\mathscr{L}L; \mathbb{K})$.

On the cohomology level, $\widetilde{\mathrm{CL}}$ induces a map

$$[\widetilde{\mathrm{CL}}] : \mathrm{SH}^*_{S^1}(M) \to H^{S^1}_{n-*}(\mathscr{L}L; \mathbb{K}).$$

This enables us to interpret a result of Davison [20, Corollary 6.4.4] as providing obstructions to Lagrangian embeddings in Liouville manifolds with cyclic dilations; see Proposition 5.

Proof of Proposition 5. Let M be a Liouville manifold with a cyclic dilation, and assume that there is an exact Lagrangian submanifold $L \subset M$ which is hyperbolic. It follows from Propositions 21 and 39 that there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SH}^*_{S^1}(M) & \xrightarrow{\ \mathbf{B}\ } & \mathrm{SH}^{*-1}(M) \\
{\scriptstyle[\widetilde{\mathrm{CL}}]}\downarrow & & \downarrow \\
H^{S^1}_{n-*}(\mathscr{L}L; \mathbb{K}) & \xrightarrow{\ \mathbf{B}\ } & H_{n-*+1}(\mathscr{L}L; \mathbb{K})
\end{array}
\qquad (5.8)
$$

where the vertical arrow on the right is the usual Viterbo map (5.1). By our assumption, there is a class $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ whose image under the Connes map $\mathbf{B}$ is an invertible element $h \in \mathrm{SH}^0(M)^\times$. By the commutativity of (5.8), and our assumption that L is a $K(\pi, 1)$ space, such a class induces an exact Calabi–Yau structure on the fundamental group algebra $\mathbb{K}[\pi_1(L)]$, which contradicts the main result of [20].

This completes the proof of Proposition 5, under the additional assumption that L is Spin. In general, we can use (S^1-equivariant) symplectic cohomologies with local coefficients and argue as above. Since Davison's result holds for any closed, orientable L, this enables us to remove the Spin assumption on L. ∎

It would also be interesting to take a look at the special case when $h = 1$ in the definition of a cyclic dilation, which leads to the following generalization of [67, Corollary 6.3].

Corollary 40. *Suppose that the marking map $\mathbf{B} : \mathrm{SH}^1_{S^1}(M) \to \mathrm{SH}^0(M)$ hits the identity $1 \in \mathrm{SH}^0(M)$, then M cannot contain a closed exact Lagrangian submanifold L which is a $K(\pi, 1)$ space.*

Proof. Let $L \subset M$ be an exact Lagrangian submanifold which is topologically a $K(\pi, 1)$ space. Since T^*L is a Weinstein manifold, so Corollary 37 applies. It follows from the proof of [20, Theorem 6.1.3] that the marking map $\mathbf{B} : \mathrm{SH}^1_{S^1}(T^*L) \to \mathrm{SH}^0(T^*L)$ cannot hit the identity. Suppose that M admits a cyclic dilation with $h = 1$; one can then use the commutative diagram (5.8) to get a contradiction. ∎

Let $\mathbb{K} = \mathbb{Q}$. It is proved in [79, Theorem A] that the Milnor fibers $M_{a,\dots,a} \subset \mathbb{C}^{n+1}$ associated to the Brieskorn singularities

$$z_1^a + \dots + z_{n+1}^a = 0 \quad \text{with } n \geq a$$

admit cyclic dilations over $\mathbb{Q}$ with $h = 1$. By Corollary 40, it implies the following corollary.

Corollary 41. *The Milnor fibers $M_{a,\dots,a}$ do not contain exact Lagrangian tori.*

The non-existence of exact Lagrangian tori has been proved for many Milnor fibers. For a recent account, see [50].

Corollary 40 can also be applied to deduce non-existence results concerning cyclic dilations. For example, consider the Weinstein 4-manifold $T_{1,1,0} \subset \mathbb{C}^3$ defined by the equation

$$x + y + xyz = 1,$$

which is the complement of a nodal elliptic curve $\Sigma \subset \mathbb{CP}^2$. This manifold is studied in [12, Section 4.1], and it follows from the computation loc. cit that

$$\mathrm{SH}^0(T_{1,1,0}) \cong \mathbb{K}[x, y, z]/(x + y + xyz - 1)$$

as $\mathbb{K}$-algebras. Since the polynomial $x + y + xyz - 1$ is irreducible over $\mathbb{K}$, the only invertible element in $\mathrm{SH}^0(T_{1,1,0})$ is the identity. If $T_{1,1,0}$ admits a cyclic dilation, then $\mathbf{B} : \mathrm{SH}^1_{S^1}(T_{1,1,0}) \to \mathrm{SH}^0(T_{1,1,0})$ hits the identity. On the other hand, from the perspective of Legendrian surgery, $T_{1,1,0}$ can be constructed by attaching two 2-handles to the disk cotangent bundle D^*T^2, so there is an exact Lagrangian torus $L \subset T_{1,1,0}$. Now, Corollary 40 shows that $T_{1,1,0}$ does not admit a cyclic dilation.

One can attach one more 2-handle to D^*T^2 to get the Liouville domain associated to the affine surface $T_{1,1,1} \subset \mathbb{C}^3$ defined by the equation

$$x + y + z + xyz = 1.$$

Since $\bar{T}_{1,1,0}$ embeds in $\bar{T}_{1,1,1}$ as a Liouville subdomain, we conclude that $T_{1,1,1}$ does not admit a cyclic dilation. In general, let $\bar{M}_0 \subset \bar{M}_1$ be a Liouville subdomain; there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SH}^*_{S^1}(M_1) & \xrightarrow{\ \mathbf{B}\ } & \mathrm{SH}^{*-1}(M_1) \\
{\scriptstyle [\tilde{v}^!]}\downarrow & & \downarrow{\scriptstyle [v^!]} \\
\mathrm{SH}^*_{S^1}(M_0) & \xrightarrow{\ \mathbf{B}\ } & \mathrm{SH}^{*-1}(M_0)
\end{array}
\tag{5.9}
$$

generalizing (5.8), from which we see that if M_1 admits a cyclic dilation, then so is M_0. In (5.9), the map

$$
\tilde{v}^! = \sum_{i=0}^{\infty} v^!_k u^k : \mathrm{SC}^*(M_1) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!]
$$
$$
\to \mathrm{SC}^*(M_0) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!]
\tag{5.10}
$$

is the S^1-equivariant enhancement of Viterbo functoriality $v^! = v^!_0$. We refer the reader to [77, Appendix C] for its detailed construction.

More generally, attaching 2-handles to D^*T^2 yields a sequence of Weinstein 4-manifolds $T_{p,q,r}$, with $p \geq q \geq r \geq 0$. When $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$, these are the Milnor fibers of parabolic and hyperbolic unimodal singularities studied by Keating in [38, 39]. Our discussions above imply the following proposition.

Proposition 42. *The Weinstein manifold $T_{p,q,r}$ admits a cyclic dilation if and only if $q = r = 0$.*

Proof. Note that $T_{0,0,0}$ is symplectomorphic to T^*T^2, so it admits a quasi-dilation. $T_{1,0,0}$ is symplectomorphic to $\mathbb{C}^2 \setminus \{xy = 1\}$; it follows from [56, Corollary 19.8] that there is a quasi-dilation in $\mathrm{SH}^1(T_{1,0,0})$. Alternatively, one can compute its wrapped Fukaya category explicitly using the techniques developed in [8], whose endomorphism algebra turns out to be formal, and is quasi-isomorphic to the associative algebra $\mathbb{K}[x, y][(xy - 1)^{-1}]$; see Example 16 for its superpotential description. The case

when $p > 1$ can be argued similarly, since there are Lefschetz fibrations $T_{p,0,0} \to \mathbb{C}^*$ whose smooth fibers are T^*S^1. In fact, $T_{p,0,0}$ is symplectomorphic to the $\widetilde{A}_p$ plumbing of T^*S^2's. On the other hand, we have seen in the above that $T_{1,1,0}$ and $T_{1,1,1}$ do not admit cyclic dilations. Since any Weinstein manifold $T_{p,q,r}$ with $p \geq 1$ and $q \geq 1$ contains $\overline{T}_{1,1,0}$ as its Liouville subdomain, the non-existence of cyclic dilations follows from the commutative diagram (5.9). ∎

Observe that, among the examples $T_{p,q,r}$ considered above, the existence of a cyclic dilation is in fact equivalent to the existence of a quasi-dilation. This is not surprising in view of Proposition 11. More interestingly, Proposition 42 implies the following corollary.

Corollary 43. *Let M be any 4-dimensional Milnor fiber associated to a non-simple singularity; then M does not admit a cyclic dilation.*

Proof. This follows from the commutative diagram (5.9) and the adjacency of singularities. The latter implies the existence of some triple (p, q, r) with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ so that M contains $\overline{T}_{p,q,r}$ as a Liouville subdomain. A detailed explanation of this fact can be found in [38, Section 2.2]. However, it follows from Proposition 42 that any such $T_{p,q,r}$ cannot admit a cyclic dilation. ∎

5.2. Parametrized closed-open maps

Let M be a $2n$-dimensional Liouville manifold. As is observed by Seidel in [58]; if one considers only closed exact Lagrangian submanifolds $L \subset M$ satisfying Assumption 38, then one can define a chain map

$$\text{CO}_{\text{cpt}} : \text{SC}^*(M) \to \text{CH}^*(\mathcal{F}(M)).$$

This is usually referred to as the *closed-open string map* in literature. On the cohomology level, it is obtained by composing the closed-open map $\text{CO} : \text{SC}^*(M) \to \text{CH}^*(\mathcal{W}(M))$ considered in [27] with the restriction morphism induced by the cohomologically full and faithful embedding $\mathcal{F}(M) \hookrightarrow \mathcal{W}(M)$. Ganatra shows in [28, Proposition 14] that this map also admits an S^1-equivariant enhancement, which can be written as a homomorphism between S^1-complexes

$$\widetilde{\text{CO}}^\vee_{\text{cpt}} \in R\text{Hom}_{S^1}(\text{CH}^{\text{nu}}_*(\mathcal{F}(M)) \otimes \text{SC}^*(M), \mathbb{K}[-n]),$$

where the complex $\text{CH}^{\text{nu}}_*(\mathcal{F}(M)) \otimes \text{SC}^*(M)$ is equipped with the diagonal S^1-action (3.2), while $\mathbb{K}[-n]$ is regarded as a trivial S^1-complex. The construction of $\widetilde{\text{CO}}^\vee_{\text{cpt}}$ is completely parallel to the cyclic open-closed string map $\widetilde{\text{OC}}$ recalled in Section 4.3. In particular, it consists of an infinite sequence of maps

$$\text{CO}^{k,\vee}_{\text{cpt}} := \widetilde{\text{CO}}^{k,\vee}_{\text{cpt}} \oplus \widehat{\text{CO}}^{k,\vee}_{\text{cpt}}$$

for each $k \geq 0$, where $\widetilde{\mathrm{CO}}_{\mathrm{cpt}}^{k,\vee}$ and $\widehat{\mathrm{CO}}_{\mathrm{cpt}}^{k,\vee}$ are maps acting on the check factor

$$\mathrm{CH}_*(\mathscr{F}(M)) \otimes \mathrm{SC}^*(M)$$

and the hat factor $\mathrm{CH}_*(\mathscr{F}(M))[1] \otimes \mathrm{SC}^*(M)$, respectively.

Here, instead of using the full construction of Ganatra, only the check components $\{\widetilde{\mathrm{CO}}_{\mathrm{cpt}}^{k,\vee}\}_{k \geq 0}$ of the cyclic refinement of $\mathrm{CO}_{\mathrm{cpt}}$ will be relevant for our purposes. For the sake of readability, it seems to be appropriate to recall here the definitions as well as the basic properties of the first few of these maps. Since the reader should already be familiar with the operations associated to parametrized moduli spaces from our discussions in Section 4, our exposition here will be very sketchy, focusing mainly on the underlying TCFT (Topological Conformal Field Theory) structures and describing primarily the domains defining various operations. We follow the general framework of [63, 67], while our setup is slightly more complicated as it involves one additional piece of data: an ordered set of interior auxiliary marked points, whose flexibility constitutes the parameter spaces of our families of Riemann surfaces.

Let $S = \bar{S} \setminus \Sigma$, where $\bar{S}$ is a bordered Riemann surface, and $\Sigma \subset \bar{S}$ is a finite set of points. Write $\Sigma = \Sigma^{\mathrm{op}} \cup \Sigma^{\mathrm{cl}}$, where $\Sigma^{\mathrm{op}} \subset \partial \bar{S}$ is the set of boundary marked points, and $\Sigma^{\mathrm{cl}} \subset S$ is the set of interior marked points. There is also a strictly ordered set of auxiliary marked points $\Sigma^{\mathrm{aux}} \subset S$ lying in small neighborhoods of the interior marked points Σ^{cl}. For simplicity, we assume that all of the points in Σ^{aux} lie in a small disk centered a particular point $\zeta_\bullet \in \Sigma^{\mathrm{cl}}$ of radius $\varepsilon > 0$. The points in Σ^{aux} are ordered according to their distances to $\zeta_\bullet$. With respect to a fixed choice of complex coordinate near $\zeta_\bullet$, they should be strictly radially ordered, i.e., satisfying

$$0 < |p_k| < \cdots < |p_1| < \varepsilon \tag{5.11}$$

if $\zeta_\bullet$ is an input, and

$$0 < |p_1| < \cdots < |p_k| < \varepsilon \tag{5.12}$$

if $\zeta_\bullet$ is an output. As a convention, when $\bar{S}$ is the closed unit disk, we will take $\varepsilon = \frac{1}{2}$. At each point of Σ^{cl}, there is a preferred tangent direction ℓ_ζ, which is fixed if $\zeta \in \Sigma^{\mathrm{cl}} \setminus \{\zeta_\bullet\}$, and points toward the point in Σ^{aux} which is closest from ζ if $\zeta = \zeta_\bullet$. Furthermore, we can divide the sets Σ^{op} and Σ^{cl} into inputs and outputs, namely, $\Sigma^{\mathrm{op}} = \Sigma^{\mathrm{op,in}} \cup \Sigma^{\mathrm{op,out}}$ and $\Sigma^{\mathrm{cl}} = \Sigma^{\mathrm{cl,in}} \cup \Sigma^{\mathrm{cl,out}}$.

Additionally, $\bar{S}$ comes with a sub-closed 1-form $\nu_S \in \Omega^1(S)$ which satisfies

$$d\nu_S = 0$$

near Σ and $\nu_S|_{\partial S} = 0$ near Σ^{op}. One can associate a real number λ_ζ to every point $\zeta \in \Sigma$ by integrating ν_S along a small loop around ζ if $\zeta \in \Sigma^{\mathrm{cl}}$ or by integrating ν_S along a small path connecting one component of ∂S to the other one, if $\zeta \in \Sigma^{\mathrm{op}}$. For $\zeta \in \Sigma^{\mathrm{cl}}$, we require that $\lambda_\zeta \notin \mathscr{P}_M$.

For each boundary component $C \subset \partial S$, we want to have a label L_C, which is a Lagrangian submanifold in M satisfying Assumption 38. For any point $\zeta \in \Sigma^{\mathrm{op}}$, this determines a pair of Lagrangian submanifolds $(L_{\zeta,0}, L_{\zeta,1})$. The convention is that if $\zeta \in \Sigma^{\mathrm{op,in}}$, then $L_{\zeta,0}$ is the Lagrangian submanifold associated to the boundary component preceding ζ with respect to the orientation of $\partial \bar{S}$, and $L_{\zeta,1}$ is the successive one. When $\zeta \in \Sigma^{\mathrm{op,out}}$, then one uses the opposite convention. The operation associated to the marked bordered Riemann surfaces $(\bar{S}; \Sigma; \Sigma^{\mathrm{aux}})$ is a map

$$\bigotimes_{\zeta \in \Sigma^{\mathrm{cl,in}}} \mathrm{CF}^*(\lambda_\zeta) \otimes \bigotimes_{\zeta \in \Sigma^{\mathrm{op,in}}} \mathrm{CF}^*(L_{\zeta,0}, L_{\zeta,1})$$

$$\to \bigotimes_{\zeta \in \Sigma^{\mathrm{cl,out}}} \mathrm{CF}^*(\lambda_\zeta) \otimes \bigotimes_{\zeta \in \Sigma^{\mathrm{op,out}}} \mathrm{CF}^*(L_{\zeta,0}, L_{\zeta,1}) \tag{5.13}$$

of degree

$$n(-\chi(\bar{S}) + 2|\Sigma^{\mathrm{cl,out}}| + |\Sigma^{\mathrm{op,out}}|) - 2|\Sigma^{\mathrm{aux}}|.$$

We remark that, due to the presence of Σ^{aux}, (5.13) is in general *not* a chain map.

We give a quick sketch of the definition of (5.13). Denote by $_k \mathcal{R}_\Sigma$ the moduli space of domains $(\bar{S}; \Sigma; \Sigma^{\mathrm{aux}})$, where

$$\Sigma^{\mathrm{aux}} = \{p_1, \ldots, p_k\}.$$

It parametrizes the (potential) variations of the conformal structure on $\bar{S}$, the positions of the points in Σ and Σ^{aux}, up to automorphism. For any representative of an element of $_k \mathcal{R}_\Sigma$, we can assign it with a Floer datum, which consists of

- a cylindrical end $\varepsilon_\zeta^\pm$ for each point $\zeta \in \Sigma^{\mathrm{cl}}$, which is positive if $\zeta \in \Sigma^{\mathrm{cl,in}}$, and is negative if $\zeta \in \Sigma^{\mathrm{cl,out}}$. The cylindrical ends $\varepsilon_\zeta^\pm$ are required to be compatible with the tangent directions ℓ_ζ specified above at $\zeta \in \Sigma^{\mathrm{cl}}$,

- a strip-like end $\tau_\zeta^\pm : \mathbb{R}_\pm \times [0,1] \to S$ for each point $\zeta \in \Sigma^{\mathrm{op}}$, which is positive if $\zeta \in \Sigma^{\mathrm{op,in}}$ and is negative if $\zeta \in \Sigma^{\mathrm{op,out}}$,

- a sub-closed 1-form $\nu_S \in \Omega^1(S)$ such that

$$(\varepsilon_\zeta^\pm)^* \nu_S = (\tau_\zeta^\pm)^* \nu_S = dt;$$

- a domain-dependent Hamiltonian $H_S : S \to \mathcal{H}_\ell(M)$ which satisfies

$$(\varepsilon_\zeta^\pm)^* H_S = H_{\lambda_\zeta, t}, \quad (\tau_\zeta^\pm)^* H_S = H_{L_{\zeta,0}, L_{\zeta,1}, t},$$

for some $H_{\lambda_\zeta, t} \in \mathcal{H}_{\lambda_\zeta}(M)$, and $H_{L_{\zeta,0}, L_{\zeta,1}, t}$ is a time-dependent Hamiltonian which takes the form $\lambda_\zeta r + C$ on the conical end of M so that all the time-1 chords of $X_{H_{L_{\zeta,0}, L_{\zeta,1}, t}}$ between $L_{\zeta,0}$ and $L_{\zeta,1}$ are non-degenerate, but now the parameter $t \in [0,1]$. Here, the λ_ζ's are real numbers associated to cylindrical and strip-like ends, as mentioned above,

• a domain-dependent almost complex structure $J_S : S \to \mathcal{J}(M)$ such that

$$(\varepsilon_\zeta^\pm)^* J_S = (\tau_\zeta^\pm)^* J_S = J_t$$

for some $J_t \in \mathcal{J}(M)$.

The moduli space $_k\mathcal{R}_\Sigma$ admits a well-defined compactification $_k\bar{\mathcal{R}}_\Sigma$, which is usually a real blowup at $\zeta_\bullet$ of the corresponding Deligne–Mumford compactification. The codimension 1 boundary components of $_k\bar{\mathcal{R}}_\Sigma$ are covered by the degenerations of domains (as we will see in the case when $\bar{S}$ is an annulus), the real blowup loci, the loci where two of the marked points p_i and p_{i+1} in Σ^{aux} share the same modulus, and the locus where p_k (when $\zeta_\bullet$ in the case of an output) or p_1 (when $\zeta_\bullet$ is an input) goes to the boundary of the disk centered at $\zeta_\bullet$ (the last three strata correspond to (5.2), (5.3), and (5.4), respectively, when S is a half-cylinder). A universal and consistent choice of Floer data is an inductive choice of Floer data for each $k \geq 1$ and each element of $_k\bar{\mathcal{R}}_\Sigma$ so that it varies smoothly over $_k\bar{\mathcal{R}}_\Sigma$ and has specified behaviors along the boundary strata of $_k\bar{\mathcal{R}}_\Sigma$. In order to construct the moduli spaces defining the operations (5.13), we need to fix such a choice.

Fix a set of Lagrangian labelings (L_C) for the boundary components in ∂S and asymptotics $(\vec{x}, \vec{y})$, where $\vec{x} = (x_\zeta)$ is a set of time-1 chords of $X_{H_{L_{\zeta,0},L_{\zeta,1}},t}$ between $L_{\zeta,0}$ and $L_{\zeta,1}$, one for each $\zeta \in \Sigma^{\text{op}}$; and $\vec{y} = (y_\zeta)$ is a set of time-1 Hamiltonian orbits $y_\zeta \in \mathcal{O}_{M,\lambda_\zeta}$, one for each $\zeta \in \Sigma^{\text{cl}}$. Define the moduli space

$$_k\mathcal{R}_\Sigma(\vec{y}; \vec{x}) \tag{5.14}$$

to be the space of pairs $((\bar{S}; \Sigma; \Sigma^{\text{aux}}), u)$, where $(\bar{S}; \Sigma; \Sigma^{\text{aux}}) \in {}_k\bar{\mathcal{R}}_\Sigma$, and $u : S \to M$ is a solution of

$$\begin{cases} (du - X_{H_S} \otimes \nu_S)^{0,1} = 0, \\ u(C) \subset L_C & \text{for each } C \subset \partial S, \\ \lim_{s \to \pm\infty} u(\varepsilon_\zeta^\pm(s, \cdot)) = y_\zeta, \\ \lim_{s \to \pm\infty} u(\tau_\zeta^\pm(s, \cdot)) = x_\zeta. \end{cases}$$

For generic choices of perturbation data, the Gromov bordification $_k\bar{\mathcal{R}}_\Sigma(\vec{y}; \vec{x})$ has boundary components coming from semi-stable breakings, together with maps from the boundary strata of $_k\bar{\mathcal{R}}_\Sigma$. A signed count of rigid elements of $_k\bar{\mathcal{R}}_\Sigma(\vec{y}; \vec{x})$ for varying asymptotics $(\vec{x}, \vec{y})$ defines the operation (5.13).

The simplest example of interest for us is a closed disk $\bar{S}$ with $\Sigma^{\text{op}} = \Sigma^{\text{aux}} = \emptyset$ and an interior marked point $\zeta_{\text{out}} \in \bar{S}$ which is an output. The boundary ∂S is labeled with a Lagrangian submanifold $L \subset M$ satisfying Assumption 38. By counting solutions $u : S \to M$ of the corresponding Floer equation (with boundary and asymptotic

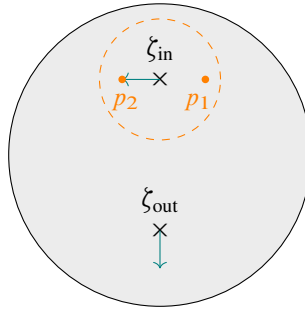**Figure 2.** Domain of the map $\phi_L^{2,0;2}$.

conditions), this defines a Floer cocycle

$$\phi_L^{1,0} \in \mathrm{CF}^n(\lambda) \tag{5.15}$$

for any prescribed real number $\lambda \notin \mathcal{P}_M$, whose cohomology class will be denoted by $\llbracket L \rrbracket \in \mathrm{HF}^n(\lambda)$. A noteworthy fact here is that the only difference between the domains defining the operation $\phi_L^{1,0}$ and the zeroth Cieliebak–Latschev map CL_0 which appeared in Section 5.1 is that the interior puncture of S is now treated as an output instead of an input.

For the next example, let $\bar{S}$ be a closed disk with two interior marked points; i.e., $\Sigma^{\mathrm{cl}} = \{\zeta_{\mathrm{in}}, \zeta_{\mathrm{out}}\}$, where ζ_{in} is an input and ζ_{out} is an output. There is no boundary marked point, and ∂S is labeled by a single Lagrangian submanifold L. The marked points ζ_{in} and ζ_{out} are equipped with asymptotic markers ℓ_{in} and ℓ_{out} so that ℓ_{out} points away from ζ_{in}. Furthermore, there are k additional auxiliary marked points $p_1, \ldots, p_k$ lying in a small neighborhood of ζ_{in}, and they are strictly radially ordered as in (5.11) with respect to the local complex coordinate near ζ_{in}. When $k = 0$, the asymptotic marker ℓ_{in} is required to point toward ζ_{out}, but when $k \geq 1$, we require that ℓ_{in} is pointing toward p_k. The associated cochain-level operation is a map

$$\phi_L^{2,0;k} : \mathrm{CF}^{*+2k}(\lambda_1) \to \mathrm{CF}^{*+n-1}(\lambda_0),$$

where $\lambda_0, \lambda_1 \notin \mathcal{P}_M$. See Figure 2 for the domain defining the operation $\phi_L^{2,0;2}$.

We now recall the definitions of the check components of the cyclic closed-open map (cf. [28, Section 5.6.2]). Let $\bar{S}$ be a closed disk with d boundary marked points $\Sigma^{\mathrm{op}} = \{\zeta_1, \ldots, \zeta_d\}$ ordered anticlockwise, among which ζ_d is the only output. There is a unique interior marked point ζ_{in} at the origin, which is an input. Moreover, there are k auxiliary interior marked points, $\Sigma^{\mathrm{aux}} = \{p_1, \ldots, p_k\}$, which are ordered so that (5.11) is satisfied. This gives rise to a moduli space of domains, which will be denoted by $_k\check{\mathcal{R}}_{d,\mathrm{cpt}}^1$. Label the components of ∂S with the Lagrangian submanifolds

$L_1, \ldots, L_d$, and fix an ordered set of Hamiltonian chords $\vec{x} = (x_1, \ldots, x_d)$, where x_i is a time-1 chord of $X_{H_{L_i, L_{i+1 \bmod d}, t}}$, together with a Hamiltonian orbit $y_{\text{in}} \in \mathcal{O}_{M, \lambda}$. As in (5.14), one can build a moduli space

$$_k \check{\mathcal{R}}^1_{d, \text{cpt}}(y_{\text{in}}; \vec{x}). \tag{5.16}$$

A signed count of rigid elements in the moduli space (5.16) for varying asymptotics $\vec{x}$ and y_{in} defines a map

$$\phi^{1, d; k}_{L_1, \ldots, L_d} : \text{CF}^*(\lambda) \otimes \text{CF}^*(L_{d-1}, L_d) \otimes \cdots \otimes \text{CF}^*(L_1, L_2)$$
$$\to \text{CF}^*(L_1, L_d)[1 - d - 2k]. \tag{5.17}$$

Note that, when $k = 0$, these operations reduce to the usual closed-open string maps $\phi^{1, d}_{L_1, \ldots, L_d}$ considered in [63, 67].

Let us write down explicitly the first few of the maps (5.17) and try to understand their basic properties. The first one of these maps is

$$\phi^{1, 1; k}_L : \text{CF}^{* + 2k}(\lambda) \to \text{CF}^*(L, L), \tag{5.18}$$

which is defined using a closed disk $\bar{S}$ with one interior marked point ζ_{in}, which is an input and carries an asymptotic marker ℓ_{in}, and one boundary marked point ζ_1, which is an output. Parametrization is given by k interior auxiliary marked points $p_1, \ldots, p_k$, whose positions satisfy the constraint (5.11). The asymptotic marker ℓ_{in} is required to point toward p_k. One can also treat the boundary marked point ζ_1 of $\bar{S}$ as an input, which results in a map

$$(\phi^{1, 1; k}_L)^\vee : \text{CF}^{* + 2k}(\lambda) \otimes \text{CF}^{n - *}(L, L) \to \mathbb{K}.$$

Going one step further, adding a marked point $\zeta_2 \in \partial \bar{S}$ which serves as an input to the previously considered marked bordered Riemann surfaces defining $\phi^{1, 1; k}_L$ gives rise to a map

$$\phi^{1, 2; k}_{L_1, L_2} : \text{CF}^*(\lambda) \otimes \text{CF}^*(L_1, L_2) \to \text{CF}^*(L_1, L_2)[-2k - 1].$$

Consider the family of Riemann surfaces parametrized by $\mathbb{R}$ as shown in Figure 3, with the marked points $p_1, \ldots, p_k$ being fixed, so that (5.11) holds with $\varepsilon = \frac{1}{2}$. When $k = 0$, this yields a homotopy $\phi^{1, 2}_{L_1, L_2}$ between

$$\mu^2(\phi^{1, 1}_{L_2}(y_{\text{in}}), x) \quad \text{and} \quad (-1)^{|x||y_{\text{in}}|} \mu^2(x, \phi^{1, 1}_{L_1}(y_{\text{in}}));$$

see [67, (2.12)]. When the points $p_1, \ldots, p_k$ are allowed to vary in the disk centered at ζ_{in} with radius $\frac{1}{2}$, one obtains a $(2k + 1)$-dimensional family $\mathcal{Y}_k$ of marked Riemann surfaces fibering over $\mathbb{R}$, whose $2k$-dimensional fibers come from the freedom
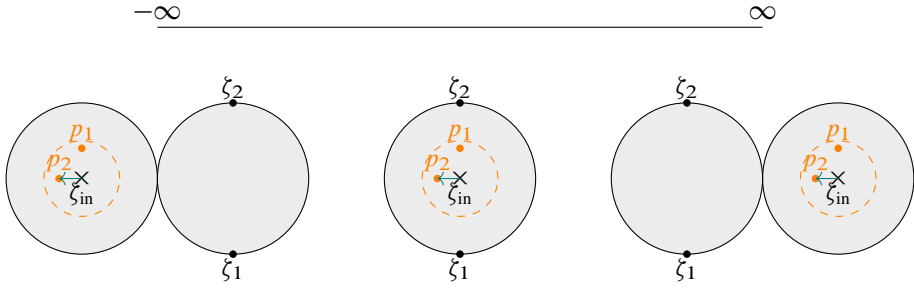
Figure 3. A section of the family $\overline{\mathcal{Y}}_2$ obtained by fixing the positions of p_1 and p_2.

of moving the points $p_1, \ldots, p_k$. The fiberwise compactification $\overline{\mathcal{Y}}_k$ of this family involves the strata coming from real blowups, which appear when $|p_{k-j+1}| \to 0$ for some $1 \le j \le k$. Note that this forces $|p_l| \to 0$ for every $k - j + 1 \le l \le k$, so there will be cylinder bubbles at the origin ζ_{in} of S containing the j-marked points $p_{k-j+1}, \ldots, p_k$, which define elements of the moduli spaces $\mathcal{M}_j$; see Figure 4. The two ends of the 1-parameter family in Figure 3 still define two ends in the fiberwise compactification $\overline{\mathcal{Y}}_k$, and they contribute to the first two terms on the right-hand side of (5.19). Moreover, in the fiber direction, there are two additional boundary strata corresponding, respectively, to the loci where $|p_i| = |p_{i+1}|$ for some $1 \le i \le k - 1$ and $|p_1| = \frac{1}{2}$. Using a similar argument as in the proof of Proposition 78, one can show that the boundary strata corresponding to $|p_i| = |p_{i+1}|$ do not contribute. On the other hand, the stratum $|p_1| = \frac{1}{2}$ gives rise to an operation, which we denote by

$$\hat{\phi}_{L_1, L_2}^{1,2;k-1} : \mathrm{CF}^*(\lambda) \otimes \mathrm{CF}^*(L_1, L_2) \to \mathrm{CF}^*(L_1, L_2)[-2k - 2].$$

As a consequence, we have

$$\mu^1(\phi_{L_1,L_2}^{1,2;k}(y_{\text{in}}, x)) + \sum_{j=0}^{k} \phi_{L_1,L_2}^{1,2;k-j}(\delta_j(y_{\text{in}}), x) + (-1)^{|y_{\text{in}}|}\phi_{L_1,L_2}^{1,2;k}(y_{\text{in}}, \mu^1(x))$$

$$= \mu^2(\phi_{L_2}^{1,1;k}(y_{\text{in}}), x) - (-1)^{|x||y_{\text{in}}|}\mu^2(x, \phi_{L_1}^{1,1;k}(y_{\text{in}})) + \hat{\phi}_{L_1,L_2}^{1,2;k-1}(y_{\text{in}}, x). \quad (5.19)$$

As we will see later, the appearance of the additional term $\hat{\phi}_{L_1,L_2}^{1,2;k-1}(y_{\text{in}}, x)$ in the above identity is the reason for the additional complexity that rises in our situation compared to Seidel–Solomon's original construction [67], see Section 5.3.

For completeness, we end this subsection by recalling some known operations constructed in [63], which are defined by considering a 1-parameter family of annuli $\overline{S}$. The outer boundary of $\overline{S}$ is labeled by the Lagrangian submanifold L_0, and the inner boundary of $\overline{S}$ is labeled by the Lagrangian submanifold L_1. There is a unique
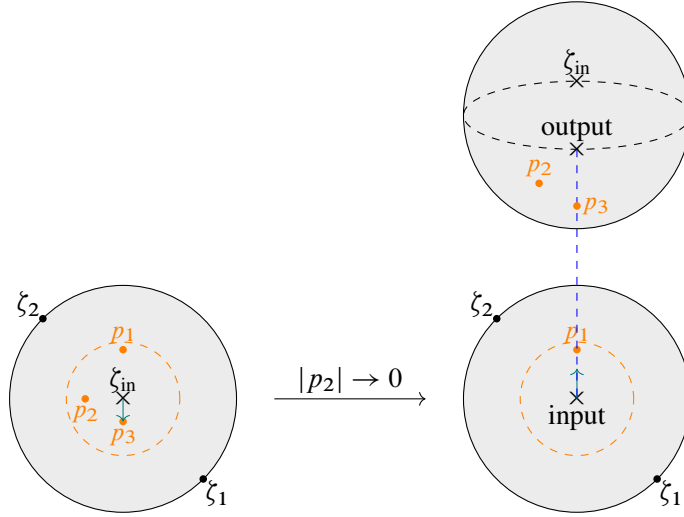
Figure 4. A representative of an element of the moduli space $_3\check{\mathcal{R}}^1_{2,\mathrm{cpt}}$ and the sphere bubble at the origin when $|p_2| \to 0$.

boundary marked point ζ_1 on $\partial\overline{S}$, which is an input. See Figure 5. Depending on whether ζ_1 lies on the boundary component labeled by L_1 or L_0, there are operations

$$\psi^{0,1}_{L_0,L_1} : \mathrm{CF}^1(L_1,L_1) \to \mathbb{K},$$
$$(\psi^{0,1}_{L_0,L_1})^\vee : \mathrm{CF}^1(L_0,L_0) \to \mathbb{K},$$

and they satisfy

$$\psi^{0,1}_{L_0,L_1}(\mu^1(x)) = (-1)^{n(n+1)/2}\,\mathrm{Str}(\mu^2(x,\cdot)) - (-1)^n \langle \phi^{1,0}_{L_0}, (\phi^{1,1}_{L_1})^\vee(x)\rangle, \tag{5.20}$$
$$(\psi^{0,1}_{L_0,L_1})^\vee(\mu^1(x)) = (-1)^{n(n+1)/2}\,\mathrm{Str}(\mu^2(\cdot,x)) - \langle \phi^{1,0}_{L_1}, (\phi^{1,1}_{L_0})^\vee(x)\rangle, \tag{5.21}$$

where by Str we mean the supertrace. In the second term on the right-hand side of (5.20), we take $\phi^{1,0}_{L_0} \in \mathrm{CF}^n(-\lambda)$ and pair it with the result of

$$(\phi^{1,1}_{L_1})^\vee : \mathrm{CF}^*(L_1,L_1) \to \mathrm{CF}^{*+n}(\lambda).$$

This is a version of the *Cardy relation*; see [63, Section 4c] for details.

5.3. Seidel–Solomon's construction

Let M be a $2n$-dimensional Liouville manifold, and let $\lambda \notin \mathcal{P}_M$ be a real number. Fix an arbitrary cohomology class $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$, with its cochain-level representative
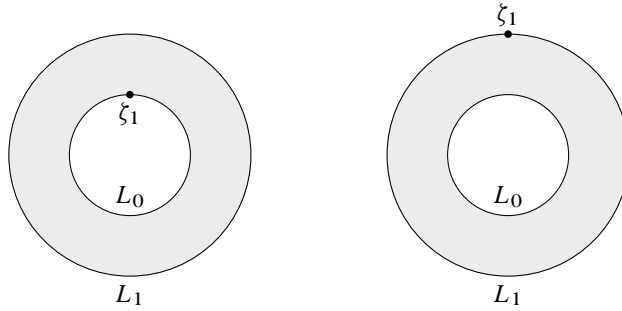
Figure 5. Domains defining the operations $\psi_{L_0,L_1}^{0,1}$ and $(\psi_{L_0,L_1}^{0,1})^\vee$.

given by

$$\widetilde{\beta} := \sum_{k=0}^{\infty} \beta_k \otimes u^{-k}. \tag{5.22}$$

Note that the sum on the right-hand side of (5.22) is actually finite since only finitely many cochains $\beta_k \in \mathrm{CF}^{2k+1}(\lambda)$ are non-zero. When $\beta_k = 0$ for all $k \geq 1$, Seidel and Solomon constructed in [67] a derivation on Floer cohomologies $\mathrm{HF}^*(L_0, L_1)$ of any simply connected Lagrangian submanifolds. In this subsection, we study a higher-order analog of Seidel–Solomon's construction. This is in general not well defined and only works in very special cases, say, when

$$L_0 = L_1 = L$$

is a Lagrangian sphere of some odd dimension $n \geq 3$, which turns out to be enough for the purpose of proving Theorem 13.

For any Lagrangian submanifold $L \subset M$ satisfying Assumption 38, consider the Floer cochain

$$\sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k) \in \mathrm{CF}^1(L, L) \tag{5.23}$$

obtained by applying the closed-open maps (5.18). This is in general not a cocycle, due to the fact that the boundary of the moduli space $_k\overline{\breve{\mathcal{R}}}_{1,\mathrm{cpt}}^1(y_{\mathrm{in}}; x)$ contains the stratum $_{k-1}\overline{\mathcal{R}}_{1,\mathrm{cpt}}^{S^1}(y_{\mathrm{in}}, x)$, whose contribution cannot be ignored in general.

From now on, let L be a Lagrangian sphere of some odd dimension $n \geq 3$. In this case, one can arrange so that

$$\mathrm{CF}^2(L, L) = 0,$$

which means that (5.23) defines a cocycle. However, for our purposes, we will need to show that, when $\tilde{b}$ is a cyclic dilation, this is the case regardless of the Floer data used in the definition of $\mathrm{HF}^*(L, L)$.

Lemma 44. *The image of the cyclic dilation $\tilde{b} \in \mathrm{SH}^1_{S^1}(M)$ under the S^1-equivariant Viterbo functoriality $[\tilde{v}^!] : \mathrm{SH}^*_{S^1}(M) \to \mathrm{SH}^*_{S^1}(T^*L)$ (cf. (5.10)) is of the form*

$$\alpha_L \cdot \mathbf{I}(b) \bmod u^{-1}, \tag{5.24}$$

where $\alpha_L \in \mathbb{K}^\times$ and the class $b \in \mathrm{SH}^1(T^*S^n)$ is a dilation.

Proof. It follows from the diagram (5.9) that $[\tilde{v}^!](\tilde{b}) \in \mathrm{SH}^1_{S^1}(T^*L)$ is a cyclic dilation. On the other hand, according to the computation of [19, Theorem 2 (2)], we have the isomorphism

$$H_{n-*}(\mathcal{L}S^n; \mathbb{K}) \cong \frac{\mathbb{K}[x]}{(x^2)} \otimes \mathbb{K}[y],$$

where $|x| = n$, $|y| = 1 - n$, and n is odd. This implies that $\mathrm{SH}^0(T^*S^n) \cong \mathbb{K}$, so the class $\tilde{b}$ satisfies $\mathbf{B} \circ [\tilde{v}^!](\tilde{b}) = \alpha_L$ for some $\alpha_L \in \mathbb{K}^\times$. It follows from the computations in [70, Section 4.1] that

$$H^{S^1}_{n-1}(\mathcal{L}S^n; \mathbb{K}) \cong \mathbb{K}\langle \mathbf{I}(x \otimes y) \rangle \oplus \mathbb{K}\left\langle \frac{\gamma^{\frac{n-1}{2}}}{(\frac{n-1}{2})!} \right\rangle,$$

where γ is the degree 2 generator dual to $x \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$, which gives rise to the map $\mathbf{S} : H^{S^1}_{*+1}(\mathcal{L}S^n; \mathbb{K}) \to H^{S^1}_{*-1}(\mathcal{L}S^n; \mathbb{K})$ in the Gysin sequence (4.15). Moreover, the marking map

$$\mathbf{B} : H^{S^1}_*(\mathcal{L}S^n; \mathbb{K}) \to H_{*+1}(\mathcal{L}S^n; \mathbb{K})$$

sends $\frac{\gamma^{\frac{n-1}{2}}}{(\frac{n-1}{2})}!$ to 0, and

$$\mathbf{B} \circ \mathbf{I}(x \otimes y) = \Delta(x \otimes k) = 1.$$

Since $x \otimes y$ defines a dilation under the BV algebra isomorphism

$$H_{n-*}(\mathcal{L}S^n; \mathbb{K}) \cong \mathrm{SH}^*(T^*S^n)$$

established in [4], and the u^0 part of $\mathrm{SH}^*_{S^1}(T^*L)$ is given by the image of the erasing map; we have proved (5.24). ∎

Proposition 45. *Let $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$ be a cyclic dilation; then (5.23) defines a Floer cocycle for any choice of Floer data which defines $\mathrm{CF}^*(L, L)$.*

Proof. The idea of the proof is to regard the operation $\phi^{1,1;k}_L$ as (up to homotopy) the composition of the (parametrized version) of the Viterbo functoriality relating the symplectic cochain complex $\mathrm{SC}^*(M)$ of the ambient space to that of a Weinstein neighborhood $\mathrm{SC}^*(T^*L)$ and the ordinary closed-open map $\phi^{1,1}_L$ defined in the Weinstein neighborhood of L. Since the image of the cyclic dilation under the Viterbo map has been determined by Lemma 44, this allows us to conclude that $\sum^\infty_{k=0} \phi^{1,1;k}_L(\beta_k)$ is a cocycle. The proof is divided into four steps.

Step 1: Realizing $\mathrm{SH}^*(T^*L)$ *as the Floer cohomology of a Hamiltonian on* M. Consider a family of marked bordered Riemann surfaces

$$\left(\bar{S}_q; \zeta_{\mathrm{in}}, \zeta_1; \frac{p_1}{q}, \ldots, \frac{p_k}{q}\right) \tag{5.25}$$

parametrized by $q \in [1, \infty)$, where $\bar{S}_q$ is the closed unit disk centered at the origin, whose center ζ_{in} is an interior puncture serving as an input, and ζ_1 is a boundary puncture serving as an output. The marked points $p_1, \ldots, p_k$ are ordered so that (5.11) holds. As part of our Floer data, denote by

$$\varepsilon_q^+ : [-\log q, \infty) \times S^1 \to S_q$$

the family of positive cylindrical ends fixed at ζ_{in}. Note that, for $q = 1$, (5.25) is just the domain defining $\phi_L^{1,1;k}$ considered in Section 5.2. We equip these domains with a family of Hamiltonians (H_{S_q}) so that $H_{S_1} : S_1 \to \mathcal{H}_\ell(M)$ is part of the Floer data defining the operation $\phi_L^{1,1;k}$. In particular, $(\varepsilon_1^+)^* H_{S_1} = H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ for some $\lambda \notin \mathcal{P}_M$. For the purpose of our argument, we need to work with the Floer cochain complex $\mathrm{CF}^*_{(a,b]}(\lambda)$ truncated in the finite action window $(a, b]$, where $-\infty < a < b < \infty$. This is defined as the quotient complex $\mathrm{CF}^*_{>a}(\lambda)/\mathrm{CF}^*_{>b}(\lambda)$. Assume that there are sequences of real numbers $\{\nu_i\}$, $\{r_i\}$, $\{\eta_i\}$ and $\{\lambda_i\}$ so that

- $\nu_i > 0$ is smaller than the distance from η_i to $\mathcal{P}_{T^*L}$, and $\nu_i \to 0$ as $i \to \infty$;
- $r_i > \max\{1, \frac{\eta_i - a}{\nu_i}\}$;
- $\frac{\eta_i}{4} < \lambda_i < \frac{\eta_i}{2}$ and $\lambda_i \notin \mathcal{P}_M$.

Let $H_{\eta_i, \lambda_i, t}$ be a small time-dependent perturbation of a one-step Hamiltonian H_{η_i, λ_i} which vanishes on the Liouville subdomain $D^*L \subset \bar{M}$; linear and has slope η_i on $[1, r_i] \times \partial D^*L$ for some $r_i > 0$; constant and equal to $\eta_i(r_i - 1)$ on $\psi^{\log r_i}(\bar{M} \setminus D^*L)$, where $\psi^{\log r_i}$ is the Liouville flow at time $\log r_i$; and equals $\eta_i(r_i - 1) + \lambda_i(r - r_i)$ on $[r_i, \infty) \times \partial\bar{M}$. Consider the Floer cochain complexes $\mathrm{CF}^*_{(a,b]}(H_{\eta_i, \lambda_i, t})$; it follows from [17, Lemma 5.1] that there is an identification

$$\mathrm{CF}^*_{(a,b]}(H_{\eta_i, \lambda_i, t}) \cong \mathrm{CF}^*_{(a,b]}(T^*L, \eta_i) \tag{5.26}$$

between the Floer cochain complexes defined on M and those on its Liouville subdomain D^*L. In particular,

$$\varinjlim_i \mathrm{HF}^*_{(a,b]}(H_{\eta_i, \lambda_i, t}) \cong \mathrm{SH}^*_{(a,b]}(T^*L) := \varinjlim_i \mathrm{HF}^*_{(a,b]}(T^*L, \eta_i).$$

Step 2: Interpreting Viterbo functoriality as continuation maps. For each Hamiltonian of the form $H_{\eta,\lambda,t}$, there exists a Hamiltonian $H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ such that

$$H_{\lambda,t} \leq H_{\eta,\lambda,t}.$$

Choose a monotone homotopy h_s between the Hamiltonians $H_{\lambda,t}$ and $H_{\eta,\lambda,t}$ so that the continuation maps

$$\kappa_k : \mathrm{CF}^{*+2k}_{(a,b]}(\lambda) \to \mathrm{CF}^*_{(a,b]}(H_{\eta,\lambda,t}),$$

which are variants of the components $\kappa_k^{\lambda_1,\lambda_2}$ in (4.9) and well defined for all $k \geq 0$, and altogether they form the (truncated) S^1-equivariant Viterbo functoriality

$$[\tilde{v}^!] : \mathrm{SH}^*_{S^1,(a,b]}(M) \to \mathrm{SH}^*_{S^1,(a,b]}(T^*L)$$

after passing to direct limits. We will denote by

$$\tilde{v}^!_k : \mathrm{SC}^{*+2k}(M) \to \mathrm{SC}^*(T^*L)$$

the components of the S^1-complex morphism $\tilde{v}^!$. Let ρ be a non-negative, monotone, non-decreasing cut-off function such that

$$\rho(s) = \begin{cases} 0, & s \ll 0, \\ 1, & s \gg 0. \end{cases}$$

We require that the family of Hamiltonians $(H_{S_q})_{q \geq 1}$ to satisfy

$$(\varepsilon_q^+)^* H_{S_q} = \rho(s+q)h_s. \tag{5.27}$$

Note that this is compatible with our requirement that $(\varepsilon_1^+)^* H_{S_1} = H_{\lambda,t}$.

Step 3: Degeneration of domain. Assume that the input ζ_{in} receives generators from the truncated complex $\mathrm{CF}^*_{(a,b]}(\lambda)$. When $q \to \infty$, the points $p_1, \ldots, p_k$ move to the origin and a cylinder with all the auxiliary marked points bubbles off at ζ_{in}. According to (5.27), this k-point angle decorated cylinder is equipped with the Floer data defining the continuation map κ_k. On the other hand, the component carrying the boundary marked point defines an operation

$$\tilde{\phi}^{1,1}_L : \mathrm{CF}^*_{(a,b]}(H_{\eta,\lambda,t}) \to \mathrm{CF}^*(L,L),$$

which is a slight variation of the usual closed-open string map

$$\phi^{1,1}_L : \mathrm{CF}^*(\lambda) \to \mathrm{CF}^*(L,L).$$

There are also boundary strata when $q \in (1, \infty)$, which correspond to the cases when a cylinder, which may contain several of the marked points $p_1, \ldots, p_k$, breaks off at ζ_{in} and when the coordinates of the auxiliary marked points satisfy $|p_i| = |p_{i+1}|$ for some $1 \leq i \leq k-1$. We know from the previous discussions that the latter strata are non-rigid by appropriate choices of Floer data, which can clearly be achieved here.

For the former ones, we can arrange the choices of Floer data so that H_{S_q} pulls back to $H_{\lambda,t}$ under the negative cylindrical end of the marked cylinder bubbled off at ζ_{in}. As a consequence, their contributions can be identified as

$$\sum_{j=0}^{k} t_L^{1,1;j}(\delta_{k-j}(y)),$$

where

$$t_L^{1,1;j} : \mathrm{CF}^{*+2j+1}(\lambda) \to \mathrm{CF}^*(L,L)$$

is the operation defined by counting rigid elements in the moduli space of triples

$$\left(q, \left(S_q, \frac{p_1}{q}, \dots, \frac{p_j}{q} \right), u \right),$$

where $u : S_q \to M$ is a solution to the equation

$$(du - X_{H_{S_q}} \otimes v_{S_q})^{0,1} = 0$$

with boundary on L and asymptotics at ζ_{in} and ζ_1 being generators of $\mathrm{CF}^*_{(a,b]}(\lambda)$ and $\mathrm{CF}^*(L,L)$, respectively.

To sum up, we have proved the cochain-level identity

$$\phi_L^{1,1;k}(y) - \tilde{\phi}_L^{1,1} \circ \kappa_k(y) = \mu^1(t_L^{1,1;k}(y)) + \sum_{j=0}^{k} t_L^{1,1;j}(\delta_{k-j}(y)) \tag{5.28}$$

for any $y \in \mathrm{CF}^*_{(a,b]}(\lambda)$.

Step 4: Applying to the cochains β_k. In view of (5.26) and the identification (obtained by rescaling the Hamiltonian perturbation) of $\mathrm{CF}^*(L,L)$ with a Floer cochain complex $\mathrm{CF}^*_{T^*L}(L,L)$ computed in the Liouville subdomain $D^*L \subset \bar{M}$, the map $\tilde{\phi}_L^{1,1}$ can be identified with the usual closed-open map

$$\phi_{L \subset T^*L}^{1,1} : \mathrm{CF}^*_{(a,b]}(T^*L, \eta) \to \mathrm{CF}^*_{T^*L}(L,L) \tag{5.29}$$

for the Liouville manifold T^*L by an appropriate version of the maximum principle. See [6, Lemma 7.4]. Choosing the action window $(a,b]$ so that $\beta_k \in \mathrm{CF}^{2k+1}_{(a,b]}(\lambda)$ for all $k \geq 1$ (which is possible since $\beta_k \neq 0$ for only finitely many k), it follows from (5.28) and (5.29) that

$$\sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k) - \phi_{L \subset T^*L}^{1,1}\left(\sum_{k=0}^{\infty} \kappa_k(\beta_k) \right)$$

$$= \mu^1\left(\sum_{k=0}^{\infty} t_L^{1,1;k}(\beta_k) \right) + \sum_{k=0}^{\infty} \sum_{j=0}^{k} t_L^{1,1;j}(\delta_{k-j}(\beta_k)). \tag{5.30}$$

The double sum on the right-hand side of (5.30) vanishes by the equivariant cocycle condition satisfied by

$$\widetilde{\beta} = \sum_{k=0}^{\infty} \beta_k \otimes u^{-k}.$$

For the reader's convenience, we will spell out the details. The equivariant differential

$$\delta_{\mathrm{eq}} = \sum_{k=0}^{\infty} \delta_k u^k$$

applied to $\widetilde{\beta}$ gives

$$\delta_{\mathrm{eq}}(\widetilde{\beta}) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \delta_{k-j}(\beta_k) \otimes u^{-j} = 0,$$

which implies that, for any fixed j,

$$\sum_{k=j}^{\infty} t_L^{1,1;j}(\delta_{k-j}(\beta_k)) = t_L^{1,1;j}\left(\sum_{k=j}^{\infty} \delta_{k-j}(\beta_k)\right) = 0.$$

After passing to the inverse limit $a \to -\infty$ (which is finite due to the existence of the lower bound of the action) and the direct limits $b \to \infty$, $\lambda \to \infty$, and $\eta \to \infty$, we see that $\sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k)$ is cohomologous to $\phi_{L \subset T^*L}^{1,1}(\sum_{k=0}^{\infty} v_k^!(\beta_k))$. By Lemma 44, the image of the cyclic dilation $\widetilde{\beta}$ under the S^1-equivariant Viterbo functoriality is $\alpha_L \cdot \beta \otimes 1 \mod u^{-1}$, where $\alpha_L \in \mathbb{K}^\times$ and $\beta \in \mathrm{SC}^1(T^*L)$ is a dilation. It follows that

$$\widetilde{v}^!(\widetilde{\beta}) = \sum_{k=0}^{\infty} v_k^!(\beta_k) \in \mathrm{SC}^1(T^*L) \subset \mathrm{SC}^1(T^*L) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!],$$

which is equal to $\alpha_L \cdot \beta$. Since $\phi_{L \subset T^*L}^{1,1}$ is a chain map and $\phi_{L \subset T^*L}^{1,1}(\alpha_L \cdot \beta)$ is a Floer cocycle, so is $\sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k)$. ∎

Since $\mathrm{HF}^1(L, L) = 0$, it follows that there is a cochain $\widetilde{\gamma}_L \in \mathrm{CF}^0(L, L)$, which satisfies

$$\mu^1(\widetilde{\gamma}_L) = \sum_{k=0}^{\infty} \phi_L^{1,1;k}(\beta_k). \tag{5.31}$$

Two $\widetilde{\gamma}_L$'s are considered to be equivalent if their difference is a degree 0 coboundary. Note that the set of equivalence classes of choices of $\widetilde{\gamma}_L$ is an affine space over

$$H^0(L; \mathbb{K}) \cong \mathbb{K}.$$

Following [67], the pair $\widetilde{L} = (L, \widetilde{\gamma}_L)$ will be called a $\widetilde{b}$-*equivariant* Lagrangian sphere.

Let

$$\tilde{L}_0 = (L_0, \tilde{\gamma}_{L_0}), \quad \tilde{L}_1 = (L_1, \tilde{\gamma}_{L_1})$$

be two odd-dimensional $\tilde{b}$-equivariant Lagrangian spheres. One can define a cochain-level endomorphism

$$\phi_{\tilde{L}_0, \tilde{L}_1} : CF^*(L_0, L_1) \to CF^*(L_0, L_1)$$

by

$$\phi_{\tilde{L}_0, \tilde{L}_1}(x) := \sum_{k=0}^{\infty} \phi_{L_0, L_1}^{1,2;k}(\beta_k, x) - \mu^2(\tilde{\gamma}_{L_1}, x) + \mu^2(x, \tilde{\gamma}_{L_0}). \tag{5.32}$$

Note that $\phi_{\tilde{L}_0, \tilde{L}_1}$ is in general not a chain map. In fact, it follows from (5.19) and the fact that $\sum_{k=0}^{\infty} \beta_k \otimes u^{-k}$ defines a cocycle in $CF_{S^1}^1(\lambda)$ such that

$$\mu^1(\phi_{\tilde{L}_0, \tilde{L}_1}(x)) = \phi_{\tilde{L}_0, \tilde{L}_1}(\mu^1(x)) + \sum_{k=1}^{\infty} \hat{\phi}_{L_0, L_1}^{1,2;k-1}(\beta_k, x).$$

Proposition 46. *Assume that $\tilde{b} \in HF_{S^1}^1(\lambda)$ is a cyclic dilation. Let $L_0 = L_1 = L$, and $\tilde{\gamma}_{L_0} = \tilde{\gamma}_{L_1}$ in the above; then $\phi_{\tilde{L}, \tilde{L}}$ is a chain map.*

Proof. The idea of the proof is similar to that of Proposition 45. Using a degeneration of domain argument, the operation $\phi_{L,L}^{1,2;k}$ can be shown to coincide up to homotopy with the composition of the (parametrized) Viterbo functoriality and the usual closed-open map $\phi_{L,L}^{1,2}$ defined in the Weinstein neighborhood $D^*L \subset M$. Lemma 44 then enables us to understand the map $\sum_{k=0}^{\infty} \phi_{\tilde{L}, \tilde{L}}(\beta_k)$.

Denote by

$$t_{L,L}^{1,2;k} : CF^{*+2k+2}(\lambda) \otimes CF^*(L, L) \to CF^*(L, L)$$

the operation defined by a family of domains

$$\left(\bar{S}_q; \zeta_{in}, \zeta_1, \zeta_2; \frac{p_1}{q}, \dots, \frac{p_k}{q} \right)$$

parametrized by the auxiliary marked points $p_1, \dots, p_k$ and $q \in [1, \infty)$, which is basically the family of domains which appeared in the proof of Proposition 45, except that now there is an additional boundary marked point ζ_2, which is an input. A similar degeneration of domain argument as in the proof of Proposition 45 implies that

$$\phi_{L,L}^{1,2;k}(y, x) = \phi_{L,L \subset T^*L}^{1,2}(v_k^!(y), x) + \mu^1(t_{L,L}^{1,2;k}(y, x)) + t_{L,L}^{1,2;k}(y, \mu^1(x))$$

$$+ \sum_{j=0}^{k} t_{L,L}^{1,2;j}(\delta_{k-j}(y), x),$$

where

$$\phi^{1,2}_{L,L \subset T^*L} : \mathrm{SC}^*(T^*L) \otimes \mathrm{CF}^*(L,L) \to \mathrm{CF}^*(L,L)[-2k-1]$$

is a closed-open map defined in the Liouville subdomain $D^*L \subset \bar{M}$. When applied to the cyclic dilation $\widetilde{\beta} \in \mathrm{SC}^1_{S^1}(M)$, we obtain by Lemma 44 that

$$\sum_{k=0}^{\infty} \phi^{1,2;k}_{L,L}(\beta_k, x) = \phi^{1,2}_{L,L \subset T^*L}\left(\sum_{k=0}^{\infty} v^!_k(\beta_k), x\right) + \mu^1\left(\sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, x)\right)$$

$$+ \sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, \mu^1(x)) + \sum_{k=0}^{\infty}\sum_{j=0}^{k} t^{1,2;j}_{L,L}(\delta_{k-j}(\beta_k), x)$$

$$= \alpha_L \cdot \phi^{1,2}_{L,L \subset T^*L}(\beta, x) + \mu^1\left(\sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, x)\right)$$

$$+ \sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, \mu^1(x)),$$

where $\alpha_L \in \mathbb{K}^\times$ and the cocycle $\beta \in \mathrm{SC}^1(T^*L)$ represents the dilation. Note that, in the computations above, the term $\sum_{k=0}^{\infty}\sum_{j=0}^{k} t^{1,2;j}_{L,L}(\delta_{k-j}(\beta_k), x)$ vanishes because $\widetilde{\beta}$ defines an S^1-equivariant cocycle; see the proof of Proposition 45. We then have

$$\phi_{\widetilde{L},\widetilde{L}}(x) = \alpha_L \cdot \phi^{1,2}_{L,L \subset T^*L}(\beta, x) + \mu^1\left(\sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, x)\right)$$

$$+ \sum_{k=0}^{\infty} t^{1,2;k}_{L,L}(\beta_k, \mu^1(x)) - \mu^2(\widetilde{\gamma}_L, x) + \mu^2(x, \widetilde{\gamma}_L)$$

$$= \alpha_L \cdot \phi^{1,2}_{L,L \subset T^*L}(\beta, x) + \mu^2\left(\sum_{k=0}^{\infty} t^{1,1;k}_L(\beta_k), x\right)$$

$$- \mu^2\left(x, \sum_{k=0}^{\infty} t^{1,1;k}_L(\beta_k)\right) - \mu^2(\widetilde{\gamma}_L, x) + \mu^2(x, \widetilde{\gamma}_L), \qquad (5.33)$$

where the second equality holds because the map $t^{1,2;k}_{L,L}$ gives the homotopy between $\mu^2(t^{1,1;k}_L(\beta_k), x)$ and $\mu^2(x, t^{1,1;k}_L(\beta_k))$. The proposition now follows from the fact that

$$\phi_{\widetilde{L},\widetilde{L} \subset T^*L} := \alpha_L \cdot \phi^{1,2}_{L,L \subset T^*L}(\beta, x) - \mu^2(\gamma_L, x) + \mu^2(x, \gamma_L), \qquad (5.34)$$

where

$$\gamma_L := \widetilde{\gamma}_L - \sum_{k=0}^{\infty} t^{1,1;k}_L(\beta_k),$$

is a chain map, which is proved in [67, Section 4]. ∎

In this way, we obtain an endomorphism

$$\Phi_{\widetilde{L},\widetilde{L}} = [\phi_{\widetilde{L},\widetilde{L}}] : \mathrm{HF}^*(L, L) \to \mathrm{HF}^*(L, L),$$

which is independent of our choice of $\widetilde{\gamma}_L$. To understand the endomorphism, $\Phi_{\widetilde{L},\widetilde{L}}$ is equivalent to understanding its action on $\mathrm{HF}^0(L, L)$ and $\mathrm{HF}^n(L, L)$.

Proposition 47. $\Phi_{\widetilde{L},\widetilde{L}}$ *acts trivially on* $\mathrm{HF}^0(L, L)$.

Proof. If we use the fact that $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$ is a cyclic dilation, then by (5.33), the action of $\Phi_{\widetilde{L},\widetilde{L}}$ can be identified with the action of $[\phi_{\widetilde{L},\widetilde{L} \subset T^*L}]$ on $\mathrm{HF}^*(L, L)$ computed in the Weinstein neighborhood $D^*L \subset M$. It is proved in [67, Section 4] that $[\phi_{\widetilde{L},\widetilde{L} \subset T^*L}]$ is a derivation. Since $\mathrm{HF}^0(L, L) \cong \mathbb{K}$, the action of $[\phi_{\widetilde{L},\widetilde{L} \subset T^*L}]$ on $\mathrm{HF}^0(L, L)$ must have weight 0. Alternatively, one can make the argument independent of the fact that $\tilde{b}$ is a cyclic dilation. This is inspired by [67, Remark 4.4]. Consider the moduli space $\mathcal{M}_{k,+}(y, L)$ of pairs $((Z^+, p_1, \ldots, p_k), u)$ studied in Section 5.1; there is a double evaluation map $\mathrm{Ev} : S^1 \times \bar{\mathcal{M}}_{k,+}(y, L) \to L \times L$ defined by

$$(t, u) \mapsto (u(0, 0), u(0, t)).$$

The formal sum of $\mathrm{Ev}_*[S^1 \times \bar{\mathcal{M}}_{k,+}(\beta_k, L)]$ for each $k \geq 0$ represents a topological cycle in $L \times L$. If we use the singular chain model of Lagrangian Floer cohomology $\mathrm{HF}^*(L, L) \cong H^*(L; \mathbb{K})$; then the action of $\Phi_{\widetilde{L},\widetilde{L}}$ on $H^*(L; \mathbb{K})$ can be equivalently interpreted as taking a cycle on the first sector L, intersecting with the cycle $\sum_{k=0}^{\infty} \mathrm{Ev}_*[S^1 \times \bar{\mathcal{M}}_{k,+}(\beta_k, L)]$, and then projecting to the second factor. In particular, the class $\Phi_{\widetilde{L},\widetilde{L}}([e_L])$ is represented by the evaluation maps $(t, u) \mapsto u(0, 0)$. Since the evaluation maps factor through the projections $S^1 \times \bar{\mathcal{M}}_{k,+}(y, L) \to \bar{\mathcal{M}}_{k,+}(y, L)$, it represents the zero homology class. ∎

To see that the endomorphism $\Phi_{\widetilde{L},\widetilde{L}}$ is non-trivial, the fact that $\tilde{b}$ is a cyclic dilation plays a crucial role.

Proposition 48. *Let* $\tilde{b} \in \mathrm{HF}^1_{S^1}(\lambda)$ *be a cyclic dilation, and let* $\widetilde{L}$ *be a* $\tilde{b}$-*equivariant Lagrangian sphere, then the action of* $\Phi_{\widetilde{L},\widetilde{L}}$ *on* $\mathrm{HF}^n(L, L)$ *is the multiplication by a non-zero scalar* α_L. *Moreover, if* $h = 1$ *in the definition of a cyclic dilation, then one can take* $\alpha_L = 1$ *for all* $\widetilde{L}$.

Proof. As noticed in the proof of Proposition 47, the action of $\Phi_{\widetilde{L},\widetilde{L}}$ on $\mathrm{HF}^n(L, L)$ coincides with the action of $[\phi_{\widetilde{L},\widetilde{L} \subset T^*L}]$ on $\mathrm{HF}^n(L, L)$ computed in the Weinstein neighborhood $D^*L \subset M$. If $h = 1$, then $\alpha_L = 1$ in Lemma 44, and $\beta \in \mathrm{SC}^1(T^*L)$ in (5.34) represents a dilation, so it follows from [56, Lemma 18.1] that the action of $[\phi_{\widetilde{L},\widetilde{L} \subset T^*L}]$ on $\mathrm{HF}^n(L, L)$ has weight 1. The general case follows from a rescaling by α_L. ∎

For convenience, we introduce the following notation:

$$\widetilde{L} \bullet \widetilde{L} := \mathrm{Str}(\Phi_{\widetilde{L}, \widetilde{L}}) \in \mathbb{K}.$$

It follows from Propositions 47 and 48 the following lemma.

Lemma 49. *Let $L \subset M$ be a Lagrangian sphere with dimension $n \geq 3$ and n is odd; then $\widetilde{L} \bullet \widetilde{L} = -\alpha_L$.*

This fact will be crucial to the proof of Theorem 13.

5.4. Odd-dimensional spheres

In this subsection we prove Theorem 13. The argument is a modification of that of Seidel in [63, Sections 3 and 4], and the main new ingredient that enters into our proof is the operation $*_k$ introduced in Appendix A. First, we want to make precise the meaning of property $(\widetilde{H})$ in the statement of Theorem 13.

Definition 50. Let M be a Liouville manifold with $c_1(M) = 0$. We say that M has property $(\widetilde{H})$ if there is a real number $\lambda > 0$ such that

(i) $2\lambda \notin \mathcal{P}_M$;

(ii) $\lambda < \min \mathcal{P}_M$. In particular, the PSS map $H^*(M; \mathbb{K}) \to \mathrm{HF}^*(\lambda)$ is an isomorphism;

(iii)   there is a cyclic dilation $\tilde{b} \in \mathrm{HF}^1_{S^1}(2\lambda)$.

The above definition is motivated by [63, Definition 2.11], where the corresponding notion deals with the special case of a dilation. As has been mentioned in Section 2.5, a lot of known examples of Liouville manifolds admitting cyclic dilations satisfy property $(\widetilde{H})$. On the other hand, it is currently unknown whether there are examples of Liouville manifolds with cyclic dilations for which the property $(\widetilde{H})$ is violated.

We take a brief look at the situation when $h = 1$ and explain what does property $(\widetilde{H})$ mean in this special case. Recall that the *Gutt–Hutchings capacities* [35] of the Liouville domain $\overline{M}$ are a sequence of symplectic capacities $\{c_k^{\mathrm{GH}}(M)\}_{k \geq 1}$ defined to be

$$c_k^{\mathrm{GH}}(M) := \inf\{a \mid \delta_{\mathrm{eq}}(x) = u^{-k+1} e \text{ for some } x \in F^{\leq a} \, \mathrm{SC}_{S^1}^{-2k+1}(M)\},$$

where $F^{\bullet}$ is induced by the action filtration on $\mathrm{SC}^*(M)$ (cf. (6.12)). It is clear from the definition and our discussions in Section 4.2, in particular Remark 31, that M admits a cyclic dilation with $h = 1$ if and only if $c_1^{\mathrm{GH}}(M) < \infty$.

Suppose that there is a sufficiently small number $a > 0$ so that there exists a cochain $x \in F^{\leq a} \, \mathrm{SC}_{S^1}^{-1}(M)$ with $\delta_{\mathrm{eq}}(x) = e$, then, according to the definition of the

action functional (cf. (6.11)), we conclude that there exists a small enough $\lambda > 0$ so that $x \in \mathrm{CF}_{S^1}^{-1}(2\lambda)$, which in turn implies that property $(\widetilde{\mathrm{H}})$ holds for M. On the other hand, knowing the geometry of M would enable us to find the largest possible a for a fixed slope λ. Denote this number by a_λ; then M satisfies property $(\widetilde{\mathrm{H}})$ with $h = 1$ as long as $c_1^{\mathrm{GH}}(M) \leq a_\lambda$. Define

$$a_M := \lim_{\lambda \to \min \mathscr{P}_M} a_\lambda.$$

Thus, as a special case of Theorem 13, we have the following corollary.

Corollary 51. *Let $\bar{M}$ be a Weinstein domain with dimension $2n \geq 6$, where n is odd, and whose first Gutt–Hutchings capacity satisfies $c_1^{\mathrm{GH}}(M) < a_M$. Then, for any Lagrangian sphere $L \subset M$, its homology class $[L] \in H_n(M; \mathbb{Q})$ is non-zero.*

From now on, let M be a $2n$-dimensional Weinstein manifold satisfying property $(\widetilde{\mathrm{H}})$. As before, we denote by $\widetilde{\beta} = \sum_{k=0}^{\infty} \beta_k \otimes u^{-k} \in \mathrm{CF}_{S^1}^1(2\lambda)$ the cochain-level representative of the cyclic dilation $\widetilde{b}$.

Let $L \subset M$ be any Lagrangian submanifold satisfying Assumption 38. Following [63, Section 4d], for every $k \geq 0$, we define a map

$$\chi_k : \mathrm{CF}^{2k+1}(2\lambda) \to \mathbb{K},$$

which is the sum of the following six expressions:

(i)$_k$ $\mathrm{CF}^{2k+1}(2\lambda) \xrightarrow{\phi_L^{2,0;k}} \mathrm{CF}^n(\lambda) \xrightarrow{\langle \phi_L^{1,0}, \cdot \rangle} \mathbb{K}$,

(ii)$_k$ $\mathrm{CF}^{2k+1}(2\lambda) \xrightarrow{\phi_L^{1,1;k}} \mathrm{CF}^1(L, L) \xrightarrow{\psi_{L,L}^{0,1}} \mathbb{K}$,

(iii)$_k$ $x \mapsto (-1)^{n(n+1)/2+1} \mathrm{Str}(\phi_{L,L}^{1,2;k}(x, \cdot))$,

(iv)$_k$ $\mathrm{CF}^{2k+1}(2\lambda) \xrightarrow{\phi_L^{1,1;k}} \mathrm{CF}^1(L, L) \xrightarrow{-(\psi_{L,L}^{0,1})^\vee} \mathbb{K}$,

(v)$_k$ $\mathrm{CF}^{2k+1}(2\lambda) \xrightarrow{\phi_L^{2,0;k}} \mathrm{CF}^n(\lambda) \xrightarrow{(-1)^{n+1}\langle \phi_L^{1,0}, \cdot \rangle} \mathbb{K}$,

(vi)$_k$ $(-1)^n \langle \cdot, \phi_L^{1,0} *_k \phi_L^{1,0} \rangle$,

where non-degenerate pairing

$$\langle \cdot, \cdot \rangle : \mathrm{CF}^*(\lambda) \otimes \mathrm{CF}^{2n-*}(-\lambda) \to \mathbb{K},$$

gives the chain-level realizations of the Poincaré duality isomorphism on Floer cohomologies. When $k = 0$, χ_0 is a chain map, and it is proved in [63, Proposition 4.11] that χ_0 is nullhomotopic, since the expressions (i)$_0$–(vi)$_0$ correspond to boundary components of a 2-dimensional family of Riemann surfaces. See also [63, Section 5b]. In general, we have the following proposition.

Proposition 52. *When n is odd, there exist choices of Floer data so that the following identity holds:*

$$\sum_{k=0}^{\infty} \chi_k(\beta_k) = 0. \tag{5.35}$$

Proof. Let $\overline{S}$ be an annulus with a unique interior puncture ζ_{in}, which is an input, and no boundary punctures. Denote by $\partial_{\text{in}} S$ the inner boundary of S and by $\partial_{\text{out}} S$ the outer boundary of S; both of them are labeled by L. Moreover, there are k auxiliary marked points $p_1, \ldots, p_k$ lying in a small neighborhood of ζ_{in}, and they are ordered so that (5.11) is satisfied with respect to the local complex coordinate near ζ_{in}. The asymptotic marker ℓ_{in} at ζ_{in} is required to point toward p_k. See the central picture of Figure 6 for a description of the domain. The associated operation will be denoted as

$$\psi_{L,L}^{1,0;k} : \mathrm{CF}^{2k+2}(2\lambda) \to \mathbb{K}.$$

When there is no auxiliary marked point, the Riemann surfaces $(\overline{S}; \zeta_{\text{in}}, \ell_{\text{in}})$, where ℓ_{in} is allowed to vary in a specific way, form a 2-dimensional family $\mathcal{T}$, which compactifies to a hexagon $\overline{\overline{\mathcal{T}}}$; see Figure 6. The construction of this 2-dimensional family, which arises as the real blowup of the KSV (Kimura–Stasheff–Voronov) compactification, is explained in detail in [63, Section 5b]. Note that the orientations of the boundary components (iii), (iv), and (v) reverse that of the usual boundary orientation of a hexagon, which leads to an additional -1 in the definitions of the operations (iii)$_k$, (iv)$_k$, and (v)$_k$ appeared in the expression of χ_k. After the points $p_1, \ldots, p_k$ are added, we get a (compactified) $(2k + 2)$-dimensional family $\overline{\overline{\mathcal{T}_k}}$ of domains $(\overline{S}; \zeta_{\text{in}}, \ell_{\text{in}}; p_1, \ldots, p_k)$ which fibers over the hexagon, whose fibers can be identified with the compactifications of the moduli space of k-point angle decorated half-cylinders $\overline{\mathcal{M}}_{k,+}$ considered in Section 5.1. Figure 6 depicts a section of the moduli space $\overline{\overline{\mathcal{T}_3}}$ obtained by fixing the positions of p_1, p_2, p_3. There are six codimension 1 boundary strata in $\overline{\overline{\mathcal{T}_k}}$ which correspond, respectively, to the boundaries of the hexagon labeled by (i)–(vi) in Figure 6, and they give rise to the operations (i)$_k$–(vi)$_k$ defined above. When the points $p_1, \ldots, p_k$ are allowed to vary, there are additional strata in $\partial \overline{\overline{\mathcal{T}_k}}$ corresponding to the loci where $|p_i| = |p_{i+1}|$ and $|p_1| = \varepsilon$.

For the boundary strata $|p_i| = |p_{i+1}|$, where $1 \leq i \leq k - 1$, it is clear from our previous discussions that one can choose Floer data so that they are non-rigid.

For the boundary stratum $|p_1| = \varepsilon$, denote its contribution by

$$\psi_{L,L;S^1}^{1,0;k-1} : \mathrm{CF}^{2k+1}(2\lambda) \to \mathbb{K}.$$

Consider the automorphism of the annulus $\overline{S}$ which switches the boundary circles $\partial_{\text{in}} S$ and $\partial_{\text{out}} S$, under which we obtain an operation which, up to homotopy, can be
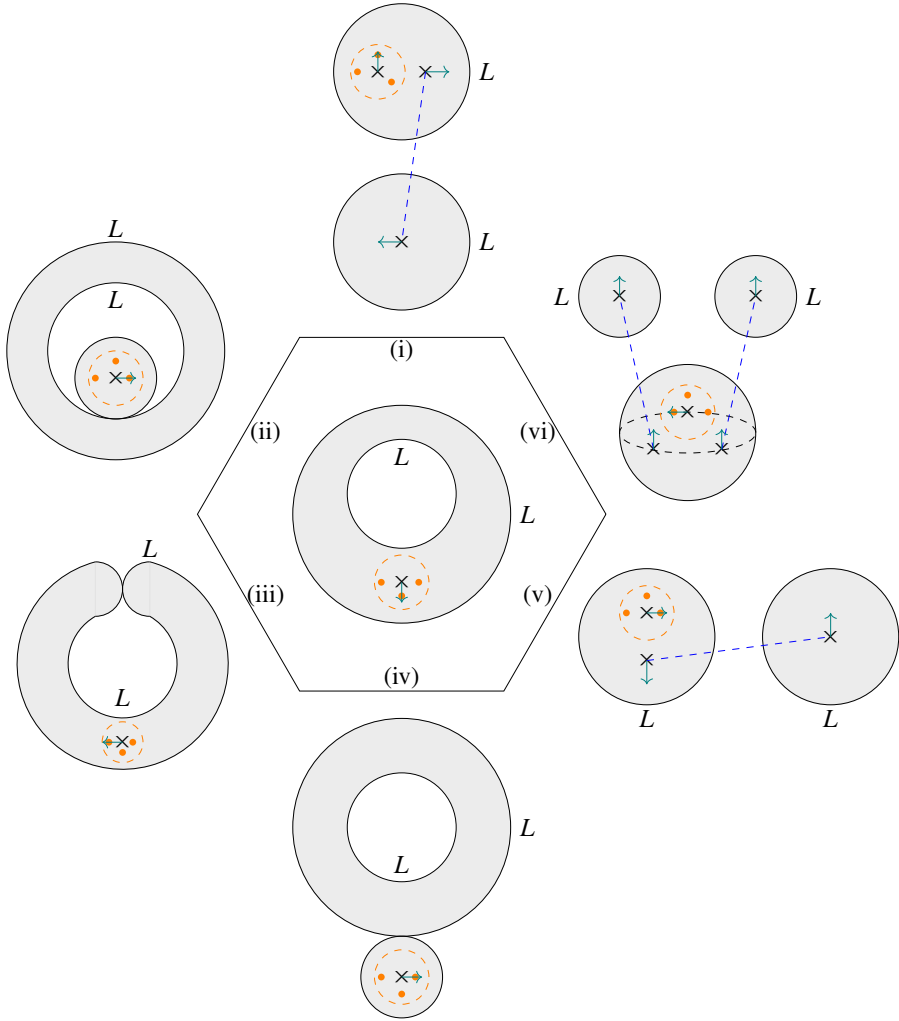
Figure 6. A section of the family $\bar{\bar{\mathcal{T}}}_3$.

identified with $\psi_{L,L;S^1}^{1,0;k-1}$. Since exchanging the order of the two boundary circles leads to a Koszul sign of $(-1)^n$ [63, Remark 5.3], we have

$$\psi_{L,L;S^1}^{1,0;k-1} - (-1)^n \psi_{L,L;S^1}^{1,0;k-1} = \text{null homotopy}. \tag{5.36}$$

This shows that the stratum $|p_1| = \frac{1}{2}$ of $\partial\bar{\bar{\mathcal{T}}}_k$ does not contribute when n is odd.

Finally, there are strata coming from real blowups in the fiber direction of the compactification $\bar{\bar{\mathcal{T}}}_k$, which correspond to sphere bubbles at the puncture ζ_{in}, containing at least one auxiliary marked point. These strata, together with the sphere bubble

at ζ_{in} without auxiliary marked point, contribute

$$\sum_{j=0}^{k} \psi_{L,L}^{1,0;k-j}(\delta_j(\beta_k)) \tag{5.37}$$

for each $k \geq 0$. Using the fact that $\widetilde{\beta} \in \mathrm{CF}_{S^1}^1(2\lambda)$ is an equivariant cocycle, we see that the sum of (5.37) over $k \geq 0$ vanishes.

Combining the above analysis, we get (5.35) when n is odd. ∎

Remark 53. Note that when $\mathrm{char}(\mathbb{K}) = 2$, the relation (5.36) holds for trivial reasons, so we cannot use it to deduce that

$$\psi_{L,L;S^1}^{1,0;k-1} = 0,$$

and the argument above fails.

Denote by $(\widetilde{C}^*, \tilde{d})$ the direct sum of two Floer cochain complexes

$$\widetilde{C}^* := \mathrm{CF}^*(-\lambda) \oplus \mathrm{CF}^*(\lambda),$$
$$\tilde{d}(\xi, x) := (d\xi, dx),$$

where d is the ordinary Floer differential, and $\lambda > 0$ is chosen as in Definition 50. The cohomology of $(\widetilde{C}^*, \tilde{d})$ will be denoted by $\widetilde{H}^*$. Note that, by item (ii) of Definition 50, we can choose the Hamiltonian $H_{\lambda,t}$ in the definition of the Floer cochain complex $\mathrm{CF}^*(\lambda)$ so that it is isomorphic to the Morse complex of some function

$$\phi : M \to \mathbb{R},$$

which computes the ordinary cohomology $H^*(M; \mathbb{K})$. Since M is a Weinstein manifold, we may assume that the ϕ is J-convex so that all the Morse critical points have index $\leq n$, and therefore $\mathrm{CF}^*(\lambda)$ is supported in degrees $* \leq n$. For similar reasons, we can arrange so that $\mathrm{CF}^*(-\lambda)$ is supported in degrees $* \geq n$.

We define a pairing on the above chain complex

$$\tilde{\iota} : \widetilde{C}^* \otimes \widetilde{C}^{2n-*} \to \mathbb{K},$$

$$\tilde{\iota}((\xi_0, x_0), (\xi_1, x_1)) = \langle x_0, \xi_1 \rangle - (-1)^{|\xi_0|}\langle x_1, \xi_0 \rangle + \sum_{k=0}^{\infty} \langle \beta_k, \xi_0 *_k \xi_1 \rangle.$$

In the above, the operations

$$*_k : \mathrm{CF}^*(-\lambda) \otimes \mathrm{CF}^{2n-*}(-\lambda) \to \mathrm{CF}^{2n-2k-1}(-2\lambda)$$

are parametrized analogs of the usual star product (A.18) on the Floer cochain complex. Their detailed constructions are recorded in Appendix A.

Lemma 54. $\tilde{\iota}$ *is a chain map.*

Proof. Picking any cochains (ξ_0, x_0) and (ξ_1, x_1) whose degrees add up to $2n - 1$, we have

$$\tilde{\iota}((d\xi_0, dx_0), (\xi_1, x_1)) + (-1)^{|\xi_0|}\tilde{\iota}((\xi_0, x_0), (d\xi_1, dx_1))$$

$$= \langle dx_0, \xi_1 \rangle + (-1)^{|\xi_0|}\langle x_0, d\xi_1 \rangle - (-1)^{|\xi_1|}\langle x_1, d\xi_0 \rangle - \langle dx_1, \xi_0 \rangle$$

$$+ \sum_{k=0}^{\infty} \langle \beta_k, d\xi_0 *_k \xi_1 + (-1)^{|\xi_0|}\xi_0 *_k d\xi_1 \rangle$$

$$= \sum_{k=0}^{\infty} \langle \beta_k, d\xi_0 *_k \xi_1 + (-1)^{|\xi_0|}\xi_0 *_k d\xi_1 \rangle.$$

Since $\mathrm{CF}^*(-\lambda)$ is supported in degrees $* \geq n$ and

$$|\xi_0| + |\xi_1| = 2n - 1$$

by our assumption, either ξ_0 or ξ_1 is zero, which in turn implies that

$$d\xi_0 *_k \xi_1 + (-1)^{|\xi_0|}\xi_0 *_k d\xi_1 = 0. \qquad \blacksquare$$

The induced pairing on cohomology will be denoted by

$$\tilde{I} : \tilde{H}^* \otimes \tilde{H}^{2n-*} \to \mathbb{K}.$$

To each $\tilde{b}$-equivariant Lagrangian sphere $\tilde{L} = (L, \tilde{\gamma}_L)$ we can associate a cocycle

$$(\xi_L, x_L) := \left(\phi_L^{1,0}, (-1)^{n+1} \sum_{k=0}^{\infty} \phi_L^{2,0;k}(\beta_k) + (\phi_L^{1,1})^\vee(\tilde{\gamma}_L) \right) \in \tilde{C}^n. \qquad (5.38)$$

To see that (ξ_L, x_L) is indeed a cocycle, notice first that $\phi_L^{1,0} \in \mathrm{CF}^n(-\lambda)$ is by definition a cocycle. On the other hand, our assumption implies that the differential of the second entry of (5.38) vanishes for degree reasons. Denote by $[\![\tilde{L}]\!] \in \tilde{H}^n$ the cohomology class of (ξ_L, x_L). As a simple observation, under the natural projection

$$\tilde{H}^n \to \mathrm{HF}^n(-\lambda),$$

the class $[\![\tilde{L}]\!]$ goes to the Floer cohomology class $[\![L]\!]$ defined by (5.15).

Theorem 55. *Let $L \subset M$ be a Lagrangian sphere of dimension $n \geq 3$ and n is odd; then*

$$\tilde{I}([\![\tilde{L}]\!], [\![\tilde{L}]\!]) = (-1)^{n(n+1)/2} \tilde{L} \bullet \tilde{L}. \qquad (5.39)$$

Proof. We have

$$
(-1)^{n(n+1)/2} \operatorname{Str}(\phi_{\widetilde{L},\widetilde{L}})
$$

$$
= (-1)^{n(n+1)/2} \left(\sum_{k=0}^{\infty} \operatorname{Str}(\phi_{L,L}^{1,2;k}(\beta_k, \cdot)) - \operatorname{Str}(\mu^2(\widetilde{\gamma}_L, \cdot)) + \operatorname{Str}(\mu^2(\cdot, \widetilde{\gamma}_L)) \right)
$$

$$
= (-1)^{n(n+1)/2} \sum_{k=0}^{\infty} \operatorname{Str}(\phi_{L,L}^{1,2;k}(\beta_k, \cdot)) - (-1)^n \sum_{k=0}^{\infty} \psi_{L,L}^{0,1}(\phi_L^{1,1;k}(\beta_k))
$$

$$
- \langle \phi_L^{1,0}, (\phi_L^{1,1})^{\vee}(\widetilde{\gamma}_L) \rangle
$$

$$
+ (-1)^n \sum_{k=0}^{\infty} (\psi_{L,L}^{0,1})^{\vee}(\phi_L^{1,1;k}(\beta_k)) + (-1)^n \langle \phi_L^{1,0}, (\phi_L^{1,1})^{\vee}(\widetilde{\gamma}_L) \rangle,
$$

where the first line follows from (5.32), and the second line follows from (5.20), (5.21), and (5.31). Proposition 52 applied to the components of $\widetilde{\beta} \in \mathrm{CF}^1(2\lambda)$ yields

$$
(-1)^{n(n+1)/2} \sum_{k=0}^{\infty} \operatorname{Str}(\phi_{L,L}^{1,2;k}(\beta_k, \cdot)) - (-1)^n \sum_{k=0}^{\infty} \psi_{L,L}^{0,1}(\phi_L^{1,1;k}(\beta_k))
$$

$$
+ (-1)^n \sum_{k=0}^{\infty} (\psi_{L,L}^{0,1})^{\vee}(\phi_L^{1,1;k}(\beta_k))
$$

$$
= - \sum_{k=0}^{\infty} \langle \phi_L^{1,0}, \phi_L^{2,0;k}(\beta_k) \rangle + \sum_{k=0}^{\infty} \langle \beta_k, \phi_L^{1,0} *_k \phi_L^{1,0} \rangle + (-1)^n \sum_{k=0}^{\infty} \langle \phi_L^{1,0}, \phi_L^{2,0;k}(\beta_k) \rangle,
$$

so one can rewrite

$$
(-1)^{n(n+1)/2} \operatorname{Str}(\phi_{\widetilde{L},\widetilde{L}}) = \left\langle (-1)^{n+1} \sum_{k=0}^{\infty} \phi_L^{2,0;k}(\beta_k) + (\phi_L^{1,1})^{\vee}(\widetilde{\gamma}_L), \phi_L^{1,0} \right\rangle
$$

$$
- (-1)^n \left\langle (-1)^{n+1} \sum_{k=0}^{\infty} \phi_L^{2,0;k}(\beta_k) + (\phi_L^{1,1})^{\vee}(\widetilde{\gamma}_L), \phi_L^{1,0} \right\rangle
$$

$$
+ \sum_{k=0}^{\infty} \langle \beta_k, \phi_L^{1,0} *_k \phi_L^{1,0} \rangle
$$

$$
= \widetilde{\imath}((\xi_L, x_L), (\xi_L, x_L)).
$$

Our assumption ensures that $\phi_{\widetilde{L},\widetilde{L}}$ is a well-defined chain map, so on the cohomology level, we get (5.39). ∎

Proof of Theorem 13. Let $L \subset M$ be any Lagrangian sphere of some odd dimension $n \geq 3$; it follows from Lemma 49 that $\widetilde{L} \bullet \widetilde{L} = -\alpha_L$ for some $\alpha_L \neq 0$. By Theorem 55, this implies that $\widetilde{I}(\llbracket \widetilde{L} \rrbracket, \llbracket \widetilde{L} \rrbracket) \neq 0$.

By definition, $HF^n(\lambda) \subset \widetilde{H}^n$ is a half-dimensional subspace, which is isotropic for the pairing $\widetilde{I}$. This implies that by projecting to the first factor $HF^n(-\lambda)$, the class $[\![L]\!]$, which is the image of $[\![\widetilde{L}]\!]$, is non-zero. By Definition 50 (ii), the dual of the PSS map $HF^n(-\lambda) \to H^n_{cpt}(M; \mathbb{K})$ is an isomorphism, and the Floer cohomology class $[\![L]\!]$ is therefore mapped to the Poincaré dual of the ordinary homology class $[L] \in H_n(M; \mathbb{K})$.   ∎

We end this section with a short remark on the signs appeared in various formulas in Sections 5.3 and 5.4. Basically, we follow the convention in Seidel's original argument; see [63, Section 5c] for details. The only difference here is that we are dealing with operations defined using an additional parametrization by k auxiliary marked points, so the orientations of the relevant moduli spaces defining the operations relating open and closed string invariants are fixed by following the original convention of Seidel and choosing a preferred orientation of the moduli space $\bar{\mathcal{M}}_{k,+}$ of k point angle decorated half-cylinders. The (relative) orientation of $\bar{\mathcal{M}}_{k,+}$ is determined inductively in [77, Lemmas 4.4.7 and 4.4.11]. This allows us to arrange the signs involved in the above computations so that they coincide with those of [63, 67].

6. Existence of cyclic dilations

Let M be a $2n$-dimensional Liouville manifold with $c_1(M) = 0$. In this section, we consider the existence questions of cyclic dilations. In Section 6.1, we use Koszul duality to show that the manifold $M_{3,3,3,3}$ admits a cyclic dilation. This example is non-trivial as $M_{3,3,3,3}$ does not admit a quasi-dilation. With the help of Lefschetz fibrations, one can produce more examples of Liouville manifolds which carry cyclic dilations starting from the known ones. This is done in Section 6.2. Section 6.3 proves the uniqueness of smooth Calabi–Yau structures on the wrapped Fukaya categories of log general-type affine varieties containing exact Lagrangian $K(\pi, 1)$'s, from which Theorem 12 follows as a corollary. The discussions in Section 6.4 are mostly speculative; they are included here merely as supplements to Section 2.4.

6.1. Koszul duality

Although for the most part of this paper we have been taking a geometric viewpoint, dealing with cyclic dilations in equivariant symplectic cohomologies instead of exact Calabi–Yau structures on wrapped Fukaya categories, this subsection is an exception. Here, we will return to the original notion of an exact Calabi–Yau structure (Definition 2) which motivates the whole paper and study it essentially from the algebraic perspective, based on a result of Van den Bergh (Theorem 56).

Before we proceed, recall that smooth Calabi–Yau structures are Morita invariant, so it makes no difference to study smooth Calabi–Yau structures on an A_∞-algebra $\mathcal{A}$ over some semisimple ring $\Bbbk$, or to consider them as Calabi–Yau structures on the A_∞-category $\mathcal{A}^{\mathrm{perf}}$. See [18, Theorem 3.1] for an explanation of this fact.

One of the main ingredients of our proof of Theorem 7 is the following theorem due to Van den Bergh [74], which enables us to characterize a large class of exact Calabi–Yau A_∞-algebras in terms of its Koszul dual. Recall that a *cyclic A_∞-algebra* $\mathcal{B}$ over $\Bbbk$ is an A_∞-algebra equipped with a chain-level perfect pairing

$$\langle \bullet, \bullet \rangle : \mathcal{B} \otimes \mathcal{B} \to \Bbbk[-n]$$

such that the induced correlation functions

$$\langle \mu_{\mathcal{B}}^k(\bullet, \ldots, \bullet), \bullet \rangle$$

are strictly (graded) cyclically symmetric for each $k \geq 1$.

Theorem 56 ([74, Theorem 11.1]). *Let $\mathcal{A}$ be a homologically smooth, complete, augmented dg algebra over $\Bbbk$ so that $H^*(\mathcal{A})$ is concentrated in degrees ≤ 0. Denote by $\mathcal{A}^! := R\mathrm{Hom}_{\mathcal{A}}(\Bbbk, \Bbbk)$ the Koszul dual of $\mathcal{A}$. Then, the following statements are equivalent:*

- *$\mathcal{A}^!$ is a proper A_∞-algebra which, up to quasi-isomorphism, carries a minimal cyclic A_∞-structure of degree n.*

- *$\mathcal{A}$ is exact n-Calabi–Yau.*

Here, by *complete* we mean that the underlying associative algebra of $\mathcal{A}$ is a quotient of the path algebra of some quiver completed at path length.

Theorem 56 should be understood in the more general framework of Koszul duality between Calabi–Yau structures, which we now describe. Recall that, over a field $\mathbb{K}$ of characteristic 0, cyclic A_∞-structures provide explicit models for the more general notion of a proper Calabi–Yau structure. Precisely, a *proper n-Calabi–Yau structure* on a proper A_∞-algebra $\mathcal{B}$ over $\Bbbk$ is defined as a degree n chain map

$$\widetilde{\mathrm{tr}} : \mathrm{CC}_*(\mathcal{B}) \to \mathbb{K}[-n]$$

whose projection to the Hochschild complex, $\mathrm{tr} : \mathrm{CH}_*(\mathcal{B}) \to \mathbb{K}[-n]$, defines a *weak proper n-Calabi–Yau structure*; i.e., it induces a perfect pairing

$$H^*(\mathrm{hom}_{\mathcal{B}^{\mathrm{perf}}}(\mathcal{P}, \mathcal{Q})) \otimes H^{n-*}(\mathrm{hom}_{\mathcal{B}^{\mathrm{perf}}}(\mathcal{Q}, \mathcal{P})) \xrightarrow{[\mu_{\mathcal{B}}^2]} H^n(\mathrm{hom}_{\mathcal{B}^{\mathrm{perf}}}(\mathcal{Q}, \mathcal{Q}))$$

$$\to \mathrm{HH}_n(\mathcal{B}) \xrightarrow{[\mathrm{tr}]} \mathbb{K}.$$

When $\mathrm{char}(\mathbb{K}) = 0$, any cyclic A_∞-algebra $\mathcal{B}$ over $\Bbbk$ has a canonically defined proper Calabi–Yau structure, and any proper Calabi–Yau structure on an A_∞-algebra $\mathcal{B}$

determines a quasi-isomorphism between $\mathcal{B}$ and a cyclic A_∞-algebra; the latter fact is due to Kontsevich–Soibelman [44]. It is proved by Ganatra [28, Theorem 2] that any full subcategory of the compact Fukaya category $\mathcal{F}(M)$ admits a geometrically defined proper Calabi–Yau structure (under certain technical assumptions which ensure that the Fukaya category $\mathcal{F}(M)$ is well defined). As a consequence, we have the following proposition.

Proposition 57 ([28, Corollary 2]). *Let M be a Liouville manifold with $c_1(M) = 0$. If $\mathrm{char}(\mathbb{K}) = 0$, then any full A_∞-subcategory of the compact Fukaya category $\mathcal{F}(M)$ admits a minimal cyclic A_∞-structure.*

If $\mathcal{A}$ and $\mathcal{B}$ are Koszul dual A_∞-algebras, then there is a duality (cf. [37, Section 4])

$$\mathrm{CH}_{*-n}(\mathcal{A}) \cong \mathrm{hom}(\mathrm{CH}_{*+n}(\mathcal{B}), \mathbb{K})$$

between their Hochschild chains, which suggests that, under Koszul duality, non-degenerate cycles in $\mathrm{CH}_{-n}(\mathcal{A})$ should correspond to maps $\mathrm{CH}_{*+n}(\mathcal{B}) \to \mathbb{K}$ which induce proper Calabi–Yau structures on $\mathcal{B}$.

Theorem 58 (Cohen–Ganatra). *Let $\mathcal{A}$ be a homologically smooth dg algebra over $\mathbb{k}$, and let $\mathcal{A}^!$ be a proper A_∞-algebra so that $\mathcal{A}$ and $\mathcal{A}^!$ are Koszul dual. Then, $\mathcal{A}$ carries a smooth Calabi–Yau structure if and only if $\mathcal{A}^!$ is a proper Calabi–Yau A_∞-algebra.*

Proof. This is a slight variant of [18, Theorem 25], where $\mathcal{A}$ is required to be *strongly smooth* (cf. [18, Definition 3]), which means that $\mathcal{A}$ is homologically smooth and $\mathbb{k}$ is perfect as a module over $\mathcal{A}$. This latter property is needed to ensure that $\mathcal{A}^!$ is proper, which we have included in the assumption. ∎

From this perspective, the content of Theorem 56 can be understood as saying that if we further impose the assumptions that $\mathcal{A}$ is complete and supported in non-positive degrees, then the Calabi–Yau structure on $\mathcal{A}$ induced by the proper Calabi–Yau structure on $\mathcal{A}^!$ is not only smooth, but also exact.

Geometrically, the A_∞-Koszul duality between the endomorphism algebras of a set of generators in $\mathcal{F}(M)$ and $\mathcal{W}(M)$ is first studied by Etgü–Lekili in [26] when M is a plumbing of T^*S^2's according to a Dynkin tree and later generalized in [23, 49] to many interesting examples of Liouville manifolds in higher dimensions. More precisely, denote by $\mathcal{F}_M$ and $\mathcal{W}_M$ the A_∞-algebras of some fixed sets of split-generators of $\mathcal{F}(M)$ and $\mathcal{W}(M)$, respectively, and assume in addition that both of these sets are indexed by the same finite set Γ. By saying that the Fukaya categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are Koszul dual, we mean that there are quasi-isomorphisms between augmented A_∞-algebras

$$R\mathrm{Hom}_{\mathcal{F}_M}(\mathbb{k}, \mathbb{k}) \cong \mathcal{W}_M, \quad R\mathrm{Hom}_{\mathcal{W}_M}(\mathbb{k}, \mathbb{k}) \cong \mathcal{F}_M,$$

where $\Bbbk := \bigoplus_{v \in \Gamma} \Bbbk e_v$ is the semisimple ring consisting of $|\Gamma|$ copies of the ground field $\Bbbk$, and it is regarded as a left $\mathcal{F}_M$-module in the first quasi-isomorphism, and a right $\mathcal{W}_M$-module in the second quasi-isomorphism above.

We will need several results from [23], which enables us to verify the completeness of the A_∞-algebra $\mathcal{W}_M$ required in Theorem 56.

Fix a finite set Γ. Let $\bar{M}_{-\Lambda}$ be a $2n$-dimensional Weinstein domain, with its Liouville form denoted by θ_M. For each $v \in \Gamma$, let $\bar{L}_v \subset \bar{M}_{-\Lambda}$ be an oriented, connected, Spin Lagrangian submanifold with vanishing Maslov class such that its boundary $\partial \bar{L}_v \subset \partial \bar{M}_{-\Lambda}$ defines a Legendrian *sphere* Λ_v with respect to the contact structure on $\partial \bar{M}_{-\Lambda}$ defined by the restriction of θ_M. Moreover, we assume that different $\bar{L}_v$'s intersect with each other transversely, and the intersections happen only in the interior of $\bar{M}_{-\Lambda}$. In particular, the Legendrian spheres Λ_v's are disjoint from each other in $\partial \bar{M}_{-\Lambda}$; together they form a link $\Lambda := \bigsqcup_{v \in \Gamma} \Lambda_v$. Attaching n-handles to $\bar{M}_{-\Lambda}$ along the Legendrian link Λ gives rise to a new Weinstein domain $\bar{M}$. Note that $\bar{M}$ contains a set of closed Lagrangian submanifolds $\{L_v\}_{v \in \Gamma}$, which are unions of $\bar{L}_v$ with the Lagrangian core disks of the Weinstein handles attached along Λ. Define

$$\mathcal{V}_M := \bigoplus_{v,w \in \Gamma} \mathrm{CF}^*(L_v, L_w),$$

to be the Fukaya A_∞-algebra of these Lagrangian submanifolds, which is well defined and $\mathbb{Z}$-graded with our assumptions on the $\bar{L}_v$'s. This is an A_∞-algebra over $\Bbbk$. For simplicity, we will assume that $\mathcal{V}_M$ is strictly unital. Otherwise, there is always a standard algebraic procedure which replaces it with a quasi-isomorphic A_∞-algebra which is strictly unital [61, Lemma 2.1]. On the other hand, the Legendrian link $\Lambda \subset \partial \bar{M}_{-\Lambda}$ also has an associated dg algebra, the Chekanov–Eliashberg algebra $\mathrm{CE}^*(\Lambda)$. Denote by $\mathcal{R}$ the set of Reeb chords ending on Λ; we have

$$\mathrm{CE}^*(\Lambda) := \bigoplus_{i=0}^{\infty} \Bbbk \langle \mathcal{R} \rangle^{\otimes i},$$

with the differential defined by counting anchored holomorphic disks with boundary punctures in the symplectization $\mathbb{R} \times \partial \bar{M}_{-\Lambda}$, whose boundary components lie in the Lagrangian submanifold $\mathbb{R} \times \Lambda$ and whose punctures are asymptotic to the Reeb chords in $\mathcal{R}$; see [8, Section 4.1]. Note that $\mathrm{CE}^*(\Lambda)$ can also be realized as a dg algebra over $\Bbbk$ by declaring $e_w \mathcal{R} e_v$ to be the set of Reeb chords from Λ_w to Λ_v. Since the union of the Lagrangian submanifolds $\bigcup_{v \in \Gamma} \bar{L}_v \subset \bar{M}_{-\Lambda}$ gives a filling of the Legendrian link Λ, there is an induced augmentation

$$\varepsilon_L : \mathrm{CE}^*(\Lambda) \to \Bbbk, \tag{6.1}$$

which equips $\mathrm{CE}^*(\Lambda)$ with the structure of an augmented dg algebra over $\Bbbk$.

By [23, Theorem 4], we have the following quasi-isomorphism, which should be understood as a general version of the Eilenberg–Moore equivalence:

$$R\mathrm{Hom}_{\mathrm{CE}^*(\Lambda)}(\Bbbk, \Bbbk) \cong \mathcal{V}_M. \tag{6.2}$$

Using the augmentation (6.1), we can write

$$\mathrm{CE}^*(\Lambda) = \Omega \, \mathrm{LC}_*(\Lambda), \tag{6.3}$$

where Ω is the Adams cobar construction, and $\mathrm{LC}_*(\Lambda)$ is an A_∞-coalgebra over $\Bbbk$ whose underlying $\Bbbk$-bimodule is generated by $\mathcal{R}$, which is called the *Legendrian A_∞-coalgebra* in [23], whose linear dual $\mathrm{LC}_*(\Lambda)^\#$ is quasi-isomorphic to the Fukaya A_∞-algebra $\mathcal{V}_M$ defined above. On the other hand, the *completed Chekanov–Eliashberg dg algebra* is defined to be

$$\widehat{\mathrm{CE}}^*(\Lambda) := (\mathrm{B}\mathcal{V}_M)^\#, \tag{6.4}$$

where on the right-hand side the bar construction is taken with respect to the trivial augmentation $\varepsilon : \mathcal{V}_M \to \Bbbk$ defined by projecting to the idempotents in the degree 0 part.

By (6.3), we have

$$\mathrm{CE}^*(\Lambda) = \Bbbk \oplus \bigoplus_{i=1}^{\infty} \overline{\mathrm{LC}}_*(\Lambda)[-1]^{\otimes_\Bbbk i},$$

where $\overline{\mathrm{LC}}_*(\Lambda) \subset \mathrm{LC}_*(\Lambda)$ is the submodule obtained by quotienting out the idempotents e_v of $\Bbbk$. It follows from the definition (6.4) that, as a graded algebra over $\Bbbk$,

$$\widehat{\mathrm{CE}}^*(\Lambda) = \Bbbk\langle\langle \overline{\mathrm{LC}}_*(\Lambda)[-1]\rangle\rangle,$$

which is the completed tensor algebra of $\mathbb{K}\langle \overline{\mathrm{LC}}_*(\Lambda)[-1]\rangle$, regarded as a module over $\Bbbk$. In particular, there is a completion map

$$\phi : \mathrm{CE}^*(\Lambda) \to \widehat{\mathrm{CE}}^*(\Lambda). \tag{6.5}$$

Define a quiver Q_Λ so that its vertices correspond to elements of Γ, and its arrows are in correspondence with the set of Reeb chords $\mathcal{R}$. More precisely, for $v, w \in \Gamma$, there is an arrow from v to w for every Reeb chord in $\mathcal{R}$ from Λ_v to Λ_w. In this way, the underlying $\Bbbk$-algebra of $\mathrm{CE}^*(\Lambda)$ is the path algebra of Q_Λ, while the underlying $\Bbbk$-algebra of $\widehat{\mathrm{CE}}^*(\Lambda)$ is the completed path algebra $\widehat{\mathbb{K}Q_\Lambda}$. In particular, $\widehat{\mathrm{CE}}^*(\Lambda)$ is a complete augmented dg algebra in the sense of Theorem 56.

We now apply Theorem 56 to concrete geometric situations. As a quick application, let T be a tree with vertex set T_0. For each $v \in T_0$ we associate a simply connected closed manifold L_v of dimension $n \geq 3$. For simplicity, we also assume

that L_v is Spin. Denote by M_T the result of plumbing the cotangent bundles T^*L_v according to the tree T. As a Weinstein manifold, M_T admits a handlebody decomposition, whose associated subcritical Weinstein manifold is the plumbing of $T^*\bar{L}_v$ according to T, with $\bar{L}_v$ being the manifold with boundary obtained by carving out an open disk from L_v. Denote by

$$\Lambda_T = \bigsqcup_v \Lambda_v$$

the union of the boundaries of the manifolds $\{L_v\}_{v \in T_0}$.

Proposition 59. *The wrapped Fukaya category* $\mathcal{W}(M_T)$ *carries an exact Calabi–Yau structure.*

Proof. According to [23, Theorem 83] (see also [8, 22]), there is a surgery map

$$\Theta : \mathcal{W}_{M_T} \to CE^*(\Lambda_T), \tag{6.6}$$

which induces an isomorphism on homologies. In particular, the dg algebra $CE^*(\Lambda_T)$ is homologically smooth. It suffices to show that the Chekanov–Eliashberg dg algebra $CE^*(\Lambda_T)$ is an exact Calabi–Yau algebra.

It follows from the proof of [23, Theorem 68] that $CE^*(\Lambda_T)$ is quasi-isomorphic to a dg algebra concentrated in degrees ≤ 0, and it is Koszul dual to the Fukaya A_∞-algebra $\mathcal{F}_{M_T}$ of the compact cores $\{L_v\}_{v \in T_0}$. Combined with (6.4), we have

$$CE^*(\Lambda_T) \cong \widehat{CE}^*(\Lambda_T),$$

so the completion map ϕ is a quasi-isomorphism. $CE^*(\Lambda_T)$ is therefore a complete dg algebra in the sense of Theorem 56. Since $\mathrm{char}(\mathbb{K}) = 0$, Proposition 57 implies that, up to quasi-isomorphism, $\mathcal{F}_{M_T}$ carries a minimal cyclic A_∞-structure. Now, the conclusion follows from Theorem 56. ∎

Note that this gives an alternative way of seeing that the cotangent bundle T^*Q of a simply connected manifold Q admits a cyclic dilation; compare with our discussions at the end of Section 4.2. It is an interesting question whether the Weinstein manifolds M_T admit higher dilations.

Our second application deals with the specific case of the affine hypersurface $M_{3,3,3,3} \subset \mathbb{C}^4$. Recall that the Liouville 6-manifold $M_{3,3,3,3}$ arises as the Milnor fiber associated to the isolated singularity at the origin

$$x^3 + y^3 + z^3 + w^3 = 0,$$

which is known as a *3-fold triple point*. The smoothing of this singularity has been studied by Smith–Thomas [69], according to which we know that there is a basis
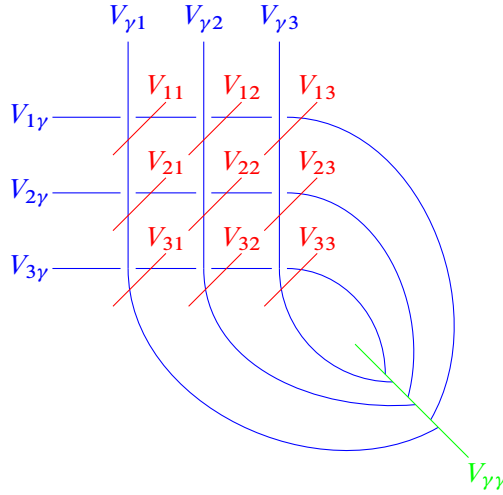
Figure 7. Configuration of vanishing cycles in $M_{3,3,3,3}$. Note that the spheres colored in blue are mutually disjoint.

of vanishing cycles in $M_{3,3,3,3}$ which consists of a configuration of 16 Lagrangian spheres, whose intersection pattern is indicated in Figure 7, where each arc represents a Lagrangian sphere.

One obtains from this the Legendrian surgery description of $M_{3,3,3,3}$.

Lemma 60. *The Milnor fiber $M_{3,3,3,3}$ is the result of attaching Weinstein 3-handles to D^6 along a Legendrian surface $\Lambda_{3,3,3,3} \subset (S^5, \xi_{\mathrm{std}})$, which is a disjoint union of 16 standard unknotted Legendrian S^2's. Up to Legendrian isotopy, the Legendrian fronts of 10 of the components in link $\Lambda_{3,3,3,3}$ are depicted in Figure 9, where each Legendrian unknot in the picture should be understood as a 2-sphere obtained by spinning the 1-dimensional unknot around along the vertical axis of symmetry of its front projection. The remaining 6 components $\Lambda_{12}, \Lambda_{13}, \Lambda_{21}, \Lambda_{23}, \Lambda_{31}, \Lambda_{32}$ are unknots linking $\Lambda_{1\gamma}$ and $\Lambda_{\gamma 2}$, $\Lambda_{1\gamma}$ and $\Lambda_{\gamma 3}$, $\Lambda_{2\gamma}$ and $\Lambda_{\gamma 1}$, $\Lambda_{2\gamma}$ and $\Lambda_{\gamma 3}$, $\Lambda_{3\gamma}$ and $\Lambda_{\gamma 1}$, and $\Lambda_{3\gamma}$ and $\Lambda_{\gamma 2}$, respectively, with all the linking numbers being ± 1. They are pairwise disjoint and disjoint from $\Lambda_{\gamma\gamma}$.*

Proof. Consider the Lefschetz fibration $t : \mathbb{C}^3 \to \mathbb{C}$ obtained as the Morsification of the polynomial $x^3 + y^3 + z^3$. The smooth fiber of t is symplectomorphic to the Milnor fiber $T_{3,3,3}$ associated to the singularity $x^3 + y^3 + z^3 = 0$, and its total monodromy is the composition of Dehn twists along a basis of 8 vanishing cycles in $T_{3,3,3}$; see [38, Section 4.2] for a detailed description of this Lefschetz fibration. By [75, Theorem 4.4], this describes D^6 as the result of attaching 8 Weinstein 3-handles
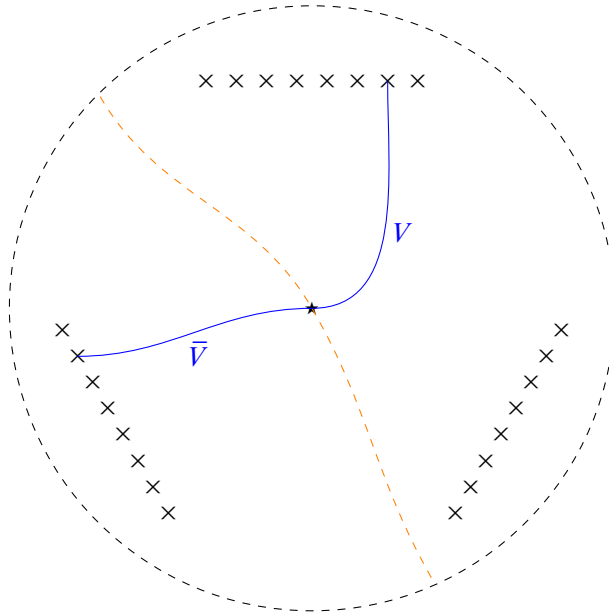
Figure 8. Base of the Lefschetz fibration $\pi : M_{3,3,3,3} \to \mathbb{C}$.

to $T_{3,3,3} \times D^2$ along a link of 8 Legendrian 2-spheres in $T_{3,3,3} \times S^1$, which restricts to the basis of vanishing cycles in $T_{3,3,3}$. Moreover, $M_{3,3,3,3}$ also carries a Lefschetz fibration $\pi : M_{3,3,3,3} \to \mathbb{C}$, with $T_{3,3,3}$ as its smooth fiber, under which the vanishing cycles of $M_{3,3,3,3}$ described in Figure 7 can be realized as Lagrangian matching spheres. Figure 8 gives a description of the base of π, where the 24 crosses are critical values, which are divided into three groups, and $\pi^{-1}(\star)$ is a smooth fiber. See [38, Section 2.5], where the detailed construction of such a Lefschetz fibration is explained. The blue arc in Figure 8 which ends at two different critical values of π is the projection of a matching sphere $V \subset M_{3,3,3,3}$. This shows that $M_{3,3,3,3}$ can be constructed by attaching 24 Weinstein 3-handles to $T_{3,3,3} \times D^2$ along a link of 24 Legendrian S^2's, whose restrictions in $\pi^{-1}(\star)$ are vanishing cycles of π. Note that it can be arranged so that the basis of vanishing cycles of π contains the afore-mentioned basis of vanishing cycles of t as a subset. Comparing with the handlebody decomposition of D^6 described above, this realizes the Weinstein domain $\bar{M}_{3,3,3,3}$ as D^6 with 16 Weinstein 3-handles attached along a link of 16 unknotted Legendrian S^2's in $(S^5, \xi_{\mathrm{std}})$. When restricting to the smooth fiber $\pi^{-1}(\star)$ of π, these Legendrian spheres form a subset of the basis of vanishing cycles of π, and every one of them lies in a matching sphere, in the fiber above $\star \in \mathbb{C}$. More precisely, consider the associated Lefschetz fibration $\bar{\bar{\pi}} : \bar{M}_{3,3,3,3} \to D^2$ on the Liouville domain (with corners) $\bar{M}_{3,3,3,3}$ obtained by cutting off the cylindrical ends of the fibers and removing the preimage
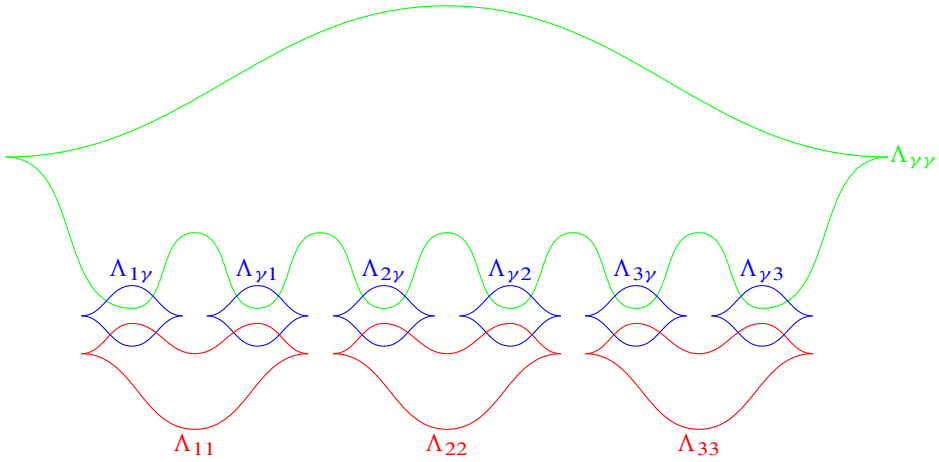
Figure 9. Front view of the Legendrian front of $\Lambda_{3,3,3,3}$, where the components Λ_{12}, Λ_{13}, Λ_{21}, Λ_{23}, Λ_{31}, and Λ_{32} are omitted since they are covered by the other components.

of the part outside of the dashed circle in Figure 8, where the fibration is locally trivial (this is the original setup of [61]). If we cut the base of π along the orange dashed arc in Figure 8, the preimage under $\bar{\pi}$ of the lower left half of the disk, which we denote by D_-, with the corners rounded off, is deformation equivalent to D^6, and the restrictions of the 16 matching spheres to $\bar{\pi}^{-1}(D_-)$ become exact Lagrangian fillings of the corresponding vanishing cycles, which are considered as Legendrian spheres in the contact boundary $\partial\bar{\pi}^{-1}(D_-)$. For example, for the matching sphere V in the figure, its restriction $\bar{V} := V \cap \bar{\pi}^{-1}(D_-)$ is a Lagrangian disk, which fills its boundary $\partial\bar{V} \subset \partial\bar{\pi}^{-1}(D_-)$, which is a Legendrian 2-sphere. In this way, the linking pattern of the 16 Legendrian S^2's in $\partial\bar{\pi}^{-1}(D_-)$ is determined by the intersection pattern of the Lagrangian matching spheres in $M_{3,3,3,3}$, which is shown in Figure 7.   ∎

In Figure 9, we have arranged so that the labelings of the components of $\Lambda_{3,3,3,3}$ coincide with that of the vanishing cycles in Figure 7, which means that for any vanishing cycle $V_{\bullet\bullet}$, the Lagrangian 3-disk $V_{\bullet\bullet} \cap D^6$ is a filling of the component $\Lambda_{\bullet\bullet}$ with the same labeling. The set of Lagrangian cocores $\{L_{\bullet\bullet}\}$ will be labeled in the same way, with $L_{\bullet\bullet}$ being the cocore of the 3-handle attached along $\Lambda_{\bullet\bullet}$. As a consequence, $V_{\bullet\bullet}$ intersects $L_{\bullet\bullet}$ non-trivially and transversely at a unique point if and only if they have the same labeling. We denote by $\mathcal{W}_{M_{3,3,3,3}}$ the Fukaya A_∞-algebra of the cocores $\{L_{\bullet\bullet}\}$.

Regarding $\Lambda_{3,3,3,3}$ as a Legendrian surface in $J^1(\mathbb{R}^2)$, one can consider its image under the base projection $p_x : J^1(\mathbb{R}^2) \to \mathbb{R}^2$. Under suitable Legendrian isotopies, the image $p_x(\Lambda_{3,3,3,3}) \subset \mathbb{R}^2$ consists of 31 circles; see Figure 10. The largest solid circle
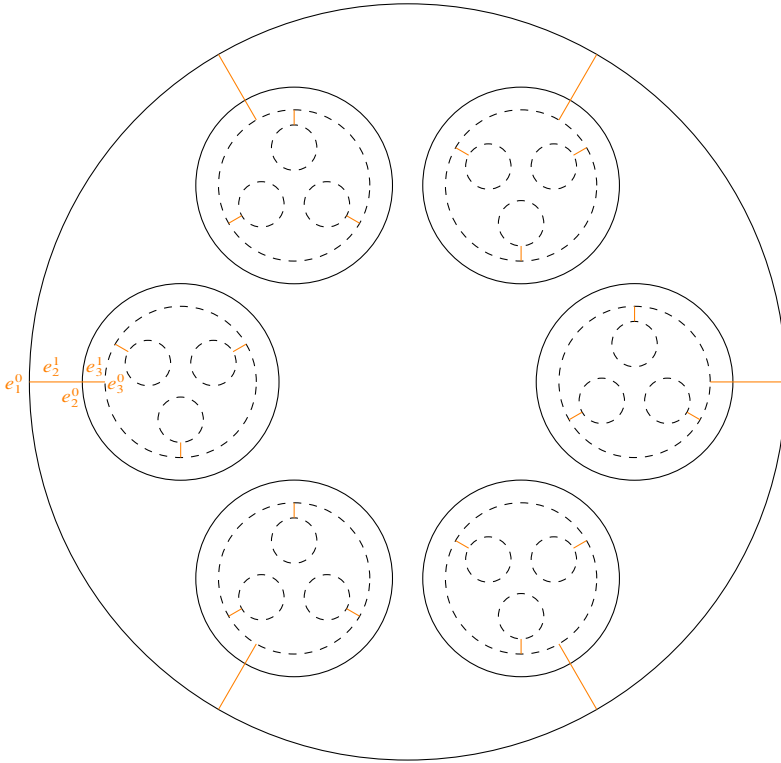
Figure 10. Base projection of $\Lambda_{3,3,3,3}$, where the orange arcs are auxiliary edges added in order to obtain a cellular decomposition.

is the projection of the cusp edges of the spheres $\Lambda_{11}, \ldots, \Lambda_{33}, \Lambda_{\gamma\gamma}$; the 6 small solid circles are projections of the cusp edges of the remaining components $\Lambda_{1\gamma}, \Lambda_{\gamma 1}, \Lambda_{2\gamma}$, $\Lambda_{\gamma 2}, \Lambda_{3\gamma}$, and $\Lambda_{\gamma 3}$, which are colored blue in Figure 9. For every one of these solid circles, there is a slightly smaller dashed circle, which is the projection of a crossing arc between a blue sphere and $\Lambda_{\gamma\gamma}$. Finally, in each of the 6 dashed circles, there are 3 small dashed circles, which are projections of the crossing arcs formed by a blue sphere and a red one. Note that for each blue sphere, say $\Lambda_{1\gamma}$, its front intersects with three red spheres $\Lambda_{11}, \Lambda_{12}$, and Λ_{13}.

Lemma 61. *Up to quasi-isomorphism, the wrapped Fukaya A_∞-algebra $\mathcal{W}_{M_{3,3,3,3}}$ is concentrated in degrees ≤ 0.*

Proof. By the surgery quasi-isomorphism (6.6), it suffices to check that the Chekanov–Eliashberg dg algebra $CE^*(\Lambda_{3,3,3,3})$ is concentrated in non-positive degrees up to quasi-isomorphism.

Equip the Legendrian link $\Lambda_{3,3,3,3}$ with a Maslov potential $\mu : \Lambda_{3,3,3,3} \to \mathbb{Z}$ as follows. For the green component $\Lambda_{\gamma\gamma}$, let $\mu = 1$ on the upper strand, and $\mu = 0$ on the lower strand. For each blue component, put $\mu = 0$ on the upper strand, and $\mu = -1$ on the lower strand. Finally, for each red component, set $\mu = -1$ on the upper strand and $\mu = -2$ on the lower strand.

To show that $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ is quasi-isomorphic to a dg algebra with all the generators concentrated in non-positive degrees, we make use of its cellular model introduced by Rutherford–Sullivan [55]. Recall that Rutherford–Sullivan's combinatorial model of $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ starts with a $\Lambda_{3,3,3,3}$-compatible polygonal decomposition associated to the base projection of $\Lambda_{3,3,3,3}$, which can be obtained by subdividing the base projection of $\Lambda_{3,3,3,3}$. For example, for the base projection shown in Figure 10, one way to obtain its $\Lambda_{3,3,3,3}$-compatible polygonal decomposition is to add the orange arcs. For each i-cell e^i_α, where $0 \le i \le 2$, the set of sheets of $\Lambda_{3,3,3,3}$ above e^i_α can be equipped with a partial ordering $\prec$ based on their heights. In particular, if the sheet S_m has larger height than S_n, then $S_m \prec S_n$, and we label the sheets so that $m < n$. There is a generator of $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ associated to each pair of sheets (S_m, S_n) above e^i_α with $S_m \prec S_n$. The generators associated to 0-, 1-, and 2-cells labeled by α are denoted, respectively, by $a^{m,n}_\alpha$, $b^{m,n}_\alpha$, and $c^{m,n}_\alpha$, and they can be assembled in strictly upper triangular matrices A_α, B_α, and C_α. The gradings of these generators are given as follows:

$$|a^{m,n}_\alpha| = \mu(S_n) - \mu(S_m) + 1,$$
$$|b^{m,n}_\alpha| = \mu(S_n) - \mu(S_m),$$
$$|c^{m,n}_\alpha| = \mu(S_n) - \mu(S_m) - 1.$$

Observe that, by our choice of the Maslov potential μ, for any sheet S_m with larger height than S_n, we necessarily have $\mu(S_m) \ge \mu(S_n)$, which implies that all the generators of the form $b^{m,n}_\alpha$ and $c^{m,n}_\alpha$ have non-positive gradings. However, there may still be generators $a^{m,n}_\alpha$ with $|a^{m,n}_\alpha| = 1$. For each such generator, consider any 1-cell e^1_β in the $\Lambda_{3,3,3,3}$-compatible polygonal decomposition which ends at e^0_α, with the other endpoint being e^0_γ for some γ. The differential d_C on cellular dg algebra is defined in completely combinatorial manner. In particular, by [55, Section 3.6.2], we have (over $\mathbb{K} = \mathbb{Z}/2$)

$$d_C b^{m,n}_\beta = a^{m,n}_\alpha + a^{m,n}_\gamma + \sum_{m<k<n} a^{m,k}_\alpha b^{k,n}_\beta + \sum_{m<k<n} b^{m,k}_\beta a^{k,n}_\gamma. \qquad (6.7)$$

By a lemma of Chekanov [55, Theorem 2.1], there is a quasi-isomorphism

$$\mathrm{CE}^*(\Lambda_{3,3,3,3}) \cong \mathrm{CE}^*(\Lambda_{3,3,3,3}) / \langle d_C b^{m,n}_\beta, b^{m,n}_\beta \rangle$$

between $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ and its quotient dg algebra, where $\langle d_C b^{m,n}_\beta, b^{m,n}_\beta \rangle$ is the ideal generated by $d_C b^{m,n}_\beta$ and $b^{m,n}_\beta$. By [49, Lemma 6.1], if one can always achieve that

$a_\gamma^{m,n} = 0$ or 1 in (6.7), then by replacing the cellular model of $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ with a quasi-isomorphic (actually, stable tame isomorphic) dg algebra if necessary, all the generators with positive degrees can be canceled out. Here, we check this explicitly for the generator $a_3^{2,3}$ associated to the 0-cell e_3^0 in Figure 10 explicitly; the verifications for the other generators $a_\alpha^{m,n}$ with $|a_\alpha^{m,n}| = 1$ are similar. Since the Legendrian front of $\Lambda_{3,3,3,3}$ above the largest solid circle consists of cusp edges, it follows that $a_2^{2,3} = 0$. Applying (6.7) to $b_2^{2,3}$, which is a generator associated to the 1-cell e_2^1, we have

$$d_C b_2^{2,3} = a_2^{2,3},$$

so after passing to the quotient dg algebra $\mathrm{CE}^*(\Lambda_{3,3,3,3})/\langle b_2^{2,3}, a_2^{2,3} \rangle$, we have by (6.7) applied to the generator $b_3^{2,3}$ associated to e_3^1 that $d_C b_3^{2,3} = a_3^{2,3}$. The dg algebra $\mathrm{CE}^*(\Lambda_{3,3,3,3})/\langle b_2^{2,3}, a_2^{2,3}, b_3^{2,3}, a_3^{2,3} \rangle$ is quasi-isomorphic to $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ and has one less generator of positive degree.

Since the Legendrian front of $\Lambda_{3,3,3,3}$ does not involve any swallowtail singularity, as explained in the proof of [49, Lemma 8.1], the cancellation of the generators does not depend on the ground field $\mathbb{K}$, so we conclude that $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ is quasi-isomorphic to a dg algebra with all the generators concentrated in degrees ≤ 0 over any field $\mathbb{K}$. ∎

Denote by τ_V the Dehn twist along the Lagrangian sphere $V \subset M_{3,3,3,3}$; it follows from [57, Lemmas 4.15 and 4.16] that

$$(\tau_{V_{11}} \circ \cdots \circ \tau_{V_{33}} \circ \tau_{V_{\gamma 1}} \circ \tau_{V_{\gamma 2}} \circ \tau_{V_{\gamma 3}} \circ \tau_{V_{1\gamma}} \circ \tau_{V_{2\gamma}} \circ \tau_{V_{3\gamma}} \circ \tau_{V_{\gamma\gamma}})^3 = [-2]. \quad (6.8)$$

Since the right-hand side of (6.8) is a non-trivial degree shift, by Seidel's long exact sequence [59], the compact Fukaya category $\mathcal{F}(M_{3,3,3,3})$ is split-generated by the Lagrangian spheres

$$V_{11}, \ldots, V_{33}, V_{\gamma 1}, V_{\gamma 2}, V_{\gamma 3}, V_{1\gamma}, V_{2\gamma}, V_{3\gamma}, V_{\gamma\gamma}.$$

Denote by $\mathcal{F}_{M_{3,3,3,3}}$ the Fukaya A_∞-algebra of these vanishing cycles. As a corollary to Lemma 61, we have the following.

Corollary 62. *The Lagrangian spheres $\{V_{\bullet\bullet}\}$ admit gradings for which the A_∞-algebra $\mathcal{F}_{M_{3,3,3,3}}$ is concentrated in degrees ≥ 0, and its degree 0 part is isomorphic to*

$$\mathbb{k} := \bigoplus_{i=1}^{16} \mathbb{K} e_i.$$

Proof. It follows from the Eilenberg–Moore equivalence (6.2) that as a $\mathbb{k}$-bimodule

$$\mathcal{F}_{M_{3,3,3,3}} \cong \mathbb{k} \oplus \bigoplus_{i=1}^{\infty} \overline{\mathrm{CE}}_*(\Lambda_{3,3,3,3})[-1]^{\otimes_{\mathbb{k}} i}.$$

It follows that any non-idempotent generator of $\mathcal{F}_{M_{3,3,3,3}}$ is of the form

$$a_1^\vee[-1] \cdots a_m^\vee[-1]$$

for some $m \geq 1$, where a_i is a generator of $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ for any $1 \leq i \leq m$. By Lemma 61, $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ is non-positively graded up to quasi-isomorphism; therefore, we may assume that $|a_i| \leq 0$ for each i, which implies that

$$|a_1^\vee[-1] \cdots a_m^\vee[-1]| = (-|a_1| + 1) \cdots (-|a_m| + 1) \geq 1. \qquad \blacksquare$$

Remark 63. In fact, a more careful study of the dg algebra $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ implies that it is quasi-isomorphic to a dg algebra freely generated by Reeb chords supported in degrees $-2 \leq * \leq 0$, which means that one can arrange the gradings so that $\mathcal{F}_{M_{3,3,3,3}}$ is concentrated in degrees $0 \leq * \leq 3$. A similar result is proved for the WKB algebra of Lagrangian 3-spheres in [68, Lemma 4.5]. We expect that the same grading property holds for the Fukaya A_∞-algebra of a basis of vanishing cycles in any Milnor fiber $M_{a_1,\ldots,a_{n+1}}$ with $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$.

The final ingredient needed for the proof of Theorem 7 is the following Koszul duality result, which is essentially due to Lekili–Ueda [48].

Proposition 64. *There are quasi-isomorphisms between A_∞-algebras over $\Bbbk$:*

$$R\mathrm{Hom}_{\mathcal{F}_{M_{3,3,3,3}}}(\Bbbk, \Bbbk) \cong \mathcal{W}_{M_{3,3,3,3}}, \quad R\mathrm{Hom}_{\mathcal{W}_{M_{3,3,3,3}}}(\Bbbk, \Bbbk) \cong \mathcal{F}_{M_{3,3,3,3}}.$$

Proof. The second quasi-isomorphism follows from the Eilenberg–Moore equivalence (6.2) and the surgery quasi-isomorphism (6.6). For the first quasi-isomorphism, we use (6.8), which implies by [48, Lemma 6.6] that the wrapped Floer cochain complex $\mathrm{CW}^*(L, K)$ is bounded above for any two objects L, K of $\mathcal{W}(M_{3,3,3,3})$. By [48, Proposition 6.5], for any object K of $\mathcal{W}(M_{3,3,3,3})$, there exists a sequence of objects $(K_i)_{i \in \mathbb{N}}$ of $\mathcal{F}(M_{3,3,3,3})^{\mathrm{perf}}$, the A_∞-category of perfect modules over $\mathcal{F}(M_{3,3,3,3})$ such that, for any fixed $j \in \mathbb{Z}$, there exists an integer $i \gg 1$ such that

$$\mathrm{Hom}^j_{\mathcal{W}(M_{3,3,3,3})^{\mathrm{perf}}}(L, K) \cong \mathrm{Hom}^j_{\mathcal{W}(M_{3,3,3,3})^{\mathrm{perf}}}(L, K_i)$$

as $\Bbbk$-vector spaces for any object L of $\mathcal{W}(M_{3,3,3,3})$. Note that, in the above isomorphism, we have identified K_i with objects of $\mathcal{W}(M_{3,3,3,3})^{\mathrm{perf}}$ via the fully faithful embedding

$$\mathcal{F}(M_{3,3,3,3})^{\mathrm{perf}} \hookrightarrow \mathcal{W}(M_{3,3,3,3})^{\mathrm{perf}}.$$

In particular, $\mathrm{CW}^j(L, K)$ of any two objects L, K of $\mathcal{W}(M_{3,3,3,3})$ is finite dimensional for any fixed j. Under the surgery quasi-isomorphism (6.6), this translates into the fact that the Chekanov–Eliashberg dg algebra $\mathrm{CE}^*(\Lambda_{3,3,3,3})$ is locally finite (finite dimensional in each degree) as a module over $\Bbbk$. It follows that the filtration on

$\mathrm{CE}^*(\Lambda_{3,3,3,3})$ by word length is complete and Hausdorff, so the completion map (6.5) is a quasi-isomorphism for the Legendrian link $\Lambda_{3,3,3,3}$. By (6.4), we get the desired quasi-isomorphism

$$\mathcal{W}_{M_{3,3,3,3}} \cong \mathrm{CE}^*(\Lambda_{3,3,3,3}) \cong (\mathrm{B}\mathcal{F}_{M_{3,3,3,3}})^{\#} \cong R\mathrm{Hom}_{\mathcal{F}_{M_{3,3,3,3}}}(\Bbbk, \Bbbk). \qquad \blacksquare$$

Remark 65. [48, Theorem 6.11] proves a general version of Koszul duality between compact and wrapped Fukaya A_∞-algebras for Milnor fibers associated to weighted homogeneous singularities $\{w(z_1, \ldots, z_{n+1}) = 0\} \subset \mathbb{C}^{n+1}$. However, they assumed that the Fukaya–Seidel category $\mathcal{F}(w)$ admits a strong full exceptional collection, which is not satisfied for the polynomial

$$w(z_1, z_2, z_3, z_4) = z_1^3 + z_2^3 + z_3^3 + z_4^3.$$

Proof of Theorem 7. It follows from Proposition 57 that $\mathcal{F}_{M_{3,3,3,3}}$ is quasi-isomorphic to a minimal cyclic A_∞-algebra. Combining (6.4) with Proposition 64, we see that A_∞-algebra $\mathcal{W}_{M_{3,3,3,3}}$ is quasi-isomorphic to a complete dg algebra, whose Koszul dual is $\mathcal{F}_{M_{3,3,3,3}}$. Moreover, Lemma 61 implies that $H^*(\mathcal{W}_{M_{3,3,3,3}})$ is supported in non-positive degrees. Applying Theorem 56 completes the proof. $\qquad \blacksquare$

As a by-product, we have the following non-formality result. A similar result is proved in [48, Theorem 7.3].

Corollary 66. *The Fukaya A_∞-algebra $\mathcal{F}_{M_{3,3,3,3}}$ is not formal over $\Bbbk$.*

Proof. Suppose that $\mathcal{F}_{M_{3,3,3,3}}$ is formal; then there is a quasi-isomorphism

$$\mathcal{F}_{M_{3,3,3,3}} \cong F_{M_{3,3,3,3}} := H^*(\mathcal{F}_{M_{3,3,3,3}}).$$

On the cohomology level, the proper Calabi–Yau structure on $\mathcal{F}_{M_{3,3,3,3}}$ induces a non-degenerate pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{CY}} : F_{M_{3,3,3,3}} \otimes F_{M_{3,3,3,3}} \to \Bbbk[-3],$$

which makes $(F_{M_{3,3,3,3}}, \langle \cdot, \cdot \rangle_{\mathrm{CY}})$ a Frobenius algebra. Denote by Δ_F the BV operator on the Hochschild cohomology $\mathrm{HH}^*(F_{M_{3,3,3,3}})$. For any class $c \in \mathrm{HH}^{1,0}(F_{M_{3,3,3,3}})$ and $a \in F_{M_{3,3,3,3}}$, we have

$$\langle \Delta_F(c), a \rangle_{\mathrm{CY}} = \langle c(a), 1_F \rangle_{\mathrm{CY}},$$

where $1_F \in F_{M_{3,3,3,3}}$ is the identity. When applied to the Euler vector field $\mathrm{eu}_F \in \mathrm{HH}^{1,0}(F(M_{3,3,3,3}))$, we obtain

$$\left\langle \Delta_F\left(\frac{1}{3}\mathrm{eu}_F\right), a \right\rangle_{\mathrm{CY}} = \langle a, 1_F \rangle_{\mathrm{CY}} = \langle 1_F, a \rangle_{\mathrm{CY}}$$

for any $a \in F_{M_{3,3,3,3}}$ of degree 3. It follows that

$$\Delta_F \left(\frac{1}{3} \mathrm{eu}_F \right) = 1.$$

Denote by $\mathrm{eu}_{\mathcal{F}}$ the image of the class eu_F under the BV algebra isomorphism

$$\mathrm{HH}^*(F_{M_{3,3,3,3}}) \cong \mathrm{HH}^*(\mathcal{F}(M_{3,3,3,3})).$$

We have $\Delta_{\mathcal{F}}(\mathrm{eu}_{\mathcal{F}}) = 1$, where $\Delta_{\mathcal{F}}$ is the BV operator on $\mathrm{HH}^*(\mathcal{F}(M_{3,3,3,3}))$.

By Proposition 64, there is an isomorphism

$$\mathrm{HH}^*(\mathcal{F}(M_{3,3,3,3})) \cong \mathrm{HH}^*(\mathcal{W}(M_{3,3,3,3}))$$

as Gerstenhaber algebras, under which $\mathrm{eu}_{\mathcal{F}}$ goes to a class $b \in \mathrm{HH}^1(\mathcal{W}(M_{3,3,3,3}))$. Since changing the Calabi–Yau structure on $\mathcal{W}(M_{3,3,3,3})$ amounts to applying the conjugation action of an invertible element $h_{\mathcal{W}} \in \mathrm{HH}^0(\mathcal{W}(M_{3,3,3,3}))^\times$ to the BV operator

$$\Delta_{\mathcal{W}} : \mathrm{HH}^*(\mathcal{W}(M_{3,3,3,3})) \rightarrow \mathrm{HH}^{*-1}(\mathcal{W}(M_{3,3,3,3})),$$

the class b satisfies $\Delta_{\mathcal{W}}(h_{\mathcal{W}}b) = h_{\mathcal{W}}$. Under the BV algebra isomorphism

$$\mathrm{HH}^*(\mathcal{W}(M_{3,3,3,3})) \cong \mathrm{SH}^*(M_{3,3,3,3})$$

established by Ganatra (cf. [27, Theorem 1.1]), the image of b defines a quasi-dilation in $\mathrm{SH}^1(M_{3,3,3,3})$.

To complete the proof, it suffices to show that $M_{3,3,3,3}$ does not admit a quasi-dilation. We follow the argument of [63, Example 2.7] to show that $M_{3,3,3,3}$ does not admit a quasi-dilation. Consider the Milnor fiber M of a 5-fold triple point, which is the affine hypersurface in $\mathbb{C}^6$ given by the equation

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 = 1.$$

Since it is the complement of a smooth divisor in the Fermat projective cubic 5-fold, there is a Morse–Bott spectral sequence [60] which converges to $\mathrm{SH}^*(M)$. Using this, one can deduce that $\mathrm{SH}^1(M) = 0$, which in particular implies that M does not admit a quasi-dilation. On the other hand, there is a Lefschetz fibration $M \rightarrow \mathbb{C}$ on M whose smooth fiber F is symplectomorphic to the Milnor fiber of a 4-fold triple point; i.e., the affine hypersurface in $\mathbb{C}^5$ defined by the equation

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 = 1.$$

Similarly, F also admits a Lefschetz fibration $F \rightarrow \mathbb{C}$ with the Milnor fiber $M_{3,3,3,3}$ as its fiber. If $M_{3,3,3,3}$ admits a quasi-dilation, then by [56, Lemma 19.5] it lifts to a quasi-dilation in $\mathrm{SH}^1(F)$. Applying the same lemma again to the Lefschetz fibration $M \rightarrow \mathbb{C}$, it follows that M also admits a quasi-dilation, which gives the desired contradiction. ∎

6.2. Lefschetz fibrations

This subsection is devoted to the proof of Theorem 6, which allows us to get new examples Liouville manifolds which admit cyclic dilations in terms of the known ones. The argument here is a slight variation of those in [67, Section 7] and [77, Section 5.3].

We use the general setup of [67, Section 7]. Let $\pi : M \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, which means that its smooth fibers F are completions of Liouville domains $\bar{F}$. More explicitly, we require the following:

- For some almost complex structure $J \in \mathcal{J}(M)$, the map π is (J, j)-holomorphic, where j is the standard complex structure on $\mathbb{C}$.

- π has finitely many isolated critical points so that each singular fiber contains at most one critical point, and the almost complex structure J is locally integrable near each of these critical points.

- There is a relatively open compact subset $\bar{M} \subset M$ so that its complement $M \setminus \bar{M}$ is identified with

$$\widetilde{M} := (\mathbb{R}_+ \times T) \cup_{\mathbb{R}_+ \times S^1 \times \mathbb{R}_+ \times \partial \bar{F}} (\mathbb{C} \times \mathbb{R}_+ \times \partial \bar{F}),$$

where $T = (\mathbb{R} \times F)/(t, x) \sim (t - 1, \mu(x))$ is the mapping torus, with μ being the total monodromy of π. By construction, $\bar{M}$ is a manifold with corners, which coincides with the Liouville domain associated to M up to deformation once the corners are rounded off.

- Fix the choice of a trivialization of the canonical bundle K_M, which induces a trivialization of K_F, the canonical bundle of the fiber.

Given such a Lefschetz fibration, consider the autonomous Hamiltonian $H_M : M \to \mathbb{R}$ defined by

$$H_M = H_F + \pi^* H_{\mathbb{C}}, \tag{6.9}$$

where $H_{\mathbb{C}}(z) = \varepsilon |z - c|^2/2$ for some $\varepsilon > 0$ is a function on the base, and $H_F : F \to \mathbb{R}$ is a Hamiltonian on the fiber which is linear on the cylindrical end $[1, \infty) \times \partial \bar{F}$ with slope $\lambda > 0$, where $\lambda \notin \mathcal{P}_F$. By [67, Lemma 7.2], for sufficiently small ε, there is a short exact sequence

$$0 \to \mathbb{K}^{\text{Crit}(\pi)}[-n] \to \text{CF}^*_{\text{vert}}(M, \lambda) \to \text{CF}^*(F, \lambda) \to 0, \tag{6.10}$$

where the Floer complexes $\text{CF}^*_{\text{vert}}(M, \lambda)$ and $\text{CF}^*(F, \lambda)$ are defined by choosing time-dependent perturbations of the autonomous Hamiltonians H_M and H_F, and the notation $\mathbb{K}^{\text{Crit}(\pi)}[-n]$ means the complex with trivial differential so that there is a copy of $\mathbb{K}$ in degree n for every critical point of π. Here, we use the notation $\text{CF}^*_{\text{vert}}(M, \lambda)$ to indicate that when $\lambda \to \infty$, the slope of our Hamiltonian H_M only increases in the

vertical direction. As a consequence, the cohomology level direct limit $\mathrm{SH}^*_{\mathrm{vert}}(M)$ is in general not isomorphic to the symplectic cohomology $\mathrm{SH}^*(M)$. The same notational convention will be used later on for equivariant Floer cohomologies.

Let M be a Liouville manifold. Recall that the action functional $\mathcal{A}_{H_t} : \mathcal{L}M \to \mathbb{R}$ of a time-dependent perturbation $H_t : S^1 \times M \to \mathbb{R}$ of some autonomous Hamiltonian $H : M \to \mathbb{R}$ is defined to be

$$\mathcal{A}_{H_t}(x) = -\int_{S^1} x^* \theta_M + \int_0^1 H_t(x(t)) dt. \tag{6.11}$$

The period spectrum $\mathcal{P}_M$ is a strictly ordered set with elements $0 < \eta_1 < \eta_2 < \cdots$, where η_1 is the minimum period of a Reeb orbit on the contact boundary $\partial \bar{M}$, and we set $\eta_0 = 0$. Let $\lambda_j = \frac{\eta_j + \eta_{j+1}}{2}$, so in particular, $\lambda_j \notin \mathcal{P}_M$ for any j, and introduce the real numbers

$$a_{\lambda_j} := -\frac{\lambda_j^2}{2} - \lambda_j, \quad j \geq 0.$$

Consider a Hamiltonian $H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ so that $\lambda \notin \mathcal{P}_M$; let $\mathcal{O}_{M,\lambda}$ be the set of 1-periodic orbits of $X_{H_{\lambda,t}}$; there is an action filtration $F^\bullet$ on the Floer complex $\mathrm{CF}^*(\lambda)$ of $H_{\lambda,t}$ given by

$$F^j \, \mathrm{CF}^*(\lambda) := \bigoplus_{x \in \mathcal{O}_{M,\lambda}, \, \mathcal{A}_{H_{\lambda,t}}(x) \geq a_{\lambda_j}} |o_x|_{\mathbb{K}}. \tag{6.12}$$

In order to analyze the compatibility between the S^1-complex structure maps $\{\delta_i\}_{i \geq 0}$ and the filtration $F^\bullet$ on $\mathrm{CF}^*(\lambda)$, we study a specific autonomous Hamiltonian $\tilde{H}_\lambda : M \to \mathbb{R}$, which has the form

$$\tilde{H}_\lambda(x) = \begin{cases} \text{some negative } C^2\text{-small Morse function}, & x \in M^{\mathrm{in}}, \\ \frac{(r-1)^2}{2}, & x \in [1, \lambda+1] \times \partial \bar{M}, \\ \lambda(r-1) - \frac{\lambda^2}{2}, & x \in [\lambda+1, \infty) \times \partial \bar{M}. \end{cases}$$

Following [77, Section 3.2.1], we define a carefully chosen small time-dependent perturbation $\tilde{H}_{\lambda,t}$ of $\tilde{H}_\lambda$. For any 1-periodic orbit $x \in \tilde{\mathcal{O}}_{M,\lambda}$ of $X_{\tilde{H}_\lambda}$, fix an isolating neighborhood $U_x \subset M$. If x corresponds to a Reeb orbit of multiplicity $k \in \mathbb{N}$, one considers a Morse function $f_x : S^1 \to [-1, -\frac{1}{2}]$ that has a unique minimum $f_x(0) = -1$ and a unique maximum $f_x(t_0) = -1/2$ for some small enough $t_0 \in S^1$. Define $h_x : \bar{U}_x \to [-1, 0]$ by

$$h_x(t, x(s)) = f_x(ks - kt)$$

on the image of x and extend it smoothly to U_x so that $h_x = 0$ on $\partial \bar{U}_x$. The time-dependent perturbation of $\tilde{H}_\lambda$ is defined to be

$$\tilde{H}_{\lambda,t} = \tilde{H}_\lambda + \varepsilon' \sum_{x \in \tilde{\mathcal{O}}_{M,\lambda}} h_x(t),$$

where $h_x(t) = f_x \circ \widetilde{\phi}^{-t}$, with $\widetilde{\phi}^{-t}$ being the time $-t$ flow of $X_{\widetilde{H}_\lambda}$, and $\varepsilon' > 0$ is a small positive number.

With our choice of $\widetilde{H}_{\lambda,t}$, an energy estimate for Floer trajectories in $\mathcal{M}_i(y^+; y^-)$ with $|o_{y^+}|_{\mathbb{K}} \in F^j \operatorname{CF}^*(\lambda)$ implies the following lemma.

Lemma 67 ([77, Lemma 3.2.4]). *For any fixed $\lambda \notin \mathcal{P}_M$, there is an $\varepsilon' > 0$ depending on λ such that*

$$\delta_i(F^j \operatorname{CF}^*(\lambda)) \subset F^j \operatorname{CF}^*(\lambda)$$

for all $i, j \geq 0$.

Proof of Theorem 6. We first show that (6.10) is a short exact sequence of S^1-complexes. Basically, (6.10) follows from the fact that the set of generators of $\operatorname{CF}^*_{\mathrm{vert}}(M, \lambda)$ consists of the following three kinds:

(i) Critical points of H_F. These generators have small negative action if we perturb H_F so that it is a C^2-small Morse function with negative values in the interior of $\overline{F}$.

(ii) Non-constant 1-periodic orbits of X_{H_F}. Writing $H_F = h_F(r)$ on the cylindrical end $[1, \infty) \times \partial \overline{F}$, such an orbit x has action

$$\mathcal{A}_{H_F}(x) = h_F(r_x) - r_x h'_F(r_x) < 0,$$

where $r_x \in [1, \infty)$ is the radial coordinate of x.

(iii) Constant orbits near the critical points of the function $\pi^* H_\mathbb{C}$, which have Conley–Zehnder index $-n$. If we choose the C^2-small Morse function appearing in (i) to be sufficiently small, the following can be achieved:

$$\mathcal{A}_{H_M}(x) = \varepsilon|\pi(x) - c|^2 + H_F(x) > 0.$$

By choosing the t-dependent perturbation of the Hamiltonian H_F (and thus H_M) carefully, Lemma 67 applies and shows that the operations $\{\delta_i\}_{i \geq 0}$ preserve the action filtration on $\operatorname{CF}^*_{\mathrm{vert}}(M, \lambda)$. This implies that the generators of $\operatorname{CF}^*_{\mathrm{vert}}(M, \lambda)$ with positive actions form the (trivial) S^1-subcomplex $\mathbb{K}^{\operatorname{Crit}(\pi)}[-n]$. Denote by $\mathcal{H}_\pi(M)$ the space of Hamiltonians which are small t-dependent perturbations of the autonomous Hamiltonians of the form (6.9) and by $\mathcal{J}_\pi(M)$ the space of compatible almost complex structures which are of contact type when restricted to the fibers F so that $\pi : M \to \mathbb{C}$ is (J, j)-holomorphic for each $J \in \mathcal{J}_\pi(M)$. The same energy estimate as in the proof of [67, Lemma 7.2] shows that for any solution $u : Z \to M$ of the Floer equation $(du - X_{H_Z} \otimes \nu_Z)^{0,1} = 0$ with asymptotics $y^\pm \in \mathcal{O}_{F,\lambda}$, where $\nu_Z \in \Omega^1(Z)$, $H_Z : Z \to \mathcal{H}_\pi(M)$, and the $(0, 1)$-part is taken with respect to some

$$J_Z : Z \to \mathcal{J}_\pi(M),$$

its image necessarily lies in the fiber F. Since universal and consistent Floer data for the operations $\{\delta_i\}_{i \geq 0}$ can be chosen among $\mathcal{H}_\pi(M)$ and $\mathcal{J}_\pi(M)$, the fact that $u(Z) \subset F$ implies that the quotient complex $\mathrm{CF}^*_{\mathrm{vert}}(M, \lambda)/\mathbb{K}^{\mathrm{Crit}(\pi)}[-n]$ can be identified with $\mathrm{CF}^*(F, \lambda)$ as an S^1-complex. This proves that (6.10) is a short exact sequence of S^1-complexes, and it follows from Proposition 21 that we have a commutative diagram

$$
\begin{array}{ccccccc}
\cdots \to & \mathbb{K}((u))/u\mathbb{K}[\![u]\!]^{\mathrm{Crit}(\pi)}[-n] & \to & \mathrm{HF}^*_{S^1,\mathrm{vert}}(M, \lambda) & \to & \mathrm{HF}^*_{S^1}(F, \lambda) & \to \cdots \\
& \downarrow{\mathbf{B}} & & \downarrow{\mathbf{B}} & & \downarrow{\mathbf{B}} & \\
\cdots \longrightarrow & \mathbb{K}^{\mathrm{Crit}(\pi)}[-n] & \longrightarrow & \mathrm{HF}^{*-1}_{\mathrm{vert}}(M, \lambda) & \longrightarrow & \mathrm{HF}^{*-1}(F, \lambda) & \to \cdots
\end{array}
$$

(6.13)

whose rows are long exact sequences of S^1-equivariant and ordinary Floer cohomology groups, which are related through the marking map $\mathbf{B}$.

Now, suppose that F admits a cyclic dilation so that, for some $\lambda \gg 0$ and $\lambda \notin \mathcal{P}_F$, there is a class $\tilde{b}_F \in \mathrm{HF}^1_{S^1}(F, \lambda)$, whose image under the composition of the marking map $\mathbf{B}$ and the continuation map $\kappa^{\lambda,\infty}$ defines an invertible element $h_F \in \mathrm{SH}^0(F)^\times$. Consider the boundary map $\mathrm{HF}^1_{S^1}(F, \lambda) \to \mathbb{K}((u))/u\mathbb{K}[\![u]\!]^{\mathrm{Crit}(\pi)}[-n]$ in the first row of (6.13). On the chain-level, it consists of an infinite sequence of maps

$$
\partial_k : \mathrm{CF}^{2k+1}(F, \lambda) \to \mathbb{K}^{\mathrm{Crit}(\pi)}[-n]
$$

for every $k \geq 0$, under which the cochain-level representative $\tilde{\beta}_F = \sum_{k=0}^\infty \beta_{F,k} \otimes u^{-k}$ of $\tilde{b}_F$ goes to $\sum_{k=0}^\infty \partial_k(\beta_{F,k})$, which lies in degree 2. However, by our assumption that $n \geq 3$, we necessarily have $\sum_{k=0}^\infty \partial_k(\beta_{F,k}) = 0$. This implies that the map $\mathrm{HF}^1_{S^1,\mathrm{vert}}(M, \lambda) \to \mathrm{HF}^1_{S^1}(F, \lambda)$ in (6.13) is surjective, and applying similar argument to the second row of (6.13) shows that there is an isomorphism

$$
\mathrm{HF}^0_{\mathrm{vert}}(M, \lambda) \cong \mathrm{HF}^0(F, \lambda),
$$

which induces an isomorphism

$$
\mathrm{SH}^0_{\mathrm{vert}}(M) \cong \mathrm{SH}^0(F) \tag{6.14}
$$

after passing to direct limits. This is in fact an isomorphism of $\mathbb{K}$-algebras, since applying the same argument as above to the pair-of-pants surface S instead of Z shows that the image of any Floer solution u with asymptotics $y_0^-, y_1^- \in \mathcal{O}_{F,\lambda}$ and $y^+ \in \mathcal{O}_{F,2\lambda}$ will be contained in F. By the surjectivity of

$$
\mathrm{HF}^1_{S^1,\mathrm{vert}}(M, \lambda) \to \mathrm{HF}^1_{S^1}(F, \lambda)
$$

and the commutative diagram (6.13), $\tilde{b}_F$ lifts to a class $\tilde{b}_M \in \mathrm{HF}^1_{S^1,\mathrm{vert}}(M, \lambda)$, whose image under $\mathbf{B}$, followed by the continuation map, is the lift of h_F in $\mathrm{SH}^0_{\mathrm{vert}}(M)$. In view of the isomorphism (6.14), this defines an invertible element of $\mathrm{SH}^0_{\mathrm{vert}}(M)$.

Finally, there is an equivariant continuation map

$$\mathrm{SC}^*_{\mathrm{vert}}(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!] \to \mathrm{SC}^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[\![u]\!],$$

under which the class $\tilde{b}_M$ goes to a cyclic dilation in $\mathrm{SH}^1_{S^1}(M)$. For its construction, one modifies the construction of the continuation map $\mathrm{SC}^*_{\mathrm{vert}}(M) \to \mathrm{SC}^*(M)$ (cf. [56, (18.25)]) in the non-equivariant case by replacing the (1-parameter family of) domain cylinders with k-point angle decorated cylinders, as in the construction of (4.9).   ∎

Remark 68. It is an easy observation in the above argument that if the cyclic dilation $\tilde{b}_F$ of the fiber F satisfies $h = 1$, then so is the total space M. In particular, according to our remarks in Section 2.5, any Milnor fiber M of a singularity of the form (2.6) has a cyclic dilation $\tilde{b}$ with $\mathbf{B}(\tilde{b}) = 1$.

6.3. Varieties of log general type

We prove Theorem 12 in this section. Our argument is based on the work of McLean [53] on the symplectic invariance of the log Kodaira dimension and the techniques in [7, 36], which allow us to produce J-holomorphic curves starting from Floer trajectories. For completeness, we will start by recalling some of the important notions and results from [53].

Let $(\bar{M}, \theta_M)$ be any Liouville domain, and let J be an almost complex structure on $\bar{M}$ which is compatible with the symplectic form $d\theta_M$. It is *convex* if there is some function $\phi : \bar{M} \to \mathbb{R}$ such that

- $\partial\bar{M}$ is a regular level set of ϕ and ϕ attains its maximum on $\partial\bar{M}$;

- $\theta_M \circ J = d\phi$ near $\partial\bar{M}$.

Every Liouville domain $\bar{M}$ has a convex almost complex structure since one can take $\phi = r$ to be the radial coordinate function in a collar neighborhood of $\partial\bar{M}$ and then extend it smoothly to the interior. The following notion plays a pivotal role in McLean's theory.

Definition 69 ([53, Definition 2.2]). Let $k \in \mathbb{Z}_{>0}$, and let $\mu \in \mathbb{R}_{>0}$. A Liouville domain $\bar{M}$ is (k, μ)-*uniruled* if, for every convex almost complex structure J and every point $p \in M^{\mathrm{in}}$ so that J is integrable in a neighborhood of p, there is a proper J-holomorphic map $u : S \to M^{\mathrm{in}}$ whose image passes through p, where S is a genus 0 open Riemann surface with $\dim H_1(S; \mathbb{Q}) \leq k - 1$, and the energy of u is at most μ.

It follows from [53, Theorem 2.3] that (k, μ)-uniruledness is a symplectic invariant of Liouville manifolds after forgetting about the energy bound μ.

We now restrict ourselves to the special case when M is an n-dimensional smooth affine variety. We say that M is *algebraically k-uniruled* if there is a polynomial

map $S \to M$ passing through every generic point $p \in M$, where S is $\mathbb{CP}^1$ with at most k points removed. This notion of uniruledness is related to Definition 69 in the following way.

Theorem 70 ([53, Theorem 2.5]). *Let M be a smooth affine variety. If the associated Liouville domain $\bar{M}$ is (k, μ)-uniruled for some μ, then M is algebraically k-uniruled.*

k-uniruledness of an affine variety is closely related to its log Kodaira dimension (2.5). In particular, we have the following lemma.

Lemma 71 ([53, Lemma 7.1]). *Let M be a smooth affine variety which is algebraically k-uniruled. If $k = 1$, then $\kappa(M) = -\infty$, and if $k = 2$, then $\kappa(M) \leq n - 1$.*

Proof of Theorem 12. We start with a summary of the main idea of the proof. In order to show that M does not admit a cyclic dilation, we argue by contradiction. Suppose that M has a cyclic dilation, we first notice that since M contains an exact Lagrangian torus, by Corollary 40, the marking map $\mathbf{B} : \mathrm{SH}^1_{S^1}(M) \to \mathrm{SH}^0(M)$ cannot hit the identity. Thus, in order for M to have a cyclic dilation, there must be some non-trivial invertible element $h \in \mathrm{SH}^0(M)^\times$. Using a limiting argument, we will show that the existence of such an element h would imply that M is uniruled by cylinders, which contradicts with our assumption that M is long general type by Lemma 71. The proof is divided into three steps.

Step 1: Existence of a Floer trajectory. Suppose that M is an n-dimensional smooth affine variety so that $\mathrm{SH}^0(M)^\times$ is not isomorphic to $\mathbb{K}^\times$, or equivalently, there is an $h \in \mathrm{SH}^0(M)^\times$ which is not a multiple of the identity. Since $hh^{-1} = 1$ holds in $\mathrm{SH}^0(M)$, there must be some $\eta \in \mathrm{SC}^0_+(M)$ so that $\alpha \cdot e + \eta$ is the cochain-level representative of h, where $\alpha \in \mathbb{K}$ and $\mathrm{SC}^0_+(M) \subset \mathrm{SC}^0(M)$ is the submodule generated by non-constant Hamiltonian orbits. Assume further that M contains an exact Lagrangian torus L; consider the Viterbo map

$$\mathrm{SH}^0(M) \to H_n(\mathcal{L}L; \mathbb{K}) \cong Z(\mathbb{K}[\pi_1(L)]) = \mathbb{K}[\pi_1(L)], \qquad (6.15)$$

where $Z(\mathbb{K}[\pi_1(L)])$ is the center of $\mathbb{K}[\pi_1(L)]$, which is just $\mathbb{K}[\pi_1(L)]$ by the assumption that L is a torus. In our case,

$$\mathbb{K}[\pi_1(L)] \cong \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \qquad (6.16)$$

is just the Laurent polynomial ring. Under the isomorphism (6.16), (6.15) maps the cocycle $\alpha \cdot e + \eta$ to a non-trivial unit of $\mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, which must be a non-zero multiple of some monomial $z_1^{a_1} \cdots z_n^{a_n}$, where $a_1, \ldots, a_n \in \mathbb{Z}$. In particular, we must have $\alpha = 0$, because the map (6.15) maps the identity to $\iota_*[L]$, where $\iota : L \to \mathcal{L}L$ is the

inclusion of constant loops, and $\iota_*[L]$ corresponds to 1 under the isomorphism (6.16). It follows that the pairing

$$\mathrm{SC}^0_+(M) \otimes \mathrm{SC}^0_+(M) \xrightarrow{\smile} \mathrm{SC}^0(M) \xrightarrow{\mathrm{pr}} C^0(M; \mathbb{K}) \cong \mathbb{K}$$

defined by composing the pair-of-pants product with the natural projection to the subcomplex $C^0(M; \mathbb{K}) \subset \mathrm{SC}^0(M)$ does not vanish, so there must be some $y_0^+, y_1^+ \in \mathcal{O}_M$ such that $y_1^+ \smile y_0^+ = \alpha' \cdot e + \zeta$, where $\alpha' \in \mathbb{K}^\times$ is some non-zero scalar and $\zeta \in \mathrm{SC}^0_+(M)$. Without loss of generality, we may assume that $y_1^+ \smile y_0^+ = e + \zeta$ for convenience.

This implies the existence of a map $u : S \to M$, with S being a 3-punctured sphere, which satisfies the Floer equation $(du - X_{H_S} \otimes \nu_S)^{0,1} = 0$, with asymptotic conditions specified by the non-constant periodic orbits $y_0^+, y_1^+ \in \mathcal{O}_M$ at two positive cylindrical ends, and converges to the minimum y^- of some C^2-small Morse function defined on M^{in}. Here, $H_S : S \to \mathcal{H}(M)$ is a domain-dependent Hamiltonian-function so that its restriction to M^{in} is a (domain-independent) C^2-small Morse function, and $\nu_S \in \Omega^1(S)$ is a closed 1-form; they are fixed as part of our Floer data defining the pair-of-pants product $\smile$, and the $(0, 1)$-part in the Floer equation is taken with respect to some domain-dependent almost complex structure $J_S : S \to \mathcal{J}(M)$.

Step 2: Producing a pseudoholomorphic curve. Starting from the Floer trajectory u, one can apply a limiting argument of [7, 36] to produce a J-holomorphic cylinder $\bar{u}_\infty : Z \to M^{\mathrm{in}}_{1-\varepsilon}$ with finite energy which passes through y^- for any convex almost complex structure J on the slightly shrinked Liouville domain

$$\bar{M}_{1-\varepsilon} := \bar{M} \setminus (1 - \varepsilon, 0] \times \partial \bar{M}$$

containing y^- in its interior and whose completion is still deformation equivalent to M, where $\varepsilon > 0$ is a sufficiently small constant.

To do this, we work with linear Hamiltonians instead and introduce a particular 1-parameter family of domain-dependent Hamiltonians $H_{\lambda,S,\theta} : S \to \mathcal{H}_\lambda(M)$ which depend on a small parameter $\theta > 0$, where as before $\lambda \notin \mathcal{P}_M$ and $\lambda \gg 0$. Specifically, for each point $z \in S$, there is a Hamiltonian

$$H_{\lambda,z,\theta} = H_{\lambda,\theta} + F_{\lambda,z} \in \mathcal{H}_\lambda(M),$$

where $F_{\lambda,z} : S \times M \to \mathbb{R}$ is independent of s when restricted to the cylindrical ends, and since it is supported near non-constant orbits of $X_{H_{\lambda,\theta}}$, we can choose $\varepsilon > 0$ small enough so that $F_{\lambda,z}$ vanishes on $\bar{M}_{1-\varepsilon}$. Set

$$H_{\lambda,\theta}(x) = \begin{cases} -\delta_\lambda + \theta f(x), & x \in M^{\mathrm{in}}, \\ h_{\lambda,\theta}(r), & x \in \bar{M}_{1+2\varepsilon} \setminus M^{\mathrm{in}}, \\ \lambda(r - 1 - \varepsilon), & x \in M \setminus \bar{M}_{1+2\varepsilon}, \end{cases}$$

where $\delta_\lambda > 0$ is a small scalar which satisfies $\lim_{\lambda \to \infty} \delta_\lambda = 0$; f is a C^2-small Morse function which satisfies $-1 \leq f \leq 0$ when restricted to $\bar{M}_{1-\varepsilon}$, has a relative minimum at $y^- \in M_{1-\varepsilon}^{\text{in}}$, and equals $r - 1 + \varepsilon$ on $[1 - 2\varepsilon, 1] \times \partial \bar{M}$. $h_{\lambda,\theta}(r)$ is an arbitrary convex function on $[1, 1 + 2\varepsilon] \times \partial \bar{M}$ which depends only on r and whose slope varies from θ to λ as r goes from 1 to $1 + 2\varepsilon$ such that $h_\lambda(r) := \lim_{\theta \to 0} h_{\lambda,\theta}(r)$ is a smooth function.

With our particular choice of the domain-dependent Hamiltonian $H_{\lambda,S,\theta}$ as above, we get a Floer trajectory $u_{\lambda,\theta} : S \to M$ which is asymptotic to a Morse critical point y_θ^- at its negative cylindrical end, and to $y_{0,\theta}^+, y_{1,\theta}^+ \in \mathcal{O}_{M,\lambda}$ at two positive cylindrical ends. It follows from our definition of $H_{\lambda,\theta}$ that the non-constant orbits $y_{0,\theta}^+$, $y_{1,\theta}^+$ necessarily lie in the collar $[1, 1 + 2\varepsilon] \times \partial \bar{M}$. To achieve the non-degeneracies of the orbits $y_{0,\theta}^+$ and $y_{1,\theta}^+$, the perturbation $F_{\lambda,z}$ can be taken to be supported near $y_{0,\theta}^+$ and $y_{1,\theta}^+$, so we may assume (by possibly rescaling ε) that $H_{\lambda,z,\theta} = H_{\lambda,\theta}$ is domain-independent in the shrinked Liouville domain $\bar{M}_{1-\varepsilon}$. Applying the maximum principle from [6, Section 7d] to the map $u_{\lambda,\theta}$ shows that $u_{\lambda,\theta}(S) \subset \bar{M}_{1+2\varepsilon}$. To achieve transversality of the moduli space $\mathcal{P}(y_{0,\theta}^+, y_{1,\theta}^+; y_\theta^-)$ where the trajectory $u_{\lambda,\theta}$ lies in, one can start from any convex almost complex structure J on $\bar{M}$ and perturb it slightly outside of $\bar{M}_{1-\varepsilon}$ to get a domain-dependent almost complex structure $J_{\lambda,S,\theta} : S \to \mathcal{J}(M)$. Note that we have arranged so that both of $H_{\lambda,S,\theta}$ and $J_{\lambda,S,\theta}$ are domain-independent on $\bar{M}_{1-\varepsilon}$, and we denote the restriction of $J_{\lambda,S,\theta}$ on $\bar{M}_{1-\varepsilon}$ as $J_{\lambda,\theta}$.

We want to pass to the limit $\theta \to 0$. Notice that, when restricted to the Liouville domain $\bar{M}_{1-\varepsilon}$, we have $\lim_{\theta \to 0} H_{\lambda,\theta} = -\delta_\lambda$, and $\lim_{\theta \to 0} J_{\lambda,\theta} = J$ for some fixed convex almost complex structure $J \in \mathcal{J}(M)$ which does not need to depend on $\lambda \gg 0$. By [36, Proposition 5.11] (which deals with the case when S is a cylinder but extends in a straightforward way to pair of pants), one can find a sequence $\{\theta_n\}$ which limits to 0 so that the corresponding Floer trajectories $\{u_{\lambda,\theta_n}\}$ converge to a limit u_λ in $C^\infty_{\text{loc}}(S, M)$, and the energy of the limiting trajectory

$$E(u_\lambda) := \frac{1}{2} \int_S \|du_\lambda - X_{H_{\lambda,z,0}} \otimes dt\|^2_{J_{\lambda,z,0}}$$

is bounded above by some constant $\mu_M > 0$, which is independent of $\lambda \gg 0$. Denote by ϕ a biholomorphic map which identifies S with $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ so that the negative puncture ζ_{out} is mapped to the origin. The composition

$$\tilde{u}_\lambda = u_\lambda \circ \phi : \mathbb{CP}^1 \setminus \{0, 1, \infty\} \to M$$

is a map whose limit at the origin is $y^- := \lim_{\theta \to 0} y_\theta^-$ and whose image goes outside of M^{in} when approaching the other two punctures. Note that, by our choice of $H_{\lambda,\theta}$, the minimum y_θ^- of the C^2-small Morse function f is independent of $\theta > 0$, so we actually have $y^- \in M_{1-2\varepsilon}^{\text{in}}$. On the other hand, it also follows from our choice of $H_{\lambda,\theta}$ that $y_0^+ := \lim_{\lambda \to \infty} y_{0,\theta}^+$ and $y_1^+ := \lim_{\lambda \to \infty} y_{1,\theta}^+$ fall outside of M^{in}.

Pick any $R_\lambda \in (1 - 2\varepsilon, 1 - \varepsilon)$ so that $\lim_{\lambda \to \infty} R_\lambda = 1 - \varepsilon$, and consider the inverse image $\tilde{u}_\lambda^{-1}(\bar{M}_{R_\lambda})$. Since $y^- \in M_{1-2\varepsilon}^{\mathrm{in}}$ and $y_0^+, y_1^+ \notin M^{\mathrm{in}}$, $\tilde{u}_\lambda^{-1}(M_{R_\lambda}^{\mathrm{in}}) \subset \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ is an open punctured cylinder for some $\varepsilon > 0$ which can be taken to be sufficiently small. We will denote it by $Z_\lambda^* \subset Z$. Since $H_\lambda \equiv -\delta_\lambda$ in M^{in}, it follows that the map

$$\tilde{u}_\lambda : Z_\lambda^* \to M_{R_\lambda}^{\mathrm{in}}$$

is J-holomorphic. Moreover, we have

$$\int_{\partial \bar{Z}_\lambda^*} \tilde{u}_\lambda^* \theta_M \leq E(u_\lambda) \leq \mu_M.$$

In particular, the removable singularity theorem for pesudoholomorphic maps applies, which shows that $\tilde{u}_\lambda$ extends to a J-holomorphic map $\bar{u}_\lambda : Z_\lambda \to M_{R_\lambda}^{\mathrm{in}}$. Letting $\lambda \to \infty$, we get a J-holomorphic map $\bar{u}_\infty : Z \to M_{1-\varepsilon}^{\mathrm{in}}$ whose image passes through y^-.

Step 3: Uniruledness. The uniruledness of the Liouville domain $\bar{M}_{1-\varepsilon}$ follows by noticing that y^- can be taken to be any generic point in $M_{1-\varepsilon}^{\mathrm{in}}$. Alternatively, one can argue as follows.

A slight variation of the construction of the moduli space $\mathcal{P}(y_0^+, y_1^+; y_-)$ enables us to define $\mathcal{P}(y_0^+, y_1^+; \bar{M})$, which parametrizes maps

$$u : S \to M$$

satisfying Floer's equation, but are now asymptotic to $y_0^+, y_1^+ \in \mathcal{O}_{M,\lambda}$ at two positive ends, and $\lim_{s \to -\infty} (\varepsilon^-)^* u(s, \cdot)$ belongs to the relative fundamental cycle in $C_{2n}(\bar{M}, \partial \bar{M})$, where ε^- is the negative cylindrical end. The Gromov bordification of $\mathcal{P}(y_0^+, y_1^+; \bar{M})$ carries an evaluation map

$$\overline{\mathrm{ev}} : \bar{\mathcal{P}}(y_0^+, y_1^+; \bar{M}) \to \bar{M}$$

defined to be the asymptote at the negative puncture for every $u \in \mathcal{P}(y_0^+, y_1^+; \bar{M})$, and the coefficient before the identity $e \in \mathrm{CF}^0(2\lambda)$ under the pair-of-pants product $y_1^+ \smile y_0^+$ is defined by pushing forward the fundamental chain $[\bar{\mathcal{P}}(y_0^+, y_1^+; \bar{M})]$ via $\overline{\mathrm{ev}}$. Our assumption that h defines a non-trivial unit in $\mathrm{SH}^0(M)$ implies that, for appropriate choices of Floer data, there is an identification between $\bar{\mathcal{P}}(y_0^+, y_1^+; \bar{M})$ and $\bar{M}$ relative to the boundaries. Applying the same argument as above to every element u of $\mathcal{P}(y_0^+, y_1^+; \bar{M})$ proves that the Liouville domain $\bar{M}_{1-\varepsilon}$ is $(2, \mu_M)$-uniruled in the sense of Definition 69. It follows from Theorem 70 that M is algebraically 2-uniruled.

By Lemma 71, M cannot be of log general type. In other words, for any smooth affine variety M of log general type which contains an exact Lagrangian torus, we necessarily have

$$\mathrm{SH}^0(M)^\times \cong \mathbb{K}^\times.$$

Appealing to Corollary 40 completes our proof. ∎

Remark 72. A key point in the above proof is that any central unit in the fundamental group algebra of a torus has vanishing constant coefficient. This is actually true for the group algebra of any torsion-free group; see [50, Theorem 4.1]. Because of this, Theorem 12 can be generalized to log general-type affine varieties containing an exact Lagrangian $K(\pi, 1)$.

Note that our theorem provides an alternative way to understand Corollary 43. One can also try to prove a statement of similar flavor to Theorem 12 by making use of the *logarithmic PSS map* introduced by Ganatra–Pomerleano in [31, 32]. Under the assumption that

$$M = X \setminus D,$$

where (X, D) is a *multiplicatively topological pair* in the sense of [31], $\mathrm{SH}^0(M)$ is isomorphic to the *logarithmic cohomology* $H^*_{\log}(X, D)$ as a $\mathbb{K}$-algebra, while $H^0_{\log}(X, D)$ does not contain any non-trivial unit.

6.4. A conjectural picture

Although the results obtained in this paper are far from providing a complete classification of Liouville manifolds admitting cyclic dilations, in view of our discussions in Section 2.4, it seems to be reasonable to expect the following (note that we consider here only the case when $\mathrm{char}(\mathbb{K}) = 0$).

Conjecture 73. *Let M be an n-dimensional smooth affine variety.*

- *If $\kappa(M) = -\infty$, then M admits a cyclic dilation with $h = 1$.*
- *If $\kappa(M) = 0$, then M admits a cyclic dilation if and only if it admits a quasi-dilation with $h \neq 1$.*
- *If $\kappa(M) = n$, then M does not admit a cyclic dilation.*

Note that, in order for our conjecture to make sense, we need to regard manifolds with $\mathrm{SH}^*(M) = 0$ as manifolds which carry cyclic dilations.

The expectation that cyclic dilations should exist for all affine varieties with

$$\kappa(M) = -\infty$$

is probably too optimistic; it seems to be more reasonable to state the conjecture for all the Milnor fibers with $\kappa(M) = -\infty$. However, there are affine varieties with

$$\kappa(M) = -\infty$$

which are not Milnor fibers, but do admit cyclic dilations. As an example, consider the affine hypersurface $M \subset \mathbb{C}^4$ defined by the equation

$$x + y + xyz + w^2 = 1.$$

Since M carries a Lefschetz fibration $\pi : M \to \mathbb{C}$ with the smooth fiber being symplectomorphic to a 4-dimensional D_4 Milnor fiber (cf. [12, Section 4.1]), combining the argument in [49, Section 4.2] with the Lefschetz fibration method due to Seidel–Solomon [67] shows that M admits a quasi-dilation. This example is also interesting in the sense that the existence of an exact Calabi–Yau structure on $\mathcal{W}(M)$ does not follow from Van den Bergh's Theorem 56. Direct computations yield the quasi-isomorphism

$$\mathcal{W}_M \cong \mathbb{K}[x, y], \quad |x| = 1, \ |y| = -2;$$

see, for example, [49, Section 7.4]. Since $\mathcal{W}_M$ is formal and has generators in positive degrees, Theorem 56 is not applicable here.

The relation between the existence of a cyclic dilation and the finiteness of the first Gutt–Hutchings capacity were explained in Section 5.4. In view of Lemma 71, the first item of Conjecture 73 implies the following conjecture.

Conjecture 74. *Let M be a smooth affine variety which is algebraically 1-uniruled; then, as a Liouville manifold, we have $c_1^{\mathrm{GH}}(M) < \infty$.*

For related studies in the case of closed symplectic manifolds, see [52].

It seems likely that there is no exact Lagrangian tori in smooth affine varieties with $\kappa(M) = -\infty$. In view of Corollary 40, this provides evidence for the more precise expectation that the marking map $\mathbf{B} : \mathrm{SH}_{S^1}^1(M) \to \mathrm{SH}^0(M)$ should actually hit the identity.

Since there should be an exact Lagrangian torus in every smooth log Calabi–Yau variety, one expects that $h \neq 1$ in view of Corollary 40. In fact, this can be rigorously proved. By [78, Theorem L], if a smooth affine variety M admits a dilation, then $\bar{M}$ is $(1, \mu)$-uniruled for some $\mu > 0$ in the sense of Definition 69. In particular, $\kappa(M) = -\infty$. The same argument as in [78, Section 5] can be applied to prove the uniruledness of M by affine lines when it admits a cyclic dilation with $h = 1$.

Although this paper does not deal with affine varieties with $0 < \kappa(M) < n$, it is not difficult to find affine surfaces of log Kodaira dimension 1 which admit cyclic dilations. For example, since T^*S^1 admits a quasi-dilation, so do $T^*S^1 \times F_g$, where F_g is a once-punctured surface with genus $g \geq 2$. Note that these affine surfaces can be partially compactified to contractible affine surfaces of log Kodaira dimension 1, whose classification can be found in [71]. It is unclear whether these contractible affine surfaces admit cyclic dilations, although we know that there are non-trivial invertible elements in $\mathrm{SH}^0(M)$.

To prove the non-existence of cyclic dilations for affine varieties with $\kappa(M) = n$, one needs to exclude the possibility of having a cyclic dilation with $h \neq 1$. It seems that the argument in the proof of Theorem 12 would still be useful, but it is in general not clear how to show that $h \in \mathrm{SH}_+^0(M)$.

A. Construction of the operations $*_k$

The construction in this appendix is motivated by the *equivariant pair-of-pants product* introduced by Seidel [64]. Here, we need a slight variant of his construction for S^1-equivariant Hamiltonian Floer cohomologies. For each $k \geq 1$, we will introduce a chain-level operation $\smile_k$ which decreases the degree by $2k$. Our real goal here is to construct a parametrized version $*_k$ of the star product (A.18) on Hamiltonian Floer cohomologies, which played a role in our proof of Theorem 13.

Let $k \geq 1$ be an integer; we first define the operation $\smile_k$. Consider the 3-punctured sphere $S = S^2 \setminus \{\zeta_{\text{in},0}, \zeta_{\text{in},1}, \zeta_{\text{out}}\}$, with two of the punctures $\zeta_{\text{in},0}$ and $\zeta_{\text{in},1}$ serving as inputs and the remaining one ζ_{out} as an output. As a convention, we will take the representative of the punctured sphere so that $\zeta_{0,\text{in}} = e^{\frac{\pi i}{3}}$, $\zeta_{\text{in},1} = e^{\frac{2\pi i}{3}}$, and $\zeta_{\text{out}} = 1$, so they are equidistributed along the equator. For the purpose of developing a parametrized theory, we also need to introduce the auxiliary marked points $p_1, \ldots, p_k \in S$. We require that the marked points $\{p_1, \ldots, p_k\}$ lie in a disk centered at ζ_{out} of radius ε, and they should be strictly radially ordered in the sense of (5.12) with respect to the standard complex coordinate near $1 \in \mathbb{CP}^1$. Denote by $\mathcal{P}_k$ the moduli space of these punctured surfaces with k-marked points.

For any representative $(S, p_1, \ldots, p_k)$ of an element of $\mathcal{P}_k$, we fix cylindrical ends

$$\varepsilon_0^+, \varepsilon_1^+ : [0, \infty) \times S^1 \to S, \quad \varepsilon^- : (-\infty, 0] \times S^1 \to S$$

with coordinates $(s, t) \in \mathbb{R}_\pm \times S^1$, where ε_0^+ and ε_1^+ are positive cylindrical ends at $\zeta_{\text{in},0}$ and $\zeta_{\text{in},1}$, respectively, and ε^- is a negative cylindrical end at ζ_{out}. The choices are made here so that none of the cylindrical ends ε_0^+ and ε_1^+ contain any of the auxiliary marked points $\{p_i\}_{1 \leq i \leq k}$, and ε^- is chosen so that the negative s-direction is given by $\theta_1 = \arg(p_1)$, where again the argument is taken with respect to the local complex coordinate near the origin. In other words, the corresponding asymptotic markers $\ell_{\text{in},0}$ and $\ell_{\text{in},1}$ at $\zeta_{\text{in},0}$ and $\zeta_{\text{in},1}$ are fixed, pointing, respectively, along the arcs $\{\varepsilon_0^+(s, 0)\}$ and $\{\varepsilon_1^+(s, 0)\}$, while the asymptotic marker ℓ_{out} at ζ_{out} is allowed to vary freely, since it is required to point toward p_1. To further fix conventions, we will require that $\ell_{\text{in},0}$ points toward ζ_{out}, and $\ell_{\text{in},1}$ points toward $\zeta_{\text{in},0}$, or equivalently, these two arrows are arranged so that they point clockwise along the equator. We say that the choices of cylindrical ends ε_0^+, ε_1^+, and ε^- are *compatible with* the asymptotic markers in the sense that the positive (resp., negative) s-directions of the cylindrical ends coincide with the directions of $\ell_{\text{in},0}$ and $\ell_{\text{in},1}$ (resp., ℓ_{out}), i.e.,

$$\lim_{s \to \infty} \varepsilon_0^+(s, 1) = \ell_{\text{in},0}, \quad \lim_{s \to \infty} \varepsilon_1^+(s, 1) = \ell_{\text{in},1},$$

$$\lim_{s \to -\infty} \varepsilon^-(s, 1) = \ell_{\text{out}}.$$

The codimension 1 boundary strata of the Deligne–Mumford compactification $\bar{\mathcal{P}}_k$ is covered by the images of the natural inclusions of the following strata:

$$\bar{\mathcal{P}}_j \times \bar{\mathcal{M}}_{k-j}, \quad 0 < j \le k, \tag{A.1}$$

$$\bar{\mathcal{P}}_k^{i,i+1}, \quad 1 \le i \le k-1, \tag{A.2}$$

$$\bar{\mathcal{P}}_{k-1}^{S^1}, \tag{A.3}$$

where the strata $\mathcal{P}_k^{i,i+1}$ are the loci where $|p_i| = |p_{i+1}|$ for some i, and the stratum $\mathcal{P}_{k-1}^{S^1}$ is the locus where $|p_k| = \frac{1}{2}$. Abstractly, the moduli space $\mathcal{P}_{k-1}^{S^1}$ can be identified with $S^1 \times \mathcal{P}_{k-1}$, so its compactification is given by $S^1 \times \bar{\mathcal{P}}_{k-1}$. However, this identification is not compatible with the choice of cylindrical ends and holds only on the topological level.

Analogous to (4.6), there is a forgetful map

$$\pi^i : \mathcal{P}_k^{i,i+1} \to \mathcal{P}_{k-1} \tag{A.4}$$

for each $1 \le i \le k-1$, which forgets the auxiliary marked point p_{i+1}. Since π^i is compatible with our choices of the cylindrical ends, it extends as a map $\bar{\pi}^i : \bar{\mathcal{P}}_k^{i,i+1} \to \bar{\mathcal{P}}_{k-1}$ on the compactifications.

One can also consider the map

$$\pi^{S^1} : \mathcal{P}_{k-1}^{S^1} \to \mathcal{P}_{k-1}$$

which forgets the marked point p_k. Under the identification $\mathcal{P}_{k-1}^{S^1} \cong S^1 \times \mathcal{P}_{k-1}$, π^{S^1} is the natural projection to the second factor. However, since π^{S^1} is not compatible with the cylindrical end ε^- when $k = 1$, the identification fails when taking the choices of Floer data into account.

In order to write down the appropriate Floer equations, we need to specify our choices of Floer data on the domains. For later purposes, we will work here with Hamiltonians of the form (4.7) on the cylindrical end instead of the quadratic ones.

Definition 75. A Floer datum for a representative $(S, p_1, \ldots, p_k)$ of an element of $\mathcal{P}_k$ consists of the following choices:

- cylindrical ends ε_0^+, ε_1^+, and ε^- which are compatible with the asymptotic markers $\ell_{\mathrm{in},0}$, $\ell_{\mathrm{in},1}$, and ℓ_{out} specified above;

- a closed 1-form $\nu_S \in \Omega^1(S)$ which pulls back to dt via the maps ε_0^+, ε_1^+, and ε^-;

- a surface-dependent Hamiltonian $H_S : S \to \mathcal{H}_\ell(M)$ which is compatible with the cylindrical ends in the sense that

$$(\varepsilon_0^+)^* H_S = H_{\lambda_0,t}, \quad (\varepsilon_1^+)^* H_S = H_{\lambda_1,t}, \quad (\varepsilon^-)^* H_S = H_{\lambda_0+\lambda_1,t}$$

for some fixed choices of Hamiltonians $H_{\lambda_0,t} \in \mathcal{H}_{\lambda_0}(M)$, $H_{\lambda_1,t} \in \mathcal{H}_{\lambda_1}(M)$, and $H_{\lambda_0+\lambda_1,t} \in \mathcal{H}_{\lambda_0+\lambda_1}(M)$, where λ_0 and λ_1 are real numbers so that $\lambda_0, \lambda_1, \lambda_0 + \lambda_1 \notin \mathcal{P}_M$;

- a surface-dependent almost complex structure $J_S : S \to \mathcal{J}(M)$ which is compatible with the cylindrical ends, meaning that

$$(\varepsilon_0^+)^* J_S = (\varepsilon_1^+)^* J_S = (\varepsilon^-)^* J_S = J_t$$

for some fixed $J_t \in \mathcal{J}(M)$.

Definition 76. A universal and consistent choice of Floer data for the operations $\{\smile_k\}$ is an inductive choice of Floer datum for each $k \geq 1$ and each marked surface $(S, p_1, \ldots, p_k)$ representing a point of $\bar{\mathcal{P}}_k$, varying smoothly in $(S, p_1, \ldots, p_k)$ such that the following conditions are satisfied:

- Along the boundary strata (A.1), the Floer data should be chosen to agree with the product of Floer data chosen previously on $\bar{\mathcal{P}}_j$ and $\bar{\mathcal{M}}_{k-j}$ up to conformal equivalence. Moreover, the choices vary smoothly with respect to the gluing charts.

- Along the boundary strata (A.2), the Floer data are conformally equivalent to the ones pulled back from $\bar{\mathcal{P}}_{k-1}$ via the forgetful map $\bar{\pi}^i$.

In the above, the conformal equivalence of Floer data is defined similarly as before. More precisely, given two Floer data $(\varepsilon_{0,i}^+, \varepsilon_{1,i}^+, \alpha_{S,i}, H_{S,i}, J_{S,i})$, where $i = 1, 2$ for $(S, p_1, \ldots, p_k)$, we say that they are *conformally equivalent* if the choices of cylindrical ends coincide, and there is a constant $c > 0$ such that

$$H_{\lambda,t,1} = \frac{H_{\lambda,t,2}}{c} \circ \psi^c \quad \text{and} \quad J_{t,1} = (\psi^c)^* J_{t,2}$$

on the cylindrical ends, where the value λ is determined by the corresponding cylindrical end, namely, $\lambda = \lambda_0$ for ε_0^+, $\lambda = \lambda_1$ for ε_1^+, and $\lambda = \lambda_0 + \lambda_1$ for ε^-.

Remark 77. Unlike the boundary strata (A.1) and (A.2), we did not impose any requirements on the behavior of universal and consistent Floer data along the stratum (A.3) in the above definition. This is mainly due to the fact that, for the purposes of this paper, we do not need to analyze it and identify the contribution of (A.10) below. The same can be said for the stratum (A.16). Compare with the way we dealt with the boundary stratum (5.7) in Section 5.1.

Inductively, since the space of choices of Floer data at each level is non-empty and contractible, universal and consistent choices of Floer data exist. From now on, fix such a choice. For every $k \geq 1$, and Hamiltonian orbits $y_0^+ \in \mathcal{O}_{M,\lambda_0}$, $y_1^+ \in \mathcal{O}_{M,\lambda_1}$ and $y^- \in \mathcal{O}_{M,\lambda_0+\lambda_1}$, we can define the moduli space $\mathcal{P}_k(y_0^+, y_1^+; y^-)$ of pairs

$$((S, p_1, \ldots, p_k), u),$$

where $(S, p_1, \ldots, p_k) \in \mathscr{P}_k$, and $u : S \to M$ is a map satisfying the Floer equation

$$(du - X_{H_S} \otimes \nu_S)^{0,1} = 0$$

with respect to the domain-dependent almost complex structure J_S, which has been fixed as part of the Floer datum for $(S, p_1, \ldots, p_k)$, together with asymptotic conditions

$$\lim_{s \to +\infty} (\varepsilon_0^+)^* u(s, \cdot) = y_0^+, \quad \lim_{s \to +\infty} (\varepsilon_1^+)^* u(s, \cdot) = y_1^+,$$

$$\lim_{s \to -\infty} (\varepsilon^-)^* u(s, \cdot) = y^-.$$

The moduli space $\mathscr{P}_k(y_0^+, y_1^+; y^-)$ admits a Gromov bordification $\bar{\mathscr{P}}_k(y_0^+, y_1^+; y^-)$, whose codimension 1 boundary $\partial \bar{\mathscr{P}}_k(y_0^+, y_1^+; y^-)$ is covered by the inclusions of the following strata:

$$\bar{\mathscr{P}}_{k-j}(y_0^+, y_1^+; y) \times \bar{\mathscr{M}}_j(y; y^-), \quad 1 \le j \le k, \tag{A.5}$$

$$\bar{\mathscr{M}}(y_1^+; y) \times \bar{\mathscr{P}}_k(y_0^+, y; y^-), \tag{A.6}$$

$$\bar{\mathscr{M}}(y_0^+; y) \times \bar{\mathscr{P}}_k(y, y_1^+; y^-), \tag{A.7}$$

$$\bar{\mathscr{P}}_k(y_0^+, y_1^+; y) \times \bar{\mathscr{M}}(y; y^-), \tag{A.8}$$

$$\bar{\mathscr{P}}_k^{i,i+1}(y_0^+, y_1^+; y^-), \tag{A.9}$$

$$\bar{\mathscr{P}}_{k-1}^{S^1}(y_0^+, y_1^+; y^-), \tag{A.10}$$

where the boundary strata (A.6), (A.7), and (A.8) come from the semi-stable breaking, and the strata (A.5), (A.9), and (A.10) correspond to the boundary strata (A.1), (A.2), and (A.3) of $\partial \bar{\mathscr{P}}_k$, respectively.

For generic choices of Floer data, the moduli space $\bar{\mathscr{P}}_k(y_0^+, y_1^+; y^-)$ is a compact manifold with corners of dimension

$$\deg(y^-) - \deg(y_0^+) - \deg(y_1^+) + 2k.$$

Every rigid element of $\bar{\mathscr{P}}_k(y_0^+, y_1^+; y^-)$ gives rise to an isomorphism

$$\mu_u : o_{y_1^+} \otimes o_{y_0^+} \to o_{y^-}$$

of orientation lines. Define the operation

$$\smile_k : \mathrm{CF}^*(\lambda_1) \otimes \mathrm{CF}^*(\lambda_0) \to \mathrm{CF}^{*-2k}(\lambda_0 + \lambda_1)$$

by

$$y_1^+ \smile_k y_0^+ = \sum_{|y^-| = |y_0^+| + |y_1^+| - 2k} \sum_{((S,p_1,\ldots,p_k),u) \in \bar{\mathscr{P}}_k(y_0^+, y_1^+; y^-)} (-1)^{|y_0^+| + |y_1^+|} \mu_u,$$

where the notational convention of Remark 25 has been applied, and we have used the abbreviation $|\cdot|$ for $\deg(\cdot)$.

Similarly, by considering the same pair-of-pants surface, but now allowing the auxiliary marked points $p_1, \ldots, p_k \in S$ to vary in a small neighborhood of $\zeta_{\mathrm{in},1}$, we can define an operation

$$\smile^k : \mathrm{CF}^{*+2k}(\lambda_1) \otimes \mathrm{CF}^*(\lambda_0) \to \mathrm{CF}^*(\lambda_0 + \lambda_1).$$

Denote the moduli space of the corresponding domains by $\mathscr{P}^k$. When analyzing the boundary strata of $\bar{\mathscr{P}}^k$, note that, in order for a $(k-j)$-point angle decorated cylinder to break "above" the surface S at $\zeta_{\mathrm{in},1}$, the points $p_1, \ldots, p_k$ should be ordered so that (5.11) holds with respect to the local complex coordinate near $\zeta_{\mathrm{in},1}$. The codimension 1 boundary $\partial \bar{\mathscr{P}}^k$ is covered by the strata corresponding to (A.1), (A.2), and (A.3), and, as a consequence, the codimension 1 boundary of the Gromov compactification $\bar{\mathscr{P}}^k(y_0^+, y_1^+; y^-)$ is covered by the inclusions of the strata

$$\bar{\mathcal{M}}_j(y_1^+; y) \times \bar{\mathscr{P}}^{k-j}(y_0^+, y; y^-), \quad 1 \le j \le k, \tag{A.11}$$

$$\bar{\mathcal{M}}(y_1^+; y) \times \bar{\mathscr{P}}^k(y_0^+, y; y^-), \tag{A.12}$$

$$\bar{\mathcal{M}}(y_0^+; y) \times \bar{\mathscr{P}}^k(y, y_1^+; y^-), \tag{A.13}$$

$$\bar{\mathscr{P}}^k(y_0^+, y_1^+; y) \times \bar{\mathcal{M}}(y; y^-), \tag{A.14}$$

$$\bar{\mathscr{P}}^k_{i,i+1}(y_0^+, y_1^+; y^-), \tag{A.15}$$

$$\bar{\mathscr{P}}^{k-1}_{S^1}(y_0^+, y_1^+; y^-), \tag{A.16}$$

which are analogous to (A.5) to (A.10), respectively. A signed count of rigid elements in the moduli space $\bar{\mathscr{P}}^{k-1}_{S^1}(y_0^+, y_1^+; y^-)$ gives rise to an operation

$$\smile^{k-1}_{S^1} : \mathrm{CF}^{*+2k}(\lambda_1) \otimes \mathrm{CF}^*(\lambda_0) \to \mathrm{CF}^{*+1}(\lambda_0 + \lambda_1),$$

and we have the following proposition.

Proposition 78. *As chain-level operations,*

$$\sum_{j=0}^{k} \delta_j(y_1^+) \smile^{k-j} y_0^+ = d(y_1^+ \smile^k y_0^+) - (-1)^{|y_1^+|} y_1^+ \smile^k dy_0^+$$

$$+ y_1^+ \smile^{k-1}_{S^1} y_0^+. \tag{A.17}$$

Proof. As in (A.4), there is a forgetful map $\pi_i : \mathscr{P}^k_{i,i+1} \to \mathscr{P}^{k-1}$, which leaves out the point p_{i+1} and extends to a map $\bar{\pi}_i$ defined on the compactifications. The universality and consistency of our choices of Floer data when defining the operations $\smile^k$ imply that the Floer data chosen for elements $(S, p_1, \ldots, p_k) \in \bar{\mathscr{P}}^k_{i,i+1}$ depend only on their images under $\bar{\pi}_i$. Since the forgetful map $\bar{\pi}_i$ has 1-dimensional fibers, given a
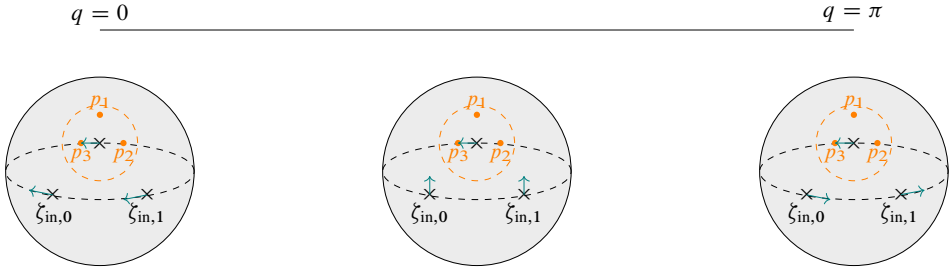
$q = 0$ $q = \pi$

Figure 11. By fixing the points p_1, p_2, p_3, we get a slice of the moduli space $\mathcal{P}_{q,3}$.

representative $((S, p_1, \ldots, p_k), u)$ of an element of $\bar{\mathcal{P}}^k_{i,i+1}(y_0^+, y_1^+; y^-)$, any other point $(S', p'_1, \ldots, p'_k) \in \mathcal{P}^k_{i,i+1}$ in the same fiber as $(S, p_1, \ldots, p_k)$ defines another representative $((S', p'_1, \ldots, p'_k), u)$ of an element of $\bar{\mathcal{P}}^k_{i,i+1}(y_0^+, y_1^+; y^-)$. This shows that the elements of the moduli space $\bar{\mathcal{P}}^k_{i,i+1}(y_0^+, y_1^+; y^-)$ are never rigid.

Thus, we only need to consider the boundary strata (A.11), (A.12), (A.13), (A.14), and (A.16). Standard breaking analysis then implies (A.17). ∎

Given a family of Riemann surfaces (S_q) parametrized by $q \in [0, \pi]$ so that each S_q is a sphere with three fixed punctures $\zeta_{\text{in},0}$, $\zeta_{\text{in},1}$, and ζ_{out} equidistributed along the equator. As above, these punctures are equipped with asymptotic markers $\ell_{\text{in},0}$, $\ell_{\text{in},1}$, and ℓ_{out}. When $q = 0$, $\ell_{\text{in},0}$ points toward ζ_{out}, $\ell_{\text{in},1}$ points away from ζ_{out}, and ℓ_{out} points toward $\zeta_{\text{in},1}$. As q varies from 0 to π, all of the asymptotic markers perform a half-turn: anticlockwise for the output and clockwise for the inputs. These are domains defining the operation

$$* : \mathrm{CF}^*(\lambda_1) \otimes \mathrm{CF}^*(\lambda_0) \to \mathrm{CF}^{*-1}(\lambda_0 + \lambda_1). \tag{A.18}$$

By symmetrizing it, one gets the familiar Lie bracket $[\cdot, \cdot]$ on Hamiltonian Floer cohomologies, which, after passing to direct limit, equips $\mathrm{SH}^*(M)$ with the structure of a Gerstenhaber algebra.

In view of the discussions above, one may expect to have higher-order analogs $*_k$, where $k \geq 1$ of the operation $*$. To define these maps, one considers the domains $(S_q, p_1, \ldots, p_k)$, the family of punctured spheres (S_q) together with k auxiliary marked points $p_1, \ldots, p_k$ lying in a small neighborhood of ζ_{out}. If we fix ζ_{out} at the origin, these points should be strictly radially ordered as in (4.22). Moreover, when q goes from 0 to π, the asymptotic markers $\ell_{\text{in},0}$ and $\ell_{\text{in},1}$ are required to rotate clockwise by an angle of π, while ℓ_{out} is required to point toward p_1, so it is free to vary. Figure 11 describes the family of domains defining $*_3$ with the positions of the marked points p_1, p_2, p_3 being fixed. If we allow the marked points $p_1, \ldots, p_k$ to vary under the constraint (4.22), we will get a $(2k + 1)$-dimensional family of marked surfaces, which fibers over $[0, \pi]$ and whose fibers are topologically $\mathbb{R}^k \times T^k$.

For a fixed $q \in [0, \pi]$, the moduli space of marked surfaces $(S_q, p_1, \ldots, p_k)$ will be denoted by $\mathcal{P}_{q,k}$. The choice of Floer data on its Deligne–Mumford compactification $\overline{\mathcal{P}}_{q,k}$ will be essentially the same as that on $\overline{\mathcal{P}}_k$, with the only exception that the choices of the cylindrical ends $\varepsilon_{q,0}^+$ and $\varepsilon_{q,1}^+$ differ for different q. It is not hard to arrange the choices of the asymptotic markers at $\zeta_{\mathrm{in},0}$ and $\zeta_{\mathrm{in},1}$ so that the moduli space $\mathcal{P}_{0,k}$ is identical to $\mathcal{P}_k$ together with their Floer data.

Recall that, as part of our Floer datum chosen for $(S_0, p_1, \ldots, p_k)$, there are a domain-dependent Hamiltonian function

$$H_{S_0} : S_0 \to \mathcal{H}_\ell(M)$$

and a domain-dependent almost complex structure $J_{S_0} : S_0 \to \mathcal{J}(M)$. On the cylindrical ends, the Floer datum pulls back to

$$(\varepsilon_{0,0}^+)^* H_{S_0} = H_{\lambda_0, t}, \quad (\varepsilon_{0,1}^+)^* H_{S_0} = H_{\lambda_1, t}, \quad (\varepsilon_0^-)^* H_{S_0} = H_{\lambda_0 + \lambda_1, t},$$
$$(\varepsilon_{0,0}^+)^* J_{S_0} = (\varepsilon_{0,1}^+)^* J_{S_0} = (\varepsilon_0^-)^* J_{S_0} = J_t,$$

where $H_{\lambda_0, t}, H_{\lambda_1, t}, H_{\lambda_0 + \lambda_1, t} \in \mathcal{H}_\ell(M)$, and $J_t \in \mathcal{J}(M)$.

Definition 79. In general, the Floer datum for a representative $(S_q, p_1, \ldots, p_k)$ of $\mathcal{P}_{q,k}$ consists of

- the choices of two positive cylindrical ends $\varepsilon_{0,q}^+$, $\varepsilon_{1,q}^+$ and a negative cylindrical end ε_q^-, which are compatible, respectively, with the asymptotic markers $\ell_{\mathrm{in},0}$, $\ell_{\mathrm{in},1}$, which depend on $q \in [0, \pi]$, and ℓ_{out}, which points to p_1;

- a closed 1-form $\nu_{S_q} \in \Omega^1(S_q)$ which pulls back to dt under $\varepsilon_{0,q}^+$, $\varepsilon_{1,q}^+$ and ε_q^-;

- a surface-dependent Hamiltonian $H_{S_q} : S_q \to \mathcal{H}_\ell(M)$ which satisfies

$$(\varepsilon_{0,q}^+)^* H_{S_q} = H_{\lambda_0, t+q}, \quad (\varepsilon_{1,q}^+)^* H_{S_q} = H_{\lambda_1, t+q}, \quad (\varepsilon_q^-)^* H_{S_q} = H_{\lambda_1 + \lambda_2, t+q};$$

- a surface-dependent almost complex structure $J_{S_q} : S_q \to \mathcal{J}(M)$ so that

$$(\varepsilon_{0,q}^+)^* J_{S_q} = (\varepsilon_{1,q}^+)^* J_{S_q} = (\varepsilon_q^-)^* J_{S_q} = J_{t+q}.$$

Note that, for general $q \in [0, \pi]$, there is an obvious identification $\mathcal{P}_{q,k} \cong \mathcal{P}_k$ given by rotating the asymptotic markers $\ell_{\mathrm{in},0}$ and $\ell_{\mathrm{in},1}$ by a certain angle. Although the identification does not preserve Floer data, this shows that the codimension 1 boundary strata of $\overline{\mathcal{P}}_{q,k}$ correspond exactly to those given in (A.1) to (A.3).

The moduli space of domains $\mathcal{S}_k$ defining the operation $*_k$ has an additional parameter q, namely,

$$\mathcal{S}_k := \bigsqcup_{q \in [0,\pi]} \mathcal{P}_{q,k}.$$

Abstractly, S_k can be identified with $[0, \pi] \times \mathcal{P}_k$, but as we have explained above, this identification is not compatible with Floer data. Denote by $\bar{S}_k$ the Deligne–Mumford compactification of S_k. The previous identification extends to one on the compactifications

$$\bar{S}_k \cong [0, \pi] \times \bar{\mathcal{P}}_k.$$

Definition 80. A universal and consistent choice of Floer data for the moduli spaces $\{\bar{S}_k\}_{k \geq 1}$ is an inductive choice of Floer data for each $k \geq 1$, each $q \in [0, \pi]$, and each representative $(S_q, p_1, \ldots, p_k)$ of an element of $\bar{\mathcal{P}}_{q,k}$ in the sense of Definition 79 so that

- when restricted to the slice $\bar{\mathcal{P}}_{q,k} \subset \bar{S}_k$, the Floer data should be universal and consistent in the same sense as in Definition 76;

- the Floer data on $\bar{\mathcal{P}}_{q,k}$ should vary smoothly with respect to the parameter q;

- the Floer data on $\bar{\mathcal{P}}_{0,k}$ coincides with those on $\bar{\mathcal{P}}_k$ under the obvious identification $\bar{\mathcal{P}}_{0,k} \cong \bar{\mathcal{P}}_k$;

- the Floer data on the moduli spaces $\bar{\mathcal{P}}_{q,k}$ should be chosen so that for any representative of an element $(S_0, p_1, \ldots, p_k) \in \mathcal{P}_{0,k}$, the Floer datum coincides with the one obtained by pulling back of that on $(S_\pi, \mathrm{sw}(p_1), \ldots, \mathrm{sw}(p_k)) \in \mathcal{P}_{\pi,k}$ via the automorphism $\mathrm{sw} : S^2 \to S^2$ of the sphere which swaps the two inputs $\zeta_{\mathrm{in},0}$ and $\zeta_{\mathrm{in},1}$ and preserves the output ζ_{out}.

Pick a universal and consistent choice of Floer data. For $y_0^+ \in \mathcal{O}_{M,\lambda_0}$, $y_1^+ \in \mathcal{O}_{M,\lambda_1}$, and $y^- \in \mathcal{O}_{M,\lambda_0+\lambda_1}$, we define for each $k \geq 1$ the moduli space $S_k(y_0^+, y_1^+; y^-)$, which parametrizes triples $(q, (S_q, p_1, \ldots, p_k), u)$, with $q \in [0, \pi]$, $(S_q, p_1, \ldots, p_k) \in \mathcal{P}_{q,k}$ and the map $u : S_q \to M$ is a solution of Floer's equation

$$(du - X_{H_{S_q}} \otimes \nu_{S_q})^{0,1} = 0,$$

whose behavior at infinity is controlled by the asymptotic conditions

$$\lim_{s \to +\infty} (\varepsilon_{0,q}^+)^* u(s, \cdot) = y_0^+, \quad \lim_{s \to +\infty} (\varepsilon_{1,q}^+)^* u(s, \cdot) = y_1^+,$$

$$\lim_{s \to -\infty} (\varepsilon_q^-)^* u(s, \cdot) = y^-.$$

Denote by $\bar{S}_k(y_0^+, y_1^+; y^-)$ the Gromov bordification of $S_k(y_0^+, y_1^+; y^-)$. For generic choices of Floer data, $\bar{S}_k(y_0^+, y_1^+; y^-)$ is a compact manifold with corners of dimension

$$\deg(y^-) - \deg(y_0^+) - \deg(y_1^+) + 2k + 1.$$

A signed count of rigid elements of $\bar{S}_k(y_0^+, y_1^+; y^-)$ defines the map

$$*_k : \mathrm{CF}^*(\lambda_1) \otimes \mathrm{CF}^*(\lambda_0) \to \mathrm{CF}^{*-2k-1}(\lambda_0 + \lambda_1).$$

Just as the usual star product $*$ measures the homotopy commutativity of the usual pair-of-pants product $\smile$, with our choices of Floer data as in Definition 80, there is a relation between the operations $*_k$ and $\smile_k$ as well, but we are not going to discuss it here.

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